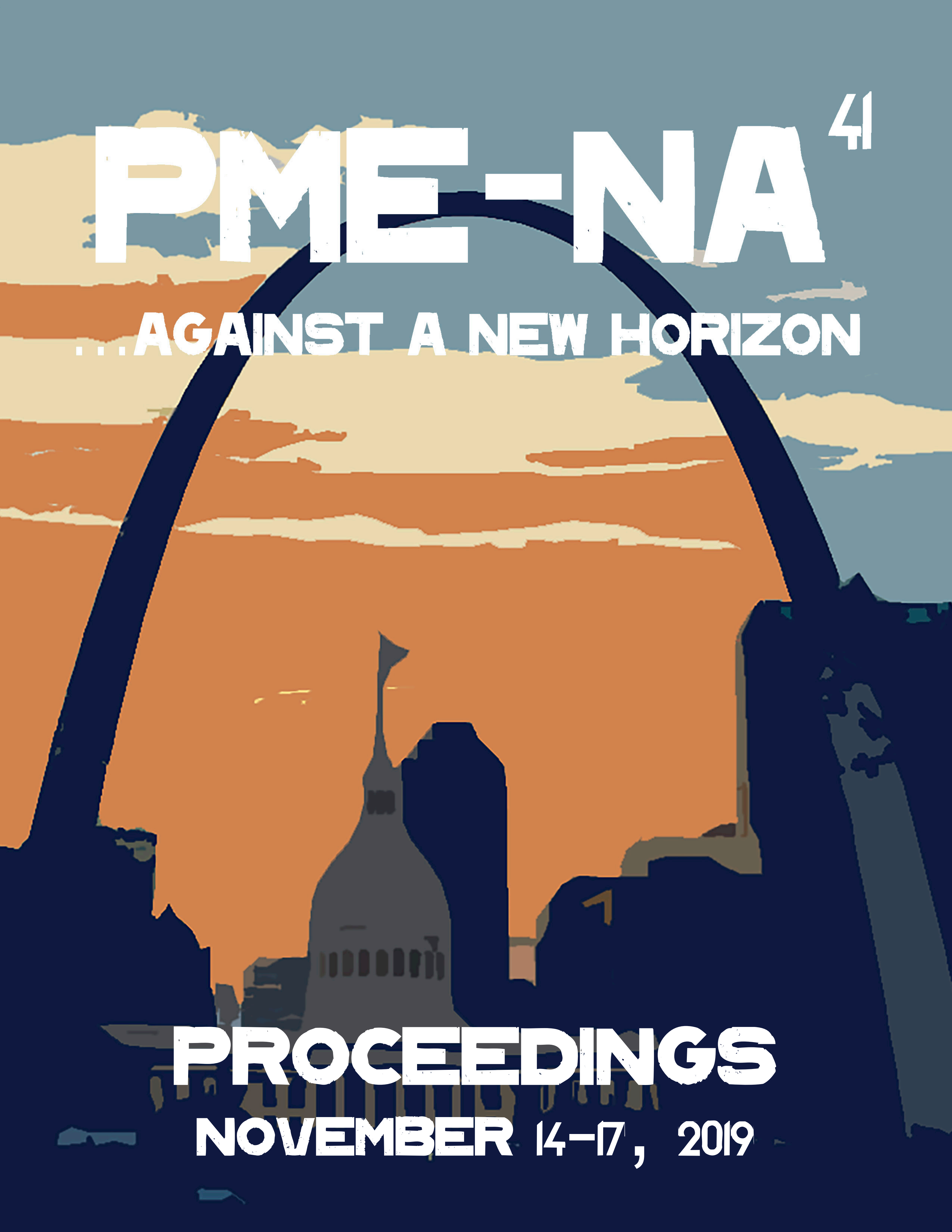
PME-NA 41
...AGAINST A NEW HORIZON
PROCEEDINGS
NOVEMBER 14-17, 2019

Proceedings of the Forty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education

...against a new horizon

St Louis, MO, USA
November 14–17, 2019

Editors

Samuel Otten
University of Missouri at Columbia
ottensa@missouri.edu

Amber G. Candela
University of Missouri at St Louis
candelaa@umsl.edu

Zandra de Araujo
University of Missouri at Columbia
dearaujoz@missouri.edu

Cara Haines
University of Missouri at Columbia
carahaines@mail.missouri.edu

Charles Munter
University of Missouri at Columbia
munterc@missouri.edu

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Land Acknowledgment

The Local Organizing Committee acknowledges the Osage, Sioux, and Miami Tribes, among others, on whose ancient and sacred land we held this conference. We note that the state colonially known as “Missouri” has no federally-recognized Native American reservations, underscoring the profound extent of the genocide, forced displacement, and cultural erasure of indigenous peoples. As a PME-NA community we recognize the ever-present systemic inequities that stem directly from past wrongdoings and we commit ourselves indefinitely to respecting and reconciling this long history of injustice.

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PME came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsrühe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction. PME-NA is the North American Chapter of PME. The first PME-NA conference was held in Evanston, Illinois in 1979. Since their origins, PME and PME-NA have expanded and continue to expand beyond their psychologically-oriented foundations.

The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers; and
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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amt23@clemson.edu

George Roy
University of South Carolina
roygj@mailbox.sc.edu

Tommy Hodges
University of South Carolina
hodgeste@mailbox.sc.edu

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Samuel Otten
University of Missouri at Columbia
ottensa@missouri.edu

Amber G. Candela
University of Missouri at St Louis
candelaa@umsl.edu

Zandra de Araujo
University of Missouri at Columbia
dearaujoz@missouri.edu

Cara Haines
University of Missouri at Columbia
carahaines@mail.missouri.edu

Chuck Munter
University of Missouri at Columbia
munterc@missouri.edu

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Center for Research and Advanced Studies (Cinvestav), Mexico
asacrist@cinvestav.mx

José Carlos Cortés-Zavala
AMIUTEM / Universidad Michoacana de San Nicolaìs de Hidalgo, Mexico
jcortes@umich.mx

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PME-NA 2019 Contributors

Local Organizing Committee

Samuel Otten, *University of Missouri at Columbia*
Amber G. Candela, *University of Missouri at St Louis*
Zandra de Araujo, *University of Missouri at Columbia*
Cara Haines, *University of Missouri at Columbia*
Charles Munter, *University of Missouri at Columbia*

pmenaLOC@gmail.com

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Preface

Dear Colleagues,

On behalf of the 2019 PME-NA Steering Committee, the 2019 PME-NA Local Organizing Committee, and the University of Missouri (Columbia and St. Louis), we welcome you to Saint Louis, Missouri, USA, for the Forty-First Annual Meeting of the International Group for the Psychology of Mathematics Education – North American Chapter, held at the Hyatt Regency St. Louis at the Arch.

This year's conference theme is **"...against a new horizon."** We decided on an intentionally intriguing and possibly provocative motto, taken from Vice President Hubert Humphrey's 1968 dedication of the St. Louis Arch. Originally, the phrase was intended to invoke the backdrop of "westward expansion" behind the newly constructed and picturesque arch. But for the Indigenous people of the continent, that development meant displacement and death. And the very erecting of a monument on the banks of the Mississippi River also displaced many African American residents and businesses in St. Louis. From perspectives within these nations and communities, the expansion was likely something to be "against."

These differing views on "expansion" mirror the differing views in relation to many issues our country generally, and our field of mathematics education in particular, face today. Events that some view as a positive new horizon may actually constitute a form of oppression for others... something to be against. We acknowledge our work is not done in isolation but rather in spaces with others. We acknowledge the complexity of the space this conference occupies and encourage attendees to learn more about St. Louis's past and present as we connect those events to our respective contexts. (For further information on Missouri's history in relation to Civil Rights, visit http://mohistory.org/exhibits/1civilrights/.)

We hope this conference serves to provoke open and critical dialogue from which we all will learn. To this end, we have reconceptualized the plenary sessions. Rather than using the plenary sessions and communal meals to share and discuss empirical research, we are inviting a variety of unique voices to help spur innovation and reflection on what contested ideas such as "expansion" and "growth" might mean in mathematics education from different perspectives.

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

This year's conference will be attended by more than 500 researchers, faculty members, and graduate students from around the world including Canada, Mexico, Turkey, Korea, and nearly every state in the USA. Each paper was reviewed by multiple referees in a double-blind process. The result was an overall acceptance rate of 68%, with 32% of research report submissions accepted as research reports, 53% of brief research report submissions accepted as brief research reports, 65% of poster submission accepted as posters, and 81% of working group submissions accepted. The papers eventually accepted comprised 67 research reports, 184 brief research reports, 126 Posters, and 13 Working Groups.

For this conference we created new strands and reframed others. We reintroduced equity and justice as a strand (in addition to its presence in the submissions as keywords) and precalculus, calculus, and higher mathematics as well as newly adding instructional leadership, policy, and institutions/systems. We merged technology with curriculum and statistics & probability with geometry. We also added miscellaneous topics to the theory and research strand. The most popular strand continues to be preservice teacher education.

We thank the many people who generously volunteered their time over the past year in preparation for this conference. We thank Helen Novielli for her assistance with formatting the conference proceedings and we have extreme gratitude for the work of Kate Stottle in helping us secure the conference venue, lead the registration process, create the conference app, and generally making everything run smoothly.

We hope this conference will establish a space to look forward in our field, perhaps toward a new horizon that we can all be *for*, clear-eyed and in solidarity. :-P

Cheers,
Sam Otten, Amber Candela, Zandra de Araujo, Cara Haines, & Chuck Munter

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Chapter 1:
Plenaries

BUILDING REGIONAL BRAIN REGIMES TO SUPPORT MATHEMATICS ATTAINMENT

William F. Tate IV
Washington University – St. Louis
wtate@wustl.edu

Author's note: This paper is based in part on "Beyond Education Triage: Building Brain Regimes in Metropolitan America," in *Facing Segregation: Housing Policy Solutions for a Stronger Society*. This paper was adapted from an address given during Facing Segregation: Building Strategies in Every Neighborhood, the 2019 annual conference of the Metropolitan St. Louis Equal Housing and Opportunity Council, on April 12, 2019, at Central Baptist Church, St. Louis, Missouri. The paper was presented through a partnership between the Center for Social Development at Washington University in St. Louis and the council. An earlier version of this paper appeared in Tate (2019a). A version of this paper served as the foundational remarks for the opening plenary session of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

Thank you for the opportunity to speak with you briefly. I begin my remarks by defining a term. Usually we hear the term *regime* and we think of something negative. But regimes vary. Positive and negative regimes exist. Here in St. Louis, we experience an example of a negative regime. We call it a segregation regime. A regime equates with a public–private arrangement organized to make and execute governing decisions (Stone, 1989). In my opinion, this city and many other urban cities put together public–private partnerships arranged and orchestrated in such a way that they are elite and effective. They influenced the community in a negative fashion, but in light of their goals proved effective. It's possible to harm, yet produce the desired result. That's extremely important to understand.

We also have here a great sports regime; in fact, we have public–private partnerships that support the advancement of our teams. Recognized across the United States and the globe, the Cardinals offer an example of an exemplary sports franchise. It's an amazing sports regime. We even had a football team here called the Rams. And if you recall, we organized a public–private partnership to generate close to $500 million to keep a losing football team in St. Louis. They were losing then, and it amazed me how quickly that financial package was done—how well it was done. To create what? A giant house. We organized a public–private partnership, a sports regime, to create a giant house to keep a sport associated with brain injury (Mez et al., 2017). Brain injuries result from more than physical contact. Our environment harms and protects the brain. Ultimately, I raise this question for every city across this country: Why can't we organize public–private partnerships to protect and to nurture the brains of our children? It should be easy. We do it for sports. We are elite at segregation regime building. Why can't we do it to protect the babies and children?

Theoretical frames guide. I'm going to illustrate my guiding theory in quick fashion: communities matter. Communities interact with schools, and there's a symbiotic relationship between schools and communities. On average, high performing and functional schools exist in "good" communities and low performing schools operate in less desirable communities.

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

Everybody knows it; the real estate agents preach it; and that's the way it goes, right? And if students reside in a community associated with fewer protective factors, there's a residual product. They experience increased risk of school dropout, unemployment, poor health, and other less-positive developmental outcomes. These factors increase the likelihood of ending up subsidized by the government's social welfare or criminal justice programs. This pathway operates in circular fashion feeding back into the community. That's the negative pattern of Figure 1.

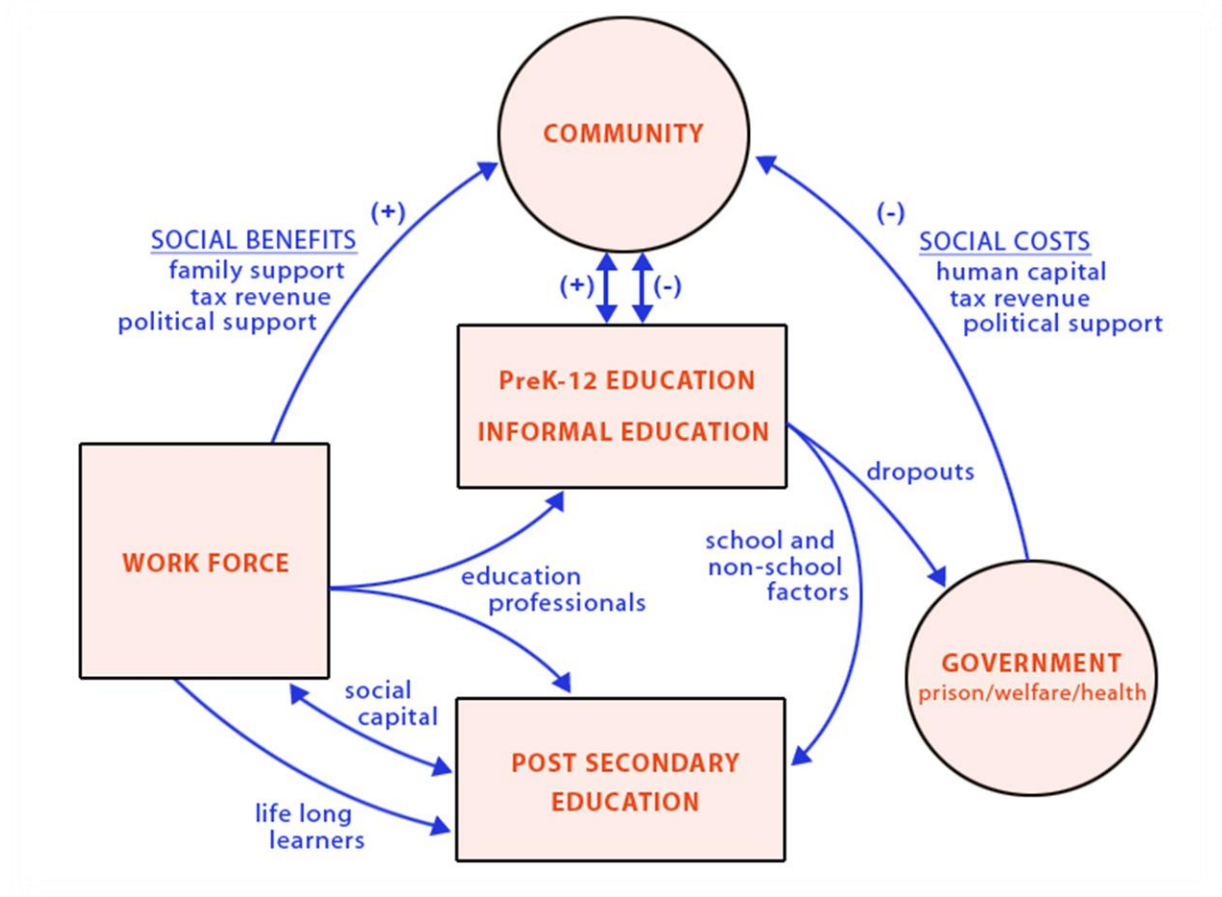

Figure 1. Conceptual model of community, education, and life course outcomes

In contrast, the other pathway generates a net positive result. And these communities stay intact over time. In the communities where the institutions work well, the schools work well; the young persons end up in postsecondary education or network into jobs; they become politically active; they generate positive tax revenue; and the residents with this experience return to live and to maintain the same kind of communities. This pattern replicates itself over and over. These two—communities and schools—are married. Rarely does the pattern change without massive intervention (Johnson, 2012).

From my perspective, this matter involves geography. I'm interested in how geography works and whether or not the first law of geography captures how opportunity organizes in our lives. Some of you might say, What is the first law of geography? It's straightforward. The first

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law of geography is that near things are more similar than distant things (Tobler, 1970). Things that are close together are very much alike. It's a very, very clean conceptual idea. What does that mean for residents in Missouri and St. Louis?

Figure 2 is a map with an equation at the top:

$$y_1 = B_o(u_1, v_1) + B_1(u_1, v_1)x_{i1} + B_k(u_1, v_1)x_{ik} + \varepsilon_i$$

To understand the idea behind the equation, we need Siri. Has anyone here asked Siri what the weather is in Missouri? You can do it right now.

What did she say? Did she tell you the weather in Jefferson City, in Kansas City, or in another metro region? I have never heard Siri give a weather report for the state of Missouri. You know why? State weather reports are less useful for most decisions. And Siri operates just like my equation; it gives you a local report on conditions.

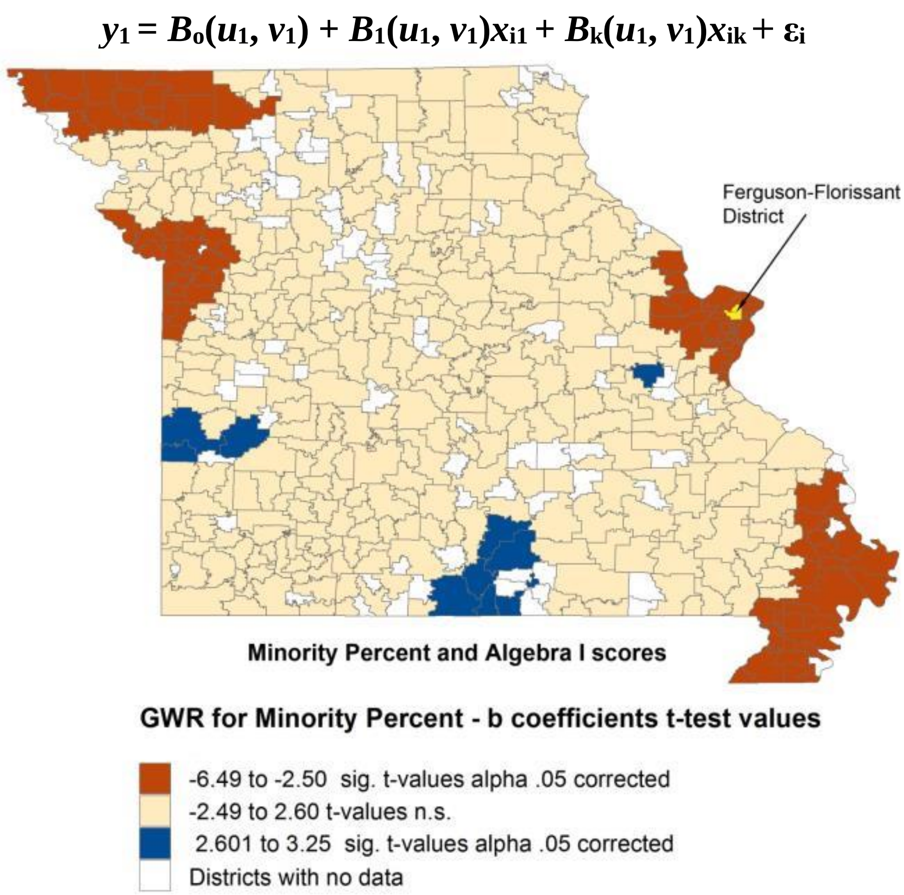

Figure 2. Minority percentages and Algebra 1 scores. GWR = geographically weighted regression. Adapted from "Place, Poverty, and Algebra: A Statewide Comparative Spatial Analysis of Variable Relationships," by M. C. Hogrebe and W. F. Tate IV, 2012, Journal of Mathematics Education at Teachers College, 3, p. 18. Copyright 2012 by the Program in Mathematics and Education, Teachers College, Columbia University in the City of New York. Adapted under Creative Commons Attribution 4.0 International Public License.

I want to know how my neighbor is doing. My equation tells us how we're doing in our neighborhood. Initially, I examined two variables. My interests include mathematics attainment. I investigated the relationship between the percentage of minority students in schools and algebra scores, and I modeled the relationship using my equation to get the local report in the context of the state of Missouri. And I discovered that the first law of geography captures the relationship—similar relationships cluster. The map illustrates that in the St. Louis metro region. There is a negative relationship between the percentages of minority students in schools and algebra performance: the more minorities, the lower the score.

And notice that, across the state, Kansas City looks the same way. And strikingly, so does our rural neighbor, the Bootheel. I don't have time to talk about the Bootheel, but I want to point out that the Bootheel and many of our urban communities have many similarities. We just haven't figured out politically how to operate together. The urban and rural divide represents a political shortcoming in light of the empirical evidence.

I'm very interested also in how students experience school discipline. Research demonstrates a relationship between school discipline and mathematics achievement (Lacoe & Steinberg, 2019). I sought to determine the relationship between school discipline and mathematics attainment in Missouri. Does clustering exist? Do these things happen in tandem? We have to think about youth in terms of layers: They're in our schools; they're in residential housing; they're in communities; and so on. And here in the state of Missouri, we have one of the largest gaps between Blacks and Whites in suspensions. In fact, as Figure 3 shows, our region experiences many of the largest racial disparities in the country.

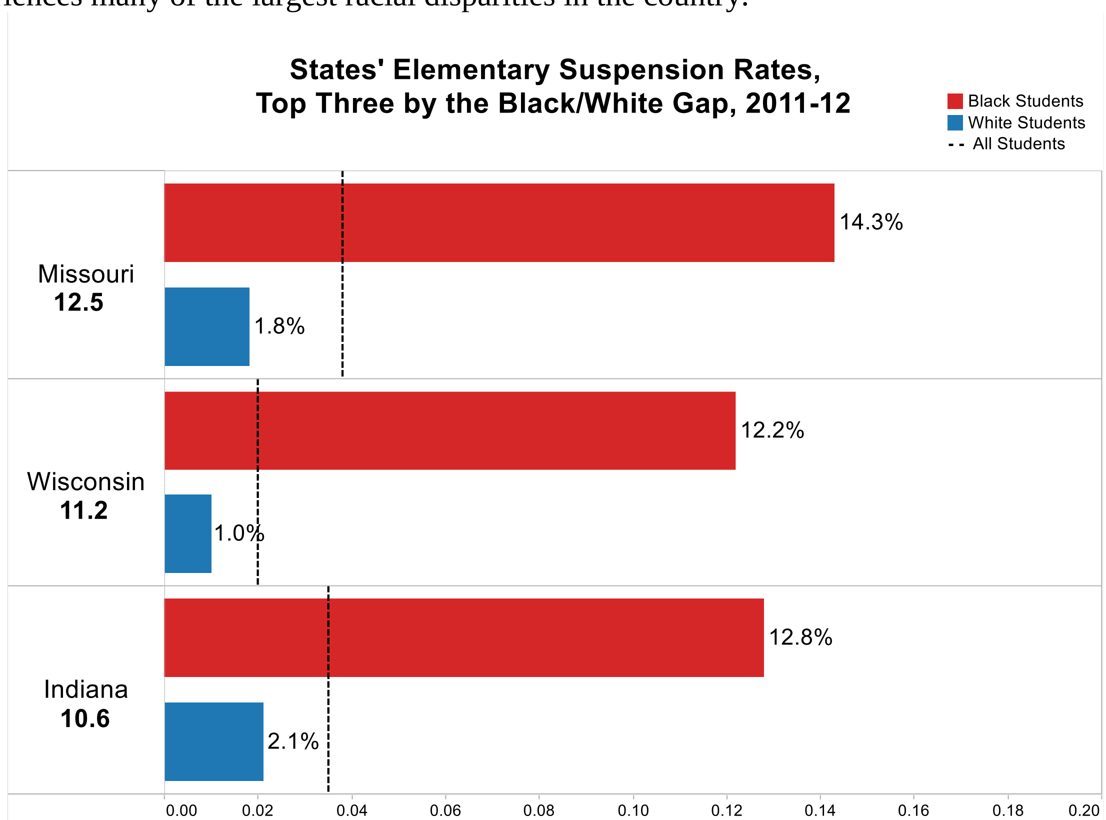

Figure 3. Adapted from "Ferguson and Beyond: A Descriptive Epidemiological Study Using Geospatial Analysis," by B. D. Jones, K. M. Harris, and W. F. Tate IV, 2015, *Journal of Negro Education*, 84, no. 3, p. 235. Copyright 2015 by the *Journal of Negro Education*.

Of the six highest suspending districts in the United States of America, three are in metropolitan St. Louis. That's important because it's geographically aligned. It's clustered. Are you following the pattern? The first law of geography applies. Near things appear more similar.

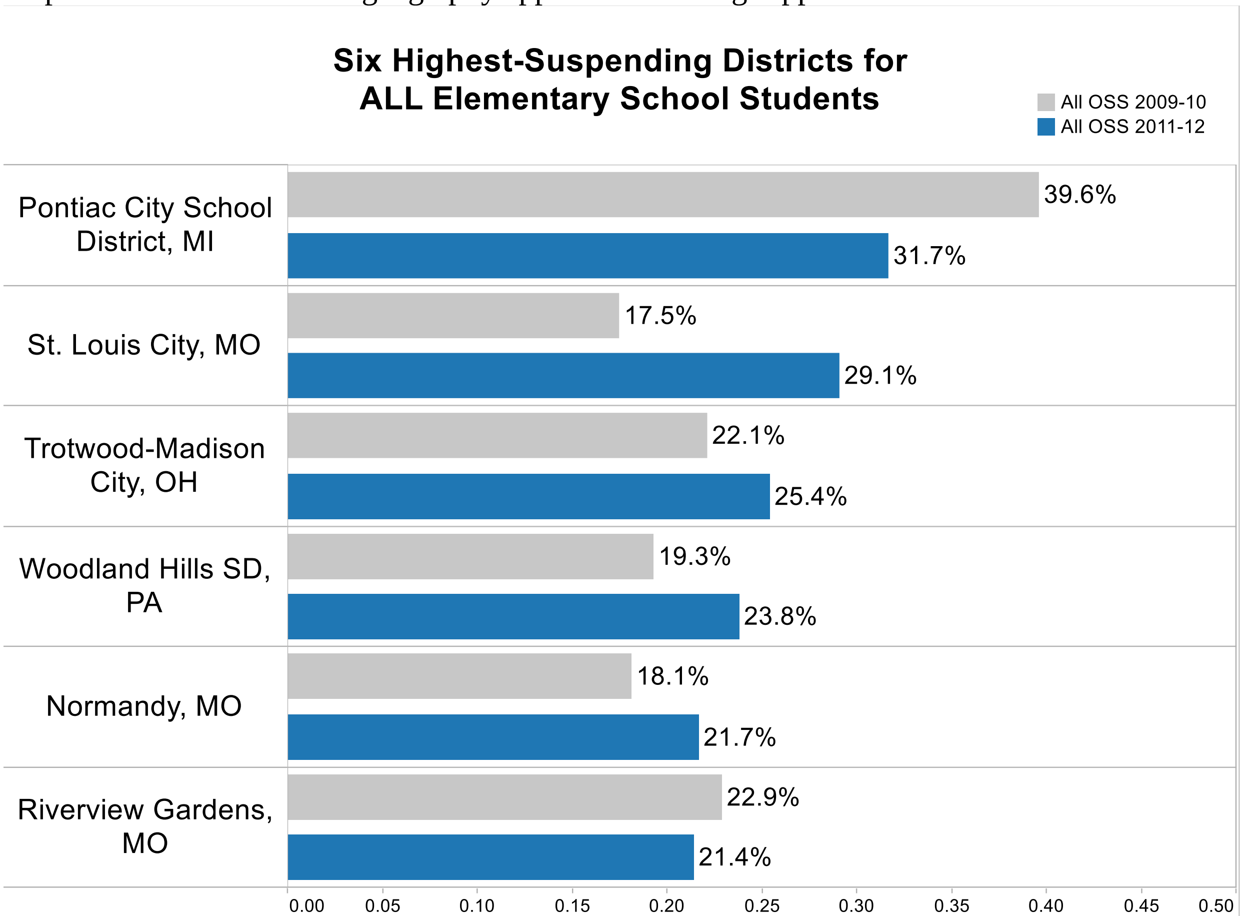

Figure 4. OSS = out of school suspensions. Adapted from "Ferguson and Beyond: A Descriptive Epidemiological Study Using Geospatial Analysis," by B. D. Jones, K. M. Harris, and W. F. Tate IV, 2015, *Journal of Negro Education*, 84, no. 3, p. 235. Copyright 2015 by the *Journal of Negro Education*.

Now remember my question of interest focuses on mathematics attainment, so I want to know whether a relationship between learning algebra and discipline exists. Figure 5 illustrates the relationship between discipline in schools and algebra performance in Missouri. A negative relationship exists with clustering in metropolitan St. Louis-the higher the discipline rates, the lower the algebra scores in the school.

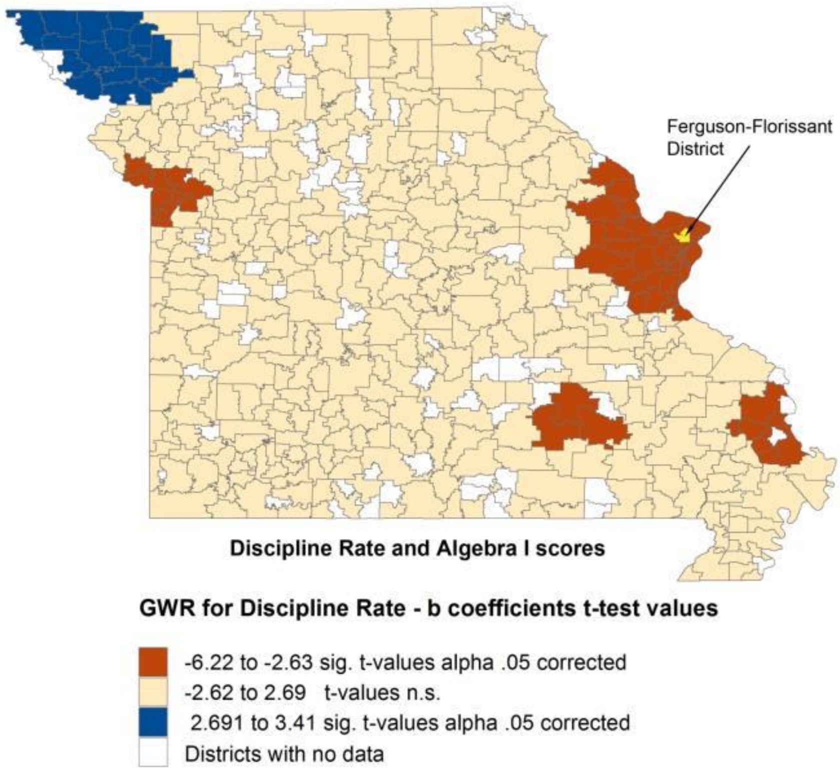

Figure 5. Adapted from "Place, Poverty, and Algebra: A Statewide Comparative Spatial Analysis of Variable Relationships," by M. C. Hogrebe and W. F. Tate IV, 2012, *Journal of Mathematics Education at Teachers College*, 3, p. 19. Copyright 2012 by the Program in Mathematics and Education, Teachers College, Columbia University in the City of New York. Adapted under Creative Commons Attribution 4.0 International Public License.

This relationship requires intervention. The *For the Sake of All* project described solutions consistent with sound developmental science (Purnell, Camberos, & Fields, 2015). Space does not permit a review of the recommendations here. My point is that we have a challenge. Housing is related to our challenge. Geography captures our challenge. And based on my initial model, if this continues, those young people won't learn algebra; the region's workforce and their academic opportunity pathways require an understanding of fundamental algebraic concepts. Our research indicates that algebraic understanding clustered geographically and without intervention will cause these young people and their communities to continue to experience economic stress. So let's think about it further. What happens when we think about the share of students receiving free- and reduced-price lunches—whether receipt is related to graduation rates? The relationship clusters. The higher the share of students that receive free and reduced lunch, in terms of distribution within the school, the lower the graduation rate. Metropolitan St. Louis and metropolitan Kansas City are the only places in the state of Missouri where that relationship is statistically significant (Tate & Hogrebe, 2015). Be clear, we should attend to all students throughout the state. However, the two metro regions offer an opportunity for targeted policy.

My colleagues and I sought to understand better the relationship between socioeconomic status and dropout rates. We examined the association between free and reduced lunch and dropout rates. Again we found that similar pattern in metropolitan St. Louis, Kansas City, and the Bootheel: a positive relationship between free and reduced lunch and the dropout rate. The higher the share of students with free and reduced lunch, the higher the dropout rate—those three areas represent the only places where that relationship is statistically significant in the state (Tate & Hogrebe, 2015).

I offer my recommendations in succinct fashion. First, we need prenatal care at scale in St. Louis. We know differentiation by race exists. It's extremely important to intervene. Second, we need a consumer report on preschool quality. Financial resources and tax incentives support; yet tax credits don't help consumers distinguish good preschools from less robust learning environments. Finding a good preschool proves especially difficult if the child's caregiver lacks the benefit of the right social networks. Third, we need to expand health insurance to families. Health insurance for children supports the care that fosters brain development and positive cognitive outcomes (Cohodes et al., 2016). I don't understand why the economic calculus is so misunderstood. The cost–benefit on this investment offers clarity. We know that if we give insurance to children only, it falls short of the robust effect associated with family coverage. These recommendations offer evidence-based practices to guide a positive regime change. We don't have to speculate.

Fourth, we need to build our teacher workforce. I work in education, and the shameful state of education saddens me. In some places, permanent substitute teachers teach students. And the students are being assessed with high stakes accountability. We know that teacher disparities abound and teacher effects are greater than school effects. If I showed you the distribution of teacher quality across the region, you would see that the disparities are stark and geographically cluster in a fashion consistent with the first law of geography (Schultz, 2014). We need to intervene.

Fifth, we need to attend to the role of artificial intelligence and the distribution of health and human services. I wrote about this topic for the *St. Louis American* (Tate, 2019b), and I expanded on it in *Diverse Issues in Higher Education* (Tate, 2019c), noting that the emerging area of algorithmic justice represents the new frontier of civil rights. Housing and residential patterns loom large in computational models informing the distribution and access to social benefits and economic opportunity. We need robust K–12 science and math education to develop stronger citizen scholars. People need to be prepared to understand how algorithms guide policy and to be taught to engage in thoughtful debate. Bob Moses called math and science the new civil rights (Moses & Cobb, 2001). He was correct. The massive funding disparities in education must be dealt with, and it's not just what the government distributes. We know that affluent families invest more money into their children than less wealthy families; and that their resource hoarding represents a foundational challenge to the democratic project (Reeves, 2017).

Sixth, we need to reclaim all the people who have been pushed out of school. We have pushed millions of people out of school in the United States, and we just ignore them.
And finally, the food deserts. My colleagues at *For the Sake of All* made some extremely important contributions (see, e.g., Cambria, Fehler, Purnell, & Schmidt, 2018; Purnell, Camberos, & Fields, 2015). I was happy to be on that team. They recognized the importance of food and nutrition as foundational to cognition, development, and human health. So if we don't deal with food deserts, the cycle of negative social outcomes articulated in my theory continues.

All of this leads me back to the point that we need a brain regime. And it's fitting that we're meeting in the backdrop of the arch landing and the Mississippi riverfront. We're going to need an effort akin to the engineering of the arch and the power of the great river-i.e., a great design and a strong will respectively.

Fannie Lou Hamer is my hero. In *Beyond the Big House,* my colleague Gloria Ladson-Billings characterized various people, including Hamer, and how they challenged the segregation regime of our society (Ladson-Billings, 2005). I will end by saying all of us need to get a little bit of the Fannie Lou. Thank you.

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CHALLENGING IDEOLOGIES, CULTIVATING COGNITION, AND REKINDLING COMMUNAL-BONDEDNESS: ADDENDUM TO "BUILDING REGIONAL BRAIN REGIMES AND SUPPORTING MATHEMATICS ATTAINMENT"

Jerome E. Morris
University of Missouri - St. Louis
morrisjer@umsl.edu

Author's note: This paper is part of an opening keynote address at the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. This paper represents the second part of a scholarly conversation with Dr. William F. Tate IV, who authored the first paper, "Building Regional Brain Regimes to Support Mathematics Attainment." Participants convened in St. Louis, Missouri, from November 14–17, 2019.

I want to thank the organizers for the opportunity to engage in this intellectual and scholarly exchange about enhancing education and specifically supporting mathematics attainment with Dr. William F. Tate--someone that I admire for his intellectual insights and ongoing contributions to the academy, the public, and the development of future scholars. As an interdisciplinary researcher whose own research examines the nexus of race, social class, and the geography of educational opportunity with a focus on the U.S. South, the examples and focus that Dr. Tate presented resonated with aspects of my research. Also, I would like to note that students from a course that I teach at the University of Missouri - St. Louis (UMSL) are in attendance. I want to acknowledge them.

A conversation about building regional brain regimes and mathematics attainment must first offer a critical understanding of rigid notions of perceived intellectual ability and the brain. Although not specifically stated in Dr. Tate's paper, enduring ideological beliefs about Black and Brown people's intellectual abilities--whether locally or globally--serve as a major impediment to building racially and gender inclusive regional brain regimes. These beliefs must be challenged. If one examines the historical literature concerning intelligence, an inference may be drawn that the fields of psychology and mathematics have maintained an allegiance to racist and sexist themes that have perpetuated the perception that Black people, Native Americans or First Nations People, women, and other groups are innately intellectually inferior to White men. In many instances, psychological research legitimated (however subjectively) overt racist themes concerning the presumed intellectual inferiority of non-European groups.

As Dr. Tate described, regimes that are effective, but not necessarily good, do in fact exist. He illustrated how sports serves as one example. Dr. Tate made the point that if political, business, and civic stakeholders can garner resources and muster the will to support sports regimes such as what was done in St. Louis in order to keep the St. Louis Rams football team in the city, then building regional brain regimes that empower people should be pursued, especially in regions such as Saint Louis. In terms of the sports example that he used, however, I would like to add that an important variable that must not be overlooked is that sports regimes are tied to economics. This kind of regime building, especially as it relates to major revenue generating spectator sports such as basketball and football, is predicated on the accrual of wealth via the disproportionate reliance on Black people's bodies (Hawkins, 2010).

Sports regimes are pervasive in the U.S. and play a critical role in creating economic prosperity for some and illusions of opportunities for others. For example, although schools, colleges, and universities are supposed to support students' brain development, administrators and systems eagerly support collision sports such as football that contribute to brain injury. Why? Because high schools and universities use the publicity from major revenue generating spectator sports such as football and basketball to heighten institutions' branding, and thereby, increase student enrollment and alumni donations (Adeyemo & Morris, in press; Donnor, 2005). On the other hand, Black males who disproportionately play revenue generating spectator sports at many of these top academic and athletic institutions graduate at lower rates than their athletic and non-athletic peers (Harper, 2016). An article by philosophy and education professor Randall Curren and law professor J.C. Blokhuis note how:

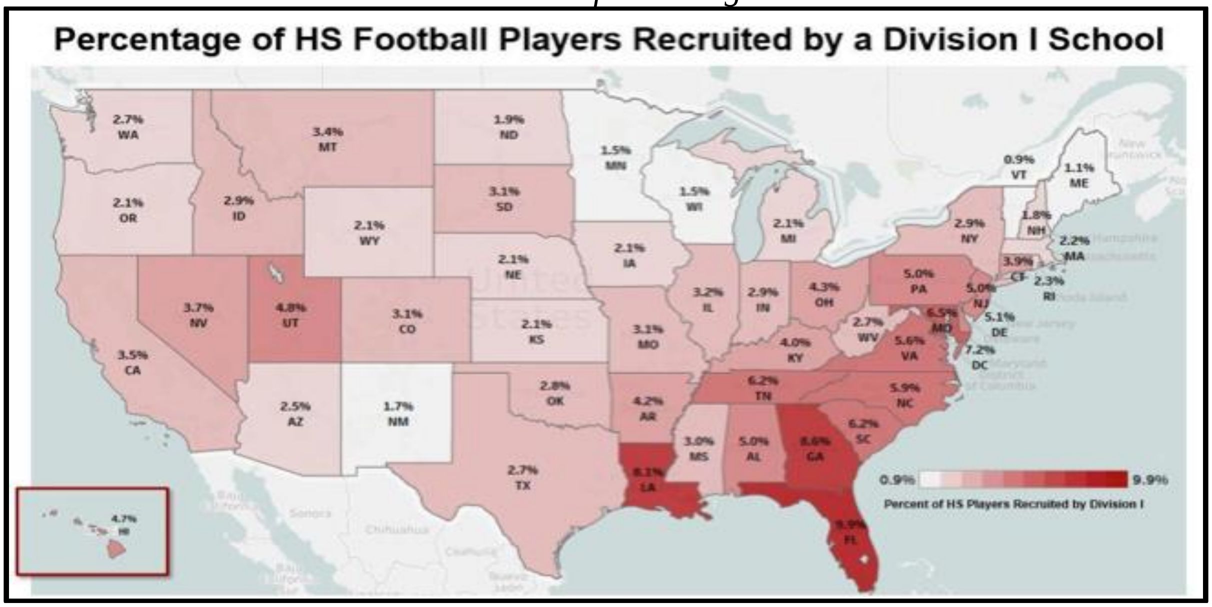
> U.S. public schools' sponsorship of tackle football is ethically indefensible and inconsistent with their educational aims. Their [Randall & Blojuis'] argument relies on three ethical principles and a growing body of evidence that many students who play football suffer traumatic brain injury and cognitive impairment that undermine their academic success and life prospects, whether or not they suffer concussions (p. 141, Curren & Blokhuis, 2018).

Despite such ethical dilemmas, these sports regimes experience astronomical economic growth. While racial achievement gaps often fall along gender lines and Black males find themselves struggling academically, sports regimes have found ways to ensure that significant numbers of Black males go through an educational and social system to ensure that a so called product--in the form of an athlete--becomes displayed on the field or the basketball court. This is not unlike that which occurred during enslavement. As a former college quarterback who grew up in Birmingham, Alabama, I know first hand, as well as scholarly, about the sports regime.

A former graduate student of mine who worked with me when I was on the faculty at the University of Georgia, Dr. Adeoye Adeyemo who is now a visiting assistant professor at the University of Illinois-Chicago, and I have researched how geography and race are shaping the experiences and beliefs of Black male high school students who play sports in the South and Midwest. In our forthcoming *Teachers College Record* article, we note the imperativeness of developing young Black male students who become valuable members of society as a result of their academic promise and achievements, and not solely because of their potential and ability to play sports. Too often, these students end up participating in an industry that generates billions of dollars for others, while leaving them feeling exploited once their playing days are over. I would like to further note that Dr. Adeyemo is doing some wonderful research around the beliefs and aspirations of Black male high school students who play sports.

The following map illustrates how geography and history are implicated in high school and college football sports regimes today. As Map 1 reveals, in states such as Georgia, Florida, and Louisiana, almost 10 percent of male high school students who play football become recruited by a Division I football program. What do we know about these states as they relate to geography, history, and race? First, we see the point that Dr. Tate made about the first law of geography and how nearer things are more similar than distant things (Tate, 2019; Tobler, 1970). We see that the southern states are disproportionately darker red in comparison to the other regions of the United States. A general understanding of history and geography informs us that these represent former slave states in the South and once served as vital cash crop states in order to build the economic base of the United States. It is not coincidental then that universities in these same former slave states now rely on Black male students who play football to increase universities' student enrollment and further grow the wealth of these sports programs. College sports represent a new form of a plantation system as Hawkins (2010) asserts.

Map 1: High School Football Players, Division I Recruits, 2015-2016
Source: National Federation of State High School Associations



Notes: Percent derived from the number of DI football recruits from 2013-2016 divided by the number of boys football participants per state in 2015-2016 as reported by the National Federation of State High School Associations. Alaska (1.2%) not pictured.

On the other hand, poverty is disproportionately along racial lines in the South, as well as throughout the United States. As I have written elsewhere, the U.S. South continues to play a unique role in understanding the intersection of race, place, poverty, and the experiences for Black people in a range of areas (Morris & Monroe, 200). It is the region where most Black people historically have lived and continue to live. It is the poorest region of the country and the majority of its public school children are minority and low-income (Southern Education Foundation, 2007, 2013). Whereas White children living in the South are also more likely to be poor in comparison to other regions, as Figure 1 illustrates, we see that Black children living in Deep South states are much more likely to live in poverty than White Children (see Fass &

Cauthen, 2005; Morris, 2008). Black children attending public schools in the South are four times more likely to live in poverty in comparison to White children. Black children also are more likely to experience extreme forms of segregation (known as hypersegregation) along multiple dimensions such as where they live, the extent of their interactions with other racial groups, and their concentration in neighborhoods (Massey & Tannen, 2015). The South, as a region, has the most hypersegregated metropolitan areas.

The South, as a region, is not alone in terms of understanding how place adversely shapes social and educational opportunities and outcomes for Black people. Although the South has the majority of hypersegregated metropolitan areas for Black people, many of the metropolitan areas with the *most extreme* black hypersegregation are located in the Midwest. These metropolitan areas include Chicago, Cleveland, Detroit, Milwaukee, and St. Louis. St. Louis remains one of the most hyper-segregated cities in the United States (Gordon, 2009; Massey & Tannen, 2015).

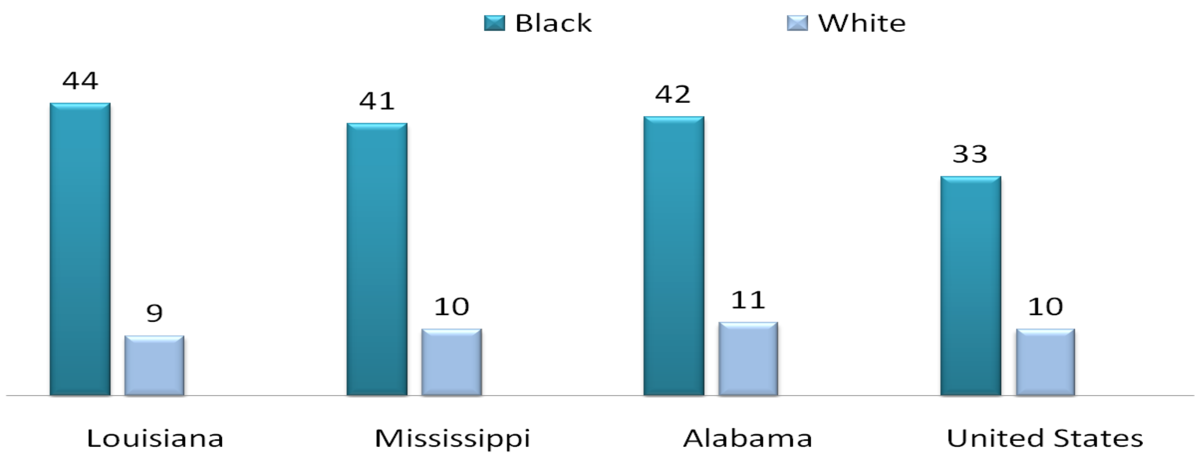


Figure 1: Poverty percentage for Black and White children in selected southern states and in the United States as a whole, 2004. Data are from the National Center for Children in Poverty (www.nccp.org). Copyright 2005 by National Center for Children in Poverty. Adapted with permission.

Shifting Ideology

The above highlighted how race, place, and poverty buttress sports regimes, and how such regimes rely on Black bodies as a source of revenue. Building regional brain regimes within a world view that relegates Black people to a subordinate status, and that views Black people as primarily physical, but not necessarily intellectual, will require the disruption of such ways of thinking. How can one disrupt these rigid notions of race and intellectual ability, along with entrenched racial patterns of schools and communities? On a micro-level, an example from my Black achievement research study in the U.S. South captures how one Black family challenged such rigid ideas (Morris, 2006-2011).

Jahmel, a third-grade student from a working class family in Metropolitan Atlanta, Georgia, took an entrance exam to attend the public school district's high achievers elementary school--which was a racially diverse magnet school. Jahmel's exam score did not meet the cutoff score set by the school. Committed to ensuring that Jahmel benefited from the excellent teaching that is available to high achieving students in such schools, Jahmel's mother and his uncle demanded a meeting with school administrators in order to discuss his overall academic record, beyond the entrance exam. Jahmel was later allowed to enroll into the high achievers' school and earned mostly A's throughout elementary and middle school. Six years later, Jahmel, a 9th grader, enrolled in the district's high-achievers' high school. Jahmel eventually graduated from high school with a 3.8 unweighted GPA, scored in the 92nd percentile on the SAT, and received the most distinguished scholarship given to an incoming freshman at a major state university. In my ethnographic interview with Jahmel and his mother, Jahmel believed that a third grade admission test could not adequately measure his academic potential or aptitude. He went on to describe how the intense preparation he received in school, as well as his determination and persistence, were instrumental in his academic achievement in school and performance on the SAT.

Another example, this one from my college classroom teaching experience, further highlights the importance of challenging ideological beliefs. In one particular course, a White male teacher, who was a student in my university-level course, described how his misperceptions of students' academic abilities shaped the way he responded to various students by race:

> In one of my classes, an Asian-American student did not perform well on most of his math homework. When he would turn it in, I allowed him to re-work the questions and would often say to him, "I know you can do this!" So, I was lenient on him because I had high expectations, honestly, because he was Asian. However, when I think about it and my view of Asian-Americans in this way, I do not know if I would have allowed a Black student to have had so many opportunities to prove that he was smart.

Although Asian-American students are often perceived as high-achievers and model minorities, many face academic and social challenges within schools and classrooms (Lee, 2009, 1994). This particular Asian-American student's academic gaps were not attributed to his intellectual abilities, but to an inability to grasp certain mathematical concepts at that time. Unfortunately, too many educators simply give up on Black students and assume that they are incapable of doing high-level academic work, particularly in mathematics. This example further demonstrates how the assumptions and mindset that some educators bring to their interactions with Black students can limit academic options, and ultimately, opportunities to fully develop academic talent. An ideological shift about the nature of race and intellectual ability is a first step in envisioning the building of a regional brain regime that reflects inclusivity and equity.

While I presented how one Black mother and an uncle pushed back against ideological views of one students' academic abilities, the reality is that a myriad of structural factors constrain the opportunities for poor Black children in U.S. public schools. Too often, where students live and go to school, as well as their race and economic status, limit their academic opportunities. Quite often, children of color are disproportionately represented among those living in poverty (Aud, Fox, & KewalRamani, 2010) and attending high-poverty schools. For example, forty-six percent of Hispanic students and 44 percent of Black students attend high-poverty schools--schools in which more than 75 percent of the students are eligible for free or reduced-priced lunch. Moreover, Black children are significantly more likely than any other group of students to experience extreme segregation by neighborhood and school (Massey & Tannen, 2015). Given the intransigence of racial isolation and poverty within schools and communities, what must be done? One may call for the continued pursuit of integration in U.S. schools, but will that be the case in a racially-stratified society where fears within the real-estate market and about school quality have become culpable in the maintenance of rigid boundaries between Black and White people? (Fox, Cybelle & Guglielmo, 2012). I'll return to these questions in the latter part of this paper.

A second aspect around brain development and mathematics attainment that Dr. Tate described, but often is ignored, is the role of early childhood nutrition in brain development. Dr. Tate mentioned the *For the Sake of All* research project, led by Dr. Jason Purnell and colleagues (2015), which included Dr. Tate. They focused on the imperativeness for healthy children, families and communities. I would like to expand on their thinking by asserting the importance of thinking about children's health and nutrition in utero and at birth--optimally with human mother's milk. I am co-leading a research team that is researching why some African-American mothers are successful at breastfeeding, contrary to the research that focuses heavily on disparities. Mary Muse, a public health researcher, is the lead researcher and director of the research project based in St. Louis (Muse, Morris & Dodgson, 2019). Building a brain regime will require encouraging, supporting, and sustaining optimal nutrition for infants within the region. For example, the scholarly literature has noted how human breast milk is best in a child's overall development. In fact, the benefits are there for the babies and mothers. Research has shown a positive link between breastfeeding and children's overall health and intelligence, even years after breastfeeding has concluded (Angelsen, Vik, Jacobsen & Bakketeig, 2001; Mortensen, Michaelsen, Sanders & Reinisch, 2002; Isaacs et. al, 2010). The above illustrates the importance of considering the ecological and sociohistorical context of nutrition and how structural, cultural, and historical forces contribute to the disproportionately low rates of breastfeeding among African-Americans, which then adversely affect mother infant bonding and children's early cognitive development.

A third point I will expound on is this notion of "community" in the building of a regional brain regime. Dr. Tate mentioned the importance of communities and how they are symbiotic in their relationships with schools. While that may be the case for many schools that are considered "good" schools today, we have seen how numerous educational reforms and policies--over decades--such as desegregation, charter schools, district restructuring, and state takeovers have undermined the symbiotic relationships that historically Black schools once had with Black students, families, and communities. I agree that supporting communities and schools is essential. But how do you support schools and communities that have been undermined through ongoing social and political forces? How do you support and sustain schools that continue to serve overwhelmingly low-income and Black students, families, and communities? I would like to suggest that in order to move forwards, we must look backwards. There is a West African concept called Sankofa, which captures this sense of going backwards first (The Spiritual Project, 2011). I know it probably sounds counterintuitive. Please follow me in my thinking.

Schools are part of a larger ecological structure (Bronfenbrenner, 1979) and serve as important places in communities and the lives of families and students (Driscoll, 2001; Morris, 2004; Payne & Ortiz, 2017; Warren, 2005). Schools play an important role in students' connectedness and in their overall educational experiences, outcomes, and well-being (Furrer & Skinner, 2003; Nasir, Jones & McLaughlin, 2011). Students who feel connected to schools demonstrate greater motivation than those who are not (Furrer & Skinner, 2003), and this sense of connectedness and belonging may be associated with students' improved academic achievement (Hawkins, Guo, Hill, Battin-Pearson & Abbott, 2001; Mendoza-Denton et al., 2002; Walton & Cohen, 2007, 2011; Walton & Brady, 2017). Thus, schools, as places, have also had meaning in Black people's lives. Unfortunately, there has been a deep sense of loss between Black people and their schools (Ewing, 2018; Morris, 2019b; Siddle Walker, 2000). Part of the reason is that policymakers and reformers who express interest in strengthening the relationships between schools that serve predominantly Black students, families, and communities tend to dismiss historical models that were built on trust, a sense of caring, and that reflected strong bonding in the relationships among schools, students, families and communities.

As a scholarly community, we now have a robust body of *historical research* about the "good" that existed within Black school-communities during the pre-*Brown* era (e.g., Anderson, 1988; Foster, 1997; Jones, 1981; Morris & Morris, 2002; Morris, 2008; Savage, 1998; Siddle Walker, 1996, 2000). A key focus of my scholarship has been developing theoretical understandings and conceptual frameworks that support *contemporary* education through sociological and ethnographic studies of exemplary Black schools during the post-*Brown* era — an era characterized by White resistance to *Brown v. Board* vis-a-vis White flight, and serious fiscal and social crises within urban communities. From a synthesis of my scholarship and the body of historical studies that captured the ways in which schools once solidified Black communities, and educators demonstrated care and concern for students, I have distilled the tenets of a theoretical model known as *communally-bonded* schooling: (1) intergenerational trust and cultural-bonding between educators and students; (2) critical presence of Black educators; (3) educators who reach out to families; (4) principals as academic leaders who bridge schools and communities; and (5) schools serving as pillars in communities (Morris, 1999, 2002; 2004; 2009; 2019a). Given the intransigence of racism in U.S. schools and society (Bell, 1992; Ladson-Billings & Tate, 1995), and the important roles that schools can play in communities, I put forth *communally-bonded schools* as a conceptual—but pragmatic—framework for supporting schools' relationships with Black students, families, and communities (See Figure 2).

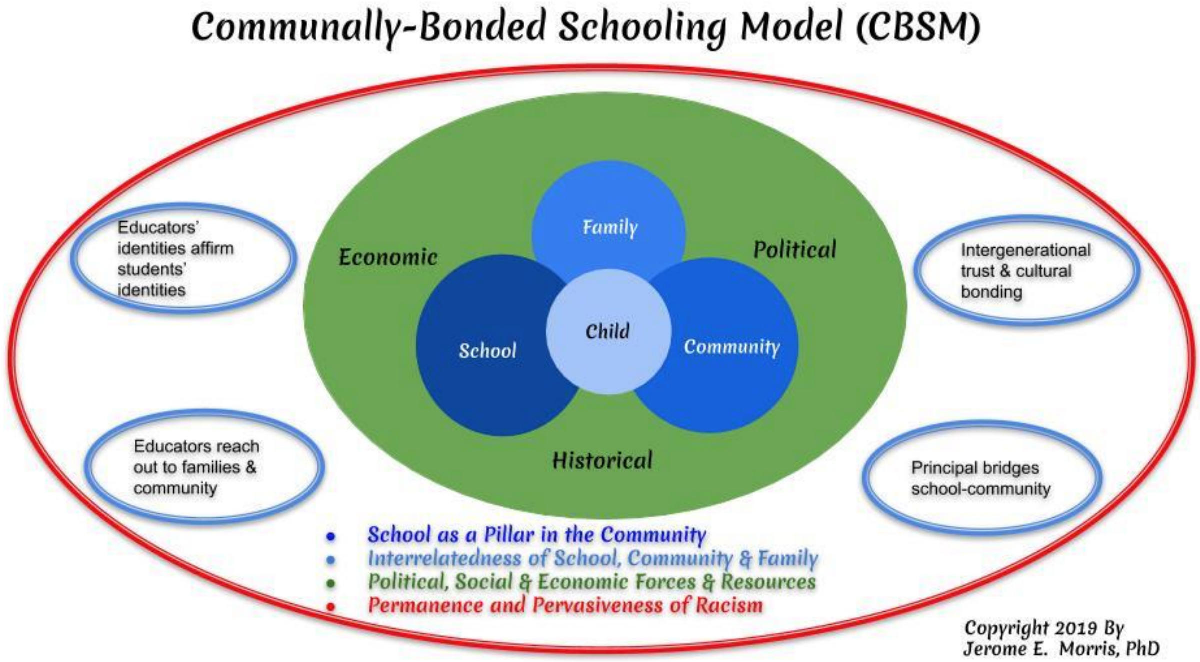


Figure 2: Communally-Bonded Model

Why St. Louis Matters: Building Brain Regimes and Rekindling Communally-Bonded Schooling

Generating the political will necessary for the building of regional brain regimes is another point to address. St. Louis is in a unique position given the international attention brought to the region as a result of the 2014 *Ferguson Unrest*, which was spurred by the killing of an unarmed Black male, Michael Brown, by Darren Wilson, a White police officer. This protest heightened the awareness of the inequalities facing Black communities, families, schools, and youth in the region. The *Ferguson Unrest* also spurred the creation of the Ferguson Commission, which developed the *Forward Through Ferguson: A Path Toward Racial Equity Report* to address the St. Louis region's deep-seated racial and economic inequity (Ferguson Commission Report, 2015). These events have generated some interest by well intentioned people in the region who desire to do something about the persistence of inequalities in education, health, employment, policing, etc. Expanding on and in concert with the numerous recommendations offered by the *Forward Through Ferguson* report, I see the building of regional brain regimes among historically marginalized communities as part and parcel of one way to structurally address lingering racial and class inequalities. Moreover, as I noted earlier, such a regime must not be viewed as an approach that targets individuals. Instead, the focus, I assert, must center families, schools, and communities. Thus, I am encouraged about the potential of building regional brain regimes within the context of also rekindling communally-bonded schooling for youth, families, schools, and communities.

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WRESTLING THE THEORY/PRACTICE GAP

David Kirshner
Louisiana State University
dkirsh@lsu.edu

From a Kuhnian perspective, multi-paradigmatic sciences like psychology attain scientific maturity if and only if a shared perspective is achieved on the foundations of the science. This underlying imperative toward unification mitigates against our accommodating education to the present reality that learning is diversely theorized in the various branches of psychology.

Keywords: Learning Theory; Teacher Knowledge

I begin this provocative plenary presentation by asking: What is good teaching?

Undoubtedly, some participants at the meeting accommodate by relating practices of teaching they find to be useful and effective, and these are duly recorded. Then someone offers a definition of good teaching with which no one disagrees: Good teaching is teaching that supports learning.

This leads, inexorably, to a next question, What is learning? Since this is a Psychology of Mathematics Education meeting, we rephrase the question: What is learning as theorized in psychology?

The conversation becomes more fragmented. Some reference one school of psychology, some another. Just the major branches—behavioral, cognitive, developmental, sociocultural—provide a dizzying variety of fundamentally different conceptions of learning. Alexander (2007) points out that some are "hard perspective" theories that brook no compromise on their unique prescription for learning, while "soft perspective" theories like social constructivism or situated cognition theory stake out a more integrative vision. But the inescapable circumstance of our time is that learning theory is fragmented across multiple branches of psychology. None of these theories, be they hard or soft, have won general acceptance; psychologists remain cloistered in their independent theoretical communities. Learning theory is fragmented.

This little exchange, brief though it is, points to an uncomfortable truth: The discourse of teaching is disjointed from the discourse of learning. For if pedagogy were truly aligned with theory, we would have to respond for each recommended practice of teaching that it is indexed to one or another of these diverse theorizations of teaching.

That we don't qualify our characterizations of good teaching in relation to a particular theorization of learning does not mean we don't care about theory, or that our pedagogical recommendations are not steeped in theoretical perspective. We do, and they are! But our discourse does not demand of us that we qualify our prescriptions for teaching as specific to a particular theorization of learning, and this means that our perspectives on teaching enter together into a theoretically undifferentiated discursive space.

Outside of design research enclaves in which a teacher works with a team of researchers who actively apply a particular theoretical perspective to inform pedagogy, it is this theoretically undifferentiated discursive space that constitutes the interface between the world of theory and world of practice. It has been pointed out that even for reform teachers committed to restructuring education "activities, as opposed to ideas, are the starting points and basic units of planning" (Windschitl, 2002, p. 138). But is there any realistic alternative in a discursive environment in which recommended practices float free of specific theoretical grounding?

Sociology of Science Context to our Discursive Construction of Pedagogy

I would like to be able to report that our discursive practice of talking about teaching practices outside of specific learning theories is the result of a deliberative process. After all, in applying psychology to education we are dealing with challenges of fragmented science that few other professions face. Medical practice is informed by many different basic sciences—anatomy, genetics, biochemistry, physiology, etc.—but these sciences share in common a basic point of view about the nature of physical reality (Weinberg, 1993), so they support and enhance each other. In contrast, we in education have been grappling with irreconcilable theorizations of learning for 130 years, since the advent of scientific psychology in the late 1800s (Lagemann, 1989). Surely, we must have sat down together as a community to deliberate about how to meet this challenges!

The historical record does not show this to be the case. In the early/middle decades of the 20th century, psychologists in various schools sought to capture the mantle of psychology for themselves, so educators flocked to the banner of behaviorism, or Gestaltism, or Functional Psychology (Dewey), declaring the others invalid—notwithstanding the resulting impoverishing and factionalizing of education. Mid-century, when cognitivism knocked behaviorism from its leading position (Gardner, 1987), some became cognitivists, and some shifted allegiances to constructivism (Piaget) or to sociocultural theory (Vygotsky) continuing the winner-take-all antagonisms of the earlier era (Cobb, 1994). In the mid-1980s, the limited ability of Information Processing theory to account for context (Lave, 1988) led to the establishment of hybrid theories like situated cognition theory (e.g., Brown, Collins, & Duguid, 1989) and social constructivism (e.g., Ernest, 1998), notwithstanding the fact that these unions of incommensurable perspectives are irreconcilable in principle (Sfard, 1998). So in the current fashion, we offer pedagogical guidance reflecting our grossly or subtly differing theoretical perspectives—no, not individually in our own teacher education courses, but in our collective contribution to the world of practice.

Psychology as a Preparadigmatic Science

Why should it matter that we have allowed psychologists to guide us through the dilemma of multiple theories of learning? The answer is that psychology is not only a multi-paradigmatic science; it is preparadigmatic in the sense of Kuhn (1970).

In Kuhn's (1970) famous theory, all sciences begin in a state of fragmentation. Science during this preparadigm period progresses in separate schools. For instance, the science of optics was pursued in "a number of competing schools and sub-schools, most of them espousing one variant or another of Epicurean, Aristotelian, or Platonic theory" (p. 12).

During this adolescent phase, progress is hampered by the absence of a shared perspective. Scientists constantly are in a position of having articulate and defend fundamental assumptions. What's more, theorists operating under different paradigms "see different things, and they see them in different relations to one another" (Kuhn, 1970, p. 150). So, paradigmatic divisions can never be resolved through rational argument, because rational argument requires shared starting assumptions.

Despite the incommensurability of its theories, it is possible for a preparadigmatic science to achieve unity of perspective and thereby break through to mature scientific status. This happens when a particular school offers such a compelling vision that members of the competing schools abandon their agendas and join with the ascendant school. In the case of optics, it was Newton's (1730) *Opticks* that led to theoretical consensus and the achievement of scientific maturity. But unlike physical sciences, social sciences like psychology remain in the preparadigmatic, fragmented, stage (Flyvbjerg, 2001; Geertz, 2000).

This Kuhnian analysis alerts us to the downside of relegating to the community of psychologists the responsibility that we have as educators to negotiate the multiplicity of learning theories. For across the broad spectrum of psychology, the common goal that underlies all efforts is the eventual unification of perspective that marks transition to scientific maturity. This doesn't mean that psychologists, intent on making their own paradigm ascendant, actively subvert education. But it does mean that psychologists will not spontaneously offer to educators a strategy that reifies separate and independent notions of learning. Thus, we have missed what is the most direct and obvious way to accommodate the fragmented state of learning theory.

Genres of Teaching

If learning is independently theorized in separate branches of psychology, and if our goal is to articulate theory-based guidance for pedagogical practice, our most direct and obvious strategy is:

- Identify the basic notions of learning pursued in psychology that together span the broad interests of educational practice;
- Construct a separate theorization for each based in the relevant learning theories; and
- Articulate a separate genre of teaching (teaching methodology) for each informed by this theorization.

Note *constructing a theorization* is not a matter of simply selecting an intact theory from psychology. In some cases, multiple theories may address the same basic notion of learning and insights from all of these should be drawn upon to inform the associated pedagogical practice. In others, the competitive imperative in psychology may induce psychologists to assert as solid accomplishments what are really only research programs (e.g., Skinner's, 1958, attempt to extend operant theory from unmediated response conditioning to verbal behavior, beaten back by Chomsky's, 1959, withering critique). So theorizations have to be assembled to ensure that boundaries between the basic notions of learning are cleanly drawn.

As articulated by NCATE (2002) and its successor CAEP (2016), education is motivated by a trio of learning goals: *skills*, *knowledge* (concepts), and *dispositions* (cultural practices). Skills are theorized in behavioral psychology, implicit learning theory of cognitive psychology, and connectionist theory. Concepts are theorized in developmental psychology (Piaget). Dispositions (understood as modes of cultural participation) are theorized in sociocultural psychology and cultural psychology. These provide a basis for articulating independent genres of teaching for skills, concepts, and dispositions, respectively (Kirshner, 2016).

Summary / Conclusions

Educators have deferred to psychology in developing strategies for dealing with the independent theorizations of learning that psychologists have generated in the various branches of their science. But psychology is driven by an underlying motive of eventual unification as needed to move from the preparadigmatic phase to full scientific maturity (Kuhn, 1970). This deference has kept us from considering a full range of strategies including what is arguably the simplest and most direct way to ground pedagogy in diverse learning theory: the genres strategy, as outlined above.

Looking at the historical record, the strategies we've adopted have been wanting. In the first half of the last century, we opted for a favored psychological school, discrediting the rest. This led to overly narrow conceptions of educational practice (e.g., behaviorism's eschewal of such basic constructs as *mind*), as well as to divisiveness when educators have opted for different psychological theories (e.g., the Reading Wars and the Math Wars). Next, we gravitated to integrative theories like social constructivism and situated cognition, heedless of the fact that these theories are incommensurable (Sfard, 1998). This produced grand visions of learning and teaching, but no real theoretical principles for realizing these grand plans. More recently, under the rubric of *design science* we assemble theories specific to a particular educational context or problem (Cobb, 2007; Lesh & Sriraman, 2005), thereby guaranteeing the impossibility of scaling up theory based pedagogical perspectives to serve the needs of mass education.

The genres approach (Kirshner, 2016) avoids all of these pitfalls. Diverse theorizations of learning give education the scope it needs, without the conflict that comes from antagonistic claims. Because each theorization is focused on a particular learning goal, we get the specificity of direction we need for strong theoretical support of practice. Agreement on a stable set of theorizations enables broad scale-up for the needs of mass education. But to finally accept and deal with the reality of fragmented learning theory would break with a century-long junior partnership (Lagemann, 2000) in which we unwittingly have joined with psychology in its ultimate quest for unification. Demonstrably, this partnership has not served us well. Do we have to courage to critically analyze the holistic and integrative visions of learning that psychology now purveys as a harbinger of the grand unification it so dearly seeks?

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CREATING A HEALTHY AND RIGOROUS CULTURE OF RESEARCH BY REVEALING OUR WORK

Eric Siy
University of Wisconsin-Madison
epsiy@wisc.edu

Graduate students experience high levels of mental health issues such as anxiety and depression. In this provocation, I focus on the imposter syndrome as one of the sources of these issues. I urge our community to chip away at imposter syndrome by revealing the messiness of research through three practices. As we face a new horizon, my hope is these practices shift the culture at conferences (and academia) so we can all be a little kinder to ourselves.

In 2019, graduate students can create online collaborative meeting spaces, access books and journals from their pockets, and use software that make typing out a reference list seem archaic. All of these resources should make the life of a graduate student in 2019 easier. There is, however, a darker trend—the growing awareness of graduate student mental health. There is no systematic, large-scale investigation of graduate student mental health (Flaherty, 2018). Most research is isolated by institution, field, and region, but there are notable similarities across these studies. Graduate students experience mental health issues at alarming rates such as high levels of anxiety, depression, and suicidal thoughts (Barreira, Basilico, & Bolotnyy, 2018; Garcia-Williams, Moffitt, & Kaslow, 2014; Levecque, Anseel, De Beuckelaer, Van der Heyden, & Gisle, 2017; Lipson, Zhou, Wagner, Beck, & Eisenberg, 2016) with women and trans individuals experiencing higher levels (Evans, Bira, Gastelum, Weiss, & Vanderford, 2018). It would be naïve of the mathematics education community to presume immunity from this trend.

An unfortunate given in academia is the normalization of imposter syndrome (IS), the feeling of getting away with fooling experts of one's own success. Clance and Imes (1978) first documented this phenomenon as a psychotherapy issue with highly successful women. They indicated the "clinical symptoms most frequently reported are generalized anxiety, lack of self-confidence, depression, and frustration related to inability to meet self-imposed standards of achievement" (p. 242). Graduate students *just have* IS as part of becoming an academic. Ergo, graduate students *just have* the mental health issues linked with IS. In this paper, I focus on how we can minimize IS in order to work towards a healthier academia.

IS can be exacerbated by academia's focus on product over process. A large part of graduate students' apprenticeship into research is reading journal articles and watching research talks. In these mediums, the researcher presents a finished product, the end of their labor (apart from writing). The journey to achieve these products, however, are concealed by terse statements and magical phrases such as "multiple rounds of coding" or "until the researchers came to a consensus." The absence of a focus on explicit methods is not only a feature of conferences but also written reports. Bikner-Ahsbahs, Knipping, and Presmeg (2015) noticed "detailed descriptions on how methodologies are substantiated in a specific project, how they are implemented to investigate a research question, and how they are used to capture the research objects are normally missing" (p. v). The self-imposed standards of achievement indicative of IS are set because of the overexposure to clean, finished products. Thus, when things go wrong, take too long, or halt, the imposter bubbles up because "I don't know what I'm doing. They'll find out." How can graduate students begin to know if they are "doing it right" if they do not even know how senior scholars worked through the murkiness of research? My multiple attempts and misfires while analyzing my dissertation data began to take a toll on me. I became withdrawn and my anxiety ran amuck. Why could I not do research? Why can't I produce?

As a mathematics education community, we can begin to change our academic culture by focusing and exposing *both* the products and process of research. As we look towards a new horizon with graduate students' mental health issues in mind, I issue the provocation:

Reveal our work.

We can begin by shifting the conversations and culture at conferences. If graduate students are expected to "have knowledge of multiple research methodologies and statistical measures" (Reys, 2017, p. 938), researchers must expose the messiness of research. This knowledge includes making pivotal research decisions ranging from recontextualizing methods from other research projects to dismantling and rebuilding entire analytical frameworks. By exposing our stumbles, we can inject misfires as part of our culture and hopefully, slowly replace the impossible standards we set for more realistic ones reflective of the research process. To begin the shift, I propose three practices we can enact during conferences and other academic spaces.

Disclose our misfires. I frequently hear the statement, "You can read more about it in the proceedings" at PME-NA and similar conferences. Our presentations mirror our written work, but this time we show videos and extended transcripts. In 2017, I was in a session where the presenter began by saying, "We would like to focus our attention today on results we have concluded since we submitted our edited proceedings." I felt I was privy to new information. It made sense. Why would we spend hundreds of dollars to watch a presentation when we can read about the results for free? We can use conference presentations to supplement our written reports and present the details behind the reports by:

- *Showing initial research questions*. When I took a literature review course in my doctoral program, the instructor always asked a set of questions. Of the set, he asked two questions that anchored our conversations: (1) What was the research question? (2) Did the researchers answer the question? As a new doctoral student, I was under the impression that a research question was pivotal and ironclad. As I worked with more researchers, however, I understood that changing the research question of a project or publication is normal. One of my mentors urged me to keep my research question at arm's length by concentrating on the affordances of my data and change the question if needed. I changed my research question four times. We can use presentations to show the evolution of research questions, especially changes after data collection.
- *Showing initial rounds of analysis.* Initial passes at analyses are meant to be refined. We make decisions to drop, refine, and add analysis. These decisions are not necessarily reported in publications and presentations. We can use presentations to show analysis before the final iteration and why the changes were made. This could provide some researchers with ideas for initial passes at analysis for similar data sets.
- *Acknowledging limitations when collecting data*. It is normal to report limitations to our projects. I have used the same limitations since writing research papers in high school such as "need more data" or "conduct the study with a different population." I continue to read similar limitations in research journals today. We can identify new ones by

> acknowledging our limitations as researchers, particularly how our identities play a role. Researchers bring their visible and enacted identities to research which influences the way participants and researchers engage with each other (Yoon, 2019). We usually address this with a researcher positionality statement, a standard practice of researchers who research equity issues. I urge *all* researchers to acknowledge their positionality and how it potentially limited their data collection. For instance, a White student participating in an interview about fractions may work differently with an Asian-American female researcher than a Black male researcher.

I want to be clear that I am not advocating for researchers to make presentations laying out every granular decision. I am aware of the limited real estate for presentations. I am suggesting specific parts of our work that are normally hidden. For instance, I attended a colloquium by a researcher who examined students' reasoning. She spoke about her position as a White researcher in a mainly non-White school, which I do not usually see in presentations from her research area. She used two minutes of her presentation to reveal the work she and her research team did to build rapport with the school and the students because of her positionality.

Make processes products. By the time my parents were my age, they owned a house and had two children—a different life than mine. I found conversations where I talk to them about why they made certain life decisions as important and, sometimes, more fruitful as the decisions themselves. I urge researchers to start similar conversations in public. In conversations with a good friend, they recounted how they outsourced their mentorship to another institution because their institution did not have faculty with similar research interests. In the same way I honor conversations with my parents about their thinking, they found conversations with their outsourced mentor about the research process as fruitful as conversations about results. They were only able to begin such conversations by going to conferences and networking. There are more graduate students who do not or cannot attend conferences but are in need of mentorship. We need to find new avenues to document and publicize our research processes.

We can begin by increasing publications and presentations about methods. I have read work or heard researchers claim there is no consensus on how to analyze the phenomenon; yet, when I find accounts of methods are limited possibly to accommodate word counts. Thus, the limited ways a particular phenomenon is studied remains in the dark. In past years, presentations in the "Theory and Methods" strand have consistently been below 10% of all reports published at PME-NA (see Table 1) and I assume reports exclusively about methods is lower. In most of our presentations, we allot one slide describing the work we did. We can expand these slides into tangible products researchers can use to understand processes. If graduate students are required to wrestle between two different theoretical approaches to intersubjectivity, there should be opportunities to wrestle with different analyses of the same concept such as pieces found in the special issues on affect in Educational Studies in Mathematics (Zan, Brown, Evans, & Hannula, 2006) and symbolic tools in the Journal of the Learning Sciences (Sfard & McClain, 2002).

Table 1: Percentage of Reports in the "Theory and Methods" Strand at PME-NA

Year	Research Reports	Brief Research Reports	Poster Presentations
2015	7.61% (7/92)	2.08% (2/96)	1.47% (2/136)
2016	4.81% (5/104)	4.9% (5/102)	3.15% (7/222)

2017	4% (3/75)	4.93% (7/142)	1.8% (3/167)
2018	7.25% (5/69)	5.84% (8/137)	3% (4/133)

Creating robust research methods and constantly changing them are not less than 10% of our work. We can create products revealing this work. I want to be clear, I am not advocating for the field to downplay results. I am advocating for a focus on creating accessible publications and presentations about the work done in conference rooms. My mentor advised me to use my research memos as data to write a rich methods chapter for my dissertation. In the end, the chapter was longer than others, but it provided a more accurate account of my work.

Practice academic humility. My dad once told me that academics are just people who are paid to say why other people are wrong. To this day, this resonates with me. At my first PME-NA, I watched a senior scholar press a graduate student to justify their use of the word *x*. The senior scholar resisted the graduate student's answer. They explained *x* means *y*. I read the scholars' work on *x* and the graduate student could not save face unless they said "*x* meant *y*."

Academic arrogance creates a hostile environment where one feels debunking is the rational response to a disagreement. Thus, diminishing participation of those who hold different views (Lynch, 2018, n.d.). Academic humility involves "owning of one's limitations, a healthy recognition of one's intellectual debts to others, a willingness to improve one's knowledge of the world, and low concern for intellectual domination and certain kinds of social status" (Lynch, n.d., p. 4). We can practice this humility by setting up spaces for scholars to see how other interpretations may enhance our explanation of teaching and learning. I suggest researchers not only be upfront and publicly acknowledge their limitations and own their biases but also acknowledge when other research areas can help explain data. There have been countless times in the past few years where I have heard researchers complain about the questions "Where's the math?" and "How does your research address equity?" We need to build epistemic trust (Lynch, 2018) by creating an environment where we can rely on others for knowledge. With technology, we can collaborate and create dialogue across research agendas to form diverse groups of researchers and support one another's work. We can create a culture where "I don't know, but I think another research area, *x*, can explain it" is as powerful statements as "Our research shows."

I hope my provocation begins a shift from normalizing imposter syndrome to normalizing the publication of the messiness of research so we can be a little kinder to ourselves. As a millennial, I may have (as a New York Times columnist described) a coddled mind with an inability to understand the way the world (of academia) works; however, many graduate student stories reveal academia as a space where the bottom-line matters. This is ironic for a field that privileges thinking, not the answer. We can either be complicit with the status quo of feeding graduate students' inner imposters or change our culture of apprenticeship. By revealing our work, we do not diminish our results. We expose the rigor of research. These accounts can support graduate students to engage in the work of research while keeping the inner imposter at bay.

Epilogue: To Those in Power

There are more factors outside of my provocation contributing to graduate students' declining mental health. Stipends hover between $15,000 and $20,000 a year (the 2019 federal poverty level for a household of one is $12,490). Most doctoral programs offer health insurance that fails to offer many necessary services (Crow, 2019). More than 12 percent of doctoral students who completed their programs will have a combined undergraduate/graduate student-loan debt around $70,000 with rates higher in the social sciences and education (Patel, 2015).

Even with the promise of a burgeoning career after graduation, 30% of graduates with a PhD in the field of education have “no definite commitment for employment or postdoctoral studies” (National Science Foundation & National Center for Science and Engineering Statistics, 2018). Moreover, one in ten female graduatestudents experience harassment and majority of the cases involve unwelcome physical contact (Cantalupo & Kidder, 2018). Graduate school should be challenging, not traumatic (Wedemeyer-Strombel, 2018). Graduate students should not have to defer medical treatment, go hungry, or put their life on hold because they simply cannot afford it.

This is an extended call to academics who have access to spaces where there are no graduate student bodies, yet policies are made and cultural norms of academia are set. Graduate students need faculty-allies in those spaces. Staunch, unrelenting allies advocate for graduate students by listening to them and acting on their behalf. It can begin with the New York subway rule: When you see something, say something. There will be times where allies are confronted with the difficult decision: Be silent and support a system launching generations of new scholars into poverty, poor mental health, and a toxic academic environment or speak truth to power and advocate for those not in the room. I urge everyone to act so we can make academia a thriving, fulfilling, and healthy space for all scholars, especially graduate students.

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TWO PATHS: TENSIONS OF AN EMERGING SCHOLAR[1]

Lauren Baucom
University of North Carolina - Greensboro

In 2018, I attended the 40th annual PME-NA conference in Greenville, SC. One of the plenary messages was jointly given by Drs. Marta Civil and Laurie Rubel. At the end of their talk, an emerging scholar of color asked for their opinions about entering the field being known as a mathematics education researcher who focuses on equity work. The two scholars on stage varied in their advice: Dr. Civil advised emerging scholars to wait to begin pursuing equity-oriented research until after they had been granted tenure to assure successful promotion. Dr. Rubel suggested that all scholars should be allowed to be wholly themselves in this field and in their work, using their identities and passions as impetus in the pursuit of their research. I was struck by the stark difference in their responses, and since then have heard similarly contrasting advice on several occasions. As a newcomer in this community and as an emerging scholar of color, the conflicting messages affected me in particular ways and made me wonder, "Can I be whole and find success as a mathematics education researcher?"

As a junior scholar, my emerging understandings about wholeness and identity situate my thinking about my own wholeness as a mathematics education researcher and member of this community, while knowing that my learning of these concepts in their complexity has just begun. Our identity-in-practice is in constant negotiation between a narrated identity-in-practice (or who I say I am) and an embodied identity-in-practice (who I bodily perform to be), authored in-the-moment, but is also set against an historical backdrop of both institutional and personal struggles (Lave & Wenger, 1991; Lave, 1996; Tan, Calabrese Barton, Kang & O'Neill 2013). This collision of historical-institutional struggles with historical-personal struggles, or one's "history-in-person", is carried across spaces and time (Holland & Lave, 2001). Our identities-in-practice are in constant negotiation due to the constraints of the communities in which we operate, and can only be fully legitimized when the more powered others in our community recognize the identities-in-practice that we are authoring.

In 2009, Gutiérrez outlined a working definition of equity and proposed that it is not enough to help students achieve in mathematics, nor is it enough for students to use the mathematics they learn to question their society. Rather, students must be able to "play the game to change the game," to find success in the K-12 educational system as it currently is while simultaneously pressing on the oppressive structures of that system to transform their world into a more just society. This framework of "playing the game to change the game" is helpful in describing each of the different responses I heard at PME-NA. Both Drs. Civil and Rubel were suggesting that educational researchers should "play the game to change the game," yet they differ in when to begin. Dr. Rubel's advice of *begin as you mean to go* echoes the "play the game to change the game" model right at the onset of the doctoral journey, where emerging scholars squarely center their work on equity to produce scholarship that pushes against systemic injustices in the field of math education within which they reside. Dr. Civil's advice of *waiting to begin* implies a "play the game to change the game" approach in which scholars who wish to focus their research on critical dimensions should first find success in academia by securing promotion and tenure with

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

work that might not be as centered on equity, in hopes of then being able to return to push against these structures after successfully securing tenure.

I am a Black, biracial, cisgender woman of color. I am a mother, raising a young daughter of color in the South. I am a mathematics educator, currently pursuing a doctoral degree in mathematics education research. I view mathematics through each of these core identities, and each core identity comes with lived experiences that are laminating my identity-in-practice into the mathematics education researcher I am becoming. As a student, I was told that I did math “really well for a black girl.” As a biracial educator, I was told that I was “not black enough” to celebrate my history for Black History Month. As a critical mathematics teacher, I was told to “stay in my lane” and quit teaching lessons that ask students to analyze issues of social justice in the mathematics classroom. As a mother, I feel a sense of urgency to address issues of systemic change as my daughter will soon enter a public school system that too often refuses to acknowledge and value her identity as a student of color. I author this statement of positionality to contextualize how I received the messages regarding these two pathways and the reflective process by which I made meaning of each.

These different messages seem to chart two distinct paths of entry into the field of mathematics education research in this area of focus. Although they converge on the idea that scholars should do work centered on justice and equity, they diverge with respect to when to begin. As an emerging scholar of color standing at that point of divergence, I see each pathway as a combination of both risk and reward, and I wonder: what will be the cost of taking either one?

Path 1: Waiting to begin

The path in which I delay centering my scholarship on equity-oriented research brings with it the reward of an increased likelihood of a “successful” navigation through the promotion and tenure process, which typically requires sufficient numbers of published articles in premier journals and an untarnished teaching record, along with adequate service contributions to the community. Although these criteria are not judged equally (Park, 1996), finding success in research, teaching, and service may be easier if one does not have a critical orientation in one’s scholarship. The chase of promotion and tenure has been documented to be more difficult for scholars of color (e.g. Martin, 2009; Stanley, 2007; Turner 1998; Turner, 2002), as they work to develop an academic identity against a historical backdrop that is traditionally resistant to change (Diggs, Garrison, Estrada & Galindo, 2009). With the knowledge that faculty of color experience additional barriers in seeking tenure and promotion than their white colleagues, choosing to begin one’s scholarship without a critical orientation promises the reward of a position that provides safe haven from termination, so that one can complete their life’s work of equity-oriented research without fear of professional retribution.

However, there are several risks that accompany this “successful” storyline. As Lave and Wenger (1991) suggest, “Who you are becoming shapes crucially and fundamentally what you ‘know.’ ‘What you know’ may be better thought of as doing rather than having something” (p. 157). Therefore, becoming a scholar is mediated by the kinds of research tools that one uses to “do” scholarly work, which authors the kind of scholar that one will become. In the process of using tools that help to define particular fields of scholarship, researchers are authoring their identities-in-practice as scholars in the field, and our identities, through recognition by others, stabilize into the researchers we will become. In essence, the identity that we embodied before the doctoral journey is no longer the same identity that we embody at the end of the doctoral journey. And if that roughly four-year pursuit of a doctoral degree is followed by a six-year chase of promotion and tenure, we might consider the scholar's "maturation" as being at least a decade-long process.

The research tools, methodologies, and theories that equity-oriented scholars use are specific and complex, and require care to assure one is not reproducing inequities in communities that have historically been marginalized. Learning to use the tools, methodologies, and theories of equity-oriented research in mathematics education requires a significant investment of time. If the first ten years of a researcher's journey are defined by using "apolitical" or "benign" research tools, methodologies, and theories that are not critically focused on equity and justice, how reasonable is it to expect that scholar to change her repertoire of tools to be able to emerge as a robustly grounded, critical scholar, when she has had little experience with critical theories and methodologies most suited to pursuing equity work?

Equity-oriented scholars who choose the post-facto "play the game to change the game" pathway must choose to mute some core identities and amplify others, with the assumption that at the end of the promotion and tenure process we will be able to "change the game", despite the fact that we would have laminated a different mathematics education researcher identity. The concomitant identity work that takes place while we engage in playing the game may inadvertently blunt our critical edge, even as we achieve "success" in this system. This system serves those in positions of power and privilege by normalizing the *waiting to begin* doctoral pathway as the journey with the most "reward" by promising safety, while inadvertently silencing would-be equity-oriented scholars for at least a decade. Further, while these junior scholars play the game, another generation of K-12 students pass through the largely unchanged US school system—a generation that will soon include my own daughter. What is the cost of my decade of "playing the game" if the scholar I become in that time does not have the skills to then effectively "change the game"?

In this pathway, a collision occurs between my narrated, embodied, and history-in-person identities-in-practice, as I will enter into negotiation with my sense of self against the backdrop of the institutional and historical narratives that constrain the work of equity-oriented scholars. The person I would be authoring myself to become would not be aligned with the person I am narrating myself to be, and I would sit in an incongruous state as what I am "doing" (Lave & Wenger, 1991) does not lead me to who I want to become.

Path 2: Begin as you mean to go

The second pathway suggests that scholars should *begin as they mean to go,* and embark on the journey of producing equity-oriented scholarship aimed at "changing the game", if this aligns with the identity-in-practice they author themselves to be, while simultaneously "playing the game" of navigating promotion and tenure. Here, "success" is twofold. First, it is defined from a source of identity, that one is first true to oneself, and that the reward is fully embodying one's lived experiences and identity-in-practice in one's scholarship and in the community, despite the known collision with one's history-in person. (Holland & Love, 2001; Tan et al., 2013). Second, success means that one is welcomed as an emerging scholar into the community by the more powered others that acknowledge the type of game that one is playing.

For me, embarking on a journey of equity-oriented work would mean focusing my scholarship on the disruption of normed messages of privilege and power structures in mathematics that are similar to the ones I have experienced in my past. As a critical scholar, this would imply that my research would foreground what Gutiérrez (2013) characterized as dimensions of a *critical* axis of equity (identity and power) as they relate to dimensions of a *dominant* axis (access and achievement). In working within the elements of identity and power, the purpose of my research would be to understand how students' and teachers' identities are narrated, performed, or backgrounded by institutional systems, while looking for opportunities to "desettle" (Bang, Warren, Rosebery, & Medin, 2012) the systems of privilege that often surround unjust social structures like education.

I would choose this pathway of becoming a critical scholar in mathematics education research because, like many, I draw from the well of my historical identities both as a mathematics learner and as a mathematics educator—salient identities that play crucial roles in shaping my emerging perspective as a woman mathematics education researcher of color. There are examples of critical scholars of color similarly responding to and drawing on their lived experiences (e.g. Fasheh, 1991; Ladson Billings, 1995; González, Andrade, Civil, & Moll, 2001, Delpit & Dowdy, 2008; Delpit, 2012; Frank, 2018), forging a possible pathway for newcomers to follow as they wrestle with the related tensions in the education field between mathematics and equity, as well as tensions between theory and practice, and the urgency to attend to systemic change (Foote & Bartell, 2011; Squire & McCann, 2018). In this way, many mathematics education researchers of color have come to do equity-oriented work out of a sense of wholeness as they balance the historicity of their core identities of being a mathematics learner and educator with being a person from an historically marginalized community.

While there are examples of those who have seemingly found success in being whole in the pursuit of equity-oriented scholarship, the risks of this pathway are grave. Dunn (2016) speaks to the real "danger" that comes with pursuing any line of research that is equity-oriented, as junior scholars navigate the complexity of departments whose mission statements may promote equity and equality, but whose actions prove equity-oriented work is supported and touted insofar as it is profitable, rather than to create a just world that humanizes people from marginalized groups. For Black emerging scholars in particular, McGee and Martin (2011) discussed the stereotypes of inferiority that high-achieving Black undergraduate and graduate students in mathematics and engineering frequently must manage and relations between that emotional work and their racial and mathematics identities. Such examples point to the risk of having to negotiate one's identity-in-practice as it is set against the backdrop of the historical narratives that certain subgroups may experience more difficulty because of the skin they are in and/or the line of work they choose to pursue. The risk of being yourself in your work is the lack of welcoming from the community at large.

Authored into the identities of many people of color is the idea that we must work twice as hard in order to gain half of the reward (DeSante, 2013). Even without a specific lens towards equity-oriented work, it has been documented that faculty of color devote more hours to both community service and mentoring, as they are producing a comparable quantity of scholarship as others (Turner, Myers, & Creswell, 1999; Villalpando & Delgado Bernal, 2002; Antonio, 2002; Turner, González, & Wood, 2008). Similarly, scholars of color teach a disproportionate number of classes centered on equity in teaching and practice—courses that are required for many teacher preparation programs, yet receive consistently low evaluations at predominantly white institutions as many students experience existential crises due to conflicting dispositions and beliefs (Turner, González, & Wood, 2008). Thus, the feat of achieving promotion and tenure is made more difficult simply by being a scholar of color, and that risk is intensified when one chooses to pursue equity-oriented work from the beginning.

An additional risk comes when the community's powerful choose not to welcome emerging scholars who choose to pursue critically-oriented research. Critics might frame such scholarship as dilution or disruption rather than a source of innovation, creativity, and expansion of the field, asking, for example, "Where's the math in this?" (Martin, Gholson, & Leonard, 2010). Such resistance works to limit the space for new ideas to be taken up in our community, and repeatedly demands that critical scholars defend the legitimacy of their scholarship on ontological rather than mere methodological grounds (Hand & Goffney, 2013). In response, alternative spaces have emerged, such as journals like the *Journal of Urban Mathematics Education,* or conferences such as Mathematics Education and Society. But even these are "risky endeavor(s)" (Matthews, 2008, p. 1), as such outlets may not rise to sufficiently "top tier" status for tenure reviews in some institutions. On top of this, the expansion for which critical scholars have pushed is not just under question inside the field; it is challenged by external sources as well, as attacks on scholars in our community (PME-NA, 2018a; 2018b) reveal the risks to one's livelihood that can unfortunately follow from engaging in equity-oriented scholarship. If those in this area of mathematics education research are having to defend their line of work rather than focus on their scholarship, what message does this send to others who would consider its pursuit?

The obvious risk to those who choose to *begin as they mean to go* is namely what the *waiting to begin* pathway is trying to assure: that scholars who wish to "change the game" are actually able to do so after the tenure chase. In 2019, the annual status report published by the American Council of Education (Espinosa, Turk, Taylor, & Chessman, 2019) reported an increase in scholars of color completing doctoral degrees between 1996 and 2016, with a larger proportion of Black and Native Hawaiian or other Pacific Islander doctoral students completing their doctoral degree in the field of education than any other group. Despite an increase in the diversity of the student body, the faculty body remained largely white. According to the Council, of all full-time faculty positions, American Indians held 0.4%, Hispanics[2] held 4.7%, Black faculty held 5.7%, and Asians held 9.3%, while white faculty held 73.2% of all full-time faculty positions (Espinosa et al., 2019). The disproportionate number of white faculty increased when delineated by the hierarchical structure of promotion. Thus, despite seeing an increase in successful graduation of doctoral candidates of color, the field is not seeing the same increase in diversity across full-time faculty ranks. These statistics paint a grave picture for an emerging scholar of color, without even considering the additional barriers of taking a critical orientation with one's scholarship. If those scholars who pursue equity-oriented research—and thus a more difficult and riskier path toward tenure—are largely scholars of color, in what ways is this community complicit in the oppression of marginalized groups?

As an emerging scholar of color in this community, I take note of the collision that occurs in this pathway between my narrated, embodied, and history-in-person identities-in-practice, and the choices I must weigh between risk and reward. While this pathway allows for me to pursue work from a legitimized sense of self from a narrated perspective, the institution writ-large, as currently configured, will likely continue to delegitimize not only my scholarship, but my core identities as a Black, biracial, woman of color, raising a daughter of color in the South. This delegitimization is no different than the messages of oppression I received as a learner, teacher, or parent of a learner of mathematics.

The Right to Wholeness in becoming a Mathematics Education Researcher

Neither of the paths described above is ideal. Each one promises rewards, but also presents risks for an emerging scholar attempting to find "successful" entry into this community of mathematics education. I have weighed these messages as I continue to author who I am becoming in this field. Others' core identities privilege them in ways that protect them from having to experience the severity of the tension between these paths, while for many the collision is nearly constant. As evidenced above, for scholars who choose to pursue equity-oriented research, there is no path that comes with taking zero risks. These risks are not the same as those inherent to the enterprise of intellectual risk-taking. Instead, these risks are rooted in the institutional and historical structures that refuse to value the core identities of those who do not align with the mainstream, dominant culture. So, what might be a research path that promotes wholeness for emerging scholars of color?

Wholeness would mean that I could engage in scholarship with my self in my work, bringing my collective experiences to bear on what I study. Wholeness would also include a sense of being welcomed into the community—welcoming not only me, but also my innovation and desire to expand the field in my scholarship. This imagined path of wholeness would require merging the definitions of success, to allow for both achievement and identity through the legitimization of our work from the community's more powerful members. Actively supporting the research pathways of emerging scholars requires this community to not only accept, but also embrace new ways of doing research.

I recognize the "risk" that I take in writing this from the position I am in within the system, as a third-year doctoral student seeking entry into and support from this community. It is often difficult to critique the very system in which we try to find individual and professional liberation. Yet, if we refuse to critique the institutional structures that continue to mirror the norms and values of the dominant culture, then we ignore the historical-personal and historical-institutional collisions occurring for emerging scholars and erase the work done by those who have come before us. How can we carve a new path – one of wholeness – for all scholars in our community of mathematics education research?

Endnotes

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[2] I use the term "Hispanics" here, as was used in the status report, despite its erasure of those who identify as Latinx, and/or those who do not share the common language of Spanish.

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SHOULD THERE BE LESS MATHEMATICS EDUCATION?

Niral Shah
University of Washington, College of Education
niral@uw.edu

Mathematics education is typically seen as intrinsically beneficial to individuals and to society. More broadly, access to STEM education is considered essential in a technology-driven modern economy, particularly for people from racially minoritized groups. In this article, I interrogate the underlying logic of these dominant narratives by asking: Should there be less mathematics education? *I argue that although mathematics and STEM education do have value, they come with significant current and future costs to society. Further, I question the net benefits they promise for people from racially minoritized groups. I conclude by provoking the field to consider what might be* gained *from having less mathematics education, and how that might better serve a vision of racial justice.*

Keywords: Equity and justice, STEM education, curriculum, policy matters

Should there be *less* mathematics education? That's not a question our field asks very often. We talk about quality (*how* we do mathematics education), but rarely about quantity (how *much* mathematics education there should be). Perhaps we don't because we assume the fundamental value of mathematics, both on its own and as a cornerstone of STEM. We see the joy that doing mathematics can bring people. We know that mathematics is required for college access and high-paying STEM jobs. We're grateful for the medicines and technologies that save the lives of our families and friends. And we acknowledge the role mathematics plays in the economy—often through STEM—as fuel for international competitiveness. To be clear, I believe we do need mathematics education. But to answer the quantity question, we must interrogate the visions we hold for society—who they center and who they exclude—and whether mathematics education can help us fully realize them.

To date, I have done mathematics education because I thought it would contribute to a vision of the world with justice for minoritized groups. Living in the United States, this means repairing the ongoing effects of the genocide and plunder wreaked by slavery and settler-colonialism. But I have become impatient, and I am skeptical of mathematics education as a viable pathway. I also now see ways that mathematics education can actually work against this vision. Everything has costs, including mathematics education.

The Costs of Mathematics Education

The costs of mathematics education vary between what happens *during* K-16 and what happens *after* K-16. For decades mathematics has played a starring role in U.S. schools. Its role expanded after the *No Child Left Behind* law in 2001, as accountability-era reforms further elevated mathematics—specifically, the arithmetic-to-algebra-to-calculus sequence that dominates U.S. education. But the curriculum is zero-sum: more time for one subject means less time for another. During this same period, we've seen cuts to science, art, and physical education in U.S. schools. Mathematics education is not solely to blame for this, but its ascendancy has imposed an opportunity cost on the rest of the curriculum.

After K-16 schooling there are other kinds of costs. Focusing for a moment on the types of jobs that come out of the STEM "pipeline," it is true that mathematics education has helped produce climate scientists and cancer researchers. But under capitalism, that's not all that comes out of the pipeline. We know that some STEM graduates end up on Wall Street as "quants"—building the mathematical models and trading algorithms that fuel predatory capitalism and that enabled the mortgage crisis and Great Recession of 2008 (O'Neil, 2017). We also know that STEM graduates gravitate toward Silicon Valley, where they might develop technologies that lead to massive job loss, or work for social media companies with the potential to destabilize democracies. As someone who worked in both the business world and tech world prior to becoming an educator, I have a unique perspective on these scenarios. One might argue that the majority of STEM graduates don't pursue such pathways, but the problem is that modern technologies wreak harm that is easily scalable. It just doesn't take that many STEM professionals to cause system-wide damage.

As a final point about costs, I believe we must also gaze far into the future and anticipate where mathematics education is taking us as a species. From the first time I saw *Star Trek* as a boy, I fell in love with space and the idea of living on other planets. One day mathematics and STEM education will take us beyond the moon to the outer reaches of the galaxy, but I worry about what we will bring with us. Could we end up with a McDonald's on Mars and racism on Jupiter? Mathematics educators may not want this future, but capitalism does not care what we want. Social studies or music education will not take us to other planets; this will be an ethical quagmire created by mathematics and STEM education.

But Can't More Mathematics Education Lead to Equity and Justice?

For decades mathematics educators have argued that access to mathematics is a matter of equity for minoritized groups (Moses & Cobb, 2001; Schoenfeld, 2002). This argument has moral elements, but it is also grounded in basic economics. If STEM jobs are among the most lucrative, then pursuing STEM is a logical response to wage stagnation and rampant income inequality: at the end of the day, people must be able to provide for themselves and their families. With respect to race, then, more racial equity in mathematics classrooms has been conceptualized as a pathway to more racial justice in society writ large. Indeed, for nearly twenty years, this view has driven my own work against racism in mathematics classrooms. But now I see certain problems with this logic.

One problem is that regardless of the amount of mathematics we make available to racially minoritized students, White-dominant systems have a way of reconfiguring themselves to re-secure advantage (Bullock, 2017; Martin, 2019). "Algebra for All" has been a decades-long equity struggle, but where does it all end when parents are lobbying for eighth graders to take calculus (Mathews, 2019)? From that perspective, the quantity of mathematics education makes little difference for racially minoritized people.

Another problem is that while greater Black and Brown participation in the STEM workforce may attenuate racial wealth gaps, it does not account for how STEM has been deployed to dehumanize people of color. We know that STEM classrooms are places where racist narratives undermine the intellectual capacity of Black and Brown students (McGee & Martin, 2011; Shah, 2017). Even Asian students—typically (and falsely) deemed "good at math"—become dehumanized as the presumption of their mathematical prowess positions them as "human calculators" (Shah, in press). A more racially just world can't only be about economics; it must also be one where people of color are recognized as full human beings.

Finally, there is the matter of representation. Perhaps we believe that more mathematics education for minoritized people will lead to greater representation in the elite sectors of society. But is our goal simply more women engineers at Facebook and a Native person running Goldman Sachs? Wouldn't it be a better world if these institutions simply didn't exist?

My point is that we should be humble about the prospects of mathematics education to effect justice at the societal level. School-level equity work in mathematics education is needed, but I now see it is an incremental project of harm reduction, not radical transformation. For example, I co-developed a tool to help teachers address implicit bias (Reinholz & Shah, 2018). This may lead to more girls of color and more emergent multilingual students participating in mathematics classrooms, but it still ends up giving these students access to mathematics and the world *as they are*. The endpoints of their educational journeys have not fundamentally changed.

What is Gained from Less Mathematics Education?

We hold deep ideological and material investments in mathematics education, which is why it's easier to focus on everything we would *lose* by having less of it. But I believe we have much to *gain* from less mathematics education. A simple but provocative question for us to ask ourselves is: If we had less mathematics education, what could we do with that newly opened space in the curriculum?

Imagine if we required every student to learn about voter suppression tactics or take a course on the politics of Indigenous sovereignty. What if every student had a chance to grapple with the details of climate policy or health care reform? What would it look like to implement a robust ethics curriculum interwoven throughout K-16 STEM education? Critical mathematics education has engaged these kinds of topics (Gutstein, 2006), albeit by making inevitable compromises within the constraints of the dominant curriculum. Ceding curricular space for more lessons and entire courses squarely focused on these issues would be a more direct route to justice through education. Lobbying for such curricula would be politically difficult, but a first step would be to open room for them.

My goal in this brief article is not to propose and debate "solutions." Engaging the question of quantity opens a gamut of new pathways for thinking about the structure of K-16 education, of which the few possibilities posed here are just a small sample. Instead, my hope is that we begin to seriously consider strategic divestments of mathematics education and their value to society. It's simply not enough for each of us to focus on our individual silos of research or practice and be content with incremental change. I argue that the field of mathematics education should prioritize a goal of justice for minoritized groups and do so with urgency, even if it means there should be less mathematics education.

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Chapter 2:
Curriculum, Technology, and Assessment

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

THE CONCEPTUAL AND PRACTICAL CHALLENGES OF TAKING LEARNING TRAJECTORIES TO SCALE IN MIDDLE SCHOOL MATH

Jere Confrey
NC State University
jconfre@ncsu.edu

Meetal Shah
NC State University
mshah2@ncsu.edu

Jennifer Persson
NC State University
jjvennes@ncsu.edu

Dagmara Ciliano
NC State University
dcilian@ncsu.edu

This paper reports on a design-based implementation study of the use of a diagnostic classroom assessment tool framed on learning trajectories (LTs) for middle grades mathematics, where teachers and students are provided immediate data on students' progress along LTs. *The study answers the question: "How can one characterize the challenges encountered when a school implements a diagnostic assessment system around learning trajectories at scale?" by identifying three explanatory themes: shifting to classroom assessment, understanding the concept and content of the LT, and seeing the results as a call to action. Each theme is discussed with references to observed activities and discussions with participants and related to the challenges connected with taking the concept of LTs to scale.*

Keywords: Learning Trajectories, Assessment and Evaluation, Teacher Knowledge

Introduction

Many believe that LTs hold great promise for widely strengthening mathematics instruction by informing teachers about the knowledge of the empirical patterns on how students learn (Daro, Mosher, & Corcoran, 2011). However, locating the disparate research contributions poses a significant risk to influencing practice at scale. Confrey and colleagues have sought to address this need by creating a software tool, Math Mapper 6-8 (MM)(sudds.co), organized around a learning map of nine big ideas, 25 relational learning clusters (RLCs) and 62 constructs. Each construct delineates a LT based on a synthesis of the related research (Confrey, 2015) that draws from the same research base as turnonccmath.net (Confrey & Maloney, 2012). MM is based on the idea of a LT as a research-based model of how students' thinking increases in sophistication relative to a domain-specific concept, in the context of instruction that is operationalized through the use of digitally-administered and scored diagnostic assessments, which return data to students and teachers immediately. These 30-minute assessments consist of items that are aligned with the levels of the LTs and to avoid excessive testing, include the content covered in an RLC. Because LTs can span multiple grades, teachers can select relevant grade-level tests (6, 6-7, 7, 7-8, 8, 6-8). Multiple equivalent forms of a test are administered in a classroom to ensure all LT-levels are assessed across students[1]. Assessment items are written by the team in consultation with inservice teachers and designed to elicit student thinking and raise issues worthy of classroom discussion. We have reported elsewhere on the validation (using item response theory (IRT)) of the trajectories based on data from students with varied demographics from our six partner middle schools (Confrey & Toutkoushian, 2018; Confrey, Toutkoushian, Shah, 2019). Our goal is to use the tool at scale across all teachers and all topics (except Algebra 1) in middle school(s) to strengthen instruction based on empirical results from our assessments.

We would argue that mathematics education needs to tackle more issues at scale and specifically how to improve learning for all students as evidenced on valid, reliable, and equitable measures.

Study after study has demonstrated the naivety of assuming that data from assessments alone sufficiently informs instruction that leads to more learning (Nelson, Slavit, & Deuel, 2012). Likewise, our studies of the use of MM reinforce the view that implementation of multifaceted learning systems is a complex activity requiring significant professional support and attention to organizational factors (Mandinach, Gummer, Muller, 2011; Tyack and Cuban, 1995). Recognizing this complexity, we asked the research question: “How can one characterize the challenges encountered when a school implements a diagnostic assessment system around learning trajectories at scale?” We recognized that the answer to this question would have conceptual and practical components.

Introducing new forms of classroom assessment in schools frequently has to overcome the barriers formed by negative reactions to the high-stakes testing required by *No Child Left Behind*. However, there remains an appetite for formative assessment practices (Black & Wiliam, 1998; Brookhart, 2015; Heritage, 2008). Classroom assessment is designed to focus on student learning and growth, rather than to view assessment as a means to measure summative accomplishment (Heritage, 2008; Wilson, 2018). Heritage describes these as including a clear statement of the learning goal, an emphasis on self-regulated learning, and focus on movement along a learning progression. She, and others, emphasize the use of assessment *for* learning (Black, Harrison, Lee, Marshall, & Wiliam, 2004) and stress that using an assessment formatively depends not on the instrument, but *how* it is used practically. It requires the focus to be on what the data show about the state of one’s understanding and how to move forward. Such assessment requires students and teachers to shift to a growth mindset (Dweck, 2006).

Math Mapper 6 - 8: A Diagnostic Classroom Assessment Tool

MM is designed to be compatible with varied curricula and scope and sequence documents. Prior to implementation of MM at a site, a few lead teachers align the assessments to relevant timepoints in the school’s scope and sequence. Any teacher can administer any assessment at any time, but a coordinated schedule of assessment supports teacher discussion, analysis, and planning at grade level. Our assessment approach involves having teachers conduct initial instruction (with or without pretesting), and about ⅔ of the way through the allotted instructional time, to give a diagnostic assessment on the relevant material using MM. After testing, teachers initiate data reviews and address topics needing further development as shown in Figure 1.

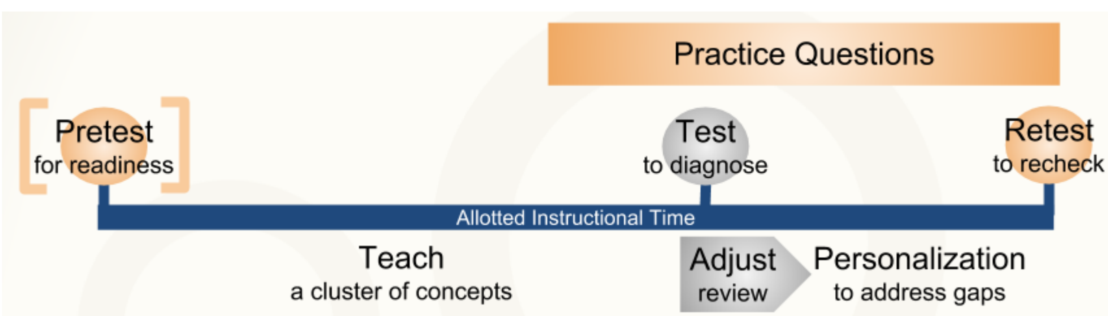

Figure 1: A Model for Implementing Classroom Assessment and Data Review

The student data are returned using a visualization of the cluster from the map with dials reporting the percent correct for each construct. Students can access a LT ladder showing the levels tested and their score by level. Students can also scroll to an item matrix showing items by level where they can review their responses and revise and resubmit answers. The teacher's display is called a heat map (Figure 2) where she sees the student performances ordered in columns from weakest to strongest[2] by construct and the levels ordered from lowest to highest in rows. The white boxes indicate untested levels for the student whose data are in that column. The other boxes are colored coded from orange (incorrect) through shades of blue to darker blue (correct). Teachers are taught to approximate Guttman[3] curves in order to identify which levels need re-teaching and which students need additional help.

The teachers have routinized two approaches to data return. In the first, using whole class instruction, they decide which levels to review based on the heat map. They tend to look for a level that is predominantly orange. Then they open it to view the item. The item can be viewed with or without the correct answer, an item analysis of student responses, and/or a report on the frequency of common misconceptions. Teachers vary substantially in the degree of student involvement in the review process, despite the research team's efforts to promote learner-centered reviews. The second approach developed by teachers is to use the data to form student groups (usually homogeneously exhibiting similar error patterns) to discuss the problems, revise, and resubmit. There is a practice feature in MM at the construct level, where individuals or groups of students can access additional items at levels of their choice and receive immediate feedback on the correctness of their responses.

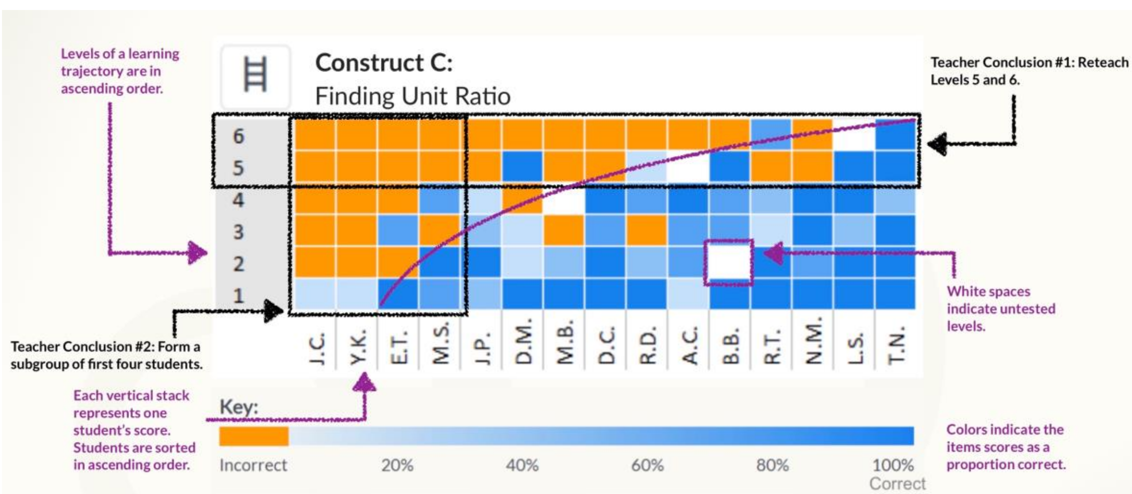

Figure 2: Sample Heat Map on "Defining and Measuring Center" With Labeled Components

Theory

The theoretical approach to the study is grounded in constructivism (Confrey and Kazak, 2006; Steffe & Gale, 1995; von Glasersfeld, 1982), with its focus on understanding how students build their knowledge gradually, working through carefully sequenced tasks in the company of peers, building gradual understanding. The process exemplifies what Piaget called "genetic epistemology" (Piaget, 1970) and Freudenthal and colleagues called "guided reinvention"

(Freudental, 1991; Gravemeijer & Doorman, 1999). It rests on the central construct of a learning trajectory (Clements and Sarama, 2004; Confrey, 2019; Simon, 1995) as a

> researcher-conjectured, empirically supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (Confrey et al, 2009, p. 347)

Secondarily, the research draws on a socio-cultural perspective on how students in classrooms engage in mathematical practice, share their approaches, cultural experiences, resources and insights, and gradually internalize the mathematical norms and expectations of a field in classroom practices (Lehrer and Schauble, 2000; Vygotsky, 1978; Yackel & Cobb, 1996). The project also draws extensively from the research on teachers' professional knowledge of content, pedagogical content knowledge (Shulman, 1986) and mathematical knowledge for teaching (Ball, Thames, Phelps, 2008), including LT-based instruction (LTBI)(Sztajn, Confrey, Wilson, & Edgington, 2012). For teachers to successfully achieve a learner-centered classroom (Confrey et al., 2017) engaging students in highly productive tasks (Stein, Engle, Smith & Hughes, 2008), they must seek to draw out student ideas and know how to orchestrate successful discussions from emergent models (Gravemeijer, 1999). Finally, our approach draws on the literature on professional growth by teachers sharing and discussing data in professional learning communities (PLCs) (Grossman, Wineburg, & Woolworth, 2001; Mandinach et al., 2011).

Methodology and Data Sources

This methodological work is situated as "design-based implementation research" (DBIR) carried out with our six demographically diverse research partner schools in 3 districts (Fishman, Penuel, Allen, Cheng, Sabelli, 2013). The schools approached the research team, wanting either to implement forms of "classroom assessment" (Pellegrino, Chudowsky, & Glaser, 2001) in which their teachers could receive data in a timely way during instructional units to revise and improve instruction and/or wanting more information about LTs. The collaboration among teachers, learning scientists, psychometricians, and software engineers involved conversations with school leadership and curriculum supervisors, from 2-4 days of summer professional development (PD) on the tool and underlying approach to learning trajectories, and then implementation of the assessments periodically during the year customized to the school's curriculum. Feedback to the research team occurs during regular grade-level PLC meetings where teachers reviewed data, discussed challenges, requested additional features and learned from peers. Data for this study were collected digitally through the use of the assessment system (n = 62000 tests), through observations and video records of classroom data returns, PLC meetings, and PD meetings meeting notes with school leadership, and monitoring ongoing participation in communication networks among teachers and researchers. Analysis of data for this paper was undertaken by the research team reviewing the video and artifacts to understand how teachers implemented the software and interpreted and acted on the data. From the classroom observations, the data review and discussions with district leaders, a set of three themes emerged to describe and explain the challenges inherent in using the software as intended to strengthen learner-centered practices and increase learning. They are summarized and discussed in terms of their conceptual and practical implications. Further, they are offered as hypotheses for future research.

Results

Theme 1: Shifting to Classroom Assessment

The observations of data returns by teachers suggest that in order to change typical views of testing by teachers and students, a shift to classroom assessment requires intentionality, explicit actions, and discussion with students. According to teachers, most students view tests as indicators of how knowledgeable, smart, and hard-working they are, and anticipate the results with trepidation and anxiety. However, classroom assessments, used formatively, are intended for feedback rather than personal evaluation and judgment. The diagnostic feedback should support informed decision-making and actions, the involvement of the students as partners, and the use of student thinking to inform next steps.

Based on our observations and teacher reports, many teachers quickly draw students' attention to their opportunity to revise and resubmit with MM. Students appreciated the opportunity to rework problems and they frequently expressed surprise that simply by rereading the problems and trying harder, they could get correct answers and experience the reward of seeing the dials immediately show their improvement. This is perhaps the most evident, simple, and direct example of the tool being used for classroom assessment.

Teachers who chose to group students together based on the heat maps seemed to be most successful in using the tool to strengthen attention to students' self-regulation, a key element of classroom assessment. One teacher, after organizing his students into groups, requested that they practice in constructs on the levels needing improvement based on their results and then return to revise and resubmit incorrect answers. His goal was to encourage them to learn the level and not just the item. Observations of his groups showed students explaining the ideas about the measurement of circles successfully to each other, calling over the teacher when more help was needed. These examples represent successful transitions to classroom assessment.

Observations also tended to reveal many teachers using the heat maps for data review by pulling up items from predominantly orange levels and simply again telling students how to solve the problems. They included admonitions to students to recall prior advice such as "I have told you to begin by drawing a T chart and building a table". These teachers seemed to view students' weak performance as needing quick and direct remediation rather than as opportunities to examine student thinking. Over time and with encouragement, teachers began to recognize that the students, having worked the problems, could provide valuable insights in their thinking. For instance, one teacher, on review of the data, recognized that she had neglected to teach percents greater than 100. She used an item at this earlier level, where the percent was 200, to reteach the concept and then relied extensively on student contributions to solve an item at a higher level involving 245% (Confrey, Maloney, Belcher, McGowan, Hennessey, & Shah, 2019). She referred to the MM items as "stretch items" and helped her students recognize their own potential to solve them. These observations have led us to recognize that even though the data provide direct evidence of students' learning needs, many teachers, especially in our lower performing schools, need additional support to learn how to orient their instruction to actively draw on students' thinking and utilize tenets of productive discourse (Stein et al., 2008).

A major challenge in shifting the orientation to a growth mindset occurred due to the weakness in overall performance by students, which may be due to the assessment's focus on conceptual understanding and reasoning. The score averages by cluster typically range from 40-60% correct, and are thus approximately 20-30% lower than on typical unit tests and quizzes. For students to understand these lower scores, teachers need to help students understand that these assessments are diagnostic and that in order to provide valuable information for all students, are designed to result in lower scores. This lower range is essential to allow for space in which to measure growth. Even so, the research team has also been concerned with the extent of the weakness in student performance, and has checked the alignment with grade level standards and asked students to judge if the material has been taught; they confirm it has. It is possible that the weak performance is an indicator of excessive procedural instruction. This would be consistent with other research which reports that middle grades students are being given excessive amounts of procedure-based materials (Dysarz, 2018) and that many teachers struggle to distinguish procedural understanding from higher conceptual levels, much less, LT-based levels (Supovitz, Ebby, and Sirinides, 2013). Further evidence for this emerged from some schools within the other themes where we discuss its implications for our future work.

How teachers responded and handled the challenge of shifting to classroom assessment and focusing on learning varied significantly by school. Schools with strong internal professional community supervisors, coaching, and district leadership transitioned more easily. In those settings, the teachers mediated the student responses, helping students to see low percentages simply meant "there was more work to do." She encouraged students to persevere by saying "our average was at 70% but this was just the first time… you'll have a chance to revise", and later, after students had revised much of their work, saying "If you're a risk-taker you can try a higher level." The strongest teachers focused on the content of the items, drawing connections to similar or related problems they had done, how to work through their reasoning, and how to coordinate the use of a variety of representations. Others focused on improvement, reporting back that "we have doubled our average score" and on refreshing the heat map to show all the students who had revised and resubmitted correct responses. In settings in which competing initiatives, especially around assessments, provided different data and direction, or mentoring and supervision were absent or weak, the initiatives encountered more problems.

These observations illustrate the complications of moving towards measurement-oriented classroom assessment. It appears that prior expectations influence the interpretation of the scale and that only if teachers explain reasons for the differences, focus on growth and the content itself, do they successfully shift the class's orientation towards using assessment for learning.

Theme 2: Understanding the Concept and Content of the LT

A second challenge of working at scale with MM comes from the need to assist teachers in understanding the conceptual foundations of the LTs in the map. The learning map in MM provides teachers access to all 62 LTs and the related misconceptions. Common Core State Standards are identified and aligned to each construct, and each level is mapped to its projected grade level. During a 2-day PD workshop, teachers are introduced to the conceptualization and research underpinning two clusters on ratio within the big idea of "compare quantities as ratio, rate or percent and operate with them". This consists of discussions of the relationship among the three constructs of ratio equivalence, base ratio, and unit ratio; and of how these form the foundation for building up, comparing ratios, and finding missing values, as well as the sequencing of the levels within each construct.

In going to scale with LTs, we have found that not reviewing all the LTs at the same level of detail and simply providing access to the LTs is insufficient for affecting practice. Observations at PLC meetings indicate that teachers seldom review LTs in planning instruction. The distinctions between levels and sequencing of levels are often overlooked by teachers. We anticipated that this would be the case, but we had hoped that providing the teachers data on students' performance would result in teachers recognizing the LT's value and relevance.

LT-based assessments differ from typical assessments that comprise of items that sample the various topics in a content domain, often referred to as "domain sampled" assessments. When teachers give domain-sampled tests, they often review only the difficult items, sometimes followed with extra explanations or practice. The item is viewed as a case of that portion of the domain. In an LT-based assessment, the item is also a case, but it is a case of likely student reasoning for that level of the construct. The meaning of an item therefore is situated in a construct for that level, and moreover, the level is situated in a sequence which delineates prior and subsequent ideas. When we observe most teachers reviewing data, the item is simply treated as an item to solve. This lack of recognition of the role and value of the trajectory became evident at one of the PLC meetings when a teacher said, the "levels [on the heat map] were random". If teachers do not recognize the significance of the LT, then much of the potential efficiency of the approach is lost.

Some teachers showed difficulty in understanding the structure of the LTs. During one of the PD sessions, the teachers complained that the ratio problems included internal multiplicative relationships that were too difficult, saying "[MM] gets too decimal-y [sic] and into fractions too quickly. We need to stick with 'numbers'." These comments showed a lack of familiarity with the Finding Missing Values in Proportions LT and strategies to find missing values in a 2 by 2 ratio box. Level 1 begins with whole number multipliers within and across the proportions, level 2 moves to combinations of multiplication and division (e.g. multiplication by 3 and division by 20, referred to as daisy chains (Confrey et al., 2014)), and level 3 tests for the resultant rational number operation (e.g. $4 \times \frac{3}{20}$). After reviewing this approach, teachers' positive reactions suggest a lack of familiarity with pedagogical approaches like daisy chains which make multiplication by a rational number more accessible to students.

Some teachers indicated that they expected the assessments items to "mirror" the items that they had taught in class. When reviewing the data, they advised students to solve the item procedurally rather than urge students to explain their thinking or engender discourse around the item. For instance, one teacher stated that, "Any time you are given three values and one unknown, that's kind of a hint that this is proportions". Such an approach is unlikely to support students in recognizing the fundamental multiplicative relations inherent in proportions (levels 1-3) and, subsequently, in distinguishing proportional from non-proportional relations (level 6).

It is becoming increasingly clear that to effectively use the tool, districts and schools will have to invest significantly in PD around the meaning of the LTs. Our experience has convinced us to begin to provide further information about the LTs and how they are situated in clusters. We see significant professional opportunities in also working with others who use other forms of evidence of student progress on LTs such as work samples (Petit, 2011; Suh & Seshaiyer, 2015).

Theme 3: Seeing the Results of LT Assessments as a Call to Action

As a diagnostic assessment tool, MM highlights issues of student understanding, and while the LTs can point out directions for movement, the effectiveness of the tool depends on the actions taken by its users (students and teachers). Responding to MM's results can be particularly challenging because classroom assessments from a robust LT-based diagnostic assessment can initially result in substantially lower student scores than other more traditional or teacher-created tests, especially if these common assessments focus primarily on procedures. We observed contrasting teachers' responses to their students' weaker performance data on these diagnostic assessments. Some teachers approached these results as a challenge, or as a call to action, encouraging their students to revise their work while simultaneously displaying the class's immediately increasing scores in real-time on a screen at the front of the classrooms, as students continued to work on these revisions. Other teachers exhibited resistance to the data.

One type of resistance that emerged was by questioning the test itself: one teacher felt that he should be able to anticipate his students' scores before they take any test, and if the scores are not what he expected, then clearly something was wrong with the test's ability to accurately assess his students' abilities. Secondly, teachers expressed beliefs that this LT-based assessment is not aligned to their curriculum, or is not aligned to "how I taught it". It is important to note that the LTs within MM and these teachers' curriculums are both aligned with Common Core State Standards, which means that these two systems are not misaligned, as some teachers claimed. A third concern of teachers is viewing the data as a form of exposure for them personally, as the results vary considerably from teacher to teacher, even within the same school. There is clear apprehension that the results will be used to evaluate them. Administrators played a key role in how this concern played out. In our highest performing school, an able mathematics supervisor kept discussions focused on the students and how to use the data to meet their needs. In a lower performing district, the administrator emphasized that low scores should not be the focus, but demonstrations of improvement should. In a third setting, with more site-based orientation, the degree of accountability ranged from strong to weak based on the instructional leadership provided by principals and other administrators.

When teachers encountered results that were lower than they expected and responded by only or excessively expressing concerns with the measure itself or of curricular/instructional misalignment, the research team noted that the response allowed them to avoid any sense of accountability for their students' LT-based data. Most often such responses to MM and the data occurred among the same teachers who expressed a preference for the use of highly procedural practices. For instance, in one school, teachers expressed a preference for using a computer system that focuses primarily on procedures. In another, teachers avoid more complex or conceptual orientations by developing their own simplified curricular materials. Thus, these observations suggest that if districts and schools want their teachers to view the results of an LT assessment as a call to action, rather than resist and reject the information, then additional supports need to be put in place to ensure their teachers understand and value an LT-based approach to learning, over a procedural approach.

Conclusions

In this paper we describe a disruptive innovation, MM, (Christensen, Raynor, & McDonald, 2015) that sits at the intersection of classroom formative assessment theory and LTs. We propose a critical goal of taking such an innovation to scale is to strengthen instruction through a model of personalization that is driven by data from valid, reliable, and equitable measures of student learning. However, taking any innovation to scale requires iterative cycles of "ramping up" toward full and successful implementation, informed by insights from classroom practice. Our DBIR study exposed important insights into the fit between MM's design and typical classroom practice. We characterized insights from our classroom observations into three preliminary themes as a means to describe the necessary shifts in practice, the need for teacher supports around LTs, and required collaborations among administrators at schools/districts. We see these descriptions as informing the development of "guardrails" to increase the likelihood of successful implementation at scale of MM. Our study also demonstrates that in order to realize the promise of LTs at scale, more resources must be devoted to helping teachers understand the foundation of each LT and cluster.

Endnotes

1 Most RLCs have 3 constructs averaging 6 levels, resulting in 18 possible levels to be tested in an assessment. With shorter tests averaging 8-10 items, not all levels are tested.

2 Teachers can display student initials to aid their own interpretation which is supported by a student matrix below or hide them for anonymity during classroom projection.

3 The display of Guttman curves and related advice on which levels to re-teach and which groups to form are currently planned for automation.

Acknowledgments

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DISSECTING CURRICULAR REASONING: AN EXAMINATION OF MIDDLE GRADE TEACHERS' REASONING BEHIND THEIR INSTRUCTIONAL DECISIONS

Shannon W. Dingman
University of Arkansas
sdingman@uark.edu

Dawn Teuscher
Brigham Young University
dawn.teuscher@byu.edu

Lisa A. Kasmer
Grand Valley State University
kasmerl@gvsu.edu

Travis A. Olson
University of Nevada-Las Vegas
travis.olson@unlv.edu

Mathematics teachers are vital components in determining what mathematics students have the opportunity to learn. There are a vast number of factors and reasons that influence a teacher's instructional decisions. As such, teachers rely heavily on their curricular reasoning (CR) to make decisions about what content to teach, how that content is taught, and the tasks to use to facilitate student learning. In this paper, we outline five strands of CR gleaned from research with middle grades mathematics teachers as they plan and implement instruction with unfamiliar curricular resources. These strands lay the foundation for our Instructional Pyramid model of CR and provide a lens through which teacher decision-making can be further understood and enhanced.

Keywords: Instructional Activities and Practices; Curriculum Enactment; Instructional Vision

Mathematics teachers are critical constituents in creating learning environments that provide students with the opportunity to learn important mathematics and that assist in developing student mathematical knowledge. One of the most important tasks required of mathematics teachers is the planning and enactment of instruction—the "what" and "how" of mathematics teaching. Figure 1 (Mathis, 2019 adapted from Stein, Smith, & Remillard, 2007) highlights the many instructional decisions during the teaching process that shape the overall mathematics lesson. While there are many decisions teachers make during the teaching process, we focus on *mathematical decisions*, defined as those decisions that influence students' opportunity to learn mathematics, and teachers' reasoning for those decisions.

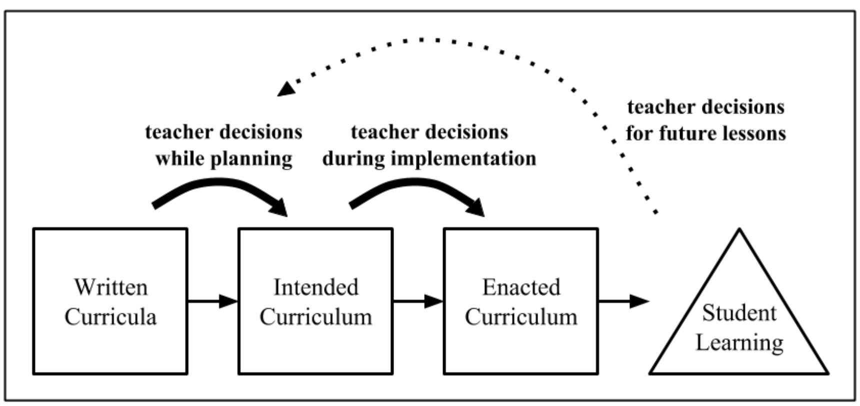

Figure 1: The Teaching Process and Teachers' Decisions That Affects Students' Opportunity to Learn (Mathis, 2019 adapted from Stein et al., 2007)

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* St Louis, MO: University of Missouri.

Researchers have sought to identify connections between teachers' implied decisions and possible factors influencing the teaching process (Bush, 1986; Graybeal, 2010; Nicol & Crespo, 2006; Remillard, 2000; Stein & Kaufman, 2010), but have rarely studied teachers' reasons for their decisions. These *factors* are not teachers' reasons for their decisions, but rather internal (e.g., mathematical knowledge, prior experiences) or external (e.g., professional development, textbooks) entities that influence or correlate with teachers' implied decisions. In contrast, *teacher reasons* are their own justifications for their decisions. Stein and Kaufman (2010) identified teachers' implied decisions during the teaching process based on learning opportunities afforded students. The authors investigated whether teachers' capacity (e.g., teacher experience, teacher education, professional development) or teachers' use of the curriculum correlated with the learning opportunities afforded students. Stein and Kaufman (2010) found that teachers' attention to the big mathematical ideas (i.e., implicit decision), which was related to the different curricula teachers used (i.e., factors), was highly correlated with students' learning opportunities. These data imply that the teachers' decisions about the big mathematical ideas within a lesson affected students' opportunity to learn mathematics. However, we do not know *why* some teachers choose to focus on the big mathematical ideas within a lesson and others did not, namely the teachers' reasons for their decisions that led to the differing student learning opportunities.

Other researchers have explicitly researched teachers' decisions and their reasoning during the teaching process; however, they hypothesized about specific factors or teacher reasons they think affect the teaching process rather than considering all potential factors or teacher reasons to give insight into teachers' decisions. Choppin (2011) hypothesized that teachers' *understanding of resources* and *attention to student thinking* would impact teachers' decisions about how they used curriculum materials; however, it may be the case that there were other factors or teacher reasons that were more prominent for why teachers used the curriculum materials in the way they did. Researchers have identified many different factors or teacher reasons that suggests we may not have a full understanding of teachers' mathematical decisions made during the teaching process if we do not consider all factors or teacher reasons.

With such a wide array of factors and teacher reasons identified that influence teachers' decisions throughout the teaching process, we suggest that teacher reasons are connected to their curricular reasoning (CR) – defined as *the thinking processes that teachers engage in and employ as they plan and enact the mathematics curriculum*. The purpose of this paper is to present a framework to characterize mathematics teachers' CR. We do this by defining and presenting five teacher-thinking processes that we refer to as CR strands. We then argue for the need to modify the Instructional Triangle (Cohen, Raudenbush & Ball, 2003) by adding a fourth dimension resulting in the Instructional Pyramid. Finally, we present the relationship of the five CR strands to the Instructional Pyramid and the interrelatedness of the CR strands in regards to teachers' mathematical decisions.

Context

As part of our NSF-funded project (#1561542, 1561554, 1561569, 1561617) that examines teachers' mathematical decisions and their reasoning as they navigate the teaching process, we developed our framework based on a sample of grade 8 mathematics teachers who taught a unit on geometric transformations (reflections, translations, rotations, and sequence of transformations). The topic of geometric transformations has historically been included in high school geometry courses; however, with the widespread adoption of the Common Core State Standards for Mathematics (CCSSM) this content was moved to grade 8 (Tran, Reys, Teuscher, Dingman & Kasmer, 2016). In addition, the authors of CCSSM use geometric transformations to build the definition of congruence, an approach rarely used in past standards nor in middle grades textbooks.

Teachers in our project were given the *University of Chicago School Mathematics Project (UCSMP)* geometry curriculum (Benson et al., 2009) to serve as the foundation for their instructional decision-making. This curriculum was chosen for two reasons: (1) its alignment with the approach to geometric transformations as found in Grade 8 CCSSM; and (2) its unfamiliarity to teachers in our study. Therefore, we aimed to identify teachers' mathematical decisions during the teaching process and their reasoning when planning with these unfamiliar curricular materials (*UCSMP*) and enacting this geometric transformation unit that was new to their grade level.

CR Strands

The teacher reasons for their decisions and reflections fell into five CR strands. In other words, we identified five key *thinking processes* that teachers used when making mathematical decisions during the teaching process. These strands are *Viewing Mathematics from the Learner's Perspective, Mapping Learning Trajectories, Considering Mathematical Meanings, Analyzing Curriculum Materials*, and *Revising Curriculum Materials.* Three of these strands—*Viewing Mathematics from the Learner's Perspective*, *Mapping Learning Trajectories*, and *Revising Curriculum Materials*—build from previous research on CR (Roth McDuffie & Mather, 2009; Breyfogle, Roth McDuffie & Wohlhuter, 2010), while the remaining two strands were identified through the open coding of our data. Below we define each CR strand and provide an example to demonstrate teachers' decisions and their reasoning for the particular CR strands. The interview excerpts below use pseudonyms for the participating teachers in the study.

Viewing Mathematics from the Learner's Perspective

As teachers make decisions or reflections, teachers reason about how their students will perceive and view the mathematics of the lesson. This thinking process is *Viewing Mathematics from the Learner's Perspective*, and defined as the teacher discussing the mathematics content of the lesson through the lens of student interpretations. Specific indicators of this CR strand are when teachers reasoned with specific details of the mathematics within the lesson and one of the following: (a) predicted (actual) student interpretations of the mathematics; (b) predicted (actual) areas of what students will do (did) with the mathematics or assessed student understanding; or (c) predicted (actual) student misconceptions. This reasoning allowed the teacher to articulate the perceived mathematical needs or mathematical knowledge held by the students. From a practical standpoint, this CR strand was typically utilized by the teacher when considering the students' prior knowledge, the students' responses to other students' thinking, the students' needs or struggles, the students' access points to the mathematics in the lesson, or students' anticipated thinking about the mathematics in the lesson.

To illustrate this CR strand, we use the following excerpt where Helen provides her reasoning for how the definition of reflection assisted her students:

> Helen: I think it helped them to recognize the pattern and recognize why those patterns were there. It also helped them, going back to that word "orientation" … and how that effects the ordered pair.

In this reflection of her lesson, Helen reasons about how the mathematics of the lesson, namely the definition of a reflection and the concept of orientation, helped her students to recognize and generalize patterns when reflecting across the different axes on the coordinate plane. To this point, she is reasoning about the lesson from the students' perspective regarding what helped them to be successful in the lesson.

Mapping Learning Trajectories

Another CR strand is related to teachers' reasoning about how a teacher maps out a lesson, either within the immediate lesson, or across units within a school year or across courses. We termed this thinking process as *Mapping Learning Trajectories*, and defined it as a teacher considering either how the mathematical concepts will unfold within a lesson, or discussing how the mathematics topics of a current lesson connect to either past or future mathematics topics students learn. Specific indicators for this CR strand are teachers' reasoning about: (1) how the mathematical concepts or skills in a lesson connected (did not connect) to past or future mathematics content; or (2) how the concepts or skills unfolded within a lesson or a unit. Therefore, the idea of a learning trajectory can take on either a short-term nature, where the teacher reasons about how the day's lesson will progress or how the given unit of lessons for geometric transformations are sequenced, or a longer-term outlook, where the teacher envisions how the lessons will connect with the mathematics content taught either in a given grade-level or across multiple grade-levels.

To illustrate this CR strand, in the following excerpt, Jill shares her reasoning for the success of her translation lesson.

> Interviewer: Do you feel like your task overall promoted student learning in the way you had hoped? How so?
> Jill: Yes, I feel that it did. It allowed the kids to explore the composition [of] reflections to determine what a translation is, and then we talked about the translation properties and which ones are preserved, and then they actually practiced. So, I feel like it did go, do what I was hoping it would do.

In her response, Jill reasons with the short-term nature of mapping learning trajectories, examining how the sequence of the lesson helped to support student learning by focusing on translations and their properties first before the students attempted to translate figures.

Considering Mathematical Meanings

The role of mathematical knowledge in the art of teaching is fundamental, and that knowledge plays a critical role of planning and enacting instruction (Ball, Thames, & Phelps, 2008). The CR strand *Considering Mathematical Meanings* is defined as the teacher's mathematical meanings of the mathematics within the lesson, or articulation of the anticipated student mathematical meanings that will be developed as a result of the lesson. Specific indicators for this CR strand are: (1) the teacher expresses his/her own mathematical meaning, which could be correct or incorrect, of the mathematics related to the lesson; or (2) the teacher expresses the mathematics students should learn during the lesson. This CR strand differs from the *Viewing Mathematics from the Learner's Perspective* in that the *Considering Mathematical Meaning* strand is from the viewpoint of the teacher and focuses on what the teacher thinks students should know, while the *Viewing Mathematics from the Learner's Perspective* strand stems from the viewpoint of the students—their misconceptions, their interpretations of the task, and their potential ways of thinking.

In the following excerpt, Judy reasons with her mathematical meaning about the relationship between reflections and rotations:

> Interviewer: Do you see a connection between reflections and rotations, besides the equal distance idea?
> Judy: …So it's really just that equal distance, but the congruent shapes are still there. And I mean, as I was talking in 2nd period, that's things they still said. But I don't know—besides that, besides them being equal and have that equal distance, I'm not sure that there's anything else I would compare those.

In this response, Judy reasons with her meaning of the properties that connect reflections and rotations. In the *UCSMP* textbook, rotations are seen as a composite of two reflections across intersecting lines (e.g., a pre-image in the first quadrant of the Cartesian coordinate system that is reflected over the *x*-axis and then over the *y*-axis results in the same image that is rotated 180° around the origin). In addition, the connection between reflections and rotations serves to highlight the properties shared by these two transformations. As this is not an approach traditionally taken in many textbooks, Judy shares her mathematical meaning of the distance preservation property, but does not discuss other properties shared by these two transformations (e.g., preservation of angle measures, collinearity of points).

Analyzing Curriculum Materials

As teachers make decisions regarding the mathematics lesson, they often use textbooks, online resources, and/or other supplementary materials at their disposal. These curriculum materials, defined as the "printed or electronic, often published, materials designed for use by teachers and students before, during, or after mathematics instruction" (Stein, Remillard, & Smith, 2007, p. 232), help to organize and structure the learning opportunities created for students. *Analyzing Curriculum Materials* is defined as when the teacher reasons about the curriculum materials by comparing the curriculum materials to other materials, providing analyses of potential strengths and weaknesses and detailing differing approaches. Specific indicators of teachers reasoning with this CR strand are (1) an analysis of a curriculum, pointing out appealing features or components that were unfavorable or that would be changed; or (2) a comparison of two or more curricula with respect to how these materials provide coverage of topics, how topics are sequenced, or for activities the teacher favored when enacting the lesson. In our analysis, we included standards documents, namely CCSSM, and state assessments in our definition of curriculum materials. Given CCSSM's important role in determining what is taught at given grade-levels as well as the prominence given to state-mandated assessments, these forms of curricula influence the decisions teachers make and play a critical role in a teacher's CR.

To illustrate this CR strand, we share a response from Ava who used the *Connected Mathematics Project* (*CMP3*) (Lappan, Phillips, Fey, & Friel, 2014) as the district adopted textbook. Given her preference for the *CMP3* curriculum, Ava was asked to compare *CMP3* to the *UCSMP* materials she was given to plan the unit on geometric transformations.

> Ava: I like the launch of them [*UCSMP*] creating their own [figure]; *CMP* just gives them a flag and tells them to reflect it. So I like that idea of starting with a white piece of paper and doing their own thing.

Ava reasons with the *Analyzing Curriculum Materials* strand as she compares the two curriculum materials, discussing features of both curriculum she liked. In subsequent data collections, Ava continued to contrast the *UCSMP* materials with what she normally taught in *CMP*, making it apparent she held a favorable disposition towards the curriculum in which she was more familiar.

Revising Curriculum Materials

The final CR strand concerns the iterative process of reflecting upon one's practice and changing parts of the lesson in order to improve the implementation or modify parts of the lesson that did not go according to plan. The *Revising Curriculum Materials* CR strand is defined as a teacher considering modifications and changes to a lesson based upon past teaching experiences. This reasoning, however, suggests a dynamic relationship between the teacher and the curriculum materials, one in which teachers reason with their CR to alter the curriculum based on experience.

In the following excerpt, Tracy is asked about her upcoming lesson that she planned, but based upon how the same lesson went in a prior class period, she is second-guessing her approach.

> Interviewer: Do you expect to get through all of the questions?
> Tracy: [Laughs] Sure, but now I'm like, NO! I mean I…no. So now I have to decide...
> Interviewer: So if you had to decide which ones you would skip or leave out, what would you decide?
> Tracy: I would probably skip the overlapping one. I did a grid one in their video of overlapping, and so I probably would skip the overlapping one. The other thing I might do is, say out of these three [questions], do two with a MIRA and do one with a protractor. So that they [students] get through the three things, but they're picking... they don't have to do it each twice. Because the protractor is going to take a little more time than the MIRA is, so if they just do one with the protractor. I mean, because I'm going to have to walk them through one, I'm going to have to go through one. I guess I need to decide which one I'm going to go through with them, because they don't know how to use that stuff. And then whichever two, they have to do one with a MIRA and one with a protractor. That probably, honestly, would be time management-wise, OK.

Based upon Tracy's reflection of the previous class period's lesson, she decided to skip some of the problems she had developed for her students. This revision of her lesson stems from student confusion during the previous class period's lesson as well as the fact that some of the problems in the planned lesson were repetitive from problems Tracy had already worked with her students.

Interplay of CR Strands

One of the predominant models regarding mathematics instruction is the Instructional Triangle (Cohen, Raudenbush, & Ball, 2003), which connects teachers, mathematics, and the students as vertices of the triangular model and where the edges of the triangle signify the interactions among these three critical components in the classroom environment (Nipper & Sztajn, 2008). This model highlights how various factors and resources influence teachers' instruction and subsequently student achievement. However, given our lens in focusing on teacher's CR and the role curriculum materials play in teachers' mathematical decisions during planning and enactment of lessons, we argue that a fourth component—the curriculum—should be added to include the various interactions that occur as teachers engage in the teaching process. Figure 2 illustrates our model that reflects the four main components of the Instructional Pyramid that influence teacher decisions during the teaching process.

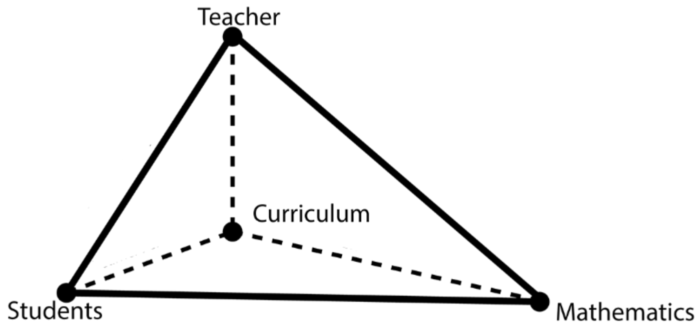

Figure 2: Instructional Pyramid for Curricular Reasoning

In the Instructional Pyramid, the edges represent CR strands, teachers' reasoning behind their mathematical decisions during the teaching process. The faces of the pyramid illustrate the use of multiple strands that interplay with one another as teachers plan and enact instruction. Figure 3 depicts the edges of the pyramid correlated to the CR strands used by teachers.

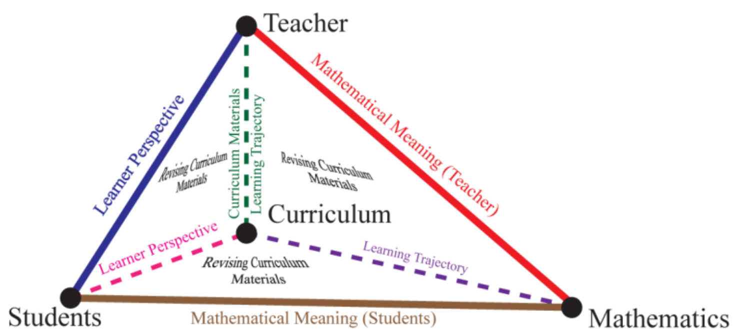

Figure 3: Model of CR Interplay

Our data set illustrates instances where teachers reason with one CR strand when making a decision, therefore existing along an edge of the pyramid. In other instances, teachers may coordinate multiple CR strands, thereby existing on multiple edges or faces of the pyramid. To highlight the interplay of these CR strands as teachers make and enact mathematical decisions, we use an interview excerpt from Judy, who planned to teach an overview lesson that would introduce her students to the language and vocabulary used with rigid transformations. Students were given a stack of geometric shapes that were translated, reflected, and rotated, and students were asked to describe what had happened to each figure.

> Interviewer: Are you planning to define any of the transformations, or are you just going to leave them in the vague terms?
> Judy: I might stay more vague today. My guess is that the words will come up, because they've heard them before. So I'm sure they'll come up. But I'm hoping that, so after this,

> I'm hoping that we'll get into days of, OK, here's a reflection. Tell me what it is. And that's when we're going to define it more. But today I think I'm going to stay a little more vague on it, and then later we'll get into more details.

In this exchange, Judy reasons with two CR strands: *Mapping Learning Trajectories* and *Considering Mathematical Meanings*. As Judy decides to "stay more vague with the definitions of each transformation" she employs the *Mapping Learning Trajectories* strand as she reasons about how the language of transformations can be more formalized over the coming days and weeks of the unit. With this reasoning, she is focusing on the Curriculum-Mathematics edge of the Instructional Pyramid. Building from this reasoning, Judy reasons with the *Considering Mathematical Meanings* strand as she delves into her thinking about staying more imprecise in this lesson. She sees this approach as fine for an introductory lesson, as she knows there will be time later in the unit to solidify the definitions of reflections, translations, and rotations. In this excerpt, Judy is imparting what she wants students to understand and gain from the day's lesson. With this reasoning, Judy is focusing on the Students-Mathematics edge of the pyramid. Taken together, the interplay of the *Mapping Learning Trajectories* and the *Considering Mathematical Meanings* strands suggests Judy's reasoning lies on the Students-Mathematics-Curriculum face of the Instructional Pyramid, as she reasons about the mathematical content of the lesson, what her students should know and understand about that content, and how that content will develop and grow throughout the unit.

Our data analysis is ongoing, but our hypothesis contends that teachers who reason with multiple CR strands when making mathematical decisions, and thus whose reasoning exists on multiple faces of the Instructional Pyramid, can provide different opportunities for students to learn mathematics. While teachers who reason with single CR strands may miss important support features in the teaching process that can improve their ability to assist student learning. The goal of the Instructional Pyramid is to provide a framework by which we can examine how teachers reason with the CR strands outlined above. This will allow researchers to examine the factors and reasons that drive teachers' decisions as they plan and enact mathematics lessons. By focusing on the CR strands teachers reason with when making mathematical decisions, teacher educators can work to support teachers' ongoing development of their use of these CR strands, thereby allowing teachers to flexibly move from the edges to the faces of the Instructional Pyramid. This will allow teachers to put into practice multidimensional CR and thus better support their abilities to make mathematical decisions that assist in promoting student learning.

Acknowledgments

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EXAMINIING MATHEATICAL MODELNG OF FIFTH GRADERS: USE OF INTERACTIVE COMPUTER SIMULATIONS

Azita Manouchehri
The Ohio State University
Manouchehri.1@osu.edu

Azin Sanjari
The Ohio State University
sanjaripirmahaleh.1@osu.edu

We analyzed 5th grade students' interactions with one computer-based modeling simulation to examine how they defined and prioritized variables in a dynamically simulated environment. Results revealed that exposure to the simulation environment helped students visualize continuous motions, interpret the different quantities present in the model, and connect how variables related to each other. Additionally, students became more precise and systematic in adjusting the variables to explore the problem, test hypotheses, and achieve desired outcomes.

Keywords: Modeling, Technology, Elementary School Education

Due to the need to establish connections between mathematics and real-world problems, studies in mathematics education have focused on mathematical modeling in recent years (Lesh et al., 2007). Calls for inclusion of modeling experiences in K-12 curriculum have been reflected in various Standards documents (CCSAM, NCTM). Despite this there is evidence that students are typically not provided with enough modelling opportunities in elementary and middle schools (Suh et al., 2016; Stohlmann & Albarracin, 2016). There has also been strong scholarly support for inclusion of technology as a "tool" for introducing learners to modeling activities Greefrath (2011), assisting them in visualizing real-world problems, and exploring their properties (e.g., Ferri, 2007). While the use of digital tools in modeling problems is studied in more detail among high school and undergraduate students, there has been limited research on their impact and the support they provide in encouraging mathematical modeling skills among elementary school learners (Greefrath et al., 2018; Geiger, 2011). In this work we aimed to address this gap by examining how interactions with one computer-based simulation influenced 5th graders' mathematical modeling process. In particular, we investigated: (1) the learners' perceptions and interpretations of a situation model concerning the impact of rate of change on distance travelled in time (2) ways that these interpretations and perceptions changed as the result of exposure to an interactive simulation depicting the same scenario.

Background Literature

Blum and Ferri (2009) define mathematical modeling as "the process of translating between the real world and mathematics in both directions" (p. 45), where real-world encompasses situations that lie outside the world of mathematics. The modeling process involves observing a real-world situation, conjecturing about it, conducting mathematical analysis, obtaining results, and evaluating the model by comparing its result with the real-world situation (Lingefjärd, 2004).

Mathematical modeling cycle describe the modeling process (Blum & Leiß, 2007; Borromeo Ferri, 2007), allow a focus on cognition, and provide a means for understanding how to trace individual thinking (Borromeo Ferri, 2007; Czocher, 2017). Blum's modeling cycle encompasses aspects of the modeling process described by Lingefjärd (2004) and serves as our theoretical framework for studying student's thought process. When encountering a real-world problem, the

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modeler initially produces a situation model. The situation model is then simplified to a mathematical model by adding structure and considering conditions and variables and restricted parameters. This formal mathematical model is analyzed, outputting mathematical results, which are interpreted in terms of the real model. The results are validated as they are checked against the real-world conditions and constraints. This process iterates until a satisfactory model is obtained (Figure 1).

Greefrath (2011) proposed an extension to Blum's modeling cycle (Figure 1) by adding technology as a bridge between the mathematical model and mathematical results. Greefrath proposes that in addition to being able to directly solve a mathematical model to arrive at mathematical results, the modeler can also build a computer model based on their mathematical model. Running a simulation is the process of executing the computer program developed to implement the mathematical model, and the outputs are denoted as computer results. The computer results are then translated back to mathematical results. The next steps are again similar to the original model, being the mathematical results and real results.

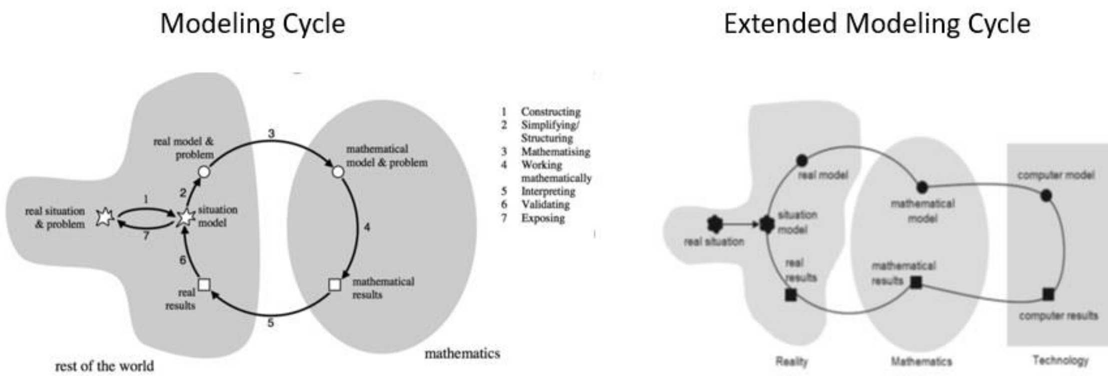

Figure 1: Modelling Cycle (Blum and Leiß 2006) and Modeling Cycle with Added Computer Model (adapted from Greefrah, 2011)

The extended cycle is advantageous as its steps do not need to be followed in order, allowing transitions between any of its stages. We used Greefrah's (2011) extended modeling cycle to track the modeling path of the students as they worked on a simulation.

Methodology

Participants

A semi-structured task-based interview (Maher and Sigley, 2014; Goldin, 2000) was used to study how 3 students examined and defined relationship among different variables as they attempted to predict specific outcomes associated with modeling a problem involving rate of change. The participants were fifth-grade students enrolled in an elementary school in the Midwest. All three were female and representing different levels of mathematical knowledge.

Procedure

Each participant (Marry, Nikki, and Tina) was interviewed three times. Each interview lasted approximately 50 minutes. During the first interview students were asked to solve The Three Runners problem (Table 1), requiring them to compare the distance between two runners, each runner's distance to the finish line, and given their speed determining which one would reach the

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finish line first. Participants' responses to this question provided baseline data on their interpretations of the task, procedures they used and factors they considered when doing so.

During the second interview participants were introduced to a computer simulation environment depicting a running scenario which paralleled the task used in the first interview. Following a free play time, they were asked to solve the same problem using the interactive simulation. During the last interview session they were asked to solve the task they had considered during the first interview without using the interactive simulation. The purpose of the last interview was to trace any shifts in their thinking as the result of exposure to the simulation.

Table 1: Interview Questions (Three Runners Problem)

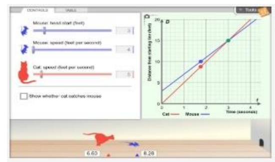
	Three Runners Problem
Step 1	Three runners are racing: a. Runner 1 is 18 meters away from the finish line, and is running towards it with the speed of 6 meters per second. How long does it take the runner to reach the finish line? b. Runner 2 is 14 meters away from the finish line, and is running towards it with the speed of 2 meters per second. How long does it take the runner to reach the finish line? c. Which runner reaches the finish line sooner? Why? d. Now suppose Runner 3 is 15 meters away from the finish line, and is running towards it with the speed of 3 meters per second. How long does it take the runner to reach the finish line? e. This time which runner reaches the finish line first? Runner 1 or runner 3? Why?
Step 2	If students' answer to part (c) or part (e) is runner 1, it means runner 1 passes runner 2 or runner 3 at some point. Following questions are: f. Can you tell when this happens? g. Can you determine where this happens?
Step 3	Irrespective of their answers to parts (c) and (e) students were asked to: h. Draw a graph of the motions of runners 1,2, and 3.

Simulation

We used the "cat and mouse" simulation during the second interview. This simulation is a part of Gizmos platform, including different interactive math and science simulations, which are designed for students in grades 3–12 aligning with the National Science Educational standards (Cholmsky, 2003). Gizmos mimics the real-world phenomena and allows the users to control several important factors while presenting information in a way that is easy to manage.

In this simulation (Table 2), students are able observe the evolution of a system over time. They explore how different objects move through time and if they satisfy a certain condition at a certain time. They also learn about acceptable parameter regions and how they should use constraints to solve the problem. Additionally, the distance between the cat and mouse, the speed of the cat, and the speed of the mouse are the parameters that students can tune. The simulation provides a continuous graph of the movement of cat and mouse running, as well as their location on the x-t plane. Table 2 summarizes the objectives and the content areas addressed in the environment.

Table 2: Modeling Simulation

Simulation	Cat and Mouse (https://www.explorelearning.com/index.cfm?method=cResource.dspDetail&ResourceID=108) 
Associated Task (Modeling Problem)	A small mouse plays on the floor, unaware of the cat creeping up on it from behind. The cat springs and the mouse desperately runs away. Will the mouse reach its hole in time to escape the cat?
Variables	Distance between cat and mouse, cat speed, mouse speed
Objectives	Inferring the effect of variables on two objectives: - The indicator of cat catching the mouse - The time it takes for the cat to catch the mouse
Content Area	Algebra-linear system

Data Analysis

Data analysis followed a two stage process. First, videos of each of the interview sessions were transcribed and reviewed to distinguish the different types of comments students made and actions they took as they worked on tasks. These comments and approaches were mapped against the phases of extended modelling cycle (Greefrath, 2011). The participants' interactions with the environment were examined to capture how exposure to the simulation shaped their modeling behaviors. Transitions between the mathematical model to the computer model, the simulation settings, and interpreting of the simulation results were of particular interest to the researchers, which were sought out amongst the data. The frequency of occurrence of each event was tallied to characterize each transition. The transitions were then analyzed in more detail, were used to study how students analyze and interpret the computer results, and were used to assess how the subsequent analyses of students regarding the problem is affected after being exposed to the simulation environment.

Findings

Table 3 offers an overview of the three participants' performance during each of the three interviews according to the answers they provided to the questions asked and their explanation of their thinking. It also describes how each student used the simulation environment and what they seemingly gained from the experience of working on the simulation to respond to a compatible context. In the following each interview session is discussed in more detail.

Interview Session 1

In the first session students answered questions (see Table 1) without using the simulation environment. The goal was to study their understanding of a problem that concerns distance/time travel in presence of rate of change, and what mathematical concepts they referenced or used. This interview served as a baseline for tracing ways that the use of simulation environment may impact their understanding of the problem or their solutions to it.

In the first session Marry showed difficulty representing the physical quantities mathematically even though she was capable of using algebraic tools (i.e., could easily carry out operations). While she correctly completed the questions in step 1, she could not justify her the algorithm she had used for computing answers for instance why she should divide distance by speed to calculate the time. She explained that she did so because she thought it is an “easy” thing to do. In an attempt to responds to questions in step 2, she drew a line to denote the location of each runner on a line, but did not correctly place the runners’ locations. Given that runner 1 is 20 minutes away from the finish line and runner 2 is 14 meters away, she put runner 1 at unit distance of the finish line, and struggled to find a way to show that runner 2 has a head start.

Figure 2: Marry's Representation of Runners’ Locations on a Number Line

Nikki seemed more comfortable with step 1 questions though part (e) seemed challenging to her. She could justify how her mathematical manipulations helped her compute physical quantities however, in response to part (e) she needed to compute the net effect of speed and distance but instead she only focused on speed and ignored distance. In step 2 she used geometric representations of physical quantities by drawing a line, and analyzed the running process in 1 second intervals to answer the question. Although she could explain how a solution to step 2 questions meant in physical quantities (for instance a runner passing another means at some time the runner is behind and in the next second is ahead), she did not know how to express this event mathematically.

Tina seemed more comfortable with algebraic operations. She completed step 1 questions easily. She correctly answered part (e) and justified her answer. She also correctly explained that in answering step 2 question she needed to check if there existed a time for which both runners were at the same distance from the finish line. She correctly used her understanding to compute the location of each runner at each time, moving in increments of 1 second, and successfully answered where and when the two runners would meet, or never meet at all. Interestingly, she did not use geometric representations, such as drawing a line, to answer these questions. She was the only students who moved to step 3 in session 1. She successfully identified the axes of the graph, correctly denoted runners’ initial location, and identified the point denoting the location of each runner after one second of movement. Although she computed the location of each runner at each given time, she did not realize that the location versus time graph is a line, and did not complete the task. Finally, she seemed more comfortable using algebra to solve the problems rather than drawing figures or graphs to explain her answers.

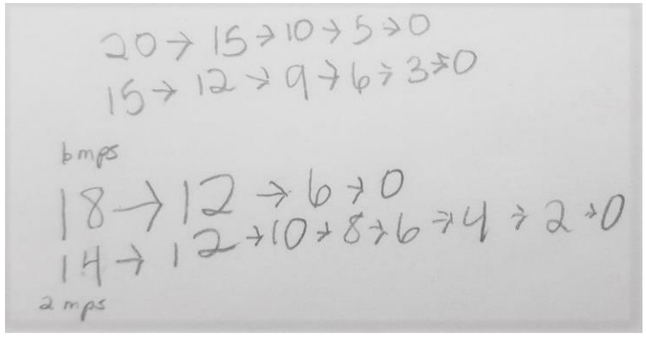


Figure 3: Tina's Computation on Runners' Locations in Time to Determine When and Where They Meet in Interview Session 1

Interview Session 2

In session 2 students were introduced to the simulation. They were given some time to explore the environment prior to the interview questions, initially all students seemed to “play” with the simulation settings. However, gradually they became more purposeful with their setting selection. In the course of the participants’ interactions with the simulation they showed a tendency to test extreme values (i.e., largest and smallest values of each variable, and moderate values) to discover the possible behaviors of the simulation with the least number of simulation trials.

Marry initially focused on the animation generated by the simulation environment, of the cat chasing the mouse, and how the location versus time graph represented this process. She spent a big chunk of her time on these outputs and used them to explain how she could represent locations, such as the head start of the mouse, on a line denoting the location to the finish line.

Nikki focused mostly on the graphical representation provided by the media and following several trials concluded that the cat catching the mouse meant that they were both at the same location at the same time. She further explained that by scaling all variables by the same value, say doubling them, the output, (i.e., if the cat catches the mouse and the time it happens) remains unaffected.

Tina seemed to follow a different approach to the use of the simulation. She immediately started with putting variables at extreme values and gradually changing them to learn about the environment. Instead of focusing on the animation or the graph, she generated ta able in the next tab explaining how this table would inform her about where the cat and mouse were at each given time point. She elaborated on how she could use this information to deduce answers. She then went back to the graph and explored how different settings affected it. She noticed the linear structure of the graph, and she extracted information from it so answer questions.

Interview Session 3

In this session students answered the same question they had encountered in session 1. After working with the simulation environment Marry correctly represented the locations on a line, distinguished the scaling of the problem, and accurately interpreted units of distance and time.

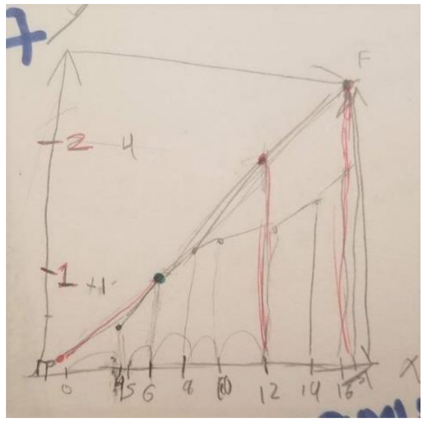


Figure 4: Marry's Representation of the Time Versus Location Graph

She successfully answered step 2 questions, and used a number line to determine if and where the runners met, similar to Tina's solution in session 1. She also completed step 3 and could explain her thinking; however, she was not as comfortable working with a graph. During the first session Nikki could only determine if the wo runners met and approximated the location where this meeting occurred. During the third interview she computed the exact values for time and location of the runners' intersecting. She also drew the location versus time graphs and explained her answers. Tina completed the graph which she could not produce in session 1, and explained how it would be affected by changing the variable values.

Table 3: Summary of Students' Performance and Use of the Simulation

	Marry	Nikki	Tina
Interview 1	-Completed step 1 but not did not correctly answered steps 2 and 3 - Mathematically representing/comparing the physical quantities	-Completed step 1 but partially answered step 2 and did not answered step 3 -Geometric representation, good understanding of the physical concepts -Taking advantage of algebraic tools to simplify and speed up the calculations	-Completed steps 1 and 2 but partially answered step 3 -Good grasp of physical quantities and mathematically expressing the relations. Using algebra to solve the problem -Geometrically and physically interpreting the results
Interview 2	-Completed steps 1, 2, and 3. In particular, successfully drew the location versus time graph.	-Completed all steps successfully. In particular, took advantage of algebraic methods as well as geometric methods.	- Successfully completed all tasks.

Interview 3	-Visualize physical quantities and study their evolution in time. Afterwards, we could mathematically describe and manipulate them	-Learn how to mathematically manipulate quantities, in particular, algebraically describe the problem and solve it using mathematical operations rather than geometric representations	-How to connect mathematical relations to physical quantities and learn about their connections.
Shifts	-Learns how to compare different distance values	-Used to draw figures and compute locations in steps of one second to arrive at a solution. Afterwards, she used mathematical manipulations without drawings to solve.	-Could not describe the graph initially, with the help of simulation gained a better understanding of the problem as a process

Discussion

At the initial stage of exposure to the simulation students shifted between the computer model and computer results for an extended period of time as a means to discover the impact of change in various variable values on the outcomes depicted on the screen. They tended to design a simulation setting, run it, and compare the computer results with what they had computed mathematically, confirming their initial ideas. By repeating this procedure, they seemed to form a more refined understanding of the problem leading to development of more precise descriptions. For instance, Nikki was comfortable using abstract algebraic methods to solve a problem in fewer steps, rather than relying on her visualization skills. On the other hand, Tina seemed to have made a connection between her abstract formulations of key physical patterns, and how they corresponded to a representation or a graph. All three participants, irrespective of their background knowledge, benefited from interacting with the simulation environment as evident in how they solved the task during the third interview. Each student used the simulation environment differently, and tested settings that helped them learn about specific aspects of the problem with which they had struggled the most. In particular, they used the simulation environment to visualize the dynamical system and its evolution, how different quantities relate to each other, how they can be represented mathematically, and if their solution is correct.

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ENACTMENT OF DIGITAL CURRICULA IN ELEMENTARY CLASSROOMS AND IMPACT ON MATHEMATICAL PRACTICES

Shannon O. S. Driskell
University of Dayton
sdriskell1@udayton.edu

Steve Rhine
Pacific University
steverhine@pacificu.edu

Ann Wheeler
Texas Woman's University
awheeler2@twu.edu

Rachel A. Harrington
Western Oregon University
harringtonr@wou.edu

Our consortium of four universities conducted case studies with five teachers in Grades 1 through 4 to explore how elementary teachers are implementing digital curricula, particularly whether they are developing the Mathematical Practices with digital curricula. Through observations with the Mathematics Classroom Observation Protocol for Practices (MCOP^2), a survey, and interviews, we found evidence that there is a broad range of implementation strategies for digital curricula—from occasional use to daily use and from supplemental curricula to full curricula. This study indicates that best practices can be identified and developed for implementing digital curricula effectively in elementary mathematics classrooms. The research has implications for teacher educators and for professional development of inservice teachers teaching with digital curricula.

Keywords: Technology, Elementary School Education, Digital Curricula, Curricula Analysis

With the transformation to digital learning, many school districts are shifting toward one-to-one technology for all students. Due to such initiatives, school budgets for digital curricula are quickly rising: from $1.8 billion in 2013 to $4.8 billion in 2014 (Cauthen, 2017). From 2015 to 2016, digital curricula expenditures increased by 25% so that digital curricula cost "now exceeds all K12 spending by $3.5 Billion" (Kafitz, 2017, n.p). The promise is that "digital devices, software, and learning platforms offer a once-unimaginable array of options for tailoring education to each individual student's academic strengths and weaknesses, interests and motivations, personal preferences, and optimal pace of learning" (Herold, 2016, n.p.). Companies are "rolling out programs in America's public schools with relatively few checks and balances" (Singer, 2017, n.p.). EdSurge (www.edsurge.com) currently lists 175 digital core curricula for elementary mathematics. These curricula are rapidly spreading into elementary classrooms. For instance, Dreambox claims that over two million students are currently using their K–8 program in the United States (Singer, 2017). With pressure on schools to improve test scores and rise to expectations of the Common Core State Standards (CCSS; National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) or individual state standards, districts are seeking solutions for differentiated learning and data-driven instructional decision making. The shift to digital curricula to personal math instruction leads to questions about its support of student learning.

For our study, we examined how teachers implement digital curricula, particularly in regard to Mathematical Practices. We adopted the definition proposed by Pepin, Choppin, Ruthven, and Sinclair (2017) for "digital curriculum materials/resources/programmes" (DCR):

> It is the attention to sequencing—of grade- or age-level learning topics, or of content associated with a particular course of study (e.g., algebra)—so as to cover (all or part of) a curriculum specification, which differentiates DCR from other types of digital instructional tools or educational software programmes (p. 657).

Further, Choppin & Borys (2017) note that "digital materials have potentially transformative features, such as enhanced interactivity, customization, and adaptive assessment" (p. 663). Accordingly, we adopted Pepin et al.'s definition and further clarify it to include only digital curricula that collect and save student data for teachers' use, as assessment is a powerful aspect of digital curricula that can impact teachers' instruction.

Related Literature

Researchers are just beginning to study the effectiveness of digital curricula and the results vary. The self-funded study conducted by the digital curriculum iXL in California found a strong positive correlation between iXL usage and state test scores (Empirical Education, 2013). A randomized control study with 4th-grade students and the Odyssey program found no significant increase in students' achievement when students used the program for one hour per week (Wijekumar, Hitchcock, Turner, Lei, & Peck, 2009). SRI International is studying learning behaviors in digital learning environments as well as the efficacy of the Reasoning Mind mathematics curriculum (SRI Education, 2018). These studies generally focus on the students' experience rather than what teachers do in the classroom.

In another study, Taylor (2013) reported that more than 29,000 classrooms in 216 countries were using Khan Academy (KA). Salman Khan, founder and executive director of KA, recommends using KA to personalize instruction, freeing up class time for engaging, high-yield activities like student discourse and meaningful collaborative projects (Khan, 2012). Khan added that, "ironically, the technology makes the classrooms more human for the teachers and students. It has also made the teachers that much more valuable" (Weltner, 2012). Contrary to Khan's recommendation, however, in a small study, Cargile and Harkness (2015) found that KA was not used to foster more active learning in the classroom; nor was instruction customized to students' progress and achievement levels. Based on the limited and conflicting research on how these programs are being implemented in schools, we have much to learn about choices teachers are making and the mathematics students are learning with digital curricula.

Theoretical Framework

Our theoretical framework consists of four factors that influence teachers' decision making with digital curricula. The first factor is teachers' self-efficacy about their mathematics and technology knowledge. "Academic self-efficacy, teacher self-efficacy, and computer self-efficacy are important predictors of the attitude toward computer-assisted learning" (Yeşilyurt, Ulaş, & Akan, 2016, p. 592). Many elementary teachers bring their own mathematics anxiety with them into the classroom (Bursal & Paznokas, 2006; Haciomeroglu, 2014; McAnallen, 2010; Swars, Daane, & Giesen, 2006). Teachers with confidence in their mathematics tend to better transform lower-cognitive-demand problems into higher-demand problems (Son & Kim, 2016). The teacher's attitude toward using technologies in the classroom is a major factor in how successful technology integration will be (Tabata & Johnsrud, 2008).

A second factor is teachers' perceptions of the balance between procedural fluency and conceptual understanding. "Procedural fluency is the ability to apply processes, techniques, and strategies accurately, efficiently, and flexibly; to transfer these methods to different problems and contexts; to build or modify procedures from other procedures; and to recognize when one strategy or approach is more appropriate to apply than another. . . Procedural fluency builds on a foundation of conceptual understanding" (NCTM, 2014).

Teachers' perceptions of the Mathematical Practices' role in learning is a third factor. Confrey and Krupa (2010) contend, "For students to become proficient in mathematics, they must internalize the eight Mathematical Practices (MP) as the means to learn and understand the content standards. The practices sustain mathematics as the content evolves" (p. 10).

Teachers' perceptions of the affordances and constraints of digital curricula that lead them to enactment patterns is the fourth factor. Studies on the teacher's role in the implementation of new curriculum have examined teachers' curriculum strategy frameworks (Remillard, 1999; Sherin & Drake, 2009), their curriculum enactment patterns (Son & Kim, 2016), and aspects of factors related to teachers' decisions on tasks or problems that they enact during class (Son & Kim, 2015). The decisions teachers make about curriculum can potentially enhance or hinder students' understanding of mathematics (Nguyen & Kulm, 2005).

Methods

Participants and Setting

Participants included five elementary classroom teachers in mostly Title I schools located in three different states that are located in three different regions of the United States: midwest, south, and west. All five teachers were using digital curricula to support student learning of mathematics. This was a purposeful sample (Patton, 2002) since we wanted to analyze the enactment of digital curricula in elementary classrooms and its impact on teaching and learning.

Data Collection

We chose to conduct case studies of five teachers in order to get a deeper understanding of how they approach the use of digital curricula in their classrooms. For our case studies, we used a survey, observations, and interviews to triangulate the data (Patton, 1999), which allowed us to develop a comprehensive understanding of these teachers' enactments of digital curricula. We developed a teacher survey to gather data about teachers' demographics; beliefs about teaching mathematics, teaching with technology, and teaching mathematics with digital curricula; use of digital curricula assessment data; beliefs about how the MPs are supported with digital curricula; and developing students' MPs. To follow up on their responses to the survey, we observed teachers' instruction that included digital curricula. We used the validated Mathematics Classroom Observation Protocol for Practices (MCOP^2) (Gleason, Livers, & Zelkowski, 2015) and field notes during classroom observations. The MCOP^2 is designed to measure the degree to which a K–16 mathematics classroom is aligned with the various practice standards set out by the Council of Chief State School Officers, NCTM, and MAA. While there are different state standards, there is a commonality across states regarding MPs, which is captured by the MCOP^2. Following classroom observations, we used a semi-structured interview (Hitchock & Hughes, 1989) to learn more about teachers' beliefs about teaching and learning with digital curricula.

Results

On the teacher survey, all five teachers strongly agreed that: a) All students need to have a range of strategies and approaches from which to choose in solving problems; b) Mathematics learning should focus on developing understanding of concepts and procedures through problem solving, reasoning, and discourse; and c) The student's role is to actively make sense of mathematics tasks by using varied strategies and representations, justifying solutions, making connections to prior knowledge or familiar contexts and experiences, and considering the reasoning of others. All five teachers disagreed that: a) Students need only to learn and use standard computational algorithms; and b) The role of the student is to memorize information that is presented and then use it to solve routine problems on homework, quizzes, and tests. For the other two questions: a) two somewhat agreed and three disagreed that mathematics learning should focus on practicing procedures and memorizing basic number combinations, and one somewhat agreed and four disagreed that students can learn to apply mathematics only after they have mastered the basic skills.

All five teachers reported that they were confident in their use of technology in the classroom. However, their confidence differed in using digital curricula to teach mathematics: one was very confident, one confident, two somewhat confident, and one not confident. These teachers also varied in their responses to the effectiveness of digital curricula to learn mathematics and to develop factual recall, procedural fluency, conceptual understanding, and mathematical reasoning/problem solving (see Table 1)(no teachers disagreed).

Table 1: Teachers' Responses to the Effectiveness of Digital Curriculum

	Mathematics Learning	Factual Recall	Procedural Fluency	Conceptual Understanding	Reasoning/ Problem Solving
Strongly Agree	1	2	1		
Agree	2	2	3	4	2
Somewhat Agree	2	1	1	1	3

Further survey results indicate that all five teachers use digital curricula for skill and practice, four use it to differentiate instruction, and four use it for acceleration of content; only one teacher uses it to support critical thinking. During their planning and instruction, four teachers often and one teacher sometimes intentionally spends time developing either the MPs or their state's own process/practice standards. See Table 2 for a summary of the varied ways the teachers report using digital curricula to assist students in developing the MPs.

Table 2: Teachers' Reported Use of Digital Curricula to Develop the MPs

	MP1	MP2	MP3	MP4	MP5	MP6	MP7	MP8
Extremely Well						1	1	1
Well	3	3	2	4	4	4	3	4
Not So Well	2	2	2	1	1		1	
Not At All			1					

To capture further evidence of teachers' enactment of digital curricula and its impact on MPs, we share case studies of each teacher.

Case Study of Allison

Allison is a 3rd-grade teacher with an emergency credential in her second year of teaching. She is a confident early-career teacher who loves teaching mathematics and is passionate about her students understanding concepts and developing procedural fluency. She has never taught any other way than with digital curricula. Her district requires her to use Go Math! daily for primary instruction and Reflex Math for practicing procedural skills. Each Go Math! lesson begins with whole-group video instruction on a particular math concept.

Allison is keenly aware of the importance of developing the MPs with her students. She is not satisfied with simply pressing "play" for the Go Math! videos and having the computer "teach" her students. Instead, she designs preliminary activities to get her students thinking about the concepts prior to watching the video. When students watch the video, Allison stops it at key moments and poses questions to her students. She finishes the lesson with an additional activity designed to enhance students' conceptual understanding. For example, when teaching a lesson on fractions, Allison had the students first think about a "fair share." While analyzing squares divided into fourths two different ways, Allison asked the students to think about whether all the pieces in the squares were fourths and to be prepared to justify their reasoning. After individual time, pairs met to discuss their thinking. She then chose certain students to explain their partner's thinking and how it was the same or different from than their own (example of MP2). She purposely chose those students to provide a range of perspectives.

After this class discussion, Allison started the Go Math! video lesson. She frequently—about every 30 seconds—stopped the video to pose purposeful questions: What do you think he will do next? (MP1). What do you think of the strategy he uses? Is there another strategy? (MP2). Can you draw a picture of what he is talking about? What would be a word problem for this number sentence? (MP4). Can you write that number sentence a different way and get the same answer? (MP4). After the Go Math! lesson, she gave each student a square from a chocolate candy bar, asked them to break the square into two or more pieces to form a new shape, and then asked, "Do you still have a fair share even though it is a different shape?" (MP4).

As Allison considered how to make teaching with digital curricula more effective for her students, she wondered: What things can I do before that will enhance the digital curricula? Where should I stop the video and discuss/predict? How can I make it more meaningful for them? She was enthusiastic about the data available on individuals' progress and eager to make use of that data to inform her teaching choices and differentiate instruction. Allison resisted when administrators insisted on fidelity of the program's implementation, arguing that she was acting in the best interest of her students' understanding. She was very concerned about other teachers in her building who were not as confident in teaching elementary mathematics and would simply press play and let the program "teach" their students. Even as an early-career teacher, she was considering ways to help enhance other teachers' work with digital curricula.

Case Study of Elise

Elise is a 4^{th}-grade teacher who has been teaching for 14 years. She is certified for Grades 4 through 9. She truly enjoys teaching mathematics, and teaches it for about 75 minutes each day. The school district adopted Eureka Math as the paper curriculum and Zearn as the digital curriculum. Although it's not required, the administration strongly encourages her to use Zearn. Each week, Elise's students use it for about 45 minutes during math class, an additional 20 minutes outside of math class, and about 45 minutes at home. The content the students explore in

Zearn typically aligns with the daily objectives. She chooses to use Zearn for acceleration, differentiation of instruction, remediation, and skill and practice.

Students typically used Zearn for 20 minutes when Elise organized different stations. All Zearn lessons began with *Number Gym* or an individually adaptive fluency experience for math skill practice. Next, students completed a fluency *Blast* that was aligned with the lesson and was the same for all students. Students then completed a guided practice and solved real-world problems that used the concrete, to pictorial, to abstract approach (MP1). Afterward, students transferred their digital learning to pencil and paper to draw representations of their thinking process (MP4). Finally, students began independent practice in the *Tower of Power*. Elise did not interact with the students while they worked on Zearn. Therefore, Elise did not personally engage the students working on Zearn in developing MPs.

During the interview, Elise expressed, "I would rather me doing instruction with them, and me working with them than having a computer program work with them." She did not feel that using digital curriculum changed her instruction: "I am still very much concrete, pictorial, abstract." She believed that it was "hard to monitor when they are on a computer because you don't even know if they are on the program half the time." As for teaching the MPs, she felt, "it is hard with Zearn. I see it like a remediation of what we have done in class." She wanted "to see their work" so that she could ask the students to analyze their work and "learn from their mistakes and have a discussion about it." She informed me that she knew students need to do math on the computer because they take the state test on the computer. She believed that there is a time and place for digital curriculum, but she did not want to be required to use it because it limits her teaching. "It's almost like you're micromanaging instead of trusting the individual teachers to make best choice for best practice for the students in their classroom." Elise did clarify that her confidence in teaching mathematics allowed her to see herself as a math teacher, but most elementary teachers "don't see themselves as a math teacher . . . they don't know the why behind the how." Those teachers, Elise felt, may rely on digital curricula more than she would, because she won't let the computer take over her teaching.

Case Study of Lindsay

Lindsay is a 1st-grade teacher who is in her third year of teaching. Her school district adopted Pearson's enVisionmath2.0(C) as the paper curriculum and Istation as the computer adaptive web-based platform, utilized for about 60 minutes during Specials time in a computer lab once a week. Istation is used during regular class time for an additional 20 minutes per week. Even though Istation covers 1st-grade content, it is not meant to align with the daily mathematics objectives. Lindsay utilizes its content for acceleration, skill practice, and Specials time.

Lindsay's two observations during class and during Specials time were noticeably different. Lindsay's typical 45 minutes of instruction began with a brief introduction to balancing equations. The class then proceeded to work in groups of 2–4 students with various mathematics manipulatives, such as Unifix cubes, two-color counters, dice, blocks, and balances, as well as dry-erase markers and whiteboards (MP4). Students were tasked with writing multiple equations that would balance (e.g., $4 + 1 = 5 + 0$) and explaining their logic to their group (MP1, MP3, and MP7). Lindsay constantly walked around to help struggling students and to encourage students to do their best. At the end of the lesson, the students came back together on the carpet to review what they had learned, with some students sharing their equations with the whole class (MP6).

As opposed to this interactive whole-class discussion and activity lesson, the Istation lesson involved students individually working on their mathematics problems. Students wore headphones and were encouraged to be quiet and work hard by themselves. If students struggled with the content, Lindsay answered their questions. Individual Istation problems varied, based on student ability. Many of the problems included interactive components, with students often singing or dragging virtual manipulatives around the screen. If students had difficulty with a concept, they were given multiple problems to help them. If they passed, they would move on in the content; if they struggled, the program would not move on to new content. Throughout this lesson, Lindsay did not engage students personally in developing MPs.

During her interview, Lindsay expressed her love for teaching mathematics face-to-face and the importance of student interactions. Lindsay felt that digital curricula were helpful as practice for procedural and conceptual fluency. She also felt that the more advanced students and those who struggled preferred digital curricula. The advanced students like to work at their own pace, and those who struggle felt the computer was sort of a safe space for them because they could make mistakes and no one would see them struggle.

Case Study of Sarah

Sarah is a 5th-grade teacher with nine years of teaching experience. She is certified for kindergarten through 5th grade and recently earned the Elementary Mathematics Instructional Leader Specialization. She truly enjoys teaching mathematics and is considered a leader in her district and school. The school district uses Eureka in 3rd-5th. The teachers also have access to IXL, which was chosen by district administration. Teachers were not given specific professional development on how to use it, and it is their discretion on how to implement it. Sarah's students typically spend one day a week on IXL, and she chooses a standard from that week's instruction for them to explore. In the interview, she shared that she sees IXL as a time for students to practice at their own pace. She also sees it as a time for students to have a "break" from Eureka and for her to have a "break" from planning. She does not use the data generated by IXL for instructional purposes, and she does not share results of student work with parents or guardians.

Sarah was observed twice: once while teaching a typical lesson and once on an IXL day. In the typical lesson, she introduced the concept of division. She had students out of their seats, grouping themselves into different animal herds. She asked the students to make observations about group size, number of groups, and remainder size (MP1, MP2, and MP4). On the IXL day, students worked quietly on tablets as she moved around the room answering individual questions. Sarah focused most of her time on a few students who needed extra support. In contrast to the first observation, Sarah was developing only MP1 with her students.

Case Study of Shelby

Shelby is a 3rd-grade teacher who is in her sixth year of teaching. She taught 8th grade for one year at a different school; since then, she's taught 3rd grade at her current school. Shelby's school district adopted Pearson's enVisionmath2.0(C) as the paper curriculum and Istation as the digital curriculum, which the students utilize for about 40 minutes during Specials time in a computer lab once a week and 1.5 hours per week during class time. The main digital curriculum for 3rd grade is Istation, but the computer lab teacher is also piloting Prodigy for potential future use instead of Istation. Students may access Istation at home, but are not required to do so. The Istation lessons are not meant to align with the daily mathematics objectives but do cover grade-level content. Shelby utilizes its content for acceleration, homework, differentiation, remediation, skill practice, and Specials time. Students independently work at their own pace with the content.

Shelby's two observations were clearly different. Shelby's 60-minute in-class observation involved students working on a warm-up as a review of previously learned mathematics concepts. Students then went to the carpet. Shelby gave certain students construction paper and students worked through scenarios as a class involving fractional relationships, such as $\frac{1}{4}$ and $\frac{3}{5}$

(MP3 and MP4). Students then went back to their seats and listened to the ebook *Give Me Half* by Stuart Murphy. Shelby asked students to write certain answers to fraction questions, and also reinforced their learning with playdough-based fraction work (MP1 and MP4). Lastly, Shelby had her class complete a fraction pizza worksheet (MP7). Shelby actively walked around the class, asking engaging mathematics questions and helping struggling students (MP6).

During the Istation lesson, students were quietly working through problems on their own. If stuck, some students used whiteboards or paper and pencil to figure out problems. Some students had difficulty with the vocabulary, in which case Shelby reassured them to try their best. Like Lindsay's students, Shelby's students often had interactive mathematics manipulatives on their screens that they could drag and drop to figure out such concepts as multiplication and rounding. Throughout this lesson, Shelby did not engage students personally in developing MPs.

During her interview, Shelby expressed her desire for more control over the content students were learning in the digital curricula, and she wanted the ability to choose the weekly math standards. Although there were downsides to Istation, Shelby felt the engagement aspect of digital curricula was good for students. She also liked that the program identified students' strengths and weaknesses, and that it provided practice. In addition, Shelby felt that unmotivated students or students who have "tuned out" the teacher learn best from digital curricula.

Discussion and Conclusion

Although there was some consistency in teachers' beliefs about teaching mathematics and in their confidence in using technology, there were some inconsistencies among their confidence in teaching mathematics with digital curricula, their beliefs about the effectiveness of digital curricula, and their use of digital curricula to assist students in developing the MPs. Several other differences were evident in how teachers enacted the digital curricula—independent work, large-group instruction, and stations. Two teachers enacted digital curricula during Specials, while three teachers enacted it during math class.

The perceived benefits of digital curricula also varied. One teacher noted the personalized instruction aspect and thought that the advanced students enjoyed being able to work ahead, while the struggling students felt safe making mistakes without being noticed. Identifying students' strengths and weaknesses, instant feedback, practice time, and students' increased success on standardized assessments were also perceived benefits. The change from interacting with peers to interacting with the computer was a concern of one teacher. Not being able to see students' work so that she could analyze their mistakes with them was a reason one teacher would rather use paper and pencil versus a computer for instruction. Several teachers expressed concern that they lacked control over what content students learned or what websites they explored, or even felt a loss of their own teaching identity.

Navigating the shift to digital learning presents several obstacles for teachers, and teachers need ways to deal with such obstacles. There is a separate knowledge base for teaching with paper and pencil than for teaching with digital curricula, and addressing this knowledge base needs to be one focus of preservice teachers' education programs and inservice teachers' professional development programs. With the advancements of innovative technologies in this digital age, digital curriculum is here to stay—at least for the near future. Teacher education and professional development programs must support teachers through the shift to digital learning. The State Educational Technology Directors Association, in *Navigating the Digital Shift 2018: Broadening Student Learning Opportunities*, recommends that "States must provide leadership as educational opportunities switch to the use of digital instructional materials to support student learning and successes" (p. 2). We echo this call and further recommend that teacher education programs address preparing preservice teachers to navigate this shift.

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INFRASTRUCTURE TO SUPPORT STUDENTS EXERCISING CONCEPTUAL AGENCY

Isaac Nichols-Paez
Vanderbilt University
isaac.t.nichols@vanderbilt.edu

Corey Brady
Vanderbilt University
corey.brady@vanderbilt.edu

Generative activities have been shown to support students to engage in space-creating play and exercise their conceptual agency to generate a mathematical space (e.g. Stroup et al. 2004), yet these studies implement generative activities only with their resonating counterpart, classroom networks, technological infrastructures that connect multiple, co-present students into a shared, digital representation. Because these technologies are in continuous redesign and still inaccessible to many classrooms, we need to understand the crucial features their infrastructure provides to the classroom system. By analyzing the strains on the classroom without classroom networks and how they relieved that pressure and revive the system, we found that the collective public displays provided students with a collective orientation and a sense of connection and individualism.

Keywords: Design Experiments, Technology, Rational Numbers

Introduction

Generative activities are activities operating at the individual, small group, and whole class within which students are actively constructing connections and relations of mathematical ideas in both prepared and emergent participation structures that reflect and build on the mathematical ideas that the group creates (Stroup, Kaput, and Ares 2002; Stroup et al. 2004; Stroup, Ares, and Hurford 2005; Ares, Stroup, and Schademan 2009). In these types of activities, the class's social group functions to explore mathematical structures together and uses their social dynamics as a purposeful resource to support their exploration. A common means for designing and developing such activities take a standard, closed-form question as a starting point, and "inverts" it, making the *answer* of the standard question into the prompt for the generative activity. For example, instead of asking students to "simplify 4(x-3)+12" (a closed-form question, with correct answer "4x"), one might ask them each to create several expressions that are "the same as 4x" (Stroup, Kaput, and Ares 2002). By inverting the traditional one-correct-answer task, generative activities provide ways for students to construct or apply mathematical principles (e.g., exploring additive inverses by repeatedly adding "+x-x" to an expression known to be equivalent to 4x. When this kind of construction is occurring in parallel across the classroom, students are able to use the diversity of their group and their ideas for experimentation to generate a mathematical *space*.

Stroup et al. (2002; Stroup, Ares, and Hurford 2005) describe the resonance of generative activities with classroom network technologies to provoke new theoretical, methodological, and design frameworks. They articulate two main principles in the flow of a generative activity: (a) space-creating play and (b) dynamic structure. Space-creating play is the idea of students generating a mathematical space via experimentation, exploration, and playfulness. Dynamic structure refers to the emergent set of connections and meanings that appear as the students produce mathematical creations and respond to each other's work, both by commenting and by imitating, expanding on, or combining work to make new creations. Dynamic structure makes use of a functional sense of activity structure that is brought into being through students' playful actions and characterizes the unfolding space students are generating. Stroup et al. use these two ideas to argue that the relationship between mathematical/scientific structures and social structures is dialectical, with each mutually building off of the other. Essential to this process is the collective, public display of students' mathematical space, either in some physical/digital inscription or through social display.

As a complementary perspective of these public displays, we can consider them a space for *conocimiento* (Anzaldúa 1987 cited by Gutiérrez 2012), or sense of becoming familiar, connecting, and receptive of others. Through students' shared solidarity in generating the mathematical space, they develop their conocimiento of both the unfolding mathematical structures and the persons engaged in the display. Additionally, public displays of their work at the whole-class level may support students' sense of *nos/otras* (Anzaldúa 1987 cited by Gutiérrez 2012), or the juxtaposition of the collective and the individual. Further connections of this perspective with generative activities and classroom networks is unexplored and possibly very fruitful because of their differences in framing knowledge but similarities in positioning participants as generators of that knowledge.

Though Stroup et al. further describe the *resonance* between generative activities and classroom networks, arguing that the networked classroom is particularly suited to support a dialectic relationship between space-creating play and dynamic structure, few studies have explored these constructs in mathematics classrooms without the technology[1]. Substantial research has shown the impact of these new networking technologies and their resonance with generative activities (e.g. Ares, Stroup, and Schademan 2009; Ares 2013; Stroup, Carmona, & Davis, 2011), but these technologies are both largely unattainable for most classrooms and still going through continuous redesign. Thus, we need to understand the specific features of the classroom network critical to fostering collective mathematics inquiry through space-creating play and dynamic structure and which are optative. Furthermore, understanding which of the features should be customizable and which are fairly generic to collaboration will both support continued technology design and strengthen the underlying theory of collective mathematics. To investigate these features of classroom networks, we investigate 1) Do generative activities and collective mathematical exploration put strain on normal classroom infrastructure? (and, how?) and 2) Which aspects of classroom networks alleviate that pressure? (and, how?).

Classroom Networks and the Group-based Cloud Computing System (GbCC).

Classroom networks have been an area active, but uneven, research and development for over 20 years (or much longer, depending on one's definition (see Abrahamson 2006; Abrahamson and Brady 2014; Roschelle, Penuel, and Abrahamson 2004)), with a varied history of research and commercialization efforts. For the purposes of this paper, a classroom network (c.f. Brady et al. 2013) is a representation and communications infrastructure (Hegedus and Moreno-Armella 2009) consisting of hardware, software, and curricular/activity components. The hardware includes a set of devices (laptops, smartphones, or other custom communications-enabled "computers"), with each student (or, less commonly, each small group), having a device. These devices are networked to communicate directly or indirectly with each other and with a teacher computer, which is connected to a public display (usually a digital projector). Software, running on the classroom computers and/or on a networked server, provides aspects of communications infrastructure by routing messages among the participating devices in configurable, activity-specific ways. Software also provides a representation infrastructure, offering students and teachers views of the activity and tools to contribute that are appropriate for the discipline, the activity, and the participants' roles. Finally, at the curricular/activity level, "documents" or other specifications of roles or goals can be sent to participants to configure their devices and displays, and to facilitate the activity in real time.

GbCC (Brady et al. 2018) is a system of this kind, emphasizing flexible programmability and rich discipline-specific representations for mathematics, science, and the social sciences. It leverages browser-based open-source tools, building upon the NetLogo Web agent-based modeling environment (Wilensky 2015), married with GeoGebra Web (https://www.geogebra.org/) as a dynamic mathematics platform for geometry and algebra in Euclidean and Cartesian representations; and several other extensions to support mapping (Leaflet, https://leafletjs.com/) and 2d physics (Box2d, https://box2d.org/). As a platform for design-based research environment, its programmability supports an open-ended array of activity structures, and it can be run on any browser-enabled device (phones, tablets, or laptops). Its flexibility, configurability, and programmability make it ideally suited to exploring our research questions.

Data and Methods

The current study was a single four-week cycle from a larger design-based research (DBR) project. The 20 participants came from a 5th grade classroom at a public middle school serving a racially (39% Black, 6% Hispanic, 4% Asian) and economically (41% free or reduced lunch) diverse population within a large metropolitan district in a midsize southern city in the USA. The class period of the DBR study was not students' normal mathematics class but a time when students were tracked based on standardized tests in order to provide individualized attention (called Personal Learning Time, PLT, in the school). The participants from the current study were considered math tier 2 students (i.e., on target but needing some extra time for mathematics). Because of the nature of standardized testing and the flexibility of this class period, students moved from tier to tier or subject to subject depending on the most current testing. Thus about half of the students in the current study had participated in a prior implementation of a design cycle with generative activities without technology[i]. The first author facilitated about 2 class sessions each week over a four week period totaling of 8 sessions, each 30-45 minutes in length, and the classroom teacher either co-facilitated or pulled specific students for individual work.

The primary data source for the current study was design and field notes taken by the first author. Audio and video recordings of each lesson were also collected and used to triangulate findings. Analysis was ongoing and continuous throughout the design where the humble theories of the class's mathematical thinking and engagement were revised after each lesson (Cobb et al. 2003), in conversations among the researchers and with the teacher. Posterior analysis took the form of reviewing the progression of the lessons contrasted with the predicted learning trajectory. We paid special attention to anticipated and unanticipated challenges and strains on the classroom system prior to introducing network technology and the nature of how those challenges and strains changed when using it.

Mathematical Context and Predicted Learning Trajectory

We chose to target 5th grade fractions standards involving equivalence, operations, and comparison for this study. Fractions have been found to be a particularly difficult concept for students, yet they can be readily used as the basis for generative activities because the mathematical space of equivalent fractions is both core to the standards and very rich. We created a sequence of generative activities, to explore equivalence for the first two weeks and then operations on fractions for the second two weeks. The activity for both topics followed a similar rough structure. The first day of each of the two weeks focused on "space-creating play" to generate the space of ways to make ½, either with equivalent fractions or with fraction operations, depending on the topic. Students worked in small groups during these times, to foster connections in their space-creating play and reflection on the dynamic structure they were creating. Following this small group work to make ½, a whole-class discussion explored the different kinds of objects in the space (to make ½) and the mathematical principles students used to generate the space. Building off this the following class session (a week later), students returned to small groups to generate ways to make a fraction of their group's choice followed by another whole-class discussion of the mathematical principles. This trajectory was supported by research both on fractions (Lamon 2012) and generative activities (Stroup, Kaput, and Ares 2002; Stroup, Ares, and Hurford 2005), the key difference from the latter was the lack of networking technology. Beyond the curricular goals, we predicted the generative activities would support students to take conceptual agency (Boaler and Greeno 2000) in the classroom to create mathematical principles of equivalence and operation and to voice their conceptual perceptions even without technology. We remained open to the question of whether these technologies would be needed, by observing the classroom system, students' engagement in the tasks, and the degree to which they exercised conceptual agency.

Results

Through our design and analysis of generative activities to support students' conceptual agency in exploring fractions *without* technology, we found that these activities put multiple strains on the classroom system for students to engage and participate. Without the technological infrastructure and additional ways to participate in the activity, the whole-class discussions led by the first author were not able to support students to have a platform to show the work they did in small groups, or to have much of a "voice" at the whole-class level. This central strain reduced students' engagement over time, and following the second whole-class discussion (week 2), the necessity of additional infrastructural support was apparent, both to the authors and to the classroom teacher. Upon the introduction of technology, students' re-engagement in the generative-activity process was visible, as usual with the introduction of any new technology. Yet more meaningfully, students' engagement was sustained through the last two weeks, and their conceptual agency increased in that time. This process contrasted significantly with the time without technology when their engagement and utilization of conceptual agency decreased over the course of the same time period. By comparing the strains on the classroom system during generative activities without technology and how the infrastructure provided by the technology relieved those strains, we can begin to identify some of the crucial features of classroom networks.

Generative Activities' Strains on the Classroom System

Progressively throughout the first two weeks of equivalent-fraction generative activities, we documented how students became less and less engaged and utilized their conceptual agency less and less. This process came to a climax when the classroom teacher requested a change in the activity in order to re-engage students at the end of week two. Upon analysis of the design, students' disengagement was progressive. Students engaged readily in the initial generative activity convening the space-creating play in almost all the small groups. Some groups even utilized their conceptual agency to recognize patterns and methods in their generation of equivalent fractions. Yet, during the whole-class discussion, students struggled to know how to participate in productive ways and see their hard work validated. Multiple students made various bids to read aloud their list of fractions in its entirety, but with upwards of over 30 fractions, this was not logistically possible. Moreover, without a means to organize or represent these contributions visibly, a reading would not have contributed to the dynamic structure. Instead, the first author focused on having students share out their methods of generating fractions and patterns they observed in their set of equivalences. While students did engage in the discussion and built multiplicative conceptual resources for fractions, field notes capture a number of students' feelings of discontent.

The following week, the first author launched another generative activity to build on students' work with ½ by generating fractions the same as a fraction of their choice. Unlike the start of the previous activity, the teacher and the first author struggled to support students to begin the activity (even to choose a fraction), and to convene space-creating play in their small groups. In the students' eyes, the small group work had lost its importance and meaning after the previous week's whole-class discussion when they perceived their work was left unchecked, ungraded, and unshared with the class. While either adult was present, students would work together to generate equivalent fractions, but their motivation reflected a perceived lack of importance of their work at the whole-class level. Thus, students' patterns and methods were much less robust during the whole-class discussion the following day, and fewer students participated. Additionally, one of the students from the previous week made another bid to read all of her fractions aloud, demonstrating a continued desire to showcase her work at the whole class level, to hear her voice as part of the group, and receive validation for the effort she had put in. Because of students' steeply declining engagement, we decided to introduce technology to re-engage students and support their sustained participation in generative activities. Our prediction was that the introduction of technology would quickly re-engage students with the task of generative activities, and that comparison in students' sustained engagement would reveal the some of the crucial features of classroom networks to support students' collective mathematics in generative activities.

Adjusted Learning Trajectory and Use of Classroom Networks

Because of the strains of the classroom system for students to see their work as meaningful at the whole-class level, we adjusted the research plan to incorporate GbCC support for the activities in the final two weeks. Since the activities designed with the technology did not strictly align with the original learning plans, we adjusted the curricular goals to target fraction comparison instead of fraction operations. We planned to use GbCC's public display to create a joint representation for students to see a reflection of themselves and their classmates as they engage with mathematics. The classroom network assembled students' fraction input as a character moving on a vertical line between a teacher-defined maximum and minimum value, with its y-coordinate corresponding to the fraction value. The class appeared as a collection of these characters moving between the max and min values. If a student's fraction input was outside of this range, their character was shown into a gray area above or below. The goal of the first week was for students to make connections from their work with equivalence within the technology as a way to begin to understand the representational forms it used and then for the class to quickly transition into comparing 'easy' fractions. We wanted students to have the chance to explore within a technologically enhanced representational world and for the class to see each other's explorations to discuss our methods and strategies. In this way, the classroom network would provide additional communicative pathways for students to feel their work and their classmates' work were meaningful at the whole-class level. We planned to end the activity sequence with supporting students to see the *density* of fractions (i.e. that between any two fractions there is another fraction). We conceptualized this as a 'zooming in' effect with the technology where the teacher could make the range a subset of the previous defined range and fractions could still be found.

The first two days of implementing GbCC went as predicted. The technology served to revive students' engagement and enthusiasm while also providing additional tools and representations to the work they were doing as a whole class. The public, anonymous display provoked a collective responsibility to fill it, positioning students to hold each other accountable during the activity, and during whole-class discussions, this public representation was a collective object for us to reference. During this space-creating play, students exercised their conceptual agency by choosing personally relevant numbers (not something seen the previous week). For example, one student found the fraction equivalent to 1/7th where the numerator was her birthday (mmddyy). Students patterns and methods extended the ideas from previous weeks using multiplicative relationships to generate equivalent fractions.

The final week of the study focused on comparing fractions, with the goal of students' having insight into the density of fractions. We started with a whole-class discussion of the previous weeks' work and asking if students had ways to know if one fraction is bigger than another (no technology). Even without technology, the class sustained a meaningful discussion, leveraging the collective perspective provided by the classroom network activities the previous week. In the following two days of activities, students sustained engagement and motivation, unlike the second week without classroom networks. Furthermore, students' utilization of conceptual agency grew as their fluency with the technology grew, compared to declining as their engagement declined, in the first two weeks. As students interacted with and in the mathematical space, a few began to use the public display as a dynamic representation - moving their characters across the screen by manipulating their fraction input successively. This type of play showcased how the classroom network became an embedded infrastructure for students to represent movement and communicate their actions to me and to others. Additionally, while these playful actions were unexpected and in fact went against the underlying goal of the activity for students to develop insight into the density of fractions, students were developing individual and share-able fluency with manipulating and comparing fractions in service of the personally-meaningful goal to predict the movement of their character up, down, and into the middle. Such spontaneous, and unpredicted, utilization of conceptual agency was not present without the classroom network's representational and communication infrastructure.

Crucial Features of the Classroom Network

The above analysis explored how generative activities strained a classroom system without adequate representational and communicational infrastructure and identified features of classroom networks that were crucial to relieving those strains and supporting students in utilizing their conceptual agency. The collective, public representation of students' work with fractions was the focal point of two such crucial features that supported collective mathematics and that were very difficult to provide without technological support. First, as demonstrated in the first week and the follow-up discussion without technology, the *public display of an aggregate representation of students' contributions* provided an essential means of discussing the activity, referring to students' work in context, facilitating activity flow, and sustaining students' attention. Leveraging this feature, we were able to facilitate whole-class discussions where students engaged in illuminating the underlying multiplicative structure of equivalent fractions and continue the conversation even when the technology was temporarily removed. These types of whole-class discussions were very different prior to implementing the technology when students did not have such a collective orientation, and they made multiple bids to reorient the discussion towards what they felt was important (e.g., their personal lists of equivalent fractions).

The second crucial feature relating to the public display was *the communal, real-time dynamic nature of the public representation*. Students displayed a sense of both collective effort and individual publicity, or *nos/otras* (Gutiérrez 2012). Simultaneously feeling both connected to the community and represented as an individual was essential for collective mathematics. The importance of this feature was demonstrated first when the classroom network was first introduced as students began to hold each other accountable to participate in the activity, and it grew further when students began utilizing their conceptual agency and publicizing their new skill of predicting the movement of their character, showing their abilities to others and sharing how they did it.

Discussion: Students Utilizing Conceptual Agency with the Technology

Understanding how the representational and communicational infrastructure of classroom networks support students' space-creating play and utilization of their conceptual agency can provide insight into these technologies' functionality and support their ongoing design. At the same time, it also can inform efforts to enact generative activities without classroom networks, identifying needs and resources for alternative supports in such classrooms. Based on our comparison here of a classroom with and without technology, two crucial features of the dynamic infrastructure emerged, in the collective orientation provided by the public representation and the simultaneous communicative avenues of collective and individual voice developing a sense of nos/otras. These aspects are vital to keep in mind as we continue to design classroom networks, infrastructure, and activities to further support students exercising their conceptual agency.

Additionally, generative activities need to be flexible enough to support students' adaptation of the task as they exercise their conceptual agency. Similar to work in microworlds (Edwards 1998), generative activities supported by classroom networks are not capsules of disciplinary learning and conceptual agency. Rather, we need to design for and encourage students to make expressive and unpredicted conceptual moves as they interact with the representations and concepts of the activities. On the other hand when the classroom system does not have the infrastructure of classroom networks, traditional infrastructures must be adjusted to foster collective orientation and nos/otras. Specifically, students need some form of collective representation of the concept to orient their individual or small-group work towards each other. Furthermore, social infrastructure must support students as they make their work public to both hear their own voice and, metaphorically, hear the voice of the choir. Over time, classroom systems can develop these types of social infrastructures through socio-mathematical norms, but classroom networks may foster more rapid development of them or a lower threshold of effort for sustaining them over time.

Implications for Further Research

Classroom networks provide a flexible space for students to interact, both with mathematical ideas and with each other, and a dynamic, public display of their work as it unfolds. This space quickly creates infrastructure in the class to foster students' prolonged engagement and utilization of their conceptual agency. Yet pragmatically, teachers, administrators, and researchers may question the necessity of this technology when compared to its cost and disruption. By observing and documenting first how a classroom group experienced strain without the technology and then was supported by it, we understand better the value of the technology, what types of additional activities may supplement it, and ideas on how we might support the classrooms without it. Additional work should compare other types of representational and communication infrastructures (Hegedus and Moreno-Armella 2009) and curriculum activity systems (Roschelle, Knudsen, & Hegedus, 2010) to better understand how students participate in collective inquiry and the necessary of these infrastructures to support students in exercising their conceptual agency. Specifically, previous studies have shown collective inquiry is possible without technology (e.g. Ball 1993; Lehrer, Kobiela, and Weinberg 2013; Fiori and Selling 2016), and exploring the infrastructure imbedded in these types of classrooms will provide insight into both the dynamics of group mathematics learning and into the design of networking technology.

Endnotes

Stroup's introduction of the construct of generative activities clarifies that their roots lie outside of mathematics, connecting to work in reading comprehension by Wittrock and in shared identity building by Freire (the identification of a community's "generative words").

[2] A disruption of losing half the participants and gaining the same number of new students caused analysis of the two design cycles to lose much of its meaning, but the class during the analyzed cycle remained intact.

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A COMPARISON OF U.S. AND CHINESE GEOMETRY STANDARDS THROUGH THE LENS OF VAN HIELE

Lili Zhou
Purdue University
Zhou756@purdue.edu

The aim of the present study is to investigate through a van Hiele theory lens the difference between Common Core State Standards of Mathematics (CCSSM, 2010) and Chinese mathematics standards—Compulsory Education Mathematics Curriculum Standards (CMCS, 2011) in descriptive geometry. A descriptive learning expectation (DLE) is defined as a standard which focuses on properties of figures and relationships between figures without considering the exact quantity. Based on the DLEs in CCSSM and CMCS, Different types for each van Hiele levels were developed and each DLE was identified into a van Hiele level and a particular type. Through examining DLEs distributions of van Hiele types, this study investigated how CCSSM and CMCS portray the development of students' geometric thinking. The findings suggest that CCSSM and CMCS emphasize different approaches in geometry teaching and learning.

Keywords: Standards, van Hiele theory, Descriptive geometry

Purpose of the Study

Through cross-national comparative studies, researchers identify differences between nations which enables them to suggest ways to improve teaching, learning and curriculum development from a unique perspective. The purpose of comparisons is to identify factors that can support student success in mathematics learning and performance. One factor influencing student learning is the curriculum. Teachers and students determine what should be taught and learned based on the previously-developed curriculums. What is taught and learned is informed by standards (Schmidt et al., 2001) in addition to other factors including teachers' beliefs and districts' or schools' policy.

This study considers standards as curriculum learning goals and focuses on a subset of geometry standards for CCSSM and CMCS. These two documents were published at roughly the same time and have been shown to have great influence on teaching and learning practices in their respective countries (Reys, 2014). Through investigating the differences and similarities of CCSSM and CMCS, this study intended to reveal how the two documents delineate the learning expectations and how standards influence other curricula. In addition, this study investigated how CCSSM and CMCS depict the development of students' geometric thinking.

Theoretical Framework

Van Hiele theory (Van Hiele, 1984) identifies five levels of students' geometric thinking and provides a framework for curriculum development, teaching and learning (Clements, 2003). Van Hiele theory have been used by many researchers in different ways. Some researchers used it to describe students' geometric thinking levels (e.g., Fuys, Gerddes & Tischler, 1988; Mayberry, 1983; Senk, 1989; Usiskin, 1982), whereas other researchers used van Hiele theory as a framework to inform curriculum development (e.g., Dingmen, Teuscher, Newton & Kasmer, 2013; Newton, 2011). In this study, I examined the expectations of geometric thinking in CCSSM and CMCS through a van Hiele theory lens. This framework makes explicit how standards portray the development of students' geometric thinking, specifically, what approaches address students' geometry levels and lead them to formal deduction. However, in terms of individual students, they have different ways to learn and think. Even though two students are in a same van Hiele level, they may not have the same geometric thinking characteristics (Wang & Kinzel, 2014). Therefore, categorizing students' geometric thinking into general levels is not convincing. In order to remedy this limitation, based on Fuys et al' s (1988) van Hiele level descriptors, I developed different types for each van Hiele level, which explicitly describe characteristics for each van Hiele level in terms of descriptive geometry standards (See Table1).

Table 1: Van Hiele Types

Van Hiele Levels	Van Hiele Types
Level 1	Recognizing figures (RF) Composing and decomposing figures (CD)
Level 2	Recognizing figures by their specified attributes (RFA) Understanding relationships between 3D and 2D figures empirically (UF) Recognizing transformation empirically (RT)
Level 3	Describing hierarchical nature of classes of figures (DH) Understanding definitions (UD) Understanding postulates (UP) Developing informal proofs (DIP) Constructing figures (CF)
Level 4	Writing formal proofs (WFP) Understanding the axiomatic system (UA)

Methods

In this study, a descriptive learning expectation (DLE) is defined as a descriptive geometry standard, while the learning expectations related to quantities and coordinates are not considered. The following Example 1 is a DLE which is included in the study, and Example 2 is a Non-DLE which is excluded in the study.

> Example 1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes. (CCSSM, p. 16)
> Example 2. Recognize area as an attribute of plane figures and understand concepts of area measurement. (CCSSM, p. 25)

Each DLE was coded into a van Hiele level and a particular type. There are two stages for coding. In the first stage, two researchers who are literate in both English and Chinese languages work together to identify the DLEs from all geometry learning expectations in CCSSM and CMCS. In this stage, the two researchers had fewer divergences. In the second stage, the two researchers coded DLEs in CCSSM and CMCS in terms of van Hiele levels and types independently. After all DLEs in CCSSM and CMCS were examined, the results were compared, and the researchers reached a more than 50% consensus. Then, discrepancies were discussed until consensus was reached.

The study performed the analysis by looking at the percentages and frequencies of DLEs in van Hiele levels and types. The distributions of van Hiele levels reflect the spirit of van Hiele theory (Newton, 2011) and the distributions of van Hiele types allowed the researcher(s) to see the respective emphases of CCSSM and CMCS.

Results

As shown in Table 2, the total number of DLEs in CCSSM and CMCS are quite different since CMCS describes DLEs in more detail, while CCSSM describes them more generally. The percentages of level 1 and level 2 DLEs in these two documents are similar with only slight differences. The numbers and the percentages in level 3 and level 4 are quite different.

Table 2: Frequencies and Percentages of DLEs of Each Level

	Level 1		Level 2		Level 3		Level 4		Total
CCSSM	5	6%	10	12%	48	59%	18	22%	81
CMCS	7	5%	23	16%	62	43%	51	36%	143

When digging into types in each level, the differences become discernable. As appearing in Table 3, CMCS contains more formal definitions and postulates than CCSSM. CMCS has 19 formal definitions and 10 postulates, whereas CCSSM has 10 formal definitions and no postulates. The two documents share five common definitions: *congruence*, *angle*, *circle*, *perpendicular line*, and *parallel line*. Both documents define the common definitions in the same way except for *congruence*. CCSSM gives the precise definition in high school level "Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations*"* (CCSSM, 2010, p. 55).

In the CMCS, *congruence* specifically refers to triangle congruence; the definition is given in middle school level as the equality of all corresponding pairs of sides and all corresponding pairs of angles.

Table 3: Percentages and Frequencies of Types in Level 3

Level 3	CCSSM		CMCS	
Describing hierarchical nature of classes of figures (DH)	4	8%	2	3%
Understanding definitions (UD)	10	21%	19	31%
Understanding postulates (UP)	0	0	10	16%
Developing informal proofs (DIP)	20	42%	13	21%
Constructing figures (CF)	14	29%	18	29%
Total	48		62	

In terms of developing informal proofs (DIP), CCSSM provides a 42% proportion of DLEs as DIP, compared to CMCS which only provides 21% DLEs as DIP. The results show that CCSSM and CMCS have significant differences, supporting van Hiele theory in their own ways. CCSSM emphasizes intuitive thinking and an inductive approach, hence CCSSM expects students to make adequate preparation in informal proofs to reach formal deduction. On the other hand, CMCS emphasizes logical thinking and a deductive approach, expecting students to develop theoretical reasoning through constructing definitions, postulates, and theorems systems.

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Discussion and Conclusion

The analyses of the data collected from distribution of DLEs consistently showed that CCSSM and CMCS have significantly different learning goals in geometry. Both CCSSM and CMCS emphasize the construction of figures, but they expect this to be accomplished using different approaches. CCSSM holds open attitudes towards tools and suggests a variety of approaches such as: freehand drawing, technology, compass and straightedge, reflective devices, paper folding, and dynamic geometric software. In contrast, CMCS only expects using a compass and a straightedge or a ruler to construct figures. In addition to the different approaches, DLEs in these two documents also have different aims. In CCSSM, some DLEs for constructing figures make preparation for further learning. For example, CCSSM addresses expectations of constructing triangles by three measures of angles or sides to prepare the criteria of triangle congruence. In CMCS, each DLE for constructing figures is corresponding to a definition, a postulate or a theorem. Instead of preparing for further learning, the aim of construction is to enhance comprehension of definitions, postulates and theorems. These results are consistent with the difference in U.S and Chinese textbooks' emphases which are already identified by the existing research.

Additionally, the results of this study resonate some differences in mathematics education in the U.S and China. For instance, the tendency for Chinese teachers is to expect students to choose generalized and abstract strategies, and for U.S teachers to encourage students to use particular and concrete strategies to solve problems (Cai, 2004; Cai & Hwang, 2002). Moreover, the DLEs in CMCS emphasize formal definitions and formal proofs which indicates that CMCS expects the theoretical achievement more than practical achievement. This finding is in line with the previous studies which found that Chinese students performed better in procedure tasks than U.S. students, but underperformed than U.S. students in creative tasks (Cai, 2000). Understanding the differences in teaching and learning in mathematics in the U.S and China can shed light on understanding the differences in standards which are set for different teaching and learning goals in the two nations.

Compared to CMCS which is concerned with the abstract and logic, the DLEs in U.S geometry curriculum focus on visual and manipulation geometry. As van Hiele-Geldof (1957) in her dissertation pointed out, for visual and manipulation geometry, cutting and pasting can help students acquire better insight into the basic concepts of geometry. Students need to have sufficiently mastered manipulations, because they can accomplish conscious perception in a geometric sense. However, to achieve a higher logical level, students should not rest on understanding the relations through concrete example visual and manipulation geometry as well as logic geometry are indispensable aspects of geometry. On the one side, visual and manipulation cannot ensure logical thinking and deduction. On the other side, only emphasizing logical thinking may fall into depending on memories instead of developing deductive ability. In order to success in geometry, students need to develop both of them. Standards should address and balance learning expectations for these two aspects of geometry. Therefore, for the further revision of CCSSM and CMCS, I suggest adding appropriate DLEs on logic geometry for CCSSM while addressing more DLEs on manipulation, especially on using dynamic geometry software for CMCS.

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ANALYZING THE TRANSITION AWAY FROM MONTESSORI WITH A MATHEMATICS POINT OF VIEW: THREE PARTICULAR FACTORS

Zachariah Benton Hurdle
Southern Arkansas University
ZBHurdle@saumag.edu

Angela Stanford
Southern Arkansas University
AGStanford@saumag.edu

Our study investigates three specific factors hypothesized to affect the transition period from third-grade Montessori math to fourth grade non-Montessori math experiences. These aspects are 1) the change in pacing and structure of the classroom, 2) the removal of manipulatives in favor of handwriting methods, and 3) the reversal of roles that teachers and students occupy. The effect of this transition on problem-solving skills of students is analyzed through personalized math metacognitive tools. Overall results show that students identify alternatives strategies when uncertain on unfamiliar problems, often reverting to object-centered methods such as manipulatives, drawings, or finger-counting. The declining use of manipulatives in the classroom is a strong influence in the transition, followed by a clear shift in student and teacher roles. Meanwhile, while pacing and structure of the classroom has minimal effect.

Keywords: Metacognition, Problem solving, Elementary school education

Introduction

Often, Montessori programs focus on “students [who] can be described as self-regulated to the degree that they are metacognitively, motivationally, and behaviorally active participants in their own learning process” (Zimmerman, 1989, p.329). Metacognition, then, is an appropriate framework when approaching self-motivated thinkers as encouraged by how the Montessori system operates because they can describe their thoughts and processes strongly for the study; and while motivation and belief impact performance, persistence, and creativity (Ryan & Deci, 2000), the Montessori program promotes reflection, organization, and planning skills (Bagby & Sula, 2011). A gap in literature regarding Montessori research exists because standardized guidelines and lack of traditional assessment make the comparison with other programs difficult (Mak, 2006; Yen & Ispa, 2000). But while this type of research is indeed rare and inconsistent, particularly involving manipulatives (Laski, Jor’dan, Daoust, & Murray, 2015), the transition from a Montessori system that often serves as early education is vital, and so the transition away from Montessori requires research to improve the knowledge base of educators.

Further, the importance of elementary mathematics education, where this transition often takes place, is considered a vital learning period in student success (Clements & Sarama, 2007). Third-grade is the primary focus of our study and emphasizes independent thinking and confidence development through problem solving (Kamii, 1994). This is vital in mathematics education because students who memorize steps are often setting themselves up for failure in higher mathematics levels (Kloosterman & Stage, 1992). Further, NCTM (2014) stresses that “tasks that encourage students to use procedures, formulas, or algorithms in ways that are not actively linked to meaning, or that consist primarily of memorization or the reproduction of previously memorized facts, are viewed as placing lower-level cognitive demands on students.”

Problem Statement

The purpose of this study was to explore three specific factors related to documentation of difficult student transitions away from a Montessori mathematics program at a particular school. During the prior school year, an issue was identified concerning students exiting the Montessori program after third-grade and entering a more direct-instruction environment in fourth grade, somewhat indicative of the typical transition experience by these types of students. These conclusions can be applied to many students experiencing a similar shift in their own education. Our study followed students in primary grades at a single campus serving PreK-12 (PreK – 3rd is Montessori, and 4th – 12th is traditional), eliminating external variables such as changing settings, peers, and institutions—variables that are commonly referenced in typical post-Montessori studies (Anderman, Maehr, & Midgley, 1999).

Research Questions

The following research questions help evaluate the transition period and how it affects students in mathematics. While metacognition is the framework, problem-solving skills is the context. These questions are elaborated as follows:

1. To what teaching practices and learning opportunities are third- and fourth-grade students exposed, and how are these related to the Montessori approach?
2. How are three specific aspects of the fourth-grade mathematics classroom perceived by students and teachers compared to their previous exposure in the Montessori style?
3. How, and to what extent, does changing the teaching practices and learning opportunities affect the problem-solving strategies of students?

Based on the related research, the three factors referenced in the second research question were: 1) the implemented learning pace and discovery style of the class, 2) the shifting focus away from materials toward handwriting methods, and 3) the difference in the roles students and teachers occupy during the transition. The purpose of the first and third research questions was to determine teaching strategy alignment to theory and provide a measurement to view the effects on student problem solving and mathematical approaches. The conclusion will then appropriately focus on the second research question, supplemented by the information provided through taking the first and third questions into consideration.

Methodology

This study took place in Central America at an International Baccalaureate-certified (IB) school, with a high school English-based math curriculum matching the global, IB curriculum. The lower grades are taught for students to be prepared for this secondary IB curriculum. The school is private, and served approximately 135 students during the time of this study, ranging from pre-kindergarten to 12th grade. Total attendance numbers were approximated because of various relocations of international students mid-year. During this study, more than twenty countries represented the diverse population of the school across the grades, and there were no comparable schools in the surrounding area. This school used a trimester system; the first trimester ran from September 1 to December 14, the second trimester from January 3 to March 31, and the third trimester from April 24 to June 29. This study took place during the first and second trimesters in order to maximize the collection of data during peak student transition.

During the academic year, there were 12 third-grade students and 16 fourth-grade students for our sample, each with a mixed amount of experience in the school's curriculum. Five fourth-grade students were identified from the group of sixteen for further problem-solving analysis.

This analysis was conducted using metacognitive math tools with the goal of targeting the third research question concerning effects on problem solving skills. Participant selection was based on cross-referencing average academic math performance with at least three to four years of experience in the Montessori program, plus teacher recommendations. These parameters lead to our sample of five students, from the population of 16 fourth-grade students.

Analysis Tools

Ten observations were conducted in each of the third- and fourth-grade classrooms, using a modified version of Piburn & Sawada's (2000) Reformed Teaching Observation Protocol (RTOP), adapted to reflect the Montessori and non-Montessori classrooms. These protocols contained 12 statements regarding classroom structures, with a ranking system and qualitative outlet for alignment with classroom theory. All students, both third- and fourth-grade, were interviewed once following a semi-structured interview script regarding learning preferences, classroom perceptions, mathematics favoritism, and role identification (for a total of 28 interviews). Additionally, all four teachers (two teachers co-taught in each grade) were interviewed three times each to discuss similar ideas along with curriculum strengths and weaknesses, classroom experiences, instructional details, and other opinions and observations. The combination of interviews and classroom observations were enough to satisfy the data collection regarding the first two research questions. Coding allowed us to place positive and negative phrases into the appropriate category related to the three factors we analyzed.

The third research question required extra analysis to evaluate what kind of effect this transition was having on students' problem solving skills (thus placing value on the effect that the transition had on mathematical problem solving on a cognitive level). The five selected fourth-grade students were interviewed in two additional settings with metacognitive tool problem sets: one included problems based on addition and subtraction, with some multiplication; the other focused on multiplication, division, and fractional application. These problems were taken from Empson, Junk, & Turner's (2006) "Formative Mathematics Assessments for Use in Grades K-3," closely following some modeling of Cognitively Guided Instruction (CGI) implied through Empson's prior collaboration over such material (Carpenter, Fennema, Franke, Levi, & Empson, 2015; Carpenter, Fennema, & Franke, 1996). Based on student work and dialogue, inferences were made toward their processes based on their progress within the transition and their preconceived notions of mathematics, how to be successful, preferred strategies, and much more.

Findings

The First Aspect: The Changing Pace and Structure

Third-grade students experienced an exploratory curriculum and held some leadership in a mixed classroom of first-, second-, and third-graders. Students approached problems with highly differentiated levels of depth based on ability, not age. One major difference from the Montessori system was the inclusion of a small, weekly group mini-lesson for each subject, and teachers instituted a "follow-up" strategy as a form of assessment. This strategy, combined with weekly homework benchmarks, helped push students forward. While third-graders had learning freedom, fourth-graders were taught as a group through direct instruction. These teachers implemented sequential book lessons to establish routines, where math was designated as first subject of the day. Students then individually performed drill and repeat exercises, and were discouraged from working with others. While third-grade teachers encouraged natural occurrences of collaboration, fourth-grade students were separated. The two fourth-grade teachers each agreed that the lack of differentiation was a problem but that direct instruction was appropriate, yet disagreed on which students were most detrimentally affected—students with high or low achievement performance. Many fourth-grade students appreciated the direct instruction because they perceived they were learning more efficiently than Montessori. They also thought the new structure prepared them more effectively for future grade levels.

The Second Aspect: Removing Montessori Materials

Montessori materials were not the only learning method for third grade. Montessori students used the manipulatives in the classroom as the original source of knowledge, but third-grade students were additionally tasked with transitioning their learning into handwriting with colored markers and, eventually, pencils. While the handwriting goal was to physically represent abstract ideas, many third-graders complained about working through material twice (once with manipulatives, once with handwriting), without connecting these together. They saw older children using handwriting, and believed manipulatives slowed their learning. They valued the new strategies and topics of fourth grade because they were aware of the intention to push themselves as a learner and thinker. To many students, manipulatives were not gradually phased out, but were instead entirely replaced. While third-graders had mixed opinions, many fourth-grade comments were negative toward manipulatives, though a few students appreciated their use. Many fourth-graders also said they were relieved to have moved on from the materials, because they were "boring" and "repetitive," and stopped helping them with math. Teachers agreed that perhaps they had been used for too long in the third-grade stage. Yet when the five selected students were pushed to problem solve, many reverted to them when faced with unfamiliar topics. In fact, the students who found the most success with the metacognitive tool problems were the ones who verbally favored manipulatives and did not denounce them.

The Third Aspect: Reversing the Teacher and Student Roles

In third grade, teachers approached the class mostly as a guide, with brief deviations when taking the responsibility of providing weekly lessons in preparation of lesson structure in fourth grade. By moving confidently in this guide role, the third-grade teachers maintained their capacities in fostering student-led learning, while students actively gained self-motivation and awareness, shown in interviews. These roles were very different in fourth grade. Students were now placed in a system where everyone was taught with the same expectations, with the teacher as the formal source of information. Fourth-graders took the role of passive learner to the teacher's authoritarian role. These observations indicated that the fourth-grade classroom was not only non-Montessori, but also followed a direct-instruction style, particularly for mathematics. Students did not comment negatively toward this new direct-instruction style, but instead embraced the teacher as a source of knowledge for more efficient learning. The teachers actually struggled more than students with identifying their own place in the classroom, especially knowing that their students came from a Montessori learning environment previously.

Conclusions

According to the literature, making the connection from the concrete manipulatives to the abstract symbols is a process that usually involves drawing visual aids as a transitional stage; the use of manipulatives themselves is more for mathematical understanding rather than algorithmic proficiency (Stein & Bovelino, 2001). For the students in this study, this was achieved only as long as meaning and value was apparent. Students generally achieve mathematical understanding if they use the manipulatives as a tool rather than a requirement (Cope, 2015). Therefore, students need to build upon what they learn to grow mathematically (D'Ambrosio, 2003), and the repetitive methods beyond mastery at this school did not reflect such values. This provides insight into correct implementation of Montessori manipulatives in the most effective way possible. Montessori programs may provide students with solid mathematical understanding with physical materials, modeling, and drawing. In fact, this study showed that students tend to revert back to these methods when approached with problems they had less algorithmic built-in strategies to accomplish, as unfamiliar exercises required different processes to be successful. The issues arise when moving into the procedural stages while battling student perceptions, and while our situation was certainly a case study, future studies may explore this stage more fully.

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WHO IS TEACHING WHO? INTERGENERATIONAL LEARNING AND CODING

Brandon A. Dickson
Huron University
bdickso9@uwo.ca

Lauren M. Harris
Ivey School of Business
lharr9@uwo.ca

Taylor Boyd
Huron University
tboyd22@uwo.ca

Sagar Jiwani
Ivey School of Business
sjiwan2@uwo.ca

Donna Kotsopoulos
Huron University
dkotsopo@uwo.ca

In this research, we explored intergenerational computational thinking in Elementary Schools in a large urban centre. Research has shown that, intergenerational learning between seniors and youth advances the speed at which both parties develop their skills. These studies however analyze learning in environments outside of the classroom, and often emphasize the benefits of the younger cohort learning from the older. This study seeks to analyze intergenerational computational thinking in the context of mathematics learning in a classroom setting. We seek to determine whether the benefits of intergenerational learning still exist between youth of different ages and look at the benefits for both groups of learners in this setting. We seek to demonstrate the impact of this learning on mathematical learning. Directions for further research will be discussed.

Keywords: Technology, Problem Solving, Instructional Activities and Practice

Introduction

Putting learners of different ages together to promote learning in a classroom setting has a long tradition. Specifically, intergenerational learning can be defined as “an interactive process that takes place between different generations resulting in the acquisition of new knowledge, skills and values” (Ropes, 2013). The aim of putting students together who differ in age is about exposing students to different levels of knowledge and experience which allows all those involved to see things from a different perspective. Intergenerational learning is more than people of different ages, its “how differences in their ages can be framed in ways that contribute to content” (Sánchez & Kaplan, 2014). This knowledge transfer offers the potential for a deeper learning than that between peers. However, many studies in this field frame this kind of learning as one group imparting knowledge onto another (Aemmi & Moonaghi, 2017; Findsen & Formosa, 2011). In a classroom setting, this often results in programs such as reading buddies, strong programs as they can be, which make older students help younger students learn how to read. Our study will take a qualitative approach to the learning that was undergone. Implications for curriculum implementation and directions for areas for further research will be discussed. This study is part of a larger research study on computational thinking and spatial learning in middle school students.

Theoretical Perspectives

While a majority of intergenerational studies examine the learning that occurs between seniors and youth, the present study focuses on intergenerational learning with a narrower age range: Grade 3’s and Grade 7’s. We are defining the learning between these groups of students as intergenerational because of the findings by Charles and Runco (2001) that there is a peak of divergent thinking in students around Grade 4. Other research shows there is a drop in creativity from Grade 6 to 7 (Lau & Cheung, 2010). As such, we posit that the groups of learners that we were working with are at essentially different points in their academic careers. This makes our study intergenerational.

Many studies analyzing the transmission of knowledge between generations have shown that intergenerational learning can be bidirectional. However, while some of these studies are set in a classroom, few are set in the context of a computer programming classroom. However, this appears to be the perfect place for such bidirectional learning. For the purpose of our study we define computational thinking as “an approach to solving problems, designing systems and understanding human behaviour that draws on concepts fundamental to computing” (Kotsopoulos et al., 2017). As Resnick et al. (2009) discuss, younger learners are able to benefit from the “low floor” of computational thinking, or the lower bar to entry required for learning while experienced learners benefit from “high ceiling” which allows for learners to continue to find new types of learning. In one study, seniors (aged 60-89) and high school students participating in a high school English class both reported learning from the experience of intergenerational learning (DeMichelis, Ferrari, Rozin, & Stern, 2015). This cooperative learning is a key finding in intergenerational studies. Sánchez and Kaplan (2014) argues that multigenerational classrooms provide opportunity for meaningful learning; participants mutually benefit from exposure to diversity of knowledge and experiences.

Methods

This research was completed in the context of a wider study analyzing the relationship between computational thinking and spatial reasoning. Our specific study examined intergenerational computational thinking between Grade 3 and Grade 7 students at a private elementary school in London Ontario. The study began with the Grade 7 students being taught various forms of coding by the first and second authors from November to March. The classroom teacher was present for each lesson but was not involved in teaching. Initially researchers followed the Pedagogical Framework for Computational Thinking (PFCT), introducing the four pedagogical experiences: unplugged, tinkering, making and remixing (Kotsopoulos et al., 2017) using the online block coding program Scratch. Next the Grade 7 students used Spheros (Figure 1), robotic spheres which are programmable using iPads to roll in any direction.

Figure 1: An Example Scratch Block Programming

Once the Grade 7 students had mastered these activities, they were tasked with teaching the Grade 3 students to code over the course of two, hour-long lessons. Grade 7s were not specifically instructed on how to teach the younger students. For the activities, each Grade 3 learner was randomly paired with a Grade 7 instructor. The first activity was “My Friend Robot” in which one partner would give instructions to direct the other partner, the ‘robot’ though a

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

maze. For the other unplugged activity "Order the Instructions", all students had to number a set of picture instructions which the other partner had to follow to fold a paper airplane. Next, students participated in Sphero activities, including a curling game which allowed students to learn how to estimate distance on Sphero. Students also learned to program Sphero to travel paths of different shapes such as a square. The final activity of each session was programing Sphero to travel through a maze which was created on the floor with a tape outline (Figure 2).

Data collected during these sessions included blog post by Grade 7 students, questionnaire responses by Grade 3 and 7 students, as well as observations recorded by the first author and the classroom teachers to collect data. Students were asked to respond to questions in the form of blogs during class time following both of the coding buddies' lessons. Questions were designed by the first author and asked the Grade 7s such things as, "What was the hardest part of teaching the grade 3's to code?" and "What are the grade 3's better at when it comes to coding?". Grade 3s were asked "What was the hardest part of coding?" and were asked to self identify with such phrases as shy and silly, as well as statements such as, I like school, and I am good at coding.

Results and Discussion

Student responses indicated that coding activities were both engaging and educational for participants. The overall responses to coding were very positive: 95% of the Grade 3 students and 67% of the Grade 7 students responded in surveys that they enjoyed the coding activities. One Grade 3 student wrote "I liked coding because it gave me an understanding of how robots work". A Grade 7 student was also enthusiastic about coding saying, "I find coding fun and really cool". The Grade 7's had to assist the Grade 3's most with left and right as the 'robot' was facing many different directions during the first unplugged activity. Of the Grade 7's, about half believed that the Grade 3's learned coding faster then their class did. One Grade 7 stated "Once you get them down to work they seemed to really enjoy it and know what they were doing."

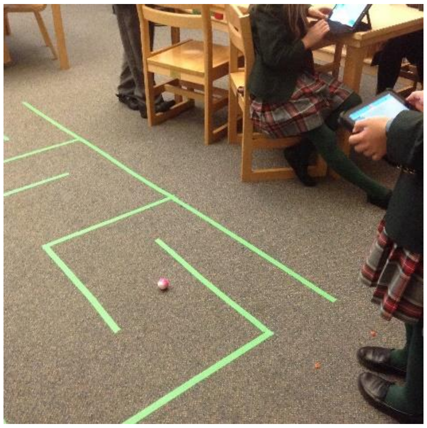

Figure 2: An Example of a Sphero Maze Activity

With the one-on-one instruction, the Grade 3s' were very quick to gain a basic understanding of coding. More interestingly, they were able to demonstrate creative critical thinking that the Grade 7s quite frequently were missing. Once the Grade 7s' taught their Grade 3 buddies the basic controls to operate the Sphero, many Grade 3's inquired about other ways to advance the use of a Sphero. One example of this came when, after being taught by the Grade 7's to make the Sphero to move in a square, one Grade 3 student asked why the loop function wasn't used to simplify the inputs. This innovative thinking was noticed by the grade 7's in a response; "[the

Grade 3's] have a great imagination so they come up with great ideas". Similar sentiments were expressed throughout by the Grade 7 students. Evidently the Grade 7's noticed the creative methods the Grade 3's discovered to program Sphero differently to complete the same task. Here it is important to look at the first time that the Grade 7s were taught coding. Notes made by the first and second author demonstrated that when taught coding by the first and second author, even when Grade 7s were given time to explore and experiment, the vast majority of students simply asked what to do and followed the instructions as they were given. This varies from the inquisitive approach taken by the Grade 3 students, who even during one-on-one instruction wanted to explore alternative approaches. This differences in learning approaches corroborates the difference in divergent thinking typically found in students of these age groups (Charles & Runco, 2001; Lau & Cheung, 2010). This difference in the approaches was noted by Grade 7 students where the claim that their strong imaginations led to more unique ideas than is present for other students.

Conclusion and Implications

These results suggest that intergenerational learning between youth relatively close in age is beneficial for both learners and teachers. The Grade 3's learned important computational thinking skills with one-on-one instructions from students who were better able to understand their learning styles and preferences. The Grade 7's learned from the Grade 3's originality to incorporate creativity into coding as the younger students demonstrated multiple wasys of reaching the same goal. A Grade 7 student said "I think my problem-solving skill is improved because when I have problems or question about coding games I would have more choices than just asking teachers or [the first author]. Instead I would use other parts of the code and work towards that to solve my questions". In the present study, intergenerational learning proved to be very useful and an effective way to promote fluid learning, rather than simply instruction, which translates into a variety of applications in a school setting as there is always multiple ages of students who could benefit from learning together. Intergenerational learning should be leveraged for a variety of academic subjects to enrich the learning of the students involved. Moreover, the fact that the students were teaching each other promoted interest which teachers may not have been able to trigger.

Acknowledgments

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THE IMPACT OF MATHEMATICALLY CAPTIVATING LEARNING EXPERIENCES

Leslie Dietiker
Boston University
dietiker@bu.edu

Meghan Riling
Boston University
mriling@bu.edu

Miriam Gates
Boston College
gatesmd@bc.edu

Secondary students do not often have positive experiences with mathematics. To address this challenge, this paper shares findings of a design-based research project in which a mathematical story framework was used to design mathematically captivating lesson experiences ("MCLEs"). We provide evidence that designing lessons as mathematical stories shows promise. That is, students reported improved experiences in MCLEs when compared to randomly-selected lessons. The MCLEs also impacted the students' descriptions of their experience.

Keywords: Curriculum; Curriculum enactment; Affect, emotion, beliefs, and attitudes

How can mathematics be taught so that students describe the subject as amazing, surprising, and full of wonder? Unfortunately, most evidence to date suggests that these descriptors do not describe the typical student experience in mathematics in the United States; students have increasingly poor attitudes in mathematics as they advance into higher grades (Mullis, Martin, Foy, & Hooper, 2016). One way to respond to these poor mathematical experiences has been to explore how lessons can be aesthetically enhanced to increase student engagement and interest by drawing on the affordances of what makes literary stories compelling and pleasurable. Analyses of high school mathematics lessons with heightened positive aesthetic responses (e.g., student exclamations of "Wow!) have linked narrative moves (e.g., misdirection) with positive student aesthetic reactions, such as anticipation, curiosity, and surprise (e.g., Dietiker, Richman, Brakoniecki, & Miller, 2016).

Building on this work, this present study is focused on learning whether lessons that are intentionally designed as mathematical stories improve student experiences. Working with six high school teachers, 18 *mathematically captivating learning experiences* ("MCLEs") have been designed and tested using the mathematical story framework. This paper reports on the results of the first design-and-test cycle of a three-year project of design-based research (Edelson, 2002), addressing the question *What impact, if any, do lessons designed as mathematical stories have on student aesthetic experiences?* Our results suggest that even in their first iteration, the design and enactment of mathematical stories shows promise.

Theoretical Framework

For this study, *aesthetic experiences* describe the way in which experiences move or compel an individual to act, such as by asking a question, persevering through difficulty, or even laughing or gasping. Therefore, the study of aesthetic dimensions of an experience examines how a particular experience enabled the compelling effects (or lack thereof) to occur. Since the audience of an enacted mathematical story is arguably the students, the students' evaluation and description of the experience is an appropriate measure of its aesthetic value.

Here we take the perspective that mathematical sequences can be interpreted as a form of *mathematical story*—a designed sequence of mathematical events (such as tasks or discussions) experienced by students connecting a beginning with its end (Dietiker, 2015). Mathematical stories have a plot, which enables the description of how a sequence can generate suspense (by setting up anticipation for a result) and surprise (by revealing a different result than the one anticipated). In this study, we use the math story framework as a conceptual resource for design.

Methods

Three pairs of experienced teachers were recruited from three high schools in the Northeast of the United States to design and test MCLEs. Each high school was selected to offer contrasts: (1) a small independent charter school with mostly Latinx students and a subject-specific curriculum, (2) a large public school with a very diverse student body (representing multiple ethnic groups) and an integrated curriculum, and (3) a large public school with a majority white student body and a subject-specific curriculum.

To prepare teachers to design MCLEs, the six teachers attended a two-week professional development in Summer 2018 at which lesson design was studied along three dimensions: captivation, coherence, and cognitive demand. The captivation and coherence dimensions were directly addressed through the mathematical story framework. A focus on the cognitive demand framework (Stein, Smith, Henningsen, & Silver, 2000) was also included to encourage to incorporate mathematical complexity as they designed MCLEs for curiosity and/or suspense.

Following the professional development, the teachers met weekly in pairs with researchers to design three MCLEs per teacher. Courses of focus were chosen to provide a wide array of topics, grade levels, and tracked levels (i.e., honors or non-honors). Teachers selected content based on topics that they had difficulty motivating in the past. Non-mathematical aspects (e.g., real-world contexts, games) that would likely influence student interest were avoided. In addition, the entire group of teachers and researchers met three times throughout the school year to share the emerging MCLE designs and get feedback from teachers from different schools.

At the start of each school year, all students were given a disposition survey using Likert items from TIMSS (2016) and the TRIPOD (Ferguson & Danielson, 2015) to measure captivation (e.g., do you like math?) and student perceptions (e.g., does the teacher care?) on a scale of 1 to 4 (see Riling, Dietiker, Gibson, Tukhtakhunov, & Ren, 2018 for more information). An aggregate "captivation" measure was then generated for each student using this instrument.

To learn if lessons designed as mathematical stories can improve the experiences of high school students, we compared the reported student experiences for MCLEs with randomly selected lessons taught in the same classrooms. In addition to 3 MCLEs per teacher, between 2 and 4 non-MCLEs per teacher were also observed. A single protocol was used to collect data for all enactments so that students would not know which lessons had a special design. After each lesson, we collected Lesson Experience Surveys (LES) for each participating student. The LES asks students to select three descriptors (e.g., intriguing, dull), displayed in a random order, and asks them to rate how they felt during the lesson from very bored (1) to very interested (4) (see Riling, Dietiker, & Gates, 2019 for more information). It also asks students to rate whether they found the content of the lesson challenging or not and to indicate to what level they agreed with statements such as "time flew by" and "the content of this lesson was relevant to my life."

Through prior work, we found that the more a student reports liking mathematics overall, independent of any particular lesson, the more they are likely to report positive experiences with a mathematics lesson ($R = 0.423$). Because of this relationship, we factored in students' mathematical captivation level when comparing MCLE and non-MCLEs. We also controlled for the teacher in order to acknowledge that students who learn from the same teacher and learn in the same classroom have related mathematical learning experiences.

Findings

Overall, the emerging results suggest that MCLEs improve the lesson experience of students and can alter the types of experiences students report.

The Impact of MCLEs on Student-Reported Lesson-Specific Interest Measures

Of the 8 classes in this study, 7 showed an increase in average student interest (on a scale of 1 to 4) with MCLEs. Table 1 displays the distribution of classes, along with the course and grade level and the average number of students surveyed across the observed lessons.

Table 1: Distribution of Classes (where "124" = school "1," teacher "2," and period "4") with Subject, Grade Level, and Interest Measures for MCLEs and Non-MCLEs

Class	Subject	Grades	Non-MCLE Avg. Rating (n)	Avg. MCLE Avg. Rating (n)	Difference
113	Algebra 2	10	2.71 (n=17)	2.89 (n=17.5)	+0.18
116	Algebra 2	10	2.81 (n=16)	2.94 (n=18)	+0.13
124	AP Calculus	12	2.60 (n=15)	2.92 (n=13)	+0.32
215	Integrated Math 3 H	10, 11	2.85 (n=24)	3.06 (n=21)	+0.21
224	Integrated Math 1	9	3.04 (n=15.5)	3.56 (n=13.5)	+0.52
311	Algebra 2	11, 12	2.76 (n=12.3)	2.77(n=15)	+0.01
322	Geometry	10	2.67 (n=12)	2.80 (n=10)	+0.13
327	Algebra 2	10	2.33 (n=10.5)	2.62 (n=9)	+0.29

At the student level, this improvement can be modeled. The graph in Figure 1 shows the linear regressions for the student interest measure by captivation for non-MCLEs and MCLEs. The influence of an MCLE on student interest in a lesson is statistically significant when taking into account student captivation and teacher, improving student experience by 0.21 ($p<0.001$). However, we did not find a statically significant difference between non-MCLEs and MCLEs on other measures, including student perception of challenge or whether time flew by.

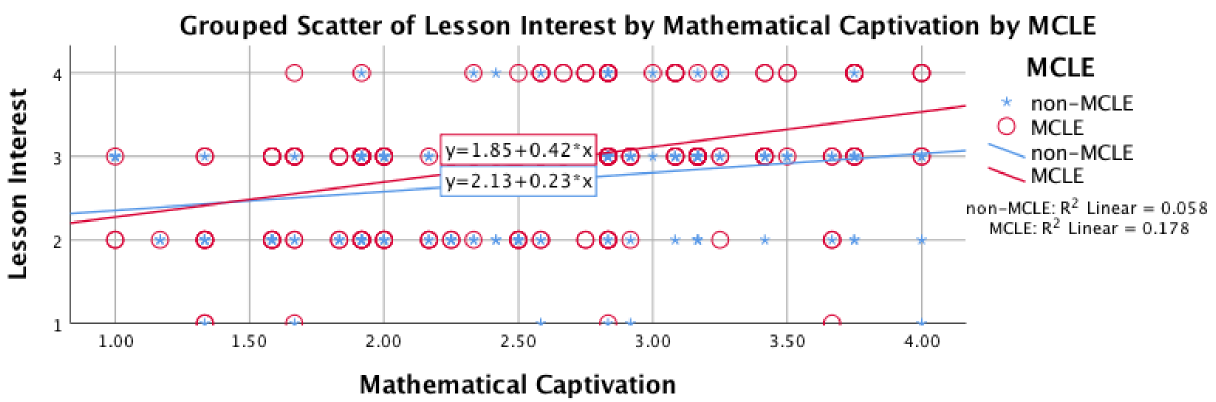

Figure 1: Scatterplot of Student Measures: Lesson Interest, from the Lesson Experience Survey, by Mathematical Captivation, from a Survey of Mathematical Disposition

Overall, MCLEs appear to differentially influence the experiences of students with different mathematical dispositions. On average, MCLEs did not change the general experience of students with the lowest lesson experience, while students with very positive views of mathematics saw the most benefit. For example, the models predict that a student with low captivation (1) will report similar interest in MCLE and non-MCLEs (non-MCLE: 2.36, MCLE: 2.27). Yet students with high captivation (4) benefit by a factor of 1.16 (non-MCLE: 3.05, MCLE: 3.53).

MCLEs were not experienced differently by students based on their gender (p=0.089). It is difficult to assess any impact of student racial identification on aesthetic reports, because the racial breakdown of each school in the study is so different that it is difficult to distinguish between the effects of a student's school and their racial identification.

The Impact of MCLEs on Student-Reported Lesson-Specific Aesthetic Descriptors

MCLEs appear to have also been successful in changing the type of aesthetic experience of the students. For example, the descriptor "intriguing" was selected on 28.0% of surveys after MCLEs, but by only 18.8% of students surveyed about non-MCLEs. On the other end of the spectrum, 6.6% of students selected "dull" to describe MCLEs, compared with 15.4% of students selecting this descriptor for non-MCLEs. In addition, when analyzing whether MCLEs received more positive, neutral, or negative aesthetic descriptors, we found that students who experienced both types of lessons used more positive descriptors when describing their experience with MCLEs and that this difference was statistically significant (p=0.001). Overall, students used positive descriptors 58.1% of the time when describing MCLEs (compared to 47.1% for non-MCLEs), while using negative descriptors 11.1% of the time for MCLEs (compared to 17.3% of the time for non-MCLEs).

Even among MCLEs, the descriptors selected by students varied. For example, students were less likely to select descriptors that have extremely positive connotations, such as "amazing" or "fascinating" than other positive descriptors, such as "thought-provoking." The students who selected these extremely positive descriptors reported higher interest levels, on average, compared to students who selected all other descriptors with the exception of "suspenseful." An example of how different MCLEs are from each other is that 23% of students surveyed about an MCLE in class 224 about geometric transformations selected "amazing," while 24% of students surveyed about an MCLE in class 215 about finding the roots of a polynomial function selected "fascinating." These classes both had high levels of student interest among both students who selected these descriptors and those who did not. For these MCLEs, the students in class 224 reported an average interest level of 3.46 and the students in class 215 reported an average interest level of 3.20; these MCLEs had a high positive impact given that the average interest level reported across all student surveys is 2.84.

Discussion

With persistent negative views of mathematics by students, particularly at the secondary level, there is a pressing need to identify ways to improve the experiences students have with mathematics. Already from our early results, we are encouraged by evidence that designing for a more positive experience is possible, and are hopeful that providing students with more captivating lesson experiences such as these can impact their views of mathematics as a whole.

In addition to designing for improved reactions overall, our work thus far suggests that it is possible to design for specific aesthetic opportunities, such as suspense and surprise. We are also heartened that students find MCLEs to be more "intriguing." As the MCLEs go through more design cycles, we are interested in learning if more students will continue to report these aesthetic responses along with others.

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LEARNING FROM NOVICES: CURRICULUM CAPACITY IN A CHANGING CURRICULUM LANDSCAPE

Corey Drake
Michigan State University
cdrake@msu.edu

How and whether to teach prospective elementary teachers about mathematics curriculum materials has been a persistent tension in mathematics teacher education. This tension has been exacerbated in recent years by the rapid development of an even wider range of instructional materials, including new curriculum series purporting to be aligned with the Common Core, open resource materials, and materials available on websites including Teachers Pay Teachers and Pinterest. In this paper, I present novice elementary teachers' perspectives on the range of curriculum contexts in which they work, as well as novices' responses to these contexts. I conclude with implications for teacher educators, curriculum developers, and school and district leaders.

Keywords: Curriculum, Teacher Education-Preservice

Purpose

How and whether to teach prospective elementary teachers about mathematics curriculum materials has been a persistent tension in mathematics teacher education. Two common reasons for this ongoing tension have been 1) the wide range of available curricula in terms of quality, content, and design and questions about whether there exist "curriculum-proof" (Taylor, 2016) strategies that would be useful for novice teachers across the range of curriculum contexts and 2) the persistent perception that well-prepared teachers create their own lessons instead of using curriculum materials (e.g., Ball & Feiman-Nemser, 1988). This tension has been exacerbated in recent years by the rapid development of an even wider range of instructional materials, including new curriculum series purporting to be aligned with the Common Core, open resource materials, and materials available on websites including Teachers Pay Teachers and Pinterest. In this paper, I explore the perspectives of novice teachers on their curriculum contexts and the implications of these contexts for understanding the curriculum capacity of preK-12 school systems and for the preparation of elementary teachers.

Theoretical Perspectives

Curriculum capacity refers to the ability of an educational system to accomplish instructional goals and provide high-quality, responsive instruction to students through the interactions of teachers, curriculum materials, and contextual supports (Drake, Land, & Gichobi, 2009). Elements of curriculum capacity include teachers' knowledge, beliefs, and practices; the quality and features of the curriculum materials; and the availability of professional development and material and social resources to support curriculum use. An important aspect of curriculum capacity for the current study is the finding that different districts make different choices about where in the system to invest to develop curriculum capacity. For example, while some districts choose to invest in high-cost comprehensive curriculum materials that provide multiple supports for teachers within the materials, other districts choose to adopt free on-line curriculum resources and spend their available funding on professional development for teachers in using the materials

(Drake, Land, & Gichobi, 2009). As a consequence, the types of knowledge and skills needed for teachers to be productive and responsive curriculum users in the first context differ significantly from those needed to be successful in the second district. Given the recent changes and expansions in the curricular landscape, as described above, the range of knowledge, skills, and curriculum use strategies with which teachers need to be prepared to fit with the range of curriculum contexts in which they might work as novice teachers seems to be expanding and requires further study.

The construct of curriculum capacity builds on two lines of teacher-curriculum research. First, Remillard (2005) describes the participatory relationship between teachers and curriculum materials, in which both the characteristics of teachers and the features of materials contribute to this relationship. Remillard (2005), as well as Remillard and Heck (2014), situate these participatory interactions between teachers and curriculum materials within specific school and district contexts that also present both supports for and constraints on the interactions between teachers and curriculum materials. Here, I am drawing on these accounts to explore the ways in which individual teacher characteristics, features of curriculum materials, and aspects of school and district context interact to produce to different types and levels of curriculum capacity. Second, Brown (2009) introduces the construct of pedagogical design capacity, or teachers' capacity to use curriculum materials to accomplish instructional goals. In this study, I build on Brown's conceptualization of capacity, but expand the construct to be a function of the interactions across individual teachers, curriculum materials, and contexts. Students are also a key element of these interactions; in this paper, they are considered as aspects of the context contributing to curriculum capacity.

Methods

The findings reported here are part of a larger study of novices' enactment of ambitious instructional practices in elementary mathematics and language arts. The larger study investigates the interactions among individual characteristics, opportunities to learn in teacher preparation programs, and the contexts of novice teaching in supporting or constraining the enactment of ambitious instruction among graduates from five different teacher preparation programs. In other words, the study investigates what works, for whom, and under what conditions in terms of novices' ambitious instruction. Data sources include surveys, interviews, and classroom observations with novice teachers in their first three years of teaching, as well as interviews with teacher preparation program personnel and surveys of cooperating teachers, student teaching supervisors, mentor teachers (during novice teaching), and principals.

For this paper, I analyzed interviews from 61 participants during their first year of teaching to understand the range of elementary mathematics curricular contexts in which novice teachers are working and the ways they are responding to those contexts. These interviews were transcribed and analyzed using a series of "big bin" codes, including ambitious instruction, opportunity to learn, challenges, resources, and context. I then conducted secondary analysis of the excerpts within the "challenges" code that related to the use of curriculum materials and used this secondary analysis to characterize the range of curriculum contexts identified by first-year teachers (FYTs). This secondary analysis included both further detailing of the nature of the challenge, as well as coding for the type of response FYTs enacted to address the challenge, including seeking additional resources and information, adapting their practices, and exiting the context.

Findings

Curriculum Contexts

Across the 61 FYTs participating in our study, the curriculum contexts ranged from no prescribed curriculum at all, requiring teachers to locate and construct their own curricula, to highly prescribed use of a single textbook or set of materials. In between these two extremes, participants reported curriculum contexts that involved combining multiple sets of materials, either with or without guidance, or using a primary set of curriculum materials and then supplementing those materials to meet specific student needs and/or contextual needs (e.g., preparation for high-stakes tests). Across these curriculum contexts, FYTs also perceived a range of contextual supports for and constraints on curriculum capacity, as illustrated below. Three quotations illustrating this range of curriculum contexts and curriculum capacity are provided below:

> I like coming up with my own curriculum and I like doing what I want but I also need a balance of 'this is what you should be doing'… I have no curriculum… so it's frustrating to be like, alright, make every activity, every worksheet and task…

> So, we do use *Envision* and that's pretty much what we stick to. It's our first year – or the district's first year – using it, so they are pretty [much] like, 'This is what you're teaching.' They're not really like, 'Try this and add this and see what you can do here.' I do add a basic – I don't know if it's really a curriculum, but it's called *Thinking with Numbers* and it's basic computation because I found that *Envision* doesn't really teach computation.

> Or in math, we do a lot of like claim-support-question, but that's not in the curriculum. It's just something that I want them to be doing deep thinking. Or like number talks, that's not in the curriculum, but we do them all the time. I kind of just have to prioritize what I want to do and what I think is important.

These quotations illustrate the range of curriculum contexts encountered by FYTs, as well as the elements of curriculum capacity in these contexts that were both supporting and constraining FYTs' use of curriculum materials to support ambitious and responsive instruction. These elements include the FYTs' personal knowledge and skills, ideas and practices learned in teacher preparation, core and supplemental curriculum resources with various strengths and limitations, and school and district expectations for curriculum use.

FYTs' Responses to Curriculum Contexts and Challenges

In addition to describing their curriculum contexts and the challenges they faced in those contexts, FYTs also described the ways in which they responded to these contextual curriculum challenges. In particular, they focused on searching for and finding additional resources and ideas, as well as adapting their use of curriculum materials to be more responsive to students. The first two quotations below represent FYTs seeking out additional resources, while the third quotation represents an example of an adaptive and responsive curriculum use practice.

> I have to teach multiplication and the standards, they have to teach the times table 0-12, but it doesn't really say how you have to teach it and I, I guess I would like more guidance whereas I just have to do a lot of research on my own to figure out how I want to teach it.

> Sometimes these games [from *Envision*] take them like five seconds and then they are bored for the rest of it… so we have a lot of other games we put in too… honestly a lot are from Teachers Pay Teachers."
>
> It's really confusing for some of them. So, I feel like – and I don't know, this is just something I picked up this year and I don't know if it's right – but I try to start with really little numbers even though that's not anywhere in my resources at all…. And then the next day I try… I'll put a wrong problem from the day before that I looked at their worksheet and say, 'What do you notice about this? Let's talk about this.' But with this curriculum and how bulky it is, I feel like I can't do that as much because it's like the next day is a new thing and we've got to keep rolling.

In the presentation that accompanies this paper, I will more comprehensively map the range of contexts – including supports and constraints on capacity – as well as the range of responses described by the FYTs. Taken together, the supports, constraints, challenges, and responses identified by FYTs provide a robust and nuanced understanding of curriculum capacity across contexts and the ways in which novices are and are not prepared to work within these contexts.

Implications

Given the wide range of curriculum contexts in which FYTs might begin teaching, it is important to understand the kinds of knowledge and skills FYTs will need in each context. For example, what kinds of capacity do teachers need to bring to contexts with no curriculum materials as compared to contexts with highly-prescribed curriculum materials? How do teachers learn to pick and choose among available resources and put them together in ways that allow students to experience a coherent curriculum? Thus, this study has implications for teacher educators considering answers to these questions, but also for curriculum developers and school and district leaders. Given that curriculum capacity is realized through interactions across teachers, curriculum materials, and contexts (including students), both curriculum developers and policymakers might consider the ways materials are designed and supports are provided to enhance the likelihood of achieving high levels of capacity.

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THE BESSIE COLEMAN PROJECT: USING COMPUTER MODELING AND FLIGHT SIMULATION IN INFORMAL STEM SETTINGS

Jacqueline Leonard
University of Wyoming
jleona12@uwyo.edu

Will Jordan
Temple University
wjjordan@temple.edu

Roni Ellington
Morgan State University
roni.ellington@morgan.edu

This research report presents the pilot-year results of a three-year research project on computer science and technology. The Bessie Coleman Project, named for the first African-American and Native woman to receive a pilot's license, provides underrepresented students with opportunities to learn about STEM-related careers by participating in computer modeling and flight simulation. Three-dimensional computer modeling and flight simulation was used as interventions to increase underrepresented students' CT skills, motivation, and persistence. Project staff and facilitators at Boys and Girls Clubs and local schools implemented the project at six sites in Wyoming in 2018. A total of 124 participated in the pilot study: 29 (summer); 95 (fall). Descriptive evidence from survey data suggest that students had sustained interest in technology and qualitative data suggest students had interest in STEM careers.

Keywords: Computer Modeling, Flight Simulation, Computer Science, Technology

The Bessie Coleman Project (BCP) is designed to expand learning opportunities for elementary and middle grade students in the Rocky Mountain (Wyoming and Denver, Colorado) and Mid-Atlantic regions of the U.S. (Philadelphia and Baltimore) by leveraging Universal Design for Learning principles and strategies. It incorporates lessons and activities in 3D modeling, flight simulation, and piloting drones which are designed to bolster students' interest and engagement in mathematics and science learning. Indeed, the overarching purpose of the project is to create new STEM Pathways that provide unique learning opportunities for underrepresented students.

The work of the BCP involves research and development of cutting-edge professional development for teachers that is connected to culturally relevant learning activities for students. This report describes our Year 1 pilot, which allowed the research team to field test the intervention in two different types of informal settings: summer camps and afterschool programs. In this project, we engage students in project-based learning through computer modeling (i.e., upper-elementary students) and flight simulation (i.e., middle-school students) as an entrée to STEM and STEM awareness. Using computer modeling to help children learn to code (grades 4-5) and flight simulation with applications to learn how to use drones for data collection (grades 6-8) builds new knowledge and understandings of complex systems and contributes to the extant literature on STEM education and workforce readiness.

Two of the core research questions that guide the BCP project are:

1. What learning experiences involving emerging technologies (i.e., computer modeling, flight simulations, and drones) effectively enable diverse populations of students to gain familiarity and relevant competencies with these technologies, and what factors influence the outcomes of the learning experiences?

2. What culturally-responsive instructional and curricular practices and models (including place-based education) used by teachers enhance student understanding of and interest in STEM occupations, and what factors influence the outcomes of the practices and models?

Theoretical Framework

The theoretical frameworks that guide **The Bessie Coleman Project** are **Constructionism** (Papert & Harel, 1991) and **Expectancy-Value Theory** (Wigfield, 1994; Wigfield & Eccles, 2000). Constructionism focuses on students' ability to *build knowledge structures* regardless of the learning environment (Papert & Harel, 1991). In this context, the learner has a great deal of autonomy as he/she is guided by the work that proceeds rather than a protocol. The concept allows students to be active learners as they *think* about ideas, investigate that idea, discuss it with peers, and make adaptations within a complex system. The constructionist principle is well suited for teaching computer modeling and flight simulation since students learn to construct various types of models while using a project-based approach. In terms of Expectancy-Value, theorists argue that "individuals' choice, persistence, and performance can be explained by their beliefs about how well they do on [an] activity and the extent to which they value the activity" (Wigfield & Eccles, 2000, p. 68). The expectancy-value framework has been applied to the subject of mathematics and can be adapted to technology. The value a student puts on a subject or task is predictive of future intentions to participate in similar tasks (e.g., enroll in courses, pursue careers, etc.) While self-efficacy measures students' beliefs about success on very specific tasks, expectancy beliefs are measured more broadly with questions such as "How well would you expect to do if you had to learn something new about…" a given topic. Assessing ability-beliefs and values among rural and urban, African-American, Latinx, Native, and female students will add to the literature on broadening participation in STEM.

Methodology

Mixed methods (i.e., field notes, interviews, photographs, and videotapes) were used to collect qualitative data on the nuances of culture and place in after-school clubs and summer camps (Creswell, 1998). One hundred twenty-four students participated in the BCP during summer and fall 2018. The size of the sites varied from seven to 47 students. Recruitment yielded levels of diversity that were somewhat more robust than the general population in Wyoming or the racial composition of school children. About 72% of the study sample were White students and 66% were males. Combined, about 18% of the sample were Black, Latinx, and Native students. More diverse students will participate in Years 2 and 3 in Colorado and Pennsylvania. Finally, most of the participants are elementary students (grades 3, 4, and 5), who were also younger than 11 years old.

Baseline and post-intervention surveys were administered to students, prior to the beginning of the intervention and afterwards. Project staff designed and administered paper surveys for student participants which, in addition to a section on student background characteristics, contained a battery of items in the following three areas: (1) participants' use of technology; (2) participants' feelings about science; and (3) participants' opinions about technology. The various components of the survey were pulled and adapted from instruments in the public domain, including value expectancy (Eccles & Wigfield, 2002), self-efficacy in technology and science (Ketelhut, 2010), and STEM attitudes (Friday Institute, 2012).

Results

Descriptive data from student participants at six intervention sites are presented. This brief report emphasizes the use of technology survey for analysis and discussion.

Students Self-Reports of STEM

Student survey items reflected self-reports of their use of technology. Descriptive statistics are shown for each item. This includes the number of cases, baseline means, and post-intervention means, along with the mean differences between the two. However, in order to more accurately measure gains, the fourth and fifth columns subtracts baseline values from post-intervention values for the same participant. The fourth column shows adjusted gains that have valid data at both time points, while the fifth column depicts the Ns. The purpose of the tables is to examine general patterns across the survey data, rather than to reveal significant differences. There were modest pre- to post-intervention gains on most items (see Table 1).

Qualitative data were also collected during the summer STEM camps on students' perceptions and experiences. To field test the intervention, flight simulation was implemented at a Boys & Girls Club in southeastern Wyoming and computer modeling was implemented at a Boys & Girls Club in western Wyoming. Student comments during a focus group interview included comments such as: "I liked how we could 3D print things and I have never…actually got to do that and do new things on the computer;" "I liked flight simulation;" and "I liked drones." Figure 1 depicts one of the student's designs during the computer modeling camp.

Discussion

Preliminary findings in the Bessie Coleman Project reveal that students gained familiarity and competencies with computer modeling and flight simulation, especially when modeling was tied to 3D printing and flight simulation was tied to flying drones. Students were enthralled as they saw the images they created on the computer come off the print bed. Moreover, they were intrigued with the capabilities of the drones. Mean differences on the use of technology survey items ranged from M= 0.07 to M=0.47. However, students had the lowest gains on how to use data to solve problems, building computer programs, designing computer games, and problem solving in general. Culturally relevant strategies included use of guest speakers and place-based themes (i.e., animal crossovers) to help Wyoming students decide what images to create during computer modeling tasks. Field trips to museums also provided cultural and place-based experiences that students identified during focus group interviews. While this project is ongoing, these preliminary results are promising. In future study years, we will use the drones to collect data to create additional problem-solving opportunities such as geocaching. Cultural relevance will also include the history of Black and female aviators in addition to guest speakers.

Tables and Figures

Table 1: Participants' *Use* of Technology

Variable		Baseline Mean	Post-Interv. Mean	Mean Diff	Adj Gain†	Adj N†
Q1.1	Even if I try very hard, I cannot use the computer as easily as paper and pencil	2.24	1.77	0.47*	*0.53**	*104*
Q1.2	I can use a computer to graph what I found out in my experiment	3.43	3.67	0.25	*0.24*	*103*

Variable		Baseline Mean	Post-Interv. Mean	Mean Diff	Adj Gain†	Adj N†
Q1.3	I can use a computer to help me know what the results of an experiment means	3.73	3.92	0.18	*0.20*	*103*
Q1.4	When I do an experiment, it is hard for me to figure out how the data I collected answers the question	2.43	2.36	0.07*	*0.09**	*102*
Q1.5	I am good at building computer programs	2.97	3.12	0.14	*0.16*	*96*
Q1.6	I am good at fixing computer programs	2.55	2.89	0.34	*0.41*	*102*
Q1.7	No matter how hard I try, I do not do well when playing computer games	1.74	1.47	0.27*	*0.31**	*104*
Q1.8	I am very good at building things in games	3.95	4.23	0.28	*0.31*	*104*
Q1.9	I can use a computer to control toys and tools	3.51	3.85	0.34	*0.39*	*102*
Q1.10	I can learn how to design a computer game if I do not give up	4.11	4.25	0.14	*0.13*	*103*
Q1.11	When solving a problem, I can create a list of steps to solve it	3.66	3.84	0.17	*0.20*	*99*
Q1.12	When solving a problem, I can see patterns in the problem	3.31	3.53	0.22	*0.23*	*102*
Q1.13	When solving a problem, I can break the problem into smaller parts	3.73	4.05	0.32	*0.35*	*102*
Q1.14	When doing an experiment, I can use a computer to help me collect data	3.87	4.14	0.27	*0.25*	*103*
Q1.15	When solving a problem, I can figure out several ways to solve it	3.73	3.88	0.15	*0.21*	*98*
Q1.16	When solving a problem, I can figure out what is the best solution	3.89	4.07	0.18	*0.22*	*104*

Response categories: 1=Strongly Disagree; 2=Disagree; 3=Somewhat; 4=Agree; 5=Strongly Agree
** Gain represents the absolute value since some items are negatively worded.*
†Adjusted values include only participants with valid non-missing data on both baseline and follow-up surveys.

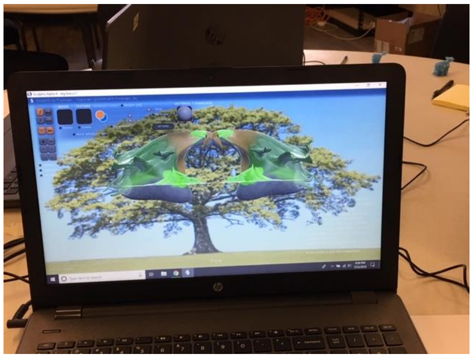

Figure 1: Computer Modeling Task

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HOW KOREAN TEXTBOOKS AND EUREKA MATH USE REPRESENTATIONS INVOLVING FRACTION MULTIPLICATION

Ji-Won Son
University at Buffalo
jiwonson@buffalo.edu

Oh Hoon Kwon
University of Wisconsin-Madison
kwon@math.wisc.edu

Ji Yeong I
Iowa State University
jiyeongi@iastate.edu

Despite of the significance of visual fraction models for teaching and learning fraction multiplication, how students are supported to learn fraction multiplication through representations remains unknown. By examining the presentation of visual representations in the contexts of word problems and computational processes in representative Korean and U.S mathematics textbook series, this study explores how the Korean and U.S. mathematics textbooks may facilitate student learning of fraction multiplication. Analyses showed that the selected U.S. mathematics textbooks provide more instances of fraction multiplication with various types of word problems and computational procedures in developing fraction multiplication. However, Korean textbooks use representations with different purposes, which in turn leads to students' explicit and meaningful understanding of fraction multiplication concepts and procedures.

Keywords: Fraction multiplication, Textbook analysis, Representations

Introduction

The Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010) highlight the importance of using "visual fraction models" for teaching and learning fractions to help students explore the underlying ideas of fraction multiplication and deepen their understanding of multiplying fractions. However, it is often reported that students and even teachers have difficulties in using representations for fraction multiplication (Webel, Krupa, & McManus, 2016). For example, although teachers and students may assume the typical area model for the multiplication of fractions is an extension of the area model for whole number multiplication, the area model—prevalently used in current U.S. curricula for fraction multiplication—seems not a direct conceptual extension of the area model for whole number multiplication (Kwon, Son, & I, 2017). It thus requires a careful investigation of the effective use of representations/models to help students deepen their understanding of multiplying fractions.

We noticed two different area models used in various contexts of fraction multiplication: *area-to-area model* and *lengths-to-area model* (Kwon, Son, & I, 2019). However, there is little research on how those models are presented in textbooks and what learning opportunities are provided with the two models to represent the fraction multiplication process. Given the key role of textbooks as the primary resource of learning, this study aims to narrow such research gap by exploring representational transition in textbooks in the contexts of word problems and computation problems through an examination of fraction multiplication in a representative Korean and U.S. textbook series. Since an international textbook analysis examines how curricula in high-achieving countries provide students with opportunities to learn, investigating Korean textbooks compared to a representative U.S. textbook series, Engage NY modules or EUREKA Math (EM) are expected to contribute to the development of students' understanding of multiplying fractions. According to Kaufman, Thompson, and Opfer (2016), more than 60% of US elementary teachers use Engage NY modules or EM in teaching mathematics. The research questions that guide this study are: (1) How is fraction multiplication introduced and developed in Korean textbooks and Engage NY modules? (2) What types of word problems are presented to introduce fraction multiplication? In the contexts of word problems, how and in what ways are representations used? (3) In the contexts of computational tasks, how and in what ways are representations used to introduce fraction multiplication?

Theoretical Perspectives

Table 1 provides four-word problem situations involving multiplication drawn from Carpenter et al. (1999) and the cognitively guided instruction (CGI) framework and three visual models recommended to be used to develop fraction concepts: area models (e.g., rectangular regions), length models (e.g., number lines), and set models (e.g., beans). These word problems and representations can be grouped into two kinds: Action on/Change of Initial Quantity and Operation/Coordination of Two quantities (Kwon et al., 2017, 2019). Action on/Change of Initial Quantity type represents an initial quantity and describe how the initial quantity changes as the operation takes place. Equal group and compare problem situations mostly belong to this type. Operation/Coordination of Two quantities type considers two quantities and illustrate how two quantities have been coordinated simultaneously; there is no distinction between two factors (quantities) because they play the same role. Combination and product of measure problem situations are Operation/Coordination of Two quantities type. Set and linear models can be categorized typically as presenting Action on/Change of Initial Quantity type whereas area models can represent both Action on /Change of Initial Quantity and Operation / Coordination of Two quantities types for fraction multiplication (Kwon et al., 2017, 2019).

Table 1: Two Underlying Problem Structures for Word Problems and Models

	Action on / Change of Initial Quantity		Operation / Coordination of Two quantities	
	Equal Group	Compare	Combination	Product of Measure
Set Model				
Linear Length Model				
Area Model	Area-to-area Model		Lengths-to-area Model	

In analyzing three types of models, we break the area model further by looking at how area models represent the fraction multiplication process: *area-to-area model* and *lengths-to-area model*. Each model reflects different aspects of a fraction and different thinking processes involved with fraction multiplication (Kwon et al., 2019). In the area-to-area model for fraction multiplication, the multiplication is used to figure out a portion of a fraction (of a whole) or a part of a part (of a whole). For example, with the problem of $\frac{1}{2} \times \frac{5}{8}$, the fraction $\frac{5}{8}$ represents the area of a piece of the unit rectangle, where a fraction is regarded as a representation of a part-whole relationship. 1/2 is an operator; thus 5/8 is a number that represents the area of the shaded pieces, and a corresponding action is taking 5 pieces out of 8 pieces equally dividing the unit rectangle. Different from the area-to-area model, with the first fraction as 'operator,' the lengths-to-area model considers both fractions as measures (i.e., 'operand'). In the lengths-to-area model with the above problem, students have a rectangle of two side lengths 5/8 and 1/2. The product of those two lengths represents the area of the shaded rectangle. To figure out the area of that rectangle in the lengths-to-area model, students need to find the unit area and count the number of unit areas inside the rectangle. In this study, we explore how two curriculum resources provide students with opportunities to learn fraction multiplication.

Methods

We chose to examine Korean textbooks (KM) published by the Ministry of Education, Science, and Technology compared to Engage NY modules (EM) due to its popularity. We particularly focused on chapters/lessons on fraction multiplication. In KM fraction multiplication is intensively addressed in Grade 5 whereas EM addresses fraction multiplication from Grade 4 to Grade 5. The data used for this study come from the relevant lessons on fraction multiplication from EM and KM. This study analyzed tasks presented in the lessons and the corresponding pages in the supplementary materials, including the teacher's manuals and student workbooks. During the first reading, coders focused on the first research question, when and how multiplication of fractions were developed. During this process, quantifiable or factual characteristics were identified, including the number of lessons, the recommended time to allocate to each topic, and the stated objectives of each lesson. In the second and subsequent readings, we focused on our second research question: ways in which the lessons support the development of conceptual understanding and procedural fluency. These reading examined qualitative characteristics, including meanings of multiplication of fractions, representations (mathematical models) used, word problem introduced, and ways in which those representations are used. After coding, we first counted the frequency of all the problem/representation types. We conducted the detailed analyses at three tiers: (1) only word problems, (2) word problems with representations, and (3) purely math problems with representation. We repeated this process for computational tasks to answer the third research question.

Selected Findings

How Fraction Multiplication is Introduced and Developed

EM introduces fraction multiplication in grade 4 and continue to develop in Grade 5. Four lessons are devoted to studying fraction multiplication in grade 4, focusing on (a whole number) × (a fraction) and (a whole number) × (a mixed number). The procedure of multiplying a whole number by a proper fraction (*n* times *a/b*) is taught as (n × *a*)/*b* using the associative property and the concept of repeated addition. The multiplication of a whole number by a mixed number is addressed using the distributive property. In grade 5, 13 lessons are devoted to developing different types of fraction multiplication in EM, which include (a fraction) × (a whole number), (a unit fraction) × (a unit fraction), and (a non-unit fraction) × (a non-unit fraction) (see Table 2). Tape diagrams have a central role to help students make sense of each multiplication.

KM introduces fraction multiplication in grade 5, one semester later than EM, and intensively develops fraction multiplication with 12 lessons. Similar to EM, KM introduce (a whole number) × (a fraction) and (a whole number) × (a mixed number) based on repeated addition with an equal group such as interpreting $3 \times \frac{2}{5}$ as 3 groups of $\frac{2}{5}$. Then, (a fraction) × (a whole number) is addressed based on "finding portions of whole units" with an "of" or "out of" idea. For example, $\frac{2}{3} \times 6$ is interpreted as finding $\frac{2}{3}$ of 6 whole units. Using the same underlying idea, KM addresses (a fraction) × (a mix number). Then, "finding portions of portions" are developed using (a unit fraction) × (a unit fraction) and (a non-unit fraction) × (a non-unit fraction) with an area-to-area model in the order. Different from EM, KM introduce (a mixed number) × (a mixed number) using a lengths-to-area model. Furthermore, multiplication of three fractions is introduced by asking students to use representations to find portion of portion of portion. Table 2 shows topic arrangement of EM and KM for fraction multiplication.

Table 2: Structure of Fraction Multiplication between EM and KM

Order	Korean Textbooks	Engage NY	Order
1	Whole number × Proper fraction	Whole number × Proper fraction	1
2	Whole number × Mixed number	Whole number × Mixed number	2
3	Proper fraction × Whole number	Proper fraction × Whole number	3
4	Mixed number × Whole number		
5	Unit fraction × Unit fraction	Unit fraction × Unit fraction	4
		Unit fraction × Non-unit fraction	5
6	Non-unit fraction × Non-unit fraction	Non-unit fraction × Non-unit fraction	6
7	Mixed number × Mixed number		
8	Multiply three fractions		

Word Problems and Types of Models in Word Problems in KM and EM

Table 3 shows word problem types involving fraction multiplication used in this study, emphasizing the role of factors. Prior research classified word problems either based on semantic structures (Smidt & Weiser, 1995), different situations (CGI research studies, CCSSM - Mathematics Glossary Table 2), or learner's mental models (Prediger, 2008). Although such classification helps identify different problem situations involving multiplication and division, roles of two factors that describe multiplication processes is not evident. Drawn from CGI's and CCSSM's whole number multiplication word problem situations that include equal groups, arrays/area, and compare, we provide an exhaustive and a combined list of fraction multiplication word problem situations. In particular, by utilizing qualitative analyses that look at various problem situations included in KM and EM, we extended CGI's and CCSSM's whole number multiplication word problem situations. Eight problem situations were identified as representing fraction multiplication situations, which can be categorized into three types, depending on the roles of two factors in the multiplication: (Operator × Operand), (Operator × Operator), and (Operand × Operand). While (Operator × Operand) and (Operator × Operator) are referred to Action on/Change of Initial Quantity type that describes how the initial quantity changes, (Operand × Operand) represents Operation/Coordination of Two quantities type.

Table 3: Fraction Multiplication Word Problem Situations

Problem Situation	Factor Role	Characteristic	Example
Equal size group	Operator × Operand	Building up the whole using a part by repeating the part.	Windsor the dog ate 2 3/4 snack bones each day for a week. How many bones did Windsor eat that week?
Comparison	Operator × Operand	Multiplicative comparison between a quantity and a referent quantity	A seamstress needs 27 2/3 yards of fabric to make a child's dress. She needs 3 times as much fabric to make a woman's dress. How many yards of fabric does she need for a woman's dress?

Unit Conversion	Operator × Operand	Converting units of measurement using dimensional analysis	Mrs. Lang told her class that class's pet hamster is ¼ ft in length. How long is the hamster in inches?
Finding portion (Partitioning)	Operator × Operand	Finding a portion of the whole	A container holds 2 liters of oil and vinegar. ¾ of the mixture is vinegar. How many liters of vinegar are in the container?
Finding portion of portion	Operator × Operator	Asking a combined operator	½ of the class are girls. ¾ of them are wearing glasses. What fraction of the class are girls wearing glasses?
Scaling (Resizing)	Operator × Operand	Scaling of the magnitude of a quantity with a scale factor, related to length, area, volume.	A company uses a sketch to plan an advertisement. The lettering on the sketch is in tall. In the actual advertisement, the letters must be 34 times as tall. How tall will the letters be on the building?
Combination (Probability)	Operand × Operand	Finding combinations among some objects with a number of possibilities	If two independent events have probabilities ⅔ and ⅘, what is the probability both will occur?
Product of measures	Operand × Operand	Finding the area of a rectangle of given dimensions	A rectangular region on a farm is ⅔ km by ⅘ km. What is its area?

Fraction multiplication word problems are similarly distributed in both curricula in terms of larger categories separated by the roles of two factors (see Table 4). Assuming (Operator × Operator) and (Operator × Operand) type problems are more related to action on or change of the initially given quantity, and (Operand × Operand) type problems are closer to operation or coordination of two given quantities, EM provided 86.6% of fraction multiplication word problems as action on or change of initial quantity type problems and 13.4% as operation or coordination of two quantities type problems, whereas KM provided 85% and 15% of each type. However, taking different problem situations in a consideration, two curricula showed different trends. KM focuses more basic situations of word problems and excluded harder situations, such as comparison, combination, or unit conversion. Emphasizing word problems with product of two measures was also notable. On contrary, EM includes a wide variety of different situations.

Table 4: Frequency Comparison of Fraction Multiplication Word Problem Situations

Factor Role	Word Problem Situations	KM	EM
Operator × Operator	Finding portion of portion	11 (27.5%)	29 (27.9%)
Operator × Operand	Equal size group (Iteration, Rate context also included)	12 (30%)	19 (18.3%)
	Finding portion (Partitioning)	8 (20%)	16 (15.4%)
	Scaling (Resizing)	3 (7.5%)	13 (12.5%)
	Comparison	0 (0%)	13 (12.5%)
	Subtotal	23 (57.5%)	61 (58.7%)
	Combination (Probability)	0 (0%)	4 (3.8%)
	Product of measures	6 (15%)	1 (1%)

Operand × Operand	Unit Conversion	0 (0%)	9 (8.6%)
	Subtotal	6 (15%)	14 (13.4%)
Total		40 (100%)	104 (100%)

Although both area models and tape diagrams are used in EM, a significant amount of tape diagrams are used in representing fraction multiplication (see Table 5). In particular, both area models and tape diagrams focus on one type of multiplying processes: action on/change of initial quantity. Different from EM, KM uses area models intensively but shows both types of multiplying processes with area models: area-to-area model (action on/change of initial quantity) and lengths-to-area model (operation/coordination of two quantities). In addition, KM presents product of two measures type problems as well as (a mixed number) × (a mixed number) type problems by the lengths-to-area model (see Figure 1). In contrast, all area models in EM rely on the area-to-area model (change of initial quantity) (Kwon et al., 2017)

Table 5: Types of Models Used in Word Problems and Computational Tasks

Representation Models	KM	EM
Area Model (Rectangular Array included)	13 (65%)	40 (26%)
Number Line	0 (0%)	1 (0.6%)
Tape Diagram	7 (35%)	113 (73.4%)
Set Model	0 (0%)	0 (0%)
Total	20 (100%)	154 (100%)
Types of Model		
Action on / Change of Initial Quantity	**15 (75%)**	**154 (100%)**
Operation / Coordination of Two quantities	**5 (25%)**	**0 (0%)**

How and in What Ways Models are Used in Computational Tasks

EM contains about 4 times as many tasks of fraction multiplication as those of Korean textbooks in computational tasks (see Table 6). When highlighting the most frequent task types involving calculation, it appears that there was a consistency of frequently used types within each textbook series. In both textbooks, calculation only is most popular, followed by calculation with representation and calculation asking explanation. However, EM additionally includes tasks that ask for both explanation and representation and those asking students to create a word problem related to fraction multiplication.

Table 6: **Problem Type in Purely Numerical Problem Contexts and its Frequency**

	Calculation only	Calculation + explanation	Calculation + representation	Calculation + explanation + representation	Calculation+ word problem	Total
KM	49 (63%)	8 (1%)	21 (27%)	0 (0%)	0 (0%)	78
EM 4 & 5	155 (55%)	20 (7%)	101 (36%)	3 (1%)	2 (1%)	281

27% of computational tasks ask for using representations in KM whereas 37% in EM demand representations. EM presents tape diagrams most frequently in representing (a whole number) × (a fraction) and (a fraction) × (a whole number), whereas KM evenly uses area models and tape

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diagrams. For (a fraction) × (a fraction), both EM and KM use area models. Figure 1 shows examples of representation use in two textbook series.

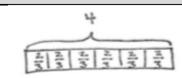
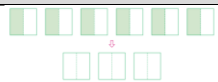
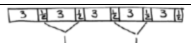
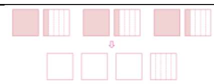
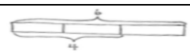
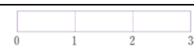
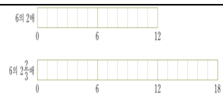
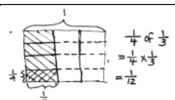
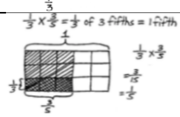
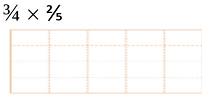
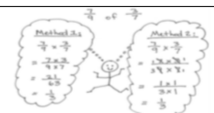
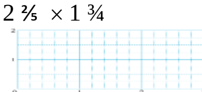
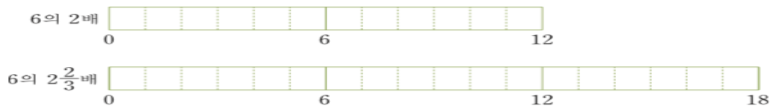
	Representation		Calculation Process	
	EM	KM	EM	KM
W × F	 4 × ⅔ = ⅔+⅔+⅔+⅔=2 2/3	 6 × ½ = 6/2=3	S1: 5 × ¾ = 5 × 3 fourths = 15 fourths S2: 5 × ¾= 5 × 3 × ¼ S3: 5 × ¾ =(5 × 3)/4	S1: 5 × ¾ =¾ + ¾+¾+¾+¾=15/4 S2: 5 × ¾ =(5× 3)/4
W × M	 5 × 3 ½ = (3 × 5)+1+1+1.2=171/2	 3 × 1 ⅕= 3+ ⅗ = 3 3/5	S1: 2 × ⅕ =(2×3)+(2×1/5)=6+⅖ S2: 2 × 31/5=6 + ⅖ =6 ⅖ S3: 2 × 3 ⅕= 2 × 16/5 =6 2/5	S1: 2 × 3 ⅕ = 31/5 + 3 ⅕ =6 ⅖ S2: 2 × 16/5=6 ⅖ S3: 2 × 3 ⅕=(2×3)+(2×1/5)=6+⅖=6 ⅖
F × W	 ⅔ × 6 = 2 × 6/3 =4	 ⅓ × 3 = 1 × 3/3=1	S1: ⅔ × 6 = 2 × 6/3 =4	S1: ⅔ × 6 = (2 × 6)/3=12/3=4
M × W	None	 2 ⅔ × 6 =	None	S1: 2 ⅔ × 6= 8/3 × 6 =48/3=16 S2: (2+⅔) × 6 = (2 × 6)+ (2/3×6)=16 S3: Reduction
UF × UF		⅓ × ¼ 	S1: ⅓ × ¼ = (1×1)/(3×4)= 1/12	S1: ⅓ × ¼ = (1×1)/(3×4)= 1/12
NF × NF UF × NF		¾ × ⅖ 		S1: ¾ × 2/4 = (3 × 2)/(4 × 4) S2: $\frac{3}{8}\times\frac{5}{6}=\frac{1\times5}{8\times2}=\frac{5}{16}$
M × M	None	2 ⅖ × 1 ¾ 	None	S1: 2 ⅖ × 1 ¾= 12/5 × 7/4 S2: $3\frac{3}{5}\times2\frac{1}{4}=\frac{18}{5}\times\frac{9}{4}=\frac{9\times9}{5\times2}=\frac{81}{10}=8\frac{1}{10}$

Figure 1: Representation Use and Calculation Processes in EM and KM

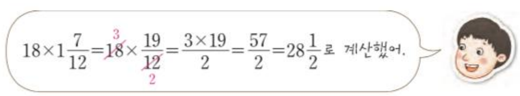
As shown in Figure 1, both KM and EM use representations as an essential means for introducing fraction multiplication and developing students' understanding about fraction multiplication. In each problem type, similar calculation strategies are introduced in KM and EM. Despite the similarity, unique features exist in developing procedures in KM and EM. EM seems to highlight the meaning of a denominator, emphasizing a role of a unit fraction and introduce a denominator as a written word. For example, for a whole number times a fraction such as 5×¾, EM asks students to represent 5×(3 fourths), which leads that (5×3) fourths is 15 fourths, 15/4. EM also asks to break down ¾ as 3×¼ to calculate 5×3×¼, which is 15/4. In contrast, KM relies on repeated addition (i.e., 5×¾ =¾+¾+¾+¾+3/4=15/4) and then address multiplying a whole number times a numerator (e.g., 5×¾). EM's approach may provide pedagogical suggestions to how to additionally address (a whole number)×(a fraction). KM introduces two additional problem types such as (a mixed number)×(a whole number) and (a mixed number)×(a mixed number), which is not addressed in EM. A unique approach in KM is using a comparison method where students are asked to think about relative sizes of two fractions and to generalize a solution method. For example, before addressing computational procedures of (a mixed number)×(a whole number), 2 ⅔×6, KM asks students to represent 2×6 using a tape diagram and then 2 ⅔×6, and compare relative size of each (see figures below).



Afterwards, KM presents two different solution methods shown below and asks students to compare and come up with a solution method for 2 ⅔×6.

$6\times\frac{8}{3}=\boxed{}$ $(6\times 2)+(6\times\frac{2}{3})=\boxed{}$

As students share their solution methods for 2 ⅔×6, KM introduces additional methods shown below with a different problem and asks students to apply the given solution method.



Based on that, KM provides an opportunity to generalize multiplying methods of (a mixed number) × (a whole number). There is a prompt that specifically asks students to explain how to calculate (a mixed number) × (a whole number) in KM at the end of each lesson. KM's guided instruction which starts from representing comparing multiplying fractions using representations, through mathematizing thinking processes using two calculation ways, to generalizing solution methods in each problem type may support students' meaningful and explicit learning of fundamental mathematical ideas embedded in fraction multiplication.

Discussion and Implications

This study contributes to the current literature because it illustrates how formal instructional resources such as textbooks may support students' learning of fraction multiplication. EM provides more instances of fraction multiplication with various situations of word problems in developing fraction multiplication. However, KM seems to provide representations with different purposes to support students' explicit and meaningful understanding of fraction multiplication. While most of tape diagrams and area models used in EM are focused on only one type: action on/change of initial quantity, KM use both types in their area models, including the lengths-to-area model (operation/coordination of two quantities). We also found that EM contains about four times as many problems of fraction multiplication as those of the KM in non-contextual situations. Although there are similarities in the use of representations and calculation strategies in the context of computation, there are unique features in developing procedures in KM and EM. The findings in both textbooks' insights and pedagogical techniques may provide rich information for textbook designers and classroom teachers to refer back to.

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THE INFLUENCE OF HIGH SCHOOL TEACHERS' CURRICULAR NOTICING ON PLANNING

Ariel Setniker
California State University Maritime Academy
asetniker2@unl.edu

Lorraine Males
University of Nebraska-Lincoln
lmales2@unl.edu

Kelsey Quigley
University of Nebraska-Lincoln
kquigley2@unl.edu

Scott Block
University of Nebraska-Lincoln
sblock4@unl.edu

This paper reports a study of two high school mathematics teachers' interactions with curriculum materials while planning lessons. Specifically, we address how their attention interacts with their interpretations and responses to the materials, and how the curriculum elements and format of each set of materials influenced these interactions. Further, we report on how teachers' noticing informs student opportunities to learn.

Keywords: Curriculum, Instructional activities and practices, Instructional vision

The concept of noticing is not unique to the study of teaching, nor is it unique in the profession of teaching to interactions that occur solely within the classroom. We expand the construct of the professional noticing of children's mathematical thinking (Jacobs, Lamb, & Philipp, 2010) to describe how high school teachers use curriculum materials during planning. Just as Jacobs, et al. (2010) focus on a "specialized type of noticing" (p. 171), the professional noticing of children's mathematical thinking, we focus on a special type of noticing: the noticing of curriculum materials by teachers for the purposes of using the materials with students in their enacted lessons. Curricular noticing is the set of skills that constitute the curricular work of mathematics teaching, namely: curricular attending, curricular interpreting, and curricular responding. Curricular attending involves "viewing information within curriculum materials to inform the teaching and learning of mathematics" (Dietiker, Males, Amador, & Earnest, 2018, p. 523), curricular interpreting involves making sense of that to which is attended, and curricular responding involves making curricular decisions based on the interpretation of curriculum materials. Figure 1 depicts the Curricular Noticing Framework.

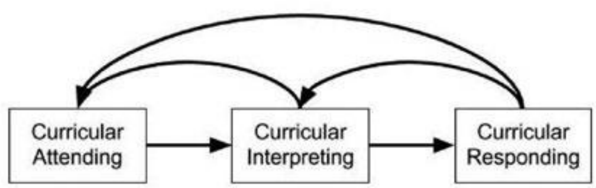

Figure 1: The Curricular Noticing Framework (Dietiker, Males, Amador, & Earnest, 2018, p. 527)

Although these definitions seem to presuppose a sequence, we argue that the process may not unfold in a strictly linear fashion. For example, while a response is dependent on a curricular

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interpretation of that to which a teacher attended, an interpretation may trigger a teachers' attention, or a decision to respond may result in the teacher attending to something new.

Purpose and Research Questions

Our purpose is to describe two high school teachers' curricular noticing during planning and the influence of curricular elements and format on this noticing. Specifically, we address:

1. What do teachers attend to in curriculum materials during planning, and what interpretations and responses do they make in relation to this attention?
2. How might curriculum elements and format influence teachers' curricular noticing?

Methods

Participants

This paper focuses on the planning of two teachers who were part of a larger study of teachers' use of curriculum materials. While both taught in the same district, they presented very different contexts for examining curriculum use. First, the two teachers had varying years of teaching experience, Alice having 9 years and Elise being in her first year. Second, Alice was piloting a new set of curriculum materials for the district, and while Elise was using what the district had been using for years, she was also new to using the materials. In this way, we had the chance to observe their curricular noticing as they were reading materials for the first time.

Curriculum Materials

The materials used by the teachers varied on multiple dimensions. For instance, Alice's materials, the CPM Educational Program's *Core Connections Algebra*, classifies as Standards-Based, meaning that it was designed to align with the NCTM (2000) Standards. Elise's materials, Pearson Education, Inc.'s *Geometry,* is classified as Conventional, meaning that it was developed from editions published prior to the release of the NCTM Standards. Comparing the curriculum use with the differing curricula may provide insight into how the curriculum elements and format might influence teachers' curricular interactions.

Data Analysis

We analyzed eye tracking recordings as well as coded each teachers' transcript. We assigned an Attend code when a teacher looked at or read aloud a section of the curriculum materials, with four subtypes: district-adopted materials, materials from previous lessons (produced by the teacher), old materials (e.g. past lesson plans), and online sources. We assigned an Interpret code when a teacher made sense of the curriculum materials, with three subtypes: the curriculum itself, students (e.g. approaches), or mathematics (e.g. working out solutions). We assigned a Respond code when teachers made a decision as to what to include in their plan, with four subtypes: using something from the materials as is, adapt it, not use it at all, or to supplement.

To address the extent to which each teacher's attention interacted with their interpretations and responses, we examined their thought processes via idea sequences. Each time a teacher focused on one big idea in their planning interview, we defined this as an idea sequence. For example, idea sequences included thinking around a particular concept or commentary around the structure of the overall curriculum materials. To explicitly represent an idea sequence, we recorded the sequence of attention and subtypes of interpretations and responses.

Results & Discussion

Idea Sequences

In order to examine teachers' curricular noticing, we identified idea sequences across each planning session. For example, when our two teachers discussed their final thoughts around selecting examples for a warm-up, we generated the idea sequences in Figure 2.

Figure 2a: Idea Sequence Example - Alice

Figure 2b: Idea Sequence Example - Elise

In her idea sequence of one minute, 57 seconds, Alice attended to what she reported as student materials from the previous lesson, discussing her interpretation of the meaning of problems near the end. She continued to attend to the student materials, ending with a discussion of doing a warm-up involving assessment, believing her students needed this opportunity. Elise, in 6 minutes, 22 seconds, attended to curriculum materials from her student teaching, searching through worksheets and interpreting the meaning of problems as she looked for a specific focus. Elise eventually found problems to use, and as she copied them into her class slides, she further interpreted their meaning and how her students would respond. Following these, Alice and Elise moved on to new ideas and thus we defined the next idea sequence of their planning periods.

Despite the different experiences of the two teachers and the different curriculum materials, Alice and Elise had a similar number of idea sequences in their planning. Alice's thought processes are represented by 7 idea sequences, Elise's by 8. This similarity is especially notable when compared to the overall planning times, with Alice spending about 18 minutes and Elise about 57 minutes. We see this more clearly by considering our two teachers' average idea sequence duration - Alice with two minutes, 10 seconds and Elise about 7 minutes. Further, we see a widely varying range of duration when it comes to the idea sequences - a range of about 9 minutes amongst Alice's idea sequences, and a range of about 23 minutes amongst Elise's.

A previous study (Males & Setniker, 2018) indicated that prospective secondary teachers always began to work with new ideas by attending to curricular resources. In this work, Elise, a secondary teacher in her first year, also began each idea sequence with attention to a curricular resource. In comparison, Alice began one of her thought processes by making interpretations about her students from past experiences, which seemed to influence what she was searching for.

Our analysis also indicates that idea sequences are longer in duration across teachers for specific curricular elements. For example, Alice's longest idea sequence is almost 9 minutes, which involved comparing closure problems. Elise's idea sequence around closure is about 6 minutes. In contrast to such similarities amongst Alice's and Elise's idea sequences, those of short duration are not focused on the same thought processes across the teachers. However, we see that while the durations of our two teachers' planning sessions and the duration of the idea sequences within were quite different, they still had similar thought processes defining separate idea sequences, though not necessarily in the same order.

Curricular Elements and Format

Each set of curricular resources used by our two teachers was comprised of numerous individual curriculum elements. In this section, we describe the attention to select elements in order to understand how curricular elements and format influenced teachers' noticing.

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Alice expressed her general appreciation for the approach of her district-adopted curriculum and furthermore, her familiarity with the format and elements through professional development and some previous collaborative planning. Elise also had a general approach to her curricular resources, in that she did not use her district-adopted curriculum very often. We see that Elise had a skeptical approach to elements in the district-adopted book, through the lens of a short class time, and yet was optimistic about almost any other element she found in other resources.

Further, the degree to which teachers' beliefs are aligned with curricular goals seems to play a large role in approach to curriculum materials and what they notice. For example, Alice had the goal of keeping her students engaged. Throughout planning, this goal was a driving force in analyzing the materials and choosing what to include in her plan.

Conclusion and Implications

Studying the two teachers' interactions while planning with their district-adopted and chosen curriculum materials provided insight into their curricular noticing and how it is influenced by the type of curriculum and also teacher predisposition and beliefs. Before teachers decided what to include in their lesson plans they attended to materials. This was evidenced by an attend code prior to a response code in each of our idea sequences. However, not every idea sequence started with an attend code - specifically, Alice began an idea sequence with an interpretation around students. This was steered by her strong beliefs around student engagement, and with this in mind began a new idea sequence focused on engagement. This was perhaps due to her years of teaching experience and/or her personal views of her students' needs.

While engaging in curricular noticing, teachers were attending simultaneously with interpreting or responding. Our idea sequences indicate that teachers were looking for particular elements that supported their personal beliefs, interpreting the meaning behind various tasks and problems while doing so. For example, Elise, while searching for a problem says "That's a fun problem...it's a converse...see why do they throw area in right there? ...no, alright, that's a good question. They actually need area this time around." Here Elise interprets the point of the question and returns to the objective sheet to confirm that her students have learned area before.

This study contributes to expanding the construct of noticing and understanding how curriculum materials influence teachers' planning and students' opportunities to learn. First, this corroborates the research that indicates that teachers' beliefs and predispositions influence their curriculum use, particularly their attention to certain curricular elements. Further, familiarity with curriculum materials aids teachers in navigating format. However, this may result in important teacher suggestions and opportunities for student learning being missed. Not all elements were attended to in each curricular resource, and even some elements which were attended to were not attended to for more than two seconds. It is likely difficult for teachers to interpret and respond to curriculum materials to plan and enact instruction if they have not attended to the curriculum materials. Therefore, this attention largely influences what students have the opportunity to learn. This underscores the importance of future work describing how curriculum developers and teacher educators may support teachers in optimizing attention to critical curriculum elements.

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LEARNING THROUGH PLAY

Dittika Gupta
Midwestern State University
dittika.gupta@msutexas.edu

Timothy Hinchman
Midwestern State University
timothy.hinchman@msutexas.edu

Jeff Hood
Midwestern State University
jeff.hood@msutexas.edu

Austin Kureethara Manuel
Midwestern State University
austin.kureetharamanuel@msutexas.ed

Marcos David Lopez
Midwestern State University
marcos.lopez@msutexas.edu

Keywords: Technology, Elementary School Education

The goal of the project is to give second grade students an exposure to hands-on engaging ways of coding, thus fostering their reasoning and critical thinking skills. The objective of this study is to examine the impact of exposing second grade students to an early coding scenario and gauge their increase in logical reasoning. The review of literature supports the effects of using play-oriented strategies to improve learning skills of students.

It is well accepted that dramatic and constructive play sequences provide opportunities to use language for questioning, monitoring and guiding the actions of one's self and others, as well as opportunities for decision-making, problem-solving and experimentation with materials and resources (Almon, J, 2003; Bennett, Wood, & Rogers, 1997; Guha, 1987; Smith, 2006; Whitebread & Jameson, 2005). The addition of a play-oriented module in a child's education improves not only their STEM-oriented learning skills, but also their communication and comprehension skills as well. This research is meant to serve to help bridge this gap so that educators can introduce certain aspects of programming in an early age.

Procedures

The participants of this study experienced the following:

1. Every 2nd grade student was given a consent form and pre-test to fill. The pre-test involving logical sequences and a maze was conducted by the classroom teacher during a regular school day as a regular activity. No researcher was present during this time.
2. School officials selected 20 students to participate in the study. The selected students met with the researchers in the school cafeteria. They were read the book “The Very Hungry Caterpillar” and then broken into random groups to play with the Code-a-Pillar®. This was repeated once a month for four months over a semester.
3. Every 2nd grade student was given a post-test, identical to the pre-test. The post-test was conducted by the classroom teacher during a regular school day as a regular activity. No researcher was present during that time.

This poster presentation will explore the results from this test and detail the progress of the participating students at month long intervals over the semester.

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BLURRING BOUNDARIES: EMERGENT TECHNOLOGICAL PRACTICES OF POST-SECONDARY STUDENTS WITH MATHEMATICS LEARNING DISABILITIES

Alayne Armstrong
University of Regina
alayne.armstrong@uregina.ca

Mirela Gutica
British Columbia Institute of Technology
mirela_gutica@bcit.ca

Keywords: technology, equity and justice, inclusive education, students with disabilities

This exploratory case study investigates the emerging practices of post-secondary students with mathematics learning disabilities (MLD) (Furlong, McLoughlin, McGilloway, Geary, & Butterworth, 2015; Willcut et al., 2013) in adapting portable electronic technologies to better suit their learning needs, examining: 1) Which technologies do they use fluently and how did they learn to use them? 2) What are the characteristics of their use and adaptations of these technologies? Working from an enactivist perspective that the systems of body, mind and environment are entangled (Merleau-Ponty, 1962), we argue that learning emerges through the interactions between these systems and that there is a potential reciprocal relationship between the use of technology and cognition in the act of learning (Li, Clark, & Winchester, 2010): technology becomes embodied within the learning system in its potential to open up and extend the learner's bodily senses to new experiences and capabilities (Söffner, 2017). Our study is situated within emerging research on the "learner's perspective" on the use of technology (Kukulska-Hulme et al., 2011) in the context of increasing learner autonomy, personal choice of devices and ways to use those devices, and used semi-structured interviews to allow maximum participation and accessibility (Alper & Goggin, 2017). We recruited nine participants with MLD from two post-secondary campuses in Western Canadian cities, enrolled in a variety of programs (e.g sciences, technology, business, social sciences). Each self-identified as being confident and knowledgeable about their use of technology, thus being potentially capable of developing technological adaptations and practices that peers might find beneficial to adopt (Kukulska-Hulme et al, 2011).

Our results reveal that participants used technology as a type of electronic "prosthesis" to effectively support their MLD: using a variety of sources of alternative on-line content to supplement in-class information about mathematic content and possible strategies; accessing aural and textual information (e.g. pausing and repeating audio and video recordings of lectures; supporting reading through electronic readers and software narrators), support for mathematical calculations (calculators to check accuracy of input; spreadsheet software formulae for repetitive calculations); utilizing structural and organizational supports (software that worked across devices; search engines; calendar alerts, etc.) which enabled them to focus their time and energy on their mathematics work. Participants described their method of learning about technology as being "trial-and-error," trying different strategies and keeping what worked. Their resulting "patchwork" practices suggested that the traditional boundaries between traditional assistive and mainstream technologies were becoming increasingly blurred for these participants. Grounded in a "learner's perspective," our work responds to a need for more research to determine how affective new assistive technologies are in supporting academic difficulties (Bryant, Bryant, & Ok, 2014). Evidence-based understandings of students' technological experiences and insights are essential to improve student access and equity, and potentially to inform the design of both assistive and mainstream technologies and their use in classroom and informal learning environments.

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UN AMBIENTE DE RESOLUCIÓN DE PROBLEMAS Y TECNOLOGÍAS DIGITALES EN UN CURSO MASIVO EN LÍNEA

William Enrique Poveda Fernández
Universidad de Costa Rica
william.poveda@ucr.ac.cr

Se describe la forma en que una comunidad en línea interactúa con actividades matemáticas y exhibe, proporciona retroalimentación y modifica sus ideas o razonamientos al utilizar diversas tecnologías digitales en un ambiente MOOC (Massive Online Open Course, por sus siglas en inglés). Los resultados muestran que los participantes compartieron y discutieron sus ideas matemáticas, como parte de una comunidad virtual, lo que les permitió formular conjeturas, explorar y buscar propiedades para sustentar relaciones y comunicar resultados. Además, consideraron un problema como punto de partida para involucrarse en la actividad de razonamiento matemático y desarrollar múltiples estrategias para resolver un problema.

Keywords: Resolución de Problemas, Tecnología, MOOC, Diseño de actividades.

Descripción de la investigación

El objetivo de la investigación fue diseñar e implementar un MOOC basado en resolución de problemas matemáticos y el uso coordinado de tecnologías digitales. El curso se estructuró en torno a tres actividades principales (Santos-Trigo 2014) que incluían brindar a todos los participantes la oportunidad de (i) mover objetos dentro de representaciones dinámicas de conceptos y problemas matemáticos; (ii) observar y analizar los atributos de los objetos matemáticos para formular conjeturas; y (iii) buscar argumentos empíricos y, luego, matemáticos para validar esas conjeturas y relaciones. El curso se estructuró de acuerdo con las ideas de Churchill (2016) y Robutti et al. (2016).

El objetivo fue el desarrollo y la práctica de un enfoque inquisitivo, donde los participantes tuvieran que examinar constantemente las propiedades y atributos de los objetos matemáticos formularan conjeturas, compartieran sus ideas, buscaran diferentes formas de justificar las relaciones y comunicaran sus resultados (Poveda, Aguilar-Magallón & Gómez-Arciga, 2018).

En el MOOC se inscribieron 2661 personas con diferentes niveles de estudios, edades, intereses profesionales o personales, ubicación geográfica, etc. Durante el desarrollo de las actividades y a través del foro de discusión, los participantes plantearon preguntas, comentarios y soluciones que fueron organizados por el equipo de diseño del MOOC de tal manera que cada integrante pudiera participar en la discusión de temas que eran importantes y consistentes con los objetivos del curso (Schoenfeld, 1992). La unidad de análisis fueron las conversaciones en los foros (Ernest, 2016).

Los resultados mostraron que los foros de discusión brindaron diversas oportunidades para que los participantes aclararan sus ideas, compartieran sus razonamientos matemáticos y participaran en un enfoque de resolución de problemas en colaboración con otros durante su trabajo en las actividades del curso. Los participantes reconocieron que el aprendizaje de las matemáticas implica enfrentarse a un dilema que necesita resolverse en términos de observar una situación, formular preguntas y buscar diferentes caminos para responderlas.

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A PROBLEM-SOLVING AND DIGITAL TECHNOLOGIES ENVIRONMENT IN A MASSIVE AND ONLINE COURSE

This research report analyzes the way in which an online community interacts with mathematical activities and exhibits, provides feedback and modifies their ideas or reasoning mathematical using various digital technologies in a MOOC (Massive Online Open Course). The results show that the participants shared and discussed their mathematical ideas, as part of a virtual learning community, which allowed them to formulate conjectures, explore and search properties to sustain relationships and communicate results. In addition, they considered a problem as a starting point to get involved in the mathematical reasoning activity and develop multiple strategies to solve a problem.

COMPARISON OF TEXTBOOKS REPRESENTATIONS OF COUNTING

Valerie N Long
Indiana University of PA
vlong@iup.edu

Stephanie Lauble
Indiana University of PA
sxmw@iup.edu

Shara Wohlgemuth
Indiana University of PA
xwjw@iup.edu

Keywords: Curriculum Analysis, Using Representations, Number Concepts and Operations

Research has shown that elementary school teachers use mathematics textbooks as their principle curriculum guide and source of lessons (Fulkerson, Campbell, & Hudson, 2013). Most mathematics textbooks were similar until the National Science Foundation (NSF) launched a major initiative to create new textbooks based on the vision of teaching espoused by the National Council of Teachers of Mathematics (NCTM) *Standards* documents (Reys, Reys, & Chavez-Lopez, 2004). Research has illustrated that "when students learn to represent, discuss, and make connections among mathematical ideas in multiple forms, they demonstrate deeper mathematical understanding and enhanced problem-solving abilities" (NCTM, 2014, p. 24). A key component of textbooks should be to include multiple representations for students to illustrate their thinking.

Due to the key role of textbooks (Fulkerson, Campbell, & Hudson, 2013) and the bedrock of conventional mathematics being learning how to count (Krasa & Shunkwiler, 2009), this research project compared the representations of counting in two kindergarten textbooks. The objectives of this study are to a) investigate which representations were used in kindergarten lessons for counting in a traditional and *Standards*-based textbook, b) which representations were more prevalent in each book, and c) on average which textbook encouraged more than one representation among its lessons on counting.

Data for this study comes from the textbooks *Everyday Mathematics* published in 2012 and a traditional textbook *Math Connects* published in 2007. They were selected because *Everyday Mathematics* is *Standards*-based and the other is traditional, not aligned with NCTM's vision for teaching. For each book, every lesson was analyzed to determine if counting was included. If counting was in the lesson, then each lesson was coded for the representations that were present: written symbols, oral language, concrete materials, pictures, or real-world context. Descriptive statistics were used to analyze which representations were being used in the texts.

Both books included all five representations. *Everyday Mathematics* had 89 lessons and all of them included counting, while *Math Connects* had 67% include counting. In *Everyday Mathematics*, 100% of all counting lessons included manipulatives, pictures, oral language, written symbols, and real-world connections. In *Math Connects*, 82.9% of all counting lessons contained manipulatives, 90% included pictures, 64.3% included oral language, 70% contained written symbols, and 30% contained real-world representations.

Teachers who use *Everyday Mathematics* as their primary source of lessons will engage their students in using all five representations for counting. When this happens kindergarten students are given the opportunity to gain a deeper understanding of counting, foundational content, through the use and connection of the different representations (NCTM, 2014). Among the representations in the textbook *Math Connects*, pictures and manipulatives were the most prevalent. Additionally, less than one-third of counting lessons in *Math Connects* contained a real-world representation. Hence, the kindergarten textbook *Everyday Mathematics* significantly encouraged more representations for counting than *Math Connects*.

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

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REVIEWING MATHEMATICS CURRICULUM EVALUATION TOOLS THROUGH AN IMPLEMENTATION LENS

Kristen R. Rolf, M. Ed.
Utah State University
kristen.rolf@usu.edu

Kaitlin Bundock, Ph.D.
Utah State University
kaitlin.bundock@usu.edu

Sarah Pinkelman, Ph.D.
Utah State University
sarah.pinkelman@usu.edu

Keywords: Curriculum, Curriculum Analysis, Policy Matters

In this session, we address the theme "Against a New Horizon…" by exploring a topic of importance to state departments of education and school districts that represents a new horizon in education research – the successful and sustainable implementation of high-quality mathematics curricula. Metz, Halle, Bartley, and Blasberg (2013) describe four stages of implementation: exploration, installation, initial, and full. The curriculum-adoption process undertaken by state departments of education and/or school districts corresponds to the exploration stage, during which decision-makers must consider issues such as specific needs, the fit of a curriculum and the feasibility of its implementation, and perspectives of stakeholders (Metz et al., 2013). The National Implementation Research Network created "The Hexagon Tool" (2018) to help individuals and teams address implementation issues related to the adoption of an innovation during the exploration stage. "The Hexagon Tool" includes six implementation-related indicators: Evidence, Usability, Supports, Need, Fit, and Capacity. Despite an overall increase in attention paid to implementation issues across the human services in recent years, relatively little attention has been paid to implementation in the field of education (National Implementation Research Network, 2018). In order to learn more about the support for implementation that states provide to their school districts during the curriculum-adoption process, we asked the following research question: How do state-created curriculum evaluation tools for mathematics intersect with "The Hexagon Tool" created by the National Implementation Research Network?

We conducted a systematic review of every state's department of education website to answer the research question. In phase one, we used Google to search for the following terms: [insert state's name] curriculum evaluation tool. Next, the first author visited each state's department of education website and searched for any curriculum evaluation tools that may have been missed during the first phase. In the third phase, we repeated the Google search using a new set of terms: [insert state's name] instructional materials evaluation tool. State-created tools with an explicitly stated purpose of evaluating curricula and instructional materials related to K-12 mathematics were included for review. The authors independently coded the intersection of the obtained curriculum evaluation tools and "The Hexagon Tool" before comparing results and coming to consensus on any discrepancies.

We compared nine state-created mathematics curriculum evaluation tools produced by seven states with "The Hexagon Tool." All of the mathematics curriculum evaluation tools addressed the Fit indicator from "The Hexagon Tool." Other indicators from "The Hexagon Tool" that were partially addressed by the curriculum evaluation tools include Support, Usability, and Capacity. Evidence and Need were not addressed by any of the reviewed evaluation tools. These findings indicate that few states attend to issues of implementation when supporting school districts to adopt high-quality mathematics curricula. Our hope is that this poster will draw attention to the new horizon of implementation research so that departments of education and school districts are better equipped to implement high-quality mathematics curricula that benefit all students.

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

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REVISITING VISUAL MODELS FOR WHOLE NUMBER DIVISION

Oh Hoon Kwon
University of Wisconsin-Madison
kwon@math.wisc.edu

Ji-Won Son
University at Buffalo
jiwonson@buffalo.edu

Ji Yeong I
Iowa State University
jiyeongi@iastate.edu

Keywords: Number Concepts and Operations, Using representations, Curriculum Analysis

Common Core State Standards (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) call attention to two interpretations of division (partitive and measurement) to help students explore the crucial ideas of whole number and fraction division. However, emphasizing the interpretations of division based on a missing factor in semantic structures sometimes hinder students from reflecting other aspects of division that are consistent regardless of number systems, especially when they are asked to explore visual models (Carpenter et al., 1999). These cognitive difficulties require a careful investigation of presentation of models to help developing students' understanding in a systematic way.

Extending previous studies (Kwon et al., 2017, 2019) on multiplication models, this study intended to explore how various models support or hinder students' understanding of whole number division. We particularly looked at whether and how two types of models (action/change-focused and comparison/coordination-focused), were presented in curricula. Two U.S. textbooks, *Engage NY* and *Math in Focus*, and one Korean textbook in Grade 2 and 3 were examined as part of a larger study for how they offer opportunities with arithmetic operation models to foster understanding of arithmetic operation. Tasks and visual models were coded and categorized according to problem situations and model focuses (see Figure 1).

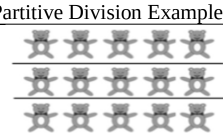
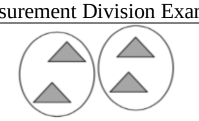
Model Focuses	Problem Structures	Descriptions	Problem Situation - Partitive Division Example	Problem Situation - Measurement Division Example
Action on /Change of Initial Quantity	Operand ÷ Operator	Divide dividend by divisor, Divide/Partition dividend into parts or portions.	 (Engage NY g3m1)	 (Engage NY g3m1)
Comparison /Coordination of Two Quantities	Operand / Operand	Distribute Quantity 1 into Quantity 2, Assign Quantity 1_1 to Quantity 2, Find the ratio/rate between Quantity 1 and Quantity 2.	 (Korean Textbook grade 3)	 (Math in Focus 2A)

Figure 1: Two Types of Division Models

We report that there were two types of division models: the action/change-focused and the comparison/coordination-focused models similar to multiplication models. Each model reflects different aspects of whole number division and different thinking processes involved with the division operation. We found that Korean and *Math in Focus* textbooks used both models frequently, whereas *Engage NY* textbook infrequently introduced the comparison/coordination-focused model for division. This suggests that without a careful conceptualization of the two models in connection with two interpretations of division (partitive and measurement), teachers and students may fail to notice the conceptual ideas of whole number division and connection to the models for whole number division. In this poster we thus will explain the two types of models representing the whole number division processes in detail and discuss how to carefully introduce those models to help students attain a better understanding of division operation.

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RELATING DIGITAL RESOURCES TO TEACHER PRACTICE

Zenon Borys
University of Rochester
zenon.borys@warner.rochester.edu

Keywords: Curriculum, Curriculum Enactment, Technology

Teachers increasingly use digital resources. Movements such as curriculum by Pinterest or the open education resources (OER) promoted by the US Department of Education (U.S. Department of Education, 2016) support the trend of teachers sharing instructional resources. While there are benefits to using digital resources in schools, there is also potential to reinforce teaching practices that provide little opportunity for students to engage in meaningful mathematical activity. This study explores teaching practices using digital resources in ways that promote meaningful mathematical activity, informing which practices are worth building on and which are worth standing against.

This study uses a *documentation genesis* framework (Gueudet & Trouche, 2009) to highlight the relationship between teachers' design of lessons and the resources they use, with an explicit focus on digital curricular resources. The documentation genesis framework posits that resources need to be examined in use. By focusing on how features of resources influence use, a process of instrumentation, and how the user influences use, a process of instrumentalization, the framework emphasizes how teachers transform curricular resources for instruction.

This study follows four teachers through the planning and implementation of a curricular unit using the *reflective investigation* methodology (Gueudet & Trouche, 2012). Using interviews and classroom observations, data captures the multiple resources teachers use in designing their unit, the reasoning surrounding didactic design decisions, and how classroom activity is organized around the resources. The focus of analysis is on how teacher uses of curricular resources align to their teaching goals. More specifically, the documentation genesis framework relates the resources, mathematics, and didactics (Trigueros and Lozano, 2012) in teachers' lesson design processes.

Results show that digital curriculum resources did not disrupt teachers' modus operandi. Rather, they supported or amplified teachers' current practices. For example, the use of digital resources did not change the emphasis on modeling and repeated practice as the mechanism of student learning. The use of digital curriculum resources changed when teachers modeled problems in the classroom. Rather than starting instruction with examples of varying difficulty, teachers waited until students were stuck to model specific problem types. The packaging of teaching changed, but the underlying process for student learning was left unchanged.

Additionally, data show teachers privileged digital resources with a capability of exporting and importing data (e.g. problems or results) that directly connected to state exams and standards, making the translation process from one program to another automatic. For instance, websites that post previous state exams in multiple file formats (e.g. .doc, .pdf, .tst) were used because test creation software could directly import .tst files, allowing teachers to randomize questions and create multiple versions of assessments. This focus on ease of translation acted as a filter to limit the resources teachers used.

This poster will present updated results from the study and discuss the merits and limitations of its analytic process.

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PROSPECTIVE SECONDARY TEACHERS' WAYS OF ENRICHING CONVENTIONAL WORD PROBLEMS

Wenmin Zhao
University of Missouri
wenminzhao@mail.missouri.edu

Samuel Otten
University of Missouri
ottensa@missouri.edu

Keywords: Word problems, Curriculum, Instructional activities and practices

Word problems are a major component of mathematics curriculum and can serve multiple purposes, such as engaging students in mathematics learning and providing opportunities to develop skills to solve real-world problems (e.g., Masingila, 2002). However, research shows that conventional word problems often fail to meet these goals (e.g., Verschaffel, Greer, & DeCorte, 2000) and are criticized for their inauthentic and oversimplified features (e.g., Niss, Blum, & Galbraith, 2007). As the implementers of curriculum materials, teachers occasionally modify textbook word problems to support students' mathematics learning and develop their problem-solving strategies. To understand the ways in which prospective secondary teachers (PTs) would enrich word problems, this study investigated 20 PTs' ways of modifying conventional word problems to become more authentic and mathematically rich.

The participants in this study were 20 PTs in their third year of a secondary mathematics teacher preparation program at a public university in the Midwestern United States. The PTs had not learned about modifying word problems or other curriculum materials prior to the study. During the study, each PT was asked to modify three word problems selected from secondary mathematics textbooks. One of the problems was: "A population in a small town is 500, and it grows by 2% each year. Determine the population size after 3 years." This problem was chosen for its unrealistic assumption (i.e., constant growth rate) and its non-mathematically-rich nature (i.e., targeting a specific answer and a particular solution strategy). After the PTs revised the three word problems, we open-coded their written responses and categorized the problem characteristics by employing constant comparisons (Glaser & Strauss, 1967).

Our findings reveal four ways in which the PTs enriched word problems: *motivation*, *personalization*, *removal of pre-determined information*, and *removal of mathematical components* (e.g., mathematical operations, representations). For example, one PT in the study modified the word problem to: "You are the mayor of a town. Your town's population is growing exponentially. What will the population be after 3 years?" This revised problem is written with a particular person identified (*personalization*) and does not present students with the initial population size of the town or the growth rate of the population (*removal of pre-determined information*). Another PT adapted the problem to: "The town is worried about when they will run out of space for the amount [*sic*] of people living there. When the town has tripled, there will not be space for any more people. Determine how many years will pass before it is at its capacity." This problem now involves a potentially relatable issue of overcrowding and allows students to consider the town capacity (*motivation*) without providing the embedded mathematical operations (*removal of mathematical components*). More categories will be detailed and exemplified on the poster.

Our study illustrates the ways in which the PTs converted conventional word problems to become more authentic as well as mathematically-richer. Since curriculum materials (or word problems) play a key role in classroom instruction, it is important for mathematics teacher educators to consider how we could support teachers and PTs to adapt word problems in a way that provides more support to students' mathematics learning.

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FEATURES OF VIDEO HOMEWORK IN FLIPPED ALGEBRA INSTRUCTION

Jaepil Han
University of Missouri
jhph6@mail.missouri.edu

Stacy Hirt
University of Missouri
smhckd@mail.missouri.edu

Jessica Kamuru
University of Missouri
jgc7vc@mail.missouri.edu

Zandra de Araujo
University of Missouri
dearaujoz@missouri.edu

Samuel Otten
University of Missouri
ottensa@missouri.edu

Wenmin Zhao
University of Missouri
wz2mb@mail.missouri.edu

Keywords: Instructional Activities and Practices, Technology, Curriculum Enactment

Flipped instruction is an instructional model in which a teacher assigns videos or other multimedia to be viewed outside of class. Many teachers report adopting this model because they want to have more collaborative time during class (de Araujo, Otten, & Birisci, 2017) and, with content delivery still an important part of their instructional vision, they move the content delivery to the video homework. Because mathematical content is typically delivered via these videos in flipped instruction, it is important that we examine them more carefully to capture how content delivery may differ (or not) between flipped and non-flipped instructional models. Thus, our study examined the following question: what are the features of lecture videos selected/created by Algebra 1 teachers utilizing flipped instruction?

We observed 13 Algebra 1 lessons. Each lesson was taught by a teacher who had adopted a flipped instructional model, however, only 11 of the teachers had assigned videos for the observed lesson. Of the 11 lessons with videos, nine had one assigned video, one had two assigned videos, and one had three assigned videos. We collected digital versions of each of these videos and analyzed them using our Flipped Mathematics Instruction Observation Protocol (Zhao, Han, Kamuru, de Araujo, & Otten, 2018).

In terms of duration, the videos ranged from six minutes and four seconds to 13 minutes and 45 seconds with an average of nine minutes and 44 seconds. All the videos were lecture videos featuring worked examples. All of the videos were created by the teachers, and all but two were created utilizing voice-over recordings of their written work. The other two videos utilized picture-in-picture formats. None involved “lecture capture” (e.g., a person recorded at a whiteboard) which has been found to be most effective in content delivery (Chen & Wu, 2015).

We also examined the instructional quality of the videos (e.g., mathematical development, visual representations, mathematics errors) and the extent to which the teachers adhered to multimedia design principles (e.g., personalization, redundancy between narration and text) (Clark & Mayer, 2008). The instructional quality of the videos varied more than the quality of multimedia design. Our analysis of these lecture videos sheds light on how content delivery may differ between flipped and non-flipped instruction and even among the teachers who have adopted flipped instruction. In addition, it provides a better understanding of current implementation of flipped instruction in Algebra 1, potentially informing supports that we may offer teachers.

Acknowledgments

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SEEING VERSUS DOING: TYPE AND PRESENCE OF PRACTICE IN DIGITAL ABACUS TRAINING

Taylyn Hulse
Worcester Polytechnic Institute

Erin Ottmar
Worcester Polytechnic Institute
erottmar@wpi.edu

Sophia Jagannathan
Worcester Polytechnic Institute
sburke@wpi.edu

Keywords: Cognition, Instructional Activities and Practices, Technology

In countries throughout Asia the Soroban abacus is used in supplemental math education to increase calculation speed and accuracy (Barner et al., 2016). Students begin abacus training with a physical abacus and progress to an internalization of the technique called "mental abacus" (Cho & So, 2018). This study explores the feasibility of abacus training using a virtual abacus and the efficacy of various types of instructional delivery including action-based practice, observational practice or the absence of practice. Results show learning gains for all students in just 20 minutes of engagement, however, there were no significant differences between instructional delivery type. This work is a first step towards designing effective instructional materials for online abacus learning.

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EDUCATIVE FEATURES IN UPPER ELEMENTARY EUREKA MATH CURRICULUM

Amy Dwiggins
University of Missouri
add39c@mail.missouri.edu

Keywords: Curriculum Analysis, Elementary School Education, Teacher Knowledge

Since Ball and Cohen's (1996) call to design curriculum materials that support both teacher and student learning and Davis and Krajcik's (2005) creation of design heuristics for the development of educative curriculum materials nine years later, mathematics curriculum designers have worked to include educative features in written curriculum that support both teacher and student learning. Opportunities for teacher learning may be further enhanced in emerging online curricula such as Eureka Math, which has quickly become the most commonly used published instructional resource for elementary mathematics teachers (Opfer, Kaufman, & Thompson, 2016). The nature of use varied considerably from teacher to teacher with the majority of teachers adapting lessons and tasks due to timing and pacing (Kaufman et al., 2017).

Online curricula have the potential for new educative features relative to traditionally published texts or the same educative features but embedded in the written curriculum in new or different ways. Drawing on Shulman's (1986) categories of content knowledge and two frameworks for analyzing educative features (Males, 2011; Quebec Fuentes & Ma, 2018), two modules of the Eureka Math curriculum were coded. Educative features in the modules, Grade 3 Module 7 *Geometry and Measurement Word Problems* (6 topics, 34 lessons) and Grade 4 Module 6 *Decimal Fractions* (5 topics, 16 lessons), were coded across categories of Subject Matter Content Knowledge, Pedagogical Content Knowledge for Mathematics Topics (Experiences, Differentiation, Activity Sequences, Representations, & Tools), Pedagogical Content Knowledge for Mathematics Practices (Discourse, Proof, Terminology, & Participation Structures), and Curricular Content Knowledge (Curricular Overview, Features, Storyline, & Goals). Each feature was also coded as providing teachers a process to implement (Enactment Guidance) or providing teachers the mathematical or pedagogical reasons for such implementation (Rationale Guidance) (Remillard, 2000).

Initial findings indicate that educative features focused on Pedagogical Content Knowledge both for Mathematics Topics and Mathematics Practices were the most prevalent across the two modules with a strong emphasis on teacher support for enacting the written curriculum at the lesson level. Notably, educative features focusing on Curricular Content Knowledge were found throughout the module in the module overview, in the introduction to each topic, and within most of the lessons rather than concentrated at the beginning of the unit. As a whole, rationale guidance across categories of content knowledge seems less prevalent than enactment guidance, a finding shared with Males (2011). Initial findings point to missed opportunities for teacher learning to be promoted, even in this online curricula where publishing restraints can be minimized. Further research on teacher use of the Eureka Math curriculum as well as comparison with other online curricula is needed.

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Chapter 3:
Early Algebra, Algebra, and Number Concepts

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

SIXTH-GRADE STUDENTS' RETENTION OF EARLY ALGEBRA UNDERSTANDINGS AFTER AN ELEMENTARY GRADES INTERVENTION

Ana Stephens
University of Wisconsin-Madison
acstephens@wisc.edu

Rena Stroud
Merrimack College
stroudr@merrimack.edu

Susanne Strachota
University of Wisconsin-Madison
sstrachota@wisc.edu

Maria Blanton
TERC
maria_blanton@terc.edu

Despina Stylianou
City College of New York
dstylianou@ccny.cuny.edu

Eric Knuth
University of Texas-Austin
eric.knuth@austin.utexas.edu

Ranza Veltri Torres
University of Wisconsin-Madison
rveltri@wisc.edu

Angela Murphy Gardiner
TERC
angela_gardiner@terc.edu

Yewon Sung
University of Wisconsin-Madison
ysung27@wisc.edu

This research focuses on the retention of students' algebraic understandings one year following a Grades 3–5 early algebra intervention. Participants included 1455 Grade 6 students who had participated in a cluster randomized trial. Approximately half of these students received an early algebra intervention as part of their regular instruction in Grades 3–5, while the other half received only their regular mathematics instruction. Results show that, as was the case at the end of Grades 3, 4 and 5, treatment students outperformed control students at the end of Grade 6 on a measure of algebraic understanding. This was despite the fact that treatment students experienced a significant decline in performance and control students a significant increase in performance after the intervention. An item-by-item analysis performed within condition revealed the areas in which students in the two groups experienced a change in performance.

Keywords: Algebra and Algebraic Thinking

Mathematics education scholars have advocated for some time that students be provided long-term experiences, beginning in the elementary grades, that can support the development of their algebraic thinking. While this view is widely accepted, research is needed to document how the algebraic understandings of students who are provided such long-term, sustained algebra experiences in the elementary grades compare to those of students who have not had these experiences once students are in middle grades. This paper focuses on a retention study in which the algebraic understandings of students who participated in a Grades 3–5 early algebra intervention were assessed at the end of Grade 6, a year after the completion of the intervention.

Theoretical Perspectives

Our work aims to develop and test a model of early algebra instruction in order to produce research-vetted curricular frameworks and instructional materials from which we can better understand early algebra's impact on students' algebraic thinking. This work has involved small scale cross-sectional (Blanton et al., 2015) and longitudinal (Blanton et al., 2019) studies and more recently a large-scale, longitudinal, cluster randomized trial (Blanton et al., in press).

In a process fully described in Blanton et al. (2018), we built a curricular framework, instructional intervention, and associated assessments around the algebraic thinking practices of generalizing, representing generalizations, justifying generalizations, and reasoning with generalizations (see also Blanton, Levi, Crites, & Dougherty, 2011) and the big algebraic ideas in which these practices can occur, namely, generalized arithmetic; equivalence, expressions, equations, and inequalities; and functional thinking.

Previous Research: The Longitudinal Study

In order to provide context, we briefly describe the longitudinal study of the Grades 3–5 intervention that preceded the retention study that is the focus of this paper. The study used a cluster randomized trial design and is fully described in Blanton et al. (in press). Participants included approximately 3000 students and took place in 46 elementary schools across three school districts, with half of the schools randomly assigned to a either a treatment or control condition. Students in treatment schools were taught our early algebra intervention in Grades 3–5, which consisted of 18 lessons per year, as part of their regular mathematics instruction by their classroom teachers. Teachers took part in professional development that addressed the big algebraic ideas and algebraic thinking practices that were the focus of the intervention.

Students completed written assessments prior to the start of the intervention in Grade 3 and at the end of Grades 3, 4 and 5. Assessments consisted of 12-14 items designed to measure understanding of the big algebraic ideas and algebraic thinking practices that formed the basis of the intervention. We found that while there were no differences between groups in performance or structural strategy use (Kieran, 2007, 2018) prior to the intervention, treatment students improved at a significantly faster rate than control students in Grade 3. While students in the two conditions made parallel gains across Grades 4 and 5, the treatment students' initial gains were maintained so that they remained significantly ahead of control students in their understanding of big algebraic ideas and thinking practices as they entered middle school (Blanton et al., in press).

Given the results of our Grades 3–5 study, we were interested in revisiting these same students one year later to assess the intervention's longer-term impact. Our research question was thus the following: *How does the performance of students who took part in a Grades 3–5 early algebra intervention compare to that of students who experienced only their regular Grades 3–5 mathematics curriculum one year after the conclusion of the intervention?*

Method

Participants

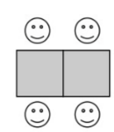
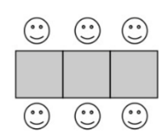
Sixth grade retention data was collected from 1455 students across 23 middle schools in the three school districts that took part in the Grades 3–5 longitudinal study. Of these students, 716 were in control schools and received only their regular instruction in Grades 3–5, while 739 were in treatment schools and were taught the intervention as part of their regular instruction.

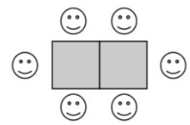
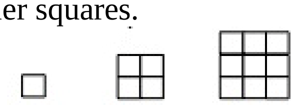
Context

While participating students were consistently in a control or treatment elementary school throughout Grades 3–5, all students moved to new (middle) schools in Grade 6 and were intermixed. The students who participated in the retention study experienced their schools' regular mathematics curriculum during Grade 6 and no instructional intervention from our project. Grade 6 teachers cited use of a variety of mathematics curricula but in all cases were expected to align their instruction with the *Common Core Standards for School Mathematics* (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center & CCSSO], 2010).

Data Collection

Students completed a written assessment at the end of Grade 6 consisting of 11 open-ended items, most of which contained multiple parts. Nine of these 11 items also appeared on the Grade 5 assessment and will be the focus of the results we share in this paper (see Figure 1).

<table>
<tr><th></th><th>Big Idea(s)</th><th></th></tr>
<tr><td>1</td><td>Equivalence, expressions, equations</td><td>Fill in the blank with the value that makes the number sentence true.
7 + 3 = ____ + 4
Explain how you got your answer.</td></tr>
<tr><td>3</td><td>Generalized arithmetic</td><td>Marcy’s teacher asks her to solve “23 + 15.” She adds the two numbers and gets 38. The teacher then asks her to solve “15 + 23.” Marcy already knows the answer is 38 because the numbers are just “turned around.”
a) Do you think Marcy’s idea will work for any two numbers? Why or why not?
b) Write an equation using variables (letters) to represent the idea that you can add two numbers in any order and get the same result.</td></tr>
<tr><td>4</td><td>Generalized Arithmetic</td><td>Brian knows that if you add any three odd numbers, you will get an odd number. Explain why this is true.</td></tr>
<tr><td>5</td><td>Equivalence, expressions, equations</td><td>Tim and Angela each have a piggy bank. They know that their piggy banks each contain the same number of pennies, but they don’t know how many. Angela also has 8 pennies in her hand.
a) How would you represent the number of pennies Tim has?
b) How would you represent the total number of pennies Angela has?
c) Angela and Tim combine all of their pennies. How would you represent the number of pennies they have all together?
Suppose Angela and Tim now count their pennies and find they have 16 all together. Write an equation with a variable (letter) that represents the relationship between this total and the expression you wrote above.</td></tr>
<tr><td>9</td><td>Functional thinking</td><td>Brady is celebrating his birthday at school. He wants to make sure he has a seat for everyone. He has square desks.
He can seat 2 people at one desk in the following way: 
If he joins another desk to the first one, he can seat 4 people: 
If he joins another desk to the second one, he can seat 6 people: 
a) Fill in the table below to show how many people Brady can seat at different numbers of desks.
<table><tr><th>Number of desks</th><th>Number of people</th></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>4</td></tr><tr><td>3</td><td></td></tr><tr><td>4</td><td></td></tr><tr><td>5</td><td></td></tr><tr><td>6</td><td></td></tr><tr><td>7</td><td></td></tr></table></td></tr>
</table>

<table>
<tr><td></td><td></td><td>b) Do you see any patterns in the table from part a? If so, describe them.
c) Think about the relationship between the number of desks and the number of people.
Use words to write the rule that describes this relationship.
Use variables (letters) to write the rule that describes this relationship.
d) If Brady has 100 desks, how many people can he seat? Show how you got your answer.
e) Brady figured out he could seat more people if two people sat on the ends of the row of desks. For example, if Brady had 2 desks, he could seat 6 people.

How does this new information affect the rule you wrote in part c?
Use words to write your new rule.
Use variables (letters) to write your new rule.</td></tr>
<tr><td>10</td><td>Functional thinking; Equivalence, expressions, equations</td><td>The table below shows the relationship between two variables, k and p. The rule $p = 2 \times k + 1$ describes their relationship.
a) Some numbers in the table are missing. Use this rule to fill in the missing numbers.
<table><tr><th>k</th><th>p</th></tr><tr><td>1</td><td>3</td></tr><tr><td>2</td><td></td></tr><tr><td></td><td></td></tr><tr><td></td><td>9</td></tr></table>b) What is the value of p when $k = 21$? Show how you got your answer.
c) What is the value of k when $p = 61$? Show how you got your answer.</td></tr>
<tr><td>14</td><td>Functional thinking</td><td>The following magic square is growing so that each day it is made up of more and more smaller squares.

Day 1 Day 2 Day 3
The following table shows a given day and the number of small squares on that day:
<table><tr><th>Day</th><th>Number of small squares</th></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr><tr><td>3</td><td>9</td></tr><tr><td>4</td><td>16</td></tr><tr><td>5</td><td>25</td></tr><tr><td>6</td><td>36</td></tr></table>a) Think about the relationship between the number of days and the number of small squares.</td></tr>
</table>

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

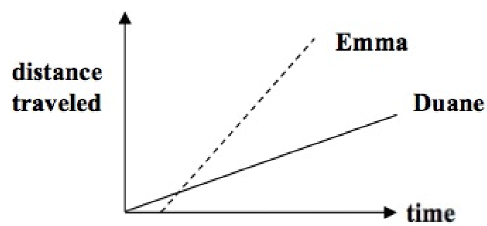
		Use words to write the rule that describes this relationship. Use variables (letters) to write the rule that describes this relationship. b) Use your rule to predict how many small squares will be inside the big square on day 100. Show how you got your answer.
21	Functional thinking	Duane and Emma each went for a bike ride. The graphs below represent the relationship between time spent riding and distance traveled for each rider. a) Who started riding first? How can you tell? b) Who rode faster? How can you tell? c) Is Emma riding the same speed on her whole trip, or is she speeding up or slowing down? How can you tell? 
23	Equivalence, expressions, equations	Do the following two equations have the same solution? Explain. $2 \times n + 15 = 31$ $2 \times n + 15 - 9 = 31 - 9$

Figure 1: Items Common to the Grade 5 and Grade 6 Assessments

Data Analysis

Coding schemes developed in previous work (see Blanton et al. [2015] for a full description of this process) were used to categorize student responses to assessment items both in terms of performance and in terms of strategy use. (In the results shared here, we focus only on performance.) Items were coded by trained coders unaware of students' treatment condition.

Changes in overall correctness over time were assessed using a repeated measures ANOVA, with treatment condition as a between-subjects factor. McNemar's test was then used to examine item-by-item changes in performance from Grade 5 to Grade 6 within each condition.

Results

The repeated measures ANOVA showed no significant impact of testing time across all students, but did reveal a significant interaction between testing time and treatment condition, $F(1, 1453) = 160.73$, $p < .001$. Overall, treatment students (M = 47.51% correct, SD = 21.54%) maintained their advantage over control students (M = 37.93% correct, SD = 19.74%) at the end of Grade 6, $F(1,1453) = 78.13$, $p < .001$. However, the gap between the two groups narrowed from Grade 5 to Grade 6, with control students experiencing an overall increase in performance from Grade 5 (M = 33.38%, SD = 19.43%) to Grade 6 (M = 37.93%, SD = 19.74%) and treatment students experiencing an overall decrease in performance from Grade 5 (M = 53.40%, SD = 24.17%) to Grade 6 (M = 47.51%, SD = 21.54%).

McNemar's test was used to compare performance (percent correct) across the two conditions on the nine individual assessment items (with a total of 24 individual item parts) common to Grades 5 and 6. The items are given in Table 1.

Table 1: Percent Correct on Assessment Items Across Conditions and Testing Times

Item	Control	Treatment

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

	Grade 5	Grade 6	Grade 5	Grade 6
1	66.76%	71.23%*	84.84%	82.68%
3a	54.05%	62.99%*	67.66%	70.09%
3b	21.65%	35.20%*	67.25%	59.40%**
4	10.34%	7.54%**	23.41%	11.91%**
5a	19.13%	36.73%*	59.95%	55.89%
5b	7.68%	10.89%*	29.50%	19.62%**
5c1	11.17%	9.36%	51.42%	16.51%**
5c2	8.52%	4.89%**	38.16%	8.93%**
9a	94.27%	96.79%*	94.86%	95.81%
9c1	21.79%	16.48%**	35.32%	23.55%**
9c2	27.23%	31.70%*	60.76%	48.17%**
9d	78.91%	79.47%	84.17%	82.68%
9e1	9.22%	6.98%	22.73%	13.80%**
9e2	16.48%	16.62%	42.49%	31.80%**
10a	56.01%	59.22%	73.61%	71.72%
10b	35.47%	48.32%*	58.46%	60.62%
10c	31.28%	38.55%*	45.33%	47.90%
14a1	8.38%	15.64%*	26.39%	24.90%
14a2	13.27%	20.25%*	49.53%	45.33%**
14b	35.89%	42.60%*	46.96%	50.34%
21a	61.59%	74.86%*	72.80%	79.70%*
21b	60.89%	70.11%*	72.67%	74.42%
21c	27.65%	34.22%*	45.60%	38.97%**
23	23.46%	19.69%	27.74%	25.58%
TOTAL	**33.38%**	**37.93%**	**53.40%**	**47.51%**

* Items on which students showed significant gain from Grade 5 to Grade 6 ($p < .05$)
** Items on which students showed significant decline from Grade 5 to Grade 6 ($p < .05$)

As shown in Table 1, control students made gains in performance on individual assessment items much more so than did experimental students. Specifically, of the 24 individual item parts, control students improved on 15 of those parts and declined on three of those parts from Grade 5 to Grade 6. Treatment students, on the other hand, improved on just one of those parts and declined on 11 of those parts from Grade 5 to Grade 6.

Of particular interest in the control condition is the identification of the areas of algebra in which students made their gains. That is, in which areas did their regular middle school instruction help to close the gap between their performance and that of their treatment counterparts? Of particular interest in the treatment condition is the identification of the areas of algebra in which students a) maintained or gained in their Grade 5 performance levels and b) declined in performance after a year without a focused early algebra education. Answers to these questions will shed light on the algebraic content and thinking practices that “stuck” with students one year after our three-year early algebra intervention—whether because the learning was supported by their “business as usual” middle school instruction or was maintained over the course of the year despite this instruction—and the areas in which this was not the case.

For control students, significant gains occurred across a range of big algebraic ideas and thinking practices. Students showed an increased understanding of equality and equations on some items (Items 1, 10b, 10c). They were increasingly able to identify one-step function rules in words (Items 9c1, 14a1) and in one of these cases make a far prediction (Item 14b). They also made gains identifying an arithmetic property (Items 3a) and representing unknowns along with arithmetic and one-step functional relationships with variables (Items 3b, 5a, 5b, 9c2, 14a2). They did not experience gains on items involving advanced understanding of equality (Item 23), two-part function rules (Items 9e1 and 9e2) or using algebraic expressions of more than one term to represent problem situations (Items 5c1, 5c2).

Treatment students maintained their Grade 5 performance levels on items addressing their understanding of the equal sign (Items 1 and 23) and equations (Items 10a, 10b, 10c). They also maintained their ability to identify an arithmetic property (Item 3a), make far predictions when working with functional relationships (Items 9d, 14b) and represent an unknown with a variable (Item 5a). Treatment students showed a decline in performance from Grade 5 to Grade 6 where they needed to identify one- or two-step functional relationships in words (Items 9c1, 9e1), provide a general argument to justify a statement about the sum of odd numbers (Item 4), and provide a variable representation of more than one term to represent unknown quantities (Items 5b, 5c1, 5c2), an arithmetic property (Item 3b), and functional relationships (Items 9c2, 9e2).

Discussion

Results from our longitudinal Grades 3–5 study (Blanton et al. [in press]) offered evidence that providing students with sustained early algebra experiences across a range of big algebraic ideas and algebraic thinking practices can in fact place them at an advantage in terms of algebraic understanding relative to students who experience a more arithmetic-focused approach to elementary mathematics as they enter the middle grades. The study reported in this paper assessed the algebraic understanding of these students one year after the conclusion of the Grades 3–5 intervention, at the end of Grade 6. The most important result we have to report is that one year post-intervention, treatment students still significantly outperformed control students on an assessment measuring their understanding of big algebraic ideas and thinking practices. That is, early algebra (i.e., the Grades 3–5 intervention) *did* make a difference in terms of helping students be better prepared for algebra in the middle grades.

Digging deeper into the item-by-item results revealed the areas in which treatment students were able to maintain their Grade 5 performance levels and the areas in which their performance dropped. Likewise, we were able to identify areas in which control students made significant gains, likely signaling areas of relative strength in their middle school mathematics instruction.

One important area in which treatment students maintained their performance was the concept of equality and the meaning of the equal sign. This was the core concept with which we started our Grade 3 lessons, and it was consistently revisited throughout the three years of the early algebra intervention. While control students made some gains in this area as well, their gains were not consistent across items and did not include growth on our most advanced item involving the recognition of equivalent equations. While the *Common Core* (NGA Center & CCSSO, 2010) does address students' understandings of the equal sign to some extent, our findings suggest a need to strengthen and maintain this focus over multiple years.

An important Big Idea in our Grades 3–5 intervention in which treatment students' learning was not robustly maintained from Grade 5 to 6 is functional thinking. While treatment students maintained their performance identifying a simple exponential function in words, they experienced a decline on all other items requesting the identification of a functional relationship, whether in words or variable notation. Control students showed some gains on these items, yet they still fell short of the performance of their treatment counterparts. This is perhaps not surprising giving the lack of a deep treatment of functional thinking in the *Common Core* (NGA Center & CCSSO, 2010) prior to Grade 8, calling into question the choice to de-emphasize the role of functional thinking in elementary grades in these standards. Our work with students across Grades 3–5 as well as the work of Blanton and colleagues (Blanton, Brizuela, Gardiner, Sawrey, Newman-Owens, 2015) with much younger students has illustrated that elementary students are capable of engaging in such thinking, beginning with very simple relationships that increase in complexity over time. The decline treatment students demonstrated in performance from Grade 5 to Grade 6 suggests not that they are unable to engage in such thinking, but that opportunities to engage in such thinking need to be sustained over time.

Only one item (Item 4) on our assessment explicitly asked students to build an argument to justify an arithmetic generalization. This was among the most difficult of tasks for Grades 3–5 students in both conditions. By the end of Grade 5, 23.41% of treatment students in the Grades 5-6 retention study were able to produce a general argument to justify that the sum of three odd numbers is an odd number. By the end of Grade 6, only half of these students were able to do so. Control students likewise showed no improvement in their ability to produce such an argument. Like functional thinking, this is an area that does not receive a great deal of attention in elementary curricula. Also, like functional thinking, argumentation is an area in which we know elementary students are capable of engaging. Our previous work (Blanton et al., 2015; Blanton et al., in press), along with the work of others (Carpenter, Franke, & Levi, 2003; Carpenter & Levi, 2000; Russell, Schifter, & Bastable, 2011a, 2011b) illustrates that while building more general arguments is challenging, students are capable of using the context of arithmetic to engage in justifying mathematical claims. In our current work, we are successfully engaging K– Grade 2 students in generalizing and justifying generalizations about even and odd numbers. We thus argue that our findings in this area suggest not a lack of ability on the part of students, but rather a lack of access to important algebraic practices.

Finally, we point to students' performance producing variable representations across a variety of big algebraic ideas to argue that this, too, is an area in need of sustained focus. Treatment students showed a decline from Grade 5 to Grade 6 in their abilities to represent an arithmetic property, functional relationships, and related unknown quantities using variable notation. While control students showed some gains in this area, the performance levels they reached were nowhere near that of their treatment counterparts. We have evidence from our Grades 3–5 work as well as the work of others with much younger students (Brizuela, Blanton, Sawrey, Newman-Owens, & Gardiner, 2015) that students are capable of representing varying quantities with algebraic notation and in fact often prefer such representations over verbal ones. We thus argue again that what is needed is a sustained focus on such experiences over time.

Conclusion

Over twenty years ago, Kaput (1998) called for an end to “the most pernicious curricular element of today's school mathematics—late, abrupt, isolated, and superficial high school algebra courses” (p. 25). He argued that we could do so by viewing algebra as a K–12 experience, integrating algebraic thinking and reasoning throughout the mathematics curriculum. Our work developing and testing a Grades 3-5 early algebra intervention coupled with our Grades 5–6 retention study lends support to Kaput's argument. First, we found that, over time, treatment students retained a significant advantage over their control peers in their understandings of worthwhile big algebraic ideas and thinking practices. Second, we believe our findings lend support to the argument that algebra must be treated as a continuous K–12 strand of thinking, not as a subject that can be infused into students' elementary curricula for a few years only to be cast aside until its more formal treatment.

Acknowledgments

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INEQUALITIES AND SYSTEMS OF RELATIONSHIPS: REASONING COVARIATIONALLY TO DEVELOP PRODUCTIVE MEANINGS

Teo Paoletti
Montclair State University
paolettit@montclair.edu

Madhavi Vishnubhotla
Montclair State University
vishnubhotm1@montlcair.edu

Mustafa Mohamed
Montclair State University
mohamedm3@montclair.edu

Systems of equations are an important topic in school mathematics. However, there is limited research examining productive ways of supporting students' understandings of systems of equations. In this paper, we first present a conceptual analysis of potential ways students may leverage their quantitative and covariational reasoning to graph systems of relationships. We then describe results from a design experiment in which we examined the potential of supporting middle-grades students in reasoning in ways compatible with this conceptual analysis. We highlight two different ways of reasoning students engaged in as they compared the relative magnitudes of two quantities with respect to a third quantity and leveraged this reasoning to graphically represent two relationships on the same coordinate system. We draw implications from these results for the teaching and learning of systems of equations.

Keywords: Algebra and Algebraic Reasoning, Cognition, Middle School Education

Systems of equations are an important topic as indicated by its presence in both U.S. and international mathematics curricula (Bergeron, 2015). Despite this, there is a dearth of research examining productive ways of supporting students' meanings for systems of equations (see Häggström, 2008). Several researchers (Olive & Caglayan, 2008; van Reeuwijk, 2001) have noted students can leverage thinking about experientially real contexts to determine solutions for systems of equations involving discrete quantities. For instance, Olive and Caglayan (2008) examined how students solved a system of equations by leveraging their quantitative reasoning (Thompson, 1995) to coordinate several quantitative units to simplify a system of two or three equations into one equation with one variable. In this report, we extend Olive and Caglayan's (2008) work by examining ways to support middle-grades students in reasoning quantitatively and covariationally to construct, graph, and determine solutions to systems of relationships. We first present a conceptual analysis (Thompson, 2008) in which we explain ways of reasoning we conjectured could be productive for students graphing systems of relationships. We then present results from a design experiment in which we highlight students reasoning in ways compatible with this conceptual analysis. Specifically, we highlight that students were able to reason quantitatively and covariationally to construct and reason about two relationships and leverage this reasoning to accurately graph both relationships on the same coordinate system. We discuss implications of these findings for school mathematics teaching and curriculum.

Reasoning Quantitatively and Covariationally to Represent Systems of Relationships

In this report, we adopt Thompson's (1993, 1995, 2008) theory of quantitative reasoning. Specifically, Thompson (1993) noted:

> A quantity is not the same as a number. A person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it... Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them. You can think of your height, another person's height,

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and the amount by which one of you is taller than the other without having to know the actual values. (p. 165)

We highlight two aspects of quantities as defined by Thompson. First, and consistent with others (Glasersfeld, 1995; Steffe, 1991), Thompson emphasizes that quantities are conceptual entities constructed by an individual in order to make sense of their experiential world. As such, understandings of a quantity can and will differ from individual to individual and it is critical for teachers and researchers to attend to *students'* conceptions of quantities. Second, quantitative reasoning may entail reasoning about numeric values, but such reasoning does not *require* numeric values (Johnson, 2012; Thompson, 2011). Instead a student may reason about the magnitude, or 'amount-ness,' of a quantity without using numeric values.

Leveraging Thompson's characterization of quantitative reasoning, Carlson et al. (2002) defined covariational reasoning as entailing a student coordinating two varying quantities while attending to the ways the quantities change together. They described five mental actions students engage in that allow for fine-grained analysis of students' activity. The mental actions include coordinating *direction of change* (surface area increases as height increases; MA2), *amounts of change* (the *change* in surface area *increases* as height *increases in successive equal amounts*; MA3), and *rates of change* (surface area *increases at an increasing rate* with respect to height; MA4-5). Although researchers (Johnson, 2012; Moore, 2014; Paoletti & Moore, 2017) have described productive ways high school and post-secondary students engage in reasoning compatible with the mental actions, there is a dearth of research examining middle-grades students' reasoning in such ways. In this study, we pay particular attention to MA3 as critical to students representing relationships between covarying quantities.

A Conceptual Analysis with an Example

In this section, we leverage one use of conceptual analysis described by Thompson (2008), namely "describing ways of knowing that might be propitious for students' mathematical learning" (p. 46). We present a conceptual analysis of ways students may leverage their quantitative and covariational reasoning in order to describe and graphically represent a system of relationships. Specifically, we conjectured we might be able to support students in reasoning about and graphically representing a system of relationships by having the students engage in a series of activities. We intended to support students to (a) construct and reason about a relationship between Quantity A and Quantity B, (b) construct and reason about a relationship between Quantity A and Quantity C, (c) compare the relative magnitudes of Quantity B and Quantity C, including when Quantity B is greater than, less than, or equal to Quantity C for all possible values of Quantity A, and (d) leverage the reasoning from (a)-(c) to graph a system of relationships and interpret the resulting graph in terms of the inequalities determined in (c).

We use the Cone/Cylinder Task, which we designed with this conceptual analysis in mind, to describe steps (a)-(d). Prior to the session addressing the Cone/Cylinder task, we spent two to three teaching episodes engaging students with the Cone Task, which entails a dynamic cone with a varying height (the same cone is seen in Figure 1). We ask students to describe quantities they could imagine in the situation to support their reasoning quantitatively. We then task students with describing how either the surface area or volume of the cone varies, without providing numeric values, as the height varies (for more on the evolution of this task see Paoletti, Greenstein, Vishnubhotla, & Mohamed, accepted). We provide students with a sheet of paper showing the cone at five equal changes in height and prompt them to describe how the surface area (or volume) of the cone changes for successive equal changes in height to support their engaging in the mental actions described by Carlson et al. (2002). After coordinating the quantities situationally, each group was able to represent the relationship they conceived between surface area (or volume) and height of the cone in a normatively correct graphical representation.

After addressing the Cone Task, we began the next session with an applet showing the same cone with a growing cylinder next to it (Figure 1). Prior to asking the students to graphically represent the surface area (or volume) of each 3D shape with respect to height, we first asked students to engage in two other activities: (1) we asked students to describe how the surface area (or volume) of the cylinder grows for equal changes in height attempting to promote their reasoning in ways described by Carlson et al. (2002), and (2) we prompted them to describe height values (as defined by the slider, *a*) such that they conceived the surface area (or volume) of the cone is greater than, less than, or equal to the surface area (or volume) of the cylinder. Consistent with quantities being conceptual entities, we were not concerned if the students' approximation for an *a*-value such that the two quantities are equal is accurate. Instead, we intended to explore how the students described the relative magnitudes of the surface areas (or volumes) before and after their determined numeric height value.

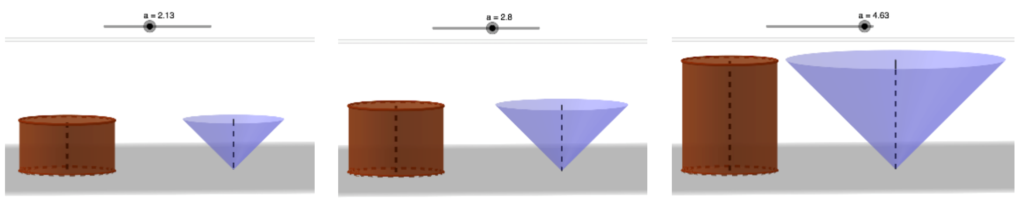


Figure 5: Several Screenshots of Cone/Cylinder Task

After addressing the two aforementioned activities, we asked students to graphically represent the relationships between surface area (or volume) and height of the two shapes on the same coordinate system. We note that since the Cone Task, involving only the cone, required several teaching episodes, each group had already constructed at least one graph representing the relationship between surface area (or volume) and height of the cone but had not yet constructed a graph representing the surface area (or volume) and height of the cylinder.

Methods, Participants, and Analysis

We conducted a design study (Cobb, et al., 2003) consisting of five small-group (one to three students) teaching experiments (Steffe & Thompson, 2000) to examine ways to support middle-grades students' (age 10-13) quantitative and covariational reasoning including the possibility of supporting their leveraging such reasoning to represent systems of relationships. We opted to engage middle-grades students as such students had not yet taken or completed Algebra I and did not have experience graphing systems of equations. The teaching experiments occurred in an underperforming school (as measured by standardized testing), which hosts a diverse student population (over 75% students of color and qualify for free or reduced-price lunch), in the Northeastern U.S. We recruited students through teacher recommendations and we engaged all students who returned consent forms.

To analyze the data, we leveraged a second use of conceptual analysis, "building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008, p. 60). Specifically, we generated, tested, and adjusted models of student's mathematics so these models provided viable explanations of each student's activity.

We used open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998) to analyze the data. First, we watched videos identifying instances that provided insights into each student's meanings. We used these instances to generate tentative models of each student's mathematics, which we compared to notes taken during on-going analysis. We tested these models by searching for supporting or contradicting instances in other activities. When evidence contradicted our models, we revised our models and returned to prior data with these new hypotheses in mind to modify previous models. This process resulted in viable models of each student's mathematics. After this, we used cross-case analysis (Yin, 2003) to compare the different students' activity to identify themes in the students' reasoning and meanings.

Results

Across the five groups, we saw two different approaches students used to graphically represent the two relationships on the same coordinate system after comparing the relative magnitudes of the surface area (or volume) of the shapes with respect to height. Four groups initially reasoned covariationally as they accurately represented each relationship on the same coordinate system without attending to the inequalities they determined (e.g., drew a linear graph to represent the relationship for the cylinder and a concave up curve to represent the relationship for cone). Two of these groups later adapted their curves in order to maintain the relationships represented by the inequalities they determined earlier in the session. The third group (a single student) reasoned about two researcher drawn curves on a coordinate system as consistent with her conceived inequalities. Due to time constraints, we were unable to probe the fifth group (a single student) in regards to how she could adjust her curves or interpret curves on a new coordinate system in relation to the inequalities she determined. To exemplify these students' activities, below we present the activities of one pair of students, Zion and Reggie.

In contrast to the other students' activities, one group, Candice and Amber, reasoned about the *difference* between the surface area of the cone and cylinder throughout their activity. By focusing on the difference in the two quantities magnitudes, the pair constructed curves that maintained the relationships they inferred regarding height values such that one surface area was greater than, less than, or equal to the other but did not accurately represent the individual covariational relationships between each shape's surface area and height. The pair adjusted their curves while continuing to reason about the difference between magnitudes to construct normatively accurate graphs.

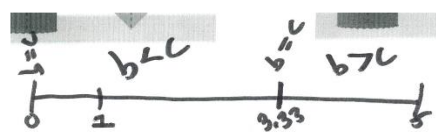
Reasoning Covariationally About Each Relationship then Adjusting: Zion and Reggie

After the sessions in which Zion and Reggie reasoned about the directional and amounts of change of surface area with respect to height of the cone (MA 2-3) and graphically represented this relationship, we presented them with the Cone/Cylinder applet and asked them to determine intervals on which the surface area of the cone (which they defined with the variable b) was greater than, less than, or equal to the surface area of the cylinder (which they defined with c). Moving the slider on the applet, they determined the two surface areas were equal at a height value of 3.33, and that for height values less than 3.33 the surface area of the cone was less than the surface area of the cylinder ($b < c$) and vice versa for values greater than 3.33 (Figure 2*a*).

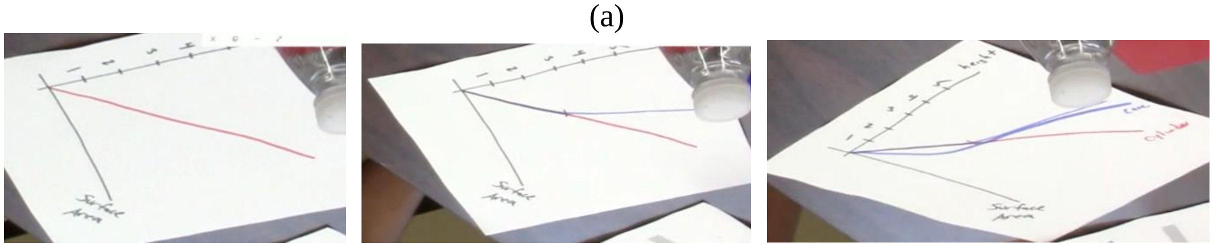
Having compared the two surface areas for all height values, the teacher-researcher (TR) asked the pair to describe how the surface areas of the cone and cylinder were increasing. Based on their activity in the Cone Task, the students spontaneously shaded in areas representing amounts of change of surface area of the cone for successive equal changes in height to argue the surface area of the cone was "growing by more and more each time." Then, the pair worked to represent via color-coordination, amounts of change in surface area of the cylinder for equal changes in height. Specifically, on a paper showing five instantiations of the cylinder and cone, the TR shaded in blue the outer surface area of the cylinder with a height of 1. In the cylinder representing the second change in height (from 1 to 2), Zion shaded in blue the same area shown in the first cylinder (with a height of 1) then shaded in brown area representing the amount the surface area changed by from a height of 1 to 2.

> TR: So that you were trying to make this brown part [*pointing to the cylinder showing the second change in height*] the same as what?
> Reggie: [*Zion and Reggie both point to the surface area representing the cylinder with a height of 1*] Same as this.
> TR: As that one?
> Reggie: 'Cause that's how much it grows every time [*pointing to the surface area of the cylinder with a height of 1*].

Shortly after this, Zion described that the surface area of the cylinder "grows constantly." We infer Reggie and Zion each understood the surface area of the cylinder increased by the same amount for equal changes of height (MA 2-3). Further, Zion sketched a linear graph representing this relationship (Figure 2*b*). We note that although the graph had the horizontal axis labeled "Surface Area," throughout their activity each student made statements that implied surface area and height were represented on the vertical and horizontal axes, respectively, and later in their activity relabeled the horizontal axis "height" when they noticed this label.



(a)



(b) (c) (d)

Figure 2: (a) Zion and Reggie's Segment Representing Height Values and (b)-(d) Several Screenshots of Their Work Graphing Each Relationship on the Same Coordinate System

Immediately after the interaction above, Reggie drew a second curve in blue to represent "the cone" (Figure 2*c*). The TR probed the students to examine if they intended to represent the surface area of the cone and cylinder as being equal for small height values:

> TR: Are they growing from 0, 0 to 3.33 [*marks a point on the curves where the blue and red curve diverge*]… are they the same, are the surface areas the same for that, from 0 to 3.33?
> Reggie: Yeah [*using his fingers to indicate the segment between (0,0) and the point on the*

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graph where the curves diverge], uh-huh.
TR: Should they be?
Zion: [*crosstalk*] I mean this one would be a little bit lower I guess [*tracing an imaginary concave up curve from (0,0) to where the curves diverge as if drawing a new blue curve*] because it's not…until we said until 3.33 the cylinder is going to be bigger. [*Zion draws a new curve in blue on their coordinate system, Figure 2d, then the TR asks him to explain why he changed the graph.*] What we said is that the, that from this point [*motioning over the interval from 0 to 3.33 on the segment in Figure 2a then pointing to b > c*] that the cu-cylinder is gonna be bigger than the cone and from this point on [*pointing to 3.33 then to 5 on the segment in Figure 2a*] the cone is gonna be bigger than the cylinder [*pointing to b < c*].
TR: How does this graph [*pointing to the graph in Figure 2d*] show you that… the surface area of the cone is less than the surface area of the cylinder?
Reggie: The cone [*tracing over the blue curve from the origin to the point of intersection*] is lower than the cylinder [*tracing over the red curve from the origin to the point of intersection*].

We highlight two aspects of this interaction. First, we infer Reggie argued the two surface areas were equivalent from 0 to 3.33 by interpreting what it would mean for the two curves to overlap on this interval (i.e., using his fingers to indicate the gap between (0, 0) and the point where the curves diverged on the graph), rather than on a conception of the relationship between the quantities' magnitudes. This inference is supported by him quickly assimilating Zion's argument regarding the need to represent the cone's surface area as being below the cylinder's surface area for height values between 0 and 3.33. Second, we highlight how Zion leveraged the inequalities they had determined earlier (i.e. explicitly referencing the inequalities they had determined in Figure 2a) to adjust their curve to create a representation that reflected their stated inequalities; Zion coordinated his activity reasoning about inequalities regarding the two surface areas in relation to height as he constructed two curves representing each surface area-height relationship.

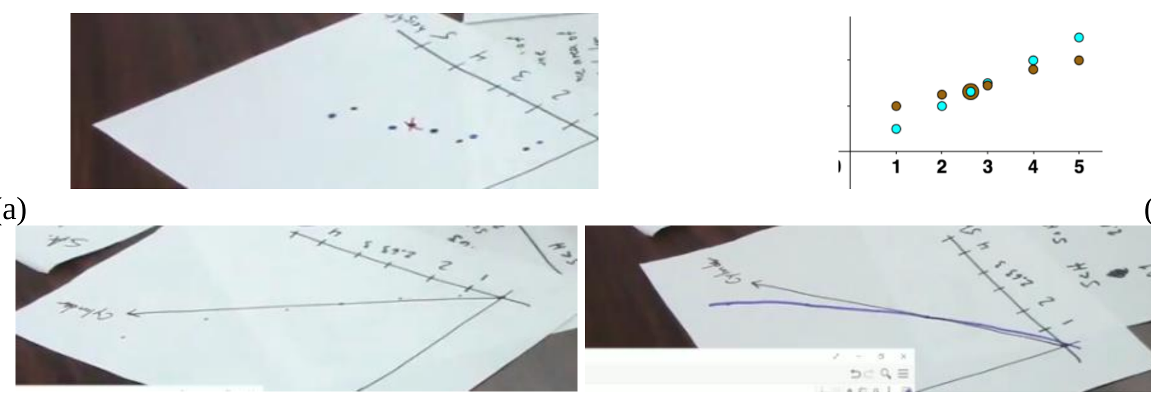
Reasoning About the Difference of Two Quantities to Represent a System of Relationships

Like Zion and Reggie, at the outset of the Cone/Cylinder Task session, Candice and Amber described the surface area of the cone increased by more for equal changes in height (MA 2-3) and the surface area of the cylinder increased by equal amounts for equal changes in height (MA 2-3). When prompted to determine a height value such that the two surface areas were equal, they approximated a height value of 2.63 and indicated for height values between 0 and 2.63 the surface area of the cone was less than that of the cylinder and vice versa for height values between 2.63 until 5. Each student had sketched a linear function to represent the relationship between surface area and height of the cylinder but had labeled their axes differently (Candice labeled the vertical axis "height" and Amber labeled the horizontal axis "height").

Attempting to ensure the students would discuss the same quantities when discussing their graphs representing both relationships, the TR created new axes with height values from 1 to 5 on the horizontal axis and asked the students to color coordinate points (blue for the cone and brown for the cylinder) representing the height and surface areas of the two shapes. Addressing this, each student considered the difference in the two surface areas for corresponding height values as they plotted points. Specifically, after plotting the brown points above the blue points at height values of 1 and 2 (Figure 3*a*/*b*), the TR asked how they had chosen to space the points. Candice maintained the points above the height value of 2 were closer than the points at a height value of 1 and Amber stated the points above the height of 1 should have been spaced farther apart. We infer each student understood that the surface area of the cylinder was bigger than the surface area of the cone and that the difference between surface areas was decreasing until 2.63.

After plotting these points, the TR moved the applet to a height value of 2.63 and each student indicated the surface areas "are going to be the same." Candice then placed a blue point above a height value of 3 and Amber placed a brown point on top of this blue point. Noticing the horizontal placement of the points, the TR motioned to the value on the axis below their points to examine if the students intended to plot points above a height value of 3. Amber quickly indicated their points should be above a point between two and three on the height axis, putting an X on top of their points (Figure 3*a*). Candice then placed a tick on the height axis representing a value of 2.63 and the students plotted overlapping points above this tick. We infer the students understood the two shapes had the same surface area at 2.63, which they understood meant the two curves would overlap at this value.

The students continued plotting points after 2.63 such that the blue point was above the brown point and the distance between points was "getting farther [apart]. Because [the difference between surface areas is] getting bigger." Throughout their activity the students focused on the difference in surface areas as they plotted points. However, they were not explicitly focusing on the locations of the points in relation to an individual shape's surface areas, which the TR conjectured could be problematic. For instance, both he and Amber perceived that the blue points (Figure 3*a*/*b*), representing the surface area and height of the cone, fell in a straight line that did not accurately represent the relationship. Hence, the TR drew new coordinate axes and a linear curve representing the surface area and height of the cylinder, which each student had produced earlier. He then asked the pair to sketch the curve representing the surface area and height of the cone. Consistent with reasoning about the difference between surface areas, Candice plotted points above and below the red line that maintained the difference in surface areas they had described previously. For instance, the spacing between their plotted points and points on the line for height values greater than 2.63 was increasing (Figure 3*c*). Candice then connected these points with a curve (Figure 3*d*). Compatible with attending to difference in surface areas rather than the individual surface areas, when Candice drew the curve, Amber exclaimed "Oh my god! It looks like the other line." Amber then explained the blue curve was similar to the curve they had constructed to represent the surface area and height of the cone in previous sessions.



(a) (b)

(c) (d)

Figure 3: Several Screenshots (and a Recreation) of Candice and Amber's Work

Throughout their activity, we infer Amber and Candice focused on the relationships they inferred creating inequality statements (i.e. a difference) rather than on the covariational relationships they had inferred regarding each individual surface area-height relationship. After plotting points that reflected how the difference between quantities was changing but not the quantities themselves, they were able to use the difference between surface areas to accurately sketch a curve representing the relationship between surface area and height of the cone on the same coordinate system with a line representing the surface area and height of the cylinder.

Discussion

In this paper, we presented a conceptual analysis of four interrelated activities, (see steps (a) – (d) in Conceptual Analysis with an Example), we conjectured could be productive for students graphing systems of relationships. While both pairs of students presented eventually constructed curves representing a system of relationships based on the relationships they inferred in the situation (step (d)), they did so in different ways. Specifically, Amber and Candice's reasoning focused on the *difference* in surface areas' magnitudes which they inferred from comparing the two surface areas (step (c)) as they initially plotted points. They continued to reason about the difference in surface areas to plot points representing the surface area and height of the cone after the TR provided a new coordinate system with a linear graph representing the surface area and height of the cylinder (which they had previously constructed addressing step (b)). This process resulted, to Amber's surprise, in a curve similar to the one they obtained when addressing step (a). In contrast, Zion and Reggie, as well as the three other groups, initially produced a graph of each individual relationship that reflected the covariational relationship they inferred from the situation (e.g. the relationships they inferred in steps (a) and (b)). During this activity, they did not attend to how the two curves may intersect. After considering the inequalities they deduced when comparing the relative magnitudes of the two surface areas (step (c)), two of these groups were able to sketch a graph accurately representing the two relationships they conceived.

The results we presented provide novel insights into students developing understandings of systems of relationships and also add to the literature on students' quantitative and covariational reasoning in two important ways. First, we highlight that the students were never presented with, nor created, values for the surface area of either shape. Rather they reasoned about quantities' magnitudes (Thompson, 1995) when reasoning about the direction and amounts of change of surface area as well as when comparing the relative size of the two shape's surface areas. The students later used height values to describe and graphically represent these varying magnitudes. Hence, the results presented here highlight ways in which middle-grade students can productively reason quantitatively and covariationally about both magnitudes and values to graphically represent relationships.

Second, the results provide empirical examples that, at least some, middle-grades students are capable of leveraging the mental actions describe by Carlson et al. (2002) to reason about and graphically represent relationships between covarying quantities. We conjecture supporting middle-grade students in developing such reasoning as a way of thinking (Thompson, 2016) could serve as the foundation for their developing notions of linear and non-linear relationships as well as rate of change more generally. Future researchers may be interested in examining ways to support middle-grades students engaging in such reasoning to develop such notions as well as to develop other relation classes (e.g., exponential, quadratic relations).

Finally, we note that our focus was broader than determining a solution to a system of equations, which is often a goal of school mathematics (e.g., Häggström, 2008). We were interested in exploring ways students can reason about systems of relationships, including how two quantities are represented when one is greater than, less than, or equal to the other with respect to a third quantity. The students in our study rarely experienced difficulties comparing two quantities magnitudes situationally or graphically. Hence, we encourage future researchers and curriculum designers to examine the possibility of supporting students explicitly comparing two quantities magnitudes or values prior to tasking them with determining unique solutions to systems of equations. That is, we conjecture reasoning about systems of relationships can serve as a foundation for students' developing productive meanings for systems of equations.

Acknowledgments

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FROM COMPUTATIONAL STRATEGIES TO A KIND OF RELATIONAL THINKING BASED ON STRUCTURE SENSE

Cesar Martínez-Hernández
Universidad de Colima
cmartinez7@ucol.mx

Carolyn Kieran
Université du Québec à Montréal
kieran.carolyn@uqam.ca

This paper provides evidence on how elementary school students from a Mexican public school move from using an operational sense, expressed in computational strategies, to a kind of relational thinking based on structure sense ideas and expressed by number decomposition. Even though the results are somewhat preliminary, they illustrate how students can leave behind their computational strategies and develop a more sophisticated mathematical reasoning regarding equivalence of numerical expressions and equalities; however the strategies they developed tended to be based on "ad hoc decomposition of one side and comparison with the initial form of the other side," rather than on compensation.

Keywords: Algebra and Algebraic Thinking

Background

Among the different approaches to developing algebraic thinking in the early grades, recent studies have begun to distinguish more clearly the notion of structure sense in arithmetic (Kieran, 2018b) and to stress its importance as a key feature of algebraic thinking that should be promoted from an early age onward. Equivalence tasks involving equalities such as $a+b=c+d$ have been shown to be particularly important (Martínez & Kieran, 2018). Among the studies dealing with structure sense, Asghari and Khosroshahi (2016) have described how certain tasks in the elementary grades that use the equal sign can provoke a kind of mathematical thinking involving both operational and structural conceptions.

Pang and Kim (2018), as well as Schifter (2018), in studies on equalities of the form $a+b=c+d$, have emphasized the importance of engaging students in discussions on the structural properties of such equalities. In particular, Schifter states that if students focus on the structure of equalities such as 57+89=56+90 in order to determine their veracity, they are thereby indicating a relational thinking with respect to equivalence tasks, which is considered a fundamental aspect of algebraic thinking (Carpenter, Franke, & Levi, 2003).

Based on Schifter (2018), and the theoretical perspective addressed in Kieran (2018b), Martínez and Kieran (2018) demonstrated that students at early ages accept equalities such as $a+b=c+d$ and are able to rewrite them in various ways when asked to identify the equivalence of both sides of the equality. However, their strategy for rewriting an equality to show equivalence (e.g., 172+18=182+8 as 100+72+10+8=100+82+8) indicates that they have a natural inclination to rewrite the equality in a manner that clearly relates to the total of each side. That is to say that they do not decompose and recompose the numbers with the aim of producing a written form where both sides look alike, for example, rewriting 172+18=182+8 as, among others, 100+72+10+8=100+72+10+8, or as 182+8=182+8. In other words, even when they are able to decompose the numbers, they do not establish a clear relation between both sides of the equality, but rather with respect to the total of each side. Therefore, according to Martínez and Kieran (2018), for students to determine the veracity of an equality (e.g., 172+18=182+8) and its various rewritten forms (e.g., 100+72+10+8=100+82+8), they resort to a computational strategy:

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

calculate the total of each side of the equality. Such a strategy and the form they use to rewrite the equality does not allow them to reason about the expressions of the equality based on structural aspects.

However, the results obtained by Martínez and Kieran (2018), especially the fact that students accept equalities of the form $a+b=c+d$, and that they are able to rewrite them, served as a basis for the current study. This study, which is reported herein, involved designing an equivalence task on numeric equalities where students were to be explicitly instructed not to calculate the total of both sides in order to determine the veracity of the equality. This design was considered to be a pivot for developing in students a sense of structure with respect to equivalence, since accepting numerical sentences as bona fide numerical objects and being able to operate on and with them as objects is viewed as fundamental in looking for and expressing structure in arithmetic (Martínez & Kieran, 2018, p. 169). Hence, the question that guided the research of this study is the following: *What is the nature of the strategies students use to determine the equality of the right and left sides of a numerical equation when they are explicitly asked to not calculate the total of each side?*

Theoretical Framework

Different theoretical viewpoints underpin discussions on the development of algebraic thinking at early ages (see Cai & Knuth, 2011; Kieran, 2018a). One such perspective focuses on seeking, using, and expressing structure in numbers and numerical operations (Kieran, 2018b).

Structure in Numbers and Numerical Operations

The main idea in this perspective lies in the notion of structure sense in arithmetic. As Kieran (2018b) points out, there are several researchers interested in the notion of structure in algebra. Linchevski and Livneh (1999) and Mason, Stephens, and Watson (2009), among others, emphasize structure as a feature of algebraic thinking. On the one hand, Linchevski and Livneh relate the development of structure in arithmetic with the development of structure in algebra. Mason et al., on the other hand, maintain that students need to develop *structural thinking*, that is, a focus on relations rather than on procedures.

According to Kieran (2018b, p. 80), the notion of structure is one of the central ideas in mathematics; however, not only are there different perspectives on it, but also it is often treated as if it were an undefined term. Nevertheless, it is strongly related to generalization in the literature on algebraic thinking. On this base, generalization involves identifying the structural, and vice versa, the structural involves identifying the general.

In contrast, Kieran (2018b) discusses aspects related to structure that might expand this notion and that, at the same time, involve more than just the basic properties of arithmetic. She proposes a focus on the notion of structure in arithmetic that draws on Freudenthal (1983, 1991, quoted in Kieran, 2018b). According to Freudenthal, the whole numbers constitute an order structure. This order structure leads to the addition structure, as well as to the multiplication structure. Following Freudenthal, Kieran points out that structure in numbers and numerical operations involves different *means of structuring*, according to, for example, factors, multiples, powers of 10, evens and odds, decomposition of primes, etc. This perspective is embodied in the idea that developing a sense of structure entails seeing through mathematical objects and drawing out relevant structural decompositions.

Based on Freudenthal´s ideas, and the various perspectives found in the literature regarding the notion of arithmetical structure, Kieran suggests promoting various experiences with equivalence that involve decomposition, recomposition, and substitution. Such experiences would allow students to become aware of different means of structuring numbers and numerical operations. The research literature suggests that students think in a structural way about equivalence when they transfer part of a number to another. For example, Carpenter et al. (2003) have shown how students indicate a relational thinking about equivalence when they implicitly use the associative property to decide about the equality of the two sides. Without writing it as such (e.g., for the numerical equality 56+47=54+49), students express that 56+47=(54+2)+47=54+(2+47)=54+49. Britt and Irwin (2011) have referred to this approach as the compensation strategy, which they express alternatively as follows: 56+47=56–2+2+47=54+49. However, the results from our pilot study (see Martínez & Kieran, 2018) suggested that the compensation strategy is not the one that underpins the approach used by the students when they determined the equivalence of the left and right sides of an equality. Consequently, we had to extend our research.

Method

Participants

For this second stage of the project, three students (S1, S2 and S3) from a Mexican public school participated, ages between 11 and 12 years. When data were collected, the students had just finished elementary school. Those three students were selected because they had taken part in the first stage of the project (see Martínez & Kieran, 2018).

Task Design

The proposed task in this second stage of the project is a redesign of Task 3, reported in Martínez and Kieran (2018). Hence, the task involves the equal sign to show the equivalence of expressions in an equality (e.g., 10+7=5+12). The task has four sections, the first three having the following structure:

- A true equality (numerical equation) is presented.
- Students are asked to show, <u>without calculating the total on both sides</u>, that the equality is true. (The task explicitly asks for them not to calculate the total.)
- They are asked for an explanation of their reasoning.

Section four asks for a generalization of the students' strategy for showing the veracity of the equalities as given in the three previous sections.

Data Collection

Data collection was carried out in one single session, eight months after the initial study (reported in Martínez & Kieran, 2018). The technique used in data collection was that of the group interview, conducted by one of the researchers (subsequently referred to as "I"). The aim of the group interview was to allow students to verbalize their reasoning. During the interview, each participant was given the printed task sheet in order to have a written record of his/her work. The 40-minute session was also video-recorded.

Results and Discussion

Due to space constraints, this report concerns only the work developed by the three students during the first part of the task, which involved the equality 10+7=5+12, within the setting described in the Task Design section above (where the equality 10+7=5+12 is presented; students are asked to show, <u>without calculating the total on both sides,</u> that the equality is true, along with providing an explanation of their reasoning). Data analysis is based on each student's written work sheets, as well as their verbalizations as evidenced by the video-recordings, and the researcher's field notes.

Spontaneous Exhibition of Operational Sense

The first item in the initial section from the task displays the expression 10+7=5+12 and explicitly asks for showing, <u>without calculating the total on both sides</u>, that this equality is true. Despite the explicit request, students spontaneously referred to the total on both sides of the given equality after individually reading and answering the question. The following verbatim, translated from the original Spanish transcriptions, reflects this moment of the interview:

> I: Would there be a way to show that this equality [is true]? [referring to 10+7=5+12, the interviewer does not complete the question because S3 interrupts]
> S3: By adding [referring to the total; the other students agree]

As can be seen, in spite of the instructions on the work sheet to not calculate the total, S3 answers spontaneously that one needs to add up each of the sides. S1 and S2 explain in a similar manner. Fig. 1 shows S1's work, where her initial written answer is observed ("It is true because if we add the two sides, the sums that are the results are the same.").

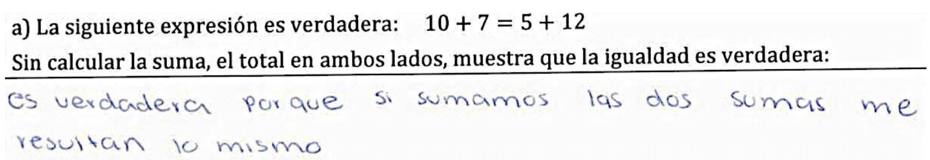
a) La siguiente expresión es verdadera: 10 + 7 = 5 + 12
Sin calcular la suma, el total en ambos lados, muestra que la igualdad es verdadera:
es verdadera porque si sumamos las dos sumas me resultan lo mismo

Figure 1: S1's Initial Answer

Based on these answers, the interviewer asks for the possibility of using a particular strategy that the students already know and exhibited in the earlier study (Martínez & Kieran, 2018) – as described in the Background section above, that is, to rewrite the given equality in another equivalent form:

> I: [The researcher writes on the board the equality 10+7=5+12] Is there another way to rewrite this equality?
> S2: As a subtraction.
> S3: As a subtraction [inaudible].
> I: All right, as a subtraction, according to S2 […] For example S2?
> S1: Twenty minus three [note that S1 answers]
> S2: Equals eighteen minus one [S2 completes the equality 20-3=18-1]

As we observe from the verbatim, the strategy followed by S1 and S2 is to search for one or more numbers in such a way as to preserve the total value for each side. These initial results, as reflected in the report of the pilot study, show a strong operational sense in students. They manage each side of the equality in an independent way, guided by the total.

Transition to a Kind of Relational Thinking

From the students' initial responses, the interviewer tries once again to see if the students would apply their earlier strategy for rewriting a given equality (Martínez & Kieran, 2018):

I: Would there be a way to write also [referring to rewriting the given equality] but using, say, these same numbers? [points to the board to the equality 10+7=5+12; see Fig. 2]
S3: Yes.
I: […] For instance, could this [points to the left side, 10+7, but S3 interrupts]?
S3: 5+5+5+2 [verbalizes the expression].
I: Ok, S3 says that this [referring to the left side of the equality 10+7=5+12] could be written as 5+5+5+2 [writes on the blackboard as expressed by S3] Is this OK?
S1 and S2: Yes [both at once].
S3: Is equal to [inaudible].
I: […] This [referring to the right side of 10+7=5+12], in which other way? Look, this [referring to 5+5+5+2] already has a form of, I mean, it [10+7] can be re-expressed in this way [Referring to 5+5+5+2] […]. Ok, Can this [the right side 5+12] be rewritten identically to this? [referring to 5+5+5+2].
S3: Yes.
I: Why?
S2: It gives the same
S1: Because it gives the same.
I: How would you rewrite it?
S3: 5+5+5+2 [verbalizes the expression].
I: Ok, 5+5+5+2 [writes the equality 5+5+5+2=5+5+5+2 as proposed by S3, on the blackboard, see Fig. 2]

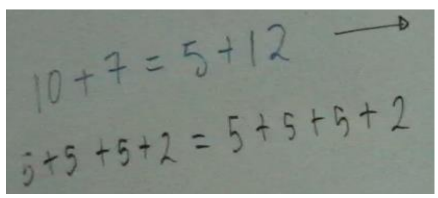

Figure 2: Equality Proposed by S3

As can be observed from the preceding verbatim and from Fig. 2, S3 is able to propose an equivalent form (5+5+5+2=5+5+5+2) of the equality 10+7=5+12, in such a way that both sides of the equality have the same form. This, however, does not necessarily mean that a kind of relational thinking is at play, since the students are working each side separately. Furthermore, S1 and S2 justify based on the preserved total. This is seen when S1 answers: *"Because it gives the same"*.

S2 and S3 are, however, showing that their initial strategies are evolving. The form proposed by S3 is in fact a decomposition of each side of the initial equality. This suggests that S3 is leaving behind the idea of looking for two or more numbers that preserve the total. The evidence for this is derived from the following verbatim:

I: Does this 5 [points to the first number 5 on the left side, 5+5+5+2, of the equality seen in Fig. 2] come from any part here [points to the initial equality 10+7=5+12]? […] Or, are you looking for two or more numbers in order to get 17 [the total]?
S3: Yes, I rewrote the two 5s [the first two numbers of 5+5+5+2] from the 10 [referring to the 10 in the expression 10+7], and from the 12.

S2: From the 7! [S2 interrupts, referring to the 7 in the expression 10+7].
S3: From the 7, we had 2 left, that is why I wrote the plus 2 [S3 explains that the 7 is rewritten as 5+2]
I: And, how would you justify these? [referring to the right side of the equality 5+5+5+2=5+5+5+2, seen in Fig. 2]
S2: The 5 comes from the 5 [of the right side of the initial equality 10+7=5+12], the two 5s and the 2 come from the 12.

As suggested by the verbatim, S2 and S3 are able to decompose each side of the given equality 10+7=5+12 in a common form. They explain, through decomposition, where each number comes from in 5+5+5+2=5+5+5+2. However, the right side in this last equality might be an *ad hoc* writing, in order to coincide with the right side (5+12) of the initial equality – a notion that is synthesized in Fig. 3 and which is discussed more fully in the paragraphs that follow.

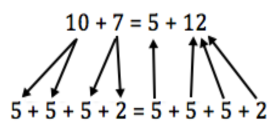

Figure 3: Rewritten Equality Through Decomposition

Based on this new strategy of rewriting each side (or at least one side) through decomposition of numbers and getting both sides of the rewritten equality to have the same form, the interviewer returns to S3's equality to inquire into the need to calculate the total:

I: Seeing it in this way [points to the equality 5+5+5+2=5+5+5+2; see Fig. 2], is it necessary to add up in order to decide if the equality is true? Would you still add [calculate the total] or is the addition no longer necessary?
S3: It is no longer necessary for me.
I: Why not, S3?
S3: Because it is easy to see what will be the result.
I: Ok [….] What is this and this expression like? [pointing to both sides of 5+5+5+2=5+5+5+2]
S2: The same
S3: The same
I: Then, is it necessary to add?
S1: Oh, no!
S2: No, because [inaudible]
S3: But to know the result of each one? [referring to the total of each side]
S2: No, but if it is the same [in the same form], obviously it will give the same [the total will be the same]. If the expression is the same, it will be equal, it will give the same.

As can be seen from the above verbatim extracts, S3 relates the decomposed equality to both sides of the initial equality, but still thinks about the total as a means to validate it. S2 and S1 support the idea that the decomposed equality of S3 is all that is needed for validation, due to the common form in which both sides could be rewritten. It is important to note how S2's speech and thinking have changed. At the beginning of the task, S2 states that the total *"is" the same,* which means that S2 has calculated the total. But now S2 justifies with another tense: *"will be" the same*, which means that S2 is aware that she will get the same total, but that it is not necessary to calculate it. It is enough to observe that both sides of the equality are written in the same form.

As of this moment, the students apply the following strategy to generate an equality equivalent to a given equality and to justify its veracity: rewrite one of the sides of the given equality based on numerical decomposition, and use this to guide the decomposition of the other side; if this decomposition fits the numbers on the other side of the initial equality, then it is not necessary to calculate the total of each side of the newly obtained equality to prove that it is true.

The above described structural approach is clearly observed when the interviewer subsequently asks the students to rewrite in another different form (different from 5+5+5+2=5+5+5+2) the initial equality 10+7=5+12, in such a way that both sides look alike. Fig. 4 shows S1's work during the group interview. However, S1 genuinely decomposes only the left side (10+7) of the initial equality to obtain 2+2+2+2+2+2+2+3, and copies this expression onto the right side. This is made obvious because S1 could not relate the expression 2+2+2+2+2+2+2+3 (i.e., the right side of the equality in Fig. 4) with the right side (5+12) of the initial equality. After a few futile attempts to relate the right side of the equality in Fig. 4, with the right side of the initial equality, S1 starts anew. She proposes a different expression for the right side, 2+2+1+5+5+2, and then guided by this expression rewrites the left side of the initial equality in the same way (see Fig. 5).

Figure 4: Rewritten Equality by S1

Figure 5: Second Rewritten Equality Proposed by S1

Now, the students are able to rewrite an equality by means of a decomposition that clearly shows equivalence; however, the truth of the equality refers to the rewritten form, not necessarily to the initial equality – perhaps because from the beginning they are told that the given equality is true. This is seen, for example, in the explanation offered by S1 (Fig. 6): *"Because if I look and compare each number, they are the same and obviously it is the same"*

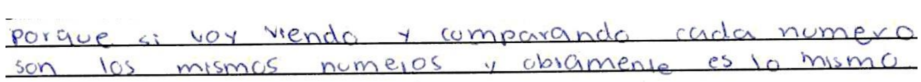

Figure 6: S1's Explanation of the Veracity of the Rewritten Equality

Summarizing, the students' strategy consists of the following: decompose the left (or right) side; copy it to the right (or left side accordingly); compare both sides of the rewritten equality with their corresponding expressions in the initial equality; if they are able to establish the equivalence relation (by pointing to where each number of the rewritten equality comes from with respect to the given equality), then they can conclude that both sides are equal (and that it is not necessary to refer to the total of each side). As previously noted, we refer to this structural approach for generating equivalent numerical equalities as "ad hoc decomposition of one side and comparison with the initial form of the other side".

Interaction Between Participants and Task Design

As could be discerned from the verbatim extracts, the participants' interactions, the task design, and, not least of all, the interviewer's interaction with the students were all crucial for the students to evolve in their mathematical reasoning.

Firstly, it was of the utmost importance to state explicitly to the students that they were not to calculate the total for each side. However, as discussed above, they spontaneously referred to the total as their initial approach for comparing right and left sides of an equality. Thus, the interviewer's role, based on the participants' impulse to calculate, was also crucial in promoting other approaches. For example, a key question that led the students to relate the initial equality with the numerical decomposition they were proposing for their equivalent equality was to have them explain where the numbers came from in their rewritten equality. This helped them focus on the given equality and its rewriting (for at least one side of the equality), based on the involved numbers (see Fig. 3). Lastly, the contribution each student provided in the first part of the task – especially the initial decomposition of the left side proposed by S3 – was crucial for the eventual consolidation of the new strategy that grew out of the students' interactions. The analyzed interview episodes illustrate clearly the significance of a sociocultural environment in the construction of mathematical knowledge.

Conclusions

The first set of results emanating from the designed task of this study support the results and conjectures put forward in Martínez and Kieran (2018). Students show a strong operational sense that leads them to use computational strategies, as a first idea, in equivalence tasks. However, based on their acceptance of equalities such as $a+b=c+d$, students are able to develop a structural sense that is anchored in their arithmetic.

If we compare, on the one hand, the strategy that the students in this study developed with, on the other hand, the compensation strategy reported in the literature, we note several differences. Applying the compensation strategy to the equality 10+7=5+12 involves a simultaneous comparison of the relation between the numbers on the left side (10+7) with those on the right side (5+12), an observation that our participating students never explicitly mentioned. The compensation strategy involves noting that the 10 on the left side is 2 less than the 12 on the right side, and the 7 on the left side is 2 more than the 5 on the right side; so everything is balanced. According to the compensation strategy, this could be expressed as 10+7=10+2-2+7=12+5=5+12. Alternatively, this kind of thinking could be expressed in written form by decomposing the left side into 10+2+5 and the right side into 5+2+10, leading to the equivalent equality, 10+2+5=5+2+10. However, our data did not provide evidence of the use of such strategies. In contrast, the students developed a different structural strategy for generating equivalent equalities – a strategy that has not been reported in the literature up to now.

Such preliminary results lead to further questions regarding whether or not students apply the same strategy for generating equivalent equalities when presented with initial equalities involving much larger numbers where they cannot immediately observe the total for each side or with equalities where they are not told at the outset that the equalities are true. Analyses bearing on such questions are being prepared for follow-up reports.

Finally, even though the students showed a noticeable evolution in the development of their mathematical thinking, we want to emphasize the importance of the role played by the teacher in this development – a role that involved helping the students to move from an operational to a structural kind of thinking in order to determine the equivalence of numerical equalities.

Acknowledgments

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DE ESTRATEGIAS DE CÓMPUTO HACIA UN PENSAMIENTO DE TIPO RELACIONAL BASADO EN UN SENTIDO DE ESTRUCTURA

Este reporte muestra evidencia de cómo alumnos de primaria, de una escuela pública de México, transitan de un sentido operativo, manifestado en estrategias de cómputo en tareas de equivalencia, hacia un pensamiento de tipo relacional basados en cierto sentido de estructura y manifestado en la descomposición de números. Aunque los resultados son preliminares, estos muestran que los alumnos son capaces de desprenderse de estrategias de cómputo y desarrollar un razonamiento matemático más sofisticado en torno a la equivalencia de expresiones e

igualdades numéricas; sin embargo, la estrategia que desarrollan se sustentan en una "descomposición ad hoc de un lado y comparación con la forma inicial del otro lado de la igualdad", más que en la compensación.

USING NUMBER SEQUENCES TO MODEL MIDDLE-GRADES STUDENTS' ALGEBRIC REPRESENTATIONS OF MULTIPLICATIVE RELATIONSHIPS

Karen Zwanch
Oklahoma State University
karen.zwanch@okstate.edu

The number sequences describe a hierarchy of students' concepts of number. This research uses two defining cognitive structures of the number sequences—units coordination and the splitting operation—to model middle-grades students' abilities to write linear equations representing the multiplicative relationship between two unknowns. Results indicate that students who have constructed a tacitly nested number sequence (TNS), second in the hierarchy, do not represent multiplicative relationships algebraically. Students who have constructed an advanced tacitly nested number sequence (aTNS), third in the hierarchy, do so inconsistently, and students who have constructed an explicitly nested number sequence (ENS), fourth in the hierarchy, do so consistently. Coordinating three levels of units in activity advantages aTNS and ENS students, while splitting further advantages ENS students' algebraic reasoning.

Keywords: Algebra and Algebraic Thinking; Number Concepts and Operations

Literature Review

Algebra is persistently characterized as difficult for students to master (e.g., Kieran, 2007), and is a staple in school mathematics. Preceding a formal algebra course, algebra can be conceptualized as an extension of arithmetic. Russell, Schifter, and Bastable (2011), for example, conclude that arithmetic ideas such as understanding operations, generalizing and justifying, extending the number system, and using symbolic notation contribute to early algebraic reasoning. With these ideas in mind, algebraic reasoning is defined by Hackenberg (2013) as "generalizing and abstracting arithmetical and quantitative relationships, and systematically representing those generalizations in some way… [and] learning to reason with algebraic notation in lieu of quantities" (pp. 541–542).

In addition to defining algebraic reasoning, it is important to consider what concepts support algebraic reasoning. Students' concept of the equal sign is one such concept (Carpenter, Franke, & Levi, 2003; Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012). Students can have two distinct notions of equality (Kieran, 1981): relational and operational. A relational concept indicates an understanding of the equal sign as indicating a balance, or equality of expressions on both sides of the equation. This is the more sophisticated of the two notions, but many children conceive of the equal sign operationally (Baroody & Ginsberg, 1983), indicating a conception of the equal sign as an operator. This is detrimental to students' ability to solve equations (Carpenter et al., 2003), and many mistakes in high school mathematics can be attributed to incorrect uses of the equal sign (Kieran, 1981). Furthermore, a relational concept of the equal sign compliments, rather than replaces, an operational concept (Matthews et al., 2012), implying that students who have constructed a relational concept may also apply an operational concept.

The present research study will examine the ability of middle-grades students to write linear equations representing the multiplicative relationship between two unknowns. Students' concept of the equal sign will be considered as it supports or limits their algebraic reasoning. Finally,

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cognitive structures that define students' number sequences (Steffe & Cobb, 1988) will be used to model students' algebraic reasoning.

Theoretical Framework

The number sequences (Steffe & Cobb, 1988) were originally developed in research with elementary students, and described a hierarchy of four concepts of number: the initial number sequence (INS), the tacitly nested number sequence (TNS), the explicitly nested number sequence (ENS), and the generalized number sequence (GNS). Ulrich (2016b) subsequently identified a student who had constructed an advanced tacitly nested number sequence (aTNS), and Ulrich and Wilkins (2017) found 36% of sixth-grade students to be operating with only an aTNS. The aTNS falls in the number sequence trajectory between the TNS and the ENS (Ulrich, 2016b), making the trajectory of number sequences for middle-grades students INS, TNS, aTNS, ENS, GNS. The present research will focus on the algebraic reasoning of middle-grades students who have constructed an aTNS, and how their algebraic reasoning compares to that of students who have constructed a TNS and an ENS.

Units Coordination

The number sequences are defined by cognitive structures, such as units coordination and construction (Ulrich, 2015, 2016a). Figure 1 is a visual representation of the levels of units students can coordinate, organized by number sequence. Students with a TNS assimilate tasks with one level of units and can coordinate or construct a second level of units in activity (Figure 1, row 1). Thus, TNS students can immediately conceptualize a number such as seven as seven individual units of one. In mental activity, TNS students can chunk those seven individual units into a single composite unit containing seven individual units (Ulrich, 2015). A composite unit provides economy in reasoning because students must only maintain one unit of seven, rather than seven units of one. However, because the composite unit is constructed in activity, it decays following mental activity (Ulrich, 2015), leaving only seven units of one (i.e., the assimilatory structure) available for reflection.

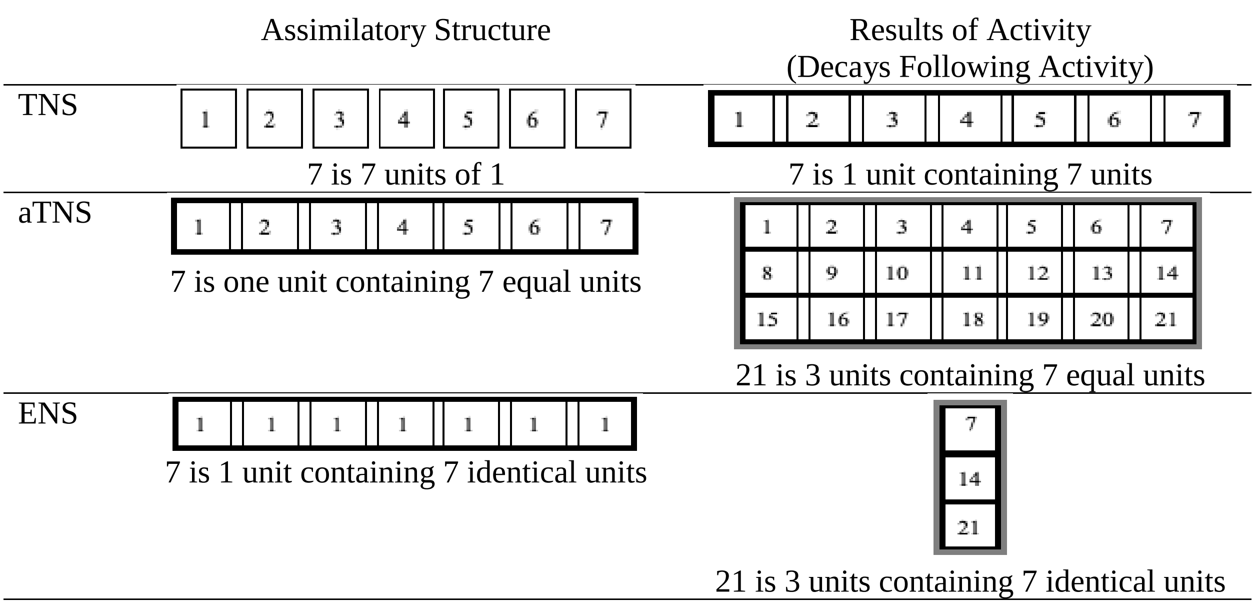

Figure 1: Depictions of 1-, 2-, and 3-levels of Units, Organized by Number Sequence

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Students who have constructed the next number sequence, an aTNS, assimilate with two levels of units, or a composite unit, and construct a third level of units in activity (Ulrich, 2016b; Figure 1, row 2). aTNS students can conceptualize seven as a single composite unit containing seven units, and can operate on a composite unit to conceptualize a three-level unit structure in activity. This allows aTNS students to conceptualize 21, for instance, as three groups of seven. But, following activity, the third-level of units decays leaving aTNS students to reflect only on a composite unit.

Similar to aTNS students, ENS students assimilate tasks with a composite unit and construct a third-level of units in activity (Ulrich, 2016a; Figure 1, row 3). However, ENS students have also constructed an iterable unit of one and a disembedding operation (Steffe, 2010a; Ulrich, 2016a). These operations advantage their reasoning over aTNS students (Ulrich, 2016b) by supporting a conception of seven individual units of one as identical rather than equal (Steffe, 2010a). ENS students can reflect on the relationships between the units contained within the composite and the composite as a whole, without destroying either quantity (Steffe, 2010a). These operations support multiplicative reasoning because ENS students conceive of the composite unit as seven times the size of each individual unit (Ulrich, 2016a).

Splitting

A splitting operation can also be understood in terms of students' number sequences. The splitting operation, defined as simultaneously partitioning and iterating (Steffe, 2010b), is within the zone of potential construction (Norton & D'Ambrosio, 2008) of students who have constructed an ENS. That is to say, ENS students can learn to split; this is not true for TNS students (Ulrich, 2016b). aTNS students, on the other hand, have been found to *solve* splitting tasks, although Ulrich (2016b) concludes that they do so by *sequentially*, rather than simultaneously, partitioning and iterating. Accordingly, this research will examine how solving splitting tasks by sequentially partitioning and iterating advantages the algebraic reasoning of aTNS students over that of TNS students, and how the simultaneity of splitting advantages the algebraic reasoning of ENS students over aTNS students.

Algebraic Reasoning

Hackenberg et al. (Hackenberg, 2013; Hackenberg, Jones, Eker, & Creager, 2017; Hackenberg & Lee, 2015) use the multiplicative concepts to model students' algebraic reasoning. The multiplicative concepts are based on students' levels of units coordination (Hackenberg & Tillema, 2009). Students who assimilated with three levels of units demonstrated "swift" equation writing when representing the multiplicative relationship between two unknowns (Hackenberg & Lee, 2015, p. 219). On the other hand, students who assimilated with two levels of units demonstrated "effortful" equation writing, and even with considerable effort, only four out of six of these students were able to write a correct algebraic representation of the multiplicative relationship between two unknowns (Hackenberg & Lee, 2015, p. 214). Only two out of six students who assimilated with one level of units represented the relationship between two unknowns algebraically (Hackenberg, 2013). Hackenberg et al. (2017) find that an unknown quantity constitutes a composite unit. This directly ties students' abilities to operate on composite units to their algebraic reasoning, and can be used to explain, at least in part, the difficulty students may have in representing the multiplicative relationship between two unknowns algebraically. The implications of these findings are that students must have constructed an assimilatory composite unit in order to operate on unknown quantities, and that students who assimilate with composite units may or may not represent the multiplicative relationship between two unknowns algebraically.

Existing research uses students' multiplicative concepts to model their algebraic reasoning. The present study expands on existing literature by using the cognitive structures that define students' number sequences to model their algebraic reasoning. The number sequence framework allows this research to distinguish between the algebraic reasoning of two groups of students – aTNS and ENS – both of whom assimilate with composite units, but only one of whom (ENS students) reason multiplicatively and split.

Methods

This report draws from the qualitative results of a larger, mixed methods study. In phase one, 326 students in grades six through nine were given a survey (Ulrich & Wilkins, 2017), the purpose of which was to attribute to each student a number sequence. In phase two, students were selected to participate in qualitative interviews based on the results of the survey. In total, 18 students were interviewed across two days, for approximately 45 minutes each day. Interviews were audio and video recorded for retrospective analysis, and students' written work was collected. The interviews included confirmation of students' number sequence attribution from the survey, characterization of their concept of the equal sign as either operational or relational, and characterization of their algebraic reasoning on 11 tasks.

Results are from the responses of 16 participants who had constructed either a TNS, an aTNS, or an ENS (Table 1). Students were assigned pseudonyms that begin with the same letter as their number sequence attribution (e.g., Ann has constructed an aTNS). The focus of the analysis is on one of the algebra problems, the phone cords problem. Students were told: Steven's cord is five times the length of Rebecca's cord. Draw a picture to represent the two cord lengths, and use algebra to represent the relationship between the two cord lengths (Hackenberg, 2013). The first part of the problem, drawing a picture to represent the cord lengths, will be referred to as the splitting task. The splitting task was considered correct if students drew a picture in which one cord length was about five times the length of the other, and the student explained that Steven's cord was the longer. The algebraic portion of the problem was considered correct if students wrote an equation equivalent to $y = 5x$ and explained that y represented the length of Steven's cord and x represented the length of Rebecca's cord.

Table 1: Participants by Grade and Number Sequence

	6^{th} Grade	7^{th} Grade	8^{th} Grade	9^{th} Grade
TNS	Tabitha	--	--	Travis
aTNS	Ann	Ava	Amanda	Alex
	Aaron	Alyssa		
	Abby	Andy		
ENS	Elle	--	Erin	Elizabeth
	Evan		Emily	Emma

Results

Concept of the Equal Sign

Tabitha and Ann were determined to only reason with a relational concept of the equal sign. On a question that asked if the equation $37 + 29 - 5 = 48 + 14$ was true or false, they both responded false. Although their response was correct, their reasoning indicated that they conceive of the equal sign as an operator rather than a relation. For example, Tabitha said, "I added 37 plus 29 and I got 66. So I subtracted 66 by 5 and I got 61. And on here, it said that 37

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plus 29 minus 5 equals 48 but actually it equals 61." Ann's response was similar. On the same question, all other students explained that the equation as false by calculating the value of each expression and indicating that they were not equal ($61 \neq 62$). This indicates a relational concept of the equal sign. Therefore, only an operational concept of the equal sign was attributed to Tabitha and Ann. A relational concept was attributed to all other students.

The Splitting Task

Neither TNS student solved the splitting task within the phone cords problem. Tabitha drew a seemingly arbitrary length to represent Steven's cord. She said, "Maybe this long?" but expressed no comprehension of a relationship between the lengths nor did the lengths have a 1:5 relationship. Travis, on the other hand, stated one correct numerical example for Steven's and Rebecca's cords, but did not generate any other correct numerical examples and did not generate a picture of the cords even at the interviewer's request. This is not surprising, given that TNS students have not been shown in previous research to split (Steffe, 2010b). Four aTNS students out of eight solved the splitting task, presumably by sequentially partitioning and iterating (Ulrich, 2016b). All ENS students solved the splitting task, which is again, not surprising, given that splitting is within the ZPC of ENS students (Steffe, 2010b).

The Phone Cords Problem

Neither TNS student solved the splitting task or represented the problem algebraically (Table 2). Both TNS students were prompted to use a numerical example to try to make sense of the relationship. Travis determined that if Rebecca's cord was five feet Steven's would be 25, but that was the extent of his progress on the problem. Tabitha did not generate multiplicatively related cord lengths. She indicated that if Rebecca's cord length was two then Steven's would be seven because, "I went up from two, and then I went up to five, and I counted. Like, 2; 3, 4, 5, 6, 7." When counting from two to seven, Tabitha raised two fingers on her left hand, and then raised one additional finger on her right hand each time that she uttered a number word. On the phone cords problem, Tabitha inappropriately applied additive reasoning to a numerical example of a multiplicative relationship.

Table 2: Number of Students by Number Sequence Who Solved the Splitting Task and Represented the Phone Cords Problem Algebraically

	Solution to the Splitting Task		
Algebraic Representation	Correct	Incorrect	Total
Correct	**7 Total**	**2 Total**	**9 Total**
	0 TNS	0 TNS	0 TNS
	1 aTNS	2 aTNS	3 aTNS
	6 ENS	0 ENS	6 ENS
Incorrect	**3 Total**	**4 Total**	**7 Total**
	0 TNS	2 TNS	2 TNS
	3 aTNS	2 aTNS	5 aTNS
	0 ENS	0 ENS	0 ENS
Total	**10 Total**	**6 Total**	**16**

Results of the ENS students will be presented next so that aTNS students' responses can be situated between those of TNS and ENS students. All six ENS students represented the phone cords problem algebraically by writing an equation equivalent to $y = 5x$, and explained the meaning of each variable (Table 2). Five of the six ENS students first wrote an incorrect equation, but used numerical examples to generate a correct equation. Elizabeth, for instance, first wrote two expressions: $5n$ and $n + 4$. She reasoned that if Rebecca's cord was three feet long then "his [Steven's] consists of, like, five little sections of that, then his would be fifteen feet long. So then five times three does equal 15… yeah, it [$5n$] works." In this excerpt, Elizabeth built on the numerical example $5 \cdot 3 = 15$ to determine the correct expression, and subsequently generated a correct equation $5n = S$.

Erin used a numerical example to decide between an equation ($5R = x$) and an inequality ($x + 5 > R$), despite being specifically asked to write an equation. She explained the inequality saying that "we don't know exactly how long they [the cords] are," so she used an inequality to indicate that Steven's (x) was longer. It was not until Erin worked through a numerical example that she determined the equation $5R = x$ was correct. Comparably, Emily used a numerical example to correct a reversal in her equation, and Evan began with a numerical example and then substituted variables for Steven's and Rebecca's cord lengths into the structure of that example.

aTNS students' solutions to the phone cords problem were inconsistent. Three aTNS students algebraically represented the phone cords problem, and there were two commonalities among these solutions. First, the successful aTNS students all began with numerical examples and substituted variables into the structures of those examples. Second, they all reverted to an operational concept of the equal sign, despite having demonstrated a relational concept earlier in the interview. Both behaviors can be observed in Alyssa's work. Alyssa initially wrote $x5 - y$.

> Alyssa: So Steven's would be *x* and Rebecca's would be *y*. And Steven's is 5 times as long as hers so you do this [multiply *x* by 5] and subtract hers [*y*] and you get the answer, I guess.
> Interviewer: OK. And what would the answer be?
> Alyssa: I don't know because there's no numbers. It just has 5.
> Interviewer: That's OK, but what would it, what would the answer represent? …
> Alyssa: The amount of cord that was timesed onto it. …
> Interviewer: So what if we say, for example, that Rebecca's cord is 3 feet long. …
> Alyssa: His would be 15, cause hers, his is 5 times as long as hers, so 3 times 5 is 15.
> Interviewer: Oh, very good. So if Rebecca's is 3 feet, Steven's is 15 feet. Does that work in our equation? …
> Alyssa: I don't think it would work with that equation [$x5 - y$] because you can't get these two [3 and 5] times each other. So it would probably be *x* equals 5 times 3. And then you would get 15 and that'd be his. … Yeah. *x* equals five times *y*.

It was not until Alyssa considered a numerical example that she rewrote her equation. Prior to that, she tried to multiply five by the length of Steven's cord, which she represented as $x5$ because the problem states that his cord is five times the length of Rebecca's. After determining numerically that Rebecca's cord should be multiplied by five, she was able to generate the equation "*x* equals five times three" and then "*x* equals five times *y*." The use of numerical examples was common among the three successful aTNS students.

The second behavior that can be observed in this excerpt is that Alyssa reasoned with an operational concept of the equal sign when she indicated that the result of $x5 - y$ is "the answer, I guess," and explained that you do not know the answer because *x* and *y* are not assigned numerical values. Alyssa was focused on finding a numerical result, and not on the equality of the expressions. Of the three successful aTNS students, all three reverted to an operational concept of the equal sign, as did Alyssa, at some point in their reasoning.

Interestingly, in Table 2 all TNS and ENS students are represented on the main diagonal—they either solved both portions of the phone cords problem (ENS students) or they did not (TNS students). The reasoning of aTNS students was less consistent. Three aTNS students represented the problem algebraically, and five did not. Of the three who correctly represented the problem algebraically, only one also solved the splitting task. Conversely, of the five aTNS students who did not represent the problem algebraically, three solved the splitting task. The five aTNS students who solved only one portion of the task are on the off-diagonal of Table 2; all students on the off-diagonal are aTNS students. Thus, despite both aTNS and ENS students assimilating with composite units, aTNS students were less successful and the results of the splitting task were disconnected from the algebraic representation.

Discussion

TNS students made very limited progress on the phone cords problem. Tabitha applied additive reasoning, Travis only generated one multiplicative numerical example, and neither student represented the relationship algebraically. All ENS students correctly represented the relationship algebraically. These results are comparable to those of Hackenberg et al., who found that four out of six students who assimilate with composite units (Hackenberg & Lee, 2015) and two out of six students who assimilate with one level of units (Hackenberg, 2013) represented the multiplicative relationship between two unknowns algebraically.

Also, five of the six ENS students interviewed used numerical examples to facilitate their equation writing. This is also consistent with Hackenberg et al.'s (2017) result that students who assimilate with composite units may build this type of equation using numerical examples. Therefore, an unknown quantity is assimilated by ENS students with a composite unit, in which Rebecca's cord length, for example, is a quantity containing an unknown number of units of one. This allows ENS students to operate on the unknown (i.e., the composite unit) in activity to form a three-level unit structure constituting the relationship between Steven's and Rebecca's cords.

Erin was the only ENS student who wrote an inequality to represent the relationship between the cords. This is similar to a student Hackenberg et al. (2017) identified, Tim, who like Erin, could assimilate with composite units. On a problem that is parallel to the phone cords problem, Tim insisted that a five times multiplicative relationship between two unknowns was "approximate" because "we don't know it [the exact lengths]" (Hackenberg et al., 2017, p. 46). Like Tim, Erin expressed a conception that while Steven's cord was longer than Rebecca's, the exact multiplicative relationship was unknown as long as the lengths were unknown. Consistent with Hackenberg et al.'s (2017) analysis, the use of an inequality to represent a multiplicative relationship between two unknowns can be explained as the manifestation of Erin's need to reduce the complexity of the unit structure on which she was operating.

aTNS students did not consistently solve the splitting task or the algebraic portion of the phone cords problem. Three aTNS students represented the problem algebraically, only one of whom solved the splitting task. Three aTNS students solved the splitting task but did not represent the problem algebraically. These results are evidence that although aTNS students may solve splitting tasks by sequentially partitioning and iterating, the sequential nature of that activity is not sufficient to support their algebraic representation of the resulting relationships. The sequential partitioning and iterating behavior of aTNS students is qualitatively distinct from the splitting in which ENS students are likely to engage. This provided ENS students with an advantage in their algebraic reasoning.

Both aTNS and ENS students used numerical examples to build an algebraic equation on the phone cords problem. However, ENS students tended to only work through one numerical example prior to writing a correct equation, while aTNS students tended to work through several. Also, whereas building an equation using numerical examples was a productive behavior for all five of the ENS students who took that approach, it was only productive for three out of eight aTNS students. Furthermore, all three aTNS students who represented the phone cords problem algebraically reverted to an operational concept of the equal sign at some point during their reasoning, despite demonstrating the ability to reason with a relational concept earlier in the interview. In contrast, no ENS students demonstrated reasoning consistent with an operational concept of the equal sign on the phone cords problem. Hackenberg and Lee (2015) conclude that students must assimilate with three levels of units, as would GNS students, before they are likely to represent multiplicative relationships between two unknowns without relying on numerical examples. This explains why all successful aTNS and ENS students, who only assimilate with composite units, relied on numerical examples to some extent. Additionally, this demonstrates a range of ways in which two groups of students who assimilate with composite units may rely on numerical examples to support their algebraic reasoning.

Conclusions

While both aTNS and ENS students assimilate with and operate on composite units, there are qualitative differences in their algebraic representations of multiplicative relationships between two unknowns. Consistent with the analysis of Hackenberg et al. (2017), both aTNS and ENS students are likely to have assimilated the phone cords problem with a composite unit consisting of Rebecca's cord, which contains an unknown number of units of one. Then, in activity, both aTNS and ENS students are assumed to operate on said composite unit to generate the relationship between Rebecca's and Steven's cord lengths; this facilitates their equation writing. Following activity, however, one level of units decays leaving both aTNS and ENS students to reflect only upon Rebecca's cord as a unit containing an unknown number of units of one.

For the three aTNS students who wrote the correct equation, the decay of Steven's cord from the unit structure resulted in their applying an operational concept of the equal sign. In other words, because the relationship between Steven's and Rebecca's cords was lost, the variable that had previously been representative of Steven's cord length was now simply "the answer." ENS students have the same limitation following activity; the third level of units decays and the relationship between the cord lengths is lost. In contrast to aTNS students, however, ENS students did not reason operationally about the equal sign. This is because ENS students are advantaged by the ability to reflect on the results of the splitting operation.

For ENS students, the results of splitting (i.e., Steven's cord being five times the length of Rebecca's) is maintained following activity. Therefore, despite the decay of the third level of units, the splitting operation allowed ENS students to maintain the relationship between Steven's and Rebecca's cords and reason normatively about their equation. In constrast, aTNS students presumably solve splitting tasks by sequentially partitioning and iterating, if they solve them at all, making the results of the splitting task unavailable for reflection following activity. As such, when the third level of units decayed following aTNS students' equation writing, they must recreate the results of the splitting task if they are to interpret the equation normatively.

Reverting to an operational concept of the equal sign has been noted in previous literature (Matthews et al., 2012), and this research provides a theoretical rationale for such a regression in the situation of writing an equation to represent the multiplicative relationship between two unknowns. aTNS students reverted to an operational concept of the equal sign to compensate for the inability to reflect on the results of the splitting task when the third level of units decayed. Therefore, although both aTNS and ENS students assimilate with and operate on composite units, the splitting operation advantaged ENS students' ability to represent the multiplicative relationships between two unknowns algebraically.

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COMPARATIVE AND CONDITIONAL INEQUALITIES: A DISTINCTION EMERGING FROM STUDENT THINKING

Teo Paoletti
Montclair State University
paolettit@montclair.edu

Mustafa M. Mohamed
Montclair State University
mohamedm3@montclair.edu

Madhavi Vishnubhotla
Montclair State University
vishnubhotm1@montclair.edu

Richard G. Cella
Montclair State University
cellar1@montclair.edu

In this paper, we draw a distinction between two uses of inequalities, comparative and conditional inequalities, that emerged from our interactions with students. We highlight that despite reasoning quantitatively to make accurate statements with both types of inequalities, conventional notations for conditional inequalities were not sensible from the students' perspective. We describe a third notation we introduced which the students were able to understand and use in later teaching episodes.

Keywords: Using representations, Algebra and Algebraic Thinking

Inequalities are an important topic in both U.S. and international mathematics (Bergeron, 1995). Some researchers (e.g., Blanco & Garrote, 2007; Tsamir & Almog, 2001) have indicated students struggle with aspects of inequalities. Hence, there is a need to better understand ways to support students in developing more productive understandings of inequalities and inequality symbols. In order to examine the potential of supporting students developing understandings of inequalities via their reasoning about varying quantities, we conducted several small-group teaching experiments. The purpose of this paper is to draw a distinction between two distinct uses of inequalities, comparative and conditional inequalities, that emerged from our interactions with middle-grades students.

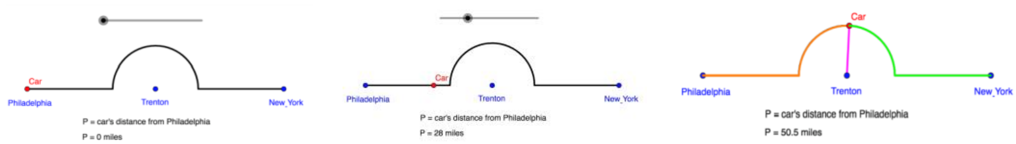
Theoretical Perspective: Mathematics of Students, and Quantitative Reasoning

We contend students' mathematics is fundamentally unknowable to us as teachers and researchers (Glasersfeld, 1995). Hence, when interacting with students, we attempt to build viable models of a *student's mathematics* that explain the student's observable words and actions. Steffe and Thompson (2000) noted, "Our practical stance is that the better we understand [student's mathematics], the better positioned we are to affect students productively…We strive to specify the mathematical concepts and operations of students and to make them the conceptual foundations of school mathematics" (p. 269). We interpret Steffe and Thompson to indicate researchers who build models of students developing productive meanings have the opportunity to support the evolution of school mathematics curricula so as to promote similar meanings for larger populations of students.

Several researchers (see Thompson and Carlson, 2017 for a review of the literature) have begun to explore ways in which students' quantitative reasoning (Thompson, 2011) can support their developing productive meanings for various mathematical ideas. Specifically, and as recommended by other researchers (Castillo, Johnson, & Moore, 2013; Johnson, 2015; Stevens et al., 2017), we asked students to construct and describe quantities changing in dynamic tasks in order to better understand the ways students may compare the relative magnitudes of their constructed quantities.

Two Uses of Inequalities: An Example and a Distinction

Consider a car traveling from Philadelphia to New York on a road that goes around Trenton (Figure 1a-b; this task is an adaptation of Going Around Gainesville as reported by Moore et al., 2018). A student may consider locations where the car's distance from Philadelphia (Figure 1c orange), which they define with the variable P, is greater than, less than, or equal to the car's distance from New York (Figure 1c green), which they define with the variable N (i.e., where $P < N$, $P > N$, and $P = N$). When a student reasons about two quantities' relative magnitudes or values to compare the two quantities, we refer to the student as reasoning about a *comparative inequality*.



(a) (b) (c)

Figure 1: (a-b) Two Instantiations of the Going Around Trenton Problem and (c) One with Distance from Philadelphia (Orange), Distance from Trenton (Pink), and Distance from New York (Green)

To further characterize the comparative inequalities, the student may use the values in the applet (bottom of Figure 1) to determine the car's distance from Philadelphia values such that $P < N$, $P > N$, and $P = N$. For example, the student may determine that for $P = 50.5$ miles, $P = N$; for 0 miles $\leq P < 50.5$ miles, $P < N$; and for 50.5 miles $< P \leq 101$ miles, $P > N$. When the student reasons about range of one quantity's magnitudes or values (e.g. $0 \leq P < 50.5$) such that a certain condition holds (e.g., $P < N$), we refer to the student as reasoning about a *conditional inequality*.

There are several differences between the two uses of inequalities. When reasoning about a comparative inequality, a student is considering a relationship between *two quantities* by comparing their relative magnitudes or values. In contrast, conditional inequalities require students to understand that some particular property is maintained for variations in *one quantity's magnitude or values*, thus necessitating images of variation.

Methods, Participants, and Analysis

To examine ways to support middle-grades students' (age 10-13) developing understandings of inequalities, we conducted a design study (Cobb, et al., 2003) consisting of five small-group (one to three students) teaching experiments (Steffe & Thompson, 2000). We chose to work with middle-grades students because we wanted to work with students who had not completed or yet taken Algebra I, and therefore did not have experience working with algebraic inequalities. To analyze the data, we used both on-going and retrospective analyses consistent with the teaching experiment methodology (Steffe & Thompson, 2000). For brevity's sake, we focus on the first pair of students we engaged in the study, Candice and Amber, as their activity prompted us to consider the two uses of inequalities during on-going analysis.

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

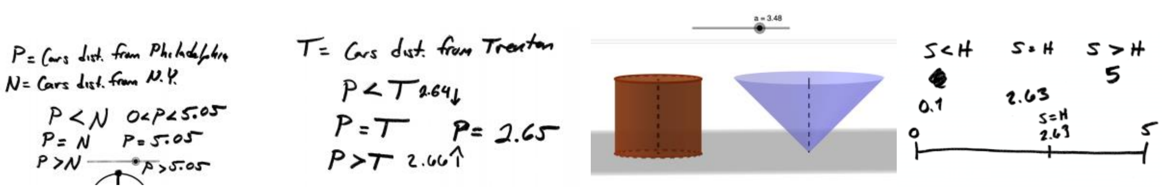
Results

We first describe Candice and Amber's initial activity addressing prompts about inequalities in the Going around Trenton Task (Figure 1). We then briefly present their activity from a later task in which the students were able to use a line segment to viably describe a situation using both comparative and conditional inequalities.

Reasoning about Quantities to Consider Comparative and Conditional Inequalities

Initially, the TR presented a dynamic applet of the Going around Trenton situation without *P*-values visible. When the TR asked the pair where the car's distances from Philadelphia and New York would be the same, Amber pointed to a location on the road directly above Trenton. Amber quickly noted that the car's distance from New York would be longer than its distance from Philadelphia prior to hitting the point directly above Trenton and the car's distance from New York would be shorter after it passed this point. We infer Amber was able to reason about the comparative inequalities involving the car's distances from Philadelphia and New York by comparing their relative magnitudes throughout the car's entire journey.

After the above interaction, the TR presented the applet showing values for the car's distance from Philadelphia (in this first iteration, the values varied from 0 to 10.1 without a specified unit). The TR then defined variables for the car's distance from Philadelphia (P) and the car's distance from New York (N) and wrote $P < N$. After a short interaction, both students understood this notation to represent the distance from Philadelphia was less than the distance from New York and identified this was true from the beginning of the trip until the car reached the point directly above Trenton. The students were then able to interpret the other statements ($P = N$, $P > N$) in terms of quantities' magnitudes (i.e. as comparative inequalities).



(a) (b) (c) (d)

Figure 2: (a) The TR Written Notations Representing Conditional Inequalities, (b) The Students' Invented Notation for Representing the Conditional Inequalities, (c) Screenshot from the Cone-Cylinder Problem, (d) The Students' Written Work for this Problem.

Immediately after this interaction, the TR intended to support the students in understanding a conventional way to represent the interval of values for which each statement was true. He wrote statements next to each of the comparative inequalities to represent the *P*-values such that the comparative inequality was true (seen in Figure 2a) while attempting to convey what he intended this notation to represent. However, both students expressed they did not understand why the TR had written $P > 5.05$ and he spent two minutes attempting to support the students understanding his intended meaning for the notation.

Inventing a Notation for Conditional Inequalities

We infer the students did not understand the conditional inequality statements the TR wrote based on their not leveraging this notation when tasked with a new prompt. When the TR asked them to determine the *P*-values such that the car's distance from Philadelphia (P) was less than, equal to, or greater than the car's distance from Trenton (T), the students first identified a location on the road, with $P = 2.65$, such that car's distances from Philadelphia and Trenton were equal. Then, Candice moved the car to the right, showing a P-value of 2.7, and both students indicated at this value the car's distance from Philadelphia was greater than its distance from Trenton. Indicative of each student understanding using the inequality symbols to compare two quantities magnitudes, the students simultaneously pointed to $P > T$ on the paper (Figure 2b) to represent this comparative inequality. After this Candice indicated this statement was true "for anything past two point six five." However, to represent this, she created a new notation, writing 2.66↑ (see Figure 2b). We infer that the students used their invented notation to represent conditional inequalities. Collectively, we infer the students were reasoning about both comparative and conditional inequalities as they reasoned about and compared different quantities in the situation. Further, the students understood the comparative inequality statements (e.g., $P < T$, $P > T$) in ways consistent with our intentions. However, they invented their own notation to represent conditional inequalities as they did not interpret these statements (e.g., $0 < P < 5.05$) in ways consistent with the TR's intentions.

Negotiating Conventions for a New Notation for Conditional Inequalities

The students' thinking steered us to notice underlying differences between these two types of inequality statements leading us to distinguish between comparative and conditional uses of inequalities. In the next session, we introduced a line segment with endpoints 0 and 101 to represent all possible values of the car's distance from Philadelphia. They were able to use this notation to describe quantities in the Going around Trenton task as well as two months later addressing the Cone/Cylinder Problem (Figure 2c). Specifically, they identified at a height value of 2.63, the two shapes had equal surface areas and, using the line segment notation (Figure 2d), discussed for which height values each shape's surface area was greater than the other.

Discussion

In this paper, we draw a distinction, based on our models of student's mathematics, between comparative and conditional uses of inequalities. Consistent with Steffe and Thompson's (2000) description of specifying "the mathematical concepts and operations of students and to make them the conceptual foundations of school mathematics" (p. 269), we conjecture this distinction has several implications for the teaching and learning of inequalities in school mathematics.

First, the students were able to reason quantitatively (Thompson, 2011) to characterize comparative and conditional inequalities. We conjecture having students engage in dynamic activities, which present changing quantities likely supported such reasoning. Second, we highlight that despite being able to accurately describe each type of inequality via quantitative reasoning, the students did not interpret statements we intended to represent conditional inequalities (e.g., $0 < P < 5.05$) in ways consistent with our intentions. However, when we introduced the line segment notation, similar to the number line notation sometimes used in school mathematics (Tsamir & Almog, 2001; McCabe, Sorto, White, Dover & Andreasen, 2010), the students were able to use this notation to describe intervals on which certain comparative inequalities were valid. This leads us to question introducing notations involving inequality symbols for conditional inequalities early in students' experiences. We concur with Thompson's (1995) recommendation regarding introducing notational conventions, "Separate what is new and conceptual from what is new and conventional, and teach for what is conceptual while postponing new conventions (especially those having to do with conventional notation)" (p. 202). Finally, we conjecture supporting students in becoming explicitly aware of these two uses of inequalities may support them in developing productive understandings of inequalities and call for more research exploring this possibility. For instance, we conjecture using different notations for comparative and conditional inequalities may support students to explicitly differentiate between the two types of inequalities and later more formal notations can be introduced when pedagogically appropriate for students.

Acknowledgments

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CONFIDENT OR FAMILIAR? THE ROLE OF FAMILIARITY AND FRACTION ESTIMATION PRECISION ON METACOGNITION

Charles J. Fitzsimmons
Kent State University
cfitzsi4@kent.edu

Clarissa A. Thompson
Kent State University
cthomp77@kent.edu

Pooja G. Sidney
University of Kentucky
pooja.sidney@uky.edu

Understanding fraction magnitudes is especially important for academic achievement, but fraction reasoning is difficult for children and adults. To accurately reason about fraction magnitudes, people may need to monitor what they know and what they do not know. However, little is known about fraction metacognition. In the current study, we examined trial-by-trial fraction estimates, confidence judgements, and ratings of fraction familiarity in adults. Adults' confidence was related to their estimation precision, suggesting accurate monitoring. However, adults judge their confidence in estimating fraction magnitudes, in part, based on their familiarity with each fraction. The role familiarity plays in judgments of confidence with fractions suggests that people may be less likely to check for errors or restudy when reasoning about highly-familiar fractions.

Keywords: Metacognition, Cognition, Rational Numbers, Number Concepts and Operations.

Across multiple countries, children's understanding of fraction magnitudes is predictive of their overall mathematics achievement, even when controlling for SES, working memory, IQ, and fraction arithmetic abilities (Siegler et al., 2012; Torbeyns, Schneider, Xin, & Siegler, 2015). However, compared to whole numbers, children and adults are less precise when estimating the magnitude of fractions (e.g., Siegler & Opfer, 2003; Siegler, Thompson, & Schneider, 2011). One source of difficulty in understanding fraction magnitudes is the *whole number bias*: when knowledge of whole number concepts interferes with fraction reasoning (Ni & Zhou, 2005). For example, children tend to estimate fractions with larger components (e.g., 15/30) as larger than equivalent fractions with smaller components (e.g., 1/2; Braithwaite & Siegler, 2017). The difficulties children face while learning and problem solving with fractions may result, in part, from the limited set of fractions they encounter in textbooks (Braithwaite & Siegler, 2018). For example, computational models make errors similar to those of children when they receive fractions from common textbooks as input (Braithwaite, Pyke, & Siegler, 2017). This limited early environmental input during learning may contribute to the pervasive errors that even adults make on fraction tasks (Opfer & Devries, 2008; Sidney, Thalluri, Buerke, & Thompson, 2018). It remains an open question whether adults are *aware* of whole number bias when reasoning about fraction magnitudes. In other words, are adults able to metacognitively monitor their fraction performance?

Awareness of performance via self-monitoring is important for academic achievement because monitoring influences control of study behaviors (e.g., Dunlosky & Rawson, 2012; Metcalfe, 2009). For example, learners may be more likely to notice and correct errors when reasoning about fraction magnitudes if they can monitor their understanding that fractions with larger components do not always have larger magnitudes than fractions with smaller components (e.g., 15/30 is not > 1/2 even though 15 > 1 and 30 > 2). As this example illustrates, monitoring may be especially important for learning difficult fraction concepts, a domain in which learners often automatically misapply familiar whole number concepts (Siegler et al., 2011), and may need to inhibit this prior knowledge (Siegler & Pyke, 2013).

One way that people can monitor their mathematics performance is through judgments of confidence in their responses (e.g., Bjork, Dunlosky, & Kornell, 2012; Wall et al., 2016). Sometimes people can be *overconfident* (high confidence despite low performance) or *underconfident* (low confidence despite high performance), both of which can undermine achievement (e.g., Dunlosky & Metcalfe, 2008). Thus, assessing students' confidence when reasoning about fractions provides insights into how they monitor their understanding.

Ideally, only a student's ability to generate an accurate answer would affect their confidence, however, familiarity may also influence confidence judgments. For example, in one study of children's whole number magnitude estimation (Wall et al., 2016), most children were more confident and precise in their estimates within smaller, more familiar numerical ranges (e.g., 0 – 100) than larger, unfamiliar numerical ranges (e.g., 0 – 1,000). However, some children's confidence was less well-aligned: they were more confident in their small-range estimates even when they were equally precise in smaller and larger ranges. Others were more confident in the smaller range even when their estimates in this range were *less* precise. Overall, students may feel more confident with smaller numbers due to familiarity with the 0 – 100 range because smaller numbers are encountered more frequently within the environment than larger numbers (Dehaene & Mehler, 1992). Similarly, some fractions may be more frequently encountered in the environment than others (see Braithwaite & Siegler, 2018). Therefore, even adults may be more confident when estimating familiar fractions that occur more frequently in the environment, regardless of their actual ability to precisely estimate.

In the current study, we examined adults' confidence, familiarity, and number line estimation precision for equivalent fractions with smaller (e.g., 1/2) and larger components (e.g., 15/30). We manipulated the size of components because fractions with large components occur less frequently in math textbooks (Braithwaite & Siegler, 2018), components affect how people process fraction magnitudes (i.e., whole number bias; Ni & Zhou, 2005), and little is known about the impact components have on familiarity and confidence. **Thus, we hypothesized that adults would be more precise, confident, and familiar with fractions that have smaller components compared to those with larger components.** We also examined whether confidence during fraction estimation had a stronger relationship to familiarity than to estimation precision. Relying on one's familiarity with fractions might result in real-world educational implications such as mis-allotting study time and failing to detect errors in one's fraction performance.

Method

Participants

A total of 91 community-based adult participants (34 females) met pre-registered inclusion criteria as part of a larger study. Participants (72.5% White) were recruited from Amazon's Mechanical Turk; most had a college degree (45.1%) or completed some college (24.2%). This sample is appropriate for investigating questions about adults' metacognitive awareness in the domain of mathematics because adults use fraction concepts in their daily life at work (Handel, 2016) and when making health decisions (Lipkus & Peters, 2010).

Tasks and Procedure

Participants completed all tasks within an online Qualtrics survey. On each number line estimation trial, participants estimated the location of a fraction and then rated their confidence.

Participants were also randomly assigned to complete number line estimates either under a time constraint or not. We control for time constraint condition in our models, but the effects of condition were not central to the current hypotheses and did not have a meaningful impact on results. After estimating fractions on number lines, participants completed four additional tasks: fraction comparison, fraction equivalence, numerical inhibition, and digit updating span. As with the time constraint manipulation, these tasks were part of a larger data collection on fraction understanding and will not be discussed further. At the end of the experiment, participants rated their familiarity with each of the fractions presented during the estimation task.

Number line estimation. Participants estimated the location of 44 fractions on a 0 to 1 number line, one at a time, by clicking or dragging a slider. The 44 fractions consisted of 11 different magnitudes within the 0-1 numerical range. Each magnitude was represented by four different equivalent fractions that had smaller (e.g., 1/2 or 2/4) or larger (e.g., 12/24 or 15/30) components. We calculated the absolute difference between the provided estimate and actual fraction magnitude and multiplied these values by 100 to transform them into percent absolute error (PAE; see Siegler & Booth, 2004).

Confidence judgments. Immediately after each number line estimation trial, participants rated their confidence on a four-point scale ranging from "not so sure" to "totally sure" (Wall, Thompson, & Morris, 2015).

Familiarity judgments. Participants rated their fraction familiarity on a six-point scale from "not familiar at all" to "very familiar" for each of the 44 fractions that they had previously estimated. Participants made judgments with all fractions presented on the screen at once at the end of the experiment after all other tasks were completed.

Results

Number Line Precision, Confidence, and Familiarity

We evaluated estimation precision by calculating participants' percentage of absolute error (PAE) for each number line estimate, then averaged across all estimates as well as estimates of small component and large component fractions, separately. As can be seen in Table 1, participants were more precise, confident, and familiar with fractions that had smaller components relative to fractions with larger components.

Table 1: Precision, Confidence, and Familiarity for Small and Large Component Fractions

	Small Component (SD)	Large Component (SD)	F-Statistic	η_p^2
Percent Absolute Error	8.15% (8.14%)	9.26% (8.05%)	10.79 **	.108
Confidence Judgments	2.92 (.60)	2.75 (.66)	59.938 **	.402
Familiarity Ratings	4.45 (.90)	3.75 (1.30)	113.32 **	.560

*Note: Percent Absolute Error is inversely related to estimation precision. ** p < .01.*

To further examine adults' ability to monitor their estimation precision, we calculated a gamma correlation for each participant. Gamma is a commonly used measure for assessing trial-to-trial monitoring abilities, and it accounts for the ordinal nature of confidence judgments (Nelson, 1984; Wall et al., 2016). We report inverse values of gamma because lower values of PAE indicate more precise estimates. Participants rated their confidence higher on estimates that were more precise, $M_{\text{gamma}} = .34$, $t(90) = 12.88$, $p < .001$, indicating adults were able to monitor their trial-to-trial estimation performance.

Role of Familiarity in Confidence Judgments

To examine the role of familiarity in confidence judgments, we calculated a gamma correlation between familiarity and confidence judgments for each participant. Participants rated their confidence higher when they were more familiar with the fractions, $M_{gamma} = .51$, $t(88) = 12.64$, $p < .001$.

Of central interest, we compared whether confidence was more strongly related to estimation precision or familiarity by comparing gammas between the confidence-PAE relationship to the confidence-familiarity relationship. Confidence was more strongly related to familiarity ($M_{\text{gamma}} = .51$, $SD = .38$) than to PAE ($M_{\text{gamma}} = .34$, $SD = .24$), $F(1, 87) = 18.07$, $p < .001$, $\eta_p^2 = .17$.

We further examined the role of familiarity in confidence judgments by regressing mean ratings of confidence onto overall PAE and condition. Although PAE (which is inversely related to precision) significantly predicted confidence judgements, $\beta = -0.50$, $t(88) = 5.49$, $p < .001$, adding familiarity in the second block of the model explained an additional 11.4% of the variance in confidence judgments, $F(1, 87) = 16.70$, $p < .001$. Thus, familiarity ratings explained variance in confidence judgements above that explained by adults' estimation precision.

Discussion

We examined adults' ability to monitor their number line estimation performance while estimating equivalent fractions with smaller and larger components. Adults demonstrated whole number bias errors when they were less accurate in their estimates of fractions with larger compared to smaller components. However, they were likely aware of these errors, because their confidence was also lower when estimating large component fractions. Although adults were able to monitor their estimation performance, confidence was more strongly related to familiarity ratings than to PAE, even though adults rated their confidence, but not familiarity, immediately after making each estimate. Fortunately, all three aspects of fraction reasoning--confidence, familiarity, and PAE--were strongly related. One limitation is that participants may have interpreted our measure of familiarity as a broad measure of their confidence. For example, participants might rate how familiar they are with each fraction based on how well they just performed on the estimation task, rather than how often they had encountered the fraction prior to the experiment. Future work should aim to distinguish familiarity and confidence and measure familiarity prior to completing any fraction tasks.

These findings have implications for mathematics education. Adults in our sample tended to be precise when estimating fractions they rated as familiar. The role of familiarity in confidence may be especially pronounced for adults who are less precise when estimating magnitudes, or for children who are just beginning to formally learn about fractions. This presents an opportunity for *growth* in future research on the role of metacognition in mathematics, but it remains to be explored whether familiarity and confidence may work *against* children and adults from lower SES backgrounds who may be less exposed to environmental input about fractions (cf. Siegler & Ramani, 2008 and Ramani & Siegler, 2008 for evidence of variability in environmental input for whole numbers across Head Start and middle income families).

Our results suggest that when adults are presented with highly familiar math content, they will be more confident in their performance. This higher confidence may, in turn, reduce the likelihood that (1) they will allot additional study time to master familiar, yet difficult course content, or (2) they will check for errors when familiar numbers are used in important decision-making scenarios in everyday life (e.g., financial or health decisions). This research is a first step towards a more comprehensive account of how familiarity and confidence with math is related to students' learning and success.

Acknowledgments

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CHILDREN'S ERRORS ON A REPEATING PATTERN TASK AND THEIR ASSOCIATIONS WITH FORMAL NUMERACY KNOWLEDGE

Emily R. Fyfe
Indiana University
efyfe@indiana.edu

Alicia Macchione
Indiana University
almacchi@iu.edu

We sought to provide insight into children's early patterning skills by examining children's errors when creating repeating patterns and how these errors relate to performance on a variety of number tasks. Children (N = 31, ages 4 to 6) completed several pattern abstraction tasks (e.g., recreate this model pattern using different objects) and several early math tasks assessing numeracy knowledge (e.g., count 15 pennies, add 3 candies to 2 candies). Children exhibited moderate performance on the pattern task, and committed six different types of errors. Further, some, but not all, of the errors on the pattern task were related to children's numeracy knowledge, even after controlling for age. Results provide insight into children's knowledge of repeating patterns and how it relates to their performance on critical number tasks.

Keywords: Algebra and Algebraic Thinking; Problem Solving; Number Concepts and Operations

Objectives and Theoretical Framework

Patterning is viewed as a central component of mathematical thinking (e.g., Charles, 2005). The Early Math Trajectories theory includes patterning as a foundational skill in preschool that provides opportunities to deduce rules and structure, which then supports formal numeracy knowledge (Rittle-Johnson, Fyfe, Hofer, & Farran, 2017). Empirical evidence supports this. For example, an intervention that focused on basic repeating patterns for 3- to 5-year-olds led to greater numeracy knowledge a year later relative to typical preschool instruction (Papic, Mulligan, & Mitchelmore, 2011). Also, longitudinal evidence shows that knowledge of patterns at age 5 predicts performance on a standardized math test at age 11 (Rittle-Johnson et al., 2017).

To provide greater insight into this association, we examined children's errors when creating patterns and how these errors related to numeracy knowledge. A previous study documented four-year-olds' errors on several repeating pattern tasks (Rittle-Johnson, Fyfe, McLean, & McEldoon, 2013). In that work, common errors included creating a pattern with a different structure than a model pattern (e.g., seeing an ABB-ABB pattern, but creating an AB-AB-AB pattern), and placing objects in a random order (e.g., seeing an ABB-ABB pattern, but creating an ABBCBACAACC sequence). However, this work was limited to four-year-olds and did not include any measures of numeracy knowledge. Here, we focus on children's patterning skills between the ages of four and six and include three types of numeracy tasks. Based on the Early Math Trajectories theory, which suggests that attention to the rules and structure of the pattern is key, we hypothesized that errors that contained *no* discernible structure (e.g., random order) would negatively relate to children's numeracy knowledge.

Method

Participants

Thirty-one children ranging from 4 to 6 years old (*M* = 5.2 years, *SD* = 0.7) participated in one-on-one sessions in a quiet laboratory space at Indiana University. Children were from the local community and approximately 59% were female and 90% were White.

Measures

During a one-on-one session with a trained tutor, children completed a seven-item measure of patterning skill and a 15-item measure of number knowledge. These are described below.

Pattern Abstraction. Children solved 7 pattern abstraction items by looking at a model pattern and then recreating the same kind of pattern using different materials (see Figure 1).

Figure 1: Example Pattern Task and Materials

The model patterns were repeating patterns made of black circles and squares with two units of repeat (e.g., ABB-ABB, ABC-ABC). Children were given brightly colored shapes to make their pattern. The tutor demonstrated the task on three examples. Further, on each of the 7 target problems, the tutor first explained the model pattern before having the child try to recreate it. Children's patterns were counted as correct if their pattern contained the same structure as the model (e.g., using heart-leaf-heart to represent circle-square-circle). We also coded the errors children made (see Table 1). Interrater agreement on error type was high (kappa = .92).

Table 1: Coding Scheme Used to Identify Children's Errors on the Pattern Task

Error Type	Example Response with an ABB-ABB Model Pattern	Frequency of Use		
		age 4	age 5-6	total
No Error	ABB-ABB recreates structure of model pattern	23%	56%	43%
Partial Correct	ABB-ABC recreates some structure of model pattern, but contains errors	5%	3%	4%
Wrong Pattern AB	AB-AB-AB creates an AB structure	29%	19%	23%
Wrong Pattern Other	ABC-ABC creates a structure that is not the model structure or AB	6%	10%	8%
Duplicating	[places stars in a circle to look like the circle in the model] duplicates the literal shapes in the model pattern	15%	0%	6%
Random Order	ABACBCCCAAAB produces a random sequence of shapes with no structure	15%	12%	13%
Off Task	[places shapes in flower arrangement] uses shapes in a way that is seemingly unrelated to the task	7%	0%	3%

Numeracy Knowledge. Children solved 15 number tasks using items adapted from the Research-Based Early Mathematics Assessment (Clements, Sarama, & Liu, 2008). Three items tapped their counting knowledge (e.g., count 15 pennies), six items tapped their magnitude knowledge (e.g., which is bigger: 3 grapes vs. 4 grapes, which is closer to 6: 9 or 4), and six items tapped their arithmetic knowledge (e.g., 3 candies plus 2 candies, add 10 to 48). Children received a total score out of 15 based on their correct selections and exact numerical responses.

Results

Children did moderately well across the seven pattern items. The average percent correct was 43% (*SD* = 40%, range = 0-100%). When children solved the problems incorrectly, they were making one of six different types of errors, three of which included some structure and three of which did not. The most common error was to produce a pattern with an AB structure rather than the structure of the model pattern. This accounted for 23% of all trials and 40% of all errors. The next most common error was to produce a random sequence of shapes, which accounted for 13% of all trials and 23% of all errors. See Table 1 for a full list of error types. Compared to younger children (age 4), older children (ages 5 to 6) were more likely to make no errors, and less likely to commit the Wrong-Pattern-AB, Duplicating, and Off-Task errors.

Children also did moderately well on the number tasks. Across all 15 problems, the average percent correct was 60% (*SD* = 24%, range = 13-100%). Performance was higher on the items that tapped counting knowledge (*M* = 75%, *SD* = 34%) and magnitude knowledge (*M* = 77%, *SD* =23%) than on items that tapped arithmetic knowledge (*M* = 34%, *SD* = 29%). There were large age differences, with older children (ages 5 to 6) scoring higher across all items (*M* = 71%, *SD* = 19%) than younger children (age 4; *M* = 41%, *SD* = 18%), $t(29) = 4.39$, $p < .001$.

Of interest was the relation between children's pattern skill and their formal number knowledge. First, we ran a linear regression model predicting children's numeracy knowledge from their patterning score and their age. With just these two factors, the model accounted for 58% of the variance in numeracy knowledge. Percent correct on the patterning task significantly predicted percent correct on the number tasks, *ß* = .32, *p* = .03, even after controlling for age. The effect of the patterning task was somewhat stronger for predicting counting knowledge (*ß* = .45) than for magnitude knowledge (*ß* = .18) or arithmetic knowledge (*ß* = .25). Second, we ran a linear regression model predicting children's numeracy knowledge from the frequency of each of the six error types, controlling for age. The model accounted for 70% of the variance in numeracy knowledge, and two error types emerged as significant predictors. The frequency of committing the Duplicate error was negatively related to children's percent correct on the number task, *ß* = –.37, *p* = .01, as was the frequency of committing the Random Order error (*ß* = –.39, *p* = .01). Similar results were found when we compared children who made a specific error at least once to children who never made that error (see Figure 2). Children who committed the Duplicate or Random Order errors had lower numeracy scores than children who never did.

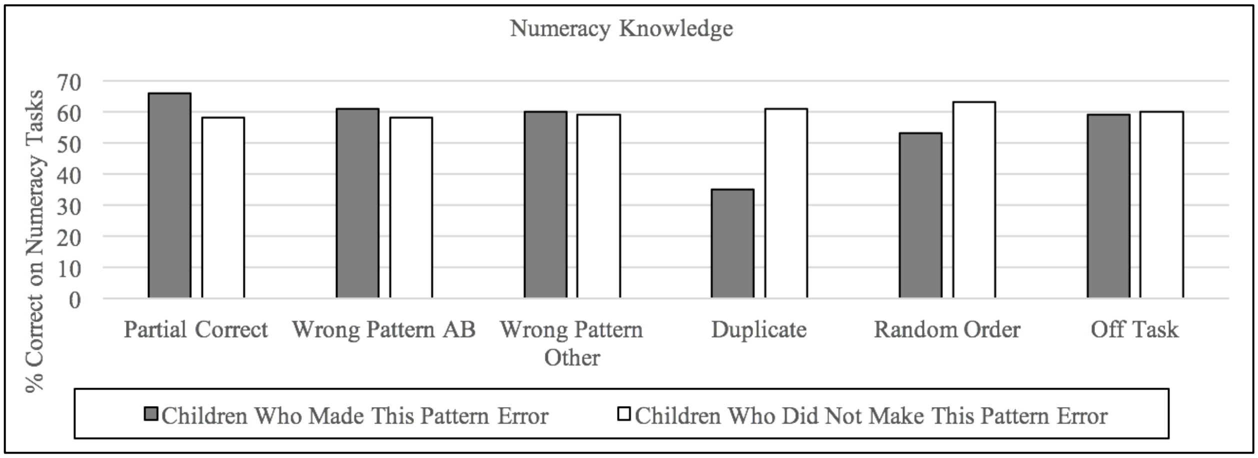

Figure 2: Numeracy Knowledge as a Function of the Pattern Errors Children Commit

Conclusion

We examined children's errors on a repeating pattern task and how these errors related to children's numeracy knowledge. Children made six different errors when trying to recreate a model pattern. Consistent with previous research (Rittle-Johnson et al., 2013), common errors included creating a simpler AB-pattern and creating a random sequence of objects. We also found that the frequency of certain error types decreased with age as patterning skill improved. Importantly, three of the six error types indicated that children's responses contained some rule or structure, and three of them indicated that children's responses did not contain any discernible structure. Consistent with our predictions, two of the three errors with no discernible structure (Duplicate and Random Order) were negatively related to children's numeracy knowledge. The three errors with some structure were not related to children's numeracy knowledge.

These findings contribute to a growing body of research investigating the associations between children's early patterning skills and their formal mathematics knowledge (e.g., Kidd et al., 2013; Kidd et al., 2014; Papic et al., 2011; Rittle-Johnson et al., 2017, see also Burgoyne et al., 2017 for a review). Consistent with this research, we found that higher performance on a pattern abstraction task was associated with higher performance on counting, magnitude, and arithmetic tasks, even after controlling for the effects of age. However, the specific type of errors that children committed also seemed to matter. Attending to the specific structure of the model pattern is important, but just noticing that there is *any* structure may be important as well. This seems consistent with the Early Math Trajectories theory, which suggests that one possible benefit of working with patterns is the opportunity to notice that certain things follow rules and have a certain structure (Rittle-Johnson et al., 2017). Indeed, many numeracy tasks require the ability to detect rules and regularities (e.g., base ten structure, the count sequence, etc.). Our error analysis provides a more fine-tuned measure of children's ability to attend to structure, which may have implications for the development of their foundational numeracy knowledge.

Given that patterning is considered a "Big Idea" in mathematics (Charles, 2005) and that mathematics has been defined as the "science of patterns" (Steen, 1988), this research on early patterning skills and errors may have implications for mathematics education more broadly.

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THE RELATIONSHIP BETWEEN ANALOGICAL REASONING AND SECOND-GRADERS' STRUCTURAL KNOWLEDGE OF NUMBER

Helena P. Osana
Concordia University
helena.osana@concordia.ca

Anne Lafay
Concordia University
anne.lafay@concordia.ca

The aim of the present study was to investigate whether analogical reasoning is predictive of children's structural knowledge of number. A group of 94 second-graders were assessed on their conceptual knowledge of place value and tested on their counting skill, arithmetic fluency, and analogical reasoning. Results showed that the most important predictor of place value knowledge was analogical reasoning after accounting for counting skill and arithmetic fluency. Furthermore, children who were classified as Structural-knowers were more proficient in analogical reasoning than children who viewed numbers on their face value. The results suggest that a structural understanding of number is not only dependent on domain-specific skills, but also on domain-general abilities, including analogical reasoning.

Keywords: Analogical reasoning, Structural knowledge of number, Place value

Understanding in mathematics has been conceptualized as facility with the conceptual structure of a domain (Mason et al., 2009; Richland et al., 2012). One goal of mathematics instruction with young children is to help them acquire a conceptual understanding of number (Mulligan & Mitchelmore, 2009). The decimal numeration system is governed by a multiplicative structure where 10 units in one denomination are grouped, or "bundled," into one unit of the next larger denomination (Ellemor-Collins & Wright, 2009). Knowledge of number structure allows children to use place value concepts to invent meaningful algorithms and develop mental computation skills (Carpenter et al., 1998; Kindrat & Osana, 2018).

Our research focuses on the cognitive processes that are predictive of children's structural knowledge of number and the instructional factors that support such knowledge. We adopted analogical reasoning as a theoretical framework for children's acquisition of number knowledge (English, 2004; Richland, 2011). Analogical reasoning is a cognitive process by which conceptual similarities in two contexts are identified and compared through "structure mapping" (Gentner & Colhoun, 2010). Conceptual understanding can be developed through structure mapping because of the attention to the structural relations between two contexts, such as a written numeral and a pictorial representation of its denominations (English, 2004; Vendetti et al., 2015). In fact, interventions based on instructional analogies have been shown to enhance students' understanding in a number of domains, including fraction division (Sidney & Alibali, 2015), proportionality (Richland & McDonough, 2010), and place value (Mix et al., 2017).

Present Study

The objective of the present study was to test the hypothesis that analogical reasoning is predictive of students' structural knowledge of number after controlling for counting skill and arithmetic fluency, two factors that are known to impact mathematics performance (Geary, 2011; Koponen et al., 2016). The data were part of a larger project investigating the physical affordances of concrete manipulatives on second-graders' numeration knowledge. Outcome measures assessed place-value understanding and number decomposition, administered before and after an instructional intervention with manipulatives. Tests of counting skill, arithmetic fluency, and analogical reasoning were administered at pretest. In the present study, we use pretest data on one specific measure of place value, the PicPVT, and examine the extent to which analogical reasoning is predictive of performance on this measure.

Method

Participants

Participants were 94 second-grade children (50 boys; age: M = 92.3 months, SD = 4.8) from 12 schools in a large, urban area in Canada.

Measures

Place value knowledge. The Picture Place Value Task (PicPVT; Kamawar et al., 2010; Osana & Blondin, 2017) was used to assess whether children understand that a digit in a numeral represents a quantity that is determined by its position. Children were presented with a written numeral with one digit underlined (either the units, tens, or hundreds digit) on a screen. A pictorial representation (i.e., sets of dots) was given below the numeral. By responding "yes" or "no," children indicated to the researcher whether the pictorial representation correctly represented the value of the underlined digit. Of the 20 items, 10 "correct display" items required a positive response from the child (Figure 1a) and 10 "incorrect display" items required a negative response (Figure 1b).

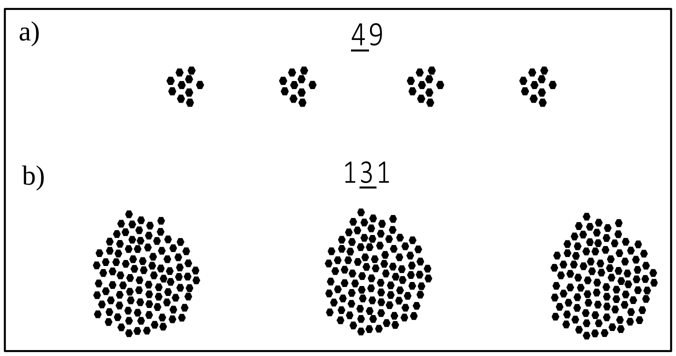

Figure 1: Sample Items on the PicPVT. Panel (a) Shows a Correct Display Item Requiring a Positive Response. Panel (b) Shows an Incorrect Display Item Requiring a Negative Response

Mathematical skills and analogical reasoning. Counting skill was assessed using the Counting and Enumeration task (CE), which required the children to count as high as possible, to skip count by 10 and 100, and to count the number of chips in a collection of 20. Arithmetic fluency was assessed using the Tempo Test Rekenen (TTR, De Vos, 1992), a timed test of mental computation in addition and subtraction. Finally, analogical reasoning was measured using the Raven's Standard Progressive Matrices (Raven et al., 1977). Children chose one picture among four that completed the missing part of a bigger pattern. On all measures, correct responses were assigned 1 point and incorrect responses 0 points.

Results

Descriptive Statistics

The means and standard deviations for all four measures are presented in Table 1 and the zero-order corrections are in Table 2.

Table 1: Means and Standard Deviations for Place Value, Mathematical Skill, and Analogical Reasoning Percent Scores

Measure	*M*	*SD*
PicPVT	.83	.19
Counting and Enumeration (CE)	.75	.19
Arithmetic Fluency (TTR)	.27	.08
Analogical Reasoning (Raven)	.74	.13

Table 2: Zero-order Correlations Between Place Value, Mathematical Skill, and Analogical Reasoning Variables

Measure	1	2	3	4
PicPVT	--	.29**	.25*	.37**
Counting and Enumeration (CE)		--	.33**	.16
Arithmetic Fluency (TTR)			--	.34**
Analogical Reasoning (Raven)				--

*p<.05, **p<.01

Predictors of Place Value Knowledge

We conducted a hierarchical multiple regression to test the relationship between place value knowledge and analogical reasoning. The criterion was PicPVT score. The first two predictors, counting skill and arithmetic fluency, were entered in the first and second steps in the analysis, and were entered together because they were similarly correlated with place value knowledge. In the third step, analogical reasoning was entered separately to allow for an independent assessment of the variance explained by analogical reasoning. The regression statistics are reported in Table 3.

Table 3: Summary of Hierarchical Regression Analysis for Variables Predicting Place Value Knowledge

Variable	R	R^2	R^2 change	Final β	*t*
Step 1	.33	.11	.11		
Counting and Enumeration (CE)				.24	2.24*
Arithmetic Fluency (TTR)				.16	1.52
Step 2	.44	.19	.08		
Counting and Enumeration				.22	2.15*
Arithmetic Fluency (TTR)				.07	0.64
Analogical Reasoning (Raven)				.30	3.01**

*p<.05, **p<.01

At Step 1, the model explained a significant 10.7% of the variance in place value knowledge, $F(2, 90) = 5.37$, $p < .01$. Introducing analogical reasoning in the model at Step 2 explained an additional 8.3% of the variance in place value knowledge, and this change was significant, $F(1, 89) = 9.07$, $p < .01$. Together, the three predictors in the model explained 19% of the variance in place value knowledge. In the final model, arithmetic fluency was not related to place value knowledge. The most important predictor was analogical reasoning, $\boldsymbol{\beta} = .30$, $t(89)=3.01$, $p < .01$.

In a second analysis, we created profiles of place value knowledge by identifying patterns of responses on the correct display and incorrect display items on the PicPVT. Children who correctly answered more than five incorrect display items and more than five correct display items were placed in the Structural Knowledge (SK) profile ($n = 74$). Children in the SK profile understood that a digit represents a quantity depending on its position in a numeral. Children who correctly answered fewer than 4 incorrect display items and more than 5 correct display items likely counted the number of groups, regardless of the number in each group. We argue that these children held a "concatenated digits" view of number (Bergeron & Herscovics, 1990), seeing numerals at their "face value" (Barnett-Clarke et al., 2010). We placed these children in a Face Value (FV) profile ($n = 16$). Finally, four children displayed no discernable patterns of responses and were likely guessing. We removed these four from the analysis.

A one-way ANCOVA was performed with profile as the independent measure (SK, FV) and analogical reasoning as the dependent measure. Counting skill and arithmetic fluency were entered as covariates. A significant effect of profile was found, $F(1, 85) = 198.32$, $p < .001$, partial $\eta^2 = .7$, with higher analogical reasoning scores in the SK profile ($M = .91$, $SD = .10$) than in the FV profile ($M = .53$, $SD = .08$).

Discussion

The findings suggest that analogical reasoning is predictive of children's number knowledge, even when accounting for counting and arithmetic skills. This suggests that a structural understanding of number is not only dependent on domain-specific skills, such counting skill and prior mathematical knowledge (Fyfe et al., 2012; Zhang et al., 2014), but also on domain-general abilities, including analogical reasoning (Geary et al., 2017). Our research thus contributes to the literature on the cognitive predictors of children's structural knowledge of number, a critical element to children's success in mathematics, beginning in the early years (Byrge et al., 2014). In particular, our findings align with those of Collins and Laski (2019), who showed that relational reasoning is predictive of numeral-quantity knowledge and magnitude comparison.

One discernable pedagogical implication that emerges from our research is that educators may wish to focus on developing young children's analogical reasoning skill because of its relationship to structural knowledge of mathematics. Although our research was correlational, the findings allow us to predict that enhancing children's analogical reasoning would increase their conceptual understanding of number and reduce the reliance on superficial aspects of mathematical representations.

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INTRODUCING MULTIPLICATION THROUGH *TOUCHTIMES*

Sean Chorney
Simon Fraser University
sean_chorney@sfu.ca

Nathalie Sinclair
Simon Fraser University
nathalie_sinclair@sfu.ca

In this research report, we work with a novel, multitouch iPad app called TouchTimes, which was designed to develop multiplicative thinking in young learners. In TouchTimes, children express multiplicative relations through two-handed gestures that emphasise the many-to-one abstraction articulated by Davydov and the functional element of Vergnaud. In this study, we analyze a pair of grade-two boys who, without any formal schooling in multiplication, are given an opportunity to experiment with the app. Our analysis suggests that the boys' express multiplication in a simultaneous way that emphasize the coordination of three quantities.

Keywords: Number concepts and operations, Elementary school education, Technology

Introduction

The Common Core characterizes children's first encounters with multiplication in terms of repeated addition. For example, the first reference to multiplication (in grade 3) states: "Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each" (Standard 1 3.OA). This characterization of multiplication in which the multiplier precedes the multiplicand is sustained through grades 3 and 4. Unfortunately, many students fail to move beyond additive reasoning, which may lead to difficulties in future mathematical learning that include topics such as proportion, ratio, and algebra (Siegler et al., 2012).

Our work is motivated in part by Davydov (1992), who identifies a core element of multiplication as the double many-to-one abstraction. For example, the first many-to-one involves the unitizing of the multiplicand (the group size) in which 3 units become one. The second many-to-one then involves the unitizing of the multiplier in which the 4 units (of 3) become one. We also draw on the work of Vergnaud (1983) on the conceptual field of multiplication, which "explicitly contrasts multiplicative reasoning with additive reasoning and addresses the difference between a scalar interpretation of multiplication and a functional one" (Askew, 2018, p. 407). In Askew's work a double number-line is used to represent doubling (1 → 2; 2 → 4; 3 → 6, etc.), which highlights the way that the multiplicand changes the product.

In this study we work with a novel, multitouch iPad app called *TouchTimes* (from now on referred to as *TT*) (Jackiw & Sinclair, 2019), which was designed to draw on both the functional and unit-based approaches described above. Specifically, in *TT*, children express multiplicative relations through gesture-based actions that promote both the many-to-one abstraction articulated by Davydov (1992) and the functional element of Vergnaud (1983). The overall research objective is to investigate how children's early use of the gesture-based, multitouch *TT* app provides children with richer experiences of multiplication.

Brief Description of TouchTimes

TT is designed with the screen split into two marked by a vertical line down the middle (Figure 1a). When a user touches one side of the screen with a finger or a number of fingers, coloured discs appear under each finger. These different coloured disks are called pips and represent the multiplicand. When the user touches the other side of the screen with a finger or

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number of fingers bundles of the pips appear under each finger. That is, each pod contains a replica of the configuration, colour and orientation of the pips from the original side; these can be seen as the multiplier. For example, if the user places three fingers on the left side (LS) of the screen, three differently coloured pips appear (Figure 1b). If the user then places four fingers on the right side (RS) while holding the three pips, four 'pods' of three pips appear (Figure 1c). If a finger is lifted from the pips side, all the pods will change accordingly, that is, each pod will have one less pip. Whenever there are pips and pods on the screen, a multiplication statement appears on the top of the screen. For the above situation, the mathematical expression is 3 × 4 = 12 (Figure 1c). When fingers are lifted completely off the screen on the pip side, all disks disappear. When fingers are lifted on the pod side, the pods remain and more can be added, or they can be individually dragged to the trash. In this way, larger numbers can be multiplied rather than being constrained by the number of fingers available. Each hand has a different function, which makes multiplication relational and functional rather than repeated addition.

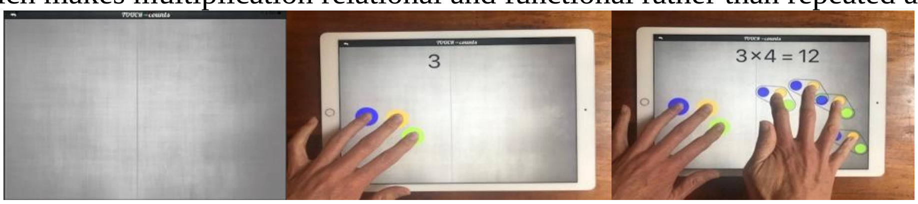


Figure 1: (a) Initial screen of *TT*; (b) Creating 3 Pips; (c) Creating 4 Pods

Theoretical Framing

The theoretical framing of this study is inclusive materialism (de Freitas & Sinclair, 2014). It adopts a monist ontological stance that attends to the learning of mathematics attending to the material context of student bodies and tools. The proposed research therefore draws significantly on theoretically-driven and empirically-tested insights into the significance of embodiment and tool-use in mathematics cognition. There is growing consensus about the significance of embodiment in mathematics education research (Radford, 2013). Existing literature explores embodiment in the context of 1) the positive impact of gesturing on student's mathematical thinking (Alibali & Nathan, 2012), 2) the high correlation of spatial reasoning (including its temporal and tangible aspects) with mathematical achievement (Newcombe & Frick, 2010); and, 3) the temporal, kinetic nature of mathematical thinking (Núñez, 2006). The question we ask in this study is what embodied coordinated gestures including both gestures and verbal expression align with multiplication properties outlined by Davydov and Vergnaud.

Methods

We have pursued a design-based research methodology which involves iterative cycles of design and implementation. We are now in our third cycle of design refinement, following initial experimentation with grade 3 children with limited school-based experiences of multiplication, which has produced some minor changes in functionality as well as specific tasks that, together with TT, enable students to develop the first level of abstraction designed in Vergnaud's theory (Bakos & Sinclair, 2019). In the research reported here, we used this same task with younger children (two grade 2 students; 8 years old) who have not had formal exposure to multiplication.

The two boys, Kevin and Joshua attended an elementary school in a rural part of British Columbia. They were two researchers present. One principally posed questions (first author) while the other video recorded. In analyzing the video data, we identified moments during which either the unit-based or functional character of multiplication emerged. We then transcribed these episodes and conducted a microgenetic learning analysis (MLA) (Parnafes & di Sessa, 2012) in order to study the unique ways in which the boys learn multiplication.

Findings

We present two episodes that highlight the emergence of multiplicative thinking with *TT*.

First Episode: A Rotating Relation

About two minutes into their exploration, both boys touch with single fingers from each hand on their own sides of the screen. The pip-holding boy, Joshua, rotates his two index fingers around, about 180 degrees, as if there was a centre of rotation between his pips. Kevin, who is holding two pods with his two index fingers, sees them also turning, even though he is not moving his fingers. The boys both laugh as Joshua continues to move his fingers. They continue this for 20 seconds; then Joshua says "I'm controlling it". After 20 seconds, Kevin touches his side with his thumb and a new pod appears. The statement 2 x 3 appears on the screen. The boys both continue laughing and then lift their fingers, which makes the pips and the pods disappear. While Joshua is laughing, Kevin's nine fingers are placed on the screen, making 9 pips. He says, "now your turn". One of Joshua's index fingers touches the screen. The statement 9 x 1 appears on the screen. Then another index finger touches the screen, thereby making two pods of 9 pips each. The statement 9 x 2 appears on the screen. Kevin starts moving his 9 fingers in little motions. The boys laugh again. They continue this moving and laughing for almost 60 seconds.

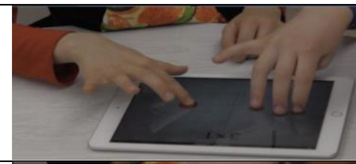
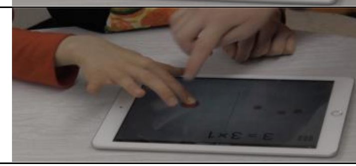
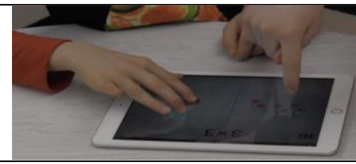
In this episode, we see the emergence of a particular gesture involving both boys and both their hands. The first time the gesture is used, Joshua has his two pip-making index fingers and Kevin has two pod-making index fingers. They maintain this particular four-handed gesture for over 40 seconds as Joshua moves his fingers, which produces changes in the orientation of the pods. Their laughter seems to emerge from their attraction to the movement that is created and perhaps to the fact that one movement is causing another one. Indeed, Joshua's assertion, "I'm controlling it" indicates that he was aware that his movement was causing the pods to change.

The gesture emerges again, but with different quantities and switched roles. This time, Kevin makes the pips and also makes 9 of them. Joshua then makes two pods with his two index fingers. Again, they laugh, perhaps attracted to the effect of the moving pips on the pods, which is slightly different this time since there are 9 pips moving instead of just 2. From this episode, we infer that the movement of the objects on the screen and the cause-effect nature of the moving fingers draws the boys' attention, sustaining their shared gesture for almost a minute. Aside from the statement "I'm controlling it" there were no other verbal exchanges—just negotiated actions. In the next episode, we examine how their emergent sense of how *TT* works is verbalized as a result of the interviewer's intervention.

Second Episode: Attending to the product

Joshua takes the iPad over and make 1 x 7 by holding one pip with his left finger and pressing multiple times on the pip-side of the screen with the other. He then makes 24 = 4 x 6. Kevin asks him to let go of his pod fingers. He does, but also lifts one of his pip fingers, thus making 20 = 4 x 5. Kevin says "oh". Then at 5:22 the interviewer asks the boys what they notice about the numbers. Joshua lets go of his pip-making fingers so they have to start over, this time Joshua makes 35 = 7 x 5. Kevin says at 5:37 "oh I get it so if you have less numbers and you have more numbers over here you get a higher number (pointing to 35) I think". It's difficult to understand his explanation, but he is certainly trying to connect how the number of pips and pods relates to the product. They experiment a little more and then at 5:50 Kevin says "if we have more of these (pointing to the pips) well we can have a higher number". At 5:57, the interviewer says, "I wonder if you can make the number 6". The boys make 9 = 3 x 3 then let go.

Table 1: Attending to the Product

	Voice	Hands	iPad
6:07		J. Places one pip-finger K. Places three pod-fingers simultaneously.	
6:08	K. Wait three this is three	K. points to the pip-side	
6:09	K. And this is three	K. Points to pod side	
6:10	J. That makes nine	J. Adds two more fingers sequentially to the pip-side K. Still pointing to the pods	

Joshua lifts his fingers, making all disappear. The boys laugh. Joshua then immediately presses three pip-fingers and Kevin says "try two". Joshua presses two pod-fingers, which produces two pods of three and the equations 2 x 3 = 6. Joshua says "yes" in an enthusiastic voice and lifts his fingers. The boys have produced their first multiplicative relation using the two-handed gestures required to make pips and pods and attending to the numerical values in the equation. It is not clear why Kevin suggested that Joshua make two pods, but since he had been conjecturing about the size of the product in relation to the number of pips and pods, it could be that he was aware that there needed to be fewer than 3 pods since 3 pods had previously produced 9.

Discussion and Conclusion

In the first episode, it would seem that the rotating four-handed gestures in which one boy moved the pips, was crucial, both affectively and cognitively, for drawing the boys' attention to the relation between the pips and the pods. The effect of moving the pips on the pods drew the boys' attention to the relation between pips and pods and also encouraged them to repeat and vary a particular kind of gesture that requires the coordination of two sets of hands. In terms of multiplicative thinking, this underlies the functional relation but it may also contribute to the sense of the pods as all being equal units. The boys also become aware that the hand that touches the screen first is the hand that controls the shape of the pods. In terms of multiplication, this relates to a recognition that the multiplicand and the multiplier play different roles and that changing the configuration of the pips changes the configuration of each one of the pods. And in the second episode, they attended also to the product and were able to find a solution to the researcher's invitation to produce 6.

We interpret this short but eventful experience as being very promising both in terms of the boys' attraction to the moving pips and pods, which seemed visually interesting and emotionally engaging, and in terms of the experiments they conducted, sometimes on their own initiative and sometimes in response to questions by the researchers.

Acknowledgement

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THE COMPLEX RELATIONSHIP BETWEEN CONCEPTUAL UNDERSTANDING AND PROCEDURAL FLUENCY IN DEVELOPMENTAL ALGEBRA IN COLLEGE

Claire Wladis
BMCC/CUNY
CUNY Graduate Center
cwladis@bmcc.cuny.edu

Jay Verkuilen
CUNY Graduate Center
jverkuilen@gc.cuny.edu

Sydne McCluskey
CUNY Graduate Center
smccluskey@gc.cuny.edu

Kathleen Offenholley
BMCC/CUNY
koffenholley@bmcc.cuny.edu

Dale Dawes
BMCC/CUNY
ddawes@bmcc.cuny.edu

Susan Licwinko
BMCC/CUNY
slicwinko@bmcc.cuny.edu

Jae Ki Lee
BMCC/CUNY
jklee@bmcc.cuny.edu

In this study we use latent class and distractor analysis, and qualitative analysis of cognitive interviews, to investigate how student responses to conceptual items may reflect different patterns of algebraic conceptual understanding and procedural fluency. Our analysis reveals three groups of students, which we label "mostly random guessing", "some procedural fluency with key misconceptions", and "procedural fluency with emergent conceptual understanding". Student responses revealed high rates of misconceptions that stem from misuse or misunderstanding of procedures, whose prevalence correlates with higher levels of procedural fluency.

Keywords: Algebra, Developmental algebra, Conceptual understanding, Concept inventory

Elementary algebra and other developmental courses have consistently been identified as barriers to college persistence and degree progress (see e.g., Bailey, Jeong, & Cho, 2010). There is evidence that students struggle in these courses because they do not understand fundamental algebraic concepts (see e.g., Givvin, Stigler, & Thompson, 2011; Stigler, Givvin, & Thompson, 2010), and many research studies have documented the negative consequences of learning algebraic procedures without any connection to the underlying concepts (see e.g., Hiebert & Grouws, 2007). However, developmental mathematics classes currently focus heavily on recall and procedural skills without integrating reasoning and sense-making (Goldrick-Rab, 2007; Hammerman & Goldberg, 2003). This focus on procedural skills in isolation may increase the probability that students use procedures inappropriately because they lack understanding of when and why the procedures work (e.g., Givvin et al., 2011; Stigler et al., 2010). In this paper we explore student responses to conceptual questions at the end of an elementary algebra course in college. We combine quantitative analysis of responses (using latent class analysis and distractor analysis) with qualitative analysis of cognitive interviews to better understand different typologies of student reasoning around some basic concepts in algebra, and to better understand how conceptual understanding and procedural fluency may relate to one another in this context.

Theoretical Framework

In this paper we use Fishbein's (1994) typology of mathematics as a human activity as a framework for analyzing student responses. Fishbein outlines three basic components of mathematics as a human activity: 1) *the formal component* (which we call *conceptual understanding*), which consists of axioms, definitions, theorems and proofs, which need to be "invented or learned, organized, checked and used actively" by students; 2) *the algorithmic component* (which we call *procedural fluency*), which consists of skills used to solve mathematical problems in specific contexts and stems from algorithmic practice; and 3) *the intuition component*, which is an "apparently" self-evident mathematical statement that is accepted directly with the feeling that no justification is necessary.

In this study we use the term conceptual understanding to denote both a formal understanding of abstract concepts (e.g. axioms), but also of how, when, and why procedures can be used. This is in contrast to procedural fluency in standard problem contexts, in which a student may be able to quickly solve particular types of standard problems correctly but may not understand of how, why, or when these methods work. Using these definitions, no question is wholly conceptual or procedural, but falls on a spectrum. In this paper we explore how procedural skills and conceptual understanding may relate to one another and how student justifications of answer choices may exhibit intuition components (either correct or incorrect), as well as how these intuitions may relate to both the processes of developing procedural fluency as well as conceptual understanding.

Methods

This study focuses on student responses to the multiple choice questions on the Elementary Algebra Concept Inventory (EACI). For details on the development and validation of the EACI, see (Wladis, Offenholley, Licwinko, Dawes, & Lee, 2018). Here we focus on 698 students who took the inventory at the end of their elementary algebra class in 2016-2017 as well as 10 cognitive interviews that were conducted towards the end of the semester with these students; these were analyzed using grounded theory (Glaser & Strauss, 1967), although a full qualitative analysis is not presented here due to space constraints. The distribution of interviewees among the three classes was not significantly different from the whole quantitative sample. In this paper we used latent class analysis (LCA) of the nine binary scored (right/wrong) multiple-choice items on the inventory (e.g., Collins & Lanza, 2010).

Description of the Classes

LCA revealed three distinct classes of students. Item response patterns, distractor analysis, and qualitative coding of cognitive interviews were then used to interpret the classes, and evidence was found among these different complementary approaches for these characterizations:

- C1 (27%): Answers to most items are indistinguishable from random guessing, likely due to low procedural/conceptual knowledge, low self-efficacy, and/or low motivation.
- C2 (28%): Some well-developed procedural skills but limited conceptual understanding.
- C3 (45%): Some well-developed procedural skills and emergent conceptual understanding.

Firstly, we consider the response patterns of students from each of the three classes. Student responses in C1 do not vary much from what would be expected for random guessing on four-option multiple choice items. C2 answers significantly worse than chance on questions 2 and 6 because of the presence of attractive distractors that likely tap into misconceptions related to the misuse of procedures. C2 and C3 are distinguished by improved performance on the items overall but different proportions of key misconceptions. Students who passed the class were most likely to be in class 3, then class 2, and least likely to be in class 1. An end-of-course standardized procedural test showed a similar outcome. To illustrate how different response patterns distinguish these three classes, we performed a distractor analysis and analyzed cognitive interviews for two exemplars: items 2 and 6, using the Bayes modal assignment to determine class membership.

Two Example Questions: Illustrating Different Class Response Patterns

First we consider item 6, which shows an interesting pattern of responses:

6. A student is trying to simplify two different expressions:

i. $(x^2y^3)^2$

ii. $(x^2 + y^3)^2$

Which one of the following steps could the student perform to correctly simplify each expression?

a. For both expressions, the student can distribute the exponent.
b. The student can distribute the exponent in the first expression, but not in the second.
c. The student can distribute the exponent in the second expression, but not in the first.
d. The student cannot distribute the exponent in either expression.

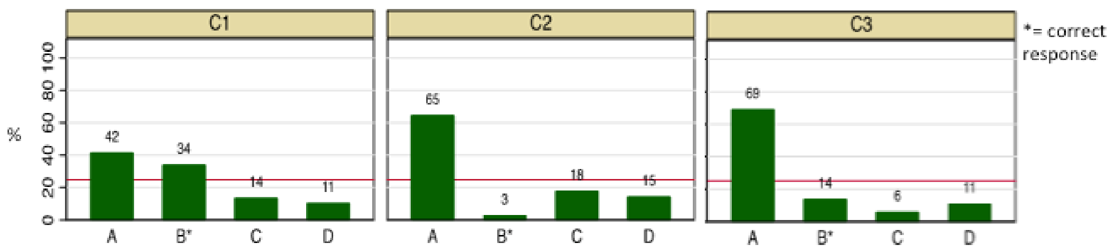

Figure 1: Item 6 Distractor Analysis

The correct answer is b. C2 and C3 were strongly attracted to option a (see Figure 1), likely because they have intuitions stemming from their experiences with procedures that use distributive properties, but they do not recognize the critical differences between distributing multiplication versus exponents—likely because they have no deeper conceptual understanding of how the distributive properties work. Selecting the correct answer is negatively correlated with scores on the procedural exam—students who selected the incorrect option a scored on average 7.1 percentage points higher on the procedural exam ($p < 0.000$) than others. Looking at student interview responses reinforces our interpretation of the three classes, and sheds light on how intuitions developed from procedural practice may impede conceptual understanding.

C1 (chose B): [The difference between the first and second equation] is that there's a plus right there [pointing to the second equation]. I think for this one [pointing to the second equation], you have to add and for this one [pointing to the first equation] you don't….

C3 (chose A): That's how you kind of get rid of the parenthesis and get rid of the outer exponents by distributing it in the inside. Whether it's with another exponent or with a number… You want to add or multiply that exponent [outside the parentheses] to the ones

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inside the parentheses but I can't remember whether you add or multiply…

Here the C1 student notices that there is a difference between the two equations and has an intuition that it is important, but doesn't actually know how to perform the distribution correctly. In contrast, none of the C2 or C3 students interviewed was able to describe when or why it is possible to distribute—they all cited different incorrect intuitions related to procedural methods.

Next we consider item 2, which reveals another interesting pattern of responses:

2. Consider the equation $x + y = 10$. Which of the following statements must be true?

a. There is only one possible solution to this equation, a single point on the line $x + y = 10$.
b. There are an infinite number of possible solutions, all points on the line $x + y = 10$
c. This equation has no solution.
d. There are exactly two possible solutions to this equation: one for x and one for y.

For this question, the correct answer is b, (the most popular choice for students in classes 1 and 3) but no examinee in class 2 chose it (see Figure 2). They were strongly attracted to option d, which was also the second most popular choice for students in other classes, although at a much lower rate. Option d is a common response from students asked to solve a system of linear equations for *x* and *y*, which may explain its popularity. Looking at student interview responses reinforces our interpretation of the three classes, and sheds light on students' reasoning.

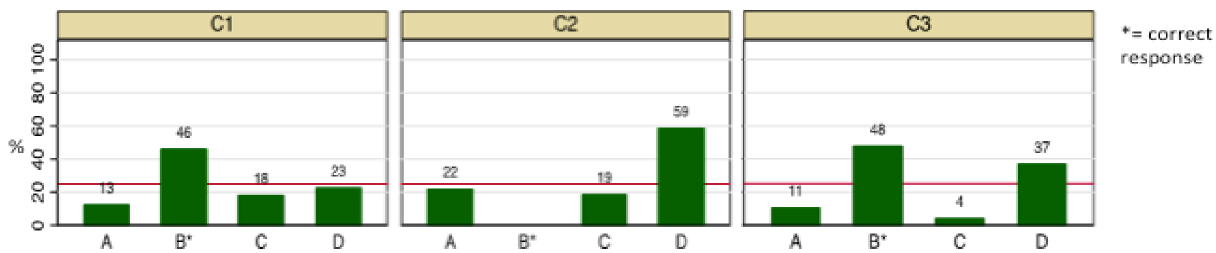

Figure 2: Item 2 Distractor Analysis

C1 (originally chose C, but drifted towards B in the interview): $x + y$ equals nothing so it can't be 10. Right?... It could be possible like it equals 10. [Option D isn't correct] maybe because *x* and *y* could be equal to anything?

C2 (chose D): I know there are certain numbers that will add up to ten, so there could be two solutions, since there's only a *x* term and a *y* term…

C3 (chose B): Ten could equal to many things. Like five plus five could equal ten. Nine plus one could equal ten. Seven plus three…it could be any number that will equal to ten.

The C1 student initially chose "no solution" because they didn't know what *x* and *y* could be, but then they started to relate this to the idea that *x* and *y* could be "anything". While their reasoning is not strictly correct, they are beginning to explore the idea that *x* and *y* may have many possible values, and they show no evidence of faulty intuitions stemming from procedural practice. The C2 student exhibits an intuition about what the equation means to find a single solution, but they do not explore whether there might be others, and they confuse the number of solutions with the number of variables in the solution set, suggesting that their intuitions about the definition of a solution set are incorrect. The student from C3 describes how this equation could have multiple solutions, demonstrating some conceptual understanding of solution sets, including the fact that they describe the relationship between the two variables.

Discussion and Limitations

This study revealed that roughly one quarter of students at the end of the course appeared to guess somewhat randomly on conceptual questions; however, cognitive interviews suggest that these students are able to make some progress towards conceptual understanding by relying initially on more naïve reasoning and that they are not typically hindered by incorrect intuitions stemming from misuse of procedures. About one quarter of students demonstrated some mastery of procedures in standard problem contexts, but demonstrated many misconceptions related to misuse of procedures on conceptual questions. In contrast, roughly half the class showed evidence of emergent conceptual understanding, with lower frequency of misconceptions related to misuse of procedures. For a number conceptual questions, particularly those that were more abstract or non-standard, conceptual understanding and procedural fluency were significantly strongly inversely related. Cognitive interviews revealed that this may happen when students develop incorrect intuitions stemming from the use of procedures.

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POST-SECONDARY STUDENTS' MISCONCEPTIONS ABOUT RATIONAL NUMBERS: IMPLICATIONS FOR K-12 INSTRUCTION

Gena Nelson, Ph.D.
Boise State University
genanelson@boisestate.edu

Sarah R. Powell, Ph.D.
The University of Texas at Austin
srpowell@austin.utexas.edu

Keywords: Post-Secondary Education, Rational Numbers, Problem Solving

A strong understanding of rational numbers, specifically fractions, is a mathematics precursor that predicts whether students experience success in algebra (Booth & Newton, 2012), and success in introductory algebra coursework has often been described as the gatekeeper to upper-level mathematics (Eddy et al., 2015). Having skills to access more advanced mathematics coursework may lead to a greater probability of enrolling in college (Byun, Irvin, & Bell, 2015) or an increase in employment earnings in adulthood (Gaertner, Kim, DesJardins, & McClarty, 2014). Given that understanding rational numbers is important for success with algebra, we aimed to investigate the misconceptions of rational numbers that post-secondary students have.

Very few studies have explored students' misconceptions and errors related to rational numbers, and the majority of studies that included college-age students focused on the narrow population of pre-service teachers as well as a focus on mastery of operations with fractions, with few items examining other processes (e.g., identifying the least common denominator) or other forms of rational numbers (e.g., decimals, percentages). To understand the misconceptions that may persist with rational numbers, we administered a 41-item assessment to 331 undergraduate students and we included decimals, percentages, and fractions. We also examined other skills, such as solving rational number word problems, finding the least common denominator, and converting between forms of rational numbers. We coded students' responses and identified error patterns. Students attempted conceptual understanding and calculations problems more often than word problems, and students made fewer errors with conceptual understanding and calculations items. Students demonstrated the most unique errors with calculations and word problems. Given that all items on the assessment required elementary or middle school knowledge of rational numbers, the number and diversity of errors with a sample of post-secondary students illustrates the difficulties that persist with rational numbers.

With this poster, participants will learn about the different types of errors post-secondary students committed with rational number problems. We also share how the types of errors committed by post-secondary students can inform instruction in college-level developmental courses as well as mathematics instruction throughout the elementary and secondary grades.

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NUMBER SENSE: THE CASE OF SARA

Valerie N. Long
Indiana University of Pennsylvania
vlong@iup.edu

Yuliya Melnikova
Indiana University of Pennsylvania
melnikov@iup.edu

Keywords: Number Concepts and Operations, Assessment and Evaluation

The number competencies that children bring to school set the stage for learning complex mathematics (National Mathematics Advisory Panel, 2008). Too many students have entered kindergarten not ready with the math skills necessary to begin formal schooling (Jordan & Levine, 2009). One skill specifically is number sense with a key component being counting. Research has found that students' mathematical difficulties are related to number sense (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Gersten, Jordan, & Flojo, 2005) and poorly developed counting procedures (Geary, Hamson, & Hoard, 2000; Jordan, Hanich, & Kaplan, 2003). This poster presentation expands on prior research of students struggling in mathematics due to issues related to counting. The purpose of this exploratory case study was to a) determine the cause of low performance on math achievement tests of a third-grade student and b) monitor the change in performance as she engaged in one-on-one instruction during her fourth-grade year.

Methods

Sara was chosen for this study because her math achievement scores revealed little to no progress during her second-grade year and a drastic drop during the third-grade year. Several interventions during her third-grade year had no effect on improvement of achievement scores. Once the cause was determined, a one-on-one instructional classroom was established at the onset of the fourth-grade with a "highly trained teacher" (Bachelor of Science in mathematics and 80+ hours early childhood methods training), where prekindergarten and kindergartner concepts were initially emphasized. Every nine weeks, Sara's progress was evaluated and revised as needed. Achievement testing, number sense testing, teacher feedback, and classroom observations were the primary means of data collection. Data were triangulated using these multiple sources to analyze the cause of low performance at the end of the third-grade year and to monitor progress during the fourth-grade year.

Results

The cause of Sara's low performance on math achievement tests was due to missing her "first lesson in mathematical language, learning how to count, the bedrock of conventional mathematics" (Krasa & Shunkwiler, 2009, p. 68). Specifically, she had difficulty entering the count sequence and counting forward and backwards for numbers less than 40. Additionally, when given 3 two-digit numbers with one missing (e.g. 35, 36), she was slow in sharing her solution because she needed to count from a familiar number. For Sara, it was as if pre-kindergarten and kindergarten never happened. Sara's change in performance during the fourth-grade showed substantial growth. Her achievement test revealed below average scores (standard score 70) at the end of the third grade and average scores at the end of fourth grade (standard score 92). In conclusion, Sara's case highlights the importance of number sense and counting.

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EQUALITY ON A SCALE VS EQUAL SIGN IN A MATHEMATICAL EQUATION

Muhammad Quadeer Haider
Southern Methodist University
Mahider@smu.edu

Lindsey Perry
Southern Methodist University
leperry@smu.edu

Leanne Ketterlin Geller
Southern Methodist University
lkgeller@smu.edu

Eloise Aniag Kuehnert
Southern Methodist University
ekuehnert@smu.edu

Introduction

Plenty of literature has highlighted elementary students' misinterpretation and difficulty in dealing with an equal sign at different grade levels (e.g., Jones, Inglis, Gilmore & Evans, 2013; Sherman & Bisanz, 2009; Stephens et al., 2013). Studies have suggested various tools to develop a relational understanding of equality (e.g., Ellis & Yeh, 2008; Jones & Pratt, 2006). Leavy, Hourigan, and McMahon (2013) reported evidence of a relational understanding of the equal sign, rather than an operational understanding, in elementary students' work when using a physical balance. In this study, we investigate how often elementary students use their relational understanding of equality using a balance to write a balanced equation of the given situation.

Data, Method, and Results

As part of the broader Measuring Early Mathematics Reasoning Skills (MMaRS) project, we conducted 32 cognitive interviews of elementary students (K-3) to confirm the ordering of skills on a learning progression focused on Properties of Operations. The interview protocol was aligned to the learning progression and included items designed to test students' understanding of various representations of equality. In this study, we focused on students' responses to three questions associated with three different skills on the learning progression. First, a picture of three apples on each side of a balance was displayed, and students were asked to write a number sentence to describe the relationship between the apples on the balance. Second, students were asked to write an equation to describe the balanced scale. Lastly, students were asked to identify if an equation, in the form of a = a, was true or not.

The analysis of 32 interviews revealed that there is a disconnection between students' understanding of equality on a pictorial scale and symbolic representation in the form of a=a. Figure 1 displays the number of students who answered correctly with the associated percentage out of 32 students. In 21 (65%) interviews, the interviewer asked student explicitly if the equation 3=3 described the situation on the balanced scale, and only 5 (24%) students correctly associated the equation with the pictorial representation.

First: Pictorial representation of apples on a balanced Scale	Second: Write an equation (a=a) for the balanced scale	Identify a true equation of form a=a shown on a card.
27 (84%)	0 (0%)	12 (37.5%)

Figure 1: Number of Correct Responses for Three Skills Out of 32

These results show a clear disconnection between pictorial and symbolic representations of equality among these elementary students. Students did not demonstrate a relational understanding of the equal sign. Rather, an operational understanding was evident, as 21 out of

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32 (66%) students wrote an equation of format a + b = c to describe the relationship between the apples on either side of the balance.

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STUDENTS' BEHAVIOR IN A DYNAMIC ALGEBRA NOTATION SYSTEM PROVIDES INSIGHT INTO THEIR ALGEBRAIC THINKING

Jenny Yun-Chen Chan
Worcester Polytechnic Institute
jchan2@wpi.edu

Taylyn Hulse
Worcester Polytechnic Institute
trhulse@wpi.edu

Katharine Sawrey
Worcester Polytechnic Institute
kbsawrey@wpi.edu

Erin Ottmar
Worcester Polytechnic Institute
erottmar@wpi.edu

Keywords: Technology, Algebra and Algebraic Thinking, Problem Solving

Algebra is a foundation for higher math, yet many students struggle with algebra notation. The abstract and symbolic nature of algebra contributes to the difficulty students encounter during learning (Greenes, 2008). In web-based Graspable Math (GM), students can explore algebraic notation by moving, combining, and substituting symbols in equations and expressions. In the system, GM completes valid transformations (e.g., turning $2x+5x$ into $7x$ when users tap the addition sign) and provides error feedback by shaking (e.g., the expression $5x\div\frac{1}{2}+2x$ shakes back and forth when users attempt to perform addition before division) and by not enacting invalid transformations. Here, we examine the relations between aspects of behavior in GM as recorded in the click-stream data and their associations with algebraic thinking.

We assessed 39 ninth-graders on their algebra knowledge (e.g., solve the equation for x. $3 = (8\text{-}6x)\div 2$; Star et al., 2016), then provided three 20-minute GM sessions on transforming expressions from a starting state to a goal state (e.g., 6+10 => 2×(3+5)). For each GM problem, we measured time spent, proportion of time to first action (i.e., time to first action / total time, an indicator of pause before acting), number of transformations made, and number of errors (i.e., invalid transformation) made in reaching the goal state; and averaged the values across problems within respective measures for individual students. Spearman correlations revealed that students who spent more time to reach the goal state in a GM problem also made more transformations and more errors during problem solving ($.41 < r\text{s} < .51$). Students who spent more time pausing before acting made fewer transformations to reach the problem goal ($r = -.39$). Only the total time spent was correlated with students' knowledge in algebra. Students with higher knowledge in algebra were faster at completing GM problems ($r = -.36$).

The findings have several implications. First, the moderate correlations between GM measures suggest that they may be distinct indicators of students' algebra skills. Second, pauses may reflect thinking and finding efficient strategies to complete GM problems. Last, aspects of students' behavior in GM may be associated with their algebra performance outside the GM environment. Future studies on the relations between GM measures and the potential mechanisms of pausing (e.g., executive functions, algebra knowledge) may provide insights into students' algebra skills.

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FOSTERING ALGEBRAIC THINKING IN ELEMENTARY SCHOOL: TEACHERS' USE OF VISUALIZATIONS

Ingrid Ristroph
The University of Texas at Austin
ingrid.ristroph@utexas.edu

Hangil Kim
The University of Texas at Austin
hangil_kim@utexas.edu

Karisma Morton
The University of Texas at Austin
karisma.morton@utexas.edu

Eric Knuth
The University of Texas at Austin
eric.knuth@austin.utexas.edu

Ryan Gertenbach
The University of Texas at Austin
rgertenbach@utexas.edu

Bethany Miller
The University of Texas at Austin
b.miller3511@utexas.edu

Research that seeks to identify and detail effective instructional practices is desperately needed in early algebra given the domain's importance to students' future mathematics learning as well as the instructional challenges the domain presents to elementary school teachers (Hohensee, 2015; Katz, 2007; National Mathematics Advisory Panel, 2008). From prior research, we know that activities accompanied with visual representations are often more effective at enhancing students' learning of content knowledge than text-only activities (Kalyuga & Singh, 2015). Elementary teachers who are oriented toward teaching mathematics conceptually are more likely to use visual representations to support students' sense-making as a regular part of their instructional practice (Jansen, Berk, & Meikle, 2017). However, little is known about how teachers use of visualizations might contribute to a more meaningful understanding of early algebraic concepts.

In this poster, we discuss analyses of classroom observations in which Grade 3 teachers taught the same set of early algebra lessons, documenting variations in their use of visual representations. The large-scale nature of this study provides a unique opportunity to identify differences in teachers' use of visual representations and to identify aspects of the practice associated with positive classroom outcomes, as the few studies that have studied instruction are often small-scale or narrowly focused studies (e.g., Jacobs et al., 2007; Webb et al., 2014). The primary questions guiding this study are twofold: *(1) How are teachers' use of visualizations differentiated across the same early algebra lessons? (2) How are the variations of teachers' use of visualizations associated with classroom early algebra learning outcomes?*

To address these questions, we focus on 28 teachers of third grade classrooms in three school districts each within an urban, suburban, and rural area in a southeastern US state. Students were taught the early algebra intervention by their classroom teachers as part of regular mathematics instruction. This study focuses on three separate lessons, each roughly one hour in duration, and that were videotaped. Lesson A addresses generalizations about even and odd numbers and was recorded in 20 classrooms. Lesson B addresses use of variables in algebraic expressions and was also recorded in 20 classrooms. Lesson C addresses identify recursive, covariational, and functional relationships and was recorded in 28 classrooms.

The poster will include several visual representations teachers used and associated classroom outcomes. For example, teachers' use of visual representations that explicitly illustrated uniqueness in a combinatorics problem was associated with a greater likelihood of addressing the conceptual goal of the lesson (e.g., recognizing functional patterns in the data set).

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ENCODING NEGATIVE SIGNS AS SUBTRACTION SIGNS: CASES OF SECOND AND FIFTH GRADERS

Laura Bofferding
Purdue University
LBofferd@purdue.edu

Mahtob Aqazade
Purdue University
maqazade@purdue.edu

Lizhen Chen
Purdue University
lizchen@purdue.edu

Keywords: Number Concepts and Operations; Cognition; Elementary School Education

Theoretical Background and Methods

The perceptual patterns that students encode when they learn whole number addition and subtraction are challenged when they encounter similar problems with negative numbers. Students may not encode negative signs when they first encounter them (e.g., Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2018; Bofferding, 2014). Based on a framework theory for conceptual change (Vosniadou, Vamvakoussi, & Skopeliti, 2008), other students might encode these new perceptual features (i.e., negative signs) but interpret them as familiar symbols, in this case subtraction signs (Bofferding, 2019; Murray, 1985; Vlassis, 2004, 2008). Students may only encode those features they deem necessary for solving a problem (McNeil & Alibali, 2004), but seeing problems and symbols in contrast can also bring new perceptual features to students' attention (Schwartz, Tsang, & Blair, 2016). Therefore, we investigate second and fifth graders' encoding of the negative sign when it appears adjacent to an operational sign, as in adding a negative (+ -) or subtracting a negative (– -), over a series of arithmetic problems. Our research question is when do students treat adjacent signs as two operations (i.e., add then subtract or subtract twice) and how and when does this change over a series of integer arithmetic problems?

We interviewed 102 second graders and 102 fifth graders on a series of 11 integer addition problems (e.g., -9 + 2, 3 + -3, -1 + -7) followed by a series of 16 integer subtraction problems (e.g., 1 – 4, 3 – -1, -7 – 3, -8 – -8) on a pretest and posttest before and after an intervention where students analyzed contrasting cases of integer arithmetic worked examples. We first identified cases where students treated negative signs as subtraction signs when adjacent to addition or subtraction signs. We then analyzed students' strategies on the other problems to identify when and how their use of the signs changed. We highlight these changes through a few cases.

Results and Discussion

In one case, a fifth grader (4.U08) treated negatives as subtraction signs, solving -9 + 2 by subtracting two from nine. The same student solved 3 + -3 by "adding three plus three…and then minus three." However, when the student solved 3 – -1, the fifth grader just subtracted one from three. Perhaps the student needed the adjacent signs to be perceptually different to do both. Another fifth grader (4.Z06) first interpreted an initial negative as nothing for -9 + 2: "Because that [-9] is not a number, like it is, but it's negative, so you just add two more, and that will be 2." For the next problem where the negative was second, 3 + -3, he did three plus three minus three but then later treated all negatives as worth zero. At an extreme, a second grader (4.G04), treated all minus signs as subtraction. If a negative number was first, the student treated it as subtracted from zero; for adjacent signs this student used the first sign with the first number and the second sign with the second number. Therefore, for 4 + -6, the student did 4 + 4 – 6 = 2.

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LANGUAGE AND NUMBER: STUDENTS' INTERPRETATION OF "LESS LOW"

Laura Bofferding	Mahtob Aqazade	Maya Cameron
Purdue University	Purdue University	Purdue University
LBofferd@purdue.edu	maqazade@purdue.edu	mcamero@purdue.edu

Keywords: Cognition; Number Concepts and Operations; Elementary School Education

When making comparisons with objects, young children have an easier time interpreting positive terms (e.g., higher, more, or hotter) as opposed to negative terms (e.g., lower, less, or colder) (Smith, Rattermann, & Sera, 1988). However, when asked to compare positive and negative numbers, children will often rely on absolute value (Bofferding, 2014; Bofferding & Farmer, 2019). Yet, when determining which temperature was *least cold*, a compound term made up with of two negative terms, second and fifth graders were better able to respond with all negative number choices (e.g., -8, -9, -1) as opposed to comparing positive temperatures (e.g., 2, 4, 7) (Bofferding & Farmer, 2019). We explored second and fifth graders' interpretations of the comparison language "less low" with and without the support of positive and negative numbers before and after an intervention focused on interpreting language such as "less negative." We examined this research question: when making comparisons about the relative positions of two cats on a staircase, how do second and fifth graders interpret the term "less low," (a) without number labels on the staircase? (b) with number labels on the staircase?

Participants included 102 second and 102 fifth graders from two public elementary schools who took a pretest, engaged in an intervention focused on language and integer operations, and completed a posttest. We focus on problems involving pictures of two cats on a staircase. The cats were placed on different steps in each problem (both low, both high, or split high and low). Students identified which cat was less low in each picture where three pictures had unlabeled stairs and three had positive and negative number labels under the stairs (as in a number line). We analyzed students' correctness, explored their pattern of responses, and identified their most common language when describing their choice and meaning given to "less low."

We found that both second and fifth graders' correct responses increased from pretest to posttest with second graders gaining slightly higher than fifth graders on problems without numbers and fifth graders gaining slightly higher on problems with numbers. Students' typical response patterns included: always choosing the higher cat; always choosing the lower cat (regardless of the cat's location or having numbers on stairs or not); some students chose the higher cat in problems without numbers and chose the lower cat in problems with numbers; and some students always chose the lower cat except for cats positioned at numbers -5 and -2. For instance, a second grader (4.C10) on the pretest problems *without numbers* said, "Because this one [lower] is less." On pretest problems *with numbers* he explained, "Because this one [4] is four and this one [-2] is 2, so this one down here [circles cat at -2]." Choosing the cat located as less low on -2 and -5, he said, "Because this one [-2] is 2 and this one [-5] is more higher than 2 [circles cat at -2]." Students' most common incorrect language explaining what less low means were "really low," "reaches to the bottom," "lowest to the floor farther down," "lower," "under the highest," "less," or "first." Equivalent correct terms for less low involved "from up to down," "would be less down," "high," "a small negative (with two negatives)," or "not low."

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Chapter 4:
Equity and Justice

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TEACHING WITH RACE IN MIND: EXPLORING THE WORK OF ANTIRACISM IN LEADING WHOLE-CLASS MATHEMATICS DISCUSSIONS

Rosalie DeFino
University of Michigan
rdefino@umich.edu

How might teachers of elementary mathematics pursue antiracism through everyday practices such as leading whole-class discussions? This paper reports on an exploratory study of one White woman teacher's efforts to challenge manifestations of structural racism in classroom interactions with students who are predominantly Black. The results include preliminary characterization of three areas of antiracist work: planned and routine practices, in-the-moment responses to student behavior, and ongoing interpretation of classroom events. Implications for research on mathematics teaching and for mathematics teacher education are discussed.

Keywords: Elementary School Education, Social Justice, Instructional Activities and Practices

Recent scholarship on the pursuit of equity in mathematics education has emphasized the importance of teacher and student identity, including racial identity, in supporting the learning of students from historically and currently marginalized groups (Aguirre, Mayfield-Ingram, & Martin, 2013; Bartell et al., 2017; de Freitas, 2008; Varelas, Martin, & Kane, 2012). Additionally, critical scholars emphasize the structural nature of educational inequities and call for broader interrogation of systems of power and social injustices (Gutiérrez, 2013; Gutstein, 2012; Martin, 2015). Mathematics educators often take up questions of identity and power and of race and racism through curricular innovations, developing mathematical investigations of sociopolitical issues like gentrification and housing displacement (Gutstein, 2012). However, there has been less attention paid to how mathematics teachers might teach with race in mind in their everyday practice, especially while working on conventional mathematics topics in elementary classrooms.

In this paper, I report on an exploratory study of one teacher's efforts to work against structural racism in her daily practice of leading whole-class mathematics discussions. I present preliminary findings pertaining to the following research question: *What does the teacher do in the context of day-to-day elementary mathematics teaching that could be considered antiracist work?* The purpose of the study is not to determine the effectiveness of the teacher's antiracist efforts nor to declare universal "best" practices. Instead, the objective is to describe and make sense of this teacher's work as she pursues antiracism in order to raise questions and considerations for the fields of mathematics education and mathematics teacher education.

Theoretical Perspectives

This project draws primarily on three theoretical perspectives: critical race theory, conceptions of teaching as complex and situated interaction, and positioning theory. I draw on critical race theory and particular conceptions of teaching to articulate foundational assumptions that undergird the project as a whole. I apply concepts from positioning theory more directly as tools in my analysis of empirical data.

Premises for Studying Race and Racism from Critical Race Theory

My attention to race and racism is driven and informed by several core concepts in critical race theory (CRT). CRT emerged in the 1970s out of several legal scholars' dissatisfaction with traditional approaches to addressing racial inequality. In the 1990s, Ladson-Billings and Tate introduced CRT to the field of education (Ladson-Billings, 1999; Leonardo, 2013). CRT scholars understand race as a social construct that has real, material consequences (Ladson-Billings, 1999). Moreover, a central idea in CRT is that racism is endemic to U.S. society, so normalized that its presence and effects often go unrecognized, especially by White people (Leonardo, 2013; López, 2003). Given this, CRT helps make visible that racism is at work in familiar contexts and activities, such as a class discussion about a math problem. In addition, CRT scholars understand racism as a system of oppression that creates and maintains a racial structure, *not* just "willful act[s] of aggression against a person based on their skin color" (Parker & Lynn, 2002, p. 9). Thus, from a CRT perspective, being an antiracist teacher must mean more than calling out or avoiding blatant personal discrimination in the classroom; instead, antiracist teaching must strive toward dismantling racist ideas and systems that structure children's lives.

Conceptualizing Teaching: Complex Interaction in Social and Historical Context

A conception of teaching as interactive and situated work is foundational to this study. I adopt Cohen, Raudenbush, and Ball's (2003) instructional triangle, which represents teaching as mutually-influencing interactions among teachers, students, content, and environments. This conception has several important implications for my analysis of antiracist efforts in mathematics teaching. First, it suggests that teaching is not unilaterally determined by any single actor or influence; teachers' intentions depend on students and the environment to be realized. Second, teaching is a complex activity that can have multiple and varied effects upon participants. For instance, there is no guarantee that all students will similarly interpret or experience a teacher's words and actions. Finally, the instructional triangle implies that environments, including endemic racism, enter into teaching and learning in several ways. For one, the institution of schooling brings a long history of social norms and expectations (Tye, 2000). Drawing again on CRT, the social and historical context of schooling includes nominally colorblind policies that disproportionately impact people of color, as well as assumptions of Whiteness as the norm (Bonilla-Silva, 2015; Yoon, 2012). Thus, by highlighting interactions between environments and teachers, students, and content, the instructional triangle model can support serious consideration of social and historical contexts in analyses of teaching and learning.

Positioning: A Conceptual Tool for Bridging the Macro and the Micro

Given CRT's emphasis on the structural nature of racism, applications of CRT in education often focus on school- and district-level issues like tracking, discipline, and funding (e.g., Blaisdell, 2016; Chapman, 2013). To bring consideration of race and racism to the level of classroom interactions, I draw on the concept of *positioning.* As conceived by Davies and Harré (1990), positioning, in part, encompasses the ways that speakers metaphorically locate each other in conversations and jointly construct *storylines*. In a mathematics classroom setting, for example, students may be positioned by themselves and others as more or less mathematically competent. Scholars of mathematics education have used the concept of positioning to examine both micro-level moment-to-moment interactions (e.g., Wood, 2013) and patterns of interaction over time (e.g., Langer-Osuna, 2011). Such studies suggest that teachers' practices have the potential to affirm or challenge storylines of students' mathematical competence. Furthermore, scholars have linked storylines being constructed in classrooms to broader societal discourses and stereotypes. For example, Shah and Leonardo (2016), Nasir et al. (2009), and Martin (2006) demonstrate that students in mathematics classrooms actively navigate racialized discourses and stereotypes about mathematical ability based on race. Such discourses and stereotypes function as possible storylines that teachers and students may contribute to or resist through classroom interaction (cf. Davies & Harré, 1990). In terms of antiracism, I explore the question of whether the storylines the teacher contributes to reflect or challenge broader racist narratives and patterns of mathematics learning experiences. Thus, I use the concepts of storylines and positioning to begin bridging the issues and identities at play in micro-level classroom interactions to macro-level narratives and patterns.

Research Questions

The purpose of this inquiry is to offer a description and exploratory analysis of one teacher's efforts to counter racist narratives and patterns of experience through her teaching of elementary mathematics. In a larger study of a teacher in a two-week summer mathematics class for rising fifth graders, I pursue three research questions:

1. How does the teacher describe her efforts to combat racism?
2. What does the teacher do in the context of day-to-day elementary mathematics teaching that could be considered antiracist work (as defined below)?
3. What tensions and challenges arise in the teacher's efforts to combat racism?

For this paper, I focus on the second research question. I use the following working definition of *antiracist work* in teaching: deliberate efforts to position students in ways that counteract racist ideas (e.g., discourses about the intellectual deficiency of Black children described by Martin, 2009) as well as efforts to disrupt racialized patterns in students' learning experiences, such as having limited access to challenging mathematics instruction (Battey & Leyva, 2018). This is not meant to be an exhaustive or exclusive definition of antiracist work, and I recognize that acting on antiracist intentions in no way guarantees that a teacher's practice is not harmful or problematic. With these limitations in mind, I use this working definition as an entry point for describing and making sense of the focal teacher's practice. In the following section, I describe the research context and methods.

Methods

This project is an interpretive qualitative study. Therefore, the perspectives I bring as a researcher necessarily inform my selection of data, analytical process, and conclusions (Hesse-Biber & Leavy, 2011; Ladson-Billings, 2000). I share my approach and rationale below.

Context and Participants

This study focuses on a two-week summer mathematics class held at a large public university in 2017 for rising fifth grade students. The mathematics class was framed as a laboratory for the study of teaching and included opportunities for adult observers to attend daily pre-brief sessions with the classroom teacher, in-person or remote viewing of the math class, and debrief sessions. All pre-brief, math class, and debrief sessions were video-recorded. Students were recruited by the program director from a neighboring school district. There were 20 students, 10 boys and 10 girls. There were 16 students of color, most of whom were Black, and 4 White students. The math class lasted about 2 hours a day and included a 15-minute break. The teacher of the math class is the focus of this study. The teacher is a White woman with extensive experience teaching elementary mathematics. She is also a prominent scholar of education who has publicly expressed her commitment to challenging racism and other forms of oppression through everyday teaching practices; I chose to study this context because of that commitment.

Data Sources

Data for this analysis were drawn from a larger collection, including video records of the 2017 mathematics class from two cameras capturing different angles of activity, daily lesson plans, and scanned classroom artifacts, such as student notebooks. I attended the 2017 program as a participant observer in a remote viewing room. I took daily field notes including descriptions of events as well as my questions and reactions for the pre-brief, math class, and debrief. I also wrote daily reflective memos. In addition, several months after the summer program, I conducted two 30-minute semi-structured interviews with the teacher about her approach to challenging racism through teaching. Both interviews were audio-recorded and transcribed.

Analytical Approach

Data reduction. I focus my analysis on whole-class discussions because that was the predominant activity structure for Days 1 through 6 of the 2017 class. I was particularly interested in discussions that surfaced tensions and complexities in the teacher's efforts to enact her antiracist commitments. To locate video segments of whole-class discussions and begin identifying particular moments of interest, I watched videos of the class in chronological order and created content logs. I selected one segment of whole-class discussion from each day of instruction, using shifts in classroom activities to bound each episode (Jordan & Henderson, 1995). I transcribed six classroom episodes, parsing discourse into one *message unit* per transcript line (Bloome, Carter, Christian, Otto, & Shuart-Faris, 2010). As defined by Bloome et al. (2010), a message unit is "the smallest unit of conversational meaning" (p. 19). Within a given talk turn, a speaker might impart multiple messages (e.g., thanking a student for contributing and then posing a question to the class). I separated such message units when transcribing to enable more nuanced coding.

Coding. Given that this paper focuses on findings related to my second research question, here I detail my process for coding classroom transcripts. I used *process coding*, or descriptive codes in the form of gerunds (Saldaña, 2016), to characterize the teacher's moves and practices in classroom interactions. I generated one process code for each line of classroom transcript. I also selectively used *versus codes*, such as "addressing distractions vs. maintaining momentum in discussion," to flag tensions and conflicts within the teacher's efforts (Saldaña, 2016). While coding, I iteratively consulted literature on teaching to inform my interpretations (e.g., Ball & Wilson, 1996; Hammerness & Kennedy, 2018; Noblit, 1993), considered insights from interviews with the teacher, and drafted analytic memos (Hesse-Biber & Leavy, 2011; Saldaña, 2016). I sorted all process and versus codes into categories to notice patterns and incongruities (Saldaña, 2016). Finally, I re-watched videos of the six classroom episodes and reread my field notes, reflective memos, and video content logs to recall relevant context, refine analytic claims, and bolster reliability (Cho & Trent, 2006).

Researcher Positionality. Recognizing that my positionality and subjectivity as a researcher inevitably shape the meanings I construct, I aim to transparently name my stances. I am a White woman and former elementary teacher who struggled to conceptualize and enact mathematics teaching that simultaneously supported student learning and challenged racism. Moreover, as a mathematics teacher educator, I am committed to describing and analyzing the complex work of teaching in ways that are pragmatically useful for supporting novice teachers to take on antiracist commitments. I am particularly invested in unpacking the tensions and challenges of antiracist work for White women, as White women make up the overwhelming majority of preservice teachers I work with and over 80% of the U.S. educator workforce is White (U.S. Department of Education, 2016). In addition, I recognize the problematic tropes and histories of well-meaning White women adopting a "savior" mentality (Matias, 2013) as well as liberal, colorblind frames in teaching children of color (Brown & Reed, 2017; Warren & Talley, 2017). To complicate and unsettle my own assumptions and inclinations as a White woman, I deliberately looked to perspectives of scholars of color. I also sought out alternate interpretations of data through discussion with critical colleagues (Jordan & Henderson, 1995).

Summary of Preliminary Results

What did the focal teacher do in the context of leading whole-class mathematics discussions that could be considered antiracist work? Based on my analysis of classroom episodes, I argue that the teacher in question engaged in antiracist work in three domains: (a) planned and routine moves and practices; (b) in-the-moment responses to student behavior; and (c) ongoing interpretation of classroom events. These results are preliminary in that they reflect initial, high-level analytic claims that will be further refined and evidenced when presented with specific classroom episodes. Given the brevity of this report, I summarize categories of the teacher's efforts rather than provide detailed illustrations of particular moves in practice. Brief examples and explanations are included to illustrate the kind of classroom interactions that led to these categorizations. I now turn to describing each domain.

Planned and Routine Work: Broadening Participation and Competence in Mathematics

One area of the teacher's work that challenged racist ideas and structures consisted of the pre-emptive, deliberate moves the teacher made to shape the learning environment and create spaces for students to engage productively with each other and the mathematics. This domain reflects the goals and considerations the teacher put forward in lesson plans, shared with adult observers during pre-brief sessions, and described in interviews. Practices in this domain include efforts to build relationships with students, such as taking care to learn and use students' preferred names, and efforts to depart from seemingly-arbitrary school rules, as with discussions of student and teacher contracts. This domain also includes the teacher's efforts to expand notions of what constitutes mathematics and mathematical smartness and to support broad participation in mathematical discourse. The teacher shared in interviews that she tries to challenge the idea that "Black children don't make mathematics" and to disrupt patterns of "experience[s] that I'm imagining that they may have had with White teachers," including not being seen as smart (Interview 2). Thus, the teacher's proactive practices are premised upon recognizing and deliberately countering racist stereotypes that link children's race with their mathematical ability and perceived smartness.

The teacher engaged in numerous practices that reflect her efforts to broaden the meaning of mathematics and mathematical smartness for her students. For example, the teacher used challenging mathematical tasks with multiple entry points and varying solution spaces, including problems with infinitely many solutions and problems with no solutions. She also emphasized constructing sound mathematical explanations over finding correct answers and invested considerable time in scaffolding and supporting mathematical explanations from students who might have otherwise been positioned by their peers as less mathematically capable. The teacher consistently positioned students of color as making important mathematical contributions, saying things like, "Can you say that again? What you said was really important and I want to make sure everyone hears it." The teacher also highlighted students' agency in deciding what made mathematical sense and how they might participate. She routinely offered multiple ways for students to participate in whole-class discussions, including the options of reading problems, interpreting tasks, posing a new problem, calling on peers, presenting solutions, explaining solutions, and restating ideas. Following my working definition of antiracist work, the teacher's planned and routine practices reflect attempts to counter racist stereotypes and depart from entrenched racial patterns in school mathematics.

Responding to Tensions and Challenges in the Moment

The previous domain consisted of things the teacher could plan for and intentionally construct over time in the classroom space. I now turn to more immediate and unanticipated interactions — how the teacher responded in the moment when student behavior posed challenges and raised tensions. Drawing from my process and versus codes, I separated tensions and challenges into two broad categories: (a) situations involving minor disruptions which the teacher generally handled using commonplace redirection strategies, and (b) situations involving more persistent or egregious disruptions that occasioned novel responses from the teacher. For both categories, I consider the teacher's responses a space for antiracist work because of the well-documented pattern of disproportionate discipline of children of color (Darby & Rury, 2018; Girvan, Gion, McIntosh, & Smolkowski, 2017; Skiba, Michael, Nardo, & Peterson, 2002; Smolkowski, Girvan, McIntosh, Nese, & Horner, 2016). Teacher-student interactions around behavior and discipline carry weighty implications for questions of racial bias and the perpetuation of racial inequalities. Through her momentary responses to challenges, the teacher had the potential to feed into or actively counter negative stereotypes of children of color as being out of control (Milner, Cunningham, Delale-O'Connor, & Kestenberg, 2019) or incapable of discussion-based, challenging mathematical work (Battey & Leyva, 2018).

Routine redirection. When students engaged in commonplace, minorly distracting behaviors like talking with peers, laughing, or commenting out of turn, the teacher responded using strategies suggestive of student-centered and culturally responsive classroom management approaches. For example, the teacher often used physical proximity to students, whole-class reminders to look at the speaker or to listen carefully, and mathematical questions to refocus students on the discussion. These strategies focus on what students should do, operating on the assumption that students want to participate successfully, rather than taking an accusatory or punitive tone. This is consistent with the student-centered *Responsive Classroom* approach to classroom management (e.g., Charney, 2002). Such redirection strategies avoid publicly reprimanding or punishing students for misbehavior, thereby protecting children's dignity (Darby & Rury, 2018). In addition, the teacher's comments in debrief sessions revealed that she consciously chose not to address behaviors if they did not interfere with other students' learning and were solely matters of teacher preference. Choices to ignore certain student behaviors, such as wearing a hood in math class, evoke the culturally responsive classroom management stance that teachers' behavior preferences reflect specific racial and cultural assumptions (Weinstein, Curran, & Tomlinson-Clarke, 2003). Thus, the teacher worked to prevent manifestations of racial bias through her routine responses to minor challenges.

More pressing challenges and complex tensions. Aside from commonplace distractions and minor disruptions, the teacher had to navigate several more complex challenges while orchestrating class discussions. For instance, at times students' side commentary positioned the student who was presenting as lacking mathematical competence, which may have caused emotional harm. Other times, the level of background noise was persistent and distracting enough to make a coherent mathematics discussion untenable. In each of these situations, the teacher's routine redirection strategies were insufficient for returning to or bringing about a productive mathematical discussion. Given that these are the sort of behavior issues that could easily escalate and contribute to racist patterns of disproportionate discipline, I think the teacher's ways of responding constitute a particularly important space for antiracist work.

One of the teacher's responses to these non-routine challenges was giving the class an impromptu task so she could speak privately with an individual student. This approach again protected the dignity of the student being addressed (Darby & Rury, 2018). Another two teacher responses were having students write a "note to self" about goals related to the student contract and making meta-comments about how the class discussion was going. When the teacher responded in these ways, she recognized students' positive contributions, such as doing "amazing thinking," alongside things students needed to work on, such as listening carefully to each other. While the teacher's responses sometimes had the potential to position individual children as troublemakers, the teacher's commentary in debriefs indicated that she was determined not to conflate students' actions in a particular moment with their mathematical abilities, effort, or overall character; she saw each interaction as just one window into students as growing, multidimensional people. This suggests a commitment to countering deficit perspectives about students of color throughout tense, challenging classroom situations.

Ongoing Interpretive Work

A third category of antiracist work was the teacher's ongoing interpretation of classroom events. As scholars of reflective teaching (e.g., Zeichner & Liston, 2014) have documented, teachers often engage in analysis and sensemaking of how lessons and activities unfold, including framing and reframing problems. Given that the focal teacher worked in a context with many adult observers, she routinely made public many of her reflections and analyses. I consider the teacher's ongoing interpretation of classroom events a space for antiracist work because of the potential for deficit framing to bring about damaging patterns in teaching and learning. Teachers' framing of problems such as why students are not volunteering to participate in a discussion, delimits the range of pedagogical solutions they consider and can have profound implications for students' learning opportunities (Horn, 2007). Thus, I argue that framing and explaining classroom events is a space where the teacher engages in antiracist work at an ideological level, but with material implications for student learning and identity development.

The teacher's navigation of racial ideologies in relation to classroom events was particularly evident in her comments during pre-brief and debrief sessions. For instance, when adult observers made suggestions or posed questions that called attention to gaps in students' mathematical understanding or concerns about student behavior, the teacher actively resisted characterizing students in deficit terms. Instead, the teacher regularly pointed to students' mathematical strengths and offered explanations for student behavior that were external to the students themselves. Specifically, the teacher spoke about conflicts between students and disruptions to whole-class discussions as instances where "the social channel" was temporarily louder than the mathematics. The teacher saw students' behavior as understandable given their prior relationships with each other, and she strategized about how to restructure class activities to tap into students' genuine interest and strengths in mathematics. The possibility that the problem was inherent to the students themselves was never entertained. This interpretation of classroom challenges notably departs from deficit ideologies that would frame children of color as lacking in intellectual ability or moral character (Darby & Rury, 2018) and suggests that part of the work of antiracism in mathematics teaching is leveraging an unyielding commitment to seeing and building on the strengths of students of color.

Discussion and Conclusions

This preliminary characterization of one teacher's antiracist efforts while leading whole-class mathematics discussions raises several issues for research on antiracism and mathematics teaching practices. One question is how to think about implications of the teacher's positionality in doing antiracist work. In other words, how does it matter that in this case the teacher is a White woman and that most of the students are Black? This is a question that many texts on equitable mathematics teaching or student-centered classroom management fail to tackle (cf. Boaler, 2016; Charney, 2002; Featherstone et al., 2011; Smith & Stein, 2018). Yet, the identities of the teacher and students invoke particular narratives, stereotypes, and societal injustices that complicate interpretations of classroom interactions, especially with regard to uses of power and authority. For example, there are widely recognized patterns of White women teachers being overly punitive and controlling of Black children's bodies (Battey & Leyva, 2018; Brown & Reed, 2017; Epstein, Blake, & González, 2017; Ferguson, 2001; Smolkowski et al., 2016). If the teacher being studied here were Black rather than White, that particular macro-level pattern would not be invoked in the same way. At the same time, stereotypes that are rooted in narrow, Eurocentric conceptions of mathematics and intelligence (Davis & Martin, 2008) could be reinforced by teachers of any race or gender. In interviews, the focal teacher explicitly described making efforts not to fall into the same patterns that many White women teachers perpetuate. Thus, the teacher's awareness of the relevance of broader patterns of injustice to the identities of herself and her students seemed to inform the particular ways she exercised power and authority in her mathematics teaching. That is, the teacher's identity mattered in her navigation of the line between oppressive and non-oppressive uses of power. Future research could explore how the tensions of antiracist mathematics teaching shift or persist in relation to teachers' identities.

In addition, considering various conceptions of power suggests different ways of navigating tensions stemming from social identities and societal patterns. Bloome et al. (2010) describe three perspectives on power: power as a product, power as a process, and power as caring relations. If exercising power includes centering care for students' well-being, then a teacher might pursue antiracist uses of power through caring. Yet, as Thompson (1998) and Rolón-Dow (2005) demonstrate, notions of care also stem from racialized assumptions and perspectives, namely, White feminist interpretations of care are often colorblind and emphasize niceness, varying significantly from Black feminist interpretations which highlight honesty and authenticity in making sense of racial realities. Thus, there is tension in figuring out how to exercise power and demonstrate care for children of color as a White woman in ways that do not reinforce Whiteness and perpetuate racism. Therefore, teachers' interrogation of their own social identities and assumptions seems central to navigating inherent tensions in pursuit of antiracism.

This exploratory study of one teacher's efforts to pursue antiracism through leading whole-class mathematics discussions suggests that there is great complexity to analyzing race and racism in everyday classroom interactions. Existing literature on mathematics teaching and on antiracism offers little support for teachers in recognizing and navigating these complexities and tensions at the level of day-to-day classroom practice. My findings suggest that additional work is needed to identify and analyze specific mathematics teaching moves and practices in relation to broader antiracist projects. Research that explores challenges that novice elementary teachers face in taking up antiracism, as well as how teaching considerations change in light of different teacher and student identities could inform justice-oriented, race-conscious approaches to mathematics teacher education. To avoid the pitfall of turning studies of teaching practice into superficial prescriptive checklists (Philip et al., 2018), researchers must continually attend to complexity and tensions stemming from identity and power structures. Furthermore, mathematics teacher educators must endeavor to support teachers in the critical analytical and interpretive work of considering how everyday teaching practices relate to macro-level narratives and patterns. The work of antiracism in leading whole-class mathematics discussions and teaching elementary mathematics is not simple; it is ongoing, socially-situated work.

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THE MATH AGENCY OF THREE EMERGENT BILINGUALS WITH IDENTIFIED LEARNING DISABILITIES

Juanita Silva
Texas State University
jsilva@txstate.edu

I present study findings that depict three Latinx emergent bilinguals' mathematical agency with identified learning disabilities in 12 discussions centered on the framework of children's mathematical thinking. I utilized qualitative analysis methods to illustrate how they exhibited agency as the power to express their mathematical thinking and make sense of story problems. I argue that when children are given opportunities to voice their thinking in any language and choose how to solve word problems, they are capable of understanding the content and are able to justify their thinking. Implications are offered for the math instruction of emergent bilingual children identified with a learning disability.

Keywords: Emergent Bilinguals, Latinx, Equity

The term "emergent bilingual" (Garcia & Kleifgen, 2010) identifies children as having the potential to become bilingual over time while still utilizing their native language. Unfortunately, the mode of instruction for these children often focuses on memorizing English vocabulary or mathematical procedures (Moschkovich, 2015). Even more problematic is that Latinx emergent bilinguals with identified labels, such as a learning disability (LD) or math difficulties, are often exposed to instruction that focuses on direct explicit instruction (e.g. Brosvic, Dihoff, Epstein, & Cook, 2006). Arguably, these methods of instruction reduce student learning and create barriers that deny them access to mathematical practices (e.g., Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993) and rich discourse (e.g. Moschkovich, 1999, 2015) that can promote mathematical understandings (e.g. Hunt, Welch-Ptack, & Silva, 2016).

This work is an alternative to reductionist instructional frameworks, positioning instruction as a vehicle to promote access for Latinx emergent bilinguals to mathematical practices utilizing their strengths (e.g. use of prior knowledge and native language) such that they promote mathematical understanding (Quintos, Civil, & Torres, 2011), bolstered achievement (e.g., Webb, Franke, Ing, Wong, Fernandez, Shin, & Turrou, 2014), and increase opportunities to enact mathematical agency (e.g., Turner, Dominguez, Maldonado, & Empson, 2013). I examine the following research question: How do three Latino/a emerging bilingual children labeled with learning disabilities and/or difficulties develop mathematical agency when exposed to mathematical practices?

Conceptual Framework: Promoting Agency for a Marginalized Population

This research is grounded in supporting and extending ways in which children from underrepresented populations can enact agency. I define *agency* as the capacity for children to make their own choices and to enact choices on the world (Holland, Skinner, Lachiotte, & Cain, 1998). Research on agency with Latinx children is of particular interest to many scholars (Gutstein, 2003; Sanchez-Suzuki Colegrove, & Adair, 2014; Turner, 2003). For example, Sanchez-Suzuki Colegrove and Adair (2014) aimed to defy deficit thinking by showing the possibilities of minority Latinx children to do mathematics when given choice. Relatedly, Turner and colleagues (2013) identified positioning moves that allowed these children to build agentic identities through participating in problem solving discussions designed to promote opportunities to share thinking.

In this study, I borrow from Turner's (2003) definition of *critical math agency* which she defines as children's capacity to "(a) understand mathematics, (b) identify themselves as powerful mathematical thinkers, and (c) construct and use mathematics in personally and socially meaningful ways" (Turner, 2003, p. 48). From Turner (2003), I take up children's capacity to "understand" and "see themselves as powerful math thinkers". Thus, I define *mathematical agency* as children's *power* to (a) *make sense* of mathematics and (b) *take ownership* of their mathematical thinking. I define making sense of mathematics as when children are able to understand their solutions and connect those solutions to the word problems they are solving. The term "power to take ownership of mathematical thinking" is defined as children's ability take action on their mathematical thinking.

One way to foster *mathematical agency* is to provide opportunities to participate in mathematical practices that allow choice and the use of prior knowledge (Jacobs & Empson, 2016; Turner et al., 2013). In this study, mathematical practices are defined as those that allow the engagement of children's (a) participation in problem solving situations, (b) invitation to discuss mathematical thinking with others in any language and (c) choice to justify and elaborate on thinking. Over the past several decades, these practices have demonstrated the potential to enhance children's learning and achievement (Webb et al, 2014). Furthermore, in a study conducted by Chval and Pinnow (2018), the teacher's positioning of children's first language as a resource and the sharing of children's thinking allowed students to thrive in building mathematical knowledge. Solving math problems in ways that make sense and discussing mathematical ideas in any language with others were important in developing agency, as are the social learning environments where children can critique the mathematical reasoning of others.

Methods and Data Sources

The case study involved three Latino/a emerging bilingual children, Julia, Martin and Gabriel, in grades three and four from an elementary school located in an urban city in the southern United States. Participants were purposively selected based on the following criteria: (a) Latino/a children who were identified as ELLs, and (b) children who had identified math difficulties (e.g. Tier 3 identification under Response to Intervention model (Fuchs, Fuchs, & Hollenbeck, 2007) or cognitively defined labels of LD. Children who had cognitively defined LD labels were those children who had individualized education goals in math (IEP), sustained, low performance on math standardized exams, and performance measures via the Woodcock Johnson test of achievement and tests of cognitive abilities.

Data collection consisted of 12 instructional sessions that lasted approximately 50 minutes for seven consecutive weeks. Sessions occurred in an after-school setting in a conference room equipped with large tables, manipulatives (i.e. unifix cubes), paper, pencils, and a whiteboard. The instructional sessions were planned by the author and a research assistant using a teaching experiment methodology (Cobb, 2000) to investigate children's enacted agency when engaged in the discussions about each child's mathematical thinking. Base ten and fractional tasks (Carpenter, Fennema, Franke, Levi, & Emspon, 2015) grounded in problem solving were used to support reasoning and sense making.

During the sessions, children were supported to solve problems in whatever ways made sense to them, communicate verbally in any language they were most comfortable, and ask questions

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about the problems. The teacher-researcher presented children with appropriate story problems, encouraged use of children's prior knowledge, and facilitated discussions and the use of teacher moves to promote agency. Some the teacher moves included were (Jacobs & Empson, 2016) ensuring that children were making sense of the word problems, and assigning competence to children's ideas (Gresalfi, Martin, Hand, & Greeno, 2009; Turner et al., 2013). The data collection process yielded three primary sources of data: video recordings of the sessions, instructor and research assistant field notes, and children's written work.

Data analysis of the sessions was collected as three embedded case studies (Yin, 2009) consistent of ongoing analysis (Powell, Francisco, & Maher, 2003) of the teaching experiment sessions and constant comparison analysis (Glass & Strauss, 1967). I select partial data segments that clearly illustrated changes in one child's agency over time in the findings below.

Findings

The iterative process of coding, comparing and refining yielded three types of mathematical agency: (a) *Limited Math Agency*; (b) *Developing math agency* and (c) *Enacting math agency*. *Limited math agency* refers to a child using procedures without making sense of the mathematical ideas and any actions that might indicate a refusal to participate in sense making. *Developing math agency* refers to a child's capacity to make sense of their mathematical ideas and might begin to share these ideas with peers and the teacher. *Enacting math agency* refers to a child's capacity to make sense and take action on their mathematical ideas. In enacting math agency children will most likely engage in discussions to take up, justify, and argue someone else's ideas.

Below, I illustrate how agency evolved for Gabriel and the teacher moves (e.g., problems written in English and Spanish; read aloud of problems in in both languages) that, in large parts, promoted children's participation in the math practices and, arguably, advanced their agency. All children exhibited limited, developing and enacting agency at one point in the twelve sessions; all had positive shifts in their agency from limited to enacting. Further details of all three children will be presented in session.

Gabriel's Agency

Gabriel began solving word problems by using standard algorithms without connections to the context. In initial sessions, Gabriel explained his final result but refused to share his reasoning behind the final result, thus exhibiting *limited agency*. At first, it was difficult to engage Gabriel in the story problems in an authentic way, but after the third session, I considered that Gabriel was not making a connection to the problem context because he was not interested in the story context that were presented. Thus, in session four, I made a significant change to the story problem where I changed the context from tamales to soccer balls. At first, he didn't seem to understand the context, because when Gabriel explained his answer he exclaimed that it was nine because he had added six and three together. I therefore suggested to Gabriel that he find a second and different strategy. As I turned my back to him, and began to attend to Julia's strategy, I noticed he began to use his fingers. Gabriel began counting six fingers one at a time, then repeated his actions two more times, as if he was counting up to six, then counted up a second set of six and one final set of six. After he counted he immediately shouted "it is 18" so I followed up with some questions.

Teacher: I saw you using your fingers.
Gabriel: No.

Teacher: Its ok to use your fingers, I use my fingers all the time. How did you use your fingers?
Gabriel: [Silent]
Teacher: Did you count 1, 2, 3, 4, 5, 6, [Pauses] 8, 9, 10…or were you doing…
Gabriel: [Shakes his head from side to side indicating a no] I was, I added 6 and 6 that's 12 and added another 6 and it was 18.

Gabriel had begun to make sense of the problem when the context was changed from tamales to soccer balls but failed to completely take ownership of sharing his finger strategy. In this instance Gabriel was exhibiting mathematical agency as *developing*. He was beginning to take ownership of his mathematical ideas, and in seeing them as valid strategies.

Later, Gabriel not only became comfortable sharing his mathematical thinking with others but also began to justify his thinking with others. For example, in session 5, Gabriel began to engage in conversations with Martin about their strategies. In this excerpt, Martin and Gabriel are sitting on the floor. While Martin worked on his strategy Gabriel looked over his shoulder. Martin had solved the problem by making six circles and four dashes in each to represent the soccer balls.

Gabriel: Six? [Gabriel use a questionable tone]
Martin: [Turns and looks at Gabriel] Yeah, he has 6 bags
Gabriel: It is five [Gabriel shows Martin five fingers and references his strategy on his paper]
Martin: Yeah [Martin smiles at Gabriel]
Gabriel: Why did you put six? [Gabriel begins counting to himself the dashes on Martin's paper to verify it is 24] 23?
Martin: No, there is 24 right here [Martin assures Gabriel there are 24 dashes not 23, and smiles at him]
Gabriel: Anyway, you have to put five on each of them [Gabriel points to his circles and dots on his paper] not four.

In this exchange of ideas, Gabriel was trying to convince Martin that his strategy was correct. Gabriel first initiated the conversation by questioning Martin's final answer of six bags. Martin seemed confident about his answer of six bags because he thought the problem said four soccer balls in each bag, instead of five. Gabriel's questioning did not discourage Martin from keeping his answer, so Gabriel took it upon himself to explore the details of Martin's strategy. After noticing the fours inside the circles, Gabriel began to argue with Martin, stating that the problem said five (soccer balls) and not four as Martin had in his strategy. Gabriel realized he had to look at the details of Martin's strategy to find the inconsistencies. Such actions produced Gabriel to take complete ownership of his ideas and succeeded in helping Gabriel make sense of Martin's strategy and locate the source for the mistake in Martin's strategy. Thus, Gabriel exhibited math agency as *enacting.*

Gabriel began to take risks in sharing his ideas about his peers' strategies and in posing mathematical arguments as the sessions progressed. Overall, Gabriel gained a sense of pride in his ideas and his methods for solving problems.

Implications and Discussion

The findings of this study reveal that these children were "able" to make sense of word problems, took ownership, and were empowered to co-construct mathematical thinking with peers. Gabriel shifted from non-explanation for solutions to taking risks in sharing his thinking. Findings point toward an alternative framing of instruction from the majority of research in special education research which supports explicit, directed instructional approaches (Woodward, 2004). If the aim of research and practice is to increase equity for all, then I suggest that special education and mathematics research integrate mathematical practices that promote choices for children to use their native language, work in small groups or individually, and have opportunities to share their thinking with peers into instruction in future studies.

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PERCEIVED ETHNICITY AND GENDER INFLUENCES ON PROFESSIONAL NOTICING OF CHILDREN'S MATHEMATICAL THINKING

Jonathan Thomas
University of Kentucky
Jonathan.thomas1@uky.edu

David Brown
University of Kentucky
David.a.brown16@uky.edu

Kristin Reeves
University of Kentucky
Kristin.reeves@uky.edu

Edna O. Schack
Morehead State University
e.schack@moreheadstate.edu

Molly H. Fisher
University of Kentucky
Molly.fisher@uky.edu

Cindy Jong
University of Kentucky
Cindy.jong@uky.edu

Keywords: Instructional Activities and Practices; Equity and Justice

Introduction

Professional noticing of children's mathematical thinking (PN), as a conceptualization of responsive teaching practice, is the subject of much focus within the mathematics education research community (Sherin, Jacobs, & Philipp, 2011). Our goal was to examine such biases within the component skills of PN and the extent to which they may be influenced by perceptions of ethnicity and gender. The research question was: *How and to what extent does bias emerge within preservice teachers professional noticing of children of differing perceived genders and ethnicities?*

Methodology

An electronic survey was fielded nationwide resulting in a total of 151 completed surveys, with participants being primarily 18-24-year-old white females. PSTs were asked to respond to three prompts (distributed randomly across four cases aimed at eliciting biases related to race/gender)– each aligned with a particular component skill of PN. We examined frequencies per component skill (i.e., attending, interpreting, deciding) rather than by case (i.e., Margaret, Shaquan, William, Miguel) to make some determination regarding the overall presence or absence of bias within participants' enactment of PN.

Findings

Interestingly, participants provided attending and deciding responses that were predominately neutral in nature while participants' interpreting responses were much more varied with respect to biased descriptions of the child's mathematics in a given case. We conducted a chi-square test of the attending ($\chi^2(3)=307.68$, $p<.001$), interpreting ($\chi^2(3)=14.54$, $p=.002$), and deciding ($\chi^2(3)=391.49$, $p<.001$) components and determined that each distribution of rating categories was statistically significant ;however, both the attending and deciding components featured predominately neutral responses while the interpreting component contained mostly non-neutral responses. As such, we disaggregated these responses per case. Examining participants' responses across perceived ethnicity reveals a relatively balanced expression of bias among asset, deficit, and both response types. While participants were slightly more likely to offer an asset-laden response to describe Shaquan's mathematical thinking and a deficit-laden response to describe William's mathematical thinking, we did not detect a disproportionate amount of a particular rating type. We note that, for all four cases, fewer participants offered neutral responses when interpreting the child's mathematical thinking. Much more often, participants interjected some form of bias into their interpretations.

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MATHEMATICS AS A CONSCIOUS RAISING EXPERINCE: A BRIDGE TOWARDS SOCIAL TRANSFORMATION

Ricardo Martinez
Iowa State University
ricardom@iastate.edu

Ji Yeong I
Iowa State University
jiyeongi@iastate.edu

This study explores preservice teachers engaging in a ratio activity connecting the wages of men and women to the number of men and women in government positions. Mathematical learning expands the critical literacy framework reading and writing the word situating mathematics as the world itself. Furthermore, the bridging of mathematics, the self and the world, was constructed by incorporating the theory of conocimiento(s) in analyzing how critical consciousness is developed within a mathematics task. This study shows mathematical learning as a conscious raising experience in investigating social injustices towards acts of social justice.

Keywords: Equity and Diversity, Social Justice, Teacher Education-Preservice

Introduction

The purpose of this study is to learn how male pre-service teachers (PSTs) experience a mathematical task that explores wages earned by women and men in various occupations. To compare wages, an activity composed of five tasks using data taken from the 2016 U.S. Bureau of Labor Statistics was used in conjunction with the use of ratios and the creation of mathematical equations. The task was created with the intention of creating a learning space where PSTs would be able to reflect upon the content being learned and were provided with an opportunity to create a figure/diagram to share out any findings. Wherein the activity looked to situate learning in/as a form of revolutionary praxis, the union of action and reflection towards social transformation (Freire, 1976). More so, revolutionary praxis requires a critical consciousness, a socio-cultural-historical-political understanding of the world with the individual acknowledgement of being able to change the world. This study explored how a mathematics activity could transform learning into a conscious raising experience. To understand consciousness development during the learning mathematics, theories of conocimiento(s) (Anzaldua, 2003) were compared with how critical conscious is developed in reading and writing the word (Freire & Macedo, 2005), both described later. Wherein it is impossible to have a critical consciousness without multiple forms of conocimientos.

Without praxis in mathematics, mathematics becomes nothing more than a mechanical process. Any mechanical process can then become a dehumanizing force, shifting the learner from a position of subject to object (Freire, 1970; 2018). Teachers, students and researchers will be unable to address critical issues in mathematics without focusing on the entire experience of learning mathematics. Hence this study focuses on PSTs because "[p]rospective teachers … need to begin to develop the political and ideological clarity that will guide [students] in denouncing discriminatory school and social conditions and practices." (Bartolome, 2004, p.119). The political and ideological clarity Bartolome spoke of is in line with critical consciousness and reflects the political power of teachers controlling learning in the classroom (Giroux, 1988). The activity created for this study uses ratios, algebraic generalization and re-representation of data to make meaning of the real world. The research question driving this study is, "how do pre-service teachers develop critical consciousness through a ratio, algebraic generalization, and representation mathematics activity?"

Literature Review

Critical Consciousness

A critical consciousness is defined as a socio-historical-political-cultural understanding of society with the self-awareness of being able to contribute to change in society (Freire, 1970). Critical consciousness is an elastic concept given society is in constantly flux (Freire, 1970; 2000; 2018). Prior to the development of a critical consciousness, is what Freire (1973) calls a transitional consciousness. Those with a sense of critical consciousness are always in between transitional and critical consciousness. Individuals not seeking critical consciousness are said to have a naïve consciousness (Freire, 1973) where they have been rendered objects due to dehumanization (Freire, 1970). Relative to developing critical consciousness, are ideas of praxis, critical literacy and conocimiento(s). Furthermore, Freire and Macedo (2005) provide a critical literacy approach in reading and writing the word (described later) which inherently allows for a socio-historical-political-cultural understanding of world. The praxis behind reading and writing the world is a revolutionary praxis (Freire, 1970) that requires a critical consciousness. Related to a transitional consciousness, Anzaldua (2004) offers conocimiento, seven stages of how consciousness changes (described later) that provide a more refined way of mapping naïve to transitional to critical consciousness development.

Identity, Mathematics, and Dehumanization in Mathematics Education

Learning how to socialize is inherent in learning mathematics because teaching mathematics is a political act that impacts students understanding of the world (Gutierrez, 2017). Knowing that mathematics is part of the socialization, and identity development of learners helps educators identify factors that contribute to students' futures (Jackson, 2010). Insofar to say the connection between identity and learning is not an unknown relationship (Martin, 2010). Wherein utilizing the identities of students in the learning of mathematics has a growing body of research (Aguirre, 2013; Celedon & Ramirez, 2012; Civil, 1998; 2014; Martin, 2010; Wood, 2013). Specifically, Aguirre, Mayfield-Ingram and Martin (2013) "define a mathematics identity as the dispositions and deeply rooted held beliefs that students develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics in powerful ways across the context of their lives" (p. 14). Although students' identities need our attention, we must also consider the role of the teacher and how their identities impact students (Macedo & Bartolome, 1997). Specifically, if we do not pay attention to students' identities then learning becomes a dehumanizing experience (Gutierrez, 2018) where students are never fully seen or heard and where teachers cannot fully meet the needs of students (Civil, 2014). Teachers (and researchers) need to focus on the development of young learners by paying attention to identity, which should be part of the daily life work of teachers (Aguirre, 2013).

The idea of humanization and dehumanization stems from the work of Paulo Freire regarding dialogue and the learning process known as a problem posing education, a humanizing approach and banking education, a dehumanizing education (Freire, 1970; 2000). A dehumanizing approach to education silences students, ignoring their identities and views students as mere objects. Shifting to mathematics education Gutierrez (2018) offers eight dimensions of humanizing practices for mathematics educators with ideas of participation/positioning, cultures/histories, creation and ownership, all having a direct connection to identity development (Nieto, 2018). Furthermore, a full discovery of identity and humanization allows for individuals to discover or (re)discover their own agency and allows for the development of a critical consciousness (Shor & Freire, 1987). As previously mentioned, viewing the identities and even roles of teachers and students as interchangeable not only negotiates the power of the classroom (Shor, 2014) but also echoes Fasheh (1982) in that, "[t]here is a price for teaching math in a way that relates it to other aspects in society and culture which may result in raising the "critical consciousness" of the learners. (p. 288)."

Critical Literacy, Mathematics, and Critical Consciousness

Mathematics as a form of critical literacy can best be summarized by Gutstein's adaptation of Freire and Macedo's (2005) *Literacy: Reading the word and the world* framework that Gutstein (2006) calls reading and writing the world with mathematics (RWWM). Freire and Macedo's framework is a way to understand the world by understanding the previous world through critical literacy (Freire, 1998). Critical literacy is the ability of communication and reflection inherent in being able to read and write (Freire, 2018). The ability to reflect and communicate is a necessity for revolutionary praxis. Gutstein (2006) developed RWWM and incorporated the role of mathematics to read the current and previous world. A "[c]ritical mathematics literacy enables the oppressed to use mathematics to accomplish their own ends and purpose" (Lenorard, 2010, p. 326). Being able to read and write the world within a critical literacy paradigm leads to empowerment and understanding of one's own agency (Freire, 1970; 2005). Knowing how to read the world and understanding one's agency directly relates to critical consciousness and (re)humanizing mathematics. Where a critical mathematics agency requires individuals to be see themselves as capable and powerful mathematical learners when participating in mathematics in personal and social ways (Aguirre, 2013).

Conceptual Framework

In welcoming the connection between mathematics and the social world, the conceptual framework of reading and writing the word (Freire & Macedo, 2005) was utilized and paired with the theory of conocimiento(s) (Anzaldua, 2003; 2015), which also acted as the analytic framework for this study putting emphasis on mathematics as a way to discover both self and the world. The teaching of mathematics should reflect the social and physical environment when learning, with such an approach augmenting a deeper understanding and imagination (Joseph, 1987). The physical environment is constantly engaged when learning math directly or indirectly, as is the political nature of learning mathematics (Gutierrez, 2018). Furthermore, as Martin states "exposing the links between mathematics and social interests should not be a threat to mathematics" (1997, p.175).

In the understanding of social and structural problems, Freire and Macedo introduced the critical literacy framework, reading and writing the word (2005). Where the reading or understanding of the world requires the reading of the word and in order to read the word one must first read the previous reading of the world (historical awareness). The writing of the world which follows the reading of the world then becomes the new world (Freire, 2005). Where the learning of mathematics is the simultaneous act of learning more about mathematics, the world and self. Insofar to say the world cannot be understood without understanding how the individual fits in said world. The bridge between the conceptual and analytic framework can be seen in the reflection on how consciousness shifts towards the development of a critical consciousness.

Methods

To observe the process of mathematical learning of PSTs while identifying stages of conocimiento(s), a single holistic (Stake, 1995) qualitative case study bounded by the duration of the activity, was conducted. The study focused on understanding how PSTs engage in a contemporary issue that does not require the control of human behavior (Yin, 1998). In this study, the case is bounded by the space time of a classroom in a large Midwest university. The unique feature of this case is the looking at wage disparities between men and women in the U.S. using ratios. The facilitator of the activity was a participant observer and main researcher behind the study. The data includes written reflections from the PSTs after each task and a culminating reflection at the end of the entire activity. Written individual reflections were written prior to group discussions and used to triangulate findings. Group discussions along with whole class discussion were recorded.

Analytic Framework: Conocimientos

Conocimiento(s) consists of seven stages and represents a change in consciousness wherein to say an individual that has experience a conocimiento(s) is no longer the same person (Anzaldua, 2015; Keating 2008). The result of no longer being the same person changes an individual's identity because gaining conocimiento(s) results in action that would be impossible without, the aforementioned, change in consciousness (Anzaldua 1999; Keating, 2016; Lara, 2005). Furthermore, conocimientos is a pivotal theory in the epistemological understanding of the interconnectedness of all people (Keating, 2006) which leads to self-discovery of one's own freewill upon the world (Anzaldua, 2015). This interconnectedness allows for the individual to be interconnected with not only others but with mathematics itself. The seven stages of conocimiento(s)—a non-linear, non-sequential and cyclical (both a cycle and/or repeating) process of consciousness (Anzaldúa, 2003)—described in Table 1, will act as the analytic framework in analyzing the data.

Table 1: Stages of Conocimiento(s) (Anzaldúa's, 2003)

Stage	Description
el arrebato	This is the stage that kicks you out of your comfort zone. This is the stage that pushes you to learn/change. It can be a positive or negative experience.
nepantla	This is the clash of worlds and or ideas both the friction and overlap of two or more thoughts. Since we are always learning we are always in nepantla but necessary not aware of it.
coatlicue	This is the pain of knowing and the realization that you need to change. The coatlicue stage is also known as deconocimientos representing that learning and growing never stops.
compromison	This is the realization that nothing in this world is fixed, which implies change is possible.
putting Coyolxuhqui together	This is where you reinvent yourself as a new person. The transformation that results from fragmentation of self, both positive and negative experiences that you learn from.
the blow up	This is the realization that change has happened and that you are a new person. It is where/when it is first internalized that change has occurred.

shifting realties	This is an internal/external spiritual transformation of being conscious of others, a shift from how this affects the individual to how this affects the world. The act of changing.

Participants

All four of the participants were White male PSTs from Iowa, taking a secondary mathematics methods course in 2017 at a large predominantly White mid-west university. All participants are traditional undergraduate mathematics majors that were one semester away from student teaching. All participants were under the age of 22. The course they were taking did not focus on the development of critical consciousness nor did any other mathematics methods course specifically focus on critical consciousness in the 4-year university-based teacher preparation program. Three of the four PSTs have previously taken one course on multicultural education that is part of the institution’s teacher preparation program. The methods course where data was collected consisted of 12 students, six male and six females.

Task

The task was developed to be part of an algebra methods course with the intent to raise critical consciousness of the learner, where “critical consciousness is brought about not through an intellectual effort alone, but through praxis – through the authentic union of action and reflection” (Freire, 1973). The role of groups was designed to allow students to interact with each other because students need to support each other in meeting their goals (Zacko-Simth, 2013). The understanding of ratios, generalization, and re-representation were the topics covered throughout the algebra methods course. Ratios were selected because the concept is frequently connected to the real world and proportional reasoning is a difficult topic to learn/teach (Lamon, 2007). Generalizations were selected to provide PSTs with the opportunity to make conclusions based on the difference of data over time (Driscoll, 1999). Generalizations allow for the extrapolation of data. Re-representation was incorporated into the activity because of its potential in being the action component of the activity. Re-representation allows learners to reword and reflect on what they have learned using visual representation. The study consisted of an activity created by the researcher comprised of five parts described in Table 2.

Table 2: Task Overview

Part	Student Engagement/Action/Reflection
1	View raw data from 2016 U.S. Bureau of Labor Statistics of weekly median incomes.
2	Construct ratios of weekly women median income to weekly men median income and find the value of ratios. Participants reflected individually then as a group on real-world context of the ratios.
3	The generalization task uses the value of the ratio(s) and creates an equation to predict what happens after a year.
4	Ratios of the numbers of men to the number of women in the US Senate and House of Representatives along with the relationships in state governorships were constructed to the value of each ratio. Participants reflected individually then as a group on real-world context of the ratios.
5	Re-representation of the entire activity.

Analysis

In Vivo coding was conducted (Creswell, 2016) due to its usefulness in thematic analysis (Saldana, 2016), wherein short phases were analyzed. Thematic coding has emerged as a rigorous approach in analyzing themes (Fereday & Muir-Cochrane, 2006) where Braun and Clarke (2006) provided six phases for thematic analysis: (1) *Familiarization with the data*, (2) *Coding*, (3) *Searching for themes*, (4) *Reviewing themes*, (5) *Defining and naming themes* and (6) *Writing up*. After *Coding* was completed in *Reviewing themes* an analytic memo was written for each code. The analytic memos were part of *Searching for themes*, specifically a generative theme emerged from each analytic memo. In Vivo coding was then conducted and accompanied with/by analytic memos for each code. The memos were then coded using the analytic framework. The reflection written on worksheets were used to triangulate and in the reviewing of coding. Next, the codebook was created, see the sample in Table 3.

Table 3: Sample of Codebook

Stage	Code Description	Example invivo codes(s)
C1 – *el arrebato*	Related to discomfort or excitement caused either by the mathematics or by the context or connection to the real world	"a lot of numbers" "You are right holy crap I don't know what to tell her that sucks"
C2 – *nepantla*	Related to the comparisons of mathematics, professions, genders, history and other structures	"so we write ratio as a fraction or what" "because women are dominantly known as teachers, have always been, way back in the day they make less than we do"

Findings

Part 1

The start of the activity provided multiple *el arrebato'*s due to the context of the problem. Mathematical shake ups range from being overwhelmed by the data, "a lot of numbers" to being overwhelmed by the difference in wages (the context) "they should be on the same pay scale" to discovering things by doing mathematics "Men make 700 more dollars … oh wow." This shows that viewing and discussing real world data provides an entry (*el arrebato*) and an environment (*nepantla*) to explore the world. What we found was that the data allowed the PSTs to find contradictions to the view that men always earn more than women by finding cases where women make more. For example, one student said "I got one I got one where women make more" being surprised and excited (*el arrebato*) to find a contradiction (*compromision*) to the fixed idea that men always earn more.

We then began to see a comparison (*nepantla*) not only of wages but of number of employees. The comparison of wages and number of employees can be seen in one PST exploring elementary/middle school teachers where "like 80% women" and another PST asking, "what's the wage" only to hear the wages and say, "wow that is a big jump." The mathematical discrepancy directly relates to the real-world disparity in wages between women and men and to the results in of PSTs trying to justify the differences. At this stage of the activity the mathematics was limited to additive comparisons of not only individual professions but across, as can be seen when one PST said, "weird it is so small when women make more [yet] when men make more it's like a thousand dollars more."

Part 2

In the second part of the activity a shift from direct additive comparisons to the use of ratios, which was expected as a direct result of the task. The shift to using ratios from direct comparisons while exploring different professions among women and men is *nepantla* on multiple levels. On one level when a PST said "so we write ratio as a fraction" ratio and fraction represent the clash of two mathematical sets or worlds. Specifically, the use of ratios was naturally *nepantla* as a ratio represents the relationship between two quantities and even how ratios are represented can be seen as *nepantla* within *nepantla* as one PST said "there are 17 females to 100 males or .17 to every one male." On other levels we see; the comparison of professions in comparing police officers to engineers to teachers; or the comparison of expertise in questions posed by the PSTs like "do you think that has to do with levels of expertise?"; or historical comparisons "because women are dominantly known as teachers, have always been, [since] way back in the day;" or by breaking the border between the mathematical world of the activity and the ones individual world as can be seen in comments like "they make less than we do." Additional *el arrebato's* occurred during task two of the activity. The first was related to limitations of being unable to generalize claims with one PST saying, "from the data we cannot make any conclusions" and another saying, "doesn't matter I am not an expert." The second manifests when unexpected outcomes occur in the real-world data, as one PST said "so why is that not the case for education" in being shocked at male teachers making more even though there are more women in the profession. In this part of the task *el arrebato* and *nepantla* leads to *coatlicue* or the pain of knowing and can be seen when it was said "that the problem [is] you only have four tenths of a teacher." Whereby directly stating a problem hints there needs to be a solution implying the realization of change.

Part 3

Part 3 of the activity began with a shake up as the group selects a profession and one individual saying, "ours is going to be awkward" where a PST's ability of estimation led to *el arrebato*. The group was already aware of the difference in the weekly median income along with the unit rate relationship between men and women but in creating a formula for "*n*" many weeks they discovered "after a year, men have made 10,920, nearly eleven thousand [dollars] more" encountering *el arrebato* once more. Furthermore, in *nepantla*, in the comparison of positive and negative values the group discussed when it would be appropriate to use a positive difference (men – women) versus a negative difference (women – men). One PST said they would always subtract the numerically larger value from the numerically smaller value to ensure a positive number with another replying that means you will always use women subtracted from men only to hear "that was a low blow." Calling it a low blow is a direct result of *coatlicue* and the pain knowing that leads to a *shift in reality* as the conversation connects negative values with disadvantages in saying "it depends on how you want to see the data."

PSTs' reflection on this activity provides an example of a shift moving from *coatlicue* to the *blow up* to the *el arrebato* stages of conocimientos. When asked how the activity makes you feel, one PST instantly said "this makes me feel like shit" this pain shows the realization of change (*the blow up*) because the next statement said "it's like [am] back at my therapist. Here the individual is connecting what was learned in this task to internal transformation. This cause *el arrebato* in others as they default to say, "I don't have feelings" and "men don't have feelings."

Part 4

The beginning of this activity was *el arrebato* as the worksheet for task 4 had 19 women senators and 71 non-women senators which does not equate to 100 senators. Knowing there should be 100 senators shows that the PSTs do have a political understanding of society.

Furthermore, a *shift in realties* can be seen as PSTs look up how many governors are women only to discover "8% of governors are women" as they declare it a "fun fact." One again this reflects the *blow up* of being in *nepantla* in discovering how women are represented in the government because "for every four men there is one woman." As the PSTs travel outside the task in *shifting realties* they discover currently there are "six women governors so the number is going up" a *compomison*, a reminder that the world is not fixed.

At the end of this activity a PST repeated the statement "I can neither deny or confirm this because I am no expert." The first time it was said it represented only *neplantla* because he was in between denial. The PST initially made the claim of not being an expert then became silent. The second time the same PST said "I can neither deny or confirm this because I am no expert" shows a changing by the PST not staying silent and offering his "guesstimation" on how he sees the percentages of women. The act of "guesstimating" represents a *blow up* and *shift in reality*, while the change from silence to speaking reveals a new person, *putting Coyolxauhqui together*.

Final Reflection

When asked how all parts are related, one PST said "it takes money to run for government, right, if all these dudes are out here making more money than there in government more" followed by another PST stating "guys are making more money, so they are donating more and who are they donating to probably guys." This brief exchange shows a *blow up* for both individuals in better understanding why more men may run for office. The following exchange occurred when PSTs were asked "would you use this activity in your future classroom?":

PST A: I would adapt it to talk about what they, the students want to be in the future"
PST D: It would be discouraging if a girl comes up and says I want to be an engineer.
Researcher: what do you tell the girl?
PST D: I say … um … Oh I don't know
PST B: You are right holy crap I don't know what to tell her that sucks
PST C: life is not fair
PST A: She could be the outlier

PST A engages *nepantla* by saying the activity would be useful because of its real-world application then shifts to a *blow up* in wanting to adapt the problem specifically to what learners want to be in the future. PST D then confronts a potential *el arrebato* in the potential pain of knowing (*coatlicue*) for young women engaging in the activity. A further shake up comes when the group was asked what they would say to the young women who discovers she will earn less compared to a man. PST D does not know what to say as he experiences *el arrbato*, while PST B and P PST C are in *coatlicue* a realization that something needs to change. PST A has a shift *in reality* in offering to tell the young women she is not in a fixed world (*compromiosn*) because she can be an outlier.

Discussion

The findings show that stages of conocimientos can be experiences throughout this mathematical activity. The activity provided PSTs with one way to understand the current world by understanding how wages and government forces interact. The tasks within the activity provide a way to read the world as PSTs write the world by engaging in the problems. In going through the task, learners experienced opportunities to gain conocimiento(s) and shifted their consciousness. In bridging ratios of wages and ratios of governmental representatives the PSTs discovered injustices. Discussion on injustices with mathematical discoveries of ratios represents a shift towards developing a critical consciousness. Where "men make more in all my categories" (*nepantla*) is both reading and writing the world in looking at men and women while doing the mathematics to show they are not making equal wages. *Nepantla* alone only represents a naïve consciousness. To say "life is not fair" (*el arrebato*) is one way to read the world, saying "you would adapt it to talk about what is … the future" (*putting Coyolxuhqui together*) enters a transitional consciousness in writing the world. The *coatlicue* and *the blow up* stages of conocimiento(s) are respectively engaged in reading the world where the activity "makes me [PST] feel like shit" while discovering "it is like the rich get richer." Still in transitional consciousness the PSTs wrote the world by saying "it depends on how you use the data" (*shifting realties*) and telling future young women that they "could be an outlier" (*compromison*). Once out of *nepantla* the other six stages only represent a transitional level of consciousness and an imperfect understanding of world. The moment all stages of conocimiento(s) are lived then critical consciousness emerges as a socio-political-historical-cultural understanding in changing reality. Wherein bridging mathematics and conocimientos provides a path to see how mathematics is inherently part of self and world towards the transformion of mathematics itself.

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RHETORICOCENTRISM AND METHODOCENTRISM: REFLECTING ON THE MOMENTUM OF VIOLENCE IN (MATHEMATICS EDUCATION) RESEARCH

David M. Bowers
Michigan State University
bowersd2@msu.edu

In this brief research report, I provide a humanist exploration of the constraints of methodocentrism and rhetoricocentrism in our field with particular attention to equity and the propagation of racial (gendered, sexualitied, etc.) violence as reified through the methodological and rhetorical norms of the field of (mathematics education) research. In short, I argue that (1) methodological norms constrain the practice of research in ways that protect White interests, and (2) rhetorical norms further constrain the reporting and dissemination of research in ways that protect White interests. I then conclude that mathematics education research will not be able to adequately address issues of social justice unless these norms are systemically broadened.

Keywords: Doctoral Education; Equity and Justice; Research Methods; Systemic Change

The 41st Annual Meeting of the North American Chapter of Psychology in Mathematics Education invites the mathematics education community to consider the expansion and growth of mathematics education research "...against a new horizon," with particular attention to equity and to the inherent contradiction between normative perceptions of growth/value and the concomitant violence of displacement and marginalization. With that in mind, this proposal seeks to create a space for reflection and conversation around the manifestations of this contradiction in mathematics education research (and research on human beings more broadly), feeding the flames of a larger meta-study of mathematics education research praxis. In particular, this manuscript takes an active stance "against" a looming horizon of uncritical *scientism* that fails to adequately grapple with the unavoidable subjectivities of research performed on human beings by human beings (Howe, 2009; Latour, 2004). Through critical analysis of our (mathematics educations researchers) own subjectivities, perhaps we can better approximate some truth (Kofman, 2018) and more productively stand against the momentum of raced (and gendered, sexualitied, etc.) violence (Kendi, 2016) echoing through the normative practices of our discipline (Bowers, 2019; Coe, 1994; Fendler, 2014; Howe, 2009).

This manuscript will focus on two specific arguments, couched primarily within the tools and paradigms of Critical Race Theory (Delgado & Stefancic, 2001) and Genre Theory (Miller, 1984) - though it is hoped that the conversation thus invited will extend beyond these two arguments and theoretical foundations: (1) Methodocentrism in our field curtails the ways we can make sense of the complexity of humanity in ways that privilege White (cis-male, hetero, etc.) perspectives (Howe, 2009), and (2) rhetoricocentrism further restricts the voices of the marginalized to appease White (cis-male, hetero, etc.) ears (Bowers, 2019; Coe, 1994) through pathosphobic norms. Though these restrictions do not exhaustively cover the sometimes unacknowledged or uninterrogated subjectivities of our practice, they are emblematic of broader norms and trends (Bonilla-Silva, 2015; Leonard, 2009) whereby localized discourses and events act as pieces of larger social narratives which construct and reify White Supremacy within and beyond mathematics education research (Martin, 2009). Following the exploration of these two arguments, I will conclude with implications for future (mathematics education) research practice.

Methodocentrism: Laplacean Dream and Scientific Nightmare

> Since the early nineteenth century, many in the Western world have dreamt of ordering life and reality around the certainty of facts, the predictability of the future, the stability of isolated phenomenon, the universalization of mathematical thinking, the necessity of hypothetical/deductive thinking. This is the Laplacean dream… For this Laplacean science, 'a description is objective to the extent which the observer is excluded and the description itself is made from a point lying de jure outside the world' (Prigogine & Stengers, 1984, p. 52). This view has led to the creation of experimental methods in the sciences which involve a reduction of 'natural' complexity in order to focus attention on measuring specific aspects, or variables, of a phenomenon. As a result, Western science becomes caught up in what we call 'methodocentrism': the belief that particular, pre-formed methods can guarantee the validity of an intellectual investigation into the world by factoring out the vicissitudes of the observer's entanglement with the world (see Said, 1979, pp. 9–10). To use a word that is crucial for Stengers's (1997) conception of science, *methodocentrism* is about trying to minimize the 'risk' of intellectual investigation, an avoidance that *ultimately produces bad science.* (Weaver & Snaza, 2017, pp. 1055-1056; emphasis added)

In the article quoted above, Weaver and Snaza (2017) critique methodocentrism, which they define as the belief that the predetermined method(ology) one chooses to guide research determines its validity, legitimacy, and trustworthiness. Their critique, grounded in science studies and posthumanism, argues that methodocentrism is deeply implicated in anthropocentric and colonialist politics. The argument I put forth here diverges from Weaver and Snaza in two notable ways: (1) I further define methodocentrism to refer to and connote the normalization of quantitative and qualitative methodologies to the exclusion of other research paradigms (Howe, 2009), and (2) I adopt a humanist rather than posthumanist stance in response to concerns that posthumanism can be recolonizing in how it potentially draws attention away from the ongoing humanist projects of social justice and equity (Gholson, 2019).

I will frame this argument using two conceptual tools: (1) the *challenges to color blindness and neutrality* tenet of Critical Race Theory (Delgado & Stefancic, 2001), and (2) application of social process genre theory (Coe, 1994; Miller, 1984) to research methodology. The tenet of color blindness prompts us to interrogate how structures, individuals, objects, or practices that seem neutral actually reinforce Whiteness and White interests; in this case, I ask how normalization of quantitative and qualitative social science methodology to the exclusion of other approaches (e.g. philosophy, history, historiography) reinforces Whiteness. Social process genre theory prompts us to examine, in this case, how the form and function of methodologies interact and interrelate in ways that encourage and/or constrain against particular ways of communicating with and interacting with the world.

As a discursive entrypoint into the broad array of observations that might be made from this theoretical foundation - a jumping-off point for any conversations this manuscript might invite - I begin where all research might be said to begin: with an idea expressed in the form of a research question. In particular, I contend that the interrelationship of form and function shaped by methodocentrism restricts the array of normalized research questions in a way that is not color-blind.

Current methodocentrist norms, which value quantitative and qualitative methodology above others, have been described as the result of attempting to reproduce the practice and perceived success of the natural sciences to the study of human behavior (Gage, 1989). As a byproduct of this history, socially-constructed value has been placed upon social science research questions which can be verified or falsified through relatively straightforward empirical testing (Howe, 2009), and consequently upon questions that do not directly wrestle holistically with the complexity of racism and White supremacy (or any other hegemony); such focus on smaller compartmentalized pieces of racism serves very real and destructive functions in hegemonic discourse, both insofar as "compartmentalizing complex wholes into disparate pieces facilitates the naming and ordering of those pieces and parts in order to have dominion over them," (Patel, 2016, p. 19) and in the sense that focus on individual parts and factors obscures more holistic views and makes it more difficult to challenge overall systems (Patel, 2016). Furthermore, social-value has been placed on research questions which might be described as *objective*, which here might be best translated as "having no visible emotional component and eliciting no conscious emotional reaction." Thus, a question like, "Is there evidence of an achievement gap between White students and students of Color" might be valued above asking "how does racism dynamically and intersectionally manifest and reify itself in the context of high stakes assessment?" even though the former question shields White interests through a failure to name the larger structure of racism as an object of inquiry. Similarly, a question like "how does student racial identity influence mathematics achievement" might be valued above asking "how does White supremacy influence mathematics achievement" due to the strength of the emotional reaction the latter might generate in a reviewer (DiAngelo, 2011), even though the former question tacitly endorses and reifies "postracial" racism through failure to identify and critically wrestle with White supremacy (Bonilla-Silva, 2015).

At this juncture, I wish to highlight that no research question can be truly objective. All research questions are constructed and expressed by people based on their own beliefs, interests, and value systems; no research question has ever been expressed truly absent of emotion, tied up as these questions are in the passions and professional livelihoods of those who ask them. Even the most sterile looking question is guilty of pulling at heartstrings, for within a system of social values that looks favorably on such questions, they will be met with goodwill. Rather than removing pathos from research questions, such articulations merely cloak it, shielding subjectivity from interrogation and critical analysis. Research questions that shield themselves from critical analysis represent *bad science*, and it is time that we as a field made conscious efforts to adopt a more transparent and critical set of methodological norms.

Rhetoricocentrism: The Tyranny of Genre

> What we learn when we learn a genre, such as the genre of academic writing, '…is not just a pattern of forms or even a method of achieving our own ends. We learn, more importantly, what ends we may have…' (Miller, 1984, p. 165). (Bowers, 2019, p. 290)

In the prior section on methodocentrism, part of what I argue is that the fruitless search for "objectivity" has created a system of social values in (mathematics education) research which shields White interests, at the very outset of asking a research question, through the encouraged veiling of researcher subjectivity and pathos. Here, I further contend that this veiling of pathos and emotion continues to constrain researchers and protect White interests at the other end of the research process: the writing up and dissemination of findings.

Consider a researcher, synthesized from the experiences of myself and others, who is conducting research that reflects the complex holism of humanity, such as work at the intersection of race and gender (e.g. Crenshaw, 1991) in the context of mathematics classrooms. This researcher, dedicated to the cause of social justice, is motivated in their work (at least in part) through either personal experience of intersectional marginalization or through empathic connection to those who have experienced such marginalization (Foote & Bartell, 2011). To draw their readers in, frame their argument, and establish an empathic foothold for their readers' shared interest in this work, they begin their manuscript with a quote that evocatively expresses intersectional violence:

> I am very glad to find that in most points I am so fortunate to be of one mind with General Armstrong, who has done more than any one else to help the enfranchised blacks on their way towards a true citizenship. I regret to differ from him in my estimate of the value to the negro of a high purely literary education. The time may come when such a training will bear the same relation to their inheritances that it does to those of the literate class of our own race, but **as a rule the little colored girl was right: 'You can't get clean corners and algebra into the same nigger.' That combination is with difficulty effected in our own blood. The world demands *clean corners*; it is not so particular about the *algebra*** [original emphasis in italics]. (Shaler, 1884, p. 709)

This quote serves several important roles for this researcher's manuscript, but it violates rhetorical norms that demand the veiling of pathos in content and reader reaction; consequently, one or more editors suggest that the quote be removed. In my conversations with researchers whose work acts in opposition to violence and injustice, I find that some version of this story is not at all uncommon. Such researchers are faced with a choice: (1) Remove the evocative quote or language, allowing for publication at the cost of contributing to postracial and other posthegemoic narratives (Bonilla-Silva, 2015), or (2) keep the evocative quote because it serves an important role, risking rejection from publication in so doing.

This particular interrelationship of social value and written form (Miller, 1984) that correlates the socially constructed ideal of objectivity with the veiling of emotion might be reasonably termed *Pathosphobia*. In Aristotelian terms, mathematics education researchers are encouraged to make visible appeals to logos (rational appeal) and ethos (ethical appeal, such as the regular use of citations to convince readers of the author's credibility), but to keep pathos (appeal to emotion) covert for fear of being perceived as biased. Sadly, this veiling of pathos tacitly endorses White supremacy, both through a failure to name it (Bonilla-Silva, 2015) and through weakening the ability of researchers to provide reader's with all-important empathic footholds (Foote & Bartell, 2011) by which to make sense of work that critiques and responds to violence.

Conclusion

To briefly recapitulate, this proposal seeks to create a space for reflection and conversation around the manifestations of conflict between normative perceptions of growth/value and the concomitant violence of silencing and marginalization in mathematics education research (and research on human beings more broadly). In particular, I have argued that methodocentrism and rhetoricentrism constrain researchers in ways that obscure rather than reduce subjectivity, ultimately serving to protect White interests.

I echo the sentiments of Patel (2016): In order to advance (mathematics) educational research, we must begin "with an intentional reckoning with the worldviews used to formulate, conduct, and share research." (p. 20) Though this broad suggestion potentially means many things, and different things for different people in different contexts, one of the things it seems to mean in light of the argument I have forwarded here is that we need to make a conscious effort to identify and reckon with methodological and rhetorical norms that restrict our fields capacity for self-critique and antiracist activism (Kendi, 2016).

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KNOW YOUR TERRAIN: A CROSS-CASE STUDY OF TEACHERS' ATTEMPTS TOWARDS EQUITABLE LEARNING

Ruth N. Urbina-Lilback
Naugatuck Valley Community College
rurbina-lilback@nv.edu

Mary P. Truxaw
University of Connecticut
mary.truxaw@uconn.edu

Scholarship on equitable math teaching has demonstrated that knowledge of students is key. But how is the practice field using this knowledge in today's educational context? This paper reports findings from a cross-case analysis on teachers' use of knowledge of their students to advance mathematical learning. Four algebra teachers in grades 9 – 14 joined us on an inquiry process to understand their practice. The teachers worked in institutions with high representation of students from underserved subpopulations and demonstrated a stance on equity. A cross-case model is presented describing the central phenomena in the teachers' use of knowledge of their students. Findings elucidate understanding of situated practice in support of equitable learning.

Keywords: Equity, Instructional activities and practices, Algebra and algebraic thinking, Teacher knowledge

Purpose

The National Council for Teachers of Mathematics (NCTM) has called on teachers to act in support of equitable teaching (NCTM, 2014). Yet, leading scholars have noted that teaching mathematics equitably is challenging (Strutchens, et al. 2012; NCTM Research Committee, 2018). Much of the work on equitable teaching showcases exemplars of frameworks and/or perspectives on equity (e.g., Bonner & Adams, 2012; Gutstein, 2016; Ladson-Billings, 1995a), but does not tell us enough about day-to-day complexities that teachers face as they attempt to support equitable outcomes in their classrooms. This paper reports findings from a cross-case analysis on four teachers' attempts to advance their diverse students' learning in grades 9 – 14 based on what they knew about their students (Urbina-Lilback, 2018). A model describing the central phenomena on teacher's use of knowledge of their students is presented.

Perspectives

Through a critical synthesis of the literature on equitable math teaching, we identified common core characteristics. We found that whether equitable math teachers were reflective and proactive about equity (e.g., Bonner & Adams, 2012; Felton & Koestler, 2015), built on students' lived experiences (e.g., Aguirre & Zavala, 2013; Rubel, 2017), and/or held high expectations for their students (Ladson-Billings, 1995b), they needed to *know their students* (Ladson-Billings, 1997). These core practices required teacher's knowledge and use of students' backgrounds and lived experiences – for example, language use and associated positioning (Civil, 2014; Moschkovich, 2002), relational patterns and interests (Gay, 2000), practices and funds of knowledge (Civil, 2016), etc. Moreover, what teachers knew and used about their students was highly dependent on their teaching context, including the particular student make-up of the classroom. This recognition is not new. Gutiérrez (2002) argued that equity in math education could be advanced by recognizing its situated nature, focusing on "what it takes to enact particular practices, especially ones that relate to certain kinds of students" (p. 171).

Based on the interdependence of the teachers' practice and its context, we employed Lave's

(1991) situated social practice framework to study the teachers' practice. We positioned the teacher, as the implementer of this practice, and studied the teacher in activity, with and arising from the socially and culturally structured world of the teacher. Additionally, we framed what teachers "knew" about their students considering both cognitive and sociocultural perspectives.

Methods

We combined ethnographic (Emerson, Fretz & Shaw, 2011) and case study methods (Yin, 2014) to explore, document, and interpret teachers' situated practice and perceptions (Yin, 2014). We purposefully recruited and selected participants – specifically, math teachers who taught algebra or algebraic applications, and worked in institutions with at least 25% representation of students from underserved populations (e.g., ethnic minorities and/or low SES). From this pool, we obtained recommendations from relevant sources and self-report data from the teachers' related to educational equity issues. Four teachers that met these criteria accepted our invitation and participated in this research. Pseudonyms are used for teachers and schools.

Participants and Context

Beth and Shannon both worked in a high school with at least 79% students of color and 56% students were eligible for free and reduced lunch. Their math program was tracked. We observed Beth in an Algebra 2 course in the lowest track during a unit on rational exponents. Shannon was the only math teacher in a Sheltered Language Instruction program. We observed one of her Algebra 1 classes in a unit on systems of linear equations. Eddy worked in a high school with at least 65% students of color and at least 70% were eligible for free or reduced lunch. The math program was tracked. We observed Eddy in two of his Algebra 1 classes, both in the lowest of placement. Dena worked in a two-year public institution where at least 28% students were of color and about 39% students were of low SES (based on Pell Grant data). We observed Dena in a Quantitative Analysis course during a unit of probability.

Data Sources and Analysis

We observed each participant for the length of a unit of study (approximately a month). Data sources included: teacher interviews and check-ins, ethnographic field notes, audio and video recordings of lessons, samples of student work, and other teacher specific artifacts the teachers used to interpret their students' learning and/or their instruction.

In the first stage of data analysis, we used multiple coding strategies to capture the teachers' use of knowledge of the student—for example: selective coding (Glaser, 1978) beginning with a start list of codes derived from the literature (Miles & Huberman, 1994) and simultaneous coding (Saldaña, 2009). We used constant comparative methods (Strauss & Corbin, 1998) and triangulation of data to uncover patterns related to knowledge of the student, teacher behaviors and learning goals. In the second stage of analysis, we conducted within-case analyses using explanation building methods (Yin, 2014), including pattern codes (Miles & Huberman, 1994).

To uncover deeper structure, we iteratively built analytic models for each case. Through a cross-case analysis, we generated a comparison profile for a cross-case model. All patterns were tested with the individual models and the cross-case model. This paper focuses only on the cross-case model, but it represents the result of the full analytical process.

Results

Analysis uncovered knowledge of students across cases, how teachers used this knowledge, challenges to their use, and the teachers' learning goals. Due to space limitations, we focus on common knowledge of the students used across the four cases and we share the cross-case model that captures the central patterns. Please access the full study for details (Urbina-Lilback, 2018).

Common Knowledge of the Student

Teachers used a wide variety of categories of knowledge of their students, ranging from mathematical aspects (e.g., student thinking) to nonmathematical aspects (e.g., competing priorities outside the classroom and family support). Some of these forms were highly situated, like teachers' understanding of students' mathematical learning development. Of this wide range of knowledge categories, ten knowledge categories were evident in all cases (see Table 1).

Table 1: Knowledge of the Student Used Across All Cases

Mathematical	Mathematical AND Nonmathematical	Nonmathematical
Mathematical Thinking	Learning Attitudes	Cultural Dissonance
Foundational needs	Ownership Over the Learning	Competing Priorities and/or SES Challenges
Language and Structure of Math		Personal Characteristics or Conditions
Mathematical Strengths		

We provide a description here for two categories that are central in the larger phenomenon across cases – mathematical thinking and ownership over their learning.

Mathematical thinking. Teachers highly prized students' mathematical thinking. Most of the classroom activities that teachers facilitated centered on students' exchange of mathematical thinking. Teachers deliberately sought students' partial understanding as opportunities to learn. Mathematical thinking often involved two other categories of knowledge, specifically – foundational needs and mathematical language and structure. Mathematical thinking was found to be both a means and an end and, thus, a central form of knowledge of the students.

Ownership over their learning. Each teacher articulated particular indicators related to students' ownership over their learning. Common indicators included: responsibility in working outside the classroom to do homework and/or study and coming in for help outside the classroom times. In all cases, ownership over the learning process was construed as a form of self-advocacy. Analysis revealed that this was a central form of knowledge of the students.

Cross-Case Model: Your Thinking Can Empower You

The cross-case model (see Figure 1) illustrates the forms of knowledge of the student used across cases, how teachers used that knowledge, the factors that challenged or supported the teachers' knowledge use in practice, and the teachers' learning goals. The model captures the central patterns across all cases. We work from left to right in the model to briefly describe its components and also the larger phenomena that it represents.

Knowledge of the student. There were ten common knowledge categories (see Table 1). In the model, we list only nine to start with. The last one – ownership over their learning – was a characteristic the teachers worked to develop as a long term goal.

Teachers' learning goals. Knowledge about students was filtered through teacher's learning goals – both short term (e.g., how to do a particular problem) and long term (e.g., developing perseverance, self-advocacy skills, organizational skills or problem solving skills).

Teacher's use. This category refers to experiences that teachers created and/or facilitated to support their students' learning. Teacher's use was highly situated and influenced by knowledge of their students and learning goals. Teachers' use of knowledge of their students both mediated and moderated the learning experience. The model depicts the moderating role with two possible outcomes – **engagement** (and/or math thinking) or **low target** (students not engaged in learning).

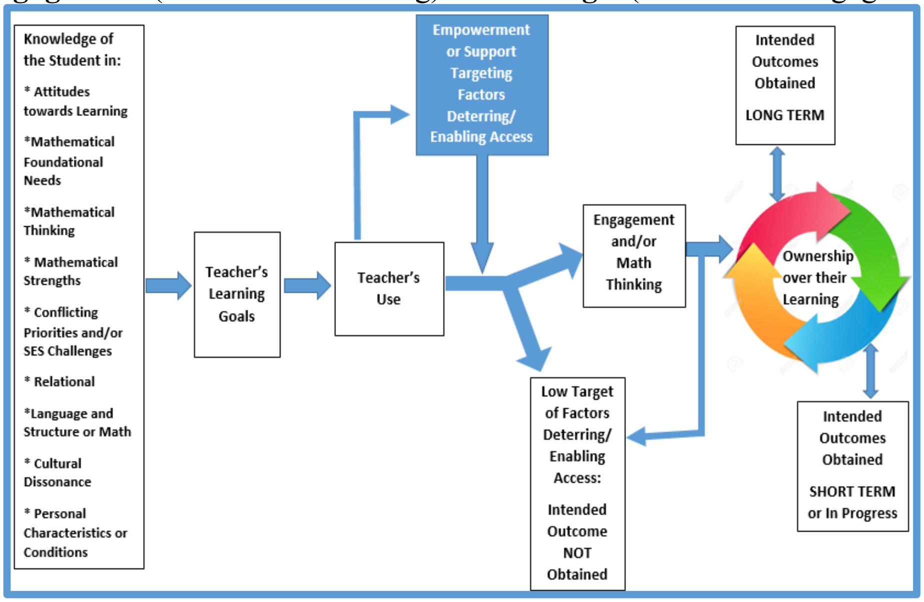

Figure 1: Cross-case Model – Your Thinking Can Empower You

Empowerment or support targeting factors deterring/enabling access. Even when teaching goals aligned with teachers' use, the effort to empower students needed to target particular student needs. Teachers' use depended on how much the teacher knew their students and also on how much the interventions targeted the particular needs. When teachers' empowerment did not target (or the target was very low), the students were less able to engage in learning.

Low target of factors deterring/enabling access. In some cases, teachers ascribed a negative or positive value (or both) to what they knew about their students, depending on whether they saw it as "deterring" or "enabling" learning. If deterring/enabling factors were not targeted (or targeted very low), students' ability to engage in math thinking was affected.

Engagement and/or mathematical thinking. Mathematical thinking was a means and an end in learning. Engagement in this central practice mediated students' development of more long-term outcomes that supported students with their ongoing learning.

Ownership over the learning. Each teacher provided descriptors and indicators of what ownership over learning meant for their students – for example, college and career readiness, mathematical muscle, organizational and self-advocacy skills, and perseverance. In some cases, intended outcomes were in the process of being developed, thus, **in progress** or **short term**. In other cases, intended outcomes were met, empowering students with a set of **long-term**, self-sustaining skills that shifted control over the learning process from the teachers to their students.

Concluding Thoughts

This study demonstrates how equity-oriented math teachers use knowledge of their students within their situated practice furthering both short term and long term learning goals. Mathematical thinking and ownership over learning are central to the cross-case model. Also relevant is that the teachers use many non-mathematical aspects about their students in order to enhance their mathematical learning. Findings are consistent with related research in multiple respects. They provide greater understanding of situated practice in support of equitable teaching and reflect the practice field's tendency towards empowerment through the culture of school.

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STRUCTURED PARTICIPATION PROMOTES ACCESS AND ACCOUNTABILITY DURING COOPERATIVE LEARNING IN MATHEMATICS EDUCATION

Nicholas Kim
University of Tennessee - Knoxville
nkim2@vols.utk.edu

Frances K. Harper
University of Tennessee - Knoxville
francesharper@utk.edu

This study investigated how the use of structured participation, specifically group presentation roles, supported more equitable participation in doing and learning mathematics. We analyzed videos of two lessons focused on radius, diameter, circumference, and area of circles in which students worked in small groups in a STEM project-based geometry class. Students' mathematics learning was assessed by the teacher through a group presentation where students were assigned individual presentation roles. We identified themes related to access and participation in small group and whole class interactions during this problem-focused lesson and found nuanced differences in both access and participation when compared to students' typical involvement in the more common unit-wide project-based tasks. Findings provide insights into effective uses of structured participation for both group work and whole class interactions to support more equitable doing and learning of mathematics in high school classrooms.

Keywords: Equity and Diversity, Instructional Activities and Practices, High School Education, Geometry and Geometrical and Spatial Thinking

Over the past thirty years, reform efforts in mathematics education have advocated for a greater focus on sense making and explanation and justification (e.g., NCTM, 1989; NCTM, 2000; NCTM, 2014). As reforms have slowly infiltrated more and more mathematics classrooms, the exclusive use of traditional direct instruction strategies has diminished in favor of greater emphasis on peer-to-peer collaboration in small groups (e.g., Boaler, 2008) and whole class discussion (e.g., Smith & Stein, 2018). Cooperative learning strategies found in complex instruction still provide some of the most effective ways of ensuring that peer-to-peer interactions in small groups positively impact student achievement in mathematics for all levels of learners. One of the key principles of complex instruction is the delegation of mathematical authority from the teacher to students, and a common way of ensuring that every student has an opportunity to take on mathematical authority is through the assignment of roles that structure the participation of group members (e.g., facilitator, recorder). The premise of roles is for all students to have important work to contribute in their groups, without which the group cannot function. Cohen and Lotan (1997) found when students take responsibility for each other and encourage everyone to contribute equally to tasks by performing their respective roles, there is an increase in student success in terms of mathematical understanding, and Boaler (2008) found group roles helped promote relational equity within the mathematics classroom. Boaler describes relational equity as "relations that include students treating each other with respect and considering different viewpoints fairly" (p. 168).

Langer-Osuna (2011), as well as Herrenkohl and Guerra (1998), found group roles served as one means of structuring student participation. Structuring participation increases the likelihood that students become positioned as competent, mathematical authorities. Students positioned with authority participate frequently in small groups and are able to gain access to and hold the floor, are able to decide what is correct, are seen as contributing meritorious ideas, and become influential in student-led discussions (Cohen & Lotan, 1995; Engle, Langer-Osuna, & McKinney de Royston, 2008; Inglis & Meja-Ramos, 2009). Erickson and Schultz (1997) defined participation structures as interactionally marked ways of speaking and listening, getting and holding the floor, and leading and following. In complex instruction, roles are typically designed to structure small group interactions, which are not specific to the mathematics work, but instead structure how the group cooperates (i.e., facilitator, not “grapher”) (Cohen, 1994). This study builds on prior investigations of how the assignment of roles for working in small groups can promote access to mathematical understanding typically reserved for higher-achieving students who take over group assignments. We extend this work by exploring a non-traditional approach to role assignment and structure, namely assigning roles that structure how group members will present their mathematics work in a whole class setting. We considered how these presentation roles influenced more equitable access and participation among group members in both the small group and whole class settings. We theorized that, while providing access to mathematics for every learner, presentation roles might also provide teachers more ways to monitor student accountability. Thus, we aimed to investigate whether group presentation roles increased student participation differently than traditional group roles such as leader, time keeper, and materials manager for students who historically let others dominate group work activities.

Theoretical Framework

We draw on *figured worlds* as the theoretical basis for this study. People use figured worlds as a way to interpret particular actor and actions. These “worlds” are socially, culturally, and politically constructed and provide a way to “figure out” the significance of certain acts or the value of particular outcomes (Holland, Lachicotte Jr., Skinner, & Cain, 1998). As various factors (e.g., actors, actions, social/cultural/political forces) come together and relate to each other in different ways, *storylines* emerge that provide a taken-for-granted sequence of events for making meaning of actors, acts, and events (Holland et al., 1998).

For example, a possible storyline for collaborative projects in the mathematics classroom might be: “At the beginning of a project, one student takes the lead.” Students take this action for granted, but depending on who takes the lead, students make sense of that action differently. If a boy takes the lead, he is likely positioned as smart, whereas a girl is likely positioned as bossy for similar actions (Langer-Osuna, 2011). The identities of ‘smart’ and ‘bossy’ are enacted by students and assigned by their peers based on specific classroom expectations for and interpretations of actors and actions, which are in turn are influenced by broader social, cultural, and political interpretations of gender identities.

In this way, the storylines constructed through figured worlds provide the context for what counts as mathematical engagement and for how students make sense of themselves as successful or not in relation to that engagement. In other words, students’ mathematics identities are shaped as they come to see themselves and are seen by others as mathematically capable (or not) in relation to storylines (Horn, 2008). We use storylines to operationalize how students negotiated (i.e., took up, resisted, or shifted) mathematics identities in one cooperative task.

Research Design and Methods

This study was taken from a subset of data from a larger study that reviewed how students took up, negotiated, shifted, or challenged an innovative mathematics teaching approach using STEM project-based learning across an academic year (Harper, 2017). The study took place at a STEM-themed magnet school, in a low-income area of a small Midwestern city, whose mission emphasizes technology-driven project-based learning. The study took place in one ninth grade geometry classroom which consisted of 16 consented research participants. There were six males (4 white, 1 Black, 1 Latino) and ten females (9 Black, 1 Asian American). The teacher participant is a White woman who was in her fourth year of teaching. She and the second author collaborated on various projects focused on equity and social justice in mathematics education for three years, and the teacher frequently attended various professional developments on technology and equitable collaboration. The study will narrow in on one focus group of three females: Rosy, an Asian American student with high status; Carley, a Black student with high status; and Monique, a Black student with low status. We characterize students based on status rather than achievement because the construct of status recognizes that abilities in mathematics are socially constructed rather than cognitively fixed. Status is an idea commonly used in complex instruction to describe the social ordering of individuals based on perceived academic ability and social standing, where everyone agrees it is better to have a higher status (Cohen, 1994). Status can change in moment-to-moment interactions (Wood, 2013), but overarching perceptions of status influence how students describe themselves and others as “good at math” and “not good at math” in more rigid ways.

Focus Lesson Overview

Across two days, students worked in groups of three on radius, diameter, circumference, and area circle problems. The assignment was not part of a larger project, as was typical in this classroom. Instead the lesson involved problem-based learning around seemingly straightforward procedural problems through collaborative group work. The assignment consisted of twelve total circle problems, divided into four groups so that each group was responsible for presenting two or three assigned problems. Instead of the typical group roles used in complex instruction, such as facilitator, materials manager, or time keeper, to maintain accountability, the teacher assigned presentation roles for the whole class presentation to encourage engagement in the activity and accountability for individual learning in small groups. The three presentation roles were: (1) explainer, (2) question answerer, and (3) connector. The explainer was responsible for the initial explanation of the three problems; the question answerer was the only person who could respond to teacher and peer questions after or during the explainer’s presentation; and the connector had to note similarities and differences to other problems presented by other groups to reinforce the big ideas of the assignment (i.e., conceptual connections and relationships across radius, diameter, circumference, and area of circles). During the first day, students worked on agreeing upon solutions to their assigned problems and preparing for their presentation. The students used a large poster board to present their work to the whole class at the end of the first day and into the second day.

Data Analysis

The first author created detailed descriptions of the actions, emotions, and body language in videos alongside a rough transcription of what the teacher and students said across two days in which the mathematics task was enacted. We analyzed this summary of the video in Nvivo (a qualitative analysis tool) by creating codes based on the strong theoretical basis of complex instruction (e.g., status, roles) (Yin, 2009). We identified initial codes independently and then compared preliminary analyses to arrive at final codes, which the first author confirmed by applying to the video and paying attention to exactly what was said along with actions, emotions, and body language. Next, we used the final codes and references to coded instances in the video of classroom observations to continue to refine themes about who and what was granted agency in the doing and learning of mathematics. These themes helped us to identify broader storylines at play during the lesson, based on talk, gestures, actions, etc., that were collectively constructed by and guided interactions among the focus group members, the teacher, and other students in the mathematics classroom figured world. For more information on how figured worlds guided analyses in the larger study, see Harper (2017).

Findings

In this section, we present our findings as the storylines that were collectively constructed by and guided interactions among the focus group members, teacher, and other students in this particular lesson. We provide illustrative evidence of group interactions and talk that show how we came to identify each storyline. These storylines shed light on how access and participation were negotiated among group members and the teacher in both small group and whole class settings by showing who and what was granted agency in the doing and learning of mathematics. More specifically, our findings show a dominant storyline of inequitable participation and access and a momentary, emerging storyline that shows promise for more equitable access and participation following the introduction of presentation roles that influenced who and what was granted agency. The first two storylines were collectively constructed as the group members worked in a small group setting, and the third storyline was collectively constructed during the group's whole class presentation.

Dominant Storyline (Before Teacher assigned presentation roles): Only the "good" Mathematics Students Do the Mathematical Work of the Task

Rosy and Carley, as the students with high status in the group, did the majority of the mathematical work while Monique, as the student with low status in the group, did not engage with the group or the mathematics. Monique's attention was not on any relevant mathematical content, but rather her laptop where she engaged in social media and computer games even after instructed by the teacher to put away all irrelevant technology. Rosy and Carley did not try to engage or encourage Monique to participate as they had become accustomed to Monique's lack of engagement in the class through previous group activities (see Harper, 2017 for evidence of this claim about how Monique was positioned in storylines outside of this specific lesson). After Rosy and Carley discussed how to solve their assigned problems, they shifted their attention to writing on the poster board for the presentation at the end of class. At this time, Monique put her laptop away to make room for the poster board at their tables. Monique began watching Carley telling Rosy what to write on the poster board for each assigned problem, but it was unclear if she was trying to understand the solutions. Rosy and Carley were granted agency for the mathematical work as they were considered high-status students. Being a high-status or "good" mathematics student did not require including other students in the mathematical work of the group.

Emergent Storyline (After Teacher Assigned Presentation Roles): "Good" Mathematics Students Include Other Students in the Mathematical Work

The teacher introduced presentation roles and assigned them to each student after allowing the groups to work for 15 minutes. Rosy and Carley maintained ownership over the mathematical work and direction of the group, based on their status as "good" mathematics students. For example, Rosy expressed how she wanted to do multiple roles because she frequently answered teacher and student questions during individual and whole class interactions:

> Rosy: Shoot, why do we got to all have different jobs, cause like I want to be the explainer and the question answer.

> Teacher: (*After overhearing Rosy's comment*) Well you got to delegate some work out, but you guys only have #9 and #11 when you go up there.

After the teacher's insistence that all group members play a role in the mathematical work of the whole class presentation, Carley and Rosy made a noticeable effort to include Monique and encourage her participation. Carley assigned roles to her group members after this short exchange between Rosy and the teacher (maintaining authority over the group's work). She assigned Rosy to be the explainer, and Rosy agreed. Carley then assigned Monique the position of question answerer. Carley took it upon herself to be the connector even though she was confused about the role and repeatedly questioned the teacher and her group members about the purpose of the role.

After roles were determined, Rosy made the first real effort to engage Monique with the mathematical work she and Carley had completed by asking, "What do you need to be the question answer?" Monique responded so softly that it was inaudible in the video. Rosy began to explain step-by-step how she and Carley solved each assigned problem, and she made additional efforts to ensure that Monique understood the mathematical work. Namely, Rosy provided additional information about circles, such as the relationship between radius and diameter, as she explained the steps:

> Rosy to Monique: (*Carley is looking at her phone but also trying to listen to Rosy)* So, if they ask you a question… so basically, I am going to explain it but like you might have to re-explain it if they ask you a question. If they don't get it, you'll have to explain it again.
>
> So, number 11 says find the radius. So, I said the purple one is radius, right? So that means the radius is half of this line, so the information they gave us is the diameter which is this part right here. We need to figure out radius, so if the radius is 4.8, we have to, we have to split that in half. Because half of the diameter is radius. So, half of 4.8 is 2.4. Okay so it's easy to divide the diameter by 2.
>
> Number 9. So basically, we are finding the circumference of the circle. The circumference is all around the whole circle (*Rosy draws it*). Umm, the given information we have the area. The area is everything inside the whole entire circle. (*Rosy models it on paper again*) You know area right. (*Monique shakes her head yes*) Umm, so basically, we know the formula to find the area is pi times r squared and r stands for the radius. So basically, what we use, we took the area and divided it by pi because that's what we times it by... pi -r squared is the formula to find the area. We divided that by pi to cancel each other out."

Monique rarely spoke and mostly watched and listened to Rosy explaining each part of the problem. Although many might not consider this active engagement on Monique's part, this level of access to and participation in the mathematical work was a notable improvement when compared to her access and participation before the introduction of presentation roles. In both Carley's assignment of a role to each group member and Rosy's "teaching" of the mathematical work to Monique, we see both Carley and Rosy maintaining their status as "good" mathematics students who have authority over the work of the group. However, their actions following the teacher's introduction of presentation roles begin to shift what it means to be "good" mathematics students to include encouraging participation by others in the mathematical work of the group.

Dominant Storyline (During Whole Class Presentation): Only the "Good" Mathematics Students Do the Mathematical Work of the Task

During the whole class presentation, Rosy and Carley enacted their presentation roles as explainer and connector, respectively. Rosy began the presentation with an explanation of problem 9 and then problem 11 when prompted by the teacher:

> Teacher: Okay, what about #11?
> Rosy: We had to find the radius of the circle which is the point in the middle to the side of the circle, but the only information given was the diameter which is one side of the circle to the other. And um so basically, that means the radius is half of the diameter. So, we had to divide the diameter in half. 4.8 divided by 2 is 2.4."

The teacher then asked Carley to make connections to other problems:

> Carley: [Our problem] was similar to all of these problems from 15 to 9.
> Teacher: How is it similar?
> Carley: Because they to do the square and division to get rid of the multiplication (Inaudible)…

These excerpts illustrate how the teacher encouraged both Rosy and Carley to explain further or in greater detail. Thus, both Carley and Rosy, as high-status students, were positioned as student capable of learning and doing mathematics. The teacher's and other classmate's positioning of Monique, however, was noticeably different when Monique enacted her role as question answerer.

> Teacher: Does anyone have any questions for this group?
> Student 1 from another group: Why did you guys use secondary colors [on your poster]?
> Teacher: Monique, do you have an answer for that?
> Monique: Because we just wanted to use them.
> Student 2 from another group: I have a question. Why did you have to find the radius on that problem?
> Rosy: It made it easier for us to calculate 2 pi r when we have r.

In our analyses of other group presentations, the teacher enforced the individual presentation roles for all group members. The above excerpt shows how the teacher's enforcement of the presentation roles changed when Monique engaged with her role. The initial question posed to Monique was a non-academic question in which the teacher required Monique to answer based on her opinion, not on her capability to learn or do mathematics. When another student posed the second question about finding the radius, Rosy immediately answered the question before Monique had a chance to answer. The teacher enforced the roles for all other students but allowed students with high status to take on the role for a student with low status in this instance. This excerpt illustrates the power of the dominant storyline that only "good" mathematics students do the mathematical work of the task and shows how the teacher and other students were complicit in positioning Monique as someone incapable of participating in that mathematical work.

Discussion and Conclusion

Much of the research on complex instruction at the secondary level in mathematics focuses on "master" teachers who are experienced with the pedagogical approach (e.g., Boaler & Staples, 2008) or contexts in which students experience complex instruction across the curriculum (e.g., Horn, 2008). This research extends what is known from previous studies by exploring one aspect of complex instruction within a classroom where the teacher is newer to using complex instruction and where complex instruction strategies are not reinforced across the curriculum by other teachers. Findings from such investigations are important because they can shed light on how the moment-by-moment negotiations that must happen among the teacher and students to shift who and what is given mathematical agency (e.g., Harper, 2019; Wood, 2013). Changes towards more equitable access and participation in mathematics do not happen instantaneously with the introduction of strategies such as structured roles for participation. Instead, these shifts are negotiated, resisted, and (hopefully) taken up over time, and this study contributions to our understanding of how such changes to access and participation may begin. In this final section, we identify and discuss two implications based on the findings presented here.

Pervasiveness of Inequitable Access and Participation

Both the first and third findings above illustrate how powerful and pervasive storylines that encourage inequitable access in and participation in mathematics learning and doing are. In small group work, Rosy and Carley took ownership of the task by discussing and solving their assigned problems, using appropriate terminology and formulas while demanding justification for each process. Unfortunately, Monique had little involvement with the discussion. During the whole group presentations, Monique seemed to encounter yet another missed opportunity for mathematical engagement. Interestingly, the teacher consistently enforced presentation roles for high-status students, and she pushed the high-status students such as Rosy and Carley for justification and more detailed mathematical explanations. She was less consistent with her enforcement of roles for student with low status. The teacher enforced Monique's role as question answerer for a non-academic question posed by her peers but then allowed Rosy to take over Monique's role when a mathematical question was asked. We might speculate as to why the teacher allowed Rosy to take over Monique's role. Perhaps the teacher worried that Monique may not know the answer, and she wanted to avoid calling her out as the question answerer for a mathematical problem. Perhaps the teacher was happy to see Monique participating more than she typically did and wanted to encourage her to feel safe participating in the future. The teacher's knowledge of her students likely played a part in her decision to allow Rosy to take over. Regardless of the teacher's exact rationale, however, one implication to consider is that we, as teacher educators and researchers, may need to better prepare teachers to navigate the gradual and complex shift towards more equitable access and participation. Recognizing how powerfully storylines that encourage inequitable access and participation might resist change may help teachers remain diligent in attempts to incorporate strategies such as structured roles more consistently.

Variety of Roles for Structured Participation

Findings from the emergent storyline show how, after presentation roles were introduced, Monique had the greatest access to and participation in doing and learning mathematics through the whole lesson. Although Carley and Rosy maintained most of the mathematical authority, Rosy used that authority to prepare Monique for her role as the question answerer. Rosy was motivated to engage Monique because she knew that Monique needed to understand the mathematical work in order to answer any questions that arose from the explanation. Monique made an effort to watch and listen to Rosy's explanation and subsequent discussions. Monique did not provide much of her own mathematical thinking, but she made the effort to understand her group's mathematical thinking. Interestingly, a role that was intended to promote more equitable participation in whole class presentations provided greater access and participation in the small group setting instead. This shows that participation structures such as presentation roles, which are not typically associated with complex instruction, might have a larger than intended effect by providing accountability across small group or whole class settings when those roles are enforced.

Although employing various types of structured roles for participation shows promise for increasing engagement, the findings here show that increased participation does not necessarily mean greater active engagement with the mathematics. Students may be motivated to participate more because they are assigned a specific role, but if those roles are not enforced by the teacher with fidelity then students may depend on their peers to do the heavy mathematical work. A combination of the assignment of group presentation roles with teacher enforcement of such might prove an effective way of using this particular structured participation format. If students know they will need to present to the whole class as individuals in a group assignment, they cannot hide completely behind high-status students. They will have to become experts in their problem to be prepared for various question types from peers or teacher. It is the role of the teacher to explain the roles extensively as well as enforcing the roles with the same rigor for every student.

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MATHEMATICS TEACHERS' BACKGROUNDS AND THEIR BELIEFS ABOUT TEACHING DIVERSE STUDENTS

Tarik Buli
University of Maryland
tbuli@umd.edu

This study investigates if there is a relationship between mathematics teachers' backgrounds and their beliefs about teaching students of color. To study this relationship, an exploratory factor analysis was conducted on a teacher beliefs survey administered to teachers who entered the New York City Teaching Fellows in the summer of 2006 or 2007. The survey items which were analyzed are rooted in culturally relevant pedagogy. Linear regression models were used to determine if there is a significant relationship between the teacher's background and each belief factor that emerged. Findings indicate that the teachers' race, college selectivity, and age at the start of their teaching career influence their beliefs about teaching students of color who are traditionally underrepresented in mathematics.

Keywords: Equity and justice; Culturally relevant teaching; Affect, emotion, beliefs, and attitudes; Social justice

In recent years there have been repeated calls to increase the talent of the teaching pool while also diversifying it. Selective alternative teacher certification programs (ATCPs) like Teach For America and Teaching Fellows programs are one mechanism for attracting academically elite talent to urban teaching, supplying large numbers of math teachers to high-poverty schools in urban districts nationally. These programs also report admitting increasingly diverse proportions of teachers of color for recent cohorts (TeachForAmerica, 2018).

A question is whether elite, generally White college graduates approach their roles as mathematics teachers differently than Black and Latinx teachers in these programs. Accordingly, what do different kinds of prospective urban mathematics teachers believe about their students of color who are traditionally underrepresented in mathematics, and those students' capacities to learn advanced mathematics? The purpose of this study is to address the research question: is there a relationship between mathematics teachers' backgrounds and their beliefs about teaching students of color in high-poverty schools?

Theoretical Framework

Esmonde (2009) explains that within the mathematics classroom, each interaction amongst members of the community, including individual students, their peers, and their teachers, contributes to constructing, challenging, and reifying their social positions. These social positions are based not only on the internal classroom dynamics, but also on the identities ascribed to individuals as part of their broader community membership. Different community memberships, including racial and cultural backgrounds, are positioned differently within the math classroom. In mathematics education literature, it is well documented that students from marginalized backgrounds, particularly African American, Native American, and Latinx students, are amongst the lowest performers on mathematics achievement tests, especially compared to White and Asian American students (Martin, 2012). Less studied is the relationship between these oft-referenced achievement gaps and the "adverse conditions under which some children are often forced to learn, the privileged conditions afforded to others, and how forces like racism are used to position students in a racial hierarchy" (Martin, 2009, p. 300).

Contrary to perspectives which hold mathematics to be a neutral, logic-based content with no social preferences, in fact mathematics "plays a unique role in producing and maintaining racial inequity in U.S. society" (Martin, 2009, p.308). In mathematics, this may manifest through racial apathy and color blindness as dominant ways to consider problems of racism and race in math environments, rather than recognizing the racialized structure in math classrooms as simply mirror images of race and racism in society at large. Racial apathy refers to the indifference towards racial inequality in society and lack of engagement with social problems related to race (Forman, 2004 [as cited by Martin, 2009]). Color blindness refers to the beliefs that societal institutions function meritocratically, so race is no longer relevant, and any race-based patterns of social inequality are due to individual or group cultural deficiencies. This ideology leads to the resulting belief that no systemic efforts are needed to respond to racialized outcomes.

Furthermore, pressure to recruit teachers who were high achievers has led to the rapid increase of alternative certification programs, which sometimes adopt "missionary-like goals relative to African-American, Latino, and Native American children and position[] White teachers as the savior of these children" (Martin, 2009, p. 305). This focus on high achievers leaves the following questions unanswered: "How do these teachers construct ideas and beliefs about mathematics ability among their students? [and] How are these constructions affected by teachers' beliefs about race?" (Martin, 2009, p. 305). Thus, while these teachers may possess mathematical knowledge, they may lack knowledge related to problems of equity and race.

This work examines how a sample of alternatively certified teachers make sense of mathematics teaching and learning for their students of color at urban NYC public schools. I examine to what degree a sample of NYCTFs agree with widely held social beliefs that may construct students of color from a deficit perspective, which in turn may limit the students' access and opportunities to succeed in mathematics. Additionally, this work considers whether there are group differences among how the sample of teachers surveyed make sense of mathematics teaching and learning for students of color. That is, do teachers' beliefs vary based on the teacher's race, age, where they grew up, and selectivity of the college they attended, with implications for teacher recruitment discussed below.

Methodology

To answer the research question, I analyzed data from a survey that included measures of mathematics teachers' sociocultural beliefs. These teacher participants entered teaching from 2006-2007 through the New York City Teaching Fellows (NYCTF), a selective ATCP that was the largest teacher training program in the United States at that time, training some 300 to 400 secondary mathematics teachers annually. Three hundred and seventy-four of the approximately 620 NYCTF mathematics teachers who entered in 2006 or 2007 took this survey in 2016 and 2017. The racial demographics of the teachers in the analytic survey were 59% White (n=219), 15% Asian (n=55), 19% Black (n=70), and 8% Hispanic (n=30).

Survey Data

A set of 37 survey items were developed to answer the study questions. These questions were rooted principally in Ladson-Billing's (1995) theory of culturally relevant pedagogy (CRP), critical race theory, and Martin's (2007, 2009) research on racialization in mathematics classrooms. They were on a 5-point Likert scale ranging from 1 (Strongly Disagree) to 5 (Strongly Agree). In developing the items, existing measures of teachers' multicultural beliefs, perceived cultural competencies, and racial privilege were referenced. Draft items were reviewed by three mathematics education scholars with relevant expertise. For face validity, the items were vetted with current urban mathematics teachers as part of cognitive interviews.

School Data

These teachers' survey history and school assignment data were reviewed from the NYCDOE. This data was used to link the participants' survey responses to demographic data (race, attendance rates, levels of poverty) on the students in their assigned schools.

Exploratory factor analysis. To study the relationship between the teacher's background and their beliefs, an exploratory factor analysis (EFA) was conducted. The *following* 5 clusters of teacher beliefs emerged from the factor analysis: Culturally Responsive Teaching (CRT), Awareness of Privilege (AP), Teaching Students of Color Rewarding (TSCR), Teaching Multicultural Education in the Math Class (TMEMC), and Teaching for Social Justice (TSJ).

Linear regression. I used five linear regression models to determine if there is a significant relationship between the teacher's background and the five factors identified as part of the EFA. In the linear regression model, the dependent variables are the 5 belief factors that emerged. The independent variables are the aforementioned teachers' background characteristics. This includes the teachers' race (RACE), which is categorized as White, Asian, Black, or Hispanic; the teachers' age (AGE) at the start of their teaching career which is categorized as younger (age 23 or younger), average (ages 24-26), or older (27 or older); where the teacher grew up (LOC), which is categorized as in NYC, NYC Suburb (within 150 miles of NYC) or outside of NYC (greater than 150 miles from NYC); and their college selectivity (SEL), ranging from 1 as very selective to 5 as not selective. Controls were used for the teaching context of teaching fellows' first school where they taught, which included continuous variables measuring the attendance rate of the school (ATT), the percentage of students receiving free and reduced lunch (FRL) and the percentage of students who are Black, Hispanic, or American Indian (BHAI).

Table 1: Regression Results for Five Belief Factors

Measure	CRT	AP	TSCR	TMEMC	TSJ
	Coefficient (SE)	***Coefficient (SE)***	***Coefficient (SE)***	***Coefficient (SE)***	***Coefficient (SE)***
Intercept	-1.576 (.98)	-.56 (.95)	-1.004 (.95)	.795 (.94)	-.436 (.97)
Attendance Rate	1.38 (.98)	-.195 (.95)	1.336 (.95)	-1.043 (.94)	.151 (.97)
Race: Asian	.112 (.17)	.111 (.17)	-.109 (.17)	.099 (.17)	-.223 (.17)
Race: Black	.249 (.16)	.098 (.16)	.279* (.16)	.233 (.16)	.414** (.16)
Race: Latino/a	.213 (.23)	.112 (.22)	.552** (.22)	-.072 (.22)	.256 (.23)
NYC Suburb	-.04 (.17)	.274* (.16)	.228 (.16)	.024 (.16)	.14 (.17)
Outside NYC	.082 (.15)	.393*** (.15)	.173 (.15)	.15 (.15)	.256* (.15)
Very Selective	-.06 (.15)	.246* (.14)	.011 (.14)	.309** (.14)	-.091 (.15)
Least Selective	.043 (.14)	-.016 (.14)	.033 (.14)	.029 (.14)	.028 (.14)

Younger	-.048 (.14)	.277** (.14)	.177 (.14)	-.01 (.14)	.012 (.14)
Older	.072 (.15)	-.156 (.14)	-.121 (.14)	-.196 (.14)	-.115 (.15)

*p<.10; **p < .05; ***p < .01

Note. The referent group is White teachers, age 24 - 26 years at time of entry, who graduated from a NYC high school, and attended a moderately selective college.

Discussion of Results

As shown in Table 1, the teachers' race was significantly associated with two of the five belief factors. In particular, Latinx (β=.552, p=.013) and Black teachers (β=.279, p=.080, significant at .10 level) rated teaching students of color as more rewarding than did White teachers. Additionally, compared to White teachers, Black teachers believed more strongly in teaching for social justice in their mathematics courses (β=.414, p=.011). These findings are consistent with literature that shows that teachers of color, particularly Black and Latinx teachers, who share cultural backgrounds with their students may leverage this connection in their instruction (Kohli, 2018; Lee, 2004).

Attending a highly selective undergraduate institution was a significant predictor of two of the belief factors. Compared to teachers who attended moderately selective colleges, teachers who attended highly selective schools believed more strongly in the importance of incorporating multicultural education into the mathematics classes they taught (β=.309, p=.030) and were most aware of their racial and class privilege (β=.246, p=.089). These teachers may be more privileged themselves, so their heightened awareness may stem from their similarly heightened privilege. Additionally, given their social capital, these teachers may have chosen to become teachers because they believe teaching is a mechanism for social change (Tamir, 2009) and may believe that teaching multicultural education is a means to disrupt the inequities that their students of color may face.

The teachers' age and where the teacher grew up in relation to NYC also significantly predicted their beliefs. The model indicates that teachers who grew up farther away from New York City, or started teaching at younger ages (23 or younger), were more aware of their racial and class privilege than teachers who grew up in New York City or started teaching at ages 24-26. Also, teachers who grew up outside of New York City believed in teaching mathematics for social justice more so than teachers who graduated from New York City high schools. Teachers from outside of NYC may believe in social justice more strongly than those from NYC for similar reasons as to why the elite college graduates more strongly believe in multicultural education. They may choose to come and teach in urban schools for their desire to effect social change.

Of the five belief factors which were analyzed, only the linear regression model for the CRP belief factor had no significant predictors from the teachers' background on their beliefs.

Significance

Because selective ATCPs like NYCTF recruit their teachers from elite colleges which are generally predominately white and tend to severely underrepresent racial minorities, these findings point towards the need to recruit from other networks of potential teachers. This may mean more intentional recruiting from moderately selective colleges which admit and graduate higher rates of students of color, than their elite counterparts. If teachers who are racial minorities themselves have funds of knowledge which enable them to better teach their students of color, and find this process rewarding, then recruitment efforts can be tailored to bring these teachers into the profession. Further study could examine whether similar trends exist among in-service and pre-service teachers trained along traditional routes, and point towards a similar shift toward recruiting more teachers who themselves come from marginalized backgrounds.

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STUDENTS' PERCEPTIONS OF OPPORTUNITIES TO LEARN IN TESTING-ORIENTED MATHEMATICS CLASSROOMS: A MIXED METHODS STUDY

Rosa D. Chavez
Stanford University
rdchavez@stanford.edu

This study examined the relationship between testing environment and perceived learning opportunities for students from historically marginalized communities using survey and video data from the Measures of Effective Teaching Longitudinal Database. Student surveys (N=24,208) were analyzed and showed that students perceived their opportunities to learn to increase as testing environment increased. However, representative videos of both high- and low-testing classrooms in which the majority were students of color showed that opportunities to learn decreased as testing environment increased. This indicates a need to shift away from raising test scores towards a focus on how high-stakes testing policies may be influencing perceived learning opportunities.

Keywords: Student Perceptions, Opportunities to Learn, Policy Matters, Classroom Practice

Introduction

Testing practices, now routine and highly consequential, claim to be rooted in equitable practices (Au, 2016; Cross, 2007; Foster, 2016) but do not address the underlying structural issues of inequality that marginalized students face in and outside the classroom (Flores, 2007; Gutierrez, 2008; 2014; Ladson-Billings, 2006). Little work has focused on how high-stakes testing policies enable or constrain opportunities to learn in mathematics classrooms.

Understanding how students perceive their opportunities to learn may offer insights into how students make sense of what they should be learning in U.S schools. Yet, these perceptions may not align with what educators and psychologists believe are robust opportunities. This study examined this tension through the following questions:

1. What is the relationship between federally mandated standardized assessments and students' perceived opportunities to learn within mathematics classrooms?
2. What are the actual learning opportunities in these classrooms as captured through representative videos?

Theoretical Framework

Research has shown that students' mathematical activity is social in nature and developed through participation in communities of practice (Cobb & Yackel, 1996) with particular patterns of interaction, understandings, assumptions, attitudes, and norms (Cobb & Bauersfeld, 1995; Cobb, Gresalfi, & Hodge, 2009; Engestrom, 1999). It is difficult to separate what students are learning from the ways in which they are learning it (Beach 1999; Boaler, 1997; Cobb & Bowers, 1999; Lave, 1988). Thus, this study analyzes students' perceived opportunities to learn in mathematics classrooms using Gresalfi & Cobb's (2006) *Opportunities to Learn* framework, which focuses on learning opportunities that foster the development of positive dispositions towards mathematics.

This paper examines how students perceived the learning opportunities in their classrooms in relation to whether the classrooms were also perceived as highly or minimally oriented towards testing. It then examines whether students' perceived learning opportunities are evidenced by examining classroom interactions across representative videos.

Methods

Data for this study comes from the Measures of Effective Teaching Longitudinal Database (MET-LDB). Through the use of linked survey and classroom video data, this dataset enables connections between the ways students are positioned during classroom interactions and their perceived opportunities to learn. This study analyzed survey responses for 24,208 students and looked at eight classroom videos. The population included 23% White, 31% Black and 35% Latinx students. Roughly 49% were eligible for free or reduced-price lunch and 14% were classified as English Language Learners.

This study looked at the relationship between testing environments and learning opportunities by aligning the student perception survey in MET-LDB to *opportunities to learn* from the literature. Testing-oriented classrooms that focus on state assessments may be leading students to develop particular dispositions towards mathematics. Using results of survey analysis, two classrooms were selected to determine characteristic practices of both high- and low-testing environments. Three additional classrooms in each category were randomly selected to determine if similar patterns exist. Using Nvivo coding software, all eight videos were logged and subsequently coded for student agency, student intellectual authority vs teacher authority, behavioral control, cognitive demand of math being constructed, student participation, teacher content knowledge error, emphasis on answer vs. process, and structure of classroom.

Results

Perceived Opportunities to Learn and Student Achievement Outcomes

Different classrooms provide different opportunities for participation. Overall, students perceived opportunities to learn to increase as their perceived testing environment increased, and the patterns varied across elementary ($\alpha = .88, se = .03, p < .0001$) and secondary ($\alpha = .77, se = .01, p < .0001$) contexts. As students progressed into middle school, the relationship between perceived testing environment and perceived opportunities to learn slightly decreases. (Figure 1).

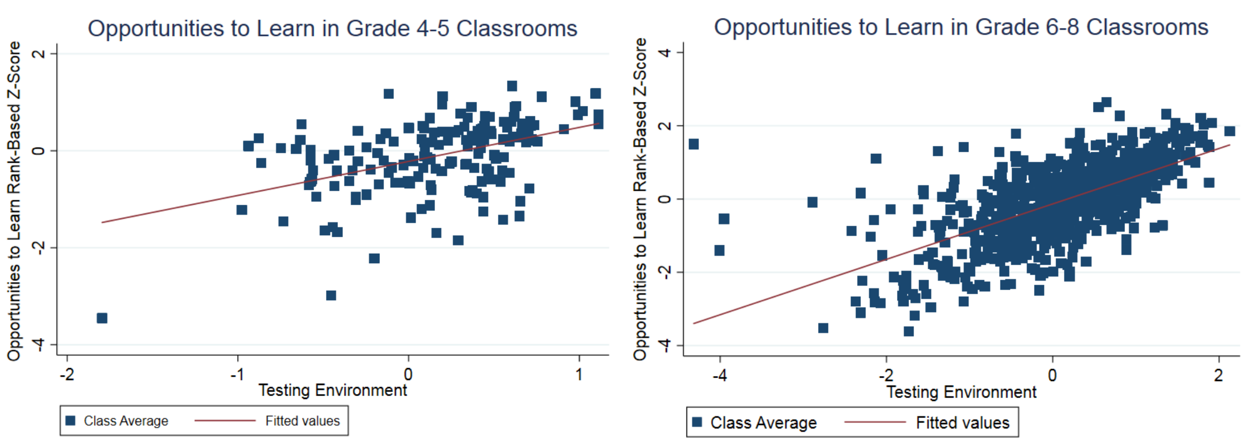

Figure 1: Perceived Opportunities to Learn Across Grades 4-8 Using OLS Regression Models

Instructional Practices in Classrooms

This section reports actual learning opportunities made available through social interactions in representative classrooms with majority students of color, and then compares this analysis to the larger quantitative trends. Four high- and low-testing classrooms were selected.

In low-testing classrooms, teachers provided more opportunities for students to engage in sense-making, procedural fluency, and encouraged students to make real world connections. Across all high-testing classrooms, the cognitive demand of the tasks that students participated in was low, and students were provided less time to meaningfully engage with the task or questions being posed.

Table 1: Code Frequencies for High- and Low-Testing Classrooms.

	Low-Testing Classrooms				High-Testing Classrooms			
Code	A	B	C	D	E	F	G	H
Student agency	3	6	1	4	1	3	4	7
Student intellectual authority	1	3	0	1	0	1	1	2
Sense-making	2	7	0	1	0	0	1	4
Teacher scaffolding	2	4	3	2	4	4	2	3
Encourages justification/rationale	0	6	3	1	1	4	0	3
Praise	2	0	1	1	4	2	0	3
Collective learning	1	0	0	0	0	1	0	1
Real world connection	2	4	0	1	0	0	0	0
Task computation	0	6	3	0	0	0	0	0
Low cognitive demand	3	0	4	8	8	7	6	4
Reduced cognitive demand	0	0	2	1	6	2	7	0
Inequitable learning opportunities	1	1	1	4	4	0	2	0
Focus on answer	3	0	4	8	5	1	3	0
Position students with incompetence	1	0	1	6	5	0	0	0
Control of student bodies	2	0	5	6	5	0	1	0
Control participation	2	0	5	4	7	1	4	0
Asserting teacher authority	5	0	1	6	5	2	5	1
Teacher error	0	0	0	3	5	0	0	0
Lack of scaffolding	2	3	2	8	4	1	0	0
Test prep	0	0	2	0	0	1	3	0
Mentions of state test	3	0	1	0	0	0	0	0
Task instructions	3	0	0	1	5	0	1	3
Resources	0	0	4	2	2	0	2	1
Other	2	0	2	0	2	0	0	0

Low-Testing Classrooms vs. High-Testing Classrooms

In low-testing classrooms teachers encouraged students to make connections and the tasks required students to practice both conceptual understanding and procedural fluency. In high-testing classrooms the math being constructed was more likely to be less cognitively demanding for students, but teachers more often praised them.

Student agency, student intellectual authority, inequitable learning opportunities, control of participation, and asserting teacher authority emerged as characteristic of both high- and low-testing classrooms. However, when classrooms D and H were removed from the sample, student agency and student intellectual authority were no longer characteristic of high- testing classrooms, and inequitable learning opportunities, control of participation, and asserting teacher authority were no longer characteristic of low-testing classrooms.

Discussion

As classroom practices continue to be informed by teachers' interpretations of policy messages (Diamond, 2007), students experience different educational opportunities. Understanding how testing policies influence the opportunities to learn that support the development of positive mathematical dispositions and how these dispositions influence student engagement with mathematics could prove useful in understanding differential achievement across student populations (Gresalfi, 2009). This study showed that different populations did indeed perceive their opportunities to learn differently across testing environments. Furthermore, there was a misalignment between how students perceived their learning opportunities and the actual learning opportunities based on a qualitative analysis of representative classroom videos.

High-testing classroom practices were characterized by teachers emphasizing use of algorithms and procedures, lowering cognitive demand for students. Class time was also dedicated to mostly teacher led instruction or individual student practice (Boaler, 2002; Boaler & Brodie, 2004; Boaler & Staples, 2008), as well as increased control of student behavior and student participation. In contrast, low-testing classroom practices were characterized by teachers encouraging students' sensemaking, agency, and intellectual authority. In these classrooms, teachers engaged students in tasks that allowed them to practice both conceptual understanding and procedural fluency. However, student perceptions of the learning opportunities in some classrooms contradicted what the video analysis showed, raising the question of what narratives students are drawing on to make sense of what counts as opportunities to learn.

While high-stakes testing can be seen as a positive influence on what is taught in contrast to what is taught when tests are not required, it becomes problematic when "*tests drive the curriculum*" (Madaus, Russell, & Higgins, 2009; Shepard, 1990). The learning opportunities that teachers create for students, particularly in mathematics classrooms, can determine whether students are willing to take risks and make mistakes, are able to ask critical questions, and see themselves as capable of learning (Boaler, 2016; Gresalfi & Cobb, 2006; Oakes et al., 1990; Surgenor, 2014).

As shown by this study, students in low-testing classrooms were provided more learning opportunities. These learning opportunities, combined with equitable access to resources, could be a step toward reducing the "opportunity gap" (Flores, 2007), which could eventually reduce the achievement gap for marginalized students.

Conclusion

As testing environment increased students perceived their opportunities to learn to increase, but a deeper analysis showed that low-testing environments in fact afforded more opportunities to learn. However, particular classrooms provided conflicting snapshots.

These findings raise the question of whether additional factors might be influencing students' perceptions of quality teaching and blurring the opportunities to learn in classrooms, convoluting opportunities to learn with increased performance on assessments for students of color. More research is needed that includes analysis of classrooms practices with larger samples in order to push back against accountability narratives that emphasize raising test scores as the solution to close achievement gaps in mathematics for marginalized student populations.

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TEACHERS' VISIONS OF (RACIALLY) EQUITABLE MATHEMATICS INSTRUCTION

Cara Haines
University of Missouri
carahaines@mail.missouri.edu

Erica N. Mason
University of Missouri
ericamason@mail.missouri.edu

This paper marks an initial effort to understand the development of mathematics teachers' visions of equitable instruction. The qualitative data on which this study draws includes annual interviews conducted with 51 secondary mathematics teachers over the course of two years. Preliminary analysis revealed that teachers often described equitable mathematics instruction by naming high-quality instructional practices to which they would expect to see students have access. How teachers described those students, however, varied across individuals and aligned with different conceptions of equity within the field of mathematics education. Implications for research and practice are discussed.

Keywords: Equitable instruction, High-quality mathematics instruction, Instructional vision

Over the past few decades, issues of (in)equity have become increasingly central to conversations in the field of mathematics education. "Equity" in mathematics has been consistently described as a matter of students' access to high-quality instruction (National Council of Teachers of Mathematics [NCTM], 1989; 2000; 2014). As a result, mathematics education researchers and professional organizations have made efforts to define practices that constitute high-quality instruction (e.g., Hiebert et al., 1997; NCTM, 2014), and have designated the development of those practices as a primary goal of mathematics teachers' professional learning (Association of Mathematics Teacher Educators [AMTE], 2017). Therefore, how teachers might develop high-quality practices has become a key focus of research.

One way in which researchers have attended to teachers' development of high-quality practices is through the examination of teachers' instructional visions (Munter, 2014). Drawing from Hammerness's (2001) notion of "teachers' vision," Munter and Wilhelm (2019) described instructional vision as "the discourse that teachers or others currently employ to characterize the kind of 'ideal classroom practice' to which they aspire but have not yet necessarily mastered" (p. 2). As a way to assess teachers' instructional visions, Munter (2014) offered a set of interview prompts and rubrics that relate to key dimensions of high-quality mathematics teaching, including the role of the teacher, classroom discourse, and mathematical tasks. Each of the three rubrics (some with subdimensions of practice) delineates a trajectory of teachers' visions along levels of sophistication—with the most sophisticated level aligning with findings from classroom research in mathematics education from the last 30 years.

One limitation of Munter's (2014) visions of high-quality mathematics instruction (VHQMI) assessment tool, as he pointed out, is that it does not account for the development of teachers' conceptions of high-quality instruction that is also *equitable*. While some may consider equitable mathematics instruction as merely access to the high-quality practices like those described in Munter's framework (e.g., NCTM, 2014), others have argued that such conceptualizations of equity (or equitable instruction) are insufficient in that they lack a historical orientation (Martin, 2003; Reinholz & Shah, 2018). Following this argument, we assert that attending to teachers' visions of high-quality instruction is not enough, and argue the need to also investigate teachers' visions of *equitable* mathematics instruction. This study marks initial steps in such an investigation, and was guided by the following questions: Do teachers consider high-quality teaching practices to be equitable in and of themselves, or do they conceptualize equitable instruction in different ways?

Conceptions of Equity in Mathematics Education

Researchers in the field of mathematics education have conceptualized equity in different ways. For example, as noted earlier, for more than 20 years, the NCTM (1989; 2000; 2014) has described equity as a matter of increasing access to high-quality mathematics instruction *for all* students. While increasing access to high-quality instruction for all students may be a necessary waypoint toward equity (Reinholz & Shah, 2018), Martin (2009) has argued that NCTM's (1989) "for all" sentiment is an instantiation of colorblind racism and ignores the systemic oppression of historically marginalized groups of students.

In its later publications, however, NCTM (2000; 2014) moved a step beyond their "for all" rhetoric by expressing concern for particular groups, including students who are "black, Latino/a, [and] American Indian"(NCTM, 2014, p. 60). While the explicit naming of students for whom increased access is required is arguably an improvement from their colorblind "for all" rhetoric, NCTM still did not call for "improving the collective conditions" of students who, historically, have been the least supported in mathematics (Martin, 2015, p. 22). In the analysis we report here, we follow Martin's (2003) argument that definitions of equity in mathematics education must "take into account the collective histories of the groups for whom equity is desired" (p. 13).

Method

The research on which our analysis is based was conducted in the context of a larger research project that took a mathematics-specific approach to decreasing a racial opportunity gap (Flores, 2007) in a Northeastern urban school district. The sample considered in this study includes a total of 51 secondary (Grades 6–12) mathematics teachers (8 of whom were teachers of color, and 43 of whom were white teachers) from two cohorts who were participants in the larger study. Teachers' participation in the project included engagement in consecutive 2-year summer workshops that focused on confronting issues of (racial) inequity in mathematics education. During these workshops, project leaders attempted to support teachers in investigating and addressing inequities in school mathematics by: re-conceiving what it means to know and do mathematics; discussing the historical marginalization of Black students; reflecting on their own (and students') racial and mathematical identities (Nasir, 2011); and developing and enacting ideas for more equitable practice (e.g., culturally relevant pedagogy, Ladson-Billings, 1995).

Data used in this analysis include 84 semi-structured annual interviews that were conducted with the teacher 51 participants over the course of 2 years. All interviews were audio-recorded and transcribed. In these interviews, teachers were asked a series of questions regarding their visions of high-quality and equitable mathematics instruction. Analyses for this study focused on teachers' responses to the following prompt: *If you were asked to observe another teacher's math classroom for one or more lessons, what would you look for to determine whether the mathematics instruction was equitable?* All interview questions pertaining to teachers' visions were structured in this way to allow participants to talk about practices they would expect to see in an ideal, equitable mathematics classroom rather than necessarily describing their own.

Analysis and Findings

For this preliminary phase of analysis, we collected and analyzed 84 responses (from the 51 teachers) to the interview question of interest. Next, for each response, we identified the instructional practices that teachers described as equitable. For some responses, we were able to apply Munter's (2014) VHQMI assessment tool to code the instructional practices teachers named. For others, however, we resorted to an open coding process, as some teachers had described dimensions of instruction that are not captured by the VHQMI (e.g., "culturally-relevant" mathematical tasks and broadening notions of valid participation). While the latter finding provides important insight into teachers' visions of equitable instruction (discussed more in the implications section), for the analysis described here, we focused only on the former cases in which teachers considered equitable and high-quality instructional practices to be synonymous. After identifying those cases, we made note of the practices that teachers named using the VHQMI as a guide. For example, if a teacher said "I would attend to the types of questions the teacher was asking," then we would assign the "teacher question" label from the VHQMI Classroom Discourse rubric. After identifying the practices teachers named, we drew on the different conceptions of equity in mathematics education discussed earlier to characterize how teachers had described students. That is, we paid particular attention *to whom* (which groups of students) the teachers attended in their descriptions and, when applicable, their rationales for naming those students. We found that teachers described students in one of three ways: 1) all students; 2) all students with particular attention to different groups of students; and 3) students with nondominant ways of knowing and doing. These differences, which we interpreted as differences in teachers' conceptions of equity, are discussed in the three subsections that follow.

Colorblind Perspectives

Most teachers described equitable mathematics instruction using "for all" rhetoric (Martin, 2009). In these descriptions, teachers suggested that equitable mathematics instruction would require "everyone" and/or "all students" to have access to particular instructional practices and learning opportunities, but did not provide a rationale for their argument. These descriptions were unique in that they did not distinguish between groups of students in any way but, rather, implied a universal standard "for all" students. As an example, consider the following quote from one participant's interview: "Um, I guess I would look for participation um, you know, from-like if all students were participating." While this participant alluded to an important issue of equity—that all students should have opportunities to participate—their description suggests that equity merely necessitates equal opportunities *for all*, a common colorblind sentiment.

Beyond Colorblind Perspectives

In other cases, teachers still called for equal access to particular instructional practices and learning opportunities, but explicitly named groups of students between which opportunity gaps typically exist (e.g., students of high and low status, girls and boys, students of color and white students). The following quote from one participant is representative of this type of response:

> So I think the key is seeing that all the students are engaged and, um, and being able to, if you feel like "Okay, wait a minute, were the boys called on more than the girls? Or the girls called on more than the boys?" Were, you know, we focused on the white males? Are we focused on the African-American females or- […]

Although this participant still invoked the "for all" narrative, their attention to different groups of students distinguishes this response from the type discussed in the previous subsection. While this type of a response may still indicate a view of equity that is “assimilation-oriented” (Martin, 2009), that participants in this category paid attention to students’ race (and ethnicity, gender, and/or status) suggests an orientation to equity that extends beyond colorblind perspectives.

Perspectives that Acknowledge (Historical) Oppression

Only two teachers conceptualized equitable instruction from a perspective that, beyond calling for equal access or an explicit naming of groups of students, alluded to and offered a response to types of oppression. Another important characteristic of these responses is that in them, teachers alluded to oppressive forces of dominant mathematics (Gutiérrez, 2007), but not to the accumulated historical oppression (i.e., systemic racism) experienced by students of color. As an example, consider the following response from one teacher participant:

> [Teachers should give students] autonomy to be able to decide how they were going to - what direction they were going to take that problem. Maybe they didn't get the right answer, but they were looking at it in a different way, and the teacher showing that- What they did was just was good as what everybody else did…

Although not explicitly, this teacher acknowledged that only particular ways of knowing and doing mathematics are typically privileged in mathematics classrooms and, consequently, other—or nondominant—perspectives are often overlooked and/or marginalized. Given that in these responses, teachers, at least, alluded to the oppressive nature of school mathematics and the students who typically experience that oppression (e.g., students who look at things in a “different way”), we argue that they are both distinct from and more—yet not sufficiently—historically-oriented than the responses that constitute the previous two categories.

Next Steps in Research and Practical Implications

Given that for this analysis we did not assess the sophistication of teachers’ visions of high-quality instruction as we analyzed their visions of equitable instruction, next steps in our research will include using Munter’s (2014) VHQMI tool to assess those visions. That is, in order to determine the degree to which a teacher’s vision of equitable instruction is indeed equitable, one would have to first examine the quality of instruction, broadly, that a teacher articulated in their vision. Such an analysis might contribute to focusing professional support for individuals with unique instructional visions. For example, if a teacher articulated a less-sophisticated vision of high-quality mathematics instruction, then professional learning opportunities aimed at improving teachers’ visions with respect to mathematical tasks, classroom discourse, and/or the role of the teacher would be a necessary step in working towards articulating a more sophisticated vision of equitable mathematics instruction.

Second, as mentioned earlier, some teachers’ visions of equitable instruction extended beyond visions that focused solely on instructional practices that constitute definitions of “high-quality” instruction, such as Munter’s (2014) VHQMI and NCTM’s (2014) Mathematics Teaching Practices. Future research might attempt to define dimensions of equitable mathematics instruction and also consider how teachers’ visions might develop along those dimensions.

Regarding practical implications, the distinctions between teachers’ visions of equitable instruction we have described have important implications for teacher learning and development. For example, if teachers describe equitable instruction by invoking “for all” rhetoric, what types of professional supports might be needed to move teachers’ visions beyond that, and what implications do such visions have for teachers’ instruction and, consequently, student learning?

More importantly, given that none of the teachers in our sample conceived of equitable instruction as a means to account for the accumulated historical oppression of marginalized students, what kinds of professional supports might be needed for teachers to envision and, ultimately, enact mathematics instruction that "improv[es] the collective conditions" (Martin, 2015, p. 22) of students who have, historically, been the least supported in mathematics?

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DESIGNING CULTURALLY RELEVANT ENGINEERING CHALLENGES: CONNECTING EQUITABLE MATHEMATICS TEACHING PRACTICE WITH STANDARDS OF MATHEMATICAL PRACTICE

Deena Khalil
Howard University
deena.khalil@howard.edu

Adrian Wayne Bruce
Howard University
adrian.w.bruce@howard.edu

Meredith Kier
College of William and Mary
mwkier@wm.edu

To address inequities within mathematics education, many educators have adopted standards-based mathematics to reform disparities in accessing mathematics content and practices. However, such reform efforts do not specify how standards-based mathematics can support the equitable mathematics teaching practices needed to improve minoritized students' learning. Through a researcher-practitioner partnership, we developed professional learning experiences to empower teachers to engage minoritized students in more equitable standards-based mathematics. Such experiences included training teachers to center minoritized communities in mathematics content and practices through engineering design challenges, thus making the teaching practices more equitable and culturally relevant. In this paper, we describe research-based equitable mathematics teaching practices aligning to standards of mathematical practice.

Keywords: Equity and diversity, Social justice, Instructional activities and practices, Culturally relevant pedagogy

Purpose

In the United States, public education systems have received criticism for their failure to teach minoritized students mathematics in equitable ways (Civil, 2007; Dunleavy, 2015; Hodge, 2006). The enactment of state-mandated policies, such as the Common Core State Standards for Mathematics (CCSSM) and Standards for Mathematical Practice (SMP), have further exacerbated the existing inequities experienced by minoritized students rather than lessen them (Bartell et al., 2017; Eppley, 2015). This is because, at least in part, mathematics education remains disconnected from the experiences and realities of minoritized students (Battey, 2013; Civil, 2007; Martin, 2007, 2009). To address such inequities, mathematics educators have sought to resist the inherent whiteness in schooling by centering the lives and experiences of minoritized students and communities into the curriculum and teaching (Skovsmose, 1994; Planas & Civil, 2010). Through such culturally relevant pedagogy, these educators seek to provide educational opportunities that may buffer against the institutional and structural inequities that students are subjected to within classrooms and schools (Ladson-Billings & Tate, 1995; Skovsmose, 1994; Planas & Civil, 2010).

Several scholars suggest problem-solving through design thinking as an approach mathematics teacher may use to model the SMP with their students (Bush et al., 2018). Thus, we created a researcher-practitioner partnership, *E-Communities*, to develop professional learning experiences that empowered teachers to engage minoritized students in designing solutions to real-world problems. Such experiences included training teachers to center minoritized communities in mathematics lessons around engineering design challenges (EDCs), thus developing more equitable and culturally relevant lessons. In this paper, we describe how these research-based equitable mathematics teaching practices link to standards of mathematical practice, thereby engaging students in rehumanizing mathematics (Kier & Khalil, 2018; Gutiérrez, 2018).

Theoretical and Methodological Framework

To centralize the need for more equitable practices, this study describes a design and implementation study that calls on tenets of critical race design (CRD; Khalil & Kier, 2017) to bring together various stakeholders, including minoritized middle school teachers, learners, STEM professionals, and teacher education scholars. Given the researcher-practitioner partnership is in one of the largest Black-majority communities, CRD draws primarily from critical race theory in education (CRT; Ladson-Billings & Tate, 1995; Solórzano & Yosso, 2002) to interrogate inequities in the perspectives, structures, methods, and pedagogies that oppress minoritized learners. Accordingly, CRD uses tenets of design-based implementation research to address inequities through iteratively designed solutions to inequities in mathematics education (DBIR; Penuel, Fishman, Cheng, & Sabelli, 2011).

Through a research-practice partnership, we sought to facilitate professional learning experiences that focus on designing culturally relevant engineering design challenges (EDCs) that connect to real-world problems faced by minoritized communities (Kier & Khalil, 2018). Specifically, we supported mathematics teachers' understanding of the design process by demonstrating its alignment with the SMP. Through this process, we emphasized the need for designers to empathize in order to devise authentically real-world problems, to then foster both the connection between minoritized learners and societal challenges and center their funds of knowledge in learning (Boykin & Noguera, 2011; Coleman, Bruce, White, Boykin, & Tyler, 2016; Moll, Amanti, Neff, & Gonzalez, 2005). Although Bartell et al. (2017) identify several equitable mathematics teaching practices that may engage learners, empirical research has yet to investigate how these practices may actually align with the teaching of the SMP. To bridge the gap, we iteratively utilized a race-conscious lens by connecting education researchers, mathematics teachers, STEM professionals of color, and minoritized students to co-construct culturally-relevant EDCs for middle school mathematics learners.

Data Sources and Analysis

This three year-project engaged 42 mathematics and science teachers in professional learning on teaching through design-based tasks. Professional learning included supporting teachers and STEM professionals to plan lessons that engaged students in culturally relevant EDCs related to social justice challenges broadly, and students' lives specifically. We analyzed 50 recorded virtual professional learning sessions in which researchers, teachers, and STEM professionals, co-planned engineering and social justice lessons, and 14 virtual interactions where students discussed culturally-relevant engineering design challenges with STEM professionals. Additionally, we analyzed six face-to-face focus groups with teachers on community building, racial and social justice, reflections around the implementation of engineering design challenges, and the limits and possibilities of researcher-practitioner partnerships.

We transcribed our data and applied a priori codes to these interactions using Bartell and colleagues' (2017) equitable mathematics teaching practices framework. Next, we searched for the intersections between equitable practices and SMP by using the Dedoose analyze feature to quantify co-occurrences. We then operationalized these frequency codes by how mathematics teachers discussed teaching mathematics through problems and designing solutions that may humanize the curriculum and connect it more closely to students' lives.

Results

Below, we highlight examples of how teachers operationalized the co-occurrence intersections between equitable mathematical teaching practices and the SMPs when instructing their students to complete design challenges.

Using Students' Funds of Knowledge to Encourage Student Voice

Numerous teachers saw the utility in drawing on students' funds of knowledge. Our preliminary analysis demonstrated that students' funds of knowledge supported them to a) make sense of problems and persevere in solving them, b) construct viable arguments and critique the reasoning of others, c) look for and make use of structure, and d) look for and express regularity in repeated reasoning. By validating shared ideas and experiences between her and her students, and connecting instruction to students' experiences and interests, Tiffany came to understand how important students' funds of knowledge is for constructing viable arguments. She recognized that most adults, including herself, are challenged by hearing the way in which students think about real-world problems because they are prioritizing their own hidden assumptions about societal problems and how students may think about them. When taking the time to uncover the ways in which students define real-world problems, she was better able to support a true co-constructed a solution with them, and "make them feel like you're really listening." Here, she elaborated further:

> [The students] give you another point of view, I mean because you're always thinking of things from your adult point of view … I tell them why from my perspective, and then I let them tell me how they think [about problems] from their perspective ... it was less tension, not me just saying do it because I said.

Tiffany surmises that by connecting instruction to students' experiences and interests, she not only validates their ideas and experiences (Aguirre et al., 2013; Bartell, 2011) but also develops a bond where students feel they matter (Dixon & Khalil, 2016). Thus, when standards for mathematical practice are used in concert with drawing on students' funds of knowledge, teachers can teach more equitably as gaining a deeper understanding of their students' backgrounds and experiences.

Recognizing Multiple Forms of Discourse and Language as a Resource

Many stakeholders in the partnership emphasized the importance of code-switching as an essential skill that students need to achieve the life outcomes they expected. Our preliminary analysis demonstrated that recognition of students' multiple modes of discourse supported them to a) construct viable arguments and b) use appropriate tools strategically. Several teachers described how prototyping encouraged students to learn to communicate across their difference of opinions. Leveraging communication styles and instruments promoted students' access to new points and modes to discuss solutions. Caryn described this:

> Prototyping helps develop empathy because learners communicate using their convincing ability in their attempt to build … For instance, a learner may do things a certain way, while another will argue that the process will not work. … As prototyping proceeds, learners realize that the group must work together to reach a common/shared goal.

Caryn describes how the problem-solving process can be frustrating for students but that the physical manifestation, or lack thereof, of what they are imagining forces her students to communicate with one another during their productive struggle. By facilitating an environment that is respectful of differences, teachers can cultivate affirming relationships among students as such endorsing of possible differences in the ways students may communicate with one another demonstrates acceptances of distinct language practices as valid methods of conveying one's mathematical understanding (Civil, 2007; Gay, 2002; Ladson-Billings, 1995; Setati, 2005).

Societal and Cultural Contexts that Connect Students to Math Practices

Our preliminary analysis demonstrated supporting development of a sociopolitical disposition helped students to address the SMP. When teachers provide opportunities for students to explore social justice topics using academic content and engage students in conversations about real-world problems and how mathematics can be used to examine them, they support development of a sociopolitical disposition (Bartell et al., 2017; Frankenstein, 2012; Gates & Jorgensen, 2009; Gutstein, 2006; Skovsmose, 1994; Tate, 1995). Hillary planned to implement a lesson on providing clean, potable water to communities in need as a way to encourage students to use structure in determining how much money they can spend on materials given budget restrictions:

> I think on each of the materials, one of the things we're probably going to do is give them a constraint of, I don't know, $100, and they're going to have to figure out, within that $100, how much you can spend on each material so you can go ahead and make your water filtration system. So, for the social justice [issue] … I think it would be interesting if we gave each of the groups different amounts of money they could spend. That would get them to think about the inequity of different countries and how they don't have the money to spend on clean up, that would be an interesting way to do that.

Following the challenge, Hillary engaged students in the timely social justice topic provided what's happening in many underserved communities across the country, like Flint, Michigan and Newark, New Jersey (i.e., filtering communities' lead-poisoned water supply), into her lesson. This allowed her to take a back seat in her students' learning as she let a relevant social issue drive instruction while students contemplated how they could work within a mathematics structure to solve a real-world problem that plagues many communities in America as well as around the world (Gutstein, 2006; Skovsmose, 1994).

Conclusion

In this paper, we described a partnership between HBCU teacher educators, STEM professionals, and middle school teachers that sought to center equity in standards-based mathematics content and processes. We theorized that teachers' use of culturally relevant and equitable engineering design challenges may empower minoritized students' engagement in the SMP by positioning their daily lived experiences as epistemological resources. By utilizing design and implementation research, within a critical race design framework (Kier & Khalil, 2018), this study facilitated professional learning experiences focused on designing solutions to real-world problems faced by minoritized communities to support teachers in engaging equitable mathematics teaching practices that connected to the SMP.

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HOW RELATIONAL CAPITAL EMPHASIZES CRITICAL REFLECTION AND EMPATHY IN DESIGN THINKING TO SOLVE COMMUNITY-BASED PROBLEMS

Deena Khalil
Howard University
deena.khalil@howard.edu

Jocabed Marquez
Texas State University
jocabedmarquez3@gmail.com

Meredith Kier
College of William and Mary
mwkier@wm.edu

In the United States, teaching mathematics equitably has been a challenge that continues to diminish minoritized learners' access and opportunities to learn. To address these inequities, researchers have conceptualized various equitable practices that center the community cultural wealth of minoritized learners and teachers. In the present study, we describe how a specific community cultural wealth, the relational capital between minoritized teachers and learners, supported their application of empathy in the design thinking process. Through professional learning experiences, teachers reflected how relational capital elicited the empathic responses needed to solve the culturally relevant engineering design tasks contextualized as community-based problems of practice. Thus, fostering teacher-student relational capital holds promise for promoting equitable teaching and learning in mathematics classrooms.

Keywords: Equity and diversity, Instructional activities and practices, Culturally relevant pedagogy, Affect, Emotions, Beliefs, Attitudes

Purpose

One criticism of standards-based mathematics is that it magnifies the enduring inequities that currently occur in American mathematics education settings (Bartell et al., 2017; Battey, 2013). Further, such reform efforts in mathematics highlight a disconnect between the experiences and realities of minoritized learners and that of their mainstream peers (Brown et al., 2016; Eppley, 2015; Forgasz & Rivera, 2012; Gutstein, 2010), which causes the minoritized students to disengage from mathematics learning (Corso, Bundick, Quaglia, & Haywood, 2013; Crumpton & Gregory, 2011). To address this concern, researchers have called for more equitable mathematics teaching to engage minoritized learners in mathematics problem-solving (Hand, 2012; Luria & Kaufman, 2017; Turner & Celedon-Pattichis, 2011; Turner & Drake, 2016). Bartell and colleagues (2017) argue one way to do this is to focus on infusing equity into standards-based mathematics through the Standards for Mathematical Practice (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). Scholars argue that because the Standards for Mathematical Practice are intended to motivate learners from *all* backgrounds to participate in mathematical problem solving, they provide more opportunities for all learners to equitably engage in learning (Anhalt, Civil, & Lesser, 2013; Brodie, 2014; Drageset, 2014; Heng & Sudarshan, 2013; Moscardini, 2014).

The design thinking process (a process that includes empathizing, defining problems, ideating, prototyping, and testing solutions to real-world investigations) can support the implementation of the Standards for Mathematical Practice in a manner that encourages empathy with others when applying mathematics to societal challenges and designing solutions (Bush et al., 2018). The literature on how teachers seek to humanize mathematics by attending to minoritized learners affect and experiences when problem-solving is sparse (Goldin, Epstein,

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Schorr, & Warner, 2011; Frank, Khalil, Scates & Odoms, 2018; Khalil, Lake, & Johnson, 2019). In this paper, we describe how a professional learning experience supported teachers to model empathy when asking students to design solutions to community-based problems in middle school mathematics classes.

Theoretical and Methodological Framework

This study describes *E-Communities,* a research-practice partnership that draws on tenets of critical race design (CRD; Khalil & Kier, 2018) to bring together the professional knowledge and cultural resources of various minoritized stakeholders including teachers, learners, engineers, and teacher education scholars intent on developing more equitable practices. Given the partnership takes place in a predominately Black community, CRD draws from two well-established frameworks, critical race theory in education (CRT; Ladson-Billings & Tate, 1995) and design-based implementation research (DBIR; Penuel, Fishman, Cheng, & Sabelli, 2011). While CRT aids in identifying and interrogating inequities in the perspectives, structures, methods, and pedagogies that oppress minoritized learners, it, along with DBIR, supports iteratively transforming these inequities by design. Thus, CRD epistemologically, methodologically, and pedagogically aims to disrupt inequities through iteratively designing opportunities that support minoritized teachers and learners to use reflective and agential practices.

Seeking transdisciplinary perspectives needed to address the complexity of social inequalities (Solórzano & Yosso, 2002), CRD promotes minoritized learners' innovation, agency, and engagement by designing spaces that center the various assets and capitals stakeholders bring to the classroom (i.e., community cultural wealth, CCW; Yosso, 2005). Based on critical race theory, Yosso (2005) describes six capitals (aspirational, familial, linguistic, social, navigational, and resistant) and explains how these capitals can empower minoritized voices by valuing the capital shared in their stories, thereby and minimizing the power differential that exists between research-practice partners (Guajardo, Guajardo, & Casaperalta, 2008). Several scholars describe these relationships based on trust and respect as relational capital (Warren, 2011, Holme & Rangel, 2012); Gamez (2017) adds that relational capital heavily relies on empathy as a vehicle for compassion and both experiential and generative praxes. Empathy has been promoted as a key application of culturally-relevant teaching as it prepares teachers to facilitate student-teacher relationships, improve communication with parents, and builds trust when empathy is genuine and focuses on interpersonal relationships and the awareness of dominant societal structures that it promotes (Warren, 2015). In this paper we demonstrate how empathy and compassion uses the hope and inspiration inherent in aspirational capital to demonstrate how teachers support students and their academic goals; how experiential praxes in design thinking can connect familial and linguistic capitals to show how the perception of an extended familial or community member may assist teachers in leveraging a positive relationship with students; and how generative praxes links social, navigational, and resistant capitals to allow teachers to negotiate instructional relationships so that students are able to maneuver successfully through educational spaces.

Data Sources and Analysis

For three years, *E-Communities* stakeholders participated in several professional learning experiences aimed at building design thinking practices that can equitably engage learners in culturally relevant engineering design challenges (EDCs; Kier & Khalil, 2017), that were contextualized by community-based problems of practice related to social justice challenges broadly, and students' lives specifically. Professional learning included supporting teachers and

STEM professionals to plan lessons that engaged students in culturally relevant EDCs. Data was collected through recorded observations of collaborative lesson planning during face-to-face (F2F) professional learning sessions, where forty-two middle school mathematics and science teachers and nearly a dozen STEM professionals of color collaborated in interdisciplinary teams to design EDCs in a way that humanized the standards-based mathematics curriculum. Additionally, we conducted six focus groups following F2F co-planning, where teachers reflected on the limits and possibilities of designing and implementing EDCs in formal middle school mathematics and science classes. Finally, we recorded and analyzed 20 virtual professional learning sessions in which researchers, teachers, and STEM professionals, continued to co-design EDCs and 14 virtual interactions where minoritized students discussed their designs with STEM professionals of color. Below we report findings we found by adopting Lichtman's (2010) three C's, which consists of coding, categorizing, and identifying themes that operationalize the relational capital mathematics teachers either demonstrated or reflexively described during either face-to-face or online professional learning sessions.

Results

Below are descriptions of how teachers' relational capital supported their ability to model empathy to students when teaching investigations through design processes. In the following section, we demonstrate how teachers use relational capital to connect to (i) their praxis through critical reflection (Lupinski, Jenkins, Beard, & Jones, 2012), (ii) one another through the shared inquiry of professional learning experiences (Biesta & Stengel, 2016), (iii) their learners through sharing power and situating their narratives as epistemological resources (Kier & Khalil, 2018). For many teachers, incorporating design into their teaching was a way to give students critical agency in addressing societal challenges. By situating design tasks as community-based problems of practice, teachers expanded their learners' agency by helping them view mathematics as a tool that can interrogate inequities and design solutions for problems minoritized communities faced both locally and globally. As sixth-grade mathematics teacher Tasha explained:

> [Design thinking] can help develop empathy in that it forces us to leave our narrow scope of thinking. We include diverse perspectives that may lie outside of our normal world. The idea is exposure. The more you are exposed to, the more aware you are of not only others but also our limited viewpoints. This should create empathy, given a broader knowledge-base and insight from additional sources.

Thus, expanding students' knowledge and awareness of real-world problems shows promise for developing students' agency, as their empathy with the problem helps them create tentative solutions based on applying their application of mathematics knowledge and practices.

Relational capital was demonstrated as a shared inquiry between teachers and with their students. Teachers recognized the importance of empathy during professional learning when they experienced personal frustration during the design process. For example, Hilary, a seventh- grade math teacher, explains how empathy connected her professional education to her learners' experiences:

> Empathy plays a big role in teaching and learning. There should be a connection between the two. That's why, in our training (i.e., professional learning experience), we acted as students so that when we will be in our classrooms and doing the hands-on activities or modeling we

> are ready to answer their questions, we are ready to handle the challenges that our students will meet in a given project.

Hillary's empathy and critical reflection helped her counter the detrimental attributes of negative emotions by empowering changes in her teaching practice, which will lead to meaningful experiences for her students. This sentiment builds upon DeBellis and Goldin's (2006) notion that when teachers are cognizant of their affect and meta-affect, they are more confident in their abilities to support learners academic and emotional needs (Goldin et al., 2011; Khalil, Lake, & Johnson, 2019).

Ned, a seventh-grade math teacher, found that students' relational capital, steeped in empathy, improved their collaborative efforts to address problems and design solutions:

> [Designing in groups] helps build a harmonious relationship among partners/members of a group or team and would lead to improving empathy. Once individuals working as a team understand each other's limitations/weaknesses and strengths, there would be a smooth working relationship. Remarkable things can happen when empathy for others plays a key role in problem-solving. Understanding each other's weaknesses/limitations and strengths would improve any work at hand.

Ned's realization suggests that he has recognized tensions that may cause barriers to learning and is likely to adapt accordingly to meet the needs of his learners, particularly as it relates group dynamics that are based on shared ideas of trust and communalism (Dixon & Khalil, 2016). Horn et al. (2018) discussed how critical this recognition is for teacher development, as it spurs teachers' reflective reasoning, as well as their adaptability in praxis.

Finally, as teachers reflected upon iteratively designing equitable classroom practices for their learners, five teachers shared inquiries on how they might continue to leverage empathy and relational capital in their classrooms. These included:

> How do we get communities and learners to genuinely and continuously commit to a demonstration of empathy? Why are people/communities not readily disposed to meet the demands initiated by empathy? Why is the modern world is not generally endeared toward being empathic? How well can I really show empathy towards my learners? What degree can we expect middle school aged children to be empathetic, and how can we help support this? Does empathy guarantee success in design? Is there another "ingredient" that is needed to ensure success in design challenges? Can we involve the home and community in the process?

Teachers utilized relational capital's empathy to develop their relational teaching and subsequently connect to their praxis through critical reflection (Lupinski, Jenkins, Beard, & Jones, 2012). As the experiences and reflections of teachers indicate, the idea of operationalizing empathy through designing *for others* in mathematics classrooms supported teachers in their professional learning, thereby developing a more equitable environment for their learners.

Conclusion

This study revealed an emerging notion of how teachers' relational capital can be an essential part of enacting equitable mathematics teaching when engaging minoritized learners in design challenges. With empathy being an integral part of design thinking, teachers can readily model ways to humanize mathematics when addressing real-world problems. We propose that drawing on relational capital should be central to teacher-student dynamics in mathematics classrooms, as it connects empathy, both cognitively and affectively, to designing solutions. Re-imagining equitable practices through relational capital promise to inform teachers' ability to serve underrepresented minoritized populations, particularly in design-based mathematics classrooms.

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USING NOVEL APPROACHES TO BETTER UNDERSTAND BLACK MATHEMATICS TEACHER RETENTION

Toya Jones Frank
George Mason University
tfrank4@gmu.edu

Jenice L. View
George Mason University
jview@gmu.edu

Marvin Powell
George Mason University
mpowel11@gmu.edu

Christina Lee
George Mason University
cleen@gmu.edu

Asia Williams
George Mason University
awillid@gmu.edu

We used a mixed method research design to address the complexity of interrogating issues related to retention of Black mathematics teachers. The research design includes oral history interviews with retired Black mathematics teachers and large-scale survey data collected and analyzed from a Critical Race Quantitative Intersectionality approach. It is our position that the contemporary reality of the dwindling numbers of Black mathematics teachers cannot be fully understood without unpacking the issue with a critical, historical lens. Otherwise, we run the risk of perpetuating the longstanding issue of lack of diversity in the field. Thus, this paper presents preliminary findings of a 3-year research project that advances knowledge of the historical influences, with focused attention to sociopolitical forces, that impede retention of Black mathematics teachers.

Keywords: Equity and Justice, Research Methods

Purpose

Within educational circles and in public discourse regarding teacher retention, stakeholders are having what appear to be two seemingly distinct conversations, at least on the surface. One conversation highlights the need to recruit mathematics teachers to improve American students' achievement outcomes all in the name of increasing global competitiveness and employment in STEM-focused professions (National Math & Science Initiative, 2016). The other conversation calls for the retention of a diverse teaching force for an increasing multi-racial, multi-ethnic and culturally-diverse student population (Casey, DiCarlo, Bond, & Quintero, 2015). These conversations are rarely held simultaneously, meaning few conversations address the need to increase the numbers of teachers of color in mathematics. We contend that these conversations are not mutually exclusive. Further, we argue that these conversations, while well meaning, often commoditize teachers of color. Conversations about "filling pipelines" often treat teachers as data points to meet diversity requirements or to simply achieve demographic matches between students and teachers. Further, the ultimate goal of increasing mathematics teachers and teacher of color tends to always rest on US economic competitiveness rather than goals related to social justice, the public good, and the wellbeing of Black teachers.

Black mathematics teachers are one-third of one percent of all teachers and approximately 6 percent of all secondary mathematics teachers (Neil, 2015). An emergent body of literature highlights the promise of learning with and from Black mathematics teachers based on their unique pedagogical practices that honor Black students' ways of knowing (Birky, Chazan, & Morris, 2013; Clark Badeschter, & Napp, 2013a; Clark, Frank, & Davis, 2013b; Johnson, Nyamekye, Chazan, & Rosenthal, 2013) and also improve achievement for Black students (Dee, 2004; Klopfenstein, 2005). Addressing retention of Black mathematics teachers is key, as Neil found that they have the highest rate of turnover among all mathematics teachers across all racial demographics.

This limited amount of scholarship with respect to Black teachers of mathematics highlights the need to establish a line of research that lies at the nexus of increasing teacher diversity and increasing the numbers of mathematics teachers entering the field. Further, we push back on the notion that recruitment of teachers of color, and specifically Black teachers, is solely a contemporary issue. We posit that the dwindling number of Black teachers noted in the current news cycles and public discourse stems from a long history of systemic marginalization. Historical analyses of Black teachers in mathematics provide a meaningful foundation for understanding the contemporary racialized experiences, perspectives and practices of Black mathematics educators.

We used a mixed method research design to address the complexity of interrogating this broad issue. The research design included oral history interviews with retired Black mathematics teachers. To capture the experiences of current Black secondary mathematics teachers, the research team administered a large-scale survey and will be conducting focus groups. The research questions that guide this work are as follows:

1. What are the mathematical, racialized, and educational experiences of Black mathematics teachers who taught during the periods of *de facto* desegregation in the United States and those who are currently teachers?
2. How do the experiences of former and currently practicing Black mathematics teachers contribute to theorizing about the role of Black mathematics teachers and the content, intellectual, cultural, and social resources they bring to their practice?
3. How do the experiences of former and currently practicing Black teachers inform how schools, policy makers, and teacher preparation programs recruit and retain a diverse mathematics teaching force?

We see this work as an effort to illuminate issues that may not be at the center of either discourse related to retention of Black teachers or retention of mathematics teachers.

This National Science Foundation-funded research project advances knowledge of the historical influences, with focused attention to sociopolitical forces, that impede the retention of Black mathematics teachers. Ultimately, this work informs research methodology in mathematics education by integrating untapped, yet appropriate methodologies suitable for challenging issues of recruitment, retention, and praxis of other underrepresented racial and ethnic groups across time periods and school contexts. The proposed project will also contribute to theories of mathematics teaching, with respect to the role of race and racialized experiences.

Theoretical Perspective

It is important that we acknowledge that race is socially-constructed and historically contingent, not biological, fixed, or causal despite harmful social discourses (Omi & Winant, 1994/2014). As race is socially constructed, humans become part of what Omi and Winant refer to as racial projects, which do the ideological work of shaping and reshaping what race means and how social structures are organized based on that meaning. In the context of mathematics teacher education, this means that Black mathematics teachers’ work is informed by the faulty discourse of Black people as mathematically inferior to their non-Black peers. Racism occurs when society buys into racial projects and reproduces structures of domination based on essentialist categories of race. Within this system, individuals have racialized experiences, i.e., social experiences shaped by racism. We posit that teaching mathematics, in a similar fashion to learning it, is a racialized experience influenced by multi-level external forces (Martin, 2000). Further, due to social, historical, political, and cultural forces, teachers at particular social intersections (e.g., racial, socioeconomic, linguistic) Black teachers may experience teaching mathematics differently than those from dominant communities (Clark, Johnson, & Chazan, 2009).

Research on both recruiting a diverse teaching force and increasing the mathematics teaching force often focus on recruitment issues such as college completion, standardized testing, financial concerns, or the need to create viable pipelines (e.g., Nettles, Scatton, Steinberg, & Tyler, 2011). With respect to retention, researchers cite site workforce conditions like autonomy over curriculum (Ingersoll, 2011; Neil, 2015). While acknowledging the importance of these issues, we assert that much of this work neglects how racism impacts each of these issues. For instance, Achinstein, Sexton, Ogawa, and Freitas (2010) noted that teachers of color highlighted issues such as “low expectations or negative attitudes about students of color, lack of support for culturally relevant or socially just teaching, and limited dialogue about race and equity” (p. 96) as deterrents to retention.

We used critical race theory (CRT) as our guiding theoretical perspective. CRT has its origins in the Derrick Bell’s (1980) legal scholarship. Bell’s work was expanded to the field of education by Drs. Gloria Ladson-Billings and William Tate (1995). CRT rests on the premise that racism is endemic, pervasive, and is normalized through social and institutional structures and practices in public spaces such as schools (DeCuir & Dixson, 2004; Ladson-Billings, 1998, 2013; Ladson-Billings & Tate, 1995; Milner, 2008, 2017). CRT tenets include (but are not limited to): (a) the permanence of racism; (b) whiteness as property (Dixson & Rousseau Anderson, 2018, Harris, 1993); and (c) intersectionality (Crenshaw, 1989; 1993; Ladson-Billings, 2013). Some of these tenets will be addressed in detail in the findings. These tenets hold explanatory power for understanding how race and racism shape mathematics teacher education and impact retention of Black teachers.

Methods

Integrating oral history and critical quantitative analysis via mixed methods research accommodated our interdisciplinary research questions. Specifically, we collected oral history interviews of retired Black mathematics teachers and large-scale survey data of currently practicing Black teachers. Combining what may seem like disparate methodologies, the collected data is helping us identify issues that are intractable across time periods and schooling contexts. We are planning focus groups for summer 2019, which we believe will further corroborate our preliminary findings.

Participants

Survey participants. The results presented in this paper are from two data sets, results from large-scale survey and oral history interviews. We surveyed 555 currently-practicing Black teachers of mathematics nationwide (women, 53%) using a survey we developed, the Black Teachers of Mathematics Perceptions Survey (BTOMPS). We were intentional in aiming to recruit a sample of Black teachers of mathematics that is representative of the Black teaching population in the United States. We also partnered with the Benjamin Banneker Association to garner participation and used social media and teacher social networks to recruit. The participants who participated in the survey ranged between the ages of 21 and 65 years (M =34.9, SD =8.6). A majority of the teachers identified as African American (98.2%). Other teachers specified their ethnicity as Caribbean (1.1%), African, Afro-Latino, and bi-racial/multi-racial (all < 0.5%). Teachers reported a minimum of a Bachelor's degree (50.1%), Master's degree (17.5%), doctoral degree (2.9%), and a specialist degree (2%). The teachers taught mathematics and/or other subjects between 0.5 and 34 years. The teachers taught mathematics and/or other subjects for less than five years (38.7%), between 5 and 10 years (45.6%), and 11 years or more (15.7%). The percentage of years teaching mathematics only are marginally different: less than five years (39.6%), between 5 and 10 years (45.4%), and 11 years or more (15%). They taught regular scheduled classes (at least one class weekly) at elementary (22.2%), middle (45.2%), and high schools (31.9%). The majority of participants (97.6%) taught at public schools, including traditional, public charter, and public magnet schools, and the rest taught at parochial/religious and independent schools (2.4%). The teachers represented more than half of the US states and the District of Columbia, with significantly less representation from the Rocky Mountain states. Most teachers in the sample (64.1%) took a traditional route to licensure for mathematics teacher certification.

Oral history participants. To date, our team has interviewed 13 retired Black female teachers who taught mathematics beginning as early as 1953. With the exception of two participants who came to teaching from other professions, all experienced teaching mathematics in (a) segregated community schools; (b) schools that were undergoing desegregation; and/or (c) schools that had been desegregated within a few years of their arrival. The majority of the participants began their careers in 1960s-1970s, with the exception of the two career changers who began in 1986 and 1991, respectively. We decided to narrow our participation regionally, opting to interview in the Washington, D.C. and Atlanta Metro areas because of the regions' rich histories of Black education due to their booming Black middle classes and thriving Historically Black Colleges and Universities (HBCUs). Snowball sampling procedures (Babbie, 2001) proved to be the most effective means of recruiting participants. We used social media and teacher social networks to recruit senior participants.

Data Collection and Analysis

Survey. All participating teachers completed the BTOMPS, which was designed to align with the literature on Black teachers of mathematics with respect to mathematics teaching and learning, beliefs and awareness, perceptions of race and racism, and working conditions. BTOMPS includes research team-written 4-point likert-style items, with responses ranging from strongly disagree to strongly agree. The research team-written questions were developed from the analysis of about two dozen interviews with Black teachers of mathematics and pre-service teachers (see Frank, Khalil, Scates, & Odoms, 2018 for more on these interviews). We included original and revised questions from the Mathematics Teachers of African American Students Beliefs Instrument ([MT-ASBI], Howse, 2006), designed to capture and measure the beliefs of Black high school mathematics teachers as they relate to teaching. Building on earlier work, the research team also included questions from a mathematical beliefs and awareness survey developed by Campbell et al. (2013). This survey measured teachers' beliefs about mathematics content and pedagogy as well as their beliefs and awareness about students' identities and dispositions toward mathematics. Finally, we amended a microaggressions scale that was initially developed by Harwood, Choi, Browne and Mendenhall (2015). These items were used to assess teachers' perception of race and racism in their school/districts. These items used a six-point Likert-type response scale ranging from '*never*' to '*once a week or more*'. For example, one question is: *People have made me feel intellectually inferior at my school/district because of my race*. To ensure content validity, the research team conducted cognitive interviews with currently practicing Black teachers of mathematics and mathematics teacher leaders. During the first wave of administration, the research team completed psychometric testing to reduce the overall number of items on the survey. Each participant participated in the survey using an online format that was administered via Qualtrics. In this report, we report descriptive statistics in this report and share implications for future quantitative analyses

By foregrounding race, racism, and intersectionality, we chose to frame this issue of attrition of Black mathematics teachers within CRT in order to disrupt how statistical data are interpreted, and to (re)tell, through the lenses of participants, the experiential knowledge and counterstories behind the data (Solórzano and Yosso 2002). Critical Race Quantitative Intersectionality (CRQI), informed by CRT, uses quantitative methodology to frame research, policy, and practice for the purpose of social justice and educational equity in a field dominated by qualitative research methods (Covarrubias & Velez, 2013, Sablan, 2019). Using a CRQI lens, we surveyed the participating teachers about their perspectives on mathematics content and pedagogy, racialized experiences as teachers, and current working conditions.

Oral history interviews. Each participant was interviewed twice, once without formal videography to help them recall their experiences and to familiarize them with the oral history process. The second interviews were recorded using a professional videographer. All interviews, formal and informal, lasted from 1.5 to 3.5 hours. In the tradition of oral history, collected data from primary sources such as newspapers, school board archives, yearbooks, mathematics texts, pictures, and other artifacts pertinent to the participants' lived experiences. Many of our primary sources were shared by the participants.

We used Firouzkouhi and Zargham-Boroujeni's (2015) 4-step analytic process for rigorously analyzing oral history data: (1) data gathering with participants and first-level, inductive coding based on researchers' impressions and memos; (2) second-level coding to determine sub-categories within the first-level codes; (3) third-level coding and determining the main categories; and (4) connecting the main categories to each other to develop a strong narrative. These steps are complementary, connected, and related in a cyclical fashion, such that the final stage connects to the first to form what the authors call an analytic "circuit." Our analysis of this data is also informed by knowledge of the literature about Black teachers from historical and contemporary perspectives. In our analysis of the oral history data, we identified themes that inform connections to, implications for, and divergent experiences from contemporary recruitment and retention issues for Black mathematics teachers.

Post-analysis, we began to identify in the qualitative data set what Sisson called "critical incidents" with respect to race and racism in the mathematics teaching experiences of our teachers, meaning instances are pivotal to the identities and lived experiences of the participants. These critical incidents highlighted the salience of race within the themes generated during the initial oral history coding process. Because we adopted a CRQI quantitative approach (Covarrubias & Valez, 2013; Sablan, 2019), we identified findings in the descriptive data that mirrored the experiences of our oral history participants, and vice versa. Seeking out these reflexive relationships between the qualitative and quantitative data allowed us to begin to put together a narrative about the intractability of racism in the experiences of Black mathematics teachers over time.

Results

Using three tenets of CRT to frame this section, we present preliminary findings and highlight how integrating novel methodologies helps to paint a more complete picture of recruiting and retaining Black mathematics teachers.

The Permanence of Racism

As noted earlier, for critical race theorists, racism is commonplace in the U.S. (DeCuir & Dixson, 2004; Ladson-Billings, 1998, 2013). It follows that it is endemic in education as well (Delgado and Stefancic, 2001). Thus, the one of the goals of critical race theorists in mathematics education is the unmasking of racism when "denotations [of racism] are submerged and hidden in ways that are offensive though without identification" (Ladson-Billings, 1998, p. 9). Our data point to the veiled permanence of racism and the inherent racist structures of district- and school-level practices contribute to the Black mathematics teacher shortage and attrition. We use examples from the qualitative and quantitative data of our study to point out instances of veiled racism. When asked about instances of microaggressive experiences in their schools/districts, the survey participants reported the following: (a) 60.4% responded that people have made them feel intellectually inferior because of their race at least once monthly; (b) 58.2% indicated that at least once per month their contributions were minimized due to their race; and (c) 54% reported having their academic ability or intelligence minimized once monthly or more frequently.

This data was collected in the spring-summer of 2018, yet it mirrors the sentiments of the retired teachers in the study who also shared microaggressive experiences related to having their intelligence questioned. Experiences included those like Mrs. Joyce Lyons, a retired teacher who taught over 40 years in the mid-Atlantic region. She shared an experience from one of her first teaching jobs in the 1960s:

> I just felt like the white teachers thought that we [Black teachers] were not smart as they were. One [fellow teacher] even told me, 'You just don't have the experience to teach math…I don't know what he meant by that, but he kept telling me that.

The qualitative data set is filled with examples such as these that mirror what the quantitative data tells us about microaggression trends related to Black teachers who are currently teaching. Other examples included being doubted and disrespected by administration and parents. These data point to the longstanding, inherent, and overlooked nature of racism that plagues mathematics teaching.

Whiteness as Property

The whiteness as property tenet states that those who are identified as white are guaranteed rights that are equal to, if not more valuable than, material resources that position them in power (Harris, 1993). Often, these rights are so inherently normalized, they are difficult to recognize (Milner, 2017). It is important to note that whiteness is not simply a fixed characteristic of a particular group of people. In the context of mathematics teacher education, teaching advanced mathematics courses is the material property that is withheld from Black teachers. In fact, Neil (2015) found that Black mathematics teachers overwhelmingly teach lower-level mathematics. In her study, she found only 3% and .3% of Black teachers taught calculus and statistics, respectively. Connecting the contemporary to the historical, the teachers in our oral histories consistently recounted being relegated to teaching lower-level mathematics when teaching in more racially-diverse teaching settings. We noted that only when their student populations were

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predominantly Black were they afforded the opportunities to teach advanced mathematics courses. We see this tenet as interconnected with the previously-discussed tenet of the permanence of racism, as the phenomenon of denied opportunities to "property" like teaching advanced mathematics courses occurs when the racial project of Black inferiority is played out in ways that lead to the minimizing of Black teachers' intelligence and contributions.

Intersectionality

Intersectionality calls for the examination of race in tandem with gender and its performances and expressions, sexuality, social class, nationality, and numerous other systems of oppression or privilege in various settings (Crenshaw, 1989, 1993; Delgado & Stefancic, 2001; Ladson-Billings, 2013). Race structures the lives of Black people in ways that often make it difficult to differentiate how multiple and overlapping systems of oppression impact lived experiences (Crenshaw 1989; 1993). Black people across the diaspora share a collective Black experience, *and*, simultaneously, the Black experience is multifaceted and diverse; i.e., it is intersectional. We knew that the experiences of teachers with respect to gender varied. For instance, Neil (2015) reported that Black female teachers make 32% less that their Asian male mathematics teaching counterparts. She also found that Black female mathematics teachers had the highest rate of turnover in her dataset, across men and women of all demographics.

With respect to the literature, researchers have pointed to how Black male teachers are expected to perform tough love and be disciplinarians, often at the expense of their content and pedagogical expertise (Frank et al., 2018; Bristol & Mentor, 2018; Brockenbrough 2012, 2015). On the other hand, Black female teachers are presented in the literature as caring surrogate mothers, coined othermothers by Dixson (2003) and Dixson and Dingus (2008). Intersectionality, and in particular, the intersection of race and gender are of particular salience to our findings. Our work points to gender differences such as We hypothesized that the BTOMPS microaggression responses would likely vary by gender given what we have learned from the literature about the dominant masculinist perspective of mathematics (Hottinger, 2016) and how it is compounded by race for Black women and girls in STEM (e.g., Joseph, Hailu & Boston, 2017). This research helped us interpret the survey finding wherein half as many Black men (10.3%), compared to Black women (20.4%) believed their intellectual contributions were minimized.

The oral history data point to comparable experiences. Gracie Kenon, retired veteran mathematics teacher of 38 years in Georgia and Washington, D.C. recounted the struggles of majoring in mathematics as a Black woman, even at an HBCU. Inherent in the all of the interviews with our oral history participants, who were all women by coincidence, was the care, both interpersonal and academic, that they poured into their students. Mrs. Gail Radcliffe, former mathematics teacher and administrator, described the othermothering (Dixson, 2003; Dixson & Dingus, 2008) techniques she and the other Black women teachers employed like buying alarm clocks to get their students to school on time and advocating for more rigorous mathematics curriculum for their students. While these women othermothered, many of them also discussed the balancing act of mothering their own children and being present for their students. We anticipate collecting oral history interviews from male teachers in the near future to further analyze how gender intersects with mathematics teaching for Black teachers.

One unanticipated finding with respect to intersectionality was how years of teaching impacted our survey results. The veteran teachers in our sample, i.e., those who taught for more than 10 years experienced significantly less microaggressive behaviors than those who had less experience in the field. For instance, over half of them reported having never felt intellectually inferior due to their race, when compared to over a quarter of teachers with 9 or fewer years reported feeling intellectually inferior by others due to race a few times a month or weekly. We have generated two plausible conjectures for why this pattern may be emerging in the data. First, we wonder if teachers who have taught for an extended number of years are simply more respected for their content and pedagogical knowledge and experience. We draw on our findings from the oral history component of our grant project to arrive at our second conjecture. In our ongoing interviews with retired Black teachers of mathematics we are finding that they do not name racism and oppression in ways that the currently practicing and pre-service teachers do (Frank et al., 2018). For instance, the retired teachers have shared incredible stories of persistence during the desegregation of schools, and yet when asked directly about how race impacted their teaching practice, many said they do not think that it did.

Second, we conjecture that, perhaps, in this current sociopolitical climate in which race is central to national discourse, the teachers in our study who have less teaching experience (and are likely, but not necessarily, younger) are able to name racism and the associated microaggressions in ways that our more senior participants do not, or maybe cannot. In essence, we wonder if generational differences are at play in the data. In all, these results from the microaggression scale highlight the necessity of unpacking quantitative through a CRQI lens to think about mathematics teachers' experiences at the intersections of interlocking systems of oppression and/or privilege. These findings warrant continuing examination of retention factors with respect to how intersecting identities, oppressions, and markers of privilege make mathematics teaching complex for Black teachers.

Conclusion

Clark et al. (2013a, 2013b) proposed that the study of Black mathematics teachers' experience is essential to the field of mathematics education as the field broadens to account for sociocultural (Lerman, 2000) and sociopolitical (Gutierrez, 2013; Nasir & McKinney de Royston, 2013) perspectives on teaching mathematics. They went on to theorize the role of Black mathematics teachers as "boundary spanner[s] with membership in multiple communities, a mathematically proficient and intellectually powerful African American person within a historically disempowered African American community with a history of inaccessibility to and underperformance in mathematics" (p. 1). To better understand the role of Black mathematics teachers as boundary spanners, we propose using mixed methods as described above. In this era of rapid technological advances, with Betsy DeVos at the helm of the U.S. Department of Education, charter schools with "No Excuses" mottos dominating the education of Black children, and a dwindling mathematics teaching force, coupling oral history with other methods of data collection and analysis holds promise for unpacking how critical points in history impact still resonate in the present and hold promise for understanding a pressing issue like retention that the field treats as a contemporary issue despite its longstanding presence in the field.

Our future work will continue to unpack the complexities facing Black mathematics teachers with a nod to intersectionality by considering how race is impacted by gender, region, years teaching etc. We are working on predictive statistical models of Black teacher attrition based on the microaggressions data. Furthermore, the study continues to gather and share the oral histories of retired Black teachers of mathematics who taught Black students in functionally segregated schools pre- and post-*Brown.* We intend to map the intractability of negative racialized experiences over the generations of Black teachers, as well as the persistence of African American Pedagogical Excellence, defined as an ideology, set of beliefs, and instructional practices held by Black teachers that affirm and uplift Black students and their families toward high levels of academic achievement (Acosta, Foster, & Houchen, 2018). Further, we will use the oral history and quantitative data with focus groups of currently-practicing Black mathematics teachers across the country to deepen our understanding of the issue they face and what the field can to address these conditions.

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WHAT DOES ATTENDING TO POWER IN TEACHING LOOK LIKE?

Erin Smith
The University of Southern Mississippi
erin.marie.smith@usm.edu

Jonee Wilson
North Carolina State University
jwilson9@ncsu.edu

Keywords: Equity; Social Justice

Although scholars have recognized that *Standards* (National Council of Teachers of Mathematics [NCTM], 2000) based mathematics instruction alone cannot eliminate differential access and opportunity gaps, little movement has been made on a national scale to effectively teach in ways that a.) sustain students' cultural resources and b.) address issues of power–both of which are necessary in order to achieve equity (Gutiérrez, 2012; Gutstein, 2003; Jackson & Cobb, 2010; Leonard, Brooks, Barnes-Johnson, & Berry, 2010). Instead, such teaching is often reserved to isolated social justice or equity-focused lessons. If large scale change is to occur, teachers must be prepared and supported to effectively integrate students' cultural resources and attend to power in their instruction (Gutiérrez, 2002). When teachers attend to power, they consider who gets to speak and whose voice is privileged to ensure equitable participation (Gutiérrez, 2002), provide opportunities to use mathematics to interrogate inequities and increase sociopolitical consciousness (e.g., Gutstein 2003, 2016; Gutiérrez, 2017), value alternative ways of knowing (e.g., non-dominant mathematical practices; Moschkovich, 2002), and understand that mathematics is humanistic (Gutiérrez, 2002). Unfortunately, exemplars of this kind of teaching are limited within the field.

One way to support teachers' refinement and implementation of equitable practices that attend to power is through professional development and resources, such as the videos provided by NCTM. Given NCTM's large membership pool and conferred authority, NCTM has an opportunity to share a vision of mathematics instruction that sustains students' cultures and creates social transformation. To examine how effectively NCTM currently proffers such instructional support to its members, we analyzed the professional videos available to members. To do this, we coded all available videos in the "Principles to Action Toolkit" to identify the ways students' culture and power (as described above) was attended to.

Preliminary findings indicate that while the videos demonstrate examples of "reform-oriented" or "*Standards*-based" instructional practices, they show minimal evidence of instruction that sustains students' cultures and addresses power. More specifically, we found little evidence of instruction that provided opportunities for students to use mathematics to interrogate inequities, valued "alternative ways of knowing," or served to increase students' sociopolitical consciousness. Subsequently, the videos further support Gutiérrez's claim that "reform mathematics alone does not ensure that issues of power in society are addressed" (2002, p. 150). We contend, based on our analysis, that more concrete examples of how to teach mathematics in ways that sustain students' culture and address issues of power are needed if teachers are to implement changes in their practice.

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CONSTRUCTING AN ANTI-RACIST MATHEMATICS PEDAGOGICAL TOOL FOR MATHEMATICS TEACHER EDUCATORS

Oyemolade Osibodu
Michigan State University
mosibodu@msu.edu

Brent Jackson
Michigan State University
bjackson@msu.edu

Missy D Cosby
Michigan State University
mcosby@msu.edu

Sunghwan Byun
Michigan State University
byunsun3@msu.edu

Beth Herbel-Eisenmann
Michigan State University
bhe@msu.edu

Keywords: anti-racism, equity, professional development

Mathematics education in the United States is a white institutional space that disguises itself as apolitical (e.g., Martin, 2009). In this poster, we share a tool developed to assist mathematics teacher educators to enact an antiracist pedagogy. After two years of doing professional development with a group of teachers working in a small urban district, we realized that we had not developed a collective understanding of what it means to engage in anti-racist mathematics education. Therefore, we sought to learn along-side five teachers through a series of meetings conducted over the 2018-2019 school year. We believe this tool will be useful to mathematics teacher educators as they seek to incorporate and antiracist stance in their professional growth, teaching, and professional development they lead.

We draw on Collins' (2009) four domains of racism: structural, disciplinary, cultural, and interpersonal. The structural domain of power is the ways in which various institutions are networked to uphold the racial hierarchy. The disciplinary domain of power considers the ways in which policies and procedures are carried out or enacted by individuals or organizations. The cultural domain of power maintains a racial hierarchy by perpetuating images and narratives in media, popular culture, and elsewhere. Lastly, at the interpersonal level, racism occurs through prejudices among individuals' interactions that are intended to have a differential and/or harmful effect on members of the non-dominant race. Often the policies and procedures are race-neutral yet have disparate and harmful impact on the non-dominant race because of the ways they are carried out. We define antiracist mathematics pedagogy as actively resisting and countering racism at all four domains within mathematics education.

In our poster, we will share the tool that poses questions to consider and provides examples of resistance in order to enact an antiracist pedagogy within each domain. For example, when considering the disciplinary domain, the tool asks: How does this task reinforce (mis)perceptions about math abilities, who can be good at math, and the ways one demonstrates competence? We hope to generate engaging conversation around this tool to continue a discussion on antiracism within our mathematics education community.

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MATHEMATICAL RESERVATIONS: THE COLONIAL PSYCHOLOGY OF MATH EDUCATION AND ITS ROLE IN FEDERAL ASSIMILATION POLICIES

José F. Gutiérrez
University of Utah
jose.gutierrez@utah.edu

Cynthia Benally
University of Utah
cynthia.benally@utah.edu

Charles Sepulveda
University of Utah
charles.sepulveda@utah.edu

Kēhaulani N. Vaughn
University of Utah
kehaulani.vaughn@utah.edu

Keywords: Equity, Social Justice, Educational History, Policy

This project is at the intersection of mathematics education, educational history, and Indigenous Studies. We begin by proposing a theoretical argument, followed by a description of an archival research project. We seek to contribute to ongoing historical debates regarding the role of mathematics education in settler colonial curricula in the United States.

Educational scholars have argued that mainstream schooling functions as a *de facto* assimilation force, where the language and practices around academic learning are used to "smuggle" English-only policy, reproduce cultural erasure, and inscribe dominant values and ideology rooted in whiteness (e.g., Benally, 2014; Darder, 1991; Delpit, 2006; Trennert, 1988). The aim of colonial Indian education was to "Kill the Indian but save the man" and to train the Native to subservience (Lomawaima, 1993, 2018). For math education, the argument continues that the contemporary math curriculum acts as an insidious vessel that laces classroom discourse with colonial logics and norms (Gutiérrez & Scott, 2019; Popkewitz, 2004; Warburton, 2015). Without critical pedagogical interventions, these oppressive logics enable "ontological imperialism" (Bamberger & diSessa, 2003) and "cognitive normalization" (J.F. Gutiérrez, 2015) that are constantly at play in the psychology of mathematics teaching and learning. Is the current environment of math education for Native students merely the "de facto" state of affairs, or was it *designed* that way from the start? We hypothesize that oppressive practices in mathematics education are traced to late 19th and early 20th century U.S. assimilation policies, for example, the use of Native American boarding schools to transform not only the appearance and behavior of Native peoples but their *psychology* too—an act of internal colonization (Lomawaima, 1993).

We are launching a collaborative historiographic project in which we will examine our emerging hypothesis. Our research team consists of four scholars from varying fields with methodological expertise, including mathematics education, Indigenous epistemologies, decolonizing methodologies, Indian Education, and Native American history. We will analyze primary and secondary resources such as documents, photographs, and other artifacts, related to math curricula and policies utilized in Native American boarding schools. The main strategy within this archival project is to identify and code connections between historiographical materials and contemporary math education policies, practices, language, and concepts.

Discovering links between settler colonialism, federal assimilation policies, and the psychology of math education is vital for understanding the damaging effects of math schooling. This project can provide additional impetus for stakeholders to re-imagine the purpose of math education and to consider critical pedagogical approaches based on humanizing mathematics, such as represented by *Mathematx* (R. Gutiérrez, 2019), among other decolonizing pedagogies.

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MATHLETE TO ATHLETE: A CASE STUDY OF A DIVISION I STUDENT-ATHLETE'S MATHEMATICAL BELIEFS AND PERSISTENCE

Brianna Kurtz
University of Central Florida
brianna.kurtz@ucf.edu

Megan Nickels
University of Central Florida
megan.nickels@ucf.edu

Keywords: Affect, Emotion, Beliefs, and Attitudes, Equity and Diversity, Post-Secondary Education

Currently in the United States, there are over 179,200 students participating in collegiate athletics at the highest level, Division I. Of these, only 2% will continue on in their collegiate sport professionally (NCAA, 2018). That being said, student-athletes often face discrimination in both the eyes of professors and students, who may view them as being academically inferior and perpetrate the "dumb jock" stereotype (Engstrom & Sedlacek, 1991). Additionally, the students may often find the balance between academics and athletic performance to be a particular challenge (Yukhymenko-Lescroart, 2018). For student-athletes with interests in STEM in general and mathematics in particular, these challenges can be even more pronounced. Student-athletes have reported surprise by their peers and professors alike at their presence in mathematics and STEM courses, even going so far as feeling as though they are "spectacles (Comeaux, Griffin, & Bachman, 2017). This type of "othering" can have severe effects on mathematical performance. In one particular study conducted by Riciputi and Erdal (2017), student-athletes who were primed with questions about demographic identity and their athletic participation before engaging in a mathematics test attempted fewer problems at a statistically significant rate than student-athletes who were primed with questions only about demographic identity.

In this case study, a series of interviews were conducted with a purposively chosen male student-athlete who entered a Division I university in a major athletic conference as a STEM major. The interviews were coded to look for themes of identity and it was found that, as the student's athletic career developed, a shift in dominant identity from STEM student to student-athlete was noticed. Furthermore, due to the demands of Division I athletics, the student ultimately switched career goals out of STEM fields altogether. While this was a case study, further implications regarding analyzing ways to encourage student-athlete persistence in higher level mathematics and STEM coursework through post-secondary education were explored as a result of these interviews.

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A FRAMEWORK FOR DEVELOPING CRITICAL STATISTICAL LITERACY

Lauren Baucom
University of North Carolina at Greensboro
lnbaucom@uncg.edu

Keywords: Data Analysis and Statistics, Equity and Justice

In the 2007 *Guideline for Assessment and Instruction for Statistics Education* Report, Franklin and colleagues established that the learning of statistics has equal importance in understanding content knowledge and nurturing the statistical literacy one needs in order to be a productive citizen in today's society. However, it was not until 2012 that many educators were required to teach courses that contained the domain of statistics for the first time due to the adoption of the Common Core State Standards (National Governor's Association Center for Best Practices & Council of Chief State School Officers, 2010) by 47 US states. As a result, data analysis and statistics has gained recent prominence in grades 6 – 12, yet many K-12 mathematics educators are not adequately prepared to teach this new content area because of lack of prior coursework in statistics (Franklin et al., 2015; Shaughnessy, 2007).

Supporting students' skepticism of statistical information is necessary for this current era of data saturation, yet few students have the opportunity to cultivate healthy statistical habits of mind (Lee & Tran, 2015) and instead opt for heavy consumption of data as absolute truth. Literature has shown that students are not learning the statistical thinking and reasoning needed to develop healthy skepticism of statistical information, and that inappropriate reasoning of youth and adults about statistical ideas is pervasive, assiduous, and immutable (Garfield & Ahlgren, 1988). This compounded by teachers' limited experiences with data statistics and analysis which have implications for promoting statistical thinking of students.

Parallel to developing a healthy skepticism in statistics is the need to develop a critical consciousness in students engaging in mathematics learning. Developing students' critical consciousness in statistical thinking requires a sociopolitical turn that empowers students to identify with and transform their world (Gutierrez, 2013). The racial demographics of student populations in the United States schools are currently shifting. Given the large surges of enrollment of students of color and the simultaneous shrinking populations of white students in our public schools (Snyder, De Brey, & Dillow, 2018), it is important that youth of color have a robust, mirroring, statistical experience in K-12 education in order to engage in and critique the arguments around the statistics that classify them. Statistical literacy has been shown to be a fruitful space of radical possibility that could prompt a student to "recognize his or her position in society" and "motivate individuals to action" (Frankenstein, 1989, 1990, 1995, 2009; Gutierrez, 2013).

With a call for statistical reasoning that centers on citizenship of all students (Franklin et al., 2007), there is a need to theorize this work of fostering statistical literacy with a critical lens. As central tenets of the framework, I will use the existing literature around developing healthy skepticism and critical consciousness to highlight progress in this area of research and hypothesize a potential framework.

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BECOMING: EXPLORING THE LATINX GRADUATE EXPERIENCE – DIALOGUE NESTED IN NEPANTLA

Heather Lindfors-Navarro
University of Missouri
hblindfors-navarro@missouri.edu

Melissa Adams-Corral
The Ohio State University
adams.2153@buckeyemail.osu.edu

Ricardo Martinez
Iowa State University
ricardom@iastate.edu

Gabriela E. Vargas
University of Illinois Urbana-Champaign
gvargas2@illinois.edu

Keywords: Social Justice, Equity, Doctoral Education

> How do we develop mutual support systems that can engage us in dealing profoundly and meaningfully with issues that are deeply emotional: how we were treated as learners, how our intelligence was interfered with, how we experienced prejudice and discrimination and saw it affecting others? (Weissglass, 1998, p. 4)

Julian Weissglass' question mirrors the impetus which began our work together as graduate students in mathematics education. As Latinx graduate students entering mathematics education, we are becoming mathematics scholars invested in critical work. Through dialogue, we grapple with questions together: What is a critical mathematics scholar? How do I become one? Our identities as Latinx graduate students provides us with a position to best experience, understand, and speak to the barriers in becoming critical mathematics scholars because we are the ones living the movement (Freire, 2005). We engage in dialogue to reflect deeply on the barriers and to develop our strategies and forms of resistance. We reveal emotion and share stories with each other and the reader. With our vulnerability lies a risk to speak and be heard.

We chose to engage in a living manuscript (Shor & Freire, 1987) in which we are in dialogue after acknowledging the insight our identities provide and our common commitment to action. Dialogues in mathematics education have raised issues about racism and mathematics (Ambrosio, Et al., 2013); we wish to utilize a similar dialectic – centering the voices of graduate students in mathematics education. Each author of this poster is at a different research institution in four different states, each arriving to graduate school from different paths, exemplifying the rich complexity of varied perspectives.

Through engaging in the living manuscript, we found becoming a critical mathematics scholar to be complex. It requires answering the questions, *What is a scholar?* and *What is mathematics?* before asking *What is critical*? More so, the overlapping and layered spaces of our experiences resulted in the opportunity to harness nepantla (Anzaldua, 2013), that space of limitless possibilities, disidentification, and transformation. Nepantla offered us an opportunity to transform and grow as scholars, separated over vast geographical distance. We thus encourage others to engage in dialogue to learn more about self and the world. We end by echoing the words of bell hooks, where "[t]he engaged voice must never be fixed and absolute but always changing, always evolving in dialogue with a world beyond itself" (p. 11, 1994).

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Chapter 5:
Geometry, Measurement, Statistics, and Probability

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

TEACHERS' ENGAGEMENT WITH A COMPETING MODELS INFORMAL INFERENCE TASK

Jennifer N. Lovett
Middle Tennessee State University
Jennifer.Lovett@mtsu.edu

Ryan Seth Jones
MTSU
Ryan.Jones@mtsu.edu

Matthew Duncan
MTSU
Matthew.Duncan@mtsu.edu

Informal inference is a critical practice for students to engage in if they are to understand formal statistical methods. However, during informal inference students often utilize complex ideas that many in-service teachers are not prepared for as they have not had the opportunity to think deeply about statistics and develop statistical knowledge for teaching (Groth, 2013). Research shows that engaging teachers in authentic inquiry of content supports the development of that content knowledge, and there is an urgent need to do so through professional developments (PDs). However, there is limited literature concerning PDs in statistics education, and a dearth of research focusing on teachers' engagement with informal inference tasks. This paper describes in detail how two teachers engaged with a seminal informal inference task during a PD, including their reasoning about variability and sample size when making inferences.

Keywords: Data Analysis and Statistics, Teacher Knowledge, Technology

Informal inference is a critically important practice for students to develop in middle and high school if they are to deeply understand more sophisticated inferential methods in later grades (Pfannkuch, 2011). Even if students do not go on to study formal inference this foundation will allow them to question claims presented to them in wider society. However, informal inference makes use of a complicated set of ideas, representations, and practices. What's more, research has shown that most teachers do not have a deep understanding of the foundational concepts related to statistical inference (Lovett & Lee, 2017; Franklin et al., 2015). These findings should not be viewed as shortcomings of teachers, but rather shortcomings of the educational experiences they received in their school mathematics and teacher preparation programs. Even though standards documents such as *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) and *Common Core State Standards* (National Governors Association Center for Best Practice & Council of Chief State School Officers, 2010) have increased the emphasis on statistics, statistics has not been prioritized in K-12 mathematics classrooms and preservice mathematics teacher preparation programs. This has created an environment where teachers, even mathematics teachers, can complete their K-16 schooling without being supported to think deeply about statistics and develop the statistical knowledge for teaching (Groth, 2013) they need to teach the statistical standards in their curriculum. Research has shown that one effective way to increase teachers' content knowledge is to engage them in authentic inquiry that uses the concepts one desires teachers to develop (Borko, 2004). Thus there is an urgent need to support in-service teachers in deepening their statistical knowledge through professional development (PD) to support student learning in this area.

Even with this urgent need there has been a lack of research on PDs in statistics education. There are a few PDs that have researched teachers' statistical knowledge (Bargagliotti et al., 2014; Huey & Weber, 2018; Nieszporek, Biehler, & Griese, 2018; Peters, 2018; Peters, Watkins, & Bennett, 2014; Wassong, 2018) but none that we found report on teachers' approaches, understandings, and misunderstandings of specific informal inference tasks. These studies report pre/post knowledge gains or a retrospective analysis of the impact of the professional development on content (Huey & Weber, 2018; Nieszporek et al., 2018; Wassong, 2018), dilemmas teachers experienced regarding distributions or measures of center (Peters, 2018; Peters et al., 2014), or misunderstandings of sampling variability and regression that emerged as a result of several tasks (Bargagliotti et al., 2014). Thus the purpose of this paper is to share teachers' approaches and understandings on an informal inference task used during a professional development aimed to increase middle school teachers' statistical knowledge for teaching.

Background Literature and Framework

Informal Inference

A number of researchers have defined or described informal inference (e.g., Garfield & Ben-Zvi, 2008; Makar and Rubin, 2009; Rossman, 2008; Zieffler, Garfield, delMas, & Reading, 2008) and the common consensus is there are three key principles of informal inference: 1) claims, generalizations, predictions, parameter estimates, and conclusions, that extend beyond describing the given data; 2) the use of data as evidence for those claims; and 3) acknowledging uncertainty in their claims through the use of probabilistic language. Within informal inference Zieffler et al. (2008) identified three types of inference tasks that allow for the development of these key aspects of inferential reasoning: 1) estimate and draw a graph of a population based on a sample (referred to as *Samples and Populations*); 2) compare two or more samples of data to infer whether there is a real difference between the populations from which they were sampled (referred to as *Comparing Groups*); and 3) judge which of two competing models or statements in more likely to be true (referred to as *Competing Models*). Zieffler et al. (2008) identified three types of *competing models* tasks that have been used sample data to choose between two models or claims. In this paper we focus on one of these, a task used by Lee, Angotti, and Tarr (2010), the Schoolopoly task, which compares data generated by a probability device and the theoretical probability to determine if the probability device was "fair."

Schoolopoly task. The Schoolopoly task is a competing models task that asks students to investigate whether a die is fair or not based on a simulation (https://s3.amazonaws.com/fi-courses/tsdi/unit_3/Schoolopoly%20Task.pdf). There were a total of six dice companies to investigate and each pair of students were assigned one company. The students are asked to "investigate whether the die sent to you by the company is, in fact, fair. That is, are all six outcomes equally likely to occur?"

Variability

Research has shown that grounding statistical inquiry in a deep understanding of variability and supporting students to represent, describe, measure, and model variation in data can support them to understand statistical methods as well as the epistemic role they play in making claims in light of variability (e.g., delMas & Liu, 2005; Peters, 2011; Peters et al., 2014). In spite of the importance of helping students understand variability, research has also shown that variability is a topic that is challenging for people of all ages to grasp and the research on teachers' understanding of variation is sparse (Sánchez, daSilva, & Countinho, 2011). Random variation in everyday life is often perceived as having no structure, mere random differences. Because of this lay understanding of randomness, students and teachers alike often do not conceptualize random variation in ways that are consistent with statistical inquiry. For example, they rarely connect random variation with a mathematical structure that leverages probability to conceive of a distribution of variable outcomes (Lehrer & Kim, 2009; Watson 2006). This challenge extends to the use of variation in order to make predictions of probability situations. Sánchez and Garcia (2008) asked six teachers to predict the number of times each number would occur if a dice was rolled 60 times. Five of the six teachers predicted an equal number of outcomes for each number on the dice; a sequence of 10, 10, 10, 10, 10, 10. These teachers were viewing variability deterministically, looking for a definite answer.

Sampling and Sample Size

When making decisions under uncertainty, most people tend to use judgmental heuristics that can cause errors in reasoning (Harradine, Batanero, & Rossman 2011). One example that applies to competing models tasks is that people tend to believe that even a small sample should reflect all of the characteristics of the population (Kahneman & Tversky, 1972). This misunderstanding can be extended to random samples. Even if a random sample is chosen in an appropriate way and of sufficient size, research has shown that students believe that the random sample is a replica of the population. This way of thinking does not take into consideration the variability across samples (Harradine et al., 2011; Saldanha & Thompson, 2002). Saldanha and Thompson (2002) found when high school students were asked to judge how representative a sample was in relation to a population parameter, students compared their sample to the population parameter and not "on how it might compare to a clustering of statistic's values" (p.265).

Professional Development Context

The task was situated within a year-long professional development program designed to blend online learning resources (*Teaching Statistics Through Data Investigations*, http://go.ncsu.edu/tsdi), an intensive summer workshop, and monthly professional learning community (PLC) meetings. The summer institute was designed to support the development of content knowledge and knowledge of student thinking using resources from a middle grades curriculum called Data Modeling (modelingdata.org). The year-long professional development framed statistical inquiry as the practice of generating knowledge using variable data, engaged teachers with statistical inquiry tasks, and supported them to analyze student artifacts and collectively plan instructional strategies for their classrooms.

The focus task in analysis occurred during day 3 of the five-day summer workshop. We began the first day of the workshop by engaging teachers in making an inference about a randomly drawn reward process to determine if the drawing was fair or biased in order to frame the activity of making model based inferences in the midst of variable systems. With this broad framing, we then engaged teachers with repeated measurements of a common object to engage with variability and distribution, and supported them to make judgements about student thinking using student artifacts from a similar activity. Teachers then explored students' approaches to measuring the center of a distribution, and teachers invented statistics to measure variability. During day 3 the teachers began by creating repeated samples of chance processes to explore sampling variability. The Schoolopoly task occurred after these activities, with the goal for teachers to use ideas about data, distribution, sampling variability, probability, and statistics to make their judgements about the dice companies.

Schoolopoly Task for Teachers

We modified Lee et al.'s (2010) original Schoolopoly task that was designed for middle school students to be used with middle school teachers during our PD (Figure 1). The first modification we made was in the choice of technology. Since Probability Explorer is not readily available to teachers, we used StatCrunch. The same six "companies" that Lee et al. (2010) used are now built into StatCrunch as an app. Secondly, the teachers were asked to investigate all six dice companies instead of just the one company that middle school students were assigned. Finally, teachers were given a limited number of trials that they could assign to the six companies as they wanted to. In Lee et al.'s (2010) study, the middle school students were investigating the role of sample size for the first time and did not have any directions regarding the number of trials. However, since teachers were aware that a larger sample size would produce a more representative sample we chose to limit the number of trials they could run. Two additional parts of the task were added as the teachers were working through the task. They were unaware of these additional parts when they began working. Part B and C of the task allowed teachers additional trials to determine the one fair company. These decisions to modify the task for teachers were made so that teachers would have opportunities to compare empirical data from different populations, and so to provoke a need to make inferences with limited numbers of samples since students in the past often ran large numbers of trials to reduce uncertainty.

Part A: Your school is planning to create a board game modeled on the classic game of MonopolyTM. The game is to be called Schoolopoly and, like MonopolyTM, will be played with dice. Because many copies of the game expect to be sold, companies are competing for the contract to supply dice for Schoolopoly. Some companies have been accused of making poor quality dice and these are to be avoided since players must believe the dice they are using are actually "fair." Each company has provided a sample die for analysis and to investigate:

Luckytown Dice Company	Dice, Dice, Baby!
Dice R' Us	Pips and Dots
High Rollers, Inc.	Slice n' Dice

1. You have 300 trials to divide up between the 6 companies. Test the companies to determine if the company is fair or biased. There is at least one fair company.
2. In a google doc, record your evidence to support your decision if the company is fair or biased and your decision of how to divide up the 300 trials.

Part B: The PD facilitator offers each group 200 additional trials to identify the single unbiased company. Trials can be used to confirm or disconfirm their previous identifications of each company as biased or unbiased.

Part C: The PD facilitator offers each group 50 additional trials if they would like them.

Figure 1: Schoolopoly Task Modified from Lee et al. (2010).

Methodology

Given the importance of developing teachers' statistical knowledge for teaching and the lack of research on teachers' approaches to informal inference tasks, warrants study in the ways that teachers engage in an informal inference task during a PD. In this study we aim to answer the following research questions:

RQ 1 How do teachers engage with the Schoolopoly task?
RQ 2 Within the Schoolopoly task, how do teachers reason about sample size and variability?

To answer our research questions, we utilized a multiple case study design (Yin, 2009) to describe the ways teachers engaged with the Schoolopoly task and the ways they reasoned about sample size and variability. Each case is represented by a pair of teachers working on the Schoolopoly task.

Participants

During the summer of 2017, 20 (2 = males, 18 = Females) middle school mathematics teachers participated in a week-long summer institute housed within year-long professional development project. Teachers were recruited from local Southeastern public schools representing seven different schools across both urban and suburban areas. Seven out of the twenty teachers were African-American and the remainder were White. As a whole, teaching experience ranged from less than one year to twenty plus years.

Data Collection and Analysis

Teachers each had access to a computer but worked in pairs to complete the Schoolopoly task. They collaborated in a google document to organize screenshots of the graphical representations created by StatCrunch as well as their identifications and justifications of each machine being fair or biased. We screen and audio recorded each pair during the task.

After the screen recordings were transcribed, each researcher individually wrote a narrative of the teachers working through the task. The narratives included portions of the transcript, justifications and graphical representations from the google document, and the researchers recording of their actions. Researchers compared their descriptions and collaboratively wrote a single description for each pair of teachers. Then we completed a cross-case comparison (Merriam, 1998) to identify themes in the ways teachers engaged with the task and how the reasoned about sample size and variability. Through this comparison three themes emerged; 1) The ways teachers articulated the two models under competition, 2) the ways teachers described and measured variability, and 3) the ways teachers accounted for sample size.

Results

Due to space limitations, we present one of the three cases of teachers, Myra and Kendall. This case was chosen because in many ways these teachers typified approaches of the other teachers in the PD, but also because of their efforts to formally quantify the variability to inform their decisions, which was less typical.

Kendall and Myra's Work on the Schoolopoly Task

Myra and Kendall began the task by dividing up their 300 roles evenly among the six companies before making any inferences about which companies were fair or biased. Through examination of the graphs Myra and Kendall immediately declared five of the six companies biased without much discussion. They made initial judgements about the status of each company by focusing on how similar the proportions are, explaining "If the dice were unbiased the results would be similar/same for each outcome."

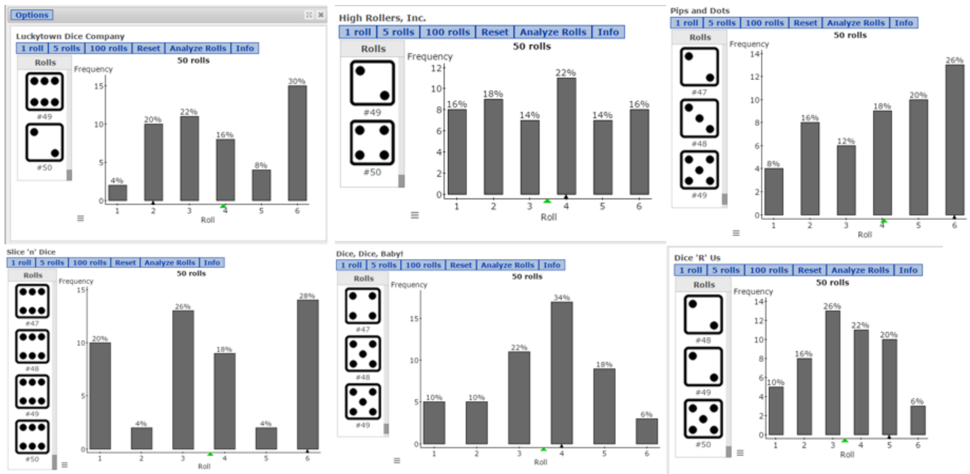


Figure 2: Myra and Kendall's Distributions After 50 Rolls

Myra and Kendall began looking for how similar outcomes were after the initial 50 rolls for each company. They then mathematized the idea of similarity by calculating the difference from the lowest and highest percentages for all six companies. For example, for Luckytown (Figure 2), Kendall and Myra wrote, "Biased. Luckytown has a range of results from 4% to 30%. This tells me that I am very unlikely to roll a 1 as compared to rolling a 6." This was their first method of quantifying the variability in the data. This approach led them to identify all companies but one as biased. The one company they declared unbiased after 50 rolls was High Rollers Inc (Figure 2). Kendall and Myra identified this company as fair since "the range of this data is 6%, making the results similar to each other (within 6% of each other) making this more likely to be the unbiased or fair option. They did not discuss sampling variability or mention what was an expected variation in the occurrences; only that six percent fell into the interval that they felt comfortable calling unbiased.

When Kendall and Myra were offered an additional 200 rolls and they first suggested to use 100 of their additional rolls on High Rollers Inc. to verify their results. Then they decided to consider other companies that may be fair even if they originally judged them as biased. To decide which companies to give additional rolls to they identified companies that had "no way" of being fair based on the first 50 rolls, referring to the difference in the highest and lowest percentages when making this decision. Kendall and Myra decided that a sample size of 50 was sufficient to make this decision as long as the difference was extreme.

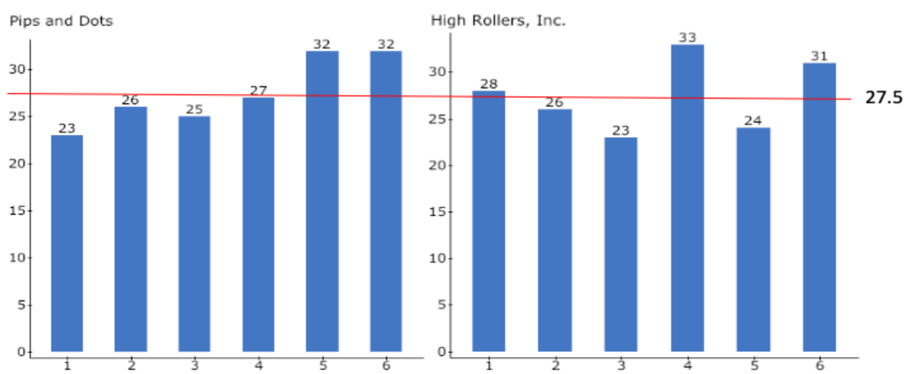
Kendall and Myra decided that the only companies that needed additional trials was Pips and Dots and High Rollers Inc. They give an additional 40 trials to Pips and Dots and High Rollers Inc. and continued with their method of comparing the difference in the percentages of the lowest to the highest occurrence. At this point they again compare all six companies, even though the two companies now have 90 trials and the other four still only have 50 trials. Kendall and Myra did not discuss how the sample size impacted the conclusions they were drawing. Instead continued to focus on companies that they "could for sure say absolutely not" unbiased,

> I feel like these percents are just so far apart like their range is four percent to 28 percent and there's two of them [referring to Slice and Dice and Dice R Us']. And this one we've got six percent to 34 percent [referring to Dice Dice Baby!]. I mean that's almost a 30 percent

difference on these ones. I mean this is 24 difference, this is almost 30. Like 28 and 11 to 22 that's only 11 difference. So are we, do we think Pips N Dots and High Rollers?

Myra and Kendall chose to only focus on High Rollers, Inc. and Pips and Dots. However, at this point in the task examining the differences in the percentages was not providing enough confirming evidence for Kendall and Myra to distinguish which one was fair, so they began looking at the expected values as a possible indication of bias. This was the first mention that the expected value of the dice, “So they should be in the 16 to 17 range so how far are they off from it? Because I feel like half of mine are pretty close to that and half of them are not.” They continued with this line of thinking and continued adding trials to High Rollers Inc. and Pips and Dots until each company had a total of 140 trials. Myra and Kendall decided “to stick with our original company [High Rollers, Inc.] because it seemed fair with Part A and Part B where Pips and dots only seemed the most fair with Part B.” However, they were still not convinced and wanted more trials.

Since Myra and Kendall were still unsure, the PD facilitator offered a final 50 trials. They decided to divide the trials evenly among High Rollers, Inc. and Pips and Dots for a total of 165 trials (Figure 3). Still trying to quantify the variability in the distributions, they determined the expected value for each number on the dice as 165/6 to be 27.5 trials. They then calculated “variability in the data by finding the average distance from 27.5.” They referred to this measure as the M.A.D. For Pips and Dots this was 3 and for High Rollers, Inc. was 3.166. This Myra to comment that Pips & Dots was less variable and Kendall asked if that matters. Mayra said “it showed the outcomes were closer to the expectation if fair”. They then agreed this measure was helpful and declared Pips and Dots a fair company.



Number	1	2	3	4	5	6
Frequency (f)	23	26	25	27	32	32
|f-27.5|	4.5	1.5	2.5	0.5	4.5	4.5
Average Absolute Deviations						3.0

Number	1	2	3	4	5	6
Frequency (f)	28	26	23	33	24	31
|f-27.5|	0.5	1.5	4.5	5.5	3.5	3.5
Average Absolute Deviations						3.17

Figure 3: Myra and Kendall’s Measure of Variability After 165 Rolls

Articulating the competing models. During the task Kendall and Myra never explicitly articulated the two models under competition but as the task progressed there was a shift in their description of their null model. Originally, Kendall and Myra were comparing the distributions their data produced to a uniform probability model. They expected all outcomes for a single company to be “similar/same for each outcome.” However, when using the difference in the highest and lowest outcome was not helping them move forward in identifying a single company, Kendall and Myra shifted their model. For the second half of the task, they described

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

their null model as the expected value either in terms of percentage or number of outcomes. This shift in their null model allowed Kendall and Myra to quantify the variability of each distribution as the dispersion from the expected value.

Describing and measuring variability. In the end, Kendall and Myra tried a total of three ways to describe the variability in the distributions: examining the difference in the highest and lowest percentage, examining the number of outcomes that fell into a middle region around the expected value, and calculated a measure by subtracting each frequency from the expected frequency of 27.5 and taking the average of those distances. These approaches increased in their sophistication when they had to identify a single company as fair. Their first two attempts at quantifying the variability were subjective by comparing the differences in the highest and lowest percentages and the number of outcomes that fell into the middle region. For both of these they subjectively determined an acceptable range in percentages or what constituted the middle region around the expected value. Their final approach, subtracting each frequency from the expected frequency of 27.5 and taking the average of those distances, provide Kendall and Myra a numerical measure to compare the variability in each companies' distributions.

Accounting for sample size. Sample size played a crucial role in the way Kendall and Myra engaged with the task and how they discussed variability. The way in which Kendall and Myra accounted for the sample size at the beginning of the task, evenly distributing the samples, was similar to the way the majority of the teachers engaged with the task. Throughout the task Kendall and Myra appeared to be ok with making a conclusion that a company was biased from a sample of 50 but wanted a larger sample to be able to determine if a company was fair. This was seen when they were comparing Pips and Dots and High Rollers Inc., that each had 90 trials, to the other four companies that only had their original 50 trials. Kendall and Myra continued to classify the other companies as biased with a smaller number of trials. Their struggle with accounting for sample size in their conclusions was also shown when they decided "to stick with our original company [High Rollers, Inc.] because it seemed fair with Part A and Part B where Pips and Dots only seemed the most fair with Part B." Even though Kendall and Myra had an understanding that a larger sample size is better they did not seem to understand that larger samples would produce less sampling variability. Overall, when Kendall and Myra were considering sample size their logic seemed to focus on rolling more trials to find evidence to support the fairness, and not how much variation is expected when a company is fair.

Discussion and Conclusion

Due to the infancy of research in statistics education, many tasks that statistics educators use with teachers in a PD have not been researched with that population. We chose to incorporate the Schoolopoly task into our summer PD to engage teachers in an informal inference task that required them to use ideas about data, distribution, sampling variability, probability, and statistics to make their judgements about the competing models. Thus, we examined the ways in which teachers engaged with the task and reasoned about sample size and variability. In doing so we identified three themes emerged; 1) The ways teachers articulated the two models under competition, 2) the ways teachers described and measured variability, and 3) the ways teachers accounted for sample size.

From our results, there are several changes we would make to our modified version of Schoolopoly task for the future. It is evident that Myra and Kendall expect that a random sample of sufficient size should replicate the population characteristics (Harradine et al., 2011). Therefore, even though teachers know that large samples are more representative of the population they need more opportunities to engage in instructional activities to draw many samples to understand the variability among samples from a population (delMas, Garfield, & Chance, 1999; Saldanha & Thompson, 2002). In the end, we would still limit the number of trials the teachers could use to explore the companies but instead of adding additional trials to their distributions, have teachers generate additional samples of 50 trials and then add to their new samples. This would allow teachers to still examine all of the companies and compare multiple samples of 50 for the same company before determining which companies are fair, providing them the opportunity to estimate the underlying probability distribution of each company, similar to Lee et al.'s (2010) study. Then teachers could continue with the task, examining larger samples and comparing those samples to make a conclusion on which company they believe is fair.

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TIME EXPRESSIONS AND ELEMENTARY STUDENTS' REASONING WITH THE ANALOG CLOCK

Darrell Earnest
University of Massachusetts, Amherst
dearnest@educ.umass.edu

John Chandler
University of Montana
john.chandler@business.umt.edu

*This paper investigates the interplay of words and expressions with students' efforts to indicate times on a clock. We consider how elementary students interpret precise times (e.g., 2:30, 4:30) as compared to relative times (e.g., half past 11) as they describe this intangible quantity using a clock. Interviews with students in grades 2 and 4 (*n*=42) revealed that precise times led to whole number descriptions whereas relative times led to descriptions of part-whole relations consistent with representations of measure. A subsequent analysis of assessment performance of students across elementary grades (*N *= 612) corroborates the differential performance based on expression used in the tasks. Findings suggest that elementary students need further support to treat this invisible quantity according to measurement properties. Implications for theory are discussed.*

Keywords: Time, Clock, Elementary math

Learning mathematics is inextricably linked with the words and expressions used to enact and communicate mathematical ideas. For this reason, theorists have offered sociocultural descriptions of learning that include explicit treatments of word meaning and use (e.g., Halliday, 1993; Sfard, 2008; Vygotsky, 1978, 1986). For the topic of time, modern practices include a variety of expressions both within and across languages around the world that reference time units in substantively different ways. The time 2:15 may be stated as "two fifteen," "fifteen past two," or "quarter past two" depending on the language and community (see Bock, Irwin, Davidson, & Levelt, 2002; Burny, Valcke, DeSoete, & Van Luit, 2013). Even within a language like English, speakers of the same language across countries may draw upon time expressions with different treatments of time unit (e.g., for 11:30, a common phrase in Ireland is "half 11," an expression unfamiliar to many in the United States). Just as a language's counting words are consequential to early understandings of numeracy (Miura, 2001; Ng & Rao, 2010), we investigate if analogous expressions for time with substantively different treatments of time unit lead to differentiated descriptions of time.

Our research from a sociocultural perspective has examined how children draw upon the analog clock as a tool to solve elapsed time problems (Earnest, 2017; Earnest, Gonzales, & Plant, 2018). We share two contentions with current approaches to time instruction. First, time is invisible and untouchable, and for this reason we contend that we must consider the consequential role of words and language as individuals describe time and interpret representations of time (Bock et al., 2002; Tillman, Marghetis, Barnet, & Srinivasan, 2016). Research has established the interplay between words and pathways of mathematical thinking, such as the role of counting words in some East Asian languages that highlight the structure of the base-ten number system (Miura, 2001; Ng & Rao, 2010). Building on Sfard's (2008) treatment of words and narratives (or, the story or description of the "relations between or activities with or by objects," p. 574) in mathematics discourse, we explore such interplay in order to generate a sociocultural description of students' thinking in relation to time words.

Second, as we further consider below in implications, we dispute the current treatment of time in elementary mathematics, which relegates the topic to early elementary grades with a focus on clock-reading procedures. Technology has advanced to the point where reading the analog clock in order to identify the time is no longer an appropriate endgoal—digital clocks easily enable this and are widely available. We do not suggest the analog clock is unimportant; rather, we propose that as a field we ought to re-think the role of time in K-12 education, in particular how representations of time such as the analog clock or the *x*-axis of a graph share structural similarities (Earnest, 2015) together with its relation to other elementary mathematics topics, such as fractions and partitioning (Earnest et al., 2018).

Researchers investigating the learning and teaching of time, including our team, have drawn attention to the minimal research conducted on learning this topic (Burny et al., 2013; Kamii & Russell, 2012). Moreover, even among those studies conducted over recent decades, experimental designs and identified data points suggest different visions for what is critical and necessary in order to investigate children's time-related ideas. Unlike the present analysis, which considers student thinking in tandem with the tool and language, much of the developmental research explored children's emerging intuitions related to duration independent from the standard tools or units we have for time (e.g., Long & Kamii; 2001; Piaget, 1969; Wilkening, Levin, & Druyan, 1987). Also from a constructivist perspective, Kamii and Russell (2012) explored children's elapsed time calculations in standard units, yet independent of an analog clock's spatial representation of time. In one of the few studies that considered how children themselves grappled with time-related ideas together with standard units of and tools for time, Williams (2012) conducted a cognitive ethnography in early elementary classrooms, identifying metaphors upon which children draw as they reason with an invisible and intangible entity (see also, Lakoff & Nuñez, 2000). We therefore see this topic within mathematics education as one in need of more research, in particular because of its underlying role in the mathematics of change that characterizes middle and high school mathematics. Because time is intangible, words and expressions for time take on a prominent role in the construal of meaning, leading us to our research questions focusing on the interplay of words and action.

We address the following research questions: (a) How do students interpret precise and relative times on an analog clock? If students do so differently, (b) is there a difference in performance on such tasks across elementary grades?

Method

Participants for individual paper-and-pencil assessment included students across grades 2-5 (N = 612) from three school districts in urban, suburban, and rural areas in Massachusetts. Students were asked to, for example, "Show 2:30 on the clock" or "Show half past 11 on the clock" (Figure 1), which were two of 35 total items. For the focal tasks, we coded the position of the drawn hands as a point between 0.0 to 11.9, with accuracy coded as $\pm$ 0.2 from the accurate position (for example, an accurate hour hand position for 4:30 pointed between 4.3 and 4.7).

Using scores on the written assessment, a subset of students in grades 2 and 4 (n = 42) participated in interviews in which they solved problems involving an analog clock with independent hour and minute hands. For two focal tasks, the interviewer presented a card that stated the problem (i.e., "Show me 4:30" or "Show me half past 11") and asked the student to describe their solution approach. Our research group coded video data for accuracy of positions for hour and minute hands as described above. Further, we open coded (Corbin & Strauss, 2008) transcripts and video to identify emerging themes drawing upon utterances and activity in problem solving. Narrative categories emerged through multiple passes through the data. We present our findings beginning with interview analysis and, based on those results, turning back to the assessment data as a way to corroborate findings based on interviews. We note two decisions made due to space constraints in the present paper. First, we limit our interview analysis to narratives in data reflecting 10% or more of interviewee responses. Second, we limit our analysis to tasks to the half hour; our presentation will feature additional assessment and interview tasks featuring additional precise and relative times.

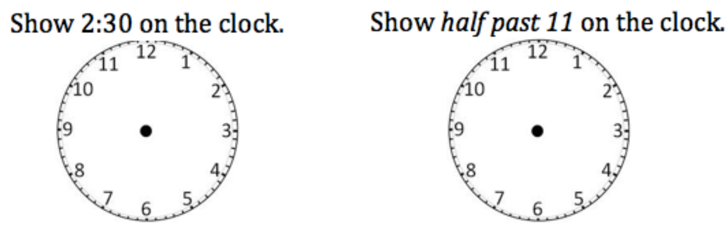

Figure 6: Focal Assessment Tasks

Results

We begin with a description of narratives (Sfard, 2008) that emerged in our open coding analysis of interviews before then detailing how such narratives emerged in students' interpretations of each time expression. Four dominant narratives emerged that reflected 10% or more of responses in interviews: Whole Number, Part-whole Relations, Conversion, and Hand Movement. Table 4 features narratives with frequencies across the 168 possible instances (2 subtasks for 2 problems for each of the 42 students)[1].

Table 1: Identified Narratives

Narrative	Number of Instances	% of total (*N*=168)
Whole Number	93	55.4%
Part-whole Relations	75	44.6%
Conversion	27	16.1%
Hand Movement	20	11.9%

Some solutions reflected ideas related to Whole Number, namely number matching and skip counting. For example, for 4:30, a second grader matched the hour hand directly (and inaccurately) on 4 and the minute hand directly (and accurately) on the 6 and stated: "It was easy, because I already knew *that* [6] was thirty, and I already knew *that* [4] was a four" (Figure 2a). Other students coded as Whole Number drew upon skip counting to count by 5s in order to position the minute hand (Figure 2b)

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* St Louis, MO: University of Missouri.

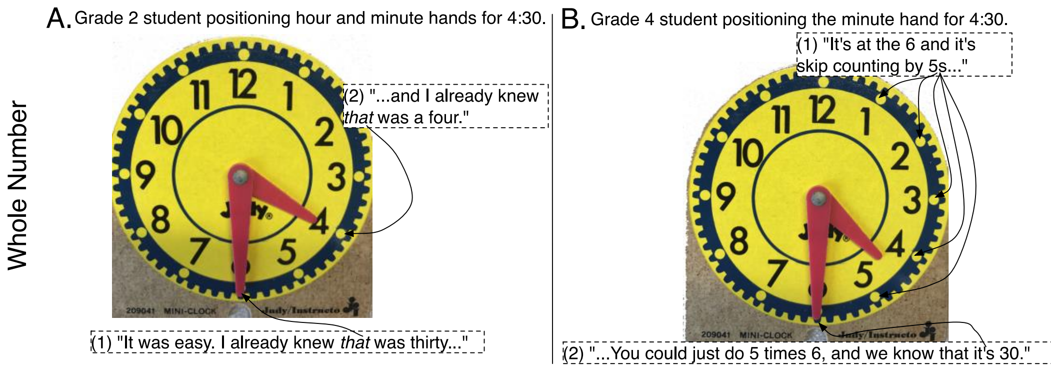

Figure 2: Whole Number Approaches to Hour and Minutes Hands

We coded approaches as Part-whole Relations (Figure 3a-c) when a student identified a unit interval or whole on the clock and described partitioning that unit or whole. For example, a second grader solving 4:30 positioned the hour hand at 4.5, stating, “It’s past the four <*sweeping his finger from the 4 to the 5*> by half, because it was 30 minutes into the hour” (Figure 3a).

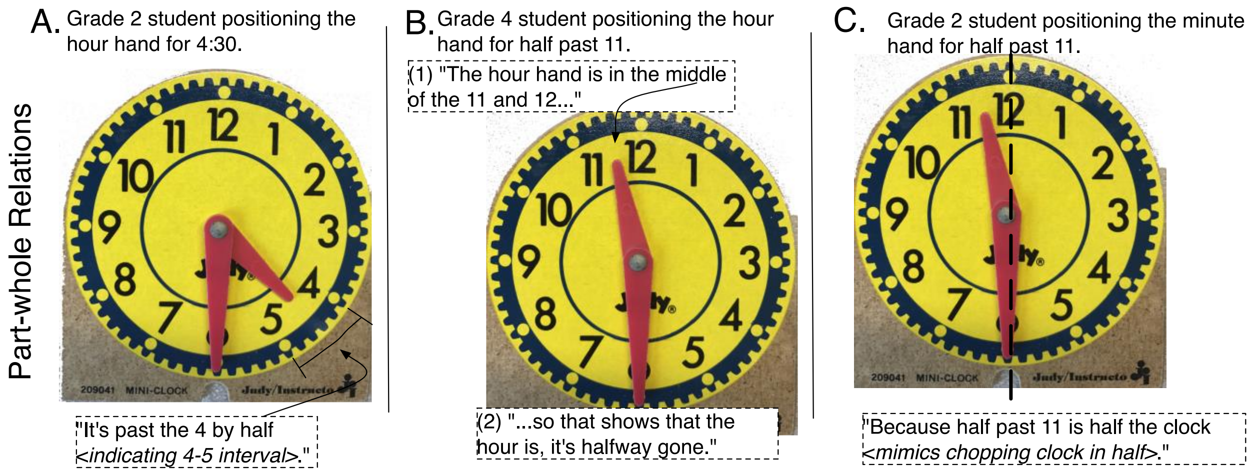

Figure 3: Part-whole Relations Approaches to Hour and Minutes Hands

We assigned the code Conversion when a student described a conversion between hours and minutes, which in our data was either one hour and 60 minutes or half an hour and 30 minutes (Figure 4a). We applied this code only 27 times out of a possible 168 solutions. Hand Movement was applied in only 20 of the 168 possible solutions. This particular code was applied when a student, first, referenced a particular numeral on the clock and, second, positioned the focal hand in tandem with a description of the moving hand; this description, however, did not include reference to a unit or whole (Figure 4b-c).

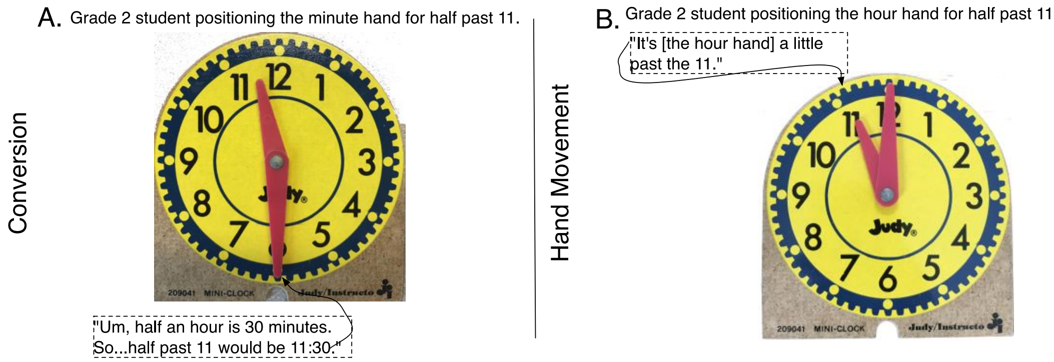

Figure 4: Conversion and Hand Movement Approaches to Hour and Minutes Hands

Expressions of Time and Corresponding Narratives

We now consider how the narratives manifested as a function of the expression in the prompt (4:30 or half past 11) and the hand (hour or minute). When presented as 4:30, students' solution approaches largely reflected ideas related to Whole Number (see left panels of Figure 5). In total, 20 of 42 students drew upon whole number ideas for the 4:30 hour hand and 36 of 42 did so for the 4:30 minute hand. For the hour hand specifically, Whole Number approaches—matching the number in the prompt with a number on the clock—were highly unsuccessful, despite being the most-represented approach for the 4:30 hour hand. Turning to half past 11 (right panels of Figure 5), students' solution approaches for both hour and minute hands largely reflected ideas related to Part-whole Relations, with 24 of 42 students drawing upon this narrative for both of the hands and mostly accurately.

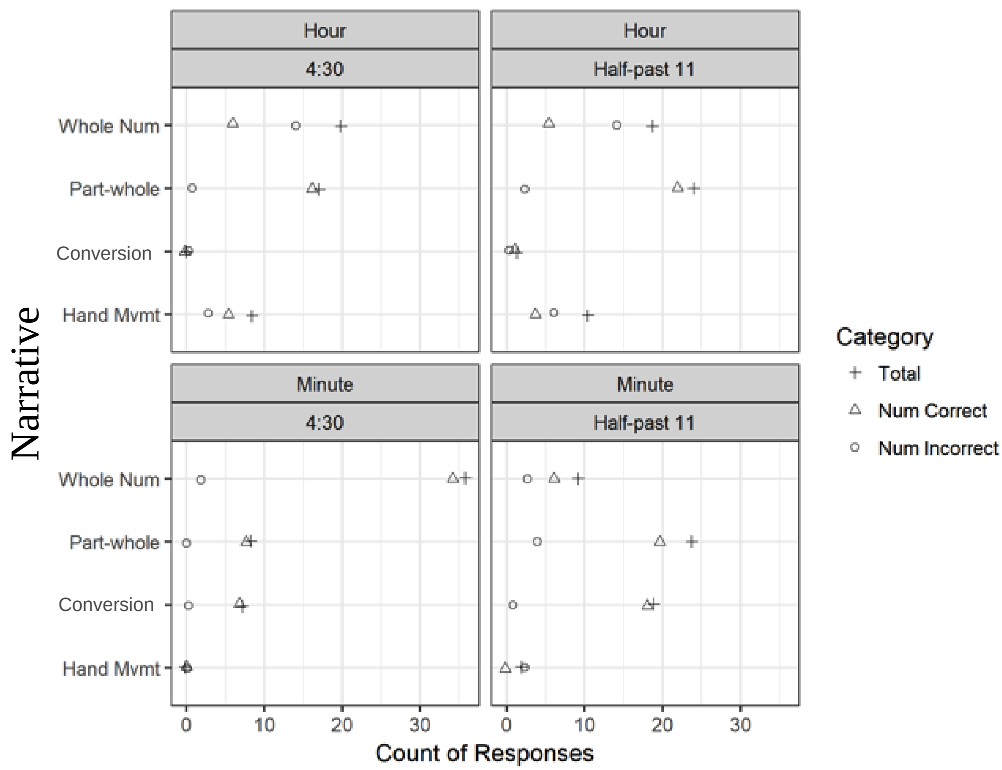

Figure 5: Narratives Emerging in Interviews

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

In sum, the Part-whole Relations narrative was deployed across hands and tasks with general success (73 instances out of 168; 90% accuracy). The Whole Number narrative was more common yet more variable with respect to accuracy (84 instances out of 168, 60% accuracy). Most students with Whole Number descriptions of the hour hand positioned it directly and inaccurately on the hour, whereas those with Whole Number descriptions of the minute hand successfully counted by 5s to position it. The units presented in the prompt were related to the narrative that emerged in children's descriptions, indicating the consequential interplay of time expression and children's descriptions of time, though as can be seen in Figure 6, those students drawing upon Part-whole Relations for the 4:30 hour hand (partitioning the hour interval) were largely successful, unlike those that drew upon Whole Number (matching the numeral 4 in the prompt with the 4 on the clock).

Given these differences in interview performances based on time units in the prompt, we turn back to assessment data for students across elementary grades to either corroborate or refute the trend in interview analysis. On the assessment, we found no statistical difference in performance on the two focal tasks when comparing the overall problems. However, when looking at hour and minute hands separately, a striking pattern emerged: accuracy for the placement of a particular hand in relation to time expression is statistically different. Across all grades, hour hand performance is better for half past 11 (Figure 7, left panel). The reverse is true for placing the minute hand; all grades perform this subtask better for the 2:30 task. Patterns in performance across subtasks are statistically significant. The minute-hand placement had lower accuracy rates relative to the 2:30 minute-hand subtask ($\widehat{\beta_s} = -0.29, \hat{\sigma} = 0.15, p < 0.05$). Hour hand position on the half past 11 subtask was more accurate than that for 2:30 ($\widehat{\beta_t} = -0.87, \hat{\sigma} = 0.12, p < 2 \cdot 10^{-13}$). Despite the fact that the two tasks featured analogous times to the half hour, accuracy in hand positioning appears to depend on which hand and on how time units are referenced in the prompt. Although the analysis involves a comparison of only two tasks and must therefore be interpreted humbly, findings corroborate qualitative trends that emerged in interview data: children are likely to interpret time meaning in relation to the time expression.

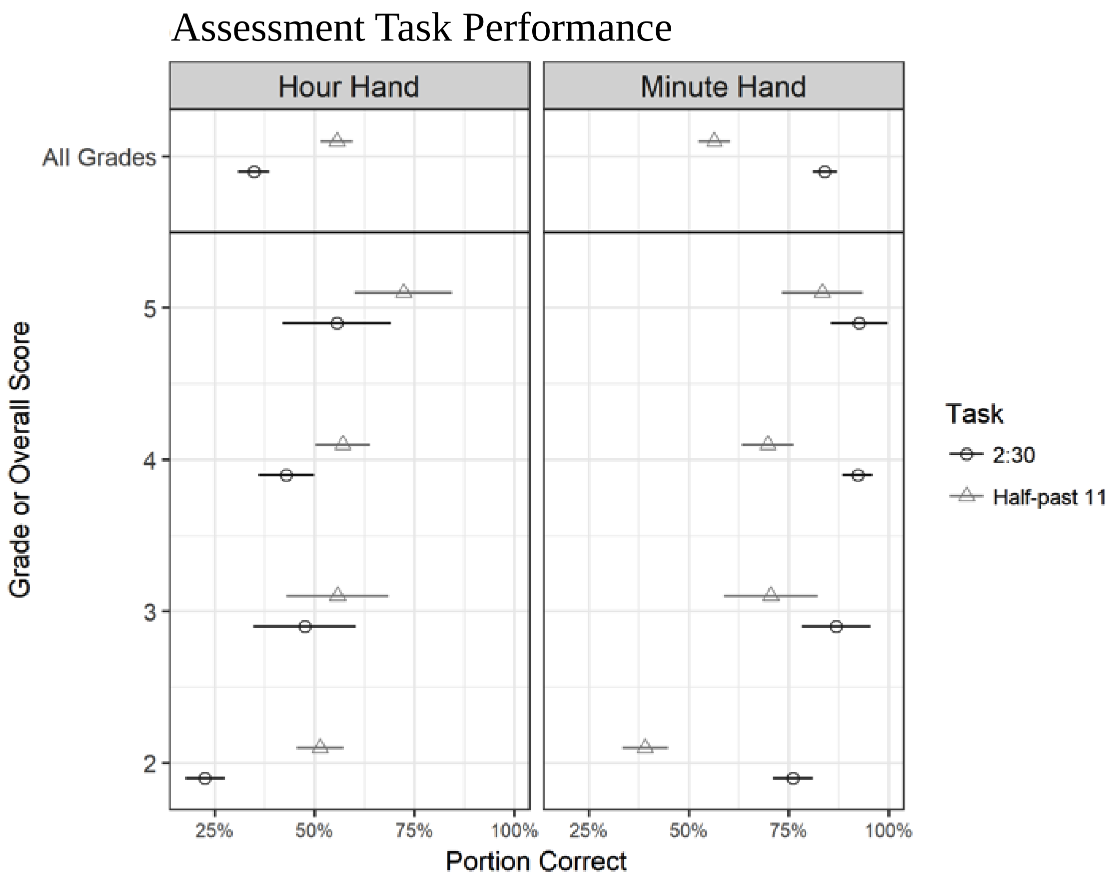

Figure 7: Performance for Hour (left) and Minute Hand (right) Placement for Each Task

Discussion

Time is a challenging topic in mathematics education due in part to its invisible character. Our mixed methods analysis indicates that expressions with substantively different treatments of time units lead to different interpretations of time. Each expression, both of which are appropriate ways to reference the time of day, encodes information about agreed-upon conventions related to unit. The times 4:30 or 2:30 are typically stated with whole number words that highlight whole number features of the clock and, in our data, were less likely to make salient clock features related to fractions and partitioning. Instead, the words are the same as those children have long used to reference whole numbers, a topic that almost assuredly has constituted the bulk of their elementary mathematics experiences thus far. Meanwhile, when presented with the expression half past *h*, many students drew upon part-whole relations, a narrative related to partitioning and measurement. This was evidenced even among the second graders on the assessment, among whom more than 50% positioned the hour hand for half past *h* accurately as compared to less than 25% for *h*:30 (see bottom left of Figure 7); of note is that both tasks are currently identified as grade 1 benchmarks (CCSSO, 2010), even though grade 2 students in our sample clearly struggled with these same ideas.

Why are these results relevant for mathematics education? Time pervades the mathematics of change in later grades as well as basic scientific investigations of the world yet, surprisingly, current time instruction highlights little more than clock-reading procedures in the early elementary grades. We attribute Whole Number narratives above as related to the over-emphasis on clock-reading procedures. Our findings contribute to emerging research with implications more broadly for a need to re-think K-12 instruction related to time. Part-whole Relations narratives—those that typically accompanied the “half past *h*” expression—well reflect measurement principles. Extrapolating these findings to later grades when students encounter other contexts involving representations of time, such as the *x*-axis of a function graph, we conjecture that a Part-whole Relations narrative for working with a clock will be generative and continue to support mathematical insights when working with such representations with structural similarities (see also, Earnest, 2015). Although further research is needed, we are less confident that Whole Number narratives well provide a foundation to support later mathematical ideas involving interval properties of time, potentially leaving students unprepared for complex mathematical ideas involving time.

A narrative is a story of relations or activities among objects (Sfard, 2008). As we found, the words for time expressions are interwoven with particular mathematical narratives that may or may not be consistent with the endorsed ideas about time and time representations. Our current instructional emphasis on clock-reading procedures may emphasize whole number properties of the clock at the expense of interval properties related to fractions and partitioning.

What should children learn about time, and when should they learn it? We do not interpret the findings above to indicate that one phrase ought to be used in curriculum and instruction absent of the other; in fact, such a claim would poorly reflect reality. Rather, our findings highlight the interplay of words and time ideas and underscore some of the complexity we as a field have been overlooking related to emerging conceptions of time. Findings suggest that we ought to be strategic and thoughtful about when and how we draw upon particular expressions and representational systems (i.e., digital and analog clock), how they may support children’s age-appropriate learning of specific time-related ideas, and what the target content goals of doing so are. Further research is necessary to determine how such expressions could be leveraged to support interval interpretations of an analog clock.

Endnotes

Note that occasionally a solution approach was given more than one code, the result of which is that the cumulative percentage in Table 4 is greater than 100%.

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DEVELOPING PRACTICAL MEASURES TO SUPPORT THE IMPROVEMENT OF GEOMETRY FOR TEACHERS' COURSES

Michael Ion
University of Michigan
mikeion@umich.edu

Patricio Herbst
University of Michigan
pgherbst@umich.edu

Claudine Margolis
University of Michigan
czmars@umich.edu

Amanda Milewski
University of Michigan
amilewsk@umich.edu

Inah Ko
University of Michigan
inahko@umich.edu

This paper reports on an ongoing project aimed at developing an inter-institutional system of professional support for the improvement of the Geometry for Teachers (GeT) courses that mathematics departments teach to preservice secondary teachers. In alignment with the literature on improvement science (see Bryk et al., 2015; Lewis, 2015), it is essential to develop and deploy practical measurement tools to inform improvement. We describe three key forms of measurement our team has been using to drive this work as well as some preliminary findings.

Keywords: Geometry and Geometrical and Spatial Thinking, Post-Secondary Education, Teacher Education-Preservice

Introduction

This paper reports on the development and use of instruments to study instruction at the college level, specifically focused on the undergraduate geometry course offered by many university mathematics departments for pre-service teachers (Geometry for Teachers, or GeT hereafter). The teaching of geometry in schools has been identified as weak and resulting in unacceptable levels of student performance (Clements, 2003; NCES, 2012). High school leaders have often described it as hard to find high school teachers who want to, and can, teach geometry. Thus, we consider increasing instructional capacity for high school geometry a systemic problem. We pose that the university GeT course is a crucial lever in preparing pre-service teachers to teach high school geometry, particularly as one of the few college mathematics courses that directly connects to high school geometry content. Yet, many mathematics education scholars (Zazkis & Leikin, 2010) and mathematicians (Wu, 2011) have questioned whether the content of university courses is sufficiently connected to what secondary teachers need to do their work. This led us to investigate the connection between participation in GeT courses and increased mathematical knowledge for teaching geometry (MKT-G).

We utilize the networked improvement communities approach described by Bryk, Gomez, Grunow, and LeMahieu (2015) to increase the capacity for geometry instruction at the K-12 level. This approach uses an organizational learning perspective in which all stakeholders are involved in the articulation of common problems as well as the design, monitoring, and continuous improvement of strategies to solve those problems. Bryk et al. (2015) indicate that achieving improvement at scale requires that particular attention be paid to variation within the system, and that understanding the sources of that variation is critical to achieving the goal of improvement research projects.

Within the scope of our project, we are currently measuring variation by gathering and analyzing data in the following ways: 1) through an assessment instrument which tests GeT students' mathematical knowledge for teaching geometry (MKT-G); 2) with end-of-term questionnaires used to collect data on course content, student composition, and technology use; and 3) through the use of instructional logs, administered at three points throughout the term, aimed at gathering more nuanced information about instructional practices in GeT courses. In this paper, we describe these measures in more detail, and explore the potential use of the data gathered from these three instruments.

Theoretical Framework

Taking a Networked Improvement Communities Approach

Bryk and colleagues (2015) set out a strategy for educational research and development called networked improvement communities (NICs), meant to harness the creative power of networked communities within an improvement science framework. One of the core principles of this approach is to "see the system that produces the current outcomes" (p. 57). The authors claim that oftentimes the traditional approach to solving complex educational problems is to quickly look for solutions to the problems without fully understanding the complex systems producing such problems. Russell and colleagues' (2017) lay out a framework for the implementation of networked improvement communities and claim that education researchers and practitioners often struggle to find effective solutions as "[the] field is not organized to learn systematically, accumulate, and disseminate the practical knowledge needed for the improvement of teaching and learning" (p. 1). A second core principle of the NIC approach is to "focus on variation in performance" (Bryk et al., 2015, p. 13), with an acknowledgement that complex systems often result in considerable variation. This focus on variation reduces the tendency to oversimplify by looking for universal solutions to the problem, and supports more realistic consideration of which solutions work, for whom, and under what set of conditions.

To study variation in performance, NICs must make use of consistent and practical measurement tools to inform the improvement efforts made by those involved (Morris & Hiebert, 2011). Additionally, a network hub is responsible for aggregating the data from those measures, as well as key insights that emerge, and feeding it back to the network so that innovations can be tested and integrated into new contexts (Bryk et al., 2015). We took on the role of network hub by creating a community of stakeholders and developing measures that would inform the community about progress toward a solution. As part of our work in this role, we facilitate opportunities for the network to engage with each other as well as the data being generated.

Variation in GeT Courses

Grover and Connor (2000) conducted a survey of 108 randomly selected U.S. colleges and universities to study the content and instructional practices of geometry courses through analysis of questionnaire responses and selected syllabi. They found considerable variation in content (e.g., geometries covered and axiomatic approaches employed), pedagogy (e.g., lecture, group work, and alignment with NCTM Professional Teaching Standards), and assessment (e.g., in-class examinations, homework, and forms of alternative assessment).

To see if this variation in GeT courses persists, we conducted an analysis of GeT course artifacts from 17 initial participants in this study (i.e., syllabi supplemented with course catalogs, interviews, and poster sessions). The results of our study of GeT courses were very well aligned with Grover and Connor's (2000) results, indicating that there still exists wide variation in GeT courses.

Methods

As the project is intended to help provide support for the instructors of university GeT courses, it is important to work collaboratively with them in order to help define the problem space. We began by locating institutions with teacher preparation programs—as we are interested in the undergraduate geometry course serving teachers, rather than geometry courses in general. Within those institutions, we looked in mathematics departments for geometry courses serving secondary mathematics pre-service teachers and identified instructors of those courses as the natural members of this community. We conducted a set of 19 initial interviews to gather their professional perspectives as instructors of GeT courses. We organized the interview data according to common challenges that many of the instructors noted as embedded in the work of teaching GeT courses. These common problems, or *tensions*, that many instructors identify in their own work we identify as inherent in the work of teaching college geometry, which are perhaps distinct from the set of problems or defects that an outsider might identify in the work of a GeT instructor (Herbst, Milewski, Ion, & Bleecker, 2018; Milewski et al., 2019).

In June 2018, we held a two-day conference which gathered more than fifty various stakeholders involved in the teaching of the GeT course (GeT instructors, high school district leaders, high school teacher leaders, and education researchers). Using what we had learned from the interviews with GeT instructors about the tensions that come with teaching the GeT course, we worked with conference attendees with a common problem in mind: to improve capacity for teaching K-12 geometry. Two of the tensions most salient in the conversations of the conference working groups were:

- *Knowledge tension:* This tension arises in contexts where GeT instructors need to consider the question “What is the *knowledge* students need to learn in the GeT course?” On one side of the tension, the course exists to provide novice teachers with the knowledge needed to teach the secondary geometry course. This includes knowledge like, “What are the common ways that students think about parallelograms?” or “How does one design good proof problems for high school geometry students?” On the other side, GeT is a university mathematics course that needs to be comparable in terms of rigor with other advanced mathematical coursework that students are expected engage with. Ideally, the course could include both kinds of mathematical knowledge, however, time is a limited resource and GeT instructors need to decide what to prioritize.
- *Experiences tension:* This tension arises in contexts where GeT instructors need to consider the question “What *experiences* can support students’ learning in the GeT course?” On one side of the tension, as GeT is a mathematics course, it would be reasonable to assume that students in the course are learning to think and act like mathematicians. Thus, students should experience opportunities to engage in reasoning, problem-solving, and other mathematical practices, which could serve them as future teachers as it would engage them in work similar to what they would have their eventual students engage. On the other side, the course acts a service course for pre-service teachers who are apprenticing into the work of a teacher, and this would suggest that they should experience opportunities to practice the work of teaching, like explaining a mathematical concept to the whole class or to each other, or interpreting fellow students’ work.

Rallying around these tensions, the conference participants collaborated to establish a set of projects to engage with over the course of the next four years that would address these tensions. Through the use of an online collaboration platform, teams of GeT instructors unite to work on a common project. An example of one of these projects is the *Geometry Knowledge Needed for Teaching* group, where 17 GeT instructors are currently working together to answer the questions: *What is the mathematical content knowledge needed for teaching high school geometry? What do we need to teach our college geometry students so they begin their teaching careers with the appropriate knowledge?* To answer these questions and discuss the tensions that come up in the work of teaching the GeT course as highlighted above, this group meets over video conference once a month, as well as throughout the month asynchronously, on an online forum.

In addition to collaborating on these projects, GeT instructors also volunteer to take and administer various measurement instruments throughout the year. The details of these instruments are outlined in the following section. We understand the potential pitfalls of large variability in those courses at the same time that we honor the academic freedom and professional judgment of instructors. The project forums, like the one described above, are meant to promote improvement by creating community. To inform the improvement efforts made by instructors who participate in the community, we track variation in course offerings and variation in outcomes (in terms of MKT-G), with the hope to ascertain whether changes in the former are related to changes in the latter. If reduction in course variability could be correlated with increases in MKT-G gains, this would lend credibility to the notion that reducing variability in the course offerings would be desirable.

Measures

Mathematical Knowledge for Teaching Geometry (MKT-G)

Herbst and Kosko (2014) have developed and validated an instrument used to measure MKT-G that targets four of the six domains of content knowledge for teaching identified by Ball, Thames, and Phelps (2008). These items are closed-ended and graded as either correct or incorrect. In a prior study, the instrument was administered to 387 inservice teachers from 47 states, as well as 195 preservice teachers from 7 institutions. Results from these tests showed that preservice teachers' scores on the MKT-G after taking a GeT course were .23 standard deviations lower than inservice teachers' MKT-G scores. While this result was unsurprising, it raised the question of whether the test could detect differences in MKT-G gained over the course of a semester of GeT instruction.

Over the course of the project, we will be collecting data from GeT students who will take this instrument at the beginning and end of their GeT course. Progress toward readiness to teach high school geometry will be measured in terms of average change in scores between pre- and post- assessments. Throughout the term, we share three sets of reports with participating GeT instructors: 1) individualized reports showing how their students performed on the pre-test compared to the national sample of practicing teachers; 2) individualized reports showing how their students performed on both the pre- and post- tests, with change in scores over the term reported in relation to the national sample of inservice teachers; and 3) aggregated reports showing each instructor how students from their institution performed on average compared to the overall average scores of all GeT students in this study. Our hope is that the instructors will use the individualized and aggregate results from these reports to reflect on and inform their course design and teaching practices, as well as spark conversations within the network about rationales, costs, and benefits of any potential changes in practice with respect to the knowledge and experiences tensions identified in their work.

Course Questionnaire and Instructional Logs

Inspired by the work of Grover and Connor (2000), we have developed a course questionnaire to be taken by GeT instructors at the end of each term in which the course is offered. In the questionnaire, instructors are asked about the course sequence at their institution, student composition, textbook use, dynamic geometry software use, as well as types of geometry taught in relation to various axiomatic approaches. Some examples of questions include:

- *Approximately what percentage of the students enrolled in the geometry course are prospective high school or middle school teachers?*
- *[What] courses are students required to take before taking this section of geometry?*
- *List the most important Euclidean geometry theorem(s) you cover.*

To better understand the experiences students are having in GeT courses, we have developed a set of instructional logs to be completed by GeT instructors three times throughout the term in which the course is offered. In each of these logs, instructors are asked to report on the planning and implementation of two lessons. As part of each log, instructors are asked to share the set of expectations for the lesson, as well as the amount of class time they devoted to each expectation. Following this, they are asked about the instructional approaches employed throughout the lesson, reporting how much class time was spent on each (e.g., whole class lecture, student presentations). These set of logs serve as one way of describing the experiences students are having in the GeT courses.

We expect these results to be useful in gauging variability across GeT courses, with the hypothesis that variability will be reduced as instructors participate in the networked improvement community. One reason that variability may be reduced over time is that GeT instructors will have access to these aggregated reports, may see evidence of other instructional practices being employed within the community, and will participate in discussions of the data all together.

Preliminary Results and Discussion

Fall 2018 MKT-G Results

In Fall 2018, 94 students from six GeT courses completed the pre- and post- MKT-G assessment. The level of MKT-G possessed by each student was provided as IRT scores derived from the IRT (item-response theory) model which takes account of the difficulty and discrimination of items (how well each item can discriminate participants' ability). In Table 1, we report on the pre- and post-IRT scores of the 94 GeT students, as well as the IRT gains from this sample. An average score of 0 for the sample of GeT students would indicate a level of knowledge equivalent to the average amount of knowledge of a national sample of in-service teachers (N=605) previously collected. The negative IRT scores reported indicate that GeT students' knowledge is below the average amount of MKT-G of the in-service teacher sample (Ayala, 2009; Crocker & Algina, 2006). The positive gain in IRT scores (0.15) between the pre- and post-test indicates that GeT students made progress over the course of the term, on average, with respect to MKT-G, and the gain was significant ($t(93)=-2.85$, $p=0.0027$). The use of the data from practicing teachers allows us to understand the GeT students' relative standing to the practicing teachers.

Table 1: Pre- and Post- test Score Comparison of GeT Students (N=94) and ISTs (N=605)

	IRT θ		
	PRE_IRT	**POST_IRT**	**GAIN_IRT**
GeT students (N=94)	**-1.29**	**-1.13**	**0.15**
ISTs (N=605)	**0.00**	**0.00**	**N/A**

Despite overall positive gains by students on average, only four of the six GeT instructors showed positive gains in their students' IRT scores. The graph below (Figure 1) represents the standing of the GeT students' IRT scores in pre-test and post-test within the distribution of IRT scores from the national sample of in-service teachers. The GeT students' post-test scores are shifted slightly to the right from their pre-test scores, indicating improvement in MKT-G over the semester of GeT instruction. Compared to the ISTs, most of the 94 students have lower than average level of MKT-G in both pre- and post- test scores and this result is consistent with the prior study.

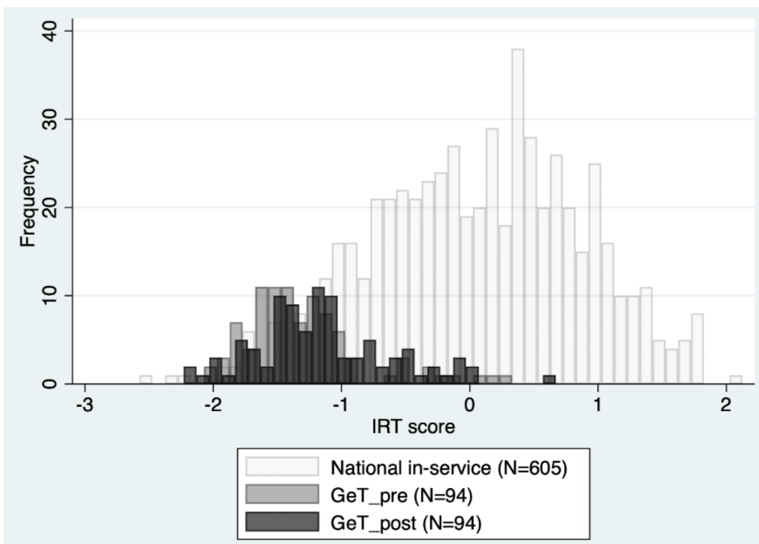

Figure 1: Histogram of the MKT-G Scores of GeT Students (N=94) and ISTs (N=605).

Course Questionnaire and Logs Results

In our first distribution of the course questionnaire, while we have a small sample of instructors (N=7), we are able to make similar observations as Grover and Connor (2000) with respect to large variability in the GeT courses. Some sample responses are shown in Table 2.

Table 2: Descriptive Results from Two Course Questionnaire Prompts (N=7)

	<=50%	51% - 80%	>81%

Q9: *Approximately what percentage of the students enrolled in the geometry course are prospective high school or middle school teachers? (N=7)*	n=2	n=2	n=3
Q31: *Indicate what percentage of the course is devoted to Euclidean geometry. (N=7)*	n=2	n=2	n=3

Also, in the Fall 2018 term, we collected a total of 20 instructional logs from five GeT instructors. An example of results from one such question is shown in Figure 2.

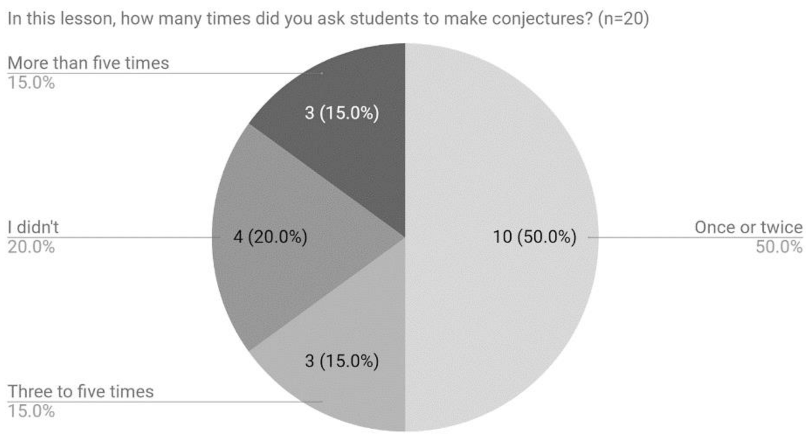

Figure 2: Aggregated Responses (N=20) to One of the Instructional Log Questions

The results from the course questionnaire and instructional logs have the potential to provide evidence of ways in which the knowledge and experiences tensions are present in the work of teaching the GeT course. We may find that GeT instructors with student populations that are less than 50% PSTs are less likely to spend considerable time on Euclidean Geometry, for example. Other uses for the data generated by these instruments is to measure variability across courses over time. The results shared here are from the first instructional term of this study, but we expect to look at how courses change over time when that data becomes available. For the time being, the instruments described here seem promising to allow us to describe the variability present across offerings of the GeT course.

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FOUR ATTENTIONAL MOTIONS INVOLVED IN THE CONSTRUCTION OF ANGULARITY

Hamilton L. Hardison
Texas State University
HHardison@txstate.edu

Quantifying angularity is critical for the study of K–12 school mathematics and beyond; yet, quantifying angularity is challenging for individuals across these grade levels. Using data from a yearlong teaching experiment with ninth-grade students, I address the role of attentional motion in quantifying angularity and present four such motions involved in the construction of angularity. Additionally, I consider implications of these motions for teaching and research.

Keywords: Geometry and Geometrical and Spatial Thinking, Quantifying Angularity

Angle measure is a critical topic in K–12 school mathematics and beyond. As a few examples, angle measure is used when characterizing shapes, constructing coordinate systems, and investigating trigonometric relationships. However, individuals' challenges with angle measure have been documented through multiple studies with students and (prospective) teachers across the educational spectrum (Akkoc, 2008; Baya'a, Daher, & Mahagna, 2017; Clements, Battista, Sarama, & Swaminathan, 1996; Crompton, 2017; Devichi & Munier, 2013; ; Fi, 2003; Keiser, 2000; 2004; Lehrer, Jenkins, & Osana, 1998; Matos, 1999; Owens, 1996; Topçu, Kertil, Akkoc, Kamil, & Osman, 2006). Yet, relatively few researchers have elaborated the mental operations necessary for measuring angles, provided empirical support for these hypothesized operations, or documented interventions resulting in productive modifications in students' thinking about angle measure. There have been some notable exceptions, two of which were reports published in roughly the last decade. Thompson (2008) presented a theoretical approach to measuring angles rooted in the principles of quantitative reasoning (Thompson, 1994; 2011) and based on multiplicative comparisons of circular arc lengths. For example, to say that an angle has a measure of *n* degrees in Thompson's view means that an angle cuts off $\frac{n}{360}$ of any circle's circumference, provided that the circle is centered at the vertex of the angle. Following Thompson's (2008) conceptual analysis, Moore (2013) empirically demonstrated the productivity of an arc-length approach to quantifying angularity with precalculus students.

Although this arc-length approach to quantifying angularity is productive, the complexity of this approach renders it an unlikely starting point for early instruction in quantifying angularity. In state and national curricular standards, angle measure is typically introduced in fourth grade (e.g., National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010); at this grade level, many students are not yet able to instantiate such multiplicative comparisons (Steffe, 2017), much less generalize them as holding invariant across a class of circles. Put simply, more research is needed to understand how individuals begin to quantify angularity (Smith & Barrett, 2017). To this end, I conducted a teaching experiment (Steffe & Thompson, 2000; Steffe & Ulrich, 2013) with ninth-grade students (see Hardison, 2018). In the present report, I elaborate some of the mental operations involved in beginning to quantify angularity with particular attention to the role of motion in the quantification process. Specifically, the primary purposes of this report are (a) to explicate the role of attentional motions in constructing an awareness of angularity, (b) to describe the nature of these motions, and (c) to consider the implications of these motions for teaching and research.

Theoretical Perspectives

The present study was informed by principles of quantitative reasoning (Thompson, 1994; 2011). A quantity is an individual's conception of a measurable attribute of an object or situation; quantities are mental constructions, and examples of quantities that individuals might construct include length, area, time, speed, and angularity. A quantity consists of three interrelated components, which are also mental constructions: (a) an object or situation, (b) an attribute or quality, and (c) a quantification, which is a set of operations an individual can enact on the attribute (e.g., a measurement process). Although all three components are critical, the second component—the attribute or quality—is of primary importance for the present study.

Taking this perspective on quantity, for an individual to construct a geometric quantity (like angularity), she must first construct an awareness of the geometric attribute. Steffe's (2013) analysis of the construction of length provides insight into how such an awareness is constructed:

> The construction of length involves motion of some kind in that it entails an uninterrupted moment of focused attention bounded by unfocused moments. It might be a sweeping of one's hand through space, walking along a path, scratching a path in the frost on a window with a fingernail, or moving ones [sic] eyes over the trunk of a rather tall tree. (p. 29)

According to Steffe (2013), attentional motion is critical in constructing an awareness of length. To be clear, it is the *attention* of the cognizing subject on the motion that is critical and not simply the motion itself. Steffe described three levels of awareness an individual might construct regarding this motion. If an individual is aware of the duration of the motion from beginning to end as well as the visual records of this motion while in the presence of (what an observer would call) a linear object, then an individual has constructed an *awareness of experiential length*. If the individual internalizes the motion—i.e., can re-present the visual records, the bounded motion, and the trace of this motion in the absence of the perceptual linear object—then an individual has constructed an *awareness of figurative length*. Finally, if an individual interiorizes the motion—i.e., can subject the re-presented experience to further mental operations (e.g., imagine the motion backwards, imagine stopping and re-starting the motion, imagine several successive iterations of the motion, etc.)—the individual has constructed an *awareness of operative length*.

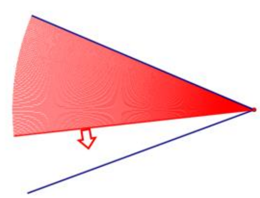

Figure 1: Radial Sweep Through the Interior of an Angle

Generalizing from Steffe's analysis of the construction of length, I consider an awareness of any geometric attribute to involve attentional motion. In particular, the construction of angularity involves attentional motion of some kind through (what an observer would call) the interior of an angular object. Attentional motion through the interior of an angle is critical because, fundamentally, angle measure is a description of the size of the interior of an angle. At the onset of the reported study, I hypothesized an awareness of a radial sweep—the trace of a rotating ray constituting the interior of an angle—would be a critical development in students' construction of angularity (Figure 1). The reason this would be such a critical development in students' quantifications of angularity is because, if the radial sweep were interiorized, then students would be able to subject the interior to further operation and produce, for example, angles three times or three-fourths as open as a given angle. Clements and colleagues (Clements et al., 1996; Clements & Burns, 2000) also argued for the significance of such a motion. However, multiple studies (Clements, et al., 1996; Crompton, 2013; Lehrer, Jenkins, & Osana, 1998; Mitchelmore, 1998; Mitchelmore & White, 1995; 1998; 2000) have shown that students tend *not* to spontaneously insert this rotational motion into non-rotational angle models (e.g., two line segments sharing a common vertex). For this reason, Mitchelmore and White (1998) recommended that angle measure as an amount of turn between two lines be abolished from elementary mathematics curricula. However, failure to insert rotational attentional motion does not imply an individual has not constructed an awareness of angularity. In other words, it is possible that an individual has constructed an awareness of angularity dependent upon some other attentional motion through the interior of an angular object. This issue raises the important questions that will be addressed in this report: what attentional motions might individuals use to constitute the interior of an angular object and what are the implications of these motions?

Methods

The data presented in subsequent sections is drawn from a teaching experiment (Steffe & Thompson, 2000; Steffe & Ulrich, 2013) conducted in the southeastern U.S. with two ninth-grade students, Camille and Kacie, over an academic year. At the time of the study, both students were enrolled in a first-year algebra course, and neither student had taken a dedicated geometry course. The overarching goal of the teaching experiment was to investigate how the students quantified angularity and how these quantifications changed throughout the study (see Hardison, 2018); the author served as teacher-researcher for all teaching sessions. Throughout the study, students engaged in mathematical tasks involving rotational angle models (e.g., rotating laser) and non-rotational angle models (e.g., hinged wooden chopsticks).

Camille and Kacie participated in 14 and 13 video-recorded sessions, respectively, which were conducted approximately once per week outside of their regular classroom instruction; each session was approximately 30 minutes in length. For each student, 2 sessions were initial interview sessions, and 1 session was a final interview session; interview sessions were conducted with each student individually to establish their ways of reasoning at the beginning and end of the teaching experiment. The remaining sessions were teaching sessions wherein the teacher-researcher worked to engender productive changes in students' ways of reasoning in addition to understanding their ways of reasoning; teaching sessions were conducted individually or in pairs. Beyond video-recordings, additional data sources included digitized student work and field notes. The records of students' observable behaviors (e.g., talk, gestures, written responses, etc.) were analyzed in detail during the teaching experiment (on-going analysis) as well as at the conclusion of the teaching experiment (retrospective analysis) via conceptual analysis (Thompson, 2008; von Glasersfeld, 1995). This report focuses on the attentional motions I abstracted from students' observable behaviors at the onset of the teaching experiment, specifically in the initial interview sessions and in the students' first paired teaching session.

Findings

From an analysis of students' activities in the teaching experiment, I abstracted three motions students enacted to account for the interior of angle models. In the subsequent sections, I illustrate each of these motions using data from the teaching experiment.

The First Motion: Radial Sweep

To investigate if students had constructed an awareness of angularity involving a radial sweep (see Figure 1), I posed a rotating laser task to each student during her initial interview. This task was purposefully posed after all other angular tasks in the initial interview to avoid leading students to use a radial sweep on other tasks. The rotating laser task involved a GSP sketch containing the image of a laser pointer. At the click of a button, the laser beam rotated 57° counterclockwise about the endpoint of the ray representing the beam. Initial and terminal positions of the laser pointer are shown in Figure 2. As I introduced the task to each student, I "turned on" the laser by clicking an action button, which showed a red ray emanating from the rightmost end of the laser. I informed the students that I would "turn off" the laser and then click the button to move the laser pointer. I asked the students to keep track of all the places on the screen the beam would have hit if the laser had been "on" as it moved; after the motion stopped, students were asked to shade the portion of the screen the beam would have hit.

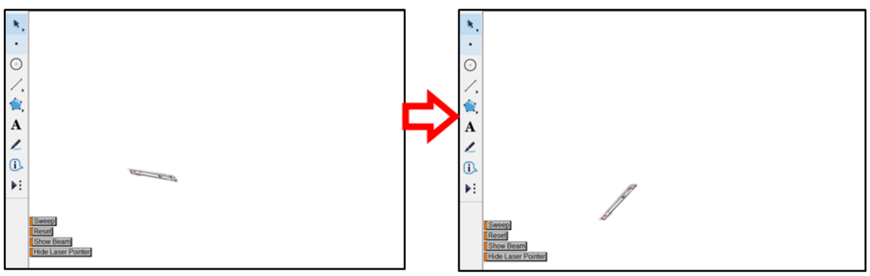

Figure 2: Initial (left) and Terminal (right) Positions of Pointer in the Laser Task

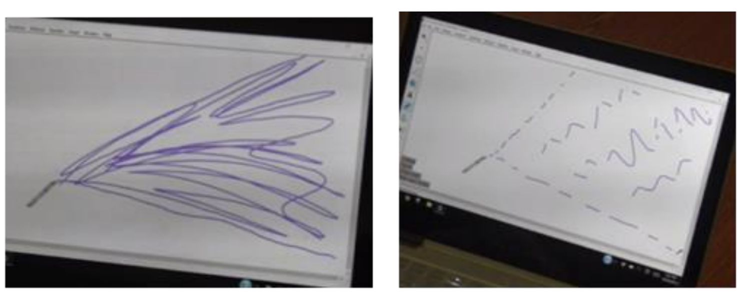

Figure 3: Camille's (left) and Kacie's (right) Shadings on the Rotating Laser Task

Both students shaded the screen accounting for three components: the initial location of the ray, the trace of the ray (i.e., the interior of the angle), and the terminal location of the ray. Camille shaded in the order that the motion occurred—shading the initial ray, then the interior, and finally the terminal ray (Figure 3 left). Kacie shaded in the opposite order (terminal ray, interior, and finally the initial ray), as if re-presenting the experience in reverse (Figure 3 right).

Both students described the boundary of the region as linear and in temporal terms, which indicated the boundedness of the imagined radial sweeping motion. For example, Camille lamented, "I didn't draw it straight but that would be like where it would stop," which indicated

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

her intent to create a linear boundary. Camille's shading of the interior in a counterclockwise motion also indicated that she was reimagining the motion of the radial sweep.

Each student re-presented the uninterrupted motion of the imagined beam, along with its trace, in visualized imagination. Therefore, I consider each student to have demonstrated an awareness of angularity via radial sweep. Camille's activities demonstrated at least an awareness of figurative angularity in that she re-presented the motion of the beam as she had observed it to occur; she had at least internalized the rotational imagery. In contrast, Kacie demonstrated an awareness of operative angularity because she indicated she could imagine reversing the direction of the sweep in re-presentation, which indicated she had interiorized the rotational imagery of the rotating beam.

Considering my hypothesis regarding the importance of a radial sweep, I viewed the students' ability to re-present the rotational motion as a promising base for developing students' angular operations. However, throughout the entirety of the teaching experiment, I saw no evidence that either student ever spontaneously inserted this rotational motion into a non-rotational angle context. In the subsequent sections, I present two other motions, which were not hypothesized, that students inserted into non-rotational angle models on comparison tasks.

The Second Motion: Re-presented Opening

The first angular task posed to each student during the initial interview session was an angular comparison task. In this task, I briefly displayed two pairs of hinged chopsticks (Figure 4) for each student and then covered them with a cloth. I asked each student to draw the chopsticks and describe the similarities or differences they noticed. A portion of Kacie's response is described in the excerpt below where Kacie's actions are described in italicized text.

Figure 4: Obtuse and Acute Chopsticks for the Drawing/Comparison Task

> Kacie: This [obtuse] one is like that [acute] one but it's farther out. [*Holds palms together and fingertips closed (Figure 5 left); then opens fingertips while keeping the bases of her palms pressed together (Figure 5 right)*]. So like they, [*gestures four times as if opening the hinged chopsticks from a closed position in the air*] uh, took it apart, I guess. But, it's still like together. And so, the same with like this [acute] one. It's just like somebody pushed it together [*holds hands open and then closes them some as if closing the obtuse chopsticks to the acute configuration*].

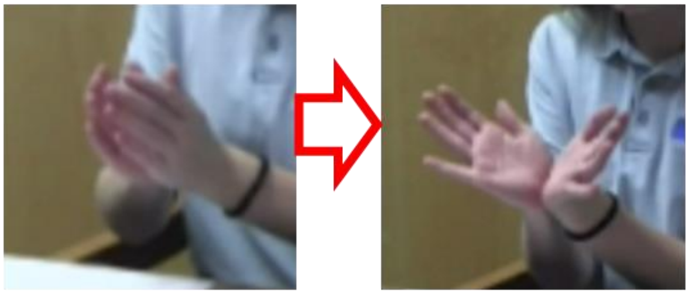

Figure 5: Kacie Imagines Opening Chopsticks from a Closed Position

During the preceding excerpt, Kacie repeatedly opened and closed her hands (Figure 5). From this, I inferred she imagined opening a pair of chopsticks while bounding this motion between two configurations: a closed configuration, which she represented with her hands, and the obtuse configuration she had drawn, which was perceptually available. I use “configurations” rather than “angles” here to emphasize the transformation of a single angle model from one state to another. From her repeated gestures, I infer that Kacie re-instantiated this motion—opening a pair of chopsticks to the obtuse configuration from a closed configuration—in visualized imagination at least four additional times. I interpret Kacie’s verbal description, “so like they, uh, took it apart,” as additional support for this inference. Kacie’s actions indicated she mentally constituted the interior of the obtuse chopsticks by re-presenting the action of opening the chopsticks to the obtuse configuration from a closed configuration. Thus, I consider Kacie to have demonstrated an awareness of angularity via *re-presented opening*.

Some readers may interpret Kacie’s re-presented opening of the chopsticks in terms of rotational motion (i.e., a radial sweep); however, I interpret Kacie’s actions more conservatively. That Kacie re-presented the opening action does not necessitate that she also held in mind a fixed position for the vertex of an angle model. Additionally, I distinguish an awareness of angularity via re-presented opening from radial sweep as the former involves motion on two distinct rays while the latter involves motion of a single ray through the interior of an angle.

The Third Motion: Segment Sweep

To illustrate a third motion used to constitute the interior of an angle, I describe and analyze Camille’s activities on an angular comparison task during the pair’s first teaching session. At this point in the session, Camille was explaining why her long chopsticks, which were set to an acute configuration from my perspective, were less open than Kacie’s short chopsticks, which were set to an obtuse configuration from my perspective.

To compare the two angle models, Camille held her thumb and index finger together over the vertex of the long chopsticks (Figure 6 left) and then dragged her thumb along one side of the chopstick and her index finger along the other (Figure 6 center and right). As she moved her fingers over the sides of the angle model, her thumb and index finger grew further apart. As she moved her hand, Camille explained, “Mine’s starting off really small and getting bigger.” Camille made a similar gesture as she referenced Kacie’s short chopsticks, “and that one’s just like – it’s just open really big,” moving her hands as if tracing out the short pair of chopsticks.

Figure 6: Camille Drags Her Thumb and Index Fingers Across the Angle Model

Camille's gestures over the long chopsticks suggested a new motion through the interior of the angle. She moved as if sweeping a growing segment, whose endpoints were determined by her thumb and index finger, through the interior of the model (Figure 7). Camille's explanation indicated she implicitly considered the experiential rate at which the segment bounded by her thumb and index fingers was lengthening as she moved her hand away from the vertex. I infer Camille compared the openness of the two pairs of chopsticks by considering which chopsticks would cause her fingers to separate more quickly as she moved her hands away from each vertex. In other words, she was not considering the duration of the motion she enacted as she moved her hands over the chopsticks; instead, she was comparing the intensity of this motion.

Because the Camille enacted these motions sequentially (i.e., one after the other) and not simultaneously, I infer she was able to make the comparison by imagining the segment sweep for one angle model while physically enacting the segment sweep for the other. Therefore, Camille demonstrated at least an awareness of figurative angularity via *segment sweep*.

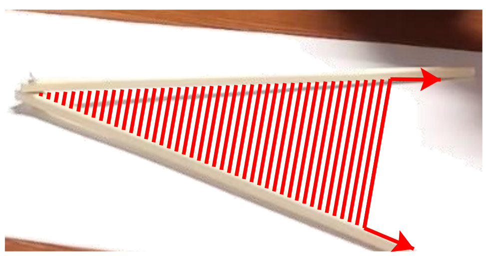

Figure 7: Segment Sweep Through the Interior of an Angle

Discussion

The construction of angularity involves attentional motion of some kind through the interior of (what an observer would call) an angular object. In my analysis of the students' activities, I have presented three such motions: radial sweep, re-presented opening, and segment sweep (Figure 8). A radial sweep involves the rotation a single ray (or segment), whose endpoint is fixed at the vertex of the angle, through an angle's interior. Re-presented opening involves imagining two distinct rays (or segments) opening from a closed position. Radial sweep differs from re-presented opening in that the former motion involves a single rotating ray while the later involves two distinct rays opening from a closed configuration. Segment sweep involves a linear segment, with one endpoint on each side of the angle, moving through the interior of the angle; the length of the segment increases at it is imagined moving away from the vertex. The motions I abstracted from students' activities were likely influenced by the angle models used in my study (i.e., chopsticks and rotating lasers); therefore, future research might investigate whether other contexts lead to other attentional motions.

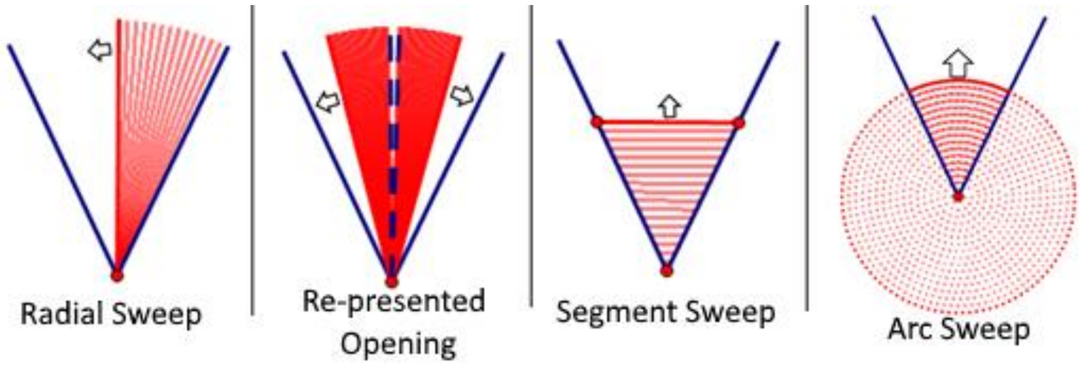

Figure 8: Four Attentional Motions Involved in the Construction of Angularity

In addition to the three motions above, I hypothesize a fourth attentional motion through the interior of an angle—an arc sweep. An arc sweep involves imagining the interior of an angle being swept out by a circular arc bounded by the sides of an angle where the circle containing the arc is centered at the vertex of the angle. Although neither student indicated imagining such an arc sweep, I hypothesize such a motion supports students to develop quantifications of angularity entailing generalized multiplicative comparisons of circular lengths holding across a class of circles (i.e., as advocated by Thompson (2008) and Moore (2013)). Through such a motion, the plane can be conceived as a collection of concentric circles and the angle conceived as a collection of arcs. Future research is needed to investigate this hypothesized attentional motion.

If an individual conceives the interior of an angle model as having an infinite area, all four of these motions can be viewed as being bounded in one sense and unbounded in another. In the case of re-presented opening and radial sweep, the motion is bounded temporally in that the motion has a beginning and an end; the extent of angularity is indicated by the duration of the motion. Re-presented opening and radial sweep are unbounded in a spatial sense, in that the length of the ray(s) moving through the interior of the angle is potentially infinite. In contrast, segment sweep and arc sweep are bounded spatially in that the segment or arc moving through the interior of the angle is always of finite length; however, segment sweep and arc sweep can be thought of as temporally unbounded: the sweeping motion starts at the vertex and continues indefinitely through the interior of the angle. In the case of segment and arc sweep, the extent of angularity is indicated by the rate at which the segment or arc grows as it moves away from the vertex and is, therefore, an intensive quantity (Piaget, 1965) resulting from a coordination of the length of the sweeping entity and this entity's distance from the vertex of the angle. All four of these attentional motions can be productively leveraged to make angular comparisons; yet, I hypothesize radial sweep, re-presented opening, and arc sweep are the most productive motions for developing normative conceptions of angle measure. In contrast, segment sweep may lead to nonnormative conceptions if operations (e.g., iteration) are applied to the sweeping segment.

Further Implications for Teaching and Research

Degrees as a unit of angular measure are often introduced via rotational imagery in terms of a 360-unit angular composite as in the CCSSM. This is a suitable approach for defining degrees if students are considering angles in rotational contexts. However, students' experiences with angles in mathematics classroom often involve non-rotational contexts, such as the interior angles of polygons. Throughout my teaching experiment, students did not spontaneously insert rotational motion into non-rotational angle contexts, which is consistent with the results of previous studies. If students do not spontaneously insert rotational motion into non-rotational angle contexts, introducing units like degrees in non-rotational contexts via rotational imagery is largely inappropriate. However, I do not share the sentiments of Mitchelmore & White (1998), who recommended that angle as turn be removed entirely from the elementary mathematics curriculum. Instead, I suggest that early instruction in angle measure involve both rotational and non-rotational contexts and different approaches for introducing units like degrees are merited depending on the context. Specifically, I hypothesize that emphasizing re-presented opening will be more pedagogically productive in non-rotational angle contexts. After students have interiorized attentional motions and subjected them to further operation (e.g., partitioning and iterating) in different contexts, teachers might work to engender students' recognition of similarities across these two contexts by drawing pictures to represent rotations or inserting rotational imagery into non-rotational angle models, for example.

Future research is needed to determine the prevalence of these attentional motions as students construct an awareness of angularity and the ways in which each motion is related to the other operations constitutive of students' quantifications of angularity. Given that many studies have found students tend to conflate linear attributes (e.g., side length) with angularity, I hypothesize segment sweep may be the most frequently considered motion if students are attending to the interior of non-rotational angle models. In closing, I offer one final hypothesis: generalized quantifications of angularity (i.e., those that are context independent) entail a recognition that different motions through an angle's interior account for the same magnitude of angularity.

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INVESTIGATING TEACHERS' PROBABILITY LITERACY

Sara Brooke Mullins
Virginia Tech
sbm3@vt.edu

Jesse L. M. Wilkins
Virginia Tech
wilkins@vt.edu

Probability literacy involves an ability to understand and interpret everyday probability events. It enables teachers to understand different views, approaches, and methods associated with probability, as well as to support students in developing a deeper understanding of probability. However, research has shown that many people do not have a fully developed probability literacy. This study investigates the level of literacy for pre-service and in-service teachers. Results show that, although the teachers in this study demonstrated some aspects of probability literacy, overall they do not have a deep probability literacy and therefore may be limited in promoting students' understanding of probability.

Keywords: Probability, Problem Solving, Teacher Education

The concept of probability has become more prominent in the K-12 curriculum over the past 30 years (e.g., CCSSI, 2010 and NCTM, 2000). Although researchers and educators have recognized probability as an important concept for students to learn, the teaching and learning of probability remains a concern today. In order to be effective citizens, students and adults must be able to read and interpret everyday probability events, thus requiring probability literacy (Gal, 2005). Therefore, it is important for teachers to have a conceptual understanding of probability, so they can support students in developing their own probability literacy. Furthermore, a strong probability literacy will ensure that teachers do not hold common misconceptions as their students, and are able to address student errors (Jones et al., 2007; Langrall et al., 2017; Shaughnessy, 1992). It is then necessary to investigate teachers' probability literacy to determine if they are being supported to develop strong literacy and the steps that may be taken to support this development. The purpose of this study is to examine pre-service and in-service secondary mathematics teachers' probability literacy. Specifically, we ask the following questions: How do teachers think about and solve probability tasks? What heuristics do teachers use to solve probability tasks? How do teachers analyze and respond to student solutions?

Review of Literature

Probability literacy involves the ability to read, interpret, and understand real-world probability events (Gal, 2005; Batanero et al., 2016). Gal (2005) defined probability literacy as "the knowledge and dispositions that students may need to develop to be considered literate regarding real-world probabilistic matters" (p. 40). He also identified five knowledge elements and three dispositional elements that comprise probability literacy. Within this study, we focus mainly on the knowledge elements. The knowledge elements consist of (1) big ideas, (2) figuring probability, (3) language, (4) context, and (5) critical questions. The *big ideas* element relates to foundational concepts such as "variation, randomness, independence, and predictability" (Gal, 2005, p. 46). *Figuring probability* focuses on the heuristics or methods used to solve probability problems. The *language* element is related to how people represent and talk about probability, including terms such as chance, random, likelihood, and certainty. *Context* focuses on probability in the real world, such as different jobs, careers, and fields of studies. *Critical questions* are those questions a person needs to ask of a situation, such as context, source, and interpretation.

Figuring probability involves strategies used to solve probability problems, such as the representativeness, availability, and anchoring and adjustment heuristics (Batanero et al., 2016; Jones & Thornton, 2005; Shaughnessy, 1981, 1992). When using the representativeness heuristic, estimates are based on the similarity between an event and its parent population (e.g., predicting the next flip of a fair coin to be tails, when a sequence of heads just occurred; Jones & Thornton, 2005; Shaughnessy, 1981, 1992). In this case, the distribution of heads is assumed to balance to 50%. For the availability heuristic, estimates are based on memory and recent occurrences (e.g., predict rolling a four on a die because other people previously rolled a four; Shaughnessy, 1981). Adjustment and anchoring are related to making estimates based on initial information given in a problem, commonly known as the conjunction and disjunction fallacies (Jones & Thornton, 2005; Shaughnessy, 1981, 1992).

A more informal method of approaching probability is through the use of intuitions; these are described as a person's beliefs and experiences with probability, either before or after formal instruction (Fischbein, 1975). Primary intuitions are those developed before formal instruction, and come from people's personal preferences or their experiences with past probability events. For example, probability may be estimated based on a person's favorite color or number, without any recognition of theoretical probability formulas (Fischbein, 1975). After formal instruction, secondary intuitions are developed, thus fostering more objective beliefs. It is important to note that secondary intuitions do not replace primary intuitions, but rather build upon primary intuitions (Fischbein, 1975). As a result, primary intuitions may appear in other contexts, and may be overextended, resulting in misconceptions and erroneous approaches.

When considering teacher's understanding of probability, research shows that many teachers do not have appropriate content knowledge for teaching probability (Stohl, 2005). Stohl further highlighted that some teachers hold negative beliefs towards probability, which results in relying on computational and procedural approaches to teaching probability. Stohl also claimed, "without specific training in probability and statistics, preservice and practicing teachers (and perhaps some teacher educators) may rely on their beliefs and intuitions, and have similar misconceptions" (p. 346), thus highlighting the need to develop strong probability literacy.

Methods

This study involved two female pre-service teachers (PSTs; Kim and Pam) and two male in-service teachers (ISTs; John and Allen). Each participant was given a pseudonym. The PSTs were enrolled in a Master's program in education at a large research university in the southeastern United States. In their early-field placements, Kim was teaching Algebra I and Pam was teaching Geometry. The ISTs were faculty at a rural, public school in the southeastern United States and had completed the same Master's program. John had taught for nine years, and was currently teaching Geometry and Algebra II. Allen had taught for four years and was currently teaching Geometry, Algebra Functions and Data Analysis, and Algebra II.

One semi-structured clinical interview, lasting about 70 minutes, was conducted with each participant. During this time, participants completed five probability tasks. After solving the five tasks, participants were asked to examine and respond to hypothetical student solutions for each problem. Then the participants were asked to explain the students' solutions and address students' errors. Interviews were videotaped and transcribed. We first evaluated the correctness of participants' responses to each task. We then determined if they correctly characterized student solutions and examined how they responded to student solutions, making notes of instances where participants appropriately addressed mistakes and how they offered support for students' "incorrect" solutions. From this, we gained insight into the five knowledge elements of probability literacy for each participant, based on how they thought about and solved probability tasks, as well as, interpreted and responded to student solutions.

Results and Discussion

John, Allen, and Kim were able to correctly solve over half the tasks, but Pam was only able to correctly solve two tasks. Table 1 presents participants' correctness of solutions before they examined student work. After examining student work, Allen changed his answer to Task 5 to the correct answer, and Pam changed her answer for Task 4 to another incorrect answer.

Table 1: Task Results

Task and Description	Kim (PST)	Pam (PST)	John (IST)	Allen (IST)	Total
Task 1: If a 6-sided die is rolled, what is the chance that it will land on a 6?	✓	✓	✓	✓	100%
Task 2: If a fair coin is tossed... which of the following is more likely to occur? Choose one. (a) HTTHTH (b) HHHHTH (c) The same chance for each of these sequences	✓		✓	✓	75%
Task 3: Imagine there are two buckets of marbles... a green bucket and a red bucket. Each of the two buckets has 100 black marbles and 100 white marbles in it, all mixed up. You will flip a coin to decide which bucket to draw from. What do you think is more likely to happen? (a) You will pull a white marble (b) You will choose the red bucket, and pull a white marble		✓	✓	✓	75%
Task 4: There are three cards in a bag. One card has both sides green, one card has both sides blue, and the third card has a green side and a blue side. You pull a card out and see that one side is blue. What is the probability that the other side is also blue?					0%
Task 5: A person must select committees from a group of 10 people. (A person may serve on more than one committee.) Would there be: (a) More distinct possible committees of 8 people (b) More distinct possible committees of 2 people (c) About the same number of committees of 8 as committees of 2?	✓		✓		50%
Total for Participant	60%	40%	80%	60%	

Overall, the ISTs demonstrated a more developed probability literacy than the PSTs. The ISTs were able to correctly solve more tasks than the PSTs. Concerning the figuring probability element, the ISTs could identify and compare different approaches and heuristics students used, whereas the PSTs struggled to compare approaches and heuristics, often falling victim to common fallacies. The ISTs also began each task by talking through the problem, discussing their intuitions, and then resorted to using formulas when they needed to. This was different from the PSTs' approaches. Kim relied solely on her formulas, whereas Pam never wrote anything down and relied on her intuitions. Examining the big ideas element, all participants were generally able to discuss some big ideas concerning probability, such as conditional probability and sample space. However, the ISTs were able to talk about these big ideas in more formal and detailed manners, indicating a better understanding of probability language. John and Allen were also more able to engage in deeper conversations about students' use of language, offering alternative ways to ask questions and present the probability tasks, thus helping students to better understand the context of probability. Kim and Pam only engaged in conversations about the meaning of 50-50, equally likely, and same chances when prompted. From this, it was evident that the ISTs had a better understanding of what critical questions to ask of students' solutions. Both John and Allen focused on the why and how questions, whereas Kim and Pam could only ask questions about the meanings of certain words.

Conclusions

Although the ISTs demonstrated a more sophisticated level of probability literacy, we argue that all participants lacked some aspects of a complete probability literacy. Each participant evaluated student responses based on their original response, which is problematic as it caused participants to incorrectly respond to student solutions. Nevertheless, John and Allen possessed some aspects of all five knowledge elements, whereas Kim and Pam possessed limited aspects of only a few elements. Kim and Pam had a limited understanding of figuring probability, big ideas, language, and critical questions, with no attempt to connect the tasks to a larger context. The difference may be due to the ISTs having more teaching experience, knowing what critical questions to ask and how to engage in dialogue with students. However, with only four participants, it is difficult to know. What this study does demonstrate is that even the more sophisticated teachers are not fully literate in their probabilistic thinking and therefore may be limited in supporting students in developing a deep understanding of probability. However, the more experienced teachers did possess a greater ability to ask critical questions, which is an important aspect of literacy to help support their students' learning.

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UNDERGRADUATES' EXPRESSION OF CRITICAL STATISTICAL LITERACY

Nina G. Bailey
UNC Charlotte
nbaile15@uncc.edu

Allison W. McCulloch
UNC Charlotte
allison.mcculloch@uncc.edu

Using statistical literacy in a data driven society to critically examine and question is imperative for students to become active and informed citizens. Using a framework adapted from Gal (2002), this study examines the use of critical statistical literacy within online discussion board posts of eight undergraduate students taking introductory statistics. The results indicate that the majority of students (75%) exhibited critical statistical literacy across all discussion prompts. The overwhelming expression of active citizenry demonstrates the potential that discussions boards have to bolster engagement, increase interest, and stimulate reflective thinking within introductory statistics courses.

Keywords: Critical Statistical Literacy, Online Discussion Board

Today's society is data driven and number saturated. Students need to be able to navigate through the abundance of quantitative messages they will undoubtedly encounter in the real world. Perhaps more importantly, those students need to be inspired to weed through such a barrage of data, and to do so requires both statistical literacy skills and taking a critical stance. For the purposes of this paper, I define *critical* statistical literacy (CSL) as the practice of analytically examining and assessing statistical content to inform action or change.

The importance of developing statistical literacy has been noted by many (e.g., Gal 2002; Del Mas 2002; Rumsey 2002). However, what is meant by the phrase *statistical literacy* is not the same for all. While there is no consensus on the elements of statistical literacy, the literature reveals consistent emphasis on understanding the basic terminology and concepts, understanding the need for and generation of data, interpreting different representations and conclusions, and acknowledging the potential for data to generate conflicting interpretations (Del Mas 2002; Gal 2002; Garfield and Ben-Zvi 2007; Rumsey 2002).

Research is rich with work that legitimizes statistical literacy as worthy of study (e.g., Watson, 1997) and describes the components of statistical literacy (e.g., Gal, 2002). Other researchers have detailed how to assess statistical literacy (e.g., Budgett & Pfannkuch, 2010), developed hierarchical frameworks (e.g., Watson & Callingham, 2003), and described curriculum design or reform (e.g., Schield, 2004; Tishkovskaya & Lancaster, 2010). While statistical literacy has gained momentum in the field, there are few examples of research that provides rich examples of how students express *critical* statistical literacy. One exception comes from Watson and Callingham (2004) who surveyed over 600 students across grades 5 through 10 with an aim of fleshing out their hierarchical levels of statistical literacy. Their findings suggest high school students need opportunities "to question critically statistical claims from media sources or other real-world contexts in order to develop the analytical habits of mind that are needed to respond critically to quantitative claims" (p. 133).

Online discussion boards are a potential vehicle for developing students' CSL. Some of the documented advantages of using online discussion boards include increased student motivation and interest when writing, and a stronger desire to produce quality work since both peers and the instructor will read the posts (Zemelman, Daniels, & Hyde, 2005). Furthermore, students have time to think before responding in an online forum unlike classroom discussions (Murphy and Coleman 2004) which nurtures reflective thinking (Hara, Bonk, and Anjeli 2000; Newman et al. 1997) and the co-construction of knowledge by sharing ideas and evaluating classmates' arguments (Gilbert and Dabbagh 2005; Pearson 2010). As a first step toward evaluating whether online discussion boards are an appropriate means to nurture CSL, this study aims to explore how students express CSL skills within online discussion forums used to supplement a face-to-face introductory statistics course at the undergraduate level.

Theoretical Framework

When considering students' development of *critical* statistical literacy, I focused on the critical literacy elements presented within Gal's framework for statistical literacy (2002). Gal's (2002) framework includes five knowledge elements and two dispositional elements. The knowledge elements are the collective set of skills needed to interpret and analyze statistical content in today's world and include literacy skills, statistical knowledge, mathematical knowledge, context knowledge, and critical questions.

What sets Gal's framework apart is the attention to the disposition. Gal describes beliefs, attitudes, and critical stance as playing a vital role in moving from passive interpretation to informed action. Specifically, Gal described critical stance as the questioning nature that is needed to critically assess and examine statistical information which includes a list of "worry questions." Since the purpose of this paper is to examine the expression of CSL, Gal's worry questions were adapted to create a list of five components that students need to be critical thinkers when employing CSL: (1) Ability to notice potential bias (2) Ability to question the sample size, sampling methods, or a lack of information regarding the sample (3) Need for additional information to make inferences (4) Ability to acknowledge conflicting or alternate conclusions (5) Ability to identify if the information uses appropriate descriptive and inferential statistics. Through attending to the ways in which students provide evidence of these components in their work that we can get a sense of their CSL.

Methods

The participants in this study were eight students all of whom took an introductory statistics course with the author at a small private university in the southeast (i.e., the case). This particular course was offered within a learning community on the topic of gender, as such students studied introductory statistics through this lens. The course was a traditional face-to-face class in which students were asked to participate in online discussion boards within the university learning management system to think critically about statistics and connect what they were learning to the other courses within the learning community.

The participants were all between the ages of 19 and 22. Of the eight students, one was male and seven were female. Six of the students are White, one Hispanic, and one Indian. The majority of the participants were majoring in the social sciences; there were a couple students majoring in other fields such as biochemistry and music therapy.

Discussion 5: Does the Wage Gap Exist? DUE 2/13 BEFORE CLASS

Does the wage gap exist? Find some data from credible sources to back up your answer. Based on the data and statistics you find, what is your best estimate of the wage gap if you think it exists.

Figure 1: Example of Discussion Board Prompt

Written responses for five discussion board prompts made up the data for this study. The discussions were structured in an open-ended, asynchronous manner. The prompt topics varied, but many were based on the gender wage gap to align with the learning community theme (e.g., see Figure 1). Students were expected to post and reply to classmates to elicit discussion.

Content analysis (Creswell, 2014) was used to explore how students expressed CSL within online discussion boards. I began by using *directed content analysis* in which I used an initial set of codes based on the aforementioned five CSL components. While coding, one additional theme emerged: (2) increasing social awareness of inequities and motivation for action to improve inequities. Random responses were selected for coding by multiple coders until codes were being applied consistently. After which, all remaining responses were coded by the first author.

Findings

The results indicate that the majority of students (75%) exhibited CSL across all discussion prompts. There was a total of six CSL components included in the codebook. The findings indicate that all of the CSL components adapted from Gal's (2002) worry questions were exhibited (See Table 1). The most frequent components were noticing potential bias and the emergent theme of increasing active citizenry. The ability to notice potential bias appeared across all prompts and all but one student. The theme of active citizenry (i.e., social awareness of inequities and motivation for action to improve inequities) emerged during coding and was exhibited by *all* students in at least 50% of the prompts. Given that active citizenry is a component of CSL that has not been discussed in previous literature, the remainder of this brief report will focus on findings related to this construct.

Table 1: Proportion of Prompts Containing Critical Statistical Literacy by Student

Student	Code					
	Bias	Sample	Additional	Alternate	Appropriate	Active Citizen
Derek	0.3	0.0	0.2	0.2	0.0	0.7
Kylie*	0.6	0.0	0.2	0.2	0.4	0.8
Payton	0.5	0.2	0.2	0.7	0.3	0.8
Naomi	0.7	0.2	0.3	0.0	0.0	1.0
Destiny	0.3	0.0	0.7	0.5	0.0	0.5
Jessica	0.2	0.0	0.0	0.0	0.0	0.5
Charlotte	0.3	0.0	0.3	0.0	0.0	0.8
Hope*	0.0	0.0	0.4	0.6	0.0	0.8

Note: * indicates that the student only completed 5 of the 6 prompts.

Active citizenry was defined in the codebook as the ability to utilize critical statistical literacy to become more socially aware of inequities, fight against inequities, and become an active, informed citizen. While engaging in prompts on the gender wage gap, many students extended the discussion. For example, Payton commented on how race plays a role within the gender wage gap: "First- the wage gap is still ever-present whether we see it in our own lives or not… Second, the wage gap should be analyzed through not just a gendered lens, but a racial lens as well. It would be unfair to fight for ONLY gender equality in pay if there is a racial gap as well." Hope, challenged the audience to consider how to promote equity: "Because different positions require different hours, should aspiring mothers in the business field receive incentives in order to minimize the wage gap?" In her response to Charlotte, she continued to question how society can implement change: "The wage gap is a big problem that needs to be addressed... What do you believe should be done in order for the gap to be fully addressed? Who would address the gap? The government?" Hope continued to demonstrate that she was thinking about potential solutions to a systemic problem in her response to Derek's admittance that he was confused by the lack of support to aspiring mothers:

> As a woman, I do agree with you that the statistics do reveal something that is unacceptable in our current world… Similar to Naomi's thought, where would the money come from for incentives to the aspiring mothers and their desire to work. I agree that business hours are not totally compatible with the average office desk for a mother, but this does not mean that it hasn't been done before… I think it would be crazy debate over a topic that Payton brought up - who has the responsibility? The mother or the employer?

In other instances, students challenged typical gender stereotypes. Naomi stated:

> The author points out that the value of some jobs decrease when women start to work those jobs as well as men. This indicates that employers are biased against women, and that women do not simply "[choose]" lower paying jobs. I think it is interesting that people think women [choose] lower paying jobs.

While some students appear to be moved by emotional reactions, others approached being an active citizen as recognizing the role that our perceptions and personal beliefs play in interpreting quantitative messages. Hope said, "It's interesting to me that even when researchers specifically have data that proves a point about how we live [and] what we do, people will still find a reason to disagree or not try to understand the statistical findings."

Discussion and Conclusion

The students in this study were able to use CSL to increase awareness of inequity, question methodology, acknowledge conflicting conclusions and identify bias, and develop a desire for more information. Watson and Callingham (2004) called for increased opportunity to engage in CSL, and the results of this study suggest that online discussion boards using a variety of open-ended prompts with popular media and articles can provide such opportunities.

The overwhelming abundance of instances where students demonstrated the use of CSL to highlight the need to become a more active citizen points to the potential that discussions boards have to bolster engagement, increase interest, and stimulate reflective thinking which aligns with literature (Hara, Bonk, and Anjeli 2000; Newman et al. 1997; Zemelman et al., 2005). It is recommended that introductory statistics courses provide opportunities to explore data and discuss topics that increase awareness of inequity. Further research is needed to determine if prompts that elicit evidence of active citizenship correlate with increased student engagement.

While the results of this study are promising, there are some limitations. First, it is unclear whether the style of prompt had an impact on the quantity and quality of CSL components. I anticipate that the low frequency of both the sampling and alternate components are connected to the prompt content. A prompt that asks students to evaluate a study might have greater potential for students to exercise CSL. Furthermore, it is unclear if the timing of the prompt played a role.

As the semester progressed, it is possible that students naturally exhibited more evidence of CSL. Future studies should change the order of prompts, to determine if timing plays a role.

It is imperative that we encourage students to develop critical statistical literacy. This study supports the notion that online discussion boards are one such vehicle in the development of CSL. Moving forward, we as a field need to identify other potential vehicles as well as pinpoint the prompt styles that best engage students.

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MIDDLE GRADES' STUDENTS' CONCEPTIONS OF AREA AND PERIMETER

Jane-Jane Lo	Hsiu-fei Sophie Lee	Fenqjen Luo
Western Michigan University	National Taitung University	Montana State University
jane-jane.lo@wmich.edu	fei@gm.nttu.edu.tw	fluo@montana.edu

Prior studies have found the relationship between the area and perimeter to be a challenging topic for learners of all ages. Often, students believe that decreasing area for a given shape results in decreasing in perimeter of that shape, or vice versa. Using the theoretical constructs of concept definitions and concept images, this study sought to identify ideas that could be used to design mathematical tasks to develop correct conceptions of this relationship.

Keywords: Measurement, Middle school education

Area and perimeter are important geometrical measurement concepts in school curricula. However, students, including pre-service elementary school teachers, around the world find these topics to be challenging (Battista, 2007; Baturo & Nason, 1996; Livy, Muir, & Maher, 2012; Machaba, 2016; Menon, 1998; Murphy, 2012, Smith & Barrett, 2017). One widely documented misconception about the relationship between area and perimeter is the idea that an increase in area for a given shape results in an increase in perimeter for that shape, or vice versa. Tirosh and Stavy (1999) suggested that this misconception is rooted in intuitive rules (More A-More B, Same A-Same B, Less A-Less B). A common way to counter this misconception is to ask students to explore the relationship between area and perimeter with rectangles. However, working with concrete examples of rectangles of the same area but different perimeters (or vice versa) does not seem to have a long-term effect on this deeply rooted misconception. Smith & Barrett (2017) concluded that "Overall, research has not yet produced a compelling explanation for this challenge or an effective instructional response" (p. 365).

One possible reason for the inadequacy of using rectangles to explore the relationship between area and perimeter, (i.e. this is a common approach to correct the more area-more perimeter misconception for some students) might be due to its heavy reliance on numerical and logical information, while the conceptual understanding of the concepts of area and perimeter and their relationship remains rooted in intuition, spatial reasoning, and everyday experiences.

Theoretical Perspectives

When hearing the words area and perimeter, what pictures come to your mind? How would you explain these ideas to others? These pictures and words are part of your concept definitions and concept images of area and perimeter. Tall and Vinner (1981) described concept image as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). These authors defined the words used to describe area as an individual's concept definition, that is, "a form of words used to specify that concept" (p. 152). Both concept image and concept definition take time to develop and may be modified by new experiences. Taking the social-cultural perspective of learning (Cobb, 1994), language, schooling, and everyday experiences are among possible factors that might impact the formations of concept definitions and concept images. The research questions were as follows:

- What were the subjects' concept definitions and concepts images of areas and perimeters?
- What were their understanding of the relationship between area and perimeter? How might those definitions and images affect their understanding of this relationship?
- How might factors such as language, schooling and everyday experiences influence the formations of concept definitions and concept images?

Methodology

This study was conducted in two public schools, one in the US and the other in Taiwan. Participants of the study included 17 seventh-grade students from the US and 12 sixth grade students from Taiwan. Based on the teachers' judgments, both groups were good representations of the various ability levels within the respected student populations. All of them had completed the instruction on area and perimeter of rectangles, triangles, and circles prior to the study. The data for this study were collected through semi-structured interviews. Due to space limitation, this paper focuses on the data gathered from following tasks:

- If I cut a piece off this paper (pointing to a sheet of paper), what would happen to its perimeter? Give me an example of a cut that would result in (a shorter, the same, a longer) perimeter in the resulting shape. Why do you think your example has (a shorter, the same, a longer) perimeter?
- Is it possible to cut a piece off this paper but make the perimeter of the resulting shape different from what you just said? If so, give an example and explain why your example works.

Results

The initial responses for the question "If I cut a piece off this paper (pointing to a sheet of paper), what would happen to its perimeter?" were quite different between the two groups. The results are summarized in Table One

Table 1: Percentage of Different Initial Responses from Both Groups

	US (n=17)	Taiwan (n=12)
Shorter	76.5%	33.3%
No change	5.9%	25.0%
Longer	5.9%	25.0%
Shorter or no change	11.8%	8.3%
Longer or shorter	0.0%	8.3%

While the majority (76.5%) of the US participants seemed to follow the Less A-Less B intuitive rule, relatively fewer (33.3%) Taiwanese participants did the same. About 25% of the Taiwanese participants chose no change and another 25% chose longer. One student commented that the answer could not be shorter because that was too obvious. She further shared her belief that the most obvious answer on mathematics test tended to be the incorrect one.

After the initial responses, all were prompted to consider possible alternative cases and to provide support for each claim. We have grouped all the final responses in three major cases: the perimeter would get shorter, not change or get longer.

The case of a shorter perimeter. All seventeen (100%) US participants and eight (66.7%) of the Taiwanese participants were able to provide a correct example that supported the case of shorter perimeter. Figure 1 shows both the correct and incorrect examples provided by participating students from US and Taiwan.

The justification for why cutting off an entire rectangle along a side were straightforward. Students pointed to the shortened portions of the boundary. Justification for cutting off a triangle at a corner drew from two distinct concept images. The first image was based on the daily experience that the direct route from point A to point B was shorter than going around a corner. The second image was based on the property of triangle inequality. Lastly, one Taiwanese student gave an example of cutting a 1/4 circle at the corner. He explained that the circumference was about 3 times the diameter, therefore, the arc from the ¼ of the circle would be about ¾ of the diameter, which would be shorter than the sum of two radii which was the diameter. Properties of circle is a main topic in the sixth grade Taiwanese curriculum.

Both types of incorrect examples involved cutting off a rectangle or square from the side or at the corner. Note that the former created longer perimeter while the later created equal perimeter. However, when asked for why they knew their examples had shorter perimeters, all these three Taiwan students' explanations indicated that they focused only on the side length that was removed, but not the side length that was created at the same time.

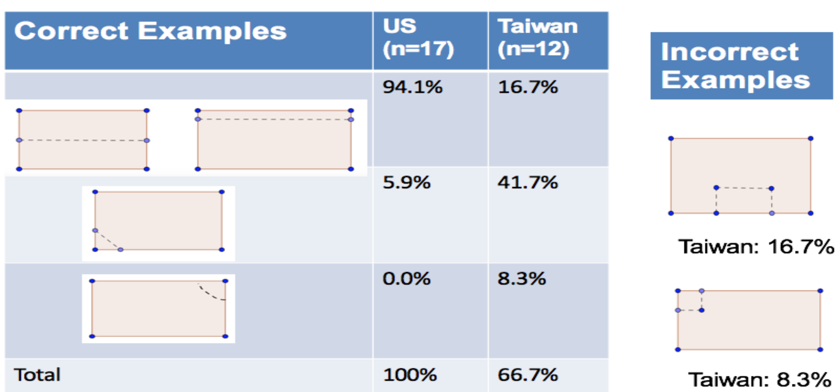

Figure 1: "Cuts" to Create New Shapes That Had Shorter Perimeters

The case of the equal perimeters. Table Two summaries the statistics of participating students' responses to the question of cutting off a piece while maintaining the same. Seven US (41.2%) and three Taiwanese (25%) participants claimed that it was impossible to reduce the area and keep the perimeter constant. While many participants thought it would be possible to do so, only some participants were able to come up with correct examples.

Table 2: Responses for the Same Perimeter Case

	US (n=17)	Taiwan (n=12)
Impossible to make an example	41.2%	25.0%
Incorrect example	17.7%	8.3%

Conceptually possible example	5.9%	0.0%
Correct example	35.3%	66.7%

The most common correct approaches were to cut either a square or a rectangle off the corner These participants were able to explain that since opposite sides of a rectangle (or square) were congruent, the new shape would have a perimeter equal to the original shape. Interestingly, the incorrect examples created by three US students and one Taiwanese all involved cutting a triangle either at the side of corner. They believed that it's possible to have the sum of the two newly created side lengths to be the same as the side length they removed. This is a direct violation of the triangle inequality property.

The case of the longer perimeter. All participants of this study found this to be the hardest case. Table Three below summarize the main results.

Table 3: Responses for Longer Perimeter Case

	US (n=17)	Taiwan (n=12)
Impossible to make an example	58.8%	33.3%
Incorrect example	17.7%	0.0%
Conceptually possible example	5.9%	8.3%
Correct example	17.7%	58.3%

Ten US (58.8%) and four Taiwanese participants (33.3%) claimed that it was impossible to reduce the area while increasing the perimeter. Only three US (17.7%) and seven Taiwanese (58.3%) participants were able to generate correct examples as seen in Figure 2 below.

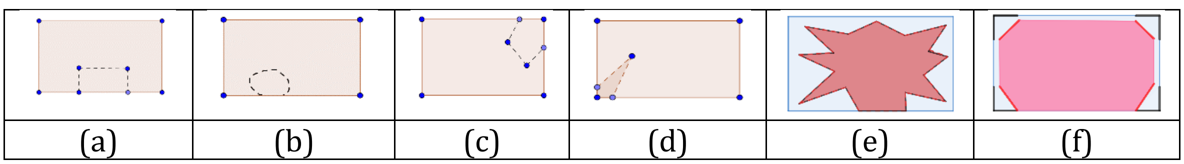

Figure 2: Sample Student Cuts Claimed to Create New Shapes that Have Longer Perimeters

The successful examples generated by students for this particular condition were based on three main concept images: 1) going around is longer than going straight (Figure 2a & 2b), 2) replacing a segment with more or longer segment(s) will result in longer perimeter (Figure 2c & 2d), and 3) shapes with more sides will have longer perimeters (Figure 3a & Figure 3b). While the first and second concept images would lead to correct examples, the third one did not always generate correct examples. Figure 2 also includes examples that were based on the same conception. However, the example seen in 2e did have a perimeter longer than original rectangle, but the example in 2f had a perimeter shorter than the original rectangle. The lengths from one point to another point around the corners (marked in red color) are shorter than the length of the four pairs of corner segments (marked in black).

Discussion and Implications

This study has deepened and expanded our horizon for concept images and definitions in geometry. First, we discovered that some students tried to support their responses with their daily life experiences, such as "direct routes from point A to point B as opposed to going around the

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corner." Additionally, some students used their mathematical knowledge learned in school, such as triangular inequality, to justify why the sum of the lengths of any two sides must be greater than or equal to the length of the remaining side. Future studies are needed to explore the impact of purposefully-designed activities in these different settings on students' concept definitions and concept images of area and perimeter.

Second, we found that when participating students were asked to justify their initial responses with an example, they started to ponder between generating an applicable and justifiable example and modifying their prior responses. As a result, a lot of insights about middle grade students' concept definitions and concept images of area and perimeters were identified. This went beyond previous findings based on the paper-and-pencil survey only.

Future study is needed to examine the effectiveness of such activities in helping students build a more solid understanding of area, perimeter, and their relationship. It would also be interesting to see if the effect of this sequence of activities can be transferred to the relationship between surface area and volume.

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COLLEGE STUDENTS' UNDERSTANING OF CONDITIONAL PROBABILITY

Marcie Tiraphatna
Virginia Tech
marciet@vt.edu

Jesse L. M. Wilkins
Virginia Tech
wilkins@vt.edu

This study investigates one undergraduate student's understanding of conditional probability to explore whether a framework (Tarr & Jones, 1997) originally designed for assessing middle school students' thinking can be successfully used with undergraduate students. A semi-structured clinical interview was conducted with tasks associated with the framework as well as additional tasks more appropriate for an undergraduate student. Findings suggest that the framework can be applied to undergraduate students; however, an additional level may be needed to differentiate more sophisticated understandings of conditional probability.

Keywords: Probability, Undergraduate-Level Mathematics

Conditional probability is the probability of an event (A) given that another event (B) has occurred, P(A|B) = P(A and B)/P(B). When dealing with dependent events or situations in which the relationship between events is not obvious, determining the conditional probability can prove to be difficult for students. Fischbein and Gazit (1984) found that students' performance in determining probabilities differed greatly depending on whether balls chosen from a bucket were replaced or not. Among the reasons that students may have difficulty in without-replacement situations is that the sample space changes or the situation seems counter to their intuition. Students often forget or do not realize a change in the sample space has occurred when the ball is not replaced.

Furthermore, using words such as "after" instead of "given," conditional probability tasks may lead students to believe that conditioning events only precede another event, when, in fact, two events could occur simultaneously or the conditioning event could happen later. An example of this is the Falk Phenomenon (Falk, 1988; Shaughnessy, 1992). When drawing two balls without replacement, (see Q3, Figure 1), finding the probability that the first ball chosen is black (B_1) given that the second ball chosen is black (B_2), $P(B_1|B_2)$, is usually more difficult than finding the probability that the second ball is black given the first one is black, $P(B_2|B_1)$. B_2 naturally seems to depend on B_1 while the converse does not seem possible. This may influence students to believe that $P(B_1)$ is the same regardless of the knowledge of B_2.

Falk (1988) also describes people's struggles with defining the conditioning event and sample space. Consider the three-card problem (see Q6, Figure 1). Most people look at the number of cards possible (two) instead of the number of sides (three). Students may also have issues with the conjunction rule, P(A and B) ≤ P(A). Tversky and Kahneman (1983) found that statistically naïve undergraduate students say the probability of a man having a heart attack was less than the probability of a man having a heart attack and being older than 55. One possible explanation for this is that they are confusing "having a heart attack *and* being older than 55" with "having a heart attack *given that* they are older than 55," P(A and B) versus P(A|B).

Theoretical Framework

Tarr and Jones (1997) created a four-level framework for assessing middle school students thinking related to conditional probability and independence. Students categorized as Level 1

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tend to use subjective reasoning and usually ignore quantitative or numerical information. Students with Level 2 thinking have a mix of subjective and quantitative thinking. However, their quantitative thinking is limited and not complete; they sometimes revert to subjective thinking if situations do not go as anticipated. Students categorized as Level 3 understand that quantities are relevant, but are not able to assign specific numbers to probabilities. Level 4 students understand how changes in a sample space are related to conditional probability and can give numerical values for probabilities. In any given classroom, students might have different levels of probabilistic thinking.

Q1) Alice, Bob, and Cody are in a class together where students can earn raffle tickets. The teacher draws a raffle ticket from a cup at the end of each day to determine who gets a piece of candy. In the cup, Alice has 4 tickets, Bob has 3 tickets, and Cody has 1 ticket.

a) Who do you think has a better chance of winning?
b) Today the teacher draws one of Alice's tickets, gives her the candy, and then throws the ticket away. Bob says, "That's ok. I have a better chance of getting candy tomorrow since now I have just as many tickets as Alice." Cody says, "I still have 1 ticket so my chance of winning hasn't changed." What are your thoughts with regards to their reasoning?

Q2) The combination for a safe is not 2-8-5, but each digit is one away from 2-8-5.
If I gave you one chance to pick, do you think you could unlock the safe?
Is it more likely that you would unlock the safe, more likely it would stay locked, or is it the same chance?
Go ahead and pick the first digit.
You are correct. Has your chance of unlocking the safe changed? Why or why not?
Pick second digit. You are correct.
Has your chance of unlocking the safe changed? Why or why not?

1	7	4
2	8	5
3	9	6

Q3) A bucket contains 3 black balls and 4 white balls. You pick a ball from the bucket without looking.

a) What is the probability of drawing a black ball?
b) What is the probability of drawing a white ball?
c) Let's say that you draw a white ball. After drawing it, you put it back into the bucket and make a second draw. What is the probability of picking a white ball again on the second draw?
d) Now let's say you draw a black ball. This time you do not return the ball back to the bucket. A second ball is drawn from the bucket. What is the probability that this ball too, will be black? Or white?
e) Draw two balls without replacement. If we know the second ball is white, what is the probability the first ball is black?

Q4) A hospital gathered information from 1000 patients about smoking and lung cancer. The data is presented in the table provided.

a) What is the probability that a person chosen at random from this sample will have lung cancer, if they are a smoker?
b) What is the probability that a person is a smoker if they don't have lung cancer?
c) What is the probability that a person is not a smoker and does not have lung cancer?
d) What is the probability that a person is a smoker and has lung cancer?

		Smoked	
	1000 total	**Yes** (400)	**No** (600)
Lung Cancer	**Yes** (450)	250	200
	No (550)	150	400

Q5) Suppose I shuffle a standard 52 card deck, pick a random card, and without showing you I tell you it is an ace. What is the probability my card is the ace of spades?

Q6) There are three cards in a hat. One card is blue on both sides. One card is red on both sides. One card is blue on one side and red on the other. A single card is dawn and placed on a table. If the visible side of the card is red, what is the chance that the other side is red? What about blue?

Figure 1: Interview Tasks

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Purpose

The purpose of this study is to investigate one student's understanding of conditional probability and attempt to categorize this student based on the conditional probability portion of the framework (Tarr & Jones, 1997). Although this framework was originally designed for middle school students, we believe it can also be applied to any academic level, specifically undergraduate students. The framework describes different types of thinking and milestones, which are unrelated to age or grade. The framework relies on students' understanding of "with and without replacement" situations which have an explicit change in sample space. The tasks used by Tarr and Jones (1997) did not differentiate temporal ordering or use contingency tables. Studies show that college students still have difficulties with the effect of the time axis and contingency tables (Fishbein & Schnarch, 1997; Watson & Kelly, 2007). Therefore, in this study we will further investigate these issues.

Data and Methods

The study involved one undergraduate student from a large public research university in the southeast United States. The participant, Steve, was a freshman in an applied mathematics major. He had previously taken AP Statistics and a probability and statistics course during his junior and senior year of high school.

We conducted a semi-structured interview involving six conditional probability tasks (see Figure 1). The interview was videotaped and transcribed for further analysis. The tasks were chosen to highlight students' conceptions of conditional probability in a variety of contexts. Q1, Q2, and Q3 minus part e were based on those used by Tarr and Jones (1997) which tested students' understanding of with/without replacement type problems. Q3e and Q5 were included to test the student's ability to work with the time axis. A contingency table, Q4, was used to identify the potential confusion with conditional and conjunctive statements. Q6 is the three-card problem described by Falk. This task was included because of its nuance of defining the conditional statement. Some questions involved little to no calculations, which allowed the student more opportunity to discuss their reasoning. Follow-up questions were asked after the tasks were solved.

Results and Discussion

Identifying the sample space and conditioning event is a crucial part of calculating conditional probabilities. Throughout the interview, Steve demonstrated his awareness and understanding of changes in a sample space, and that given a certain event, the sample space is restricted. When shown a contingency table, Q4, Steve was able to correctly identify the sample space for one event given another. For parts a and b, he pointed to the smoker column and circled the row of people without lung cancer. He even said things like "the precondition is."

Although Fischbein and Gazit (1984) showed and explained students' difficulties regarding with and without-replacement tasks, Steve did not have any issues with such basic tasks. In Q3, he was able to explicitly state the correct probability of drawing a black ball after a black ball was drawn and (not) replaced. With Q1, Steve used part-part reasoning when explaining why Bob is correct, saying, "He (Bob) has the same chance to get candy as Alice." He did not use the same reasoning to explain Cody's logic. Instead, he compared the numerical values, 1/7 and 1/8, to say Cody's chance of winning has increased. This change from part-part to part-whole thinking allowed Steve to recognize that there were fewer tickets, and thus, a better chance of winning for Bob and Cody.

Even if one is able to identify a change in sample space, they still may not be able to identify the conditioning event, as shown with the three-card problem. When given the three-card problem, Steve eliminated the possibility of choosing a blue/blue card, but did not recognize the sample space was the three possible faces of the cards not the two cards. He said, "We pick the red. It could just be that the other side is also red. That would be one possibility. The other possibility is taking the other side is blue. So one half." Here, Steve demonstrated an understanding of a change in some sample space, albeit, not the correct one.

We posed Q4 to test whether Steve confused a conjunctive statement with a conditioning one. When questioned about his reasoning for answering 400/1000 as the probability of a person being a smoker *and* having lung cancer, Steve said, "it's the entire group because it's not just saying 'of this group.' It's the entire group." This suggests conjunctions were not an issue.

Another misconception students often have is the idea that conditioning events need to happen sequentially. To see Steve's understanding of this, we asked Q1 and Q5. When drawing two balls without replacement and knowing the second ball was white, Steve said the following:

> So, if we know that the second ball is white and because we can't put the ball that we took out earlier back…we know that we didn't draw that ball. Because we know we didn't draw that ball, we can eliminate that from the equation if it wasn't in the bag then we have six. So then you have six possibilities minus one white. So then three black balls out of six.

When asked for the probability of a card being the ace of spades given that the card is an ace, Q5, Steve immediately wrote 1/4 down stating "although it says 52 cards, it's ace and there is only four aces." Both of these instances show the participant's ability to solve problems when the conditioning event happened before or during the event being calculated.

Throughout the interview, Steve demonstrated a strong use of numerical reasoning. He was able to assign numeric values to probabilities and explain his reasoning without the use of formulas. When asked to compare the likelihood of two events, Steve would use the numerical value of the probabilities to justify his answer. In all instances, he was able to recognize when and if a sample space had changed and apply it to his responses. He recognized that probabilities do not change in with-replacement situations and they do in without-replacement. Because of this, we would categorize him as someone with Level 4 understanding of conditional probability.

Conclusion and Significance

This study provides one case of an undergraduate student's understanding of conditional probability. We were successively able to categorize him within Tarr and Jones' (1997) framework. However, additional interviews of students with varying experiences, beyond mathematics majors, should be conducted to further validate the use of Tarr and Jones' framework for undergraduate students. More interviews of undergraduate students could also uncover issues with conditional probability similar to those of younger students.

Although we could categorize Steve within the framework due to his use of numeric values and understanding of sample space, he did not have a complete knowledge of conditional probability as he answered Q6 incorrectly. However, Q6 is considered a relatively difficult conditional probability task. This suggests that it may be possible to refine Tarr and Jones framework to include additional conditional probability tasks. The framework discussed in this study could be used by instructors to assess undergraduate students understanding before and during instruction to guide the instruction of conditional probability.

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EXAMINING DYNAMIC MEASUREMENT REASONING FOR AREA AND VOLUME

Nicole Panorkou
Montclair State University
panorkoun@montclair.edu

This paper describes the forms of reasoning that students developed as they engaged with tasks presenting area and volume dynamically. The analysis of a series of design experiments with twelve fourth-grade students showed that they reasoned about dynamic measurement in terms of a) the quantities involved in the generation of the space to be measured, b) the dimensional transformation of those quantities, and c) their continuous change.

Keywords: Dynamic Measurement, Technology, Design Experiments

Dynamic Measurement

Dynamic Measurement (DYME) is an approach to geometric measurement that focuses on how space is measured by the lower-dimensional objects that generate it. One way to visualize this approach is by Lehrer, Slovin, and Dougherty's (2014) generation of attributes through motion. For instance, a rectangular area can be thought of as a swipe of a line segment '*a*' for a distance '*b*' to generate an area of '*ab*' (Figure 1a). Similarly, volume can be thought of as an extrusion of a 2-D surface of area '*c*' for a height of '*d*' to generate a 3-D space '*cd*' (Figure 1b).

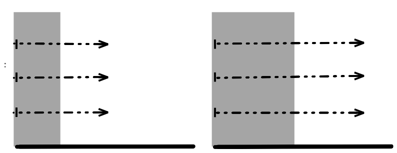

Figure 1a: DYME for Area

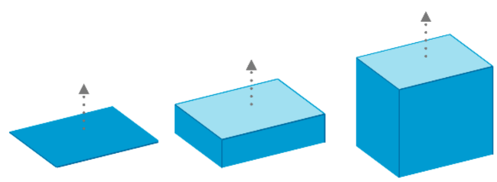

Figure 1b: DYME for Volume

DYME is built on the view that space has a dual nature as both *containment* and *generation* (Panorkou & Pratt, 2016), which is often overlooked by research and practice. As Panorkou and Pratt (2016) argued, "one can see the space as incorporating objects; in this sense, the space contains the objects. At the same time, the space can be thought of as generated by the objects" (p. 213). Typically, measurement is approached using the containment nature of space, for example, counting the number of square units needed to fill a 2-D space, or counting the number of cubic units needed to fill a 3-D object. In such approaches, the measure of area and volume is a multiplicative relationship between the unit of measure (square units, cubic units) and the object that is measured. DYME aims to bridge the two components of space into one construct. In terms of containment, DYME focuses on how the space that is measured by the lower dimensional objects (line segments, surfaces) it contains. In terms of generation, DYME focuses on the role of the lower-dimensional objects that are spanned to create the space that is measured. In this approach, the measure of area and volume is a multiplicative relationship that illustrates a transformation from two lower-dimensional quantities (e.g. *length x width* for area, *area of base x height* for volume.)

This paper reports on the findings of a series of design experiments (DEs) (Cobb et al., 2003) with twelve fourth-grade students working in pairs to describe the qualitatively different ways of

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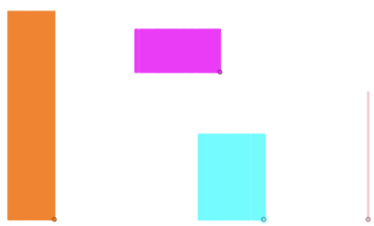
DYME reasoning about area and volume that students exhibited and discuss the means that supported their reasoning. The DE with each pair consisted of 8-10 sessions of 45-50 minutes each and the focus was the area of rectangles and the volume of right prisms and cylinders.

Characterizing Students' Dynamic Measurement Reasoning

The findings of the design experiments identified three qualitatively different but related types of reasoning that I describe in the following paragraphs by giving examples from the DEs.

Reasoning About the Quantities Involved in the Generation of Space

The early tasks in DYME for area involved dragging line segments of various lengths over various distances and reasoning about the size of the rectangle generated (Figure 2a). In terms of volume, the tasks involved dragging surfaces of various shapes and sizes over different distances and reasoning about the size of the 3-D space generated (Figure 2b).



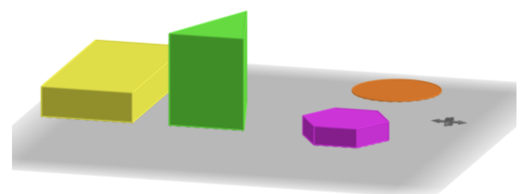


Figure 2a: Dragging Segments Over a Distance
Figure 2b: Dragging Surfaces Over a Height

By dragging the line segments and the surfaces students reasoned about the quantities involved in the measurement process. In terms of area, by dragging a line segment over a distance, students conceived three quantities that are measurable: a line segment (described as a paint roller in our task design), a dragging distance, and a generated area. For example, students argued that the "bigger the roller, the more space you can cover" and that "you can drag it [the roller] more and make a bigger one." In terms of volume, by dragging bases over a height, students conceived three measurable quantities: area of base, dragging height, and 3-D shape created. For instance, students stated that the 3-D shape created "depends on how big the base is, and it also depends on how much you can drag it."

As they dragged the surfaces to generate volume, students perceived the extrusion of 2-D shapes to create 3-D shapes as an action of "expanding," "dragging" "stretching" or "pulling" the base. For instance, Jayden argued that "it starts out as a 2-D base and when you stretch it, it goes into 3D." Students' language of *expanding, pulling, stretching,* and *dragging* shows that they recognized that there is some kind of transformation happening from 2-D to 3-D. While elaborating on their thinking further, some pairs reasoned about the extrusion as an iteration (creating identical copies) of 2-D surfaces. For example, Molly argued "It's like you are taking the base and stacking it over and over again." I noticed that students' reasoning shifted interchangeably from reasoning in terms of a transformation (e.g. stretching, pulling) to reasoning about stacks. This integrated reasoning illustrates that they were able to construct a meaning about the volume of 3-D shapes both as spaces that contain objects (multiple 2-D shapes) and as spaces that are generated by these objects. In other words, they were embracing the dual nature of space as both *containment* and *generation* (Panorkou & Pratt, 2016).

Reasoning About the Dimensional Transformation of the Quantities

The following tasks for area involved asking students to drag a line segment of 5 inches over different distances and reason about the space covered (Figure 3).

Researcher: So how much space does the first [red rectangle in Fig. 3] cover and how much does the second [orange rectangle in Fig. 3] cover?

John: 1 inch and 5 inches, 2 inches and 5 inches.

Figure 3: Dragging a Line Segment of 5 Inches Over Various Distances

Similar to John, students were able to construct a unit for measuring the space of rectangles that was a composite unit made up of the measure of the line segment and the distance of the swipe (e.g. 1 inch and 5 inches). This unit was a result of a dimensional transformation as it included two linear measures for reasoning about an area measure, considering the unit to be a composite of two dimensions. They were also able to construct various units depending on the rectangle they had to measure. The tasks that followed focused on painting rectangles using line segments of varied lengths dragged in different distances. These types of tasks helped students to reason multiplicatively about the length, width, and area. For example, in explaining why the area is 30 inches^2, Sophia argued “Because the width is 10 inches going across and the length is 3 inches. So then 3 times 10 is 30.” Similar to Sophia, for all students, reasoning multiplicatively has shown to be an intuitive way to explain the space covered. In terms of volume, the tasks involved dragging surfaces of various areas over 1-inch and 2-inch heights (Figure 4).

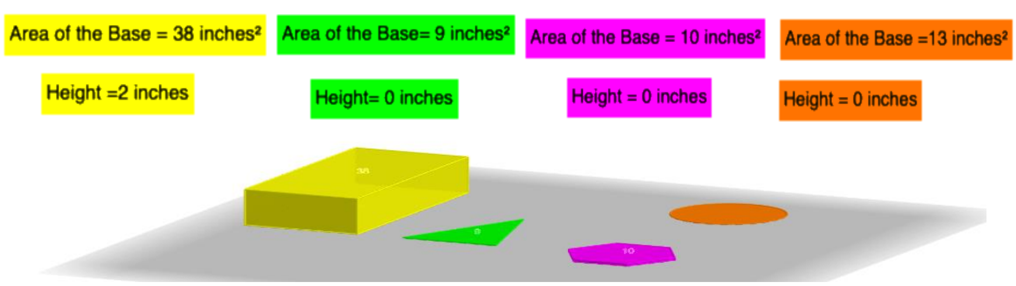


Figure 4: Students Drag Various Surfaces to Different Heights

As students increased the height of the extrusion from 1 inch to 2 inches, they reasoned multiplicatively about the relationship between area of base and height. Students explained their use of multiplication in terms of thinking about the 3-D space as consisting of stacks (copies of 1-inch drags of the specific base). For instance, Jayden argued that the size of the 3-D shape becomes two times bigger if we increase the height to 2 inches, “because it is kind of like stacking this one on top of it because this one says it is 2 inches, so we are going to make this one another inch and it is going to be the same thing.” Jayden’s reasoning shows that he viewed the iteration of 1-inch drags as making copies of one drag, and he multiplied by two to find the volume. In addition, he described this iteration in terms of "make this one another inch and it is going to be the same thing" showing that he coordinated the change in height with the change in volume. As students' thinking progressed, they were able to generalize that volume depends on the size of the base and the number of 1-inch drags of that base. For example, Molly stated, "To find the volume, you have to first find the base, how much is the area of the base and then you multiply by how many drags you have, how many drags you have done." Similar to Molly, students were able to conceptualize *base x height* as a quantitative operation (Smith III & Thompson, 2008) between two quantities (base and height) to produce a new quantity (volume).

Reasoning About the Continuous Change of Quantities

Students were asked to double, triple, and halve lengths and reason about the resulting changes in area. For example, in splitting the width of a rectangle in half a student argued that its area will now be half, stating "Because this is like half of it. And 4 times 6 equals 24, but 2 times 6 equals 12 and it is a half." Similarly, volume tasks focused on changing the linear measures of rectangular prisms of a sculpture and reasoning about the change in volume (Figure 5). Similar to the excerpt below, students were able to recognize that to make the volume *n* times bigger, they need to make the area of the base or the height *n* times bigger. They used covariational reasoning (Confrey & Smith, 1995; Saldanha & Thompson, 1998) to coordinate the two varying quantities and reason about the ways in which they changed in relation to each other.

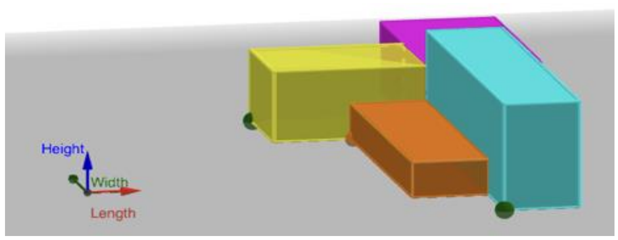


Figure 5: Students Were Asked to Change the Dimensions of the Rectangular Prisms

Researcher: OK, the first one is to double the length of the orange. Before you do it, what do you think will happen to the volume if you double the length?
Ben: It's going to double; it is going to double itself.
Researcher: Why?
Ben: Because we doubled the length. [students double the length]
Researcher: Hmm, let us see. Yes, it doubled.
Ben: Double both the length and the height of the pink one. [states the next task]
Researcher: What do you think will happen?
Ben: It will quadruple.
Researcher: Why?
Ben: Because you doubled two things. […]
Researcher: Double the length and then you half the width. What will happen?
Ben: Nothing.
Researcher: Why?
Ben: Because you halved the width. So, it's just going to go back to its original form because you cut it in half.

In sum, students reasoned about the quantities involved in DYME and how these quantities work together in generating and changing the space created. By exploring what is changing and how it is changing, they were able to conceptualize area and volume as quantities on their own and also as multiplicative relationships between length and width, and base and height, respectively. These findings aim to initiate a discussion about the nature of DYME and how it can be used for developing a conceptual understanding of measurement and other mathematical ideas, such as transformations, multiplication and covariation.

Acknowledgments

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STUDENTS' DIRECTIONAL LANGUAGE AND COUNTING ON A GRID

Sezai Kocabas
Purdue University
skocabas@purdue.edu

Laura Bofferding
Purdue University
Lbofferd@purdue.edu

Mahtob Aqazade
Purdue University
maqazade@purdue.edu

Ana-Maria Haiduc
Purdue University
ahaiduc@purdue.edu

Lizhen Chen
Purdue University
lizchen@purdue.edu

We explored 23 first and third graders' directional language and counting of spaces on a grid when debugging programming code after playing three and six sessions of Coding Awbie™. Students in both grades were more likely to describe a bunny as moving up using specific terms but more likely to describe going right using generic terms. Over half of the students made at least one counting error. Students had fewer counting errors after playing the programming game because many of them more efficiently corrected the programming code.

Keywords: Geometry and Geometrical and Spatial Thinking; Cognition; Elementary School Education

Spatial thinking is crucial for success in not only mathematics but also programming, chemistry, and medicine (Clements, 1999; Jones & Burnet, 2007; Krajewski & Ennemoser, 2009; Kyttälä et al., 2003; Lowrie & Logan, 2016; Sorby et al., 2013). Given the increased focus on students learning programming, mathematics teachers can leverage the finding that programming tasks can promote mathematical practices involving spatial thinking (Heghfield, 2003). For example, one of the Common Core Standards for Mathematical Practice is for students to reason abstractly and quantitatively; this practice involves representing situations symbolically and reasoning about quantities (National Governor's Association Center for Best Practices & Council of Chief State School Officers, 2010). Similar practices are incorporated in programming games for children, which often involve manipulating symbols that tell a character which direction to go in and how far. Therefore, the current study explored spatial thinking through students' explanations as they debugged a symbolic program written to move a bunny to a carrot on a grid.

Spatial Thinking

Spatial thinking involves perceiving and transforming the visual environment (Gardner, 1983), plays a strong role in students' ability to mentally represent and manipulate information or objects (Lowrie & Logan, 2016), and is highly correlated with academic achievement in programming, mathematics, and other fields (Clements, 1999; Jones & Burnet, 2007; Lowrie & Logan, 2016; Sorby et al., 2013). There are two dimensions of spatial thinking; spatial orientation and spatial visualization (Clements & Battista, 1992).

Spatial Orientation and Language

Spatial orientation refers to the ability to understand and interpret the relations among different positions of objects in space (Clements & Battista, 1992; Clements, 1999). Clements (1999) argued that students must have some mathematical ideas for reading and creating mental maps. Students should learn to manage abstraction, generalization, and symbolization in the mapping process. Furthermore, students should develop some ideas such as direction, distance and measurement, and location. Children naturally use vertical terms such as *up* and *down* at an early age, but *left* and *right* require more time (around age 6-8) for students to fully understand (Sarama & Clements, 2009). Teachers should help students move away from generic movement descriptions such as *over* because *over* does not extrapolate well to bidirectional spaces (Sarama & Clements, 2009). Students need to learn to use precise language, and computing programs can provide them with immediate feedback.

Spatial Visualization and Counting

On the other hand, spatial visualization involves imagining and making transformations, such as motions of an object (Sarama & Clements, 2009). Spatial visualization is strong predictor for success of counting achievement in earlier grades. For instance, spatial thinking was positively correlated with kindergarteners' ability to count 16 cubes in four rows and four columns (Kyttälä et al., 2003). Students may miscount (e.g., by double counting) if they do not find structure in the spaces they are counting (e.g., Battista et al., 1998).

Programming, Spatial Thinking, and Mathematics

Programming tasks might leverage spatial thinking because students have the opportunity to learn direction, orientation, measurement, and positional language, as with Logo turtle (e.g. forward 10 steps; Clements, 1999). For example, after using *Turtle Math*, a version of Logo for third to sixth graders, students' computation scores improved (Clements & Sarama, 1996). One current programming game for even younger elementary students (Coding Awbie™) involves tangible programming pieces that allow students to move a character along squares in a grid-like environment. This structured space could potentially support younger students' counting. Further, the programming pieces involve motions, directional arrows, and numbers that might better map to students' ideas about how to move the character. Therefore, we investigated the following research questions: How do first and third graders interpret symbolic location and movement representations and map them onto a 2-dimensional grid? How does this change after practice programming a character to move on a grid (a) in terms of the positional and directional language they use? and (b) in terms of their counting to determine the number of movements?

Method Participants and Setting

The study took place in a Midwestern public elementary school with about 46% students qualifying for free or reduced-price lunch. For this analysis, we focus on data from 23 participants who came from two first- and one third-grade class from our larger study. We selected these particular classes for the analysis in order to have an equal number of participants from each grade; however, we had to exclude one of the 12 first graders because of missing data.

Study Design

The study consisted of a pretest, three sessions playing a programming game, a presentation about programming, a midtest, three more game sessions, and a posttest. During game play, students played Coding Awbie™, a game using the Osmo™ tangible interface. Student pairs were randomly assigned to either engage in free-play for the first three sessions or explainedplay (i.e., students had to tell us their goal for each program and explain how their lines of code would help them meet their goal). During the second set of sessions after the midtest, they switched groups (those who had explained-play had free-play and vice versa).

Pretest, midtest, and posttest. Our analysis focuses on one common item from the three tests, a commenting and debugging problem. For this problem, students had to explain a pair of coding commands meant to move a bunny on a grid to get to a carrot. Students were told that the bunny could not move past or jump over houses, but they could jump over flowers. Students then identified the bug in the code and explained how they would fix the code (see Figure 1).

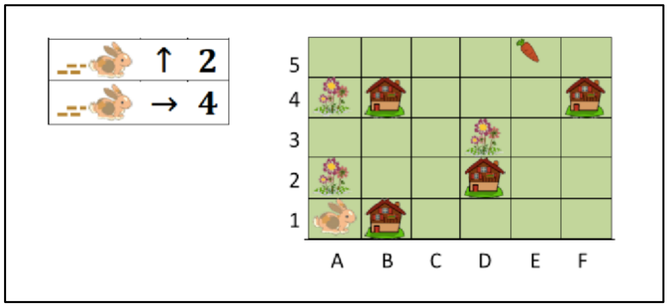

Figure 1: Programming Debugging Problem (Grid Labels Added for Ease of Reference)

Data Analysis

Positional and directional language. We analyzed the data in terms of spatial orientation for the midtest and posttest as we did not ask students to explain the program on the pretest. We expected the students to explain the first line of the program as "walk up two" (as used in the coding game) and the second line of the program as "walk right four." Therefore, we coded students' explanations of each line of code in three categories: movement, direction, and number. We focused on two categories for direction: generic or incorrect language and non-generic language. Non-generic language included the precise terms *up* and *right.* Generic language involved general statements such as *this way*, *that way*, *side*, *over*, and *forward* (incorrect).

Counting. We also analyzed the data in term of spatial visualization. When students corrected the code, they often counted spaces the bunny would move to determine if the number in the code should stay the same or change. We classified students based on whether they made a counting error on any of the three tests. For students who made any counting error, we identified on which test they made an error and whether they made an error on the *corrected* code (i.e., changed the first line of code to jump up 2) or when changing it to *new* code (i.e, changed the first line of code to jump up 1).

Results Positional and Directional Language

Table 1: Students' Generic and Non-Generic Uses of *Up* and *Right*

Students		Midtest		Posttest	
		Generic	Non-generic	Generic	Non-generic
Up	1st (n=11)	9%	73%	9%	64%
	3rd (n=12)	8%	83%	---	83%

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* St Louis, MO: University of Missouri.

Right				
1st (n=11)	82%	9%	55%	9%
3rd (n=12)	50%	42%	58%	25%
Up Explained-play (n=15)	13%	73%	7%	73%
Free-play (n=8)	---	88%	---	75%
Right				
Explained-play (n=15)	67%	27%	67%	13%
Free-play(n=8)	63%	25%	38%	25%

Overall, when describing the first line of code, students in both grades were more likely to describe the bunny as going *up* as opposed to describing thedirection with generic terms (see Table 1). On the other hand, the students were more likely to use generic terms, such as *go over* to describe the bunny's movements in the second line of code. Third graders were more likely than first graders to correctly describe this movement as going *right*, although across all students, fewer of them even referenced the direction on the posttest because some described and fixed the first line of code and then indicated the second line was okay without talking about it.

Interestingly, for students who played the game without explaining their code in the first three sessions, a higher percentage of them used the term *up* on the midtest than students who were forced to explain while they played. Further, students in this group also used less generic language for *right* on the posttest compared to the other group.

Counting

Overall, of the 23 students analyzed, 12 of them (five first graders and seven third graders; six free-play and six explained-play) made at least one counting error on this item. The majority of them, 80% of first graders who had errors and 71% of third graders who had errors, counted incorrectly on the pretest. In particular, these students changed the first line of code to have the bunny jump once from A1 to A3 and then had the bunny move right to C3 (see Figure 1). However, when counting from A1 to C3, they recounted the space A1 where the bunny would already be, suggesting the code should say *walk right 3* instead of *walk right 2*. Some students continued to double count the square the bunny occupied when adding on lines of code to walk up to C5 or over to E5. Fewer students miscounted on the midtest (none in first grade!) and posttest because they changed the first line of code to read *jump up 2*. Interestingly, they accepted the correct second line of code and did not double count in these situations (except for one third-grader who did this on the posttest). Only 33% of miscounters from the original freeplay group miscounted on the posttest compared to 67% of miscounters from the explained-play group.

Discussion

Similar to previous reports, we found that students were more likely to use the specific term *up* when describing vertical movement but less likely to use the term *right* (Sarama & Clements, 2009), choosing instead to use generic directions. Especially when giving directions to others, providing explicit directions is important. Therefore, the results suggest teachers should emphasize precise directional terminology as early as kindergarten (when the Common Core

Standards for Mathematics suggest students learn positional language), because playing the game without reinforcement of the specific language did not increase the use of *right* nor did being forced to explain their programming movements. On the other hand, students had fewer counting errors after playing the programming game because many of them more efficiently corrected the programming code and accepted the final direction to *move right 4* without experiencing the counting error they had when changing the code. This suggests that using worked examples could be used to scaffold their counting. There was a slight trend for students who initially had free-play to use more specific and less generic language as well as have fewer counting issues. If this pattern holds for the larger data set, it may suggest that having the opportunity to play before explaining helped students better attune to what they were explaining.

Acknowledgments

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CAN PEER-LED TEAM LEARNING IMPROVE STUDENT OUTCOMES IN AN UNDERGRADUATE INTRODUCTORY STATISTICS COURSE?

Sandy Spitzer
Towson University
sspitzer@towson.edu

Melike Kara
Towson University
mkara@towson.edu

Felice Shore
Towson University
fshore@towson.edu

This paper describes an intervention to improve student success in an undergraduate Introductory Statistics course. Research suggests that students' success in general education mathematics courses (such as Introductory Statistics) is a strong predictor of their eventual degree completion; thus, improving student outcomes in these courses has potential to influence overall retention. This paper presents the results of one effort to improve these outcomes, using a Peer-Led Team Learning structure. Peer coaches (former students in the course) led a series of voluntary, weekly structured tutoring sessions centered on inquiry-based lesson plans. Results indicate that students who attended the sessions outperformed their peers on several key indicators, including pass rate and course grade. We discuss the structure of the intervention, evidence of its success, and implications for undergraduate mathematics education.

Keywords: Data Analysis and Statistics; Policy Matters; Post-Secondary Education

A pressing problem in undergraduate mathematics education is student struggle in so-called "gateway" mathematics courses: those courses that satisfy a general education credit and are often required by students' majors. Research suggests that students who pass their first credit-bearing mathematics class are significantly more likely to complete their undergraduate degrees (e.g., Adelson, 2006). Therefore, interventions that target student success in these courses have the potential to increase overall student retention. The large number of students in these courses makes it difficult for faculty members to support students directly, but also provides an opportunity to shift outcomes for many students. One solution that has been extensively used and tested in other STEM disciplines, particularly Chemistry, is a structure for collaborative student tutoring called *Peer-Led Team Learning* (PLTL). In this paper, we describe one institution's efforts to construct a PLTL intervention to support student success in Introductory Statistics. This gateway course has a high historic failure rate (including students earning a grade of D or F, or withdrawing before course completion) at our institution. These failure rates have ranged up to 65% in some sections and have a historical mean of approximately 24%. Given that annual enrollment in the course generally includes 800 or more students, reducing this failure rate could meaningfully improve overall retention.

Perspectives and Literature Review

Both our choice of the course to include and structure of the intervention were informed by previous research on undergraduate student success. Our choice to study Introductory Statistics was based on our concern about the historical failure rates as described above, but also by nationwide calls to improve pathways to success for students in non-algebra intensive majors (e.g. Charles A. Dana Center, 2015). After a review of the literature, we noted that PLTL structures have been used successfully across the STEM disciplines and particularly in Chemistry courses (see, e.g. Grosser et al., 2008; Wilson & Varma-Nelson, 2016). Although less work has been done to study this model in undergraduate mathematics, there is some evidence that PLTL session can improve mathematics students' learning (Merkel et al., 2015) and attitudes toward mathematics (Curran, Carlson, & Celotta, 2013). In this paper, we attempt to extend these findings to the context of Introductory Statistics and investigate whether a PLTL structure can improve student outcomes in that course.

Methodology

Program Structure

Our intervention was based in part on descriptions of successful PLTL structures in the literature (e.g. Carlson et al., 2016). Each week, peer coaches led a set of weekly collaborative learning sessions. The peer coaches were selected from students who had earned an "A" in the course during the previous semester and chosen based on their performance in an interview. The peer coaches were trained weekly to implement inquiry-based lesson plans designed by a faculty member in mathematics education. These lesson plans aimed to deepen student conceptual understanding of the course ideas (e.g., distributions, sampling, and interpretation of inferential tests). Each coach conducted 1-2 sessions per week, each lasting approximately 2 hours. During these sessions, they would deliver the lesson plan and engage in collaborative problem-solving with up to 12 student participants. A total of 10 weeks' worth of sessions were held during the semester under study. Participation was voluntary for students, with a modest grade incentive offered based on the number of sessions attended.

Participants, Measures, and Data Analysis

During the semester under study, PLTL sessions were offered to all students (n = 682) that were enrolled in Introductory Statistics. We recorded the number of sessions which each student attended, which could range from 0 to 10. We also requested that instructors provide a percentage score on the first exam and the final exam for all enrolled students, and collected students' final assigned grade for the course. In this paper, we present the results from three different analyses of the data: *gain scores, course success,* and *course grade*. For all analyses, we tested two threshold levels for attendance, comparing students who had attended 2 or more sessions to those who attended 1 or fewer, and those attending 4 or more compared to 3 or fewer.

Our first method of data analysis was to examine students' changes from their first exam score to their final exam score (we refer to this change as a *gain score*, although the mean gains may be positive or negative). Examining students' gain from their first exam to the final exam allows us to reveal the effects of the intervention since the PLTL sessions began after the first exam. Note that the mean gains presented in the results are all negative in value, probably due to the increase in difficulty of the material between the first exam and final exam. For one section of the course, the instructor provided no exam data; students in that section are excluded from the *gain score* analysis. There were a further 131 students for whom either their first exam score or final exam score were missing. The majority of this missing data (128 students) was from students who did not take the final exam, generally because they had already withdrawn from the course. These students are also excluded from the *gain score* analysis. We compared the gain scores for the remaining 524 students using independent sample t-tests.

In contrast, the final two methods of data analysis include the entire population of 682 students in 24 sections. This allows us to examine the results of the intervention on all students, including those who withdrew from the course before the final exam. Indeed, since the primary purpose of the intervention was to improve overall student success and particularly to reduce the DFW rate, it is important to consider whether students who attended the PLTL sessions were more likely to earn a passing grade. For the second analysis, we compared the rates of course failure for students attending and not attending PLTL sessions using a chi-square analysis. This comparison is complicated by the fact that students who withdrew would certainly not attend any additional PLTL sessions, so attendance at more sessions is likely negatively correlated with withdrawal (even absent an effect on learning). Our final comparison takes into account not just whether students passed the course, but their final course grade. We computed each student's "quality points" (e.g. A=4.0, A-=3.7, B+=3.3, B=3.0, etc.) and compared the mean quality points for attenders vs. non-attenders using independent sample *t*-tests.

Results and Discussion

Results indicate that students who attended the PLTL sessions outperformed their peers on a variety of measures; however, it is difficult to generalize based on potential differences between the students who chose to attend and those who did not. This difficulty notwithstanding, results indicate that the program was successful in improving student outcomes. Below, we present our results for each of the three data analysis methods described above.

Gain Scores for PLTL Attenders vs. Non-Attenders

For each of the 524 students for whom first and final exam data was available, we computed their score *gain* as their final exam score minus first exam score. For students who attended at least two of the PLTL sessions (n = 62), the mean gain score was -4.40 (S.D. = 13.51), whereas for students who attended 0 or 1 session (n = 462), the mean gain score was -8.51 (S.D. = 13.83); this difference was statistically significant with $p < .05$. To better understand the difference in mean gain between the two groups, we computed Hedge's g to be 0.3491. The Hedge's g statistic measures the difference in means in units of pooled standard deviation. In our case, the mean gain for students who attended two or more PLTL sessions was 0.3491 pooled standard deviations higher than the mean gain for students who attended one session or none. Similarly, students who attended four or more sessions (n = 44) had significantly higher mean gain scores (mean gain = -3.26 with S.D. = 15.1) than those who attended three or fewer sessions (n = 480; mean gain = -8.46 with S.D. = 13.66; $p < .05$). It appears that students who participated in the PLTL sessions thus outperformed their classmates, at least in terms of the change between their performance before the sessions started and at the conclusion of the semester.

Course Success for PLTL Attenders vs. Non-Attenders

Our second method of data analysis compared course outcomes (pass vs. fail, with fail defined as students receiving a D or F grade or withdrawing). Table 1 presents passing rate date for students who participated in four or more PLTL sessions, compared with those who attended three or fewer; this comparison was near significant with $X^2(1, n = 682) = 4.53$, $p = .09$. This data indicates that attendance at the PLTL sessions might have somewhat improved students' course outcomes, but some of this effect is likely an artifact of the correlation between withdrawal and session attendance (because students who withdrew stopped attending sessions).

Table 1: Course Passing Rates by Attendance at PLTL Sessions

	Attended 3 or Fewer Sessions	Attended 4 or More Sessions	Total
Students (Percent) who passed the course (grade of A, B, C)	431 (69%)	43 (80%)	474
Students (Percent) who failed the course (grade of D, F, or W)	197 (31%)	11 (20%)	208
Total	628	54	682

Final Grade and GPA Comparisons for PLTL Attenders vs. Non-Attenders

Our final comparison takes into account not just whether students passed the course, but their final course grade. We computed each student's "quality points" (e.g. A=4.0, A-=3.7, B+=3.3, B=3.0, etc.) and compared the mean quality points for attenders vs. non-attenders. This comparison was near significant for students attending four or more sessions (n = 54; mean grade = 2.49; S.D. = 1.27) compared to those attending three or fewer (n = 628; mean grade = 2.15; S.D. = 1.36; p = .068). However, a statistically significant difference was revealed for students who attended five or more sessions (n = 44; mean grade = 2.56; S.D. = 1.21) compared to those attending four or fewer (n = 638; mean grade = 2.16; S.D. = 1.36; $p < .05$). This adds to the data indicating that students who attended PLTL sessions outperformed their classmates on a variety of comparisons, thus suggesting that the intervention was helpful to student success.

Other Differences Between Groups

Participation in PLTL sessions was voluntary, which raises the concern that factors other than PLTL session attendance could explain the observed difference in gain between the two groups. We proceeded to check for large differences in other factors between the group of students who attended one or fewer PLTL sessions and the group of students who attended two or more PLTL sessions. We investigated GPA prior to the Fall 2017 term and first exam score. We normalized differences in each factor by computing the Hedge's g statistic. For GPA prior to the Fall 2017 semester, we observed a Hedge's g of 0.1250. This shows that the mean GPA for students attending one or fewer PLTL sessions was 0.1250 pooled standard deviations higher than the mean GPA for students who attended two or more PLTL sessions. For first exam grade we observed a Hedge's g of 0.4474. This indicates that the mean first exam score for students who attended one or fewer PLTL sessions was 0.4474 pooled standard deviations higher than the mean first exam score for students who attended two or more PLTL sessions. These comparisons suggest that the students who attended the PLTL sessions were overall weaker in mathematics compared to those who did not. Although this limits the generalizability of our results in some ways (because we cannot infer what the effect of the intervention might have been on the students who did not attend), it does suggest that the program was successful in recruiting students who were at risk of failure and improving the outcomes of students who chose to attend.

Conclusion

These data indicate, on the whole, that students who attended the PLTL sessions outperformed their peers on several key indicators, including their final course grade, pass rate, and their final exam score (in comparison with their first exam score). These results indicate that the intervention was successful in improving individual student outcomes. However, our comparisons were complicated by the fact that attendance at the PLTL sessions appeared to be correlated with students' backgrounds as well as other factors not related to their learning (such as students who stopped attending sessions because they had already withdrawn). These complications limit the generalizability of our results, but are expected in any intervention study including an optional opportunity. The finding that students who attended the PLTL program began the semester somewhat weaker but still showed significant growth suggests that the program was successful. Given that faculty were involved only in the creation of the lesson plans and training the peer coaches, this provides evidence that the PLTL model can provide a sustainable path for supporting student learning and success in gateway mathematics courses.

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COMBINATORICS PROBLEMS: A CONSTRUCTIVE RESOURCE FOR FINDING VOLUMES OF FRACTIONAL DIMENSION?

Erik Tillema
Indiana University
etillema@indiana.edu

Jinqing Liu
Indiana University
jinqliu@iu.edu

Pavneet Kaur Bharaj
Indiana University
pkbharaj@iu.edu

Volumes of fractional dimension are a 7th grade standard in the Common Core State Standards in Mathematics (CCSS-M). Relatively little research has been conducted on how students' reason about volumes with fractional dimension. To address this dearth of research, we designed a teaching experiment based on a central conjecture that combinatorics problems could be a constructive resource in the development of volumes with fractional dimension. In this paper, we outline the reason for this conjecture. In our presentation, we elaborate on this conjecture by providing four cases of how pre-service secondary teachers (PSSTs) reasoned first with combinatorics problems, and then with volumes of fractional dimension. One contribution of the study is it expands and combines in novel ways empirically grounded theoretical constructs.

Keywords: Fraction Scheme, Combinatorics Problems, Volume, Multiplicative Concepts

Fractions and rational number knowledge is regarded as one of the most challenging domains at both the elementary and secondary levels (e.g., Behr, Wachsmuth, Post, & Lesh, 1984; Hackenberg, 2013; Norton, Wilkins, & Xu, 2018; Steffe, 2003) in part because many problems involve coordinating multiple units and multiple wholes (Lamon, 2007; Steffe & Olive, 2010). At the same time, the results of large-scale assessments (e.g., NAEP, TIMSS) also indicate that geometry and measurement are complex for both elementary and secondary students (Battista, 2007; Smith & Barrett, 2018). Fractions and volume initially come together in the Common Core State Standards in Mathematics (CCSS-M) in the 7th grade where students are asked to explore the volume of rectangular prisms with fractional dimensions.

To date, relatively little research has investigated how students reason about volumes of fractional dimension. The majority of research on fractions has focused on reasoning with one-or two-dimensional models (e.g., length, number line, circles, or rectangular area) (Lamon, 2007), while the majority of research on volume has been conducted in whole number contexts (Smith & Barrett, 2018; e.g., Clements, Sarama, & Van Dine, 2017). To address the dearth of research, we designed a teaching experiment for pre-service secondary teachers (PSSTs) whose purposes were to investigate: a. how they reasoned about volumes of fractional dimension; and b. how they would use what they learned to design lessons during their student teaching placement.

A central conjecture of the study was that combinatorics problems could support the PSSTs to develop discrete three-dimensional units as the product of three one-dimensional units (Tillema & Gatza, 2016, 2017), and that doing so would serve as a constructive resource in the development of volumes of fractional dimension. This conjecture was grounded in two results from prior research: a. students find it challenging to establish a "unit cube" as a multiplicative composition of one length, one width, and one height unit (Battista, 2007; Smith & Barrett, 2018); and b. students' multiplicative schemes for discrete quantity can be a constructive resource (rather than distractor) in their construction of fraction schemes (Hackenberg & Tillema, 2009; Steffe & Olive, 2010). The following research question guides this paper: How do PSSTs reason about volume of fractional dimension after they have used combinatorics problems to establish 3-D arrays?

Theoretical Framework

Broadly speaking, we use operations, schemes and concepts to provide an account of students' reasoning (Piaget, 1970; Von Glasersfeld, 1995). An operation is a mental action (e.g., iterating or partitioning) where operations are the "motor" of schemes. Schemes consist of three parts, an assimilated situation, an activity, and a result. The assimilatory mechanism of a scheme involves a person making an interpretation of a problem situation; the activity of a scheme entails a person using operations that transform the assimilated situation to the result. A person has constructed a concept when he or she no longer has to implement the activity of a scheme to take the result of the scheme as given in a problem situation.

Multiplicative Concepts, Volume, and Combinatorial Problem Situations

To investigate students' reasoning about volume, we build on Hackenberg's (2015) work in which she identifies three qualitatively distinct multiplicative concepts students use (see also, Steffe, 1992); the primary difference among each multiplicative concept is the number of embedded units that students take as a given prior to reasoning about problem situations. Cullen et al. (2017) have used these differences in their analysis of how students reason about volume problems involving whole numbers. They found that some students conceptualize volume as composed of individual unit cubes (i.e., units of one, akin to Hackenberg's first multiplicative concept); other students conceptualize volume as composed of rows of units cubes (i.e., treat five unit cubes as one row, akin to Hackenberg's second multiplicative concept); while other students conceptualize volume as composed of layers (i.e., treat four rows of five cubes as one layer, akin to Hackenberg's third multiplicative concept). Cullen and colleagues (2017) have also stressed the importance of students establishing a spatial structure in their construction of volumes of rectangular prisms with whole number dimension (see also, Battista, 2004). Here a spatial structure means a mental organization of volume units so that there are no gaps or overlaps.

The analysis of embedded units in Cullen and colleagues as well as in Hackenberg's work takes the "basic unit" as a unit of one. Smith and Barrett (2018) have pointed out that this elides the issue that individual volume units are themselves supposed to be a multiplicative composition of one length, one width, and one height unit. This distinction is theoretical in nature, but empirical findings suggest that students may not treat volume units as if they are a multiplicative composition of one length, one width, and one height unit (Battista, 2007). The CCSS-M approaches this issue by defining one cubic unit of volume as a unit that is one length unit by one width unit by one height unit. In our work, we have contended that this definition is appropriate for people who have already constructed a cubic unit of volume but is insufficient for supporting a person's construction of it (Tillema & Gatza, 2016). Thus, for this study, we followed an approach that we have used previously with high school students in which we presented them with combinatorics problems to help them establish discrete three-dimensional units from three discrete one-dimensional units (Tillema & Gatza, 2016, 2017). We have called these discrete three-dimensional units triples where, for example, one triple might be an outfit that a student has created in the Outfits Problem.

> *Outfits Problem*. You have 4 shirts, 3 pants, and 2 belts. An outfit is one shirt, one pants, and one belt. How many outfits could you create?

In creating outfits to solve this problem, a student might first pair one shirt with one pants, and then pair the shirt-pant outcome with one belt to create what we call a triple. We call such a unit a triple because it contains three units, but is considered a single unit, one outfit. We consider these operations to entail students in establishing discrete three-dimensional units from three discrete one-dimensional units.

Moreover, in prior studies, high school students have used such problems to establish a spatial structure for 3-D arrays by locating individual triples and regions (e.g., all triples that have the first shirt) (Tillema & Gatza, 2016, 2017). The reason combinatorics problems can be conducive for establishing a spatial structure for 3-D arrays is that they can involve ordering the units within a composite unit (e.g., a first shirt, a second shirt, etc.), and ordering the triples (e.g., shirt in the first position; pants in the second position; belts in the third position). Ordering the units within a composite unit can result in an order for the units along a single axis of an array and ordering the positions in a triple can result in an order for the axes of an array.

Fraction Schemes with Length Quantity

Our analysis also uses two key constructs from research on students' fraction schemes—their construction of iterative fraction schemes and their construction of fraction composition schemes. Steffe and Olive (2010) have found that students using the third multiplicative concept can construct an iterative fraction scheme with length quantities (see also, Hackenberg, 2015). Students who have constructed an iterative fraction scheme (IFS) with length quantities can, for example, partition a length into five parts, disembed (remove from the whole) one of those parts, and iterate it four times to create four fifths as a length. IFS students interpret the relationship between one fifth of a length and four fifths of a length to be multiplicative (four fifths is four times one fifth). Moreover, IFS students are able to establish this multiplicative relationship for improper fractions like six fifths; a length that is six-fifths is six times one fifth. This conceptualization of improper fractions is not easy for students to establish and is a hallmark of students who have constructed an IFS with length quantities (Hackenberg, 2007).

Steffe (2003) identified that a central operation in students' construction of fraction composition schemes with length quantities is recursive partitioning. He defined recursive partitioning as partitioning a partition in service of a non-partitioning goal. For example, a student who engages in recursive partitioning might respond to a request to find one third of one half by partitioning one unit of length into two parts, and then partition each part into three mini-parts. She might then respond that one mini-part is one-sixth because she could iterate it six times to make one unit of length. These constructs were central to our analysis of students' reasoning about volumes of fractional dimension.

Methods and Methodology

The data for this study was collected using teaching experiment methodology (Confrey & LaChance, 2000; Steffe & Thompson 2000). This methodology involves a researcher engaging students in problematic situations aimed at helping them in the construction of schemes and operations (Hackenberg, 2010). A researcher designs initial problems for students based on her prior experience working with students in the domain, and her understanding of prior relevant research. This initial design work is supported by guiding conjectures. Once a researcher begins interactions with students her goal is to develop working models of students' reasoning (Steffe & Weigel, 1994). Throughout the experiment, the researcher refines these working models by establishing conjectures about the affordances and constraints of students' current reasoning and designing problem situations to test these conjectures. A researcher can establish these conjectures both in between teaching episodes and in the moment of interacting with students.

The broad goal of the teaching experiment was to investigate how the PSSTs developed a combinatorial understanding of common algebraic identities (e.g., $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$). As part of supporting this work the researchers used a quantitative approach in which students represented sets of outcomes in combinatorics problems as 3-D arrays, and subsequently represented volumes of fractional dimension for these algebraic identities. The PSSTs used snap cubes as a representational tool throughout the teaching episodes (Figure 1).

The four participants for the study were enrolled in the second semester of a secondary mathematics methods class at a Midwestern university during the fall of 2018. They were selected based on willingness to participate in regular meetings over the course of the 2018-2019 academic year. During Fall 2018, they worked as a pair in 13 teaching episodes; each episode lasted for 60-90 minutes and was video recorded using three cameras—one to capture the interaction between the researcher and the PSSTs and two to capture student work.

For data analysis, all three authors watched relevant video from the entire data set. They used a four-column table as a guide for analysis. The four-column table included timestamp, brief characterization of 3-D reasoning, low inference description of what happened, and inferential analysis of the PSSTs' reasoning (Saldana, 2013). Each row in the table was a single utterance, a turn of talk, or a period of talk focused around a specific idea. The authors met weekly to discuss ongoing conjectures and reconcile differences in their interpretations of the data. These discussions were aimed at developing internal consistency and coherence with the data, and for consistency and coherence with prior research findings.

	

Figure 1: A Single Snap Cube (left) and Eight Snap Cubes Configured as a Volume (right)

Findings

The PSSTs' solution of combinatorics problems revealed differences in the way they established discrete three-dimensional units. All four of PSSTs entered the teaching experiment having constructed a recursive scheme for the solution of three-dimensional combinatorics problems. The recursive nature of their schemes meant that they could take the result of their scheme as input for a further application of their scheme; for example, they were able to take each pants and pair it with all shirts to produce all shirt-pants outcomes, and then take each shirt-pants outcome and pair it with all belts to produce the set of all possible triples.

However, not all of them produced the same multiplicative relationships in this process. One PSST had interiorized a multiplicative relationship that can be symbolized as, for example, 1 x 4 = 4. This can be thought about as one discrete 1-dimensional unit times four discrete 1-dimensional units produces four discrete 2-dimensional units. She could then iterate these operations to produce all of the possible 2-dimensional units in her solution of a problem. Once she produced all possible 2-dimensional units, she took each individual 2-dimensional unit (i.e., a 1 x 1), and paired it with all 1-dimensional discrete units from the third composite unit. This can be symbolized as (1 x 1) x 5 = 5, where the 1 x 1 in the statement symbolizes a pair. These operations were sufficient to establish a row in a 3-D array as a unit of three dimensional discrete units. The other three PSSTs established different multiplicative relationships as they created three-dimensional discrete units, which will be discussed in the presentation.

During their solution of combinatorics problems all four PSSTs worked to coordinate the multiplicative relationships that they produced with a spatial structuring for a 3-D array. None of the four entered the teaching experiment able to locate points and regions in three-dimensional space in situations where they needed to coordinate orthogonal "planes" to determine the location of a point or region. This issue, along with how these problems supported students to construct volumes of fractional dimension, will be discussed in the presentation.

Acknowledgments

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PROMOTING AN AUTHENTIC EXPERIENCE OF STATISTICAL PRACTICES IN STATISTICS EDUCATION

Layla Guyot
Texas State University
l_g244@txstate.edu

Alexander White
Texas State University
whiteale@txstate.edu

The demand for statistical skills is growing in many different fields and sectors, and the employment of statisticians is expected to increase drastically. However, employers are experiencing difficulties in hiring mainly because there is a misalignment between the skills acquired in academic settings and the skills required of a statistician at the workplace. Indeed, statisticians develop practices that grow out of experience (Pfannkuch & Wild, 2000). In order to investigate these practices, statisticians engaged in a reflection on their experience. Preliminary results revealed predominant skills from the perspective of the practitioners and can inform how to develop curriculum materials that better promote authentic practices in statistics education.

Keywords: Data Analysis and Statistics, Curriculum

From actuarial science to zoology, the field of statistics has applications in many diverse disciplines and in the era of big data, statistical skills are increasingly valuable on the job market. Even though the number of students seeking degrees in statistics have been increasing over the past few years, the supply of professionals with statistical skills still does not match the thriving demand (American Statistical Association, 2015) and employers are experiencing difficulties in hiring statisticians across disciplines and sectors. In addition, most employers are looking for experienced statisticians which makes the transition is even more challenging for graduates in statistics.

Purpose

The purpose of this research study is to promote an authentic experience of statistical practices in education to ease the transition to the workplace. Previous research has shown that the transition to the workplace is challenging for graduates in general (Grosemans, Coertjens, & Kyndt, 2017), and in particular for graduates in statistics because there is a misalignment between what they learn in academic settings and what is required of them at the workplace (Van der Berg, 2017). Indeed, statisticians develop practices that grow out of experience and are relevant to their discipline and context (Pfannkuch & Wild, 2000). We need to understand how statisticians develop these practices at the workplace and what practices should be included in academic settings to facilitate the transition. The role of mentors was found to be crucial to support recent graduates in their transition (Grosemans et al., 2017) that is why mentors of statisticians such as managers, senior statisticians or educators, need also to be involved. More specifically, the following questions guide this research:

1. What practices are important for the role of statistician?
2. How do statisticians learn and develop these practices while in transition to the workplace?
3. What key statistical practices should be included in statistics education as recommended by statisticians and their mentors?

Background and Theoretical Framework

The transition to the workplace often presents challenges because it involves adapting to a new environment, with a new role, different rules, and learning new practices. An example of difficulties encountered by individuals in transition is the misalignment between the learning outcomes in academic settings and the expectations of the workplace.

Transition to the Workplace for Statisticians

Harraway and Barker (2005) followed 913 recent bachelor's, master's and doctoral degree holders into their early career and identified discrepancies between statistical techniques taught at the university and the ones used at the workplace. Graduates agreed they needed more training in particular for regression, multivariate methods, research design, and power analysis.

Through a survey, Van der Berg (2017) investigated the transition experienced by 95 interns at the end of their training. Most interns agreed that they acquired the appropriate *statistical knowledge* before entering the workplace but did not acquire the appropriate *statistical skills* needed at the workplace. Therefore, practices acquired in academic settings do not align with practices required at the workplace. Interns listed 21 skills required at the workplace that were not taught: for example, data collection, questionnaire design, communication, writing skills, and using statistical software. In addition, mentors who accompanied interns during their transition agreed that even though interns had some theoretical knowledge, they were lacking practical skills and had to learn how to perform most of the statistical tasks. In particular, they recommended that statisticians should primarily develop skills before entering the workplace.

Therefore, a misalignment between statistical practices acquired in academic settings and statistical practices required at the workplace has been identified. We need to understand how statisticians develop practices to adapt to the workplace.

Theoretical Framework

To understand how statisticians develop practices at the workplace, this study draws on activity theory (Vygotsky, 1978; Engeström, 1987; Konkola, Tuomi-Gröhn, Lambert, & Ludvigsen, 2007) and more precisely on the concept of boundary crossing and boundary objects (Akkerman & Bakker, 2011; Star & Griesemer, 1989). Academic settings and the workplace can be conceptualized as *activity systems* whose goal is to develop statistical practices (the *object* of the activity). The activity is mediated by *tools* and regulated by *rules*, and the *division of labor* ensures the distribution of tasks and authority within a *community* and a *subject*, or individual of focus. For example, in academic settings, a student (*subject*) develops statistical practices (*object*) using curricula materials (*tools*). The *community* of students, teachers, and advisors work together ensuring that the student meets the degree requirements (*rules*). The teacher defines the responsibilities (*division of labor*) of each participant. At the workplace, a statistician (*subject*) performs statistical practices (*object*) using statistical software (*tools*) for example. The collaboration between statisticians, managers, and clients forms a *community*, following regulations (*rules*) and a specific organization of the different tasks (*division of labor*). Thus, statistical practices are considered from different perspectives (student and statistician) and are not necessarily aligned between the two activity systems of academic settings and the workplace.

The misalignment between practices acquired in academic settings and statistics required at the workplace creates challenges that statisticians learn to overcome. The concept of *boundary crossing* represents the process of establishing continuity between two activity systems and the involved challenges, or boundaries. *Boundary objects* can function as bridges between systems (Akkerman & Bakker, 2011; Star & Griesemer, 1989) and assist boundary crossing. In the context of this study, statistical practices are characterized as boundary objects because they could be utilized to facilitate the transition boundary crossing and should be coordinated between the two activity systems. Figure 1 represents academic settings and the workplace as activity systems and statistical practices as boundary objects.

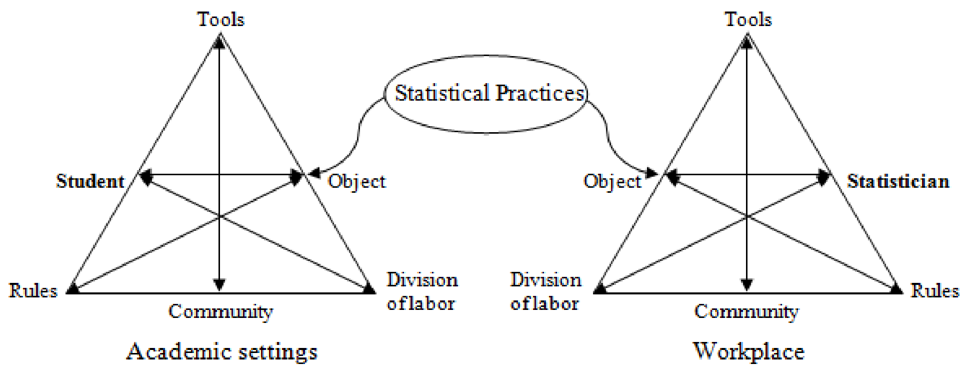

Figure 1: Activity Systems and Boundary Objects

Through the lens of activity theory with the concepts of boundary crossing and boundary objects, this study explores the transition experienced by statisticians as they move between the activity system of academic settings and the activity system of the workplace. We need to analyze how participants identify and coordinate practices from each activity system, and how they can eventually transform practices. The collaboration of statisticians, mentors, and educators will help build boundary objects, namely statistical practices, to facilitate the transition between academic settings and the workplace.

Methods

In order to investigate what important statistical practices are performed at the workplace, statisticians and their mentors reflected on their own experiences. Statisticians and mentors were recruited during a workshop presented at conferences focusing on statistical practice. During the workshop, participants engaged in a sorting task, derived from Q-methodology which aims at exploring inner perspectives and the possible differences and similarities of perspectives within a community (Brown, 1980). Participants sort a list of 24 practices into an array of 9 columns (see an example of the shape in Figure 2) from the least (rank 1) to the most (rank 9) important according to their own perspective of the role of statistician. The list of practices is based on a review of the literature about what statistician do and a review of required skills on job offers (Table 1). Participants could add up to 6 practices relevant to them.

Table 1: List of 24 Practices Classified in 5 Categories

Process of data analysis			Traits	
Design	Focus on data	Techniques	Interpersonal skills	Personal skills
- Using knowledge of the context - Translating a real problem into a statistical form - Designing studies - Preparing sampling frames	- Collecting data - Creating / Maintaining databases - Cleaning data - Using statistical software - Producing visual representations - Interpreting data	- Researching appropriate statistical methods - Developing new statistical methods - Applying statistical methods - Using advanced mathematics	-Working independently - Participating in teams / Collaborating - Communicating in writing - Communicating orally - Consulting with a client - Communicating interpretations of statistics to non-statisticians	- Being curious / willing to learn - Being skeptical / critical - Meeting deadlines - Considering ethical issues

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At the time of writing this paper, the sorting task had been conducted at four conferences in the US, France, and Japan. Overall, a total of 63 completed sorting tasks were collected from participants with different experiences (statisticians, educators, students). Each sorting task represents a perspective from a participant, arranging practices in a specific order. The analysis of sorting tasks in Q-methodology is based on data reduction, reducing all of the perspectives to identify patterns of thought. To reduce the data we used principal component analysis (PCA) where participants are correlated instead of correlating variables as it is the case in regular PCA. The results give a smaller number of perspectives that summarize different patterns of thoughts among the participants.

Preliminary Results and Discussion

We identified 3 different perspectives among our participants (Figure 1). Perspective 1 summarizes the sorting tasks of 24 participants who emphasized the importance of traits, in particular on interpersonal skills (dark orange) with "Communicating interpretations of statistics to non-statisticians" ranked as the most important (rank 9). The 24 participants contribute positively to this perspective, meaning that they all shared similar views.

The sorting tasks of 20 participants were summarized by Perspective 2 which focuses on the process of data analysis, in particular on techniques (dark blue) with the practices of "Researching" and "Applying statistical methods" being ranked as 8. One participant contributed negatively to this perspective, meaning that their view opposed the views of the other 19 participants which are represented by Perspective 2.

The third perspective is a contrast between the process of analysis (shades of blue) and traits (shades of orange). There are 7 participants contributing to this perspective, 3 contributing negatively and 4 contributing positively, meaning that they have opposite views for the importance of the process of analysis (in particular, focusing on data and designing studies) versus traits (especially personal skills).

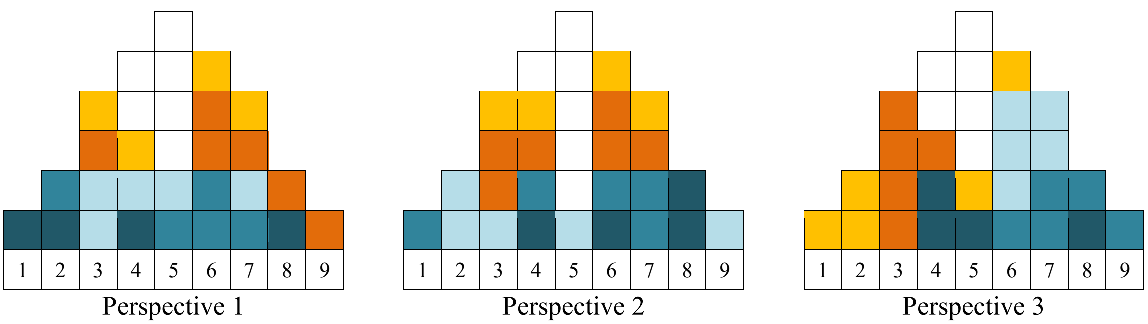

Figure 2: Statistical Practices Ranked from Highest to Lowest Mean

Additional practices mentioned by the participants included: training constantly and reading articles (added by 10 participants), mentoring junior statisticians and providing feedback (added by 10 participants), and managing projects (added by 9 participants).

Conclusion

As educators of future statisticians, we need to be aware of practices required at the workplace. In particular, we need to develop curricula materials that give students an authentic experience of important practices as identified by statisticians.

Next, interviews of statisticians and their mentors will reveal how the important practices identified through the sorting task were developed at the workplace. The tools, rules, community, and division of labor involved for each practice will be identified and will help define how these practices can be included in academic settings to ease the transition to the workplace for statisticians.

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TEACHERS' STRATEGIES FOR GENERATING A MATHEMATICAL DEFINITION FROM A LIST OF ATTRIBUTES

Susanna Molitoris Miller
Kennesaw State University
smolitor@kennesaw.edu

Writing definitions is a useful tool for mathematical learning. This study examines pre-service and in-service teachers' responses to an interview task which asks them to generate a mathematical definition based on a given list of attributes rather than providing a memorized definition for a named shape type. The strategies employed by participants reveal three main kinds of informal reasoning used when synthesizing multiple pieces of information into a single concise definition; identifying and defining, building-up attributes, and attribute thinning. These strategies were used by various participants either as exclusive approaches or in conjunction with one another. Findings provide implications concerning the role participants' concept images may play in the defining process.

Keywords: Geometry and Geometrical and Spatial Thinking, Reasoning and Proof, Teacher Knowledge

Writing mathematical definitions has been identified as a useful tool for the professional development of teachers (Leikin & Wimicki-Landman, 2001). However, it may be challenging to provide an authentic defining experience for concepts such as quadrilateral types because most teachers enter professional development programs with pre-existing ideas about these shapes. This study explores how teachers in professional development settings employ strategies to generate an accurate mathematical definition based on a list of attributes which are true of an unnamed shape type.

Literature and Theoretical Framework

Tall and Vinner (1981) describe two ways of knowing a mathematical concept, *concept image* and *concept definition.* The *concept definition* is "a form of words used to specify that concept" (p. 152, Tall & Vinner, 1981). The *concept image* is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes… built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (p. 152, Tall & Vinner, 1981). Students rely more heavily on their concept images rather than concept definitions when solving problems (Tall & Vinner, 1981), but textbooks often introduce geometric concepts with a concept definition rather than opportunities to develop rich concept images (Usiskin & Griffin, 2008). The act of defining is considered an important component of geometric reasoning that precedes formal proof (Burger & Shaughnessy, 1986). Preservice teachers have demonstrated a mismatch between their concept images and concept definitions (Cunningham & Roberts, 2010).

de Villiers, Usiskin and Patterson, describe a mathematical definition as having, *necessary*, *sufficient*, and *minimal* conditions (p. 193, 2009). *Necessary* conditions "appl[y] to all elements of the set," (de Villiers, Govender, & Patterson, 2009, p. 193). *Sufficient* conditions describe "indications of the concept," (Winiki-Landman & Leikin, 2000, p.17). And *minimal* conditions do not include any "redundant properties" (de Villiers, Govender, & Patterson, 2009, p. 196). While a definition may function containing only necessary and sufficient conditions, definitions which are also minimal are considered “economical,” (de Villiers, Govender, & Patterson, 2009, p. 196).

Methods

The ideas of concept image and concept definition suggest students should learn about many different aspects of a geometrical concept before being presented with a definition. The challenge of teaching a topic in light of previous exposure is not specific to teacher education as the presence of shape types spans the entire PreK-12 curriculum. Thus, the majority of learners arrive in geometry classrooms with some pre-conceived notions about the content In this study I explore what kinds of informal reasoning may be employed to deduce a mathematical definition of an unknown shape type from a list of known attributes. I also consider what role, if any, the learner’s pre-existing concept image may play in this process.

Participants included thirty-two pre-service teachers majoring in early childhood education, two pre-service teachers majoring in secondary mathematics education, and five in-service teachers pursuing a master’s degree in mathematics education. All were enrolled in one of two geometry content courses being taught by the investigator. All students received instruction which unpacked the necessary, sufficient, and minimal characteristics of a mathematical definition as described by de Villiers, Govender, and Patterson (2009) as well as opportunities to become familiar with all attributes included in the task. The data were collected in one-on-one task-based interviews which were video recorded and transcribed, including gestures, for analysis. Participants were given a pen, paper, and a set of sample quadrilaterals they may use when completing the defining task. Data from all participants were combined for analysis because the limited sample size and no observed differences across groups.

The task began by providing the participant with a list of attributes for “Shape Type 1.” The participant was told that all attributes were true for a specific group of shapes and that this group of shapes may be a new or a familiar group. The properties given were: is simple, is closed, is a polygon, is a quadrilateral, has two pairs of opposite congruent angles, has two pairs of parallel sides, has two pairs of opposite congruent sides, has two pairs of adjacent congruent sides, has at least one line of reflectional symmetry, and has at least 180° rotational symmetry. Collectively, these properties describe members of the rhombi family. This particular shape was chosen because it was identified as one of the quadrilateral types with which students were least familiar (Molitoris Miller, 2018).

Participants were asked to read the list and invited to ask clarifying questions about the given properties. Next the participant was asked to write a “best approximation” of a definition for “shape type 1.” Participants were reminded of the tools at their disposal, as well as the opportunity for revisions as their thinking evolves. Participants were encouraged to talk through their thinking out loud. The interviewer offered no guidance in completing the task but would note things the student said and ask for clarification or elaboration in order to better understand the participant’s thinking.

After composing a definition, participants were asked to verify if that was their final response and then allowed to make revisions as desired until they felt they reached a final answer. The interviewer asked them to once again briefly describe their thought process as they wrote the definition. After this description, the interviewer repeated back a summary of her understanding of the described solution process for the participant to verify or provide clarification. Interviews lasted ten to thirty minutes.

Interview data was analyzed and a-priori coding was preformed to describe the strategies employed by participants as they completed this task. Initial inspiration for the codes developed as the investigator noticed trends during data collection, and more detailed consideration of each individual interview ultimately revealed three main strategies for developing a definition from a list of attributes.

Results

Participants exhibited three primary strategies for developing a mathematical definition from the given list of attributes; identifying and defining, building-up, and attribute thinning. Some participants used exclusively one strategy for the entirety of their interview, while others created their final answer using a combination of techniques.

Identifying and Defining Strategy

Some participants approached the task by attempting to determine what the mystery shape type was, name it, and provide a known definition of this named type. This approach had mixed results. Some participants read through the list and almost immediately declared the same was a square. When asked how they knew it was a square, participants drew or located an image of a square and verified that all of the properties given were true for a square. One such student stated "If you draw a square and you were to label each part, then you can see by labeling it, and showing your students physically how you found this out, and being like, 'it is,' you have to physically show it." Next they provided what seemed to be a commonly known definition for a square, most often a variation of "a quadrilateral with all right angels and all sides equal." Other participants correctly identified the "Shape Type 1" as rhombi. Some of these students did not articulate the thought process that lead to this conclusion. Other participants identified that mystery shape type as a rhombus by using a sieve sorting and then provided a correct definition for a rhombus, such as, "A quadrilateral with all sides equal."

Building-Up Strategy

Students who employed the building-up strategy crafted their definition by adding properties to the definition one-by-one and considering which other properties could be deduced from the property they put into the definition. Participants most often used this strategy for condensing properties such as "quadrilateral," "simple," "closed," and "polygon." In these cases, such as the work done by Macy, participants added "a quadrilateral that…" to their definition and then crossed out the other named attributes saying, "If it's a quadrilateral then it has to be a polygon [crossed out polygon from list]. And if it's a polygon then it must be simple and closed [crossed out simple and closed]." This first deduction is based exclusively on properties nested within one another's definitions and is therefore relatively straight-forward.

Deductions were also made for more complicated relationships. These represented very informal reasoning which seemed to be based on intuition and gut feelings rather than formal mathematical rigor. For example:

> I'm thinking that these two also have to have something to do with each other. With if they have opposite congruent angles and they have opposite congruent sides like they would have to be the same length here and here [noting opposite pairs of sides on sketch of rhombus] so they would join at the point here [showing a pair of opposite congruent angles] and make two of the same angles [gesturing toward other pair of opposite angels] if they are oppositely congruent.

A participant engaging in the building-up strategy would continue adding properties to their definition until they felt all the properties in the list had in some way been accounted for by one or more of the properties in their definition.

Property Thinning Strategy

Some participants began the task by writing a long and complicated definition which was basically a litany of all of the properties provided on the sheet. The participant then considered each property on the list and if or how the definition would function differently if that property were removed. This strategy often employed very similar logical arguments to those unpacked in the building up strategy: some removed properties based on their inclusion as part of anther property's definition, and some removed properties based on some intuitive reasoning or feeling about the relationship between two or more seemingly unrelated properties.

There were two notable distinctions between the enactment of these strategies. (1) The property thinning strategy seemed to be more effective for removing properties which were not directly related to others, such as those concerning symmetry which were not needed in the definitions but were harder to logically connect to the other side and angle based properties. (2) On the other hand, the property thinning strategy did not lend itself to situations where more than one related property could be synthesized into a third shorter, but not previously listed property. One such example is the combined effects of "two pairs of opposite congruent sides" and "two pairs of adjacent congruent sides," which can be combined to deduce that all sides must be congruent.

Combinations of Strategies

As noted earlier, not all participants' solution strategies conformed strictly to one of the strategies described above. Some participants named the mystery shape as a square, but rather than providing a definition of a square from memory, they returned to the given list of attributes and employed one of the other strategies for condensing the property list into a definition. Others began by building up the definition as best they could, and then wrote a definition which contained the built-up properties as well and any that were not yet accounted for and thinned that. The majority of participants engaged in some form of property thinning when revising their definitions before settling on a final submission.

Implications and Conclusion

These results provide a few insights into the defining process experienced by the participants. The majority of definitions produced contained more than a minimal final list of attributes. This suggests that the relationship between the complex collection of attributes which are true about a particular shape family and the concise concept definition which names that shape family is not always obvious to learners. For example, some learners may not realize that having all congruent sides implies that the shape must have two pairs of parallel sides. This challenge would undoubtedly also hinder the recognition of hierarchical relationships among shape type such as the fact that all rhombi are parallelograms, a noted challenge in the study of geometry among both teachers and students (Fujita, 2012). Likewise, participants who struggled to condense, "two pairs of opposite congruent sides," and "two pairs of adjacent congruent sides," by recognizing that they collectively imply that all sides must be congruent, would likely struggle to fully grasp the relationship between parallelograms and kites, namely that their intersection forms the group known as rhombi.

We also see the powerful role played by participants' pre-existing conceptions of shape types and their attributes. The identifying and defining strategy highlights a strong urge to name the shape being considered. Naming the shape type was not a required or prompted part of the interview task, yet many participants began their work by attempting to name the shape type which in turn seemed to elicit strong ideas about what that shape should be and distracted from the list of properties.

All strategies rely on the characteristics of mathematical definitions requiring a necessary, sufficient, and minimal set of attributes. Elements of these structures of definitions drive each of the individual approaches. When employed effectively, the identifying and defining strategy depends on the given list of properties as a necessary and sufficient collection of attributes in order to identify the shape type. The building-up strategy seeks to create a minimal definition by strategically assembling a short set of properties. And the attribute thinning strategy strategically removes attributes from the given necessary and sufficient set to achieve a more minimal set of properties in the final result. All three of the strategies used rely on these key characteristics of mathematical definitions and thus engaging with these strategies promises to effectively support and develop further understanding of both the nature of shapes being defined and that of mathematical definitions in general.

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WHERE TO START? THIRD GRADERS' MEASUREMENT CRITIQUES

Laura Bofferding
Purdue University
LBofferd@purdue.edu

Ana-Maria Haiduc
Purdue University
ahaiduc@purdue.edu

Mahtob Aqazade
Purdue University
maqazade@purdue.edu

Lizhen Chen
Purdue University
lizchen@purdue.edu

Sezai Kocabas
Purdue University
skocabas@purdue.edu

Keywords: Measurement; Cognition; Elementary School Education

When measuring with nonstandard, discrete units, some early elementary students will leave gaps between the units (Clements & Sarama, 2009). When measuring with rulers, some students report the number at the end of the object as its length regardless of the starting position of the object (Bragg & Outhred, 2004; Nunes, Light, & Mason, 1993), interpret the numbers on the ruler as points as opposed to accumulating units, and count from one instead of zero (Bragg & Outhred, 2004; Clements & Sarama, 2009). One method that can illuminate students' conceptions about mathematics problems and potentially help them confront their confusions is analyzing incorrect worked examples (e.g., Booth, Lange, Koedinger, & Newton, 2013; Durkin & Rittle-Johnson, 2012). Therefore, in this study, we investigated the following research questions: When analyzing worked examples of incorrect measurements, what did third graders identify as incorrect? How did this relate to their ability to identify the correct measurements?

Participants included 37 third graders who saw pictures of items incorrectly measured with broken rulers (or cubes) and were asked to identify and correct the mistakes: (#1): egg spanning 4 to 7 on the ruler, incorrect answer: 7 units; (#2): screwdriving spanning 2 to 4, incorrect answer: 4 units; (#3): screwdriver spanning 1 to 9, incorrect answer: 9 units; (#4): screwdriver spanning the edge of ruler to 4.5, incorrect answer: 4.5 units; (#5): screwdriver measured with four blocks with gaps, incorrect answer: four blocks long. The starting points of the objects on the rulers were varied to probe students' conceptions of where to start measuring. We recorded the mistakes they identified and their reasoning for what the measurements should be.

Students' performance in identifying the length of the objects improved as the starting point of the object being measured was positioned closer to zero on the ruler. Students who thought measurements should start at one on the ruler thought the answer for (#2) would be three units, because if the screwdriver had started at one on the ruler, it would end at three. However, students who originally said the object should start at *one* or *the edge* (which never aligned with zero) when measuring, changed their answer when they saw a screwdriver positioned at one; in these cases, they said it should be positioned at zero. Surprisingly, 54% of students, many of whom said the screwdriver positioned at one should start at zero in the previous problem (#3), thought the screwdriver in (#4) correctly showed a measurement of 4.5 units. With the blocks, only 54% of the students were able to correctly determine that the screwdriver would be longer than four blocks once the gaps between the four blocks were removed; six students said that after the gap or alignment problem was fixed, the screwdriver would still be four blocks long, and surprisingly, four students said it would be less than four blocks long! Overall, students' conceptions of zero seem tied to the ruler's edge even when it was not marked at the edge.

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ESTRATEGIAS CUALITATIVAS EN LA RESOLUCIÓN DE SITUACIONES DE COMPARACIÓN DE PROBABILIDADES

Maribel Aguas-Hidalgo Ricardo Quintero-Zazueta
Centro de Investigación y de Estudios Avanzados (Cinvestav-IPN, México)
aguashidalgomm@yahoo.com.mx quintero@cinvestav.mx

Palabras clave: números racionales, probabilidad

En problemas de comparación de probabilidades están implicadas las proporciones. Para Piaget e Inhelder (1969) "…la noción de proporción se inicia siempre de una forma cualitativa y lógica, antes de estructurarse cuantitativamente". Por lo expuesto, consideramos que al tratar situaciones de probabilidad clásica con o sin proporcionalidad en un contexto de urna es necesario analizar no sólo las estrategias cuantitativas como las presentadas por Piaget e Inhelder (1951), Falk (1980), Green (1988) o Alatorre (1994), sino también las estrategias de corte cualitativo que se derivan de componentes lógicos que subyacen en las comparaciones cuantitativas para la toma de decisiones.

Las formas de razonamiento cualitativo usadas en la resolución de problemas son medios alternativos a los procedimientos cuantitativos (López, 2003). Estos medios alternativos, de corte cualitativo, son los que nos proponemos analizar desde los argumentos lingüísticos que estudiantes de educación secundaria ofrecen en la resolución de problemas de comparación de probabilidades, y que incluyen connotaciones como: 'más que', 'menos que', 'mayor que', 'menor que', 'más grande que', 'más pequeño que', sin explicitar por cuánto o cuántos. Es decir, desde un análisis de los componentes lingüísticos planteamos hacer una valoración del pensamiento cualitativo que ponen en acción estos estudiantes.

Para el desarrollo de esta investigación educativa se adopta un método cualitativo con un caso instrumental (Stake, 1999). El caso lo conforma un grupo de 35 estudiantes de tercer grado de educación secundaria. Para el diseño de las actividades, además de incluir el modelo de urna, se analizaron los tipos de problemas de razonamiento proporcional. En cuanto a la clasificación de Lamon (1993), los problemas diseñados corresponden a los de parte-parte-todo, que Özgün-Koca (2009) incluye en su estudio dentro de los de comparación numérica. Para el análisis de los resultados, una vez determinadas las relaciones de orden que se establecen con las variables que se implicaron: casos favorables, casos desfavorables o casos posibles, se identificaron las estrategias "dentro" o "entre" de solución para hacer la elección, centrando nuestra atención en los argumentos lingüísticos que no denotaban la cantidad sino la cualidad (más que, menos que,…) de los conjuntos comparados.

Con los resultados que arroje el estudio de las estrategias cualitativas queremos sustentar la pertinencia de incluir problemas de tipo cualitativo en la educación secundaria que, si bien subyacen a los problemas cuantitativos, no se contemplan en los Programas de estudio de este nivel educativo. En otras palabras, consideramos necesario valorar al razonamiento cualitativo como una herramienta de análisis, distinta a la numérica pero complementaria, que puede llevar a la simbolización algebraica de la expresión verbal, más allá del pensamiento aritmético mediático y que se podría favorecer; por ejemplo, desde situaciones de comparación de probabilidades cualitativas.

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QUALITATIVE STRATEGIES IN THE RESOLUTION OF SITUATIONS OF COMPARISON OF PROBABILITIES

This paper analyzes the influence of qualitative thinking and its relevance in decision-making in situations involving the comparison of probabilities. To this end, classic probability scenarios were designed with the urn model. In each situation two ballot boxes were raised with simple extraction, involving or not relationship of proportionality, to third-degree students of secondary education. The analysis was carried out through the categorization of the results based on the relationships established between the components of the ballot boxes (favorable, unfavorable and possible cases). The study showed that, when comparing probabilities, students not only resort to quantitative comparisons, but to other relationships where a qualitative thinking is involved influencing the decisions they take.

IDENTIFYING U.S. ELEMENTARY STUDENTS' STRENGTH AND WEAKNESS IN GEOMETRY LEARNING FROM ANALYZING TIMSS-2011 RELEASED ITEMS

Jinqing Liu
Indiana University
Jinqliu@iu.edu

Keywords: Geometry Learning, U.S. Elementary Students, Strength and Weakness, TIMSS

Rather than focus on results showing that U.S. elementary students have consistently fallen behind East Asian students in in geometry based on the Trends in International Mathematics and Science Study (TIMSS) (Mullis, Martin, Foy, &Hooper, 2016), this poster session focuses on identifying the U.S. students' strengths and weaknesses in geometry as shown in TIMSS scores so as to use their strengths to leverage their achievement in areas in which they need support.

The major data in this study are the 24 Grade 4 TIMSS released items that fall in the geometry content domain (retrieved from https://timssandpirls.bc.edu/timss2011/international-released-items.html). To define U.S. students' strengths and weaknesses in geometry, I employed the International Average percent correct and the percent correct of Hong Kong students (consistently ranked in the top 3 in TIMSS) as benchmarks, with which I compared the U.S. percent correct. I then grouped the 24 items into three categories: items on which U.S. students performed weakly (lower than the international average percent correct or below 50%), items on which they performed normally (slightly higher than the international average correct percent, 50% to 70%), items that U.S. students performed strongly (slightly lower than Hong Kong student' performance, 70% to 100%). At last, I analyzed the content and cognitive domain of each item in each group to identify emerging themes.

Initial results indicate that U.S. students performed weakly on five geometry items, three that assessed application of knowledge and two that assessed reasoning. These items involved measurement, such as figuring out the length of a folded string (item ID: M031004; 20% correct), the distance Ruth walk around the edge of a square land (Item M041155; 43% correct, compared to the international average of 50% and the Hong Kong average of 82%). U.S. students performed normally on 10 geometry items, six testing for knowledge and the rest application of knowledge. These items assessed students' knowledge of 2-D and 3-D shapes, such as identifying shapes in a given picture (Item M041143) or drawing according to specifications (e.g., Item M031325: drawing an obtuse angle). U.S. students performed strongly on nine items that focus on testing students' knowledge about symmetries and locations, five of which assessed their knowing and the rest assessed application of these knowledge. Among these items, U.S. students far surpassed the international benchmark and nearly matched Hong Kong students' performance on items such as justifying which line is a symmetry line (Item M031093), describing the movement of Jamie's counter on a board game (Item M031088). U.S. students performed best (92%) on Item M0411608 where they were asked to mark the location of a house on a map. In general, results indicate U.S. students preferred practical geometrical knowledge and were challenged by abstract geometric concepts and reasoning.

Based on these initial findings, I suggest that U.S. math teachers and educators should focus on promoting U.S. students' conceptual understanding of measurement (e.g., length, angle) and their flexibility in applying such geometrical knowledge. We should also explore the factors that contribute to U.S. students' strengths in location and symmetries related knowledge to see if we can extend these strengths to other geometrical topics.

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CORRESPONDING COLOR CODING FACILITATES LEARNING OF AREA MEASUREMENT

Jenny Yun-Chen Chan
Worcester Polytechnic Institute
jchan2@wpi.edu

Pooja G. Sidney
University of Kentucky
pooja.sidney@uky.edu

Martha W. Alibali
University of Wisconsin- Madison
mwalibali@wisc.edu

Keywords: Geometry and Geometrical and Spatial Thinking, Instructional Activities and Practices

Understanding formulas for area measurement requires making connections between the symbols in a formula and the corresponding elements of a pictorial diagram. Unfortunately, many geometry textbooks do not provide explicit links between these two representations (e.g., Kennedy, Charles, & Bragg, 2005). The absence of explicit connections between two representations combined with the instructional emphasis on algorithmic computations (e.g., plugging numbers into a formula *area*=*base*×*height*; Zacharos, 2006) may lead to misconceptions and incorrect formula applications, such as using a side instead of height to find the area of an obtuse triangle (Fuy, Geddles, & Tischler, 1988). Previous studies have shown that using colors to highlight corresponding components of text and illustrations supports learning of complex science concepts in high-school and college students (Ozcelik et al., 2009; Kayuga et al., 1999). Here, we propose that using distinct colors to highlight corresponding elements of diagrams and formulas during instruction may help students make connections between the two representations, and we test the effects of corresponding color-coding on learning of triangle area measurement.

Using a pretest-lesson-posttest design, we assessed 38 sixth-grade students' knowledge of area measurement before and after a video lesson on calculating area for an obtuse triangle. For half of the students, corresponding elements of the formula and the diagram were highlighted in distinct colors during the lesson (e.g., the base was highlighted in green in both representations). The other half of the students viewed an identical lesson without the color coding (e.g., the base of the triangle in both representations were in black).

Overall, students improved in identifying both base (pretest: 41.5%; posttest: 62%) and height (pretest: 22.5%, posttest: 43.5%) of obtuse triangles from pretest to posttest, *p*s< .001. The effect of condition emerged only in students' performance on base items. Specifically, students in the color-coding condition improved more in identifying bases (pretest: 48%; posttest: 72%) than students in the control condition (pretest: 35%; posttest: 52%), $F(1, 37) = 4.46$, $p = 0.04$. The findings suggest that corresponding color-coding of diagrams and formulas may be an effective instructional approach to promoting learning of area measurement.

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INCLUSIVELY RESPONSIVE INSTRUCTION TO ADVANCE SPATIAL REASONING AND NUMBER SENSE

Rebecca Ambrose
University of California-Davis
rcambrose@ucdavis.edu

Keywords: Geometry and Geometrical and Spatial Thinking, Elementary School Education, Design Experiment

Geometry continues to be overlooked in elementary school mathematics despite its potential to support spatial reasoning which has been shown to have a role in overall mathematics performance (Sinclair & Bruce, 2015). Moreover, when it is taught, the focus tends to be on classifying 2-dimensional shapes according to their sides and angles. To address these short-comings, I have been designing and testing activities that engage children in building polyhedra and measuring their volumes. I hope that combining attention to shape and number within the same lessons will make it more likely that geometry will be addressed with greater frequency in elementary school classrooms. The work has taken place in a 5th grade classroom with 27 11-year-olds. The class is ethnically diverse with a range of mathematical abilities and achievement levels.

Lesson design was informed Murata's model of Inclusively Responsive Instruction (2013) which assumes that students' zones of proximal development will differ. This diversity of thinking can be an asset to the classroom when students have access to the thinking of their classmates giving them opportunities to incorporate new ways of thinking with their own. When this occurs "each learner will feel valued for his or her own thinking as different ideas interact to become something collectively meaningful" (Murata, 2013, p. 18). My previous work in the domain of 3D geometry (Ambrose & Kenehan, 2009) and my understanding of the ways in which children coordinate units (see Ulrich, 2015 for example) helped me to anticipate the levels of thinking that might be evident in the class enabling me to be strategic about the student work to bring up during the class discussion.

Activities tended to include children building polyhedra using polydrons and measuring the volume of their constructions. An ample supply of cubic centimeter manipulatives was available to measure the volume of polyhedra. Children varied in what they built and their strategies for as determining the volume differed and reflected the degree to which they spatially structured (Battista, 2007) their polyhedron. Whole class discussions were crafted to ensure that different levels of thinking were available to the students so they could learn from one another. Spatial structuring emerged from the analysis of shapes as well as discussions of volume. In particular students varied in terms of attending to the number of cubes in a row, the number of rows in a "layer" and the number of layers in the shape.

In all lessons the children were positioned as knowers, builders and designers, giving them a sense of agency. Ball, Goffney and Bass (2005) argued, "mathematics instruction can be designed to help students learn that differences can be valuable in joint work, and that diversity in experience, language, and culture can enrich and strengthen collective capacity and effectiveness (p 5)." In this way the lessons provided children with an opportunity to participate in a democratic discussion space where all individuals' contributions were valued.

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Chapter 6:
In-Service Teacher Education and Professional Development

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

EXPLORING THE DISCURSIVE DIFFERENCES OF MATHEMATICS COACHES WITHIN ONLINE COACHING CYCLE CONVERSATIONS

Ryan Gillespie
University of Idaho
rgillespie@cdaschools.org

Julie Amador
University of Idaho
jamador@uidaho.edu

Jeffery Choppin
University of Rochester
jchoppin@warner.rochester.edu

We present the results of our analysis regarding the discursive tendencies of four mathematics coaches during planning and debriefing conversations within online coaching cycles. Guided by the Content-Focused model of coaching (West & Cameron, 2013), the coaching cycles are a single component of a larger online professional development model for middle school mathematics teachers in rural areas (Choppin, Amador, & Callard, 2015). This paper explores the different ways coaches talk with teachers during coaching conversations. Building on prior studies from literacy coaching (Ippolito, 2010), we found five different discursive moves for how coaches talk with teachers: invitation, suggestion, explanation, description, and evaluation. We use these moves to identify similarities and differences in the discursive tendencies of coaches. The implications of these discursive tendencies are provided.

Keywords: Inservice Teacher Education/Professional Development, Online Learning, Coaching

Introduction

Coaching teachers to support their mathematics instruction is a promising practice to improve pedagogical implementation and content knowledge (Campbell & Griffin, 2017). However, not all coaching processes are the same. There are three commonly referenced models: Content-Focused Coaching (West & Cameron, 2013), Instructional Coaching (Knight, 2007), and Cognitive Coaching (Costa & Garmston, 2016). Although each model approaches the process of coaching differently, they all articulate the same three sequential cycle components: a) a pre-conference discussion to plan a lesson, b) a collaborative lesson implementation, and c) a post-conference discussion to debrief the lesson (Bengo, 2016). The three-part coaching cycle is a prominent professional learning activity mathematics coaches use when working with individual teachers to improve their practice (Mudzimiri, Burroughs, Luebeck, Sutton, & Yopp, 2014). Despite the use of a similar three-part cycle, the way these cycles are delivered varies depending on the type of coaching and the personnel involved.

The research on literacy coaching highlights two competing stances for how coaches talk with teachers: reflective or directive (Deussen, Coskie, Robinson, & Autio, 2007; Ippolito, 2010; Sailors & Price, 2015). Coaches using a *reflective* stance facilitate improvement of teaching practices and student learning through collaborative inquiry (Ippolito, 2010). Coaching moves associated with this stance include probing questions and low-inference, non-evaluative observations as means to catalyze teacher thinking (Costa & Garmston, 2016). In contrast, a *directive* coaching stance involves the use of advice, suggestions, and evaluative feedback to support teachers to implement new teaching practices (Ippolito, 2010). Because these different coaching stances can have significant impact on the teacher's learning and uptake of new practices (Costa & Garmston, 2016), it is crucial for researchers within mathematics education to explore the existence and impact of these stances during coaching.

We examined the existence of reflective and directive stances of four middle school mathematics coaches during their coaching cycle conversations with teachers. Specifically, the study was guided by the question: What were the different discursive tendencies of four mathematics coaches within the planning and debrief conversations of a coaching cycle? This question is relevant to mathematics education as variability within coaching has been a dominant theme in existing literature. Although coaching and coaching cycles have been shown to have the potential to improve teaching practices and student learning, results have been inconsistent (Gibbons & Cobb, 2016). Variability in coaching experience, the types of activities coaches use, and the context surrounding the coaching activities often vary dramatically (Ellington, Whitenack, & Edwards, 2017). This variability has been attributed to the inconsistent impact of coaching on improving teaching and learning (Campbell & Griffin, 2017). The purpose of this study is to open up an examination of coaching discursive tendencies to explore the variability of coaching stances.

Methods

In this study, experienced coaches, all guided by the Content-Focused Coaching model (West & Staub, 2003), took part in coaching activities as part of a larger online professional learning project (Choppin et al., 2015). The project paired coaches using Content-Focused Coaching with middle grades teachers and was designed to improve teacher practices for implementing high cognitive demand tasks and facilitating mathematical discourse (Smith & Stein, 2011). Using a cohort model, nine teachers from grades 5-8 participated in a two-year online professional development program. The fully online professional development program was specifically created for mathematics teachers working in rural areas and provided teachers with an online course, online teaching labs, and video-assisted online coaching cycles.

Participants

Three of the four coaches (see Table 1) working with the cohort teachers had more than ten years of prior coaching experience and had worked collaboratively using the Content-Focused coaching model (West & Staub, 2003) for at least three years. Each coach was assigned one to three teachers, with whom they each engaged in five total online coaching cycles over the course of two years. The data for this inquiry were collected from the planning and debrief conversations of the four project coaches and their assigned teachers.

Table 1: Coach Demographics

Coach	Years of Classroom Experience	Years Coaching	Teaching and Administrative Certifications
Alvarez	28	21	Permanent Certification in K-5, 6-8 and 9-12 Mathematics
Bishop	28	14	Secondary Mathematics State 1, Secondary Mathematics State 2
Lowery	14	13	Special Education (N-12); Elementary Education (PreK-6); Reading (N-12)
Reiss	15	0	Elementary (N-6), Special Edu (K-12), Mathematics (7-12)

Context: Coaching Cycles

Each online coaching cycle was designed to support successful implementation of new discourse practices (e.g., Smith & Stein, 2011) learned during the online course and teaching labs. Each cycle followed the same structure. First, the coach and teacher had a planning discussion using video conferencing technology, Zoom, around a proposed lesson. Based on the Content-Focused Coaching model, the focus of this planning meeting was on the mathematical lesson goals, the tasks that would be used in the lesson, anticipated student thinking, and instructional practice goals for the teacher (West & Staub, 2003). The coach used a variety of discursive moves to support the teacher during this planning discussion. Following the planning meeting, the teacher video- and audio- recorded themselves teaching the lesson using Swivl Technology (automated video camera and recording). After the lesson was taught, the coach and teacher asynchronously watched and annotated the lesson video. Annotations included written comments to the other person about the contents of the video. The coaching cycle concluded with the coach and teacher having a debrief discussion that utilized the annotations to reflect on student thinking in relation to the lesson goals and the teacher's goals for instructional practice. The planning and debriefing discussions were the focus for analysis for this project and typically lasted forty to sixty minutes.

Data Collection and Analysis

For this study, we collected and analyzed the planning and debriefing conversations of one coaching cycle for all coaches and their assigned teachers. This resulted in the analysis of 15 conversations (eight planning and seven debriefing), which were transcribed verbatim. Transcripts were parsed into stanzas which we defined as including both the coach's discursive move and the teacher's response, as well as text needed for context (Saldaña, 2013). A stanza served as the unit of analysis. We then used a comprehensive codebook created by the research team to analyze all stanzas. To create the codebook, we first open-coded coaching transcripts for our data set using constant comparative methods (Corbin & Strauss, 2008). We then synthesized our initial open codes with the discursive moves associated with reflective and directive coaching stances described in the three primary coaching models (Costa & Garmston, 2016; Knight, 2007; West & Cameron, 2013). For example, Cognitive Coaching strongly promotes a reflective coaching stance using the discursive moves of invitational questioning and descriptive paraphrasing (Costa & Garmston, 2016). Similarly, Instructional Coaching leans towards reflective coaching stances by encouraging coaches to collect and share observation data using descriptive but non-evaluative discursive moves (Knight, 2007). Finally, Content-Focused Coaching suggests a fluid balance of both coaching stances through the use of reflective questioning moves as well as directive suggestions and explanations when appropriate (West & Cameron, 2013).

The open codes, combined with the literature on coaching, resulted in a broad codebook that accounted for the discursive moves and the content of the conversations. The coaching discursive moves section accounted for five broad categories within this codebook (see Figure 1). Then we connected each of the five discursive moves to either a reflective or directive coaching stance by again returning to the coaching literature. Suggestions, explanations, and evaluations connected to a directive coaching stance as those moves all involve the coach sharing their thinking and opinions with the teacher which in turn positions the coach as an expert (Sailors & Price, 2015). Invitations and descriptions connected to a reflective coaching stance because those moves position the teacher as the thinking authority since neither move contains the thinking or opinion of the coach (Ippolito, 2010).

Coaching Stance	Discursive Move	Description	Example
Reflective	Invitation	Statement or question that invites the teacher to reflect or respond	"What might be some strategies we could use to increase student participation?"
	Description	Statements that share a direct observation and do not contain inference, interpretation, judgement or opinion	"I noticed that during the turn-and-talk, Alex did not say anything to his partner."
Directive	Suggestion	Statement that recommends an action	"I think we should use a turn-and-talk prior to the whole class discussion."
	Explanation	Statements that provide an interpretation or rationale of an event, interaction, or mathematical idea	"Turn-and-talk is a powerful strategy to use prior to a whole class discussion because it increases student participation."
	Evaluation	Statements that offer praise or critique	"I think it was a great idea to use a turn-and-talk."

Figure 1: Excerpt from the Larger Codebook Focusing on Coaching Discursive Moves

To analyze the planning and debriefing meetings between the coach and teacher, the stanzas from the transcripts were coded by pairwise teams of researchers for the discursive moves of the coach, the discursive moves of the teacher, and the content being discussed within each stanza. (Note that only the discursive moves of the coach are considered for this study.) The researchers then calculated Kappa to determine consistency with coding and met to reconcile disagreements. Kappas ranged from 0.39 to 0.65, considered moderate to strong reliability (Landis & Koch, 1977).

Two examples are provided to clarify the ways the five discursive codes were used to analyze how the coach talked to the teacher. The following is an excerpt from a coach:

> One of the really nice moves you can do if the group shares a thought about something, and it's somewhat ambiguous, is you can turn to the class and say, "Can someone else use their own words to explain what Dave is saying?"

This comment from the coach was coded as a suggestion because the coach recommended the teacher use a question to promote additional student discourse. In this instance, the suggestive discursive move implied the coach held a directive coaching stance.

It is also possible that multiple discursive moves were used by a coach within one stanza. To illustrate this, the following is an excerpt from a transcript in which Reiss (pseudonym) uses a descriptive move (lines 1 - 4) followed by an invitational move (lines 5 - 6):

> I kept hearing kids saying, "No, four, no five. You need four, you need five."
> You asked the question well, how many boxes do you need to take the bagels home
> was the question you asked.
> I heard the answers four and five come out from kids.
> What mathematical idea is at play here, and where do we think the kids understanding
> is for that idea?

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In this example from a debriefing interview, Reiss recalled the contents of the lesson and details what she heard the teacher and students say without inference or evaluation. She then questioned the teacher about the mathematics and students' understanding. Since she first described student and teacher actions and then invited dialogue, this was coded as both a description and an invitation. Both of these discursive moves suggested the coach was operating from a reflective stance at that moment in the conversation since no interpretation or opinion was provided by the coach enabling the teacher to construct their own meaning from the situation.

After we coded the transcripts, the total number of each of the five discursive moves used by a coach within a conversation was calculated. For example, during a planning conversation containing 31 stanzas, Reiss used a total of 15 invitations, nine descriptions, three suggestions, 11 explanations, and zero evaluations. These counts indicate the coach held both reflective and directive stances during the conversation. However, the higher combined frequency of invitational and descriptive moves (24 total) when compared to suggestive, explanation, and evaluative moves (14 total) suggested the coach favored a reflective coaching stance. Because coaches were assigned different numbers of teachers and thus had different numbers of total coaching conversations, we calculated an average for each move by dividing the total number of discursive moves used in all conversations by the total number of conversations. For example, Reiss used a total of 52 invitational moves during four conversations resulting in an average of 13 invitational moves for a single coaching conversation (see Table 2).

To characterize reflective coaching stances using the discursive moves, we divided the sum of the number of invitational and descriptive moves by the number of coaching conversations. This provided the average number of moves in a conversation associated with a reflective coaching stance. Similarly, we determined the average number of directive stance moves by combining suggestions, explanations, and evaluations. For example, during a single conversation, Reiss averaged 22.5 combined discursive moves connected to a reflective stance and 17 combined moves associated with directive coaching (see Table 2).

Results

In total, we analyzed 594 stanzas from 15 total conversations (eight planning and seven debriefing meetings). When examining the frequency of the five discursive moves used by each coach, we found both consistency and variability. As an example of a consistency, each of the four coaches was similar in their use of descriptive moves as the average ranged from 7 to 9.5 moves per conversation. The coaches were also relatively consistent in their use of explanation moves. However, there existed larger variability within the coaches' use of invitational, suggestive, and evaluative moves. To illustrate this variability within invitational moves, Alvarez had the highest average of 22 invitational moves per conversation compared to Reiss who averaged the lowest with 13 moves per conversation. For suggestions, Bishop had the highest average with 13.7 moves per conversations whereas Alvarez used only 3 suggestive moves per conversation. Bishop also had the highest level of evaluation moves with an average of 6.5 per conversation compared to Reiss who averaged only one evaluation move per conversation.

Reflective and Directive Stances by Coach

The differences in the number of directive (invitation and description) and reflective (suggestive, invitational, and evaluative) moves suggested the existence of variability in how the four coaches balanced the two stances during the conversations (see Table 2). For example, Alvarez averaged 29 discourse moves associated with a reflective stance compared to only 12 directive moves during a single conversation. This finding suggested the coach strongly favored a reflective stance. In contrast, Bishop averaged more directive moves per conversation (32.2) than reflective moves (25.8) indicating the coach favored a directive stance. The discursive moves for Lowery and Reiss were more evenly balanced but implied both coaches slightly favored a more reflective stance.

Table 2: Average Number of Discursive Moves Per Coaching Cycle Conversation

	Reflective		Directive			Combined Reflective Moves	Combined Directive Moves
Coach	Invitation	Describe	Suggest	Explain	Evaluate		
Alvarez	22	7	3	7	2	29	12
Bishop	18	7.8	13.7	12	6.5	25.8	32.2
Lowery	18.7	7.7	11	9.3	4.3	26.4	24.6
Reiss	13	9.5	7	9	1	22.5	17

Reflective and Directive Stances Based on Planning or Debriefing

To further explore the discursive tendencies of coaches, we separately examined the data based on the conversation type. Specifically, we analyzed the frequency of discursive moves used by coaches during planning and debriefing conversations. For example, Lowery averaged 13 suggestive moves during a planning conversation and seven during a debriefing conversation (See Table 3). Again, this analysis revealed a small number of additional consistencies and further highlighted distinct differences in discursive tendencies between coaches. As an example of one such consistency, all coaches used more descriptive moves during debriefing conversations than in planning conversations suggesting each coach made a potential shift to the use of more reflective coaching stances during debriefs.

Reflective and Directive Variability

To illustrate the additional variability in discursive tendencies, Table 3 shows the average number of discursive moves per coaching cycle conversation for each coach.

Table 3: Average Number of Discursive Moves Per Coaching Cycle Conversation

	Planning Conversation					Debrief Conversation				
	Reflective		Directive			Reflective		Directive		
Coach	Inv	Desc	Sugg	Exp	Eval	Inv	Desc	Sugg	Exp	Eval
Alvarez	20	0	3	8	1	24	14	3	6	3
Bishop	21.7	6.7	19.7	11	4	14.3	9	7.7	13	8.3
Lowery	21.5	2	13	10	1	13	19	7	8	13
Reiss	16.5	6	5	9	0	9.5	13	9	9	2

Note: Invitation (Inv), Descriptive (Desc), Suggestive (Sugg), Explanatory (Exp), Evaluative (Eval)

Bishop's highest combined frequency of suggestive, explanatory, and evaluative moves in both the planning and debrief conversations portrayed a more directive coaching stance when compared to the other three coaches. In contrast, Alvarez's low frequency of suggestive and evaluative moves combined with a higher total frequency of invitational and descriptive moves suggested this coach held a highly reflective coaching stance during planning and debrief conversations. The more balanced discursive moves of Reiss and Lowery provided evidence of an evenly balanced coaching stance. We provide three examples of the variability with the discursive moves. First, Bishop and Alvarez varied greatly with their use of suggestions during planning conversations. Bishop provided 19.7 suggestive moves per planning conversation yet Alvarez only offered three per planning conversation. As a second example from debriefing conversations, Reiss and Lowery differed greatly with their use of evaluative moves. Reiss averaged only two evaluative moves per debrief conversation compared to Lowery's 13 evaluative moves per debrief conversation. As a final example, Alvarez averaged 24 invitational moves during debriefing conversations whereas Reiss averaged only 9.5 per debrief conversation. These three examples regarding the differences in discursive moves further demonstrates the existence of variability in how each coach balanced reflective and directive coaching stances. Additionally, the examples highlight large potential differences in the learning experiences of the teachers based on their coach.

Discussion

Our findings suggest the existence of certain common discursive tendencies within the four coaches. For example, all four coaches were relatively consistent in their use of descriptive moves. All four coaches also used more descriptive moves during debriefing conversations than during planning conversations. This finding is consistent with Knight's (2007) recommendation that coaches should collect, share, and examine descriptive, low-inference data with teachers during a debrief conversation. This finding is significant as it suggests each of the four coaches enacted a coaching practice identified by existing coaching literature as productive in supporting teacher learning. However, our findings were consistent with prior literacy coaching studies (e.g. Heinke, 2013) and highlighted many substantial differences in the coach's discursive tendencies. These differences could have a significant impact on the learning experiences of the teachers (Costa & Garmston, 2016). For example, a teacher planning with Bishop received many more suggestions than a teacher planning with Alvarez. Similarly, a teacher debriefing with Lowery encountered far more evaluative moves than a teacher debriefing with Reiss.

Extant research indicates that most coaches tend to use both reflective and directive stances during coaching conversations (e.g. Deussen et al., 2007). The differences in discursive tendencies also suggest variability in how each of four coaches balanced reflective and directive coaching stances despite the fact that each coach was operating within the Content-Focused Coaching model. In fact, authors of the Content-Focused Coaching model advocate for the use of both stances (West & Cameron, 2013). However, neither the coaching model nor prior research on coaching provide guidance on how to properly balance these differing stances to best support teacher learning. Additionally, these works have not closely examined the actual discursive moves coaches are using with considerations about how this may influence teacher learning. Now that we know the types of discursive moves the coaches in our project are using, it would be beneficial to examine how this relates to teachers' responses in the moment of coaching.

Future Directions

Building from the variance in mathematics coaching described by Campbell and Griffin (2017), these findings expose an additional source of variability. Our data show that even with highly experienced coaches drawing from the same coaching model, there existed considerable variability in the discursive tendencies of coaches. Likewise, although our intent was not to compare coaches based on demographics, the coach with the least coaching experience (Riess) showed tendencies similar to Lowrey who had been coaching for over a decade. These discursive differences imply the coaches used distinct coaching stances which, like other forms of variability found within coaching, likely have an impact on teacher learning (Costa & Garmston, 2016). Further exploring the possible relationship between experiences and coaching practices could reveal interesting insights that would provide information on how to best support both coaches and teachers.

These findings are also significant in that they generate new questions for coaching in mathematics education. The existence of discursive variability between coaches, even within the favorable context found in our study, warrants further exploration into the underlying causes of these differences. It is possible that the diversity in coaching discourse is due to the coaches being responsive to the varying needs of the individual teacher. However, it also plausible that these differences result from beliefs, preferences, or personal interaction styles that are inherent of the coach. Prior research on literacy coaching has highlighted that coaching discursive patterns are primarily static and not adapted based on the varying learning needs of the teacher (Collet, 2012). However, research on the adaptive nature of coaching discourse is scarce within literacy coaching (Collet, 2012) and, to the best of our knowledge, non-existent within the specific context of mathematics coaching. To fill this gap, our future analysis will use data from additional coaching cycles to compare both the discursive moves of a single coach across multiple teachers and the way in which coaches shift their discursive tendencies across multiple coaching cycles with the same teacher.

Future studies should also build on these results to examine the relationship between the discursive tendencies of coaching and teacher learning (Heinke, 2013). Although certain coaching models make claims about the affordances and drawbacks of certain types of discursive moves (e.g. Costa & Garmston, 2016), further research is needed to better understand the how different discursive tendencies affect the teacher being coached. This understanding, combined with the results of our study, can provide practicing coaches with sound guidance about how to strategically balance and employ different discursive moves. This understanding could also lead to more strategic partnering of coaches and teachers by matching the discursive tendencies of a coach to the unique learning needs of a teacher.

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MATH TEACHERS' SENSEMAKING AND ENACTMENT OF THE DISCOURSE OF "PERSEVERANCE"

Patricia Buenrostro
Vanderbilt University
patricia.m.buenrostro@vanderbilt.edu

Nadav Ehrenfeld
Vanderbilt University
nadav.ehrenfeld@vanderbilt.edu

Students' opportunities to struggle with mathematical ideas have long been considered paramount to learning. However, there's little research on how teachers (1) draw on and make sense of the discourses of perseverance, (2) enact it in classroom, and (3) develop an expansive view of perseverance. To contribute to these lines of research, we build on a case study featuring a veteran mathematics middle-school teacher across two settings: his classroom where he facilitates students' engagement with a classical mathematical task, the Tower of Hanoi, and in a subsequent video-based debrief with his colleague and our research team. We propose a conceptualization of perseverance as upholding three dimensions of (a) persistence (b) sensemaking and (c) problem solving heuristics. We argue for its potential as a conceptual resource for operationalizing perseverance more comprehensively.

Keywords: Mathematics Teaching, Perseverance, Video-based coaching, Problem-solving

Objectives

Perseverance, grit, and productive struggle are a few of the descriptors signaling the importance of fostering students' sustained engagement in math classrooms to support mathematical sensemaking. Mathematics educators, researchers, and policymakers agree that cultivating students' dispositions towards seeing mathematics as a worthwhile and *effortful* pursuit supports students' success in engaging with conceptually-rich mathematics in flexible and fluent ways. Indeed, *make sense of problems and persevere in solving them* is the first of eight mathematical practices deemed essential for teachers to champion in their classrooms as outlined in the Common Core State Standards for Mathematics (National Governors Association, 2010). Similarly, in the influential document *Adding It Up* (Kilpatrick et al., 2001), mathematical proficiency is defined as the amalgamation of five strands, one of which is *productive disposition*, or the "habitual inclination to see mathematics as sensible and useful, coupled with a belief in one's own efficacy" (p. 5). In short, perseverance is one of the noncognitive factors that is highly consequential for academic success (e.g. SRI International, 2018, p. 1).

Despite a burgeoning discourse within education broadly and mathematics classrooms specifically, little research investigates how mathematics teachers interpret these discourses of perseverance. Yet these interpretations matter for how they take them up in their classrooms. Accordingly, inspired by Naraian (2011), we explore questions such as: What discourses do teachers draw on as they aspire to promote perseverance? How do teachers design for and enact perseverance in math classrooms? What discursive dilemmas emerge in this process? How can teacher educators support teachers' expansive view of perseverance?

In this analysis, we present an illustrative "best case" of a teacher, Ezio (all names of teachers and the school are pseudonyms), who identified perseverance as an explicit goal for his class. We analyze the discourses he drew on, his enactment of a lesson designed to support perseverance, and the discussion we conducted three days later as part of our research project. We identify the Opportunities To Learn (OTLs) that emerged in our debrief that supported the teacher's thinking about perseverance more expansively while also pinpointing where we fell short as teacher educators in our own limited understanding of perseverance, the same discourse to which we aim to contribute in this paper.

Theoretical Framework

Situating Teachers as Sensemakers

Despite strong efforts to shift teachers towards ambitious instruction, the field continues to struggle with effectively translating research-driven visions and practices so they might be taken up more frequently and more seamlessly by practitioners. We address this problem in our research, as well as in our practice as teacher educators, by taking a situated view on teaching (Greeno & MMAP, 1998; Horn & Kane, 2015). A situated view on teaching (and on teacher learning) highlights that teachers do more than simply implement pedagogical discourses, but rather they constantly make sense of their classroom context as it is negotiated by institutional norms and practices and wider socio-historical discourses (Horn & Little, 2010; Narian, 2011). In this vein, we take a situative approach in examining teachers' Opportunities To Learn (OTLs) by considering the various discourses teachers draw on such as the state standards, professional development organizations, and their own epistemic stances or beliefs about what can be known, how to know it, and why it matters (Horn & Kane, 2015). In this study, we are interested in the conceptual resources made available for teachers and teacher educators in their attempts to operationalize discourses around productive struggle. We view the two settings of classroom and video-based coaching session as informative in drawing out the discursive dilemmas in each other. In other words, classroom dilemmas inform teachers' OTLs, and pedagogical conversations with colleagues, at their best, inform classroom dilemmas. In light of that, we aim to conceptualize perseverance in a way that might support teachers and teacher educators in conceiving and enacting perseverance more expansively.

Research on Perseverance

To conceptualize perseverance, we draw on several studies (Bass & Ball, 2015; Hiebert & Grouws, 2007; Sengupta-Irving & Agarwal, 2017; Stein et al. 2017), and propose a rich and productive conceptualization of perseverance as upholding the three conditions of (a) *persistence* (b) *sensemaking* and (c) problem solving *heuristics*.

Hiebert & Grouws (2007) documented the ways that certain forms of *persistence* or student struggle can be a key enabler for math learning. However, persistence alone is an insufficient condition for learning mathematics conceptually. For example, the results of a recent study by Stein et al. (2017) revealed that teachers disproportionately create conditions for students to struggle unproductively, with little explicit attention to and few opportunities for building conceptual understanding. For students to develop conceptual understanding, they need opportunities to *make sense* of mathematical ideas, construct new knowledge, and make connections. Consequently, teachers are tasked with finding challenging problems that require the right amount of struggle within students' conceptual reach or within their zone of proximal development (ZPD; Sengupta-Irving & Agarwal, 2017; Vygotsky, 1978). Less productive challenges were described by Sengupta-Irving and Agarwal (2017) as either "unnecessary struggle" or "no struggle" (see Figure 1).

Bass & Ball (2015) add a dimension to our understanding of perseverance related to problem solving *heuristics*. Analyzing a 5th grade class engaged in collective problem solving, they presented a macro-analysis of the lesson identifying (1) six stages of students' collaborative work; and (2) the teacher's instructional moves that oriented students engagement across the

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

stages. Through their analysis of how teachers made perseverance “visible and learnable,” they underscored how the teacher made explicit for students the different heuristics they exercised in making sense of the task, identifying adequate approaches and generating possible solutions. Indeed, research has shown heuristics to be an important tool for successful problem solvers (Schoenfeld, 1985).

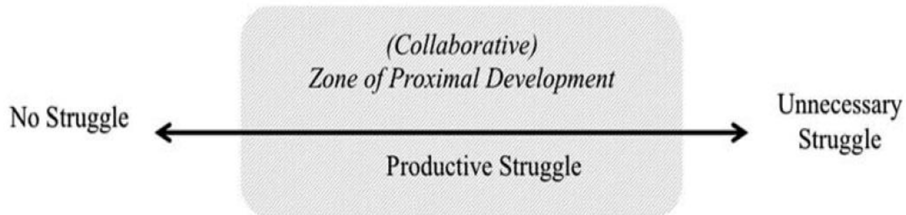

Figure 1: Productive Struggle and the ZPD (Sengupta-Irving & Agarwal, 2017)

Methods

Research Context

This analysis comes from a larger research-practice partnership aimed at understanding how video-based coaching, tied to teachers’ instruction, can support urban mathematics teachers’ development of ambitious instruction. To support and document teacher learning, our research team developed a Video-based Formative Feedback (VFF) cycle, a 4 step co-inquiry into practice. First, we prompt teachers about their goals and specific inquiries related to the class to be filmed, identifying a question we can use to inform our observation. Second, we film one classroom session using two cameras, a classroom wide-angle camera and teacher point-of-view camera. Additionally, we place four audio markers at different student groups to capture group-level conversations. These video and audio records serve as rich representations of teachers’ practice from which to focus coaching conversations. Next, our team analyzes the captured footage to identify moments of interest that may address the teachers’ inquiry questions. Finally, typically within two days of filming, we meet with the teachers to debrief the lesson, focusing on the identified video moments as a springboard for discussion. The debrief session is filmed for our research purposes.

Data sources. Primary data sources for this analysis include: (1) Focal teacher’s email describing the lesson activities and teacher’s goals; (2) the 90-minute video-taped classroom session; (3) the 81-minute videotaped debrief of the lesson with three members of the research team—the two authors of this paper and the project Principal Investigator, Ilana Horn. Patricia Buenrostro facilitated the conversation with the focal teacher, Ezio, and a collaborating teacher, Veronica. The other researchers were primarily filming (Ehrenfeld) and taking field notes (Horn) but on occasion participated in the debrief conversation. Because our relationship with Ezio extends beyond this focal event, we draw on other conversations, observations, and interviews as secondary data in confirming or disconfirming evidence for tenuous claims.

Focal participant. The VFF cycle analyzed represents a debrief with Ezio Martin, a veteran, middle-school teacher with over 17 years of experience, and his colleague, Veronica Kennedy. Ezio demonstrated strong content knowledge during his teaching and in professional development workshops, underscored by his undergraduate and graduate math degrees. His participation in a 5-year professional development program showed his commitment to his professional growth. In addition, he had strikingly warm rapport with his students, who show respect and admiration for him. As a teacher with a strong math identity, the classroom was decorated with mathematical phrases, posters and books. The class episode under study featured animated talk, boisterous laughter, and wooden manipulatives in action. In short, it appeared a vibrant environment in which to learn and do math. Ezio's queries prior to filming centered around student dynamics (e.g., status issues). Notwithstanding, in the debrief, Ezio identified perseverance as an explicit goal for the featured task, the Tower of Hanoi.

Focal task, "Tower of Hanoi." The Tower of Hanoi activity consists of three poles and varying number of rings in different sizes (see Figure 2), arranged on one pole. The goal of the task is to move all the rings, one at a time, from the initial-pole to either of the two remaining poles, while avoiding the placement of a larger ring onto a smaller one *in the least number of moves*. For example, 3 rings can be transferred in only 7 moves. The task is to find the general rule for determining the minimal number of moves for any given number of rings [$f(n) = 2^n-1$]. Ezio wanted to expose students to a nonlinear function and to experience a challenging problem with which to persevere.

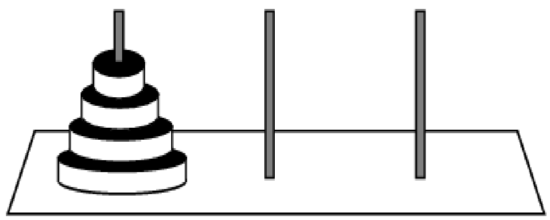

Figure 2: Illustration of Tower of Hanoi Kit with Three Poles and Four Rings

Methods of Analysis

To be clear, the centrality of Ezio's goal to have his students persevere emerged during the post-class debrief. Our overall methods represent an iterative process of analyzing both the debrief and classroom session to understand how Ezio made sense of, designed for, and enacted *perseverance*.

We looked across the data from both settings and used interaction analysis (Jordan & Henderson, 1995) with both sets of transcripts to examine the complementary and contradictory discourses that emerged. For the classroom, we looked at Ezio's design and enactment of perseverance with an eye towards the macro-analysis of lesson phases presented by Bass and Ball (2015) and expanded our definition of perseverance. For the debrief, we analyzed Ezio's conceptualization of perseverance as he discussed his teaching. Comparing the emergent views of *perseverance* arising from these analyses, we explored the discursive dilemmas (e.g., asking questions students are unable to answer) in both events—the classroom and debrief—which ultimately led to our expansive understanding and proposed characterization of perseverance described earlier.

Results

To understand the discourses that Ezio drew on in making sense of perseverance, we examined his epistemic stance on perseverance as corroborated from his email and the debrief conversation. Next, we report on the classroom events from the filmed lesson, followed by an analysis of the debrief conversation. Finally, we look across the two settings and reflect on how, in our role as coaches, we supported Ezio in re-imagining perseverance in his classroom, while also underscoring where we needed a richer conceptualization of perseverance.

Ezio's Epistemic Stance on Perseverance: "I want them to learn how to persevere"

During the debrief, Ezio interrupted the start of a video clip viewing to share the significance of having his students engage with a complex problem such as the Tower of Hanoi. He stated, "I think the Tower of Hanoi is complex enough, at least in middle school [that] I want them to learn how to persevere. It is one of the standards." He was not quite sure how to teach perseverance ("I don't know how to do it"), but he believed that posing a challenging task would create the conditions for his students to persevere. This notion is corroborated by the email Ezio sent us the day prior to filming his class in which he briefly laid out his goals and plans for the class. He noted that "most will not be able to determine the equation for Tower of Hanoi even with the hints I give them...I am willing to leave the problem 'unsolved' for those who are not able to figure it out." Ezio anticipated that "most" students would struggle, thereby, creating a situation in which they would persevere.

We learned in the debrief about Ezio's epistemic stance regarding the importance of perseverance in math learning. Ezio shared his "personal belief" that all students will experience a potentially insurmountable challenge in mathematics. He described this experience as "hit[ting] a wall," one that "everybody hits." Ezio described all math learners, including himself and the greatest of mathematicians, as eventually "hit[ting] a breaking point" but that the key is to "[learn] how to overcome [it]." Ezio distinguished those that "reach that breaking point and...give up" from those "who continue the struggle…[those] who will succeed in math." For Ezio, *hitting the wall* is a defining moment where one decides to either concede to not being capable enough or, contrastingly, persevere and "overcome that breaking point." While he said that he does not "do a good job of teaching [students] how to persevere over those obstacles," the Tower of Hanoi gives him "the opportunity to focus in on that."

Classroom Excerpt: "Maybe you haven't tried hard enough"

To understand how Ezio designed for and enacted perseverance in his classrooms, we describe what we termed as Ezio's "cycles of perseverance" in light of his epistemic stance. The cycle begins with a challenge beyond students' conceptual reach, in essence directing them to a "wall," and concludes with Ezio's scaffolding them over the wall of struggle.

The 90-minute lesson was replete with intensive mathematical work by 31 eighth graders. It was built around two mathematical tasks: a warm up with visual patterns representing a linear function (around 30 minutes), and the Tower of Hanoi (around 60 minutes). The episode discussed here is focused on the second task. It began with a playful introduction from Ezio, where he teased students about being "a bunch of babies" necessitating the use of toys to which students responded with smiles and baby voices.

Two cycles of perseverance. Students were seated in seven groups of four to six but working primarily in pairs. Ezio systematically circulated between the seven groups, conversing with students about their work, occasionally teasing them ("Maybe you haven't tried hard enough"). In our analysis of classroom events, we identified that Ezio designed for and enacted perseverance in his classroom in two cycles, each consisting of "a wall" and a scaffold (see Table 1). We highlight here two cases in which Ezio paused the groupwork to scaffold the whole class: first, to supply them with "a cheat", and second, to aid them in synthetically manipulating their empirical results to arrive at the function, $f(n) = 2^n-1$.

Table 1: Two Cycles of Perseverance, Each Consist of a "Wall" and a Scaffold

Cycle	The "wall"	The scaffold
Cycle 1	Ezio asking students repeatedly, how they know their result is the minimal possible number of moves?	"the cheat" - explains the recursive rule (i.e. how to find the answer for n rings according to the answer for n-1 rings).
Cycle 2	Ezio asking students, what would happen if they had 100 rings? (i.e. the recursive rule does not help)	the table - manipulate results to find the general function f(n)

Cycle 1. Ezio introduced the Tower of Hanoi and ensured all student pairs understood the task. As students began to collect and annotate their initial empirical results, Ezio circulated to six of the seven groups asking them "how do you know that's the fewest [number of moves]?" He repeatedly asked this question without waiting for or following up with students for a response. Moreover, the students continued to explore it empirically (i.e. trying to move rings in less moves) without discussing the question posed. We account for this question as the first "wall". Ezio followed this up with a whole-class scaffold where he explained "the cheat". The cheat represents the recursive nature of the task which allows one to easily solve for any number of rings n, given the result for n-1. For example, one can determine the least number of moves for 4 rings by building on the least number of moves for 3 rings. (we do not bring the full explanation for lack of space). We want to be explicit regarding the assumptions under which we consider this teaching move to be an over-scaffold: explaining to students the recursive rule is over-scaffolding *if* the teacher aims for student discovery of mathematical ideas. In the case where the teacher's goal is to directly teach the idea of recursive functions (which is not the case in hand), we do not consider this move to be an over-scaffold but rather an instantiation of direct instruction. Upon explaining "the cheat," Ezio prompted several groups to replicate the "cheat" or recursive rule by asking questions such as "If you know [the number of moves for] 4 [rings] can you show me 5?".

Cycle 2. After having students demonstrate the recursive rule, Ezio introduced them to the next wall: how could they determine the minimal number of moves for 100 rings? Here the recursive rule only works if students have generated the minimum for 99, 98 etc. Ezio followed up with another scaffold in the form of a table (distributed on a hand-out) where they could determine the number of minimal moves for n rings, into the function $2^n - 1$, only by way of a synthetic manipulation. As one example, for n=3 rings, students were instructed to: (a) find the minimal number of moves (7); (b) add one (8); (c) do prime factorization (2^3); and (d) write the final power form [$f(n) =2^3$]. Students were then asked to find a relationship between the final power form [$f(n) =2^n$] and the function [$f(n) =2^n - 1$]. Again, we view this as an over-scaffold in the sense that it does not center sense-making but rather guides students almost directly to generating the function.

Conclusion of classroom analysis. Our analysis shows that students moved back and forth between both edges of their collective ZPD as described by Sengupta-Irving and Agarwal (2017) as moments of "unnecessary struggle" and "no struggle" (see Figure 3). Ezio's questions (the "wall") placed students to the right edge of the ZPD students creating unnecessary struggle as students appeared to have few, if any, conceptual resources to answer the questions. Following their struggle, students were strongly scaffolded either directly by Ezio's whole-class explanation of "the cheat" (rooted in his enactment of the task) , or by the table on the handout (rooted in the design of the task). To conclude, Ezio's design and enactment of the Tower of Hanoi placed his *perseverance as persistence* at odds with sensemaking, all the while leaving out problem solving heuristics as a potential avenue for aiding students in discovery and sensemaking.

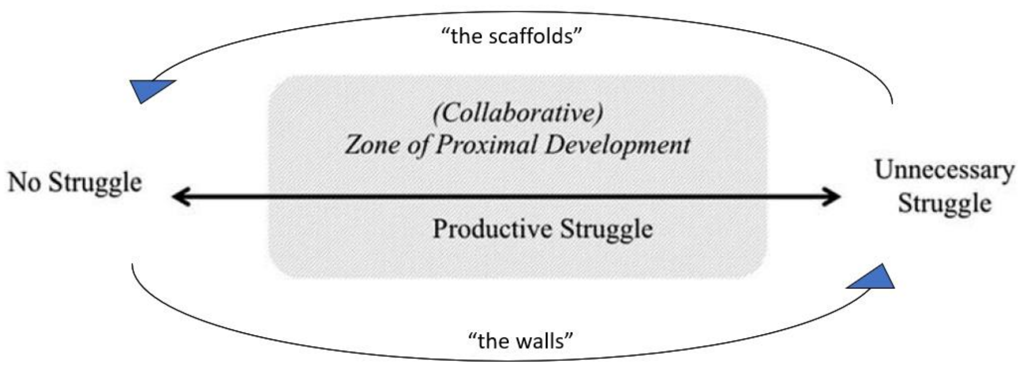

Figure 3: Cycles of "Walls" and Scaffolds Outside of Students' ZPD (adapted from Sengupta-Irving and Agarwal, 2017)

VFF Debrief Conversation

While we posit that Ezio's enactment of perseverance fell short of engaging students' sense-making in the task, we also acknowledge our own shortcomings, as teacher educators, in unpacking perseverance. During the debrief we did mention all three dimensions of perseverance, albeit in isolation from one another. For example, the coach (author one) confirmed that, in her view students were persevering (read: persisting), an observation supported by the partner teacher and the other researchers present. Secondly, she recognized that students were persisting through the activity but not verbalizing their thinking of how to reproduce the same number of (minimal) moves from one trial to the next. Consequently, the coach offered Ezio the heuristic of "solve a simpler problem first" (Polya, 1945/2004) strategy as a possible scaffold to center students' sensemaking in the lesson. The coach suggested this heuristic to Ezio as a scaffold to have students think about how and why they can consistently reproduce the transfer of 3 rings in 7 moves. By understanding the logic of the simpler problem, students could then discuss with their peers how to reproduce the transfer in the fewest moves and apply it to 4 rings, 5 rings, and so on. Essentially, they would be discovering and making sense of the "cheat."

As Ezio reflected on his "fear [that] they would waste so much time on the counting…[and not get] in-depth with anything else," he acknowledged the suggested heuristic as an important scaffold that would have accomplished students going "in-depth" with the problem. Veronica also acknowledged the value of giving challenging problems, as a first step, and the importance of teaching students problem solving heuristics. Ezio returned to the suggested heuristic several times as his big takeaway from the debrief in thinking about future ways to aid his students discovery of the "cheat" on their own.

Discussion

In the debrief, we discussed all the units of persistence, sense-making, and heuristics but did not explicitly connect the dots between them as constituting what we now propose as a conceptualization of perseverance. As researchers, we have the privilege to grapple in-depth with the in-vivo problems of practice we encounter as teacher educators. The discursive resources our analysis has generated were not available to us at the time of the debrief. Without an elucidated conceptualization of perseverance, we were also conflating perseverance as persistence as we too noted students not giving up in the absence of sense-making. Although the coach suggested a heuristic scaffold, it fell short of addressing the full mathematical practice of *make sense of problems and persevere to solve them,* by not connecting persistence to sensemaking.

Interestingly, as Ezio described the "wall" and hitting that breaking point, he reiterated that those that make it (i.e., persevere) in mathematics "learn *how* to overcome it" as opposed to those who "mentally give up on math." It bears noting here that Ezio is wondering how to help students overcome that breaking point. Learning *how* to overcome is distinct from pushing through (persistence). Overcoming a breaking point implies being able to access some internal or external resources that could shed some insight into the given hurdle, a resource such as a problem solving heuristic. Moreover, his comment about "mentally giv[ing] up on math" implicates the cognitive demand of mathematics which is more than the will and heart to continue--more than persistence. "Mentally giv[ing] up on math," seems to suggest a person not making sense of the mathematics or hitting a wall in which a path forward seems insurmountable. Although Ezio is unable to tease this out, and we are unable to support him in doing so, he does have the initial instinct and foresight to know that he wants to offer students some type of tool or heuristic that they can use in a general sense to overcome mathematical challenges.

In sum, a deeper analysis of Ezio's stance on perseverance merited more attention. As coaches, we were limited by an insufficient conceptualization of perseverance beyond persistence much like Ezio in his design and enactment of it in the classroom. While the coach was able to suggest a scaffold that served as a heuristic for the sake of making sense of the task, we were limited in explicating a formidable relationship between persistence, sense-making, and problem solving heuristics as constituting perseverance, his long term goal for his students.

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SUPPORTING BEGINNING TEACHERS TO ENGAGE IN RELATIONAL INVESTIGATIONS OF TEACHING AND LEARNING

Mallika Scott
California State University, Fullerton
malscott@fullerton.edu

This paper examines a social design-based approach to supporting beginning elementary school teachers toward ambitious and equitable mathematics teaching. First-year teachers were enlisted as co-designers of a learning community aimed at supporting participants to build classroom math communities that leverage students' diverse mathematical resources. Findings show that teachers collectively moved from thinking about teaching as fixing local problems to engaging in relational investigations of teaching and learning that centered students' mathematical experiences. This shift supported teachers to take up and make progress toward complex problems of practice in their classroom teaching. This study has implications for how we conceptualize, analyze, and design for equity-oriented learning for beginning teachers.

Keywords: Mathematics education, Instructional Vision, Equity and Diversity

> "I just feel like I have this huge moral dilemma. The math benchmarks are coming up and I have to cover all these things…it's just weighing on me. And there have been a couple periods where I haven't been the math teacher I want to be."
> Kara, Elementary School teacher, October 2016

Kara, a recent graduate of a highly-regarded teacher preparation program, is not alone in facing the moral dilemma of being committed to a vision of teaching that does not yet exist in schools on any wide scale. Schools in the US tend to be organized to sort students by various measures of "achievement" (standardized tests, "math benchmarks" etc.), searching out what students do not know rather than building on the reasoning evident in what students *are* doing (McDermott & Raley, 2011). This fixation on sorting produces winners and losers in ways that intersect with social hierarchies and inequities, resulting in students from some nondominant communities continuing to be constructed as "struggling" or "failing" (Martin, 2003, 2009). Given the high status of mathematics in the US and the ways mathematics achievement is often conflated with intelligence, math classrooms can be fertile ground for perpetuating the inequities endemic in systems of schooling preoccupied with finding deficits (Martin, 2009; Shah, 2017).

Scholarship on equity and mathematics teaching has helped to paint a picture of alternative classroom arrangements that can support robust learning for all students. In these classrooms, students are supported to make sense of complex ideas and to recognize and learn from the diverse strengths each student brings to mathematics (Boaler, 2008; Cohen, Lotan, Scarloss, & Arellano, 1999). Kara shares with other members of her cohort a deep commitment to this vision of ambitious, equity-oriented math teaching. Yet the pressure of having to give her 3rd grade students a "math benchmark" assessment led Kara to sometimes focus more on whether students were producing correct answers on pages of multiplication problems than on noticing and building on the diverse ways they were making sense of multiplication.

It is often assumed that a reasonable goal for new teachers like Kara is to "survive" their first year, not to take up an ambitious agenda. In line with this assumption, the support offered to beginning teachers tends to be geared toward generic teaching topics (Ingersoll & Strong, 2011; Mehta, Theisen-Homer, Braslow, & Lopatin, 2015). New teachers are often left entirely on their own to figure out how to navigate the "huge moral dilemmas" that arise when they attempt to counter the culture of deficit that dominates systems of schooling, leading some teachers to either give up on their commitments or to give up on teaching (DeAngelis, Wall, & Che, 2013).

This paper reports on the learning that can become possible when support for new teachers is intentionally designed to provide opportunities to take on the challenges of equity-oriented mathematics teaching in a community oriented toward a shared vision. Specifically, I investigate how a learning community, named Math Crew by participants (of which Kara was a member and I was facilitator), supported first year teachers to engage in relational investigations of teaching and learning that centered students' mathematical experiences.

Theoretical Perspectives and Prior Research

In this section I describe the vision for mathematics classrooms that formed the foundation of Math Crew. I then explain the ways that research on social design and teacher learning informed the design of this teacher learning community.

Ambitious and Equitable Mathematics Teaching

Math Crew was formed to support teachers to create classroom math communities that are both ambitious and equitable. I join other scholars in using the term "ambitious" to refer to math teaching that provides students with opportunities to engage in cognitively demanding math tasks and considers the ways students make sense of these tasks to be central to instruction (Jackson & Cobb, 2010; Kazemi, Franke, & Lampert, 2009). Further, I use the phrase "ambitious and equitable" mathematics teaching to point to a particular conception of equity underlying the shared vision held by Math Crew participants. In line with scholarship that interrogates the ways that our educational system continues to perpetuate disparate outcomes for different populations of students, I consider equity in mathematics teaching to mean not merely improving access to learning opportunities for all students, but also disrupting dominant hierarchies of power and privilege (R. Gutiérrez, 2008, 2013; Gutstein, 2003; Leonard & Martin, 2013; Martin, 2003). The vision underlying the Math Crew community is one that considers ambitious and equitable mathematics teaching to entail working toward a more just educational system where students from nondominant communities are assumed to have rich and diverse mathematical resources that benefit everyone's mathematical learning.

A Social Design Approach to Teacher Learning

Organizing classrooms to build on diverse mathematical strengths runs counter to the common discourse and practices of math schooling. Design responses to this challenge must therefore provide teachers with robust support that honors the complexity and dilemmas inherent in this work. Social design-based research approaches focus on designing activity systems that consider the tensions and contradictions participants encounter in their work to be resources for learning, rather than obstacles to be overcome (Engeström, 2011; Gutiérrez & Jurow, 2016; Gutiérrez & Vossoughi, 2010). In these approaches to design, participants are actively involved in co-constructing the supports they need to navigate the tensions they encounter (Gutiérrez & Jurow, 2016). By focusing on systems rather than individuals, social design-based research attends to the ways different aspects of design, such as the tools and artifacts provided and the participant structures used, work together to mediate learning toward a shared goal. Drawing on these principles, Math Crew was co-designed with participants to support teachers to navigate the tensions of attempting to enact ambitious and equitable math teaching while working in contexts where that vision was not widely shared.

In taking the activity system as the unit of analysis, I conceptualize teacher learning as changes in participation in teaching activities over time (Rogoff, 1994). I consider "teaching" to include classroom teaching with students as well as the planning, reasoning, and reflection that shape what happens in classrooms. For the design of Math Crew routines and my analysis of learning over time, I draw on in-depth studies of teacher learning communities that have identified generative activities for learning. These studies have found that conversational routines (e.g. replays and rehearsals of actual teaching) that support teachers to make connections between teaching choices and underlying principles of ambitious teaching can be productive sites for teacher learning and for analyzing changes in teachers' participation (Horn, 2005; Horn & Little, 2010). This line of research has also highlighted the importance of interactional norms that govern teacher activities, by noting, for example, the extent to which discussions function to either open investigation into "problems of practice" or close conversations to further analysis (Horn & Little, 2010; Little, 2002). Drawing on this research, the intention of the design of Math Crew was to create a space that invited deep collective investigations of problems of practice and supported participants to make connections to their shared vision of teaching.

In this paper, I analyze teacher learning in Math Crew by examining changes in participation over time. I find that participants moved from approaching teaching by trying to fix local problems to engaging in relational investigations of teaching and learning that centered students' mathematical experience as it connected to different aspects of the classroom learning ecology. As participants shifted to investigating from this relational perspective, they took up complex problems of practice in their classroom and worked on them over time in ways that were consequential for their students. Their learning coevolved across the Math Crew activity system and their classroom activity systems, with the travel across systems providing new resources for learning toward ambitious and equitable mathematics teaching.

Methods

This study grew out of my experience teaching prospective elementary teachers in a math methods course at a large public university. During their preparation, many of these teacher candidates developed a deep commitment to organizing their classrooms to build on students' diverse mathematical strengths. At the same time, they were daunted by the realities of working toward a vision for math classrooms that differed substantially from the focus on finding and fixing deficits present in many schools. With the aim of learning more about how to support beginning teachers toward ambitious and equitable math teaching, I invited six of these teachers to join me in creating a teacher learning community to provide support during the first year of teaching. These teachers were selected because they had each expressed commitment to equity-oriented math teaching during the methods course and would be teaching in local K-5 classrooms during the 2016-2017 school year. All six enthusiastically accepted, and Math Crew was created.

Table 1: Participants, Schools, Districts (all pseudonyms)

Teacher	Grade	School	School District
Selina	1st grade	Connections Community School	Hamilton Unified School District (HUSD)
Marie	1st grade	Rise Up Charter School	Charter School in Hamilton

Kara	3rd grade	Connections Community School	Hamilton Unified School District (HUSD)
Maritza	4th grade	Glenshire Elementary School	Logan Unified School District (LUSD)
Tina	5th grade	Cleveland Elementary School	Sullivan Unified School District (SUSD)
Lauren	5th grade	Taylor Elementary School	Hamilton Unified School District (HUSD)
Facilitator: Author			

Math Crew Design

The design of Math Crew was grounded in two structures: monthly learning community meetings and classroom visits (4-6 over the course of the year) by me as group facilitator. Underlying both structures was a set of design principles drawing from literature on social design and teacher learning communities: 1) Participants must be actively involved in design decisions, 2) Tools, artifacts, routines, and participation structures should be oriented toward student and teacher strengths rather than deficits, 3) Conversational routines and participation structures should support collective investigation of problems of practice.

As participants were actively involved in shaping and reshaping Math Crew, monthly meetings and classroom visits were responsive to the needs of the community. Different routines developed over the course of the year, including sharing success stories and dilemmas from the classroom, doing math together to support our thinking about specific content, choosing focal students and discussing questions about these students, and looking for strengths in student work and discussing how to build on them. During the first two Math Crew meetings, we worked together to formulate our shared vision for ambitious and equitable math teaching. Out of these discussions, the group articulated their shared commitment as “building a classroom community where every student meaningfully contributes to the mathematical work of the classroom.”

Data Collection

Primary data sources include monthly two-hour, video-recorded community meetings, ethnographic field notes from classroom visits, audio-recordings of a focus group interview mid-way through data collection, and audio-recordings of individual closing interviews with each participant. For the purposes of this paper, video-recordings of Math Crew meetings serve as the primary data source, with field notes serving as a supporting data source.

Analytic Methods

To reduce the data, I created activity logs for the nine Math Crew meetings, breaking them into 8-10 minutes episodes and writing a summary of the activity with observer comments. During this process, the routine of starting each meeting with one participant sharing a teaching story about their classroom math community (suggested by a participant at our first meeting) emerged as an activity to investigate for changes in participation over time. This routine happened at six of the nine meetings and followed the same protocol. I created more detailed activity logs for these stories, chunking them in 3 to 5 minute episodes and writing detailed notes about the content of the episode and observer comments about emerging patterns. These conversations were transcribed and coded using an open coding process to capture teacher participation. I worked with a research assistant to begin to group codes and to identify patterns in the ways conversations evolved over time. Through this process we decided on a subset of

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

codes that best captured the changes we were seeing in the video-recordings. This analysis revealed two phases of activity marked by substantial differences in participation. The table below shows our condensed coding scheme.

Table 2: Coding Scheme to Capture Change Over Time

Code	Description	Example
Sharing students' mathematical actions	Describing mathematical participation of students	"He was able to, you know, interpret his own data and then come up with a question which is really hard and then answer his own question."
Connecting learning ecology with math participation	Relating how students are participating to an aspect of the classroom learning ecology	"It reminds me of Kara's story about Ahmed, like having different content kind of opening up different kinds of opportunities for kids."
Connecting to ambitious and equitable math teaching	Relating a teaching action to their shared commitment	"At the end of the day the kids are out of my room, I can take a breath and then I can actually look at their work and look at what they are doing and notice their strengths"

Findings

Over the course of the school year, participants brought different teaching stories to discuss in Math Crew as they tried to enact their commitment to ambitious and equitable math teaching. Analysis of the activity of making sense of teaching stories over the course of the school year revealed two phases of the activity: Phase 1 in which teaching was minimally investigated, where relatively simple solutions were offered to complex teaching issues with little connection to a broader vision for teaching, and Phase 2 in which Math Crew teachers began to investigate students' mathematical experiences as part of a complex classroom learning ecology. As teachers engaged in these relational investigations of teaching and learning, they increasingly connected their choices in the classroom, what they saw students doing in response to these choices, and their shared commitment to creating ambitious and equitable classroom math communities.

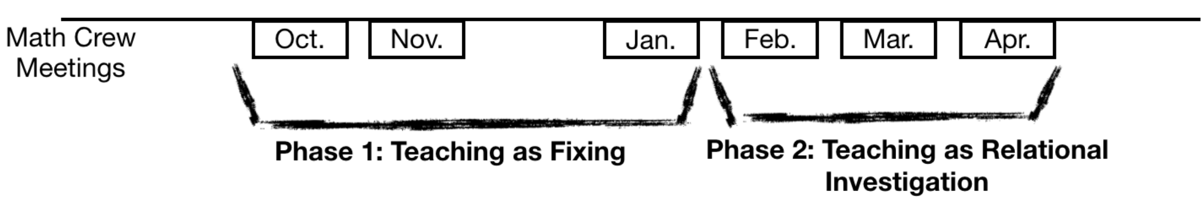

Figure 1: Change Over Time in Math Crew

During Phase 1, when teachers engaged in the routine of sharing a teaching story, both the telling of the story and the conversation that followed focused on relatively simple ideas or solutions. Teachers tended to talk mainly about the perspective of the teacher and to minimally investigate the reasons students may be acting in particular ways or to connect their actions to aspects of the classroom learning ecology. During Phase 2, stories or issues were still sometimes initially framed from the teacher perspective, but subsequent discussions of the classroom began to include investigating students' mathematical experiences and making connections between student actions and the classroom learning ecology. These discussions touched on many larger themes of teaching related to the Math Crew shared commitment to ambitious and equitable math teaching and often led the group to consider complex problems of practice such as how to support students to see value in each other's mathematical contributions or how to support students to recognize many different ways to be good at mathematics.

To provide a more detailed look at the changes in these conversations over time, I examined the ways teachers participated in these conversations, analyzing how teachers talked about mathematical participation and the connections they made to the learning ecology and to their shared vision. Table 3 shows the frequency and nature of these discursive practices during the story telling routine.

Table 3: Discursive Practices over Time

	Phase 1			**Phase 2**		
	Oct.	Nov.	Jan.	Feb.	Mar.	Apr.
Sharing students' mathematical actions	1	3	1	6	5	6
Connecting learning ecology with math participation	0	2	0	1	5	10
Connecting to ambitious and equitable math teaching	0	0	1	7	9	11

As is evident in the table above, during the first three Math Crew meetings, discussions of teaching stories were not well connected to students' mathematical actions, to how their actions related to the classroom learning ecology, or to their shared commitment to ambitious and equitable math teaching. The conversations during phase 2 included many more instances of connecting to the specifics of what students were doing in the classroom as well as connecting to themes and questions related to the Math Crew shared commitment of building a math community where every student meaningfully contributes to the mathematical work of the classroom. These discussions led to different articulations of teaching issues. Whereas in early meetings issues tended to remain local problems related to a particular classroom or lesson or to particular students, in later meetings even if issues were initially framed in local terms, participants made connections to larger problems of practice related to the shared vision of Math Crew and considered the issues from the perspective of how they might shift aspects of the classroom learning ecology.

Phase 1: Teaching as Fixing Local Problems

An example of this early form of discussing teaching stories where complexity was left relatively unexplored, occurred in the October Math Crew meeting when Tina shared a story from her 5th grade math classroom. She described a moment when a student shared an incorrect answer to a division problem, commenting, "another student turned around and said, 'WHAT?!?' And so right there and then I was like 'okay we need to refocus, this is not the way we talk to each other in math.'" Tina continued, "I had to remind myself just cuz I tell my students once, they are not going to immediately change their mindset and be able to be the kindest and nicest people in the world." In Tina's telling of this story, she described the issue as students not being kind to each other, with no exploration of why students might be responding critically to incorrect answers. Kara responded, "it kind of takes on this icky feeling of kids like really being critical of each other's thinking…And I really get on them for that, but I feel like there hasn't been a full change of attitude." Here, Kara continued with Tina's description of the issue as students being critical and then described her insufficient attempts to solve that problem.

The conversation lasted for nine minutes and the teaching responses generated were to "keep repeating yourself" because it takes time and to "write out our norms" so that everyone has an explicit visual reminder. These ideas are sensible, but they are limited and these limitations are directly related to how teachers made sense of Tina's story. The conversation focused on the perspective of the teachers, with the issue being understood as getting students to be kinder to each other. If the issue had been investigated in more depth and teachers had considered possible reasons *why* students might be responding negatively to incorrect answers such as students not seeing what they can learn from incorrect answers or not seeing value in each other's ideas (as were discussed in later Math Crew meetings), very different sorts of conversations about teaching could have been possible. These conversations might have made space for teachers to consider issues of status or of narrow cultural notions of what math is, which may have led to inquiry and teaching ideas more responsive to the complexity of the issue.

Phase 2: Teaching as Relational Investigation

During the February Math Crew Meeting, Tina brought a different story to the group that led to relational investigation and the identification of a complex problem of practice. Tina explained to the group that one of her students, Albert, had asked her if the reason they did math at the end of the day was because it wasn't important. She then decided to have her students respond to the writing prompt, "Do you think math is important? When do we use math outside of math class?" Albert was the only student who wrote that he didn't think math was important. She read his response to the group, and then commented, "so I have the question now of how I can continue to show the class that I do value math and that it is very useful…a lot of it has to do with growth mindset too with this particular student, if he's willing to open himself up to liking math." In Tina's initial description of the issue, she described it as a relatively simple issue of showing her students that "I do value math" and of Albert being "willing to open himself up."

In the 15-minute conversation that followed, participants investigated Tina's initial framing to consider possible reasons why Albert might not yet name math as important. Lauren commented, "What stood out to me was that the one reason he gave for being good at math was because he wanted to be smart which means he associates being good at math with being smart. And since he doesn't associate himself with math then he doesn't think he's smart." Maritza added, "Yeah, I kinda thought that he only associates math with calculations. He seems to think calculating numbers is boring so therefore, math is boring." Marie picked up on the theme of math and smartness, adding "what's so sad is that sometimes they are not the ones that told themselves they are dumb… just like recognizing that it is a bigger issue that we have in our culture about math and the value of it…It's just like a symptom of a larger problem." Here Marie pushed on what Tina offered as an issue of showing that she valued math and Albert being "willing" to like math by suggesting that Albert's writing could be understood as sensible given pervasive cultural narratives about what mathematics is and what it means to be good at it. The conversation then turned to how teachers might support their students to see math as a broader more inclusive space that is not just about calculation and how they might support individual students like Albert to be recognized as competent within that broad space. This problem of practice of upending cultural notions of what math is and what it means to be good at it is a much more generative space for equity-oriented teacher learning than the local problem of Tina needing to show she values math or of Albert needing to be willing to like math.

Summary of Shift to Relational Investigation

In Phase 1 of the activity of making sense of teaching in Math Crew, classroom stories were shared and discussed in ways that offered some new teaching ideas. These conversations provided teachers with space to make sense of teaching together and offered new resources for classroom teaching such as posting norms and giving students new ways to respond to each others' thinking. In Phase 2, as teachers dug deeper into teaching stories from a relational perspective that centered students' mathematical experiences, they participated in ways that created many more opportunities for equity-oriented learning. Lauren and Maritza analyzed the specifics of what Albert was saying and then connected his writing to broader issues of teaching and learning. Marie connected Albert's perceptions about himself and about mathematics to dominant cultural narratives about mathematics. Participants suggested teaching responses to shift the classroom learning ecology toward their shared vision such as providing students with opportunities to recognize each other's mathematical strengths. Throughout this conversation, participants considered Albert's perspective and connected his mathematical experience to different aspects of the learning ecology and to ambitious and equitable mathematics teaching. These ways of participating were generative for the conversation in that they led to new ways of understanding the issue and they were generative for ongoing teacher learning in that analyzing and problematizing what we think we see in the classroom is integral to creating a classroom math community that functions very differently from what is currently typical in schools.

This shift toward relational investigation supported teachers to try out new practices in their classroom teaching that were responsive to the complex classroom learning ecology. For example, the story described above about Albert was part of Tina's investigation over time into the problem of practice of expanding students' conceptions of mathematics and mathematical competence. The investigation of Albert's experience both during this conversation and during my visits to the classroom prompted Tina to implement new routines to provide students with structured opportunities to recognize each other's mathematical strengths. After the conversation described above, Tina drew on ideas offered from different participants to try out a new routine of having students write down each other's mathematical strengths at the end of partner math tasks. By the end of the year, Tina reported, and I observed, that students were spontaneously noticing and naming each other's mathematical strengths using specific mathematical language (e.g. "finding easier ways to count the cubes using multiplication and arrays", field notes 4/12/17) even when she did not use this routine. This indicates that Tina's relational investigation led to shifts in her teaching that were consequential for her students.

The shift from engaging with teaching as fixing local problems to engaging in relational investigations of teaching and learning supported Math Crew teachers to think deeply about their classroom learning ecology, to take up complex problems of practice, and to work on them over time in ways that were responsive to students' experiences of mathematics. This new equity-oriented learning became possible because participants were part of a community oriented toward a shared vision that fostered new forms of activity such as supporting teachers to come to see the mathematics classroom from their students' perspective.

Conclusion

In Math Crew, first-year teachers moved beyond focusing on the overwhelming list of day-to-day concerns that tend to dominate for most first-year teachers to dig into complex problems of practice and to consider how their teaching choices might shift the classroom learning ecology to support each student to meaningfully contribute to the mathematical work of their classroom.

The first year of teaching is often talked about in terms of "survival" rather than in terms of possibility. I offer Math Crew as a counter story to deficit narratives about the first year of teaching with the aim of raising both our sense of possibility and our sense of responsibility about supporting beginning teachers toward ambitious and equitable math teaching. If we hope to support this type of equity-oriented learning, this case suggests that it can be productive to move beyond thinking about the knowledge and practices we want teachers to master to think about the systems teachers are working within and how those systems can make available new forms of activity. Future work could build on these beginnings to investigate how we might design and support teacher communities oriented toward a shared vision across different pre-service and in-service contexts to support robust teacher learning.

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THE ROLE OF EMOTIONS IN SIMULATIONS OF PRACTICE

Amanda Milewski
University of Michigan
amilewsk@umich.edu

Emanuele Bardelli
University of Michigan
bardelli@umich.edu

Pat Herbst
University of Michigan
pgherbst@umich.edu

Technology-mediated simulations of teaching practice are becoming a more common way to introduce teachers to the dilemmas of teaching during professional development. In this paper, we show that the inclusion of markers of student emotions in cartoon-based scenarios of teaching changes teachers' appropriateness rating of the actions that the teacher took in the storyboard. Our results show that the inclusion of markers of student emotions in representations of practice could cue teachers into a particular judgments of action.

Keywords: Instructional activities and practices; Affect, Emotions, Beliefs, and Attitudes; Technology

Computer-based virtual environments, using avatars to represent students, have grown in popularity in both teacher education and educational research (Gibson, 2007; Herbst, Chazan, Chen, Chieu, & Weiss 2011; Ma et al., 2016; Straub, Dieker, Hynes, & Hughes, 2014). While such environments have been used to engage teachers in the cognitive and rational aspects of teaching, they are not frequently used to elicit teachers' reasoning related to more affective aspects of teaching (Lehtonen, Page, Miloseva, & Thorsteinsson, 2008). However, similar technology, using avatars in virtual environments, have been shown to be effective in helping individuals gain increased understandings of social situations (e.g. Cheng & Ye, 2010; Tettegah, 2005). This suggests the potential for such environments for supporting teachers' engagement with the more emotional aspects of teaching.

This is good news, given that emotions arguably play an important role in teachers' decision making (Demetriou, Wilson, & Winterbottom, 2009; Hargreaves, 1998; Zembylas, 2005). For some time now, we have known that emotional intelligence —the ability to perceive one's own emotional state—has evident value in professional circumstances (e.g. Jaeger, 2003; Lopes et al., 2006), including playing an important role in teacher's job satisfaction and organizational commitment (Naderi Anari, 2012). Beyond the many studies on teacher's emotional intelligence, there is reason to believe that the broader construct of emotional competence—the ability to recognize and understand the emotions of others—may be highly related to improved outcomes for student learning (Jennings & Greenburg, 2009). This hypothesis seems reasonable in light of recent findings that the relational aspects of teaching are as important as the cognitive aspects for student outcomes (Battey & Neal, 2018). Knowing how teachers' emotional competence shapes their work is particularly important given they often have to know what people are feeling without having to directly probe them to report their feelings—e.g., reading students' emotions when in the midst of teaching. However, learning to effectively read facial expressions is non-trivial and particularly challenging during instructional transactions where there exists a power differential between the individuals, as is the case with teachers and students (Galinsky, Gruenfeld, & Magee, 2003). Yet, little work in the field has aimed to gauge or support teachers in gaining such competencies (Hargreaves, 1998).

We see the work we describe in this paper as adding to this literature about the role of emotions, in particular teachers' emotional competence, in teaching. We have been designing scenario-based assessments to understand teachers' reasoning about mathematics instruction. In that work, we have been cautious about using facial expressions because of our own lack of certainty about how facial expressions might impact the ways in which participants' judge the instructional actions represented within such items. More specifically, we have been uncertain whether the addition of facial expressions might bias the items for individuals more capable of reading facial expressions (in both human faces as well as within cartoon representations). In that context, we have been wondering whether and how emotions represented on cartoon-based representations of students in classroom scenarios of practice play a role in the ways that teachers appraise the scenarios of teaching.

Theoretical Framework

Digital simulations of teaching use a variety of semiotic resources to represent the individuals and settings of practice. Icons, language, and dashboards (e.g., in SimSchool, Gibson, 2007) are three kinds of semiotic resources frequently used for students to respond to instruction, but for each of those a limited variety of systems of choice are available. Herbst et al. (2011) argue that graphic representations of teaching practice using two-dimensional, nondescript icons to represent teacher and students can display nonverbal aspects of teaching practice. They argue that comic-based representations can represent the same meaning as, for example, textual accounts of practice. Herbst and Chazan (2015) suggest that ***lean***, ***nondescript*** graphic elements allow the designer to easily represent practice as emergent in mathematics classroom interactions and enable viewers to project into those characters their own settings and clients—in contrast with avatars whose graphical features are used to mark high individuality. In a follow-up study, Herbst, Boileau, Clark, Milewski, Chieu, Gürsel, and Chazan (2017) show that this lean cartoon-based semiotic system can be expanded to include signifiers for complex social constructs such as race and ethnicity. In their comparison of the written story and the storyboard for the Case of Mya (Chazan, Herbst, & Clark, 2016), Herbst et al. (2017) conclude that the storyboard representation encouraged a plurality of interpretations of the same case, both expected and unexpected.

Following this line of inquiry, our lab has begun exploring whether emotions can be encoded using a similarly lean semiotic system when representing professional scenarios of teaching. One crucial foundation for that work is the well-established line of research from Ekman and colleagues who developed the facial action coding system (FACS) (Ekman & Friesen, 1978)—an analytic scheme for precisely describing various configurations of human faces as functions of muscle movements. The FACS has made possible the cataloging of human facial expressions for reliably representing human emotion interpretable by humans across cultural boundaries (Ekman & Friesen, 1986). The reliable interpretation of emotions based on facial expression representations suggests that the ability to interpret facial expressions is an important aspect of emotional intelligence (Keltner, Sauter, Tracy, Cowen, 2019). The FACS has also been used to inform the development of animated and cartoon facial expressions (McCloud, 2006; Thórisson, 1996; Spindler & Fadrus, 2009).

In this paper, we leverage the FACS to modify the default expressions of a set of cartoon characters we call the Thexpians B (Herbst & Chieu, 2011) to include lean semiotic markers of emotions (Dimmel, Milewski, & Herbst, 2015). For example, Figure 1 shows the affordances of the facial expression semiotic system to represent student emotions. In both panels of Figure 1, a student disagrees with a peer who has critiqued his shared solution. In the left panel, the student's eyes and mouth express a neutral state. In the right panel, we added to the eyes triangular-shaped eyebrows that slope inward and changed the shape of his mouth from round to triangular to represent the emotional state of anger. These changes are meant to communicate the emotions associated with that specific moment.

In previous work, we have compared these kinds of lean renderings of facial expressions with that of facial expressions as represented by human actors and demonstrated that participants are, in fact, able to interpret the cartoon-based representations of emotion with similar levels of accuracy to that of photos (see Dimmel et al., 2015). Here, we build on that work to ask how the addition of such expressions into scenarios of teaching make a difference for participants' judgment of the instructional actions taken within such scenarios. To that end, we ask: To *what extent* and *how* does the representation of emotions on student avatars in a cartoon-based representation of practice change the way that practicing teachers judge the appropriateness of instructional actions of teacher avatars?

Neutral Character	**Emoted Character**

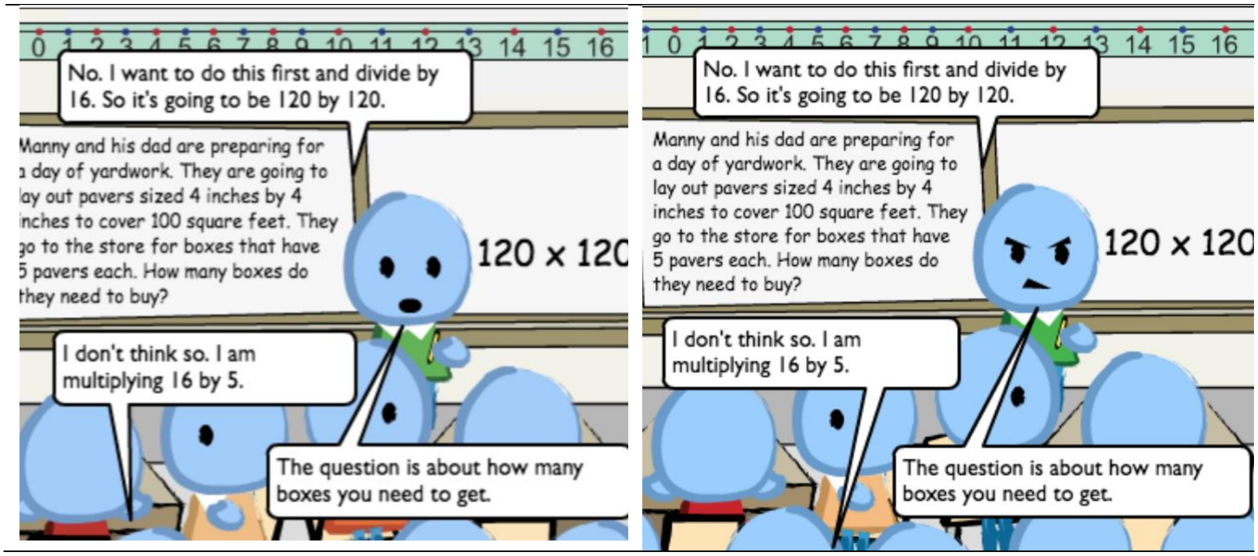

Figure 1: Neutral and Emoted Characters

Methods

Participants

In this study, we use a randomized control trial design to assess the extent to which representations of emotions impact teacher interpretations of scenarios of teaching. We recruited 69 participants from the respondent pool for the validation study of the PROSE instrument (Herbst & Ko, 2018). We randomly assigned participants to either the emoted or neutral experimental conditions. For the emoted condition, we edited the original PROSE items to include relevant facial expressions at the scenario turning point (as shown on the right pane of Figure 1). For the neutral condition, we removed all facial expressions from the original PROSE items. Instead, we represented all characters in the scenario using the neutral eye and mouth shapes (as shown on the left pane of Figure 1). Intuitively, comparing the responses to the emoted and neutral scenarios allows us to assess the extent to which the explicit representation of facial expressions in scenarios of teaching has an impact on teachers' interpretation of instructional actions.

Outcomes

The main outcome for this study is teachers' reactions to scenarios of instruction. To this end, we adapt the PROSE instrument (Herbst & Ko, 2018) to include facial expressions. The main outcome of interest of the unadapted PROSE instrument is teachers' recognition of professional obligations as a source of justification of instructional actions. In that instrument, teachers are asked to assess the extent to which a teacher instructional action was justified on the grounds of professional obligations (Herbst & Chazan, 2012). In the adaptation of this instrument to our study, we selected nine items[1] from the overall battery of twenty-four PROSE items for two possible reasons. First, following Milewski and Erickson (2015), we located items where a sizable number of participants indicated that their rating of the item depended on some assumptions they needed to make about the scenarios—we call these *high construal items*. We scanned teachers' responses to high construal items to locate items for which participants professed needing more information about the emotions of the student avatars—we call these *high emotional construal items*. For example, participants indicated their rating depended on whether or not a student in the scenario was upset. Accompanying these responses, we also found many participants that assumed that the represented student was in fact upset (even though the expression on the student's face was neutral in the item). That said, not all responses contained references to emotion and presumably some participants did not construe emotion into the item and thus the inclusion of emotional cues into the item might alter the way that participants respond to the item. Using this heuristic, we located seven items for inclusion in the study. Second, some items in the original PROSE instrument included some facial expressions which participants could have interpreted as emotions. We call these items *PROSE emoted items* and for the purposes of this study selected two such items. For all of these items, either the data or the original design of the item led us to believe that emotion may have been part of the way that participants were reasoning about their judgments. Our main outcome of interest with these adapted items from the PROSE instrument was to understand more about how the inclusion of facial expressions as markers of emotion was affecting the ways in which participants' rated the items. Figure 2 reports a brief synopsis of each item and whether it was selected for its high emotional construal or PROSE emoted items.

Participants also completed an emotion recognition questionnaire (Keltner et al., 2019). This instrument shows participants photographs of trained actors displaying emoted facial expressions and asks participants to identify the corresponding emotion. We also administered a parallel version of this instrument (developed by Dimmel et al, 2015) that uses our cartoon characters—using cartoon-based shapes for eyes, eyebrows, and mouth as well as changing their location in relation of the overall face to reproduce the facial expressions in the photographs. These two instruments allowed us to measure the extent to which our participants are able to recognize emotions in both photographs and cartoons. We use this information as a robustness check of our results.

Table 1: Description of Selected PROSE Items

Item	Description	Type
a1012	A/n (embarrassed) student shares a solution to a problem on the board. The teacher asks others to build on that solution strategy.	High Construal

[1] One item was removed from these analyses because of a mistype in the question prompt.

a3012	A student is sharing her/his solution on the board. Another student in the classroom suggests another solution strategy and the student at the board responds (with anger). The teacher compliments the students handling the disagreement.	High Construal
a3022	The teacher is asking questions to the class. One student answers all the questions, while other (frustrated) students look on. The teacher calls on students by pulling names from a cup.	High Construal
a3032	A (confused) student privately asks the teacher a question. The teacher asks the student to ask the question out loud to the class.	High Construal
a3102	A student tells the teacher that she/he is done with her work. The teacher asks her/him to help other students instead of working on the homework and she/he (begrudgingly) complies.	High Construal
a3112	A teacher ignores volunteering students to cold-call a (confused) student to publicly share a definition of a term.	High Construal
a3152	A student is solving a problem on the board. The teacher asks her/him to start over because other (frustrated) students cannot read the writing.	High Construal
a3162	Teacher offers magnets to (excited) students. When many students volunteer, the teacher decides to raffle the magnets.	Emoted PROSE

Analysis

Our data is nested both within respondent and item. That is, we asked the same participants to answer multiple items. As we are interested in the effect of facial expressions on the overall evaluation of the instructional action, we use a multilevel linear model to account for the nesting of responses within participants and items. In detail, we use a mixed effects model (Raudenbush & Bryk, 2002) with crossed participant and item random effects. We fit the equation

$$Y_{pi} = \beta_0 + \beta_1 \cdot Emoted_{pi} + u_p + u_i + e_{pi}$$

where u_p is the participant-level random effect accounts for participant-level differences in the assessment of the instructional action that is unrelated to our experimental condition, u_i is the item-level random effect accounts for item-specific differences in the assessment of the instructional action that are due to contextual factors embedded in the specific PROSE instrument. The coefficient of interest is β_1. This coefficient estimates the causal effect of including facial expressions (i.e., the difference between seeing an emoted item versus a neutral item) on participant assessment of the instructional action represented in the PROSE items.

Results

Effect of Emotions on the Assessment of Instructional Actions

Table 1 reports the estimates of the effect of facial expressions on teachers' assessment of instructional actions. Column 1 reports the results of the unconstrained mixed effects model, column 2 reports the effect on the raw assessment score (1-6 Likert scale), column 3 reports the same effect in standard deviation units. Columns 4 and 5 report the results of robustness checks testing whether our main results are sensitive to participants' recognition of emotions in photographs or cartoons.

We find that the inclusion of facial expressions in scenarios of teaching has an impact on teachers' assessment of instructional actions of, on average and in absolute value, 0.356 points on a 1-6 Likert scale or 0.280 standard deviation units. This result appears to be robust against participants' recognition of emotions in photographs or cartoons. That is, the magnitude of the effect does not change after controlling for participants' emotion recognition in photographs and cartoon images. This allows us to conclude that participants' change in their assessment of the instructional actions in emoted items is due to the emotions that are represented in the characters facial expressions. If this was not the case, we would observe a change in the estimated effect once controlling for participants' emotion recognition.

Table 2: Average Effect of Facial Expressions on the Assessment of Instructional Actions

	(1) Unconditional	(2) Pooled	(3) STD	(4) STD + Cartoon Rec	(5) STD + Photo Rec
Emoted		0.356^{**}	0.260^{**}	0.262^{**}	0.267^{***}
		(0.111)	(0.081)	(0.082)	(0.081)
Constant	1.798^{***}	1.583^{***}	0.000^{*}	0.022	0.121
	(0.239)	(0.247)	(0.063)	(0.185)	(0.213)
Var(Resp)	0.014	0.000	0.000	0.000	0.000
Var(Item)	0.430	0.430	0.000	0.000	0.000
Var(Residual)	1.633	1.617	0.844	0.841	0.840
Observations	544	544	544	544	544

Note. The unconditional model reports the results of a cross-effect mixed model that controls fro respondent- and item-level variance components. Pooled results report the difference between treatment and control groups on raw item scores. Standardized results use the control group mean and standard deviation to center the raw scores. Standard errors in parentheses. * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

How Facial Expressions Affect the Assessment of Instructional Actions

We explore how facial expressions affects teachers' assessment of instructional actions by estimating the effect of the inclusion of facial expressions on each individual PROSE item. Figure 3 visually presents the differences between the emoted and neutral conditions in their assessment of the instructional actions in each of the eight PROSE items.

We note three main findings. First, in three items (a3022, a3012, a3101) the emoted scenarios received higher agreement with the normative actions than the neutral items. For the rest of the items, participants agreed less with the normative actions in the emoted items than in the neutral items. This finding suggests that facial expressions can both increase and decrease agreement with instructional actions. Second, we notice that all the 95% confidence intervals cross the red line at zero. This suggests that none of the differences that we observe are significant at the 95% level. This result likely indicates that our experiment was underpowered to detect significant differences in participants' assessment on each individual item. Third, even if these differences are not significant, we can still make inferences about the direction of these differences. That is, most of the confidence intervals for the negative point estimates are to the left of the dashed vertical line and most of the confidence intervals for the positive intervals are to the right of the dashed vertical line. This gives us evidence that the direction of the effect of emotions on participants' assessment of instructional actions is in the same direction as the sign of the point estimate for most items.

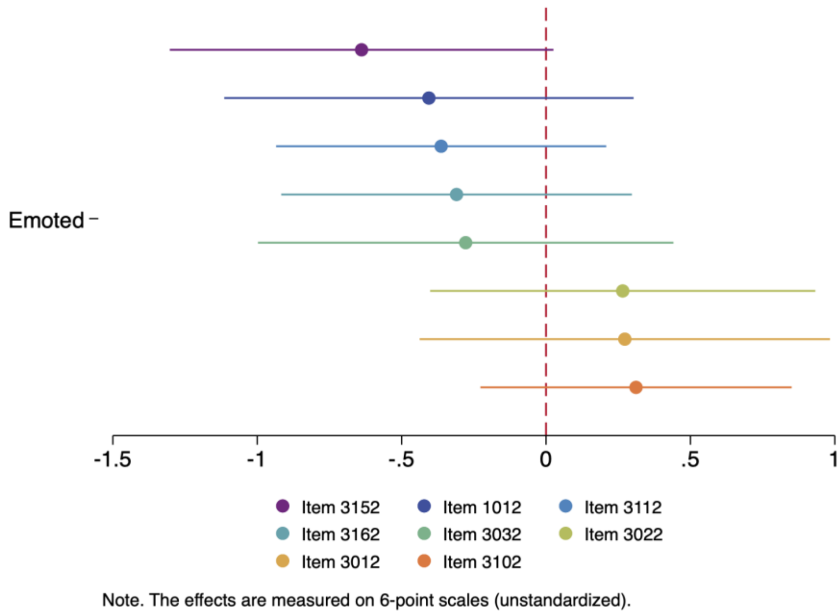

Figure 2: Effect of Facial Expressions on Each PROSE Item

Future Directions

Together these two analyses suggest that: (1) the inclusion of facial expressions for representing emotions can impact the overall participant agreement with instructional actions and (2) the direction of that effect is not uniform—that is it can cause participants to agree with the actions of the teacher more or less favorably. With that, we suggest that more work will need to be done to understand exactly how the addition of emotions makes a difference for participants overall rating of an item. To begin this work, we examine some of the more qualitative differences between the items for which the addition of emotional cues had differing effects. To frame our quantitative results, a positive difference between the emoted and neutral items means that participants who saw emoted items deemed the teacher resolution of the teaching dilemma as more appropriate than the teacher's normative (expected) action. Conversely, a negative difference between the emoted and neutral conditions means that the inclusion of emotions in the items made participants view less favorably the teacher's follow-up action.

With this in mind, we notice that the emotions in items where we have a negative difference are the direct result of the teacher's initial action, either (1) the teacher ignoring or being insensitive to a student's emotions (a1012, a3032, a3112) or (2) failing to anticipate students' emotions (a3152, a3162). For these items we see participants casting judgment on the represented teacher's actions. For example, after viewing a scenario in which a teacher overlooks students who are volunteering and cold calls a non-volunteering student with a confused expression, one participant reacted by saying "I prefer to acknowledge those who are willing to contribute *rather than embarrass those who are not. The students who did not raise their hands in this scenario look confused*." (*emphasis ours*, a3112, 4356). Similarly, after witnessing the teacher offer mementos to volunteers only to change her mind and draw sticks once students demonstrate excitement about the mementos, one participant commented "Teacher should have *planned in advance* to draw sticks, and told the students that's how they would determine who gets one" (*emphasis ours*, a3162, 5095).

For the items for which we observed a positive difference, student emotions are a consequence of other students' own actions and the represented teacher could be understood as resolving these situations (a3012, a3022, a3102). For these items, we see participants' responses supporting the teacher's actions, in spite of the students' emotional state. For example, in the context of an item in which a student prefers to move on to other homework rather than help his peers and the teacher insists he help his peers, one respondent said, "*This student is also being rude* by being bold enough to state that they are sure they are correct. I have had students like this is[sic] the past and they have often benefited from checking answers when they thought they had done all their work correctly. The idea of working in groups is so that everyone can benefit." (*emphasis ours*, a3102, 4571). These differences across the teachers' reactions to emotions in the items makes us wonder whether the source of the emotional reaction plays a role in how teachers develop an agreement with instructional actions. We plan to explore this insight in future work.

Discussion and Significance

With the increase in interest in student non-cognitive outcomes (Battey & Neal, 2018), it has become more important to understand the role of emotions in teacher decision-making (Hargreaves, 1998). In this paper, we explored how emotions play a role in teacher decision-making. We found that facial expressions communicate emotions in scenarios of teaching and that teachers are responsive to these emotions when reacting to a scenario of teaching. While the effect size of the inclusion of emotions in PROSE items is small to moderate (a quarter of a standard deviation), this effect is non-negligible. Unaddressed emotions in items, either construed or represented, could bias participants' assessment of instructional actions in ways that are difficult to predict a priori because facial expressions can both increase and decrease the direction of these assessments.

Our previous work suggests these facial expressions perform as well as photographs of emotions (Dimmel et al., 2019) This highlights the affordances of the semiotic system (composed of eyebrows, eyes, and mouths) in enabling the reading of facial expressions while dramatically reducing the number of tokens used for such denotation. This can help in designing new instruments to assess teachers' emotional competence. with the goal to support teacher professional development around the role of emotions in teaching.

These findings have important implications for the use of cartoon-based scenarios of teaching in teacher professional development. We can use facial expressions to include the role of emotions in instructional decisions. Different teachers might interpret the emotions differently. This could help in describing the variation in instructional actions due to response to emotions.

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PROFESSIONAL GROWTH IN THE MATHEMATICS TEACHER AS AN INTERCONNECTED NETWORK OF CHANGE

James C. Willingham
James Madison University
willinjc@jmu.edu

This exploratory case study investigated interactions among internal and external factors and an elementary mathematics teacher's classroom practices and learning outcomes during a longitudinal professional development program. Data from the critical case of a teacher engaged in professional change supported an examination of the interaction of these domains and provided a rich narrative regarding the importance of these interactions in sustained teaching change. The model of professional change evidenced by this case adds to an expanding body of knowledge regarding the Interconnected Model of Teacher Professional Growth.

Keywords: Affect, Emotions, Beliefs, and Attitudes; Instructional Activities and Practices; Instructional Leadership; Teacher Education-Inservice/Professional Development

Introduction and Background

Three decades ago, an important transition in the study of mathematics teacher change began when the largely ignored question of how teachers' conceptions of mathematics affected their instructional practices was first widely considered (Thompson, 1984). Questions such as this expanded the focus on effective mathematics teaching from teachers' knowledge of mathematics to their conceptions of mathematics and its teaching (Ernest, 1989; Guskey, 1986; Philipp, 2007; Thompson, 1984). This shift highlighted that constructs including teachers' beliefs, views, and attitudes about mathematics were essential components of their teaching practices, that these practices slowly evolved in response to a myriad of other factors, and that teachers in transition operated in a dual reality between their espoused and enacted conceptions (Clark & Hollingsworth, 2002; Ernest, 1989; Guskey, 1986; Pajares, 1992; Philipp, 2007).

Recently, research has considered how these dispositions impact professional development in four important ways. First, the models examining the influence of teachers' conceptions on their classroom practices have grown increasingly sophisticated and begun to account for the nonlinear relationships among the factors involved in these relationships (Clarke & Hollingsworth, 2002; Wilkins, 2008). Second, a variety of professional development programs have supported long-term changes in teachers' conceptions of teaching mathematics and their associated practices (Garet, Porter, Desimone, Birman, & Yoon, 2001; Loucks-Horsley, Love, Stiles, Mundry, & Hewson, 2003; Wilson & Berne, 1999). Third, researchers have presented a diverse range of models to describe the stages through which teachers transition as their conceptions and practices evolve (Andreasen, Swan, & Dixon, 2007; Farmer, Gerretson, & Lassak, 2003). Finally, research has explored the impact of a multitude of other factors, such as the school setting, the teacher's perspective concerning professional development activities, and the types of learning activities implemented in the classroom, on teachers' conceptions and implementation of teaching practices (Clarke et al., 2013; Opfer & Pedder, 2011).

Purpose

A complete model of mathematics teacher development must describe the teachers' motivations and dispositions for their teaching as well as the influence of these factors on areas such as the teacher's implementation of learning activities, interactions with students and the classroom environment, and interpretations of professional development experiences (Goldsmith & Shifter, 1997; Opfer & Pedder, 2011; Wagner & French, 2010). The principal purpose of this study was to explore the interaction of these factors in an elementary teacher engaged in the process of mathematics teaching change during a longitudinal professional development program. This intent led to the primary research question of the study reported here: How do factors including a mathematics teacher's knowledge, beliefs, and attitudes influence her engagement in professional development and ultimately interact with her classroom teaching practices and student outcomes?

Theoretical Framework

A well-developed model of teacher change, the Interconnected Model of Teacher Professional Growth (IMTPG, Clarke & Hollingsworth, 2002), provided a lens through which to view teacher change throughout this study. In this theoretical model (see Figure 1), four domains identified by Guskey (1986) were expanded, and the relationships among them were described in terms of enactment and reflection similar to those posited by Ernest (1989). The domains of this model included the personal domain (knowledge, beliefs, and attitudes), the external domain (outside sources of information), the domain of practice (enacted classroom experiences), and the domain of consequence (salient outcomes involving student learning). Two processes, *enactment* and *reflection* mediated interactions among these domains. Enactment is the active process of operationalizing ideas from one domain into another, while reflection is a determined consideration of the experiences of one domain as they influenced another. The authors posited that interactions within these domains occurred within a specific *change environment* consisting of a particular set of elements, unique to each teacher, that would facilitate or inhibit experiences within each of the model's domains (Clarke & Hollingsworth, 2002).

Within this change environment, two types of teacher shifts arise as specific interactions occurred between the domains (Clarke & Hollingsworth, 2002). The first of these, termed a *change sequence*, occurred any time one domain exerted influence on another. These changes were often temporary and typically enacted in some form of professional experimentation quickly abandoned by the teacher. However, change sequences occasionally lead to further interactions between the domains, resulting in a more permanent transition. These extended interactions are part of a *growth network* and represented the product of teacher development.

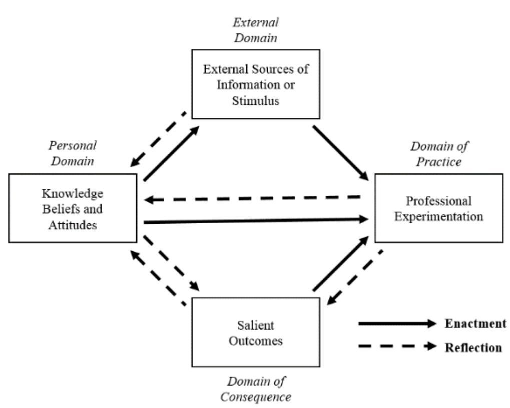

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Figure 1: The Interconnected Model of Teacher Professional Growth

Methodology

An exploratory, holistic single-case design (Yin, 2014) supported consideration of modeling teacher change with the IMTPG. The study focused on Gale Martin, a second-grade teacher, deemed significant as evidence from her engagement in an ongoing professional development project, Project Influence, indicated that she represented the critical case of a teacher displaying strong growth mindset characteristics and engaging in the processes of teaching change.

Data Collection

The data collected throughout the study focused on how Ms. Martin's experiences, beliefs, and practices influenced her described and observed mathematics teaching practices and her adaptation of a demonstration lesson for use in her classroom. Data sources included historical records of her mindset and beliefs, semi-structured interviews, classroom observations, artifacts of the observed and enacted demonstration lessons, reflective journal entries, and artifacts from Ms. Martin's lessons. The researcher collected this data across four stages, including a participant selection process, baseline classroom observations, Ms. Martin's engagement in the demonstration lesson through her professional development project, and her adaptation and enactment of the demonstration lesson in her own classroom.

Data Analysis

The researcher then organized this body of data in chronological fashion, corresponding approximately with the data collection stages described above, and completed a simple time series analysis (Yin, 2014). A holistic analysis of themes, "not for generalizing beyond the case, but for understanding the complexity of the case" (Creswell, 2012, p. 101), was performed for each stage of data through open coding with reduction of these codes into themes consistent with the theoretical framework. Themes emerging from each stage of analysis guided interpretation and coding of the next stage, with all stages revisited for completion.

The Participant

The researcher selected Ms. Gale Martin, a Caucasian female in her mid-thirties, as the critical case for the study. Rationales for this selection included historical survey data indicating persistent growth mindset characteristics, a positive record of changes in beliefs regarding the teaching and learning of mathematics, and observational records indicating a change in classroom teaching practices consistent with the mindset and belief data. Ms. Martin was an elementary mathematics teacher in her second year of teaching second grade and her fifteenth year of teaching elementary school who taught in a rural elementary school of approximately 330 students in a southeastern state, who was engaged in her third year of ongoing professional development for mathematics teaching.

The Demonstration Lesson

As part of the study, Ms. Martin observed a second-grade demonstration lesson conducted by the faculty of Project Influence with a lesson goal of "engaging students in thinking about subtraction with regrouping, while potentially representing the process symbolically" (Demonstration Lesson, October 28, 2015). The lesson involved students interacting with the following task in a problem-solving format.

> On Thursday, Tara was at home representing numbers with base-ten blocks. The value of her blocks was 304. When she wasn't looking, her little brother grabbed two longs and a flat.

> What is the value of Tara's remaining blocks? Use pictures, words, and/or symbols to describe how you solved the problem.

Students worked in pairs to solve the problem and participated in mathematical discussions across pairs, small groups, and the whole group under guidance of an expert teacher.

Results

The unique strength of a teacher's classroom practices exists in their ability to mediate the outcomes of their students' learning (Hiebert & Grouws, 2007). The IMTPG suggests that a variety of interactions among the domains described previously mediate both the practices the teacher selects and their students' outcomes, and a complete description of these interactions would be impossible in a report of this scope. Instead, this section contains a detailed description of the interactions between two sets of specific domains, the external domain to the domain of practice and the domain of practice to the domain of consequence. These exemplars then support a general description of the manner in which the IMTPG summarizes the teacher's experiences throughout Project Influence.

External Domain to Domain of Practice

Ms. Martin cited her involvement with immersion activities as one of the most impactful experiences of Project Influence and attributed much of the change she had implemented to these opportunities to experience effective teaching and learning practices first hand.

> I've told people this, and I'll continue to tell people this. Project Influence is the best professional development I've ever had, because it's so useful and it's so purposeful. It helps me be a better teacher, because I see it in action, I'm immersed in it. So it's not somebody standing in the room telling me all of these things I need to do, I'm in the middle of those practices. Us being the student with the teacher, helps us come back to our classroom and know how we need to do that with our kids. That's just been the most meaningful thing. (Selection Interview, September 9, 2015)

This description suggested that in addition to highlighting effective teaching practices these experiences provided opportunities to return to the classroom and experiment with new methods of instruction and to evaluate their success with elementary students. In addition to the practices she encountered, Ms. Martin also cited the influence of the project faculty who modeled these instructional techniques.

> [The Project Influence facilitator], just her enthusiasm and the way she ran the classroom, I really thought, "Hey, this is definitely something that I can do, I'm already doing a lot of this." So I guess it just fed into what I was kind of already doing. (Selection Interview, September 9, 2015)

This enthusiasm and affirmation of her teaching practices, both those that were effective and in place prior to involvement with the project and those adopted from the project itself, appeared to have provided continued motivation for change for Ms. Martin.

When asked to identify specific teaching practices that modeled during these activities, Ms. Martin described the facilitator allowing learners to have their own ideas and guiding conversations about mathematics from those ideas rather than toward solutions.

> The way she facilitated the classroom, the way she let us have our own ideas and never shot anybody down. That's another thing that I really like, we don't talk about answers. That was another thing that I had to change, because yeah, they want to know the answer, I want to know the answer, that's just something that you've always done. I've changed that also. (Selection Interview, September 9, 2015)

In this quote, Ms. Martin once again attributed specific changes in her classroom practices to her experiences during Project Influence and alluded to a belief about mathematics teaching, allowing learners to do the thinking about mathematics, which appeared to have evolved during her experiences with the project.

Domain of Practice to Domain of Consequence

Ms. Martin described the biggest change that had occurred in her classroom in recent years as involving a shift from problem performing to problem solving (Rigelman, 2007). She elaborated on how her changes in instruction had impacted her students' learning and described the importance of problem solving for her students.

> It is a big shift. It's probably the biggest shift that I have just seen, the impact that it has on their learning. They are learning, like the whole of problem solving is so important. . . for them to be able to have their own ideas and me not say, "Oh, this is how we're going to do it," or, me give them a thought or an idea and let them cling to that because they will. If they think that this is Ms. Martin's way, they want to do what Ms. Martin is doing. (Background Interview, September 18, 2015)

In describing this change in learning, Ms. Martin emphasized the independent problem-solving ability that had developed in her students. She contrasted this independent thinking with that of problem performers, who she described as focused on the solution to a problem rather than understanding the process of solving it.

> If you're just performing you're just giving an answer. Maybe you really don't know why you got that answer or how you came to that answer, you're just giving the answer because that's the final result. Problem solving involves so many more life skills that these kids are going to need, not just finding a solution, but there are just many different ways that problems can be solved and there's not just one right path and one right answer. (Background Interview, September 18, 2015)

The combination of independent thinking, a focus on deep understanding, and the ability to recognize that problems are solvable in more than one way formed the basis of the life skills Ms. Martin associated with her students' transition to problem solving over problem performing.

Later in the school year, Ms. Martin referenced how these changes in problem-solving approaches had caused other changes in her students' abilities to see connections between their thinking. She also described the reasons for this success.

> Well, for example today in math, I had them sharing out their examples or their solutions strategies to the test that we did at the end of class. And you would have certain kids say, "Well, my strategy looks a lot like Sarah's strategy." So they're mirroring or they're

> recognizing that their strategies looks similar to another strategy in the room. And here it is November, so we're moving along on that trajectory. And as far as continuing that, I think you just still tell them, pulling those cards and making everybody responsible for an answer. And they know that they're going to have to provide some sort of answer and some sort of discussion. (Point of View Interview, November 4, 2015)

In this quotation, Ms. Martin recognized that her students are progressing toward the goal of recognizing and connecting different mathematical ideas and attributed this success to her classroom practices related to accountability. She also expressed a desire to continue utilizing these practices in order to encourage and support these changes.

Support for the Interconnected Model of Teacher Professional Growth

The description of Ms. Martin presented here suggested a reflective practitioner deeply invested in the process of transitioning to reform-oriented instruction. However, the multitude of connections between Ms. Martin's conceptions of teaching and learning mathematics, mathematics teaching practices, experiences in Project Influence, and insights into her students' mathematical development offer substantial evidence that these change processes do not occur in isolation and require an extensive support network to initiate and maintain. Based on evidence regarding these factors and their relationships, I present a generalized growth network for Ms. Martin's change environment in Figure 2.

This growth network presents a model of the incremental changes Ms. Martin described experiencing throughout her time in Project Influence combined with the processes observed as she adapted a demonstration lesson from Project Influence for use in her classroom. In this generalized network, some conception of the teaching and learning of mathematics, such as Ms. Martin's valuation of students' thinking and communicating about mathematics or her ideas regarding the manner in which a specific piece of mathematics content should be taught, influenced her interpretations of and interactions during an activity from Project Influence (Arrow 1). Examples of these activities included immersion in a problem-solving task or the observation and debriefing of a demonstration lesson. Ms. Martin's reflections on this experience (Arrow 2) then served to either confirm or incrementally reshape the conception in question. Under the recent influence of this conception, Ms. Martin then adapted some aspect of the Project Influence experience, such as a new classroom norm, questioning practice, or mathematical task, for enactment in her classroom (Arrow 3). Reflecting on this enactment's outcomes, such as the success of a lesson or changes in her students' affect or mathematical understanding (Arrow 4), served to further reinforce or extinguish the conception (Arrow 5) and reinitiate the cycle through external interactions or further classroom experimentation. Although specific examples of this process appear later in this discussion, the current example serves to explain the general processes in play as Ms. Martin's beliefs and classroom practices changed.

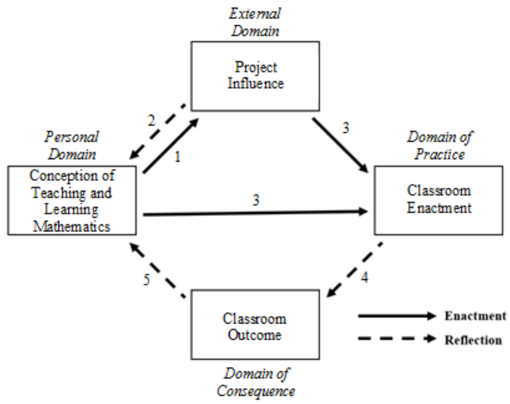

Figure 2: The Generalized Growth Network for Ms. Martin's Change Environment

Discussion and Conclusion

The generalized growth network for Ms. Martin's change environment models specific descriptions of her experiences in Project Influence combined with evidence collected as she adapted the demonstration lesson she observed as part of the study for use in her classroom. Three examples of the specific evidence for this growth network, one related to each of Ms. Martin's goal layers (Willingham, 2017), are provided here in order to illustrate their impact on the process of teacher change (see Figures 3, 4, and 5). These models each share a common pathway (Arrow 1 in each diagram) representing the influence of Ms. Martin's personal domain characteristics on her interpretations of her professional development experiences. Additionally, as the examples provide evidence of the same growth network pathways across different domain characteristics, the diagrams share common labeling of these pathways tailored to each diagram (e.g., 2a, 2b, 2c represent reflective pathways between domains at three different goal levels).

The first example, situated at the level of Ms. Martin's global goals, involves changes in Ms. Martin's beliefs and teaching practices regarding the value of students' thinking and communicating about mathematics (see Figure 3). Although the primary evidence for this growth network developed from Ms. Martin's descriptions of her changes in practices based on her earliest experiences with Project Influence, the network parallels the reinforcement of these beliefs and practices during her implementation of the demonstration lesson. In this growth network, Ms. Martin became involved with Project Influence due to her desire to continue improving her abilities as a mathematics teacher, which she attributed to her own growth mindset and love of mathematics (Arrow 1).

During her first experiences with Project Influence, Ms. Martin became aware of reform-oriented teaching practices related to the value of students' thinking and communicating about mathematics due to the modeling of Project Influence's faculty and witnessing these practices in use in a demonstration lesson. Based on these experiences Ms. Martin developed goals for her classroom aligned with these practices (Arrow 2a). Operationalizing these goals, Ms. Martin described adopting the norms and problem-solving approaches she had experienced in Project Influence to her own classroom (Arrow 3a), and noted the influence that these practices had on her students' abilities in this area, which she later described as students' classroom maturity (Arrow 4a). Ms. Martin described these changes as transformative to both her way of thinking about (Arrow 5a) and teaching mathematics (Arrow 3a) and elected to continue her involvement with Project Influence when given the chance (Arrow 1), initiating a cyclic process.

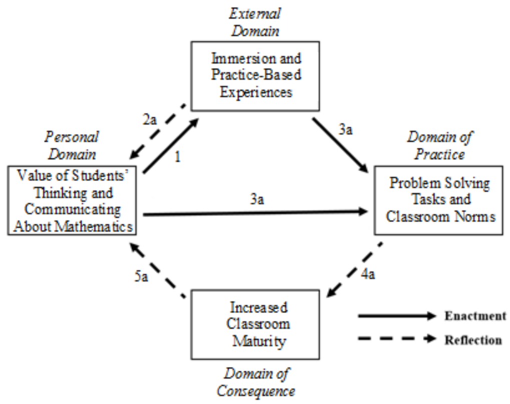

Figure 3: The Growth Network for Ms. Martin's Beliefs and Practices Regarding the Value of Students' Thinking and Communicating About Mathematics

In other iterations of this growth network, Ms. Martin's goals at different levels were involved. As an example, during her activities involving the demonstration lesson of this study, she described considering how the lesson matched with her envisioned mathematics learning trajectory (see Figure 4). Her reflections on aligning this lesson involved finding an appropriate place in the trajectory for its use (Arrow 2b) and planning for the lesson by preparing her students to engage with its content by referencing concepts from earlier in the year. She then considered the types of interactions she would use to address struggles she had witnessed from students in the observed demonstration lesson (Arrow 3b). Once the lesson was enacted, Ms. Martin reflected on its specific outcomes (Arrow 4b), and considered these outcomes in terms of the sequence of lessons and unit in which it was situated (Arrow 5b).

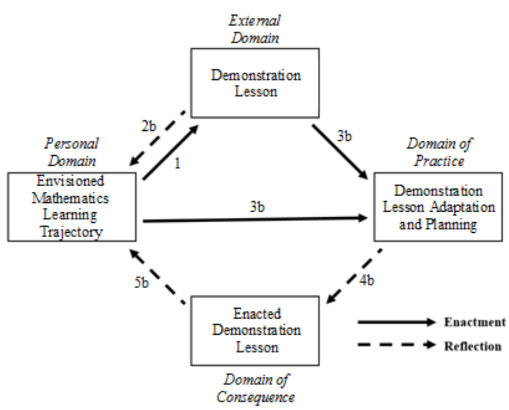

Figure 4: The Growth Network for Ms. Martin's Envisioned Learning Trajectory

The same set of contexts provide insight to the changes to the demonstration lesson Ms. Martin made based on her specific content goals for her enacted lesson (see Figure 5). In this case, her reflections from the demonstration lesson focused on the manner in which she would prepare her students to examine how different representations of numbers support thinking about numeric operations such as subtraction (Arrow 2c). These reflections, along with the core activity from the demonstration lesson, helped Ms. Martin prepare her students to work with this idea during the enacted demonstration lesson by scaffolding the idea in the lesson immediately preceding it (Arrow 3c). Immediately following this lesson Ms. Martin described monitoring her students' progress with the content goals of the enacted lesson (Arrow 4c) in order to determine the course of the remaining lessons in the sequence (Arrow 5c).

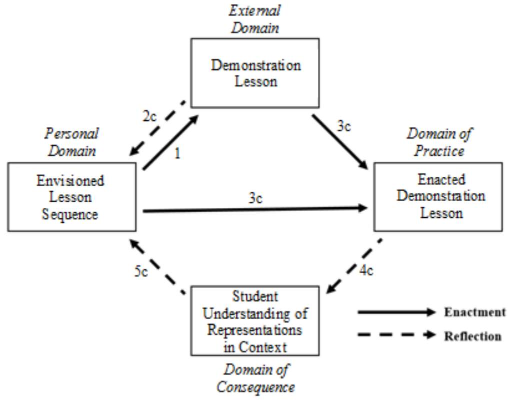

Figure 5: The Growth Network for Ms. Martin's Lesson Goals for Her Enacted Demonstration Lesson

Ms. Martin's descriptions supported interpreting the first example given here as reinforcing her current beliefs and practices due to her experiences with the demonstration lesson in this study. In this case, all three of the examples shown share the same form of initial change sequence as their first enactment pathways (Arrow 1 in Figures 3, 4, and 5). In these shared pathways, a goal-dependent conception of the teaching and learning of mathematics, each at a different goal level, influenced Ms. Martin's interpretations of the demonstration lesson. These layered change sequences then lead to unique connected growth networks in which the mediating pathways are identical while the specific domain foci are dependent on Ms. Martin's pedagogical goals. This highly connected, multidimensional growth network offers a potential explanation for why Ms. Martin's experiences in Project Influence were so influential on her beliefs and teaching practices.

Although Ms. Martin's circumstances are obviously unique, her case provides an exemplar for the manner in which a growth-oriented mindset mediates the internal and external factors of teacher change. For those involved in the development of mathematics teachers, two features of this mediation are the most instructive: the layering of mathematical goals at a variety of pedagogical levels, and the incorporation of explicit reflection in the translation of the activities and practices of professional development programs to the classroom. For Ms. Martin, these layered goals involved the personal growth of her students, the intentionality of her learning trajectory, and the core intention of a single mathematics lesson. Well-designed professional development programs should consider opportunities for participating teachers to define their own goals along similar levels, and then use these goals to frame explicit conversations regarding the manner in which key features of the program translate to the classroom to support their attainment.

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EXAMINING OBSTACLES TO MATHEMATICS GRADUATE STUDENTS' DEVELOPMENT AS TEACHERS

Mary Beisiegel
Oregon State University
mary.beisiegel@oregonstate.edu

Claire Gibbons
Pierce College
cgibbons@pierce.ctc.edu

Alexis Rist
Oregon State University
delonga@oregonstate.edu

As the field of professional development for mathematics graduate students evolves, there is a need for research studies that examine how mathematics graduate students develop as teachers. This study aims to fill this gap in the research by studying mathematics graduate students' experiences with teaching as they progress through their graduate programs. Mathematics graduate students were recruited from a large university mathematics department. They were responded to surveys and were interviewed semi-annually for two or more years. We analyzed their responses using thematic analysis and a framework that captured their development as teachers. In this report, we describe a framework for teachers' development and context and methods of the study. We present findings on how the framework elucidated mathematics graduate students' development as teachers and illustrate obstacles to their development.

Keywords: Professional Development; Post-Secondary Education; Affect, Emotions, Beliefs, Attitudes; Instructional Activities and Practices

As is common in most graduate programs in the sciences in the United States, mathematics graduate students (MGSs) receive funding to support the time they spend in their master's or doctoral programs. In exchange for this funding, they spend between 12 to 16 hours each week working for mathematics departments as instructors or teaching assistants. In some graduate programs, MGSs have the opportunity to teach their own courses for the duration of their graduate programs. In other programs, MGSs do not teach their own courses but instead have the opportunity to lead recitations or workshops that support larger lecture sections which are taught by faculty members. During a two-year master's degree program, mathematics graduate students might have the opportunity to teach as many as seven classes or lead as many as 24 recitations, and in a six-year doctoral program 15 classes or 60 recitations. This means that during their graduate programs, MGSs contribute to the learning experiences of hundreds, if not thousands, of undergraduate students. Their contribution to undergraduate students' learning experiences extends past their graduate programs, with more than 60 percent of new mathematics PhDs finding employment in post-secondary education settings in which teaching makes up a significant portion of their work (Golbeck, Barr, & Rose, 2016). Thus, in their roles as graduate students and later as faculty members in mathematics departments, MGSs exert a significant impact on undergraduate learners' trajectories in STEM fields (Belnap & Allred, 2009; Ellis, 2014).

Despite their contact with and impact on undergraduate students, MGSs receive very little preparation for teaching (Deshler, Hauk, & Speer, 2015; Ellis, 2014). Experts in MGS professional development have not yet reached consensus on the breadth and depth of PD programs that prepare MGSs to teach. Programs vary from a few hours, to an intensive week, to a seminar that spans a full academic year. Few programs extend beyond MGSs' first year in graduate school (Deshler et al., 2015; Ellis, 2014; Harris, Froman, & Surles, 2009; Kung & Speer, 2009; McGivney-Burrelle, DeFranco, Vinsonhaler, & Santucci, 2001). Several researchers who aimed to change MGSs' teaching practices over an academic semester or year discovered that their professional development (PD) programs did not improve the MGSs' practices (Belnap, 2005; DeFranco & McGivney-Burrelle, 2001; Speer, 2001). Preparation for a broad scope of evidence-based teaching practices (e.g., inquiry-based learning, active learning, equitable instruction) is crucial as most MGSs' teaching practices remain rooted in lecturing, with lecture-based mathematics courses causing significant problems for undergraduate students (Deshler et al., 2015; Miller et al., 2018; Stains et al., 2018).

Prior to developing and implementing a two- to three-year PD program for MGSs, our research team sought to understand how MGSs think about teaching, what they learn about teaching, and what type of PD might be relevant at different stages of MGSs' development. We had in mind the question – how might MGSs' needs for professional development be different in their second year of teaching compared with their fourth year of teaching? Looking to the literature, we observed that little is known about MGSs' development as teachers. In an attempt to characterize the types of research conducted regarding MGSs' growth as teachers, Miller and colleagues (2018) completed a review of the literature of professional development for MGSs. The review identified 26 peer-reviewed articles since 2005 that investigated MGSs' teaching development, only 17 of which attended to growth. Thus, the authors concluded that "[mathematics teaching assistants'] growth as teachers is a largely unexamined practice" (Miller et al., 2018, p. 2) and suggested that this area of study would benefit from longitudinal studies that make explicit a model of growth.

With this in mind, the purpose of the study described in this proposal is to investigate MGSs' development as teachers, what and how they learn about teaching, and changes in their thinking about teaching and learning longitudinally as they progress through their degree programs. The research questions that guide this study are: (1) How does an existing framework of teacher development elucidate mathematics graduate students' growth as teachers? (2) What kind of experiences do MGSs have with teaching and what impact(s) do those experiences have? (3) What features of their graduate school and teaching experiences support or hinder their learning about teaching and their development as teachers?

Theoretical Framework

Because research has not yet addressed MGSs growth as teachers, we looked to the K-12 literature, where researchers have studied schoolteachers' experiences in order to gain an understanding of teachers' growth over time. Katz (1972) described four developmental stages, which include: (1) *survival* of the first year of teaching, with particular focus on classroom management and the routines of classrooms and schools; (2) *consolidation*, in which teachers begin to understand which skills they have mastered, and what tasks they still need to master; (3) a period of *renewal*, when teachers become tired of their routines and start to think of how things might happen differently; and (4) reaching *maturity*, where teachers think more broadly about the contexts of schools and students' learning (p. 52-53). We aimed to use this lens to see whether and how MGSs might progress from thinking of teaching as lecturing (survival and consolidation) to thinking about incorporating active learning into their practice (renewal and maturity).

Context and Methods of the Study

At the beginning of the academic years in 2015-2018, participants were recruited from the mathematics department at a large, doctorate-granting institution. Approximately 5,000 undergraduate students enroll in courses such as Pre-calculus, Differential, Integral or Vector Calculus, Business Calculus, or Differential Equations each year. Most of these courses are structured as three hours of lecture with 150-250 students per class and are taught by an instructor. MGSs are generally assigned to run recitations (60-80 minute workshops each week) of smaller groups of students from the large lecture sections. MGSs are not assigned to courses based on knowledge, skill, or experience; their assignments to courses mostly depend upon scheduling.

When new MGSs first arrive to this graduate program in mathematics, they receive two to three days of professional development for their teaching assignment, with a focus on how to support active learning and student engagement in mathematics during recitations. In the first term of their graduate program, they attend a seminar for one hour each week that addresses teaching-related concerns such as grading papers, student conduct issues, and lesson planning. In the summer after their first year, they have the opportunity to teach their own course, then they return to the main MGS duty of leading recitations. Only informal mentoring happens before and during the summer sessions and into the MGSs' subsequent years.

We developed two beginning-of-the-academic-year surveys, one for new and one for experienced MGSs, and protocols for mid-year and end-of-year interviews. Surveys are used at the beginning of the year because of logistical issues. They include open-ended questions that inquire about MGSs' thoughts about teaching and learning mathematics, how they would describe a well-taught mathematics lesson, and what influenced the way they think about teaching. Mid- and end-of-year interviews allow a deeper investigation of MGSs' teaching practices, their most recent teaching experiences, whether they feel that they are receiving adequate support, and what other support they feel they need to grow as teachers. The intention of the study is to survey and interview participants for the duration of their graduate programs to study their development over time. Table 1 illustrates participation in the study.

Table 1: Study Participants

Recruitment Year	Number of Participants
2015-2016	11 new participants: 4 first year, 2 second year, 4 third year, 1 fourth year
2016-2017	11 continuing participants; 6 new participants: 4 first year, 1 third year, 1 fourth year
2017-2018	11 continuing participants; 10 new participants: 8 first year, 1 third year, 1 fifth year
2018-2019	14 continuing participants; No new participants

Our research team analyzed participants' responses to survey and interview questions in two rounds of coding using thematic analysis (Braun & Clarke, 2006). Thematic analysis has six stages which include: (1) familiarization with the data; (2) coding interesting features of the data in a systematic way and collating data that is appropriate for each code; (3) possibly combining codes into themes and collect data for each them; (4) reviewing the themes and supporting data for each theme; (5) continuing to analyze the themes, generating a clear definition for each; and (6) producing the report of the themes with selected data to provide evidence of each theme. In the first found of coding, we applied a deductive approach (e.g., using a pre-existing coding frame) where we looked for instances of the participants' experiences that could be elucidated with Katz's (1972) four-stage model of teacher development. In the second round of coding, we used an inductive approach that focused on codes we developed through the first round of analysis, when we observed issues that either provided explanation for the stages or weren't captured by Katz's framework. These codes included what had an *impact on their views about teaching* (graduate course work, their students, the instructor they are assigned to, the course they are assigned to, their previous experiences as learners, office hours, their MGS peers, and the resources they use for teaching), *issues of identity* (being a teaching assistant versus being an instructor, resignation), and *teaching* (descriptions of their teaching practices, the transitions they have made in their teaching, what changes they would make to their teaching, and what is important for their teaching).

Findings

Using Katz's Framework to Elucidate MGSs' Development as Teachers

We have found Katz's (1972) framework to be a useful lens to view MGSs' development as teachers. First-year MGSs struggled to survive as they adjusted to their roles as teachers. In the quote below, a first-year MGS described his initial experience teaching:

> By that point the quarter [midterms], it was just getting really hectic and I wasn't able to plan as much as I usually like to plan for courses. Sometimes I was looking at the material for about two hours before I started that day whereas usually I like to look at it the day before or during the weekend or something. And so sometimes, though, the classes that I went to where I was kind of doing it on the fly, where I was literally looking at it like an hour or two before class. A lot of times it's just more like get the notes done, go in, and do it.

We discovered that after they gained some experience teaching, MGSs could reflect on their teaching and think about how their teaching might evolve:

> I think previously, I was more focusing on, "I just want to survive my first teaching experiences." So, now that this is my fourth time [leading a recitation], I feel a little bit more comfortable trying to incorporate more active learning in my classroom, and trying non-traditional techniques whereas previously, when I taught, for example, my first time teaching my own class and I taught Calculus, I did mostly lecture because I just wanted to do what I felt most comfortable with – what I felt I could be successful at.

We found that a few MGSs reached the *renewal* stage after a couple of years of leading recitations and were ready to grow their teaching practices:

> Because I've already gone through three years now teaching. So I'm already comfortable with coming up to class and writing things down and grading things in a reasonable enough fashion and in good time. But, yeah, it would be really great to be able to like just take it another step further.

A participant in their third year of the graduate program acknowledged that they could now better understand what active learning meant: "maybe even if I heard the same thing, it would carry more weight now. It might be good to hear [about active learning] in the context of now having three years' experience. I would have a better understanding for the context it would fit in." We observed that a small number MGSs began to think about incorporating active learning strategies in their third and fourth years of teaching.

Out of 38 study participants, we observed that only a few reached the *renewal* stage, and only one or two of the MGSs spoke of teaching in ways that Katz (1972) would categorize as *maturity* even in their sixth year of teaching. Most MGSs appeared to be stuck in the *consolidation* stage and their descriptions of teaching were remarkably unchanged year after year. We have also found that MGSs do not pass through Katz's (1972) developmental stages linearly. In fact, they sometimes returned to the *survival* stage if their new teaching assignment varied significantly from their prior teaching assignments. Based on these findings, we investigated further what was having a significant impact on the study participants development as teachers.

Always a Teaching Assistant, Never a Teacher

The role of teaching assistant stood out as significantly problematic for the research participants. Some of the participants felt that being an instructor and a teaching assistant were very different things, with a second-year MGS stating, "Because [working as a teaching assistant] is totally different than teaching. And I didn't realize that until I got in there." She did not think her work as a teaching assistant (helping individuals or groups of students during class time) did not count as teaching:

> I guess I don't feel like I've taught yet. Right now in [instructor's] class where I just walk around and answer individual questions. And so I don't really feel like I've taught yet but I have ideas for what I'd want in my own classroom. But as a teaching assistant, like you can't tell them no cell phones. Really it's up to the instructor. And so it kinda sucks when you get super frustrated that all these students are on their cell phones and you can't do anything about it. Or you wanna be able to say, you have a week to turn in late work instead of everyone handing you stuff in the final week of classes and you have to take it because that's the instructor's policy. So, stuff like that I don't really have control over.

Several of the MGSs expressed dissatisfaction at the amount of communication they had with the instructors of the courses they were assigned to. One participant said: "I just had like zero interaction with the instructor. So, it just would have been nice to be on the same page about things that the students were learning and things that he wanted, that kind of thing." This lack of communication led to feelings of uncertainty about what mathematics they should offer students:

> I mean I'm often happy to or at least, you know, I've always taken the approach that if the instructor wants something done I'll just do it their way. Which can sometimes be hard. I guess with you know with giving concepts I go back and forth on you know if you're teaching a concept should you teach it the same way the teacher does to reinforce thinking about it a certain way. Or should you teach it in the way that you think about it so a student can see these two different ideas and maybe they'll pick up one instead of the other. And I think I have a tendency to follow the instructor if I know what they're doing and not if, well if I don't know what they're doing then it's a tossup.

Others experienced a sense of ineffectiveness in the role of teaching assistant, with little input into or control over the class:

> But I have no power over what happens in the recitation hour. There is a quiz that was written up, there's an activity that was written up. That's nice, because I appreciate the standardization, and I know that the people who were making these things care a lot, or think a lot about it. So I'm not grumpy about that, necessarily. It's the instructor's course. That's fine. But it means I'm not choosing any of the problems, or anything like that. So what can I do?

Some MGSs' experienced some detachment, disinterest, and frustration in what happened in the role of teaching assistant:

> But I haven't, you know it's not my problem I didn't put it together. I didn't it just. Yeah. I don't know I'm just less invested and then it's coupled with the material just being very difficult and the students struggling. It's hard to, it can be a lot to overcome. You know just when you just have a very small window.

The perceived lack of contribution to students' learning led some MGSs' to feel as though it didn't matter who they were:

> For the impact on their learning, I think I'm interchangeable with all the other graduate students. Every once in a while, there'll be a problem that everyone's asking questions on, and I'll be like, "Okay, let's review something that you all might enjoy having a small review of." And that's fine, but they could get that in the tutoring center. They could get that from going to anybody's office hour.

One MGS expressed the pain that resulted in being a teaching assistant and feeling forced to work with students in ways she didn't agree with: "When I am forced to overwhelm my students or otherwise have no freedom in teaching and consequently get cruel reviews... I take things personally… negative feedback is painful."

A MGS in the fourth year of his graduate program had started to see the value in teaching mathematics in ways that he felt most comfortable with, and that he might even become assertive in his thinking about the recitations he would lead:

> Things that I learned? Probably just re-emphasizing that I should probably be a little bit more assertive about how I think the class should go. Because to some extent, now that I think about it, it probably depends on me as a leader for that recitation. Rather than just you know the instructor wants it done a certain way. If there's a way that I'm comfortable doing it probably works better than what someone else would do.

Discussion and Conclusion

Given the obstacle that the role of teaching assistant played in MGSs' development as teachers, we propose that mathematics departments reconsider how the role is portrayed to MGSs and instructors. Specifically, we recommend that departments fully define the role and make explicit how teaching assistants are valued and how they contribute to students' learning. We also recommend that instructors are offered their own professional development in what and how they communicate with their teaching assistants (e.g., valuing their contributions, providing space for MGSs to contribute to the course, helping MGSs to learn about teaching). Finally, we recommend that professional developers keep in mind the obstacles to MGSs' growth as teachers that we have described in this paper. In particular, professional development programs that aim to teach MGSs about teaching should inform and empower MGSs in their roles as teachers.

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BEING A PD OPPORTUNIST: LEVERAGING TEACHERS' IDEAS TO BUILD MATHEMATICAL KNOWLEDGE AND CHALLENGE BELIEFS

Richard Kitchen
University of Wyoming
rkitchen@uwyo.edu

In this paper, I describe a process used in professional development sessions to leverage participating teachers' mathematical ideas as a means to further develop their mathematical knowledge and potentially modify beliefs teachers may hold about instruction and their students. I use an integrated theoretical framework that incorporates noticing, Whiteness and positionality. Sample instructional vignettes are provided that demonstrate how I employ teachers' solutions to mathematical problems as a means to support their mathematical thinking, challenge beliefs they may hold about teaching and their students, while also validating the mathematical ideas that they bring to the learning process.

Keywords: Equity and Diversity, Teacher Knowledge.

In my work as a Mathematics Teacher Educator (MTE), I teach content and methods courses for prospective teachers as well as lead professional development (PD) sessions for practicing teachers. One of my goals during PD sessions for practicing teachers is to leverage their mathematical ideas as a means to support them developing deeper mathematical knowledge and to potentially examine beliefs they may hold about instruction and their students. This involves noticing teachers' mathematical ideas and having them share these ideas with their peers (Sherin & van Es, 2009; van Es & Sherin, 2008). More than just noticing teachers' mathematical ideas, a primary focus of instruction in the PD sessions *is* their ideas and much of my instruction is constructed based on participating teachers' ideas. This entails working to both highlight and build on teachers' mathematical ideas during instruction as a means to support teachers reflecting on and potentially revising their mathematical thinking (Schoenfeld, 1985).

In this paper, I describe a process that I use in the "Institute," PD sessions for P-12 teachers to leverage their mathematical ideas as a means to extend teachers' mathematical knowledge and potentially modify beliefs they may hold. During the 2016-17 and 2017-18 academic years, I led monthly full-day professional development sessions for more than 30 primary and secondary-level teachers of mathematics who teach in culturally and linguistically diverse, rural schools in northern New Mexico, USA. The vast majority of the participating teachers are culturally and linguistically diverse women, primarily of Hispanic and Native American descent. In each Institute session offered, teachers regularly engaged in problem solving activities, shared their solution strategies with one another and the whole group, and examined other teachers' solutions. I work to make teachers' mathematical ideas central in PD sessions I lead for the following three reasons: (1) To champion the notion that we all bring mathematical knowledge and ideas to bear to solve problems (Schifter, 2005); (2) To endorse teachers' knowledge and ideas as mathematically valid and important; and (3) To distribute the mathematical authority in our sessions (Cobb, & Yackel, 1996). In 2016-17, the problems assigned in the sessions were aligned with the Common Core State Standards for Mathematics (CCSSM) Domain, "Operations & Algebraic Thinking" (National Governors Association Center for Best Practices [NGA] &

Council of Chief State School Officers [CCSSO], 2010). During 2017-18, problems assigned aligned with the CCSSM Domain, “Number & Operations—Fractions.”

An important consideration for me in designing the Institute sessions is the fact that many of the participants are elementary school teachers who are culturally and linguistically diverse women whose ideas have historically been marginalized in the mathematics classroom (Becker 2003). As an MTE, I work to notice what Louie (2018) refers to as “students smartnesses” (p. 59). This is an intentional strategy in which I work specifically to showcase the mathematical work of women and teachers of color who are part of the Institute. Noticing, Whiteness and positionality informed the theoretical framework that informed my moment-to-moment decision-making about how to leverage teachers’ ideas during the Institutes. Noticing and making teachers’ mathematical ideas central in PD shifts the focus away from me as the sole mathematics authority. Since the vast majority of Institute participants are culturally and linguistically diverse individuals, noticing these participants’ ideas is a means to work against my white privilege and toward the building of an inclusive professional development community. An important part of my work as a PD provider is considering white privilege and how it operates to provide differential educational opportunities based upon race and class (Battey, 2013; Martin, 2013). Whiteness in mathematics education has historically subjugated culturally and linguistically diverse students, while reproducing privilege for white students (Battey, 2013; Joseph, Haynes, & Cobb, 2016; Martin, 2013).

Through noticing and taking teachers’ ideas seriously, they begin to view themselves as more capable mathematics learners (Wager, 2014) and this is reinforcing and cyclical in nature. This leads to more academic risk taking over time in which the teachers become more willing to attempt to solve more difficult problems (Kitchen, DePree, Celedón-Pattichis, & Brinkerhoff, 2007). In addition, noticing and making culturally and linguistically diverse participants’ ideas central in our work as MTEs contests instruction in which culturally and linguistically diverse students are constructed as mathematically deficient, while positioning students from the dominant culture as the most mathematically qualified (Kitchen, et al., 2007). Hence, another goal in the Institutes was to position teachers, particularly culturally and linguistically diverse teachers, as competent problem solvers from the outset. An individual’s positionality is determined via rights and duties acquired, assumed or that are imposed upon the individual (Redman & Fawns, 2010; Varela & Harré, 1996). Once positioned, one begins to view the world from a particular perspective (Davies & Harré, 1990). It is also the case that positionality is fluid and dynamic, responsive to shifting contexts as people occupy multiple positions (Tait-McCutcheon & Loveridge, 2016).

Lastly, noticing and making the teachers’ mathematical ideas a focus of instruction was also intended to encourage teachers to consider how their students may have mathematical ideas worthy of incorporating in their instruction as well. As students become more confident participating in mathematical activities and expressing their ideas, teachers begin to place greater value on their students’ ideas (Hiebert & Carpenter, 1992). In addition, I wanted teachers to consider the value of their students’ developing positive mathematical identities. By developing their own positive mathematical identity, they will want the same for their students (Kitchen, et al., 2007). A potential result is that teachers begin to position their students as competent problem solvers (Kitchen, Burr, & Castellón, 2010; Tait-McCutcheon & Loveridge, 2016).

Research Methodology

I provide one example of how I work to leverage teachers' ideas as a means to develop their mathematical knowledge. Given the space limitations of a Brief Research Report, I do not provide examples of how I leverage teachers' mathematical ideas to explore beliefs teachers may hold. For the sample teacher solution offered below, I describe how I leveraged these solutions to further develop teachers' mathematical knowledge. The example came from copies of collected teacher work samples and from photos taken of teachers' solutions to tasks posed during Institute sessions. Throughout the Institutes, I engaged in my own journaling about what transpired during sessions, which helped me reconstruct the instructional decisions that I made at particular moments. I also examined teacher evaluations of sessions. This served to inform me about how teachers were responding to my pedagogy in general, and to my focus on their mathematical ideas in particular. All these data subsets were analysed using interpretive methods (Creswell, 2014). Each data subset was read or viewed as a whole, followed by a period of clarifying learning goals that emerged as I worked to notice, center the teachers' mathematical ideas, and then construct instruction based upon these ideas. An iterative process of reflecting upon and then clarifying my learning goals and resultant instruction followed (Miles, Huberman & Saldana, 2013). This process went through multiple revisions as the data were repeatedly read and reviewed to check the consistency of findings. After I had established how to characterize instruction that took place during the Institutes, I searched for commonalities and differences across instructional vignettes to further examine how as well as why I made particular in-the-moment instructional decisions during the Institutes.

Using Teachers' Ideas to Enrich Their Understanding and Attend to Their Beliefs

I now offer an example to address how I noticed and then leveraged teachers' ideas in-the-moment as a means to further develop their mathematical knowledge. In each example cited below, culturally and linguistically diverse Institute participants created the solution strategies shared.

In this example, I built on a participating teacher's idea to develop notions of number sense through examination of a mathematical property. The following problem was offered to participating teachers that approximates a fifth-grade CCSSM Algebra Standard: Judy says that to find 5 times 26, she can find 10 × 26 (260), and then divide this number by 2 to get 130. Write an equation that demonstrates her mathematical thinking.

As teachers worked on the problem individually and then with group members, I walked around observing and listening to how teachers explained their solutions to one another. I then selected one of the participating teachers to share her solution strategy on chart paper for the whole group to view. My goal was to then further examine her solution with the whole class and derive mathematical insights about a mathematical property. After writing her solution strategy on the chart paper, she shared this strategy with the entire class:

$(5 \times 2) \times 26 =$
$\vee$
$10 \times 26 = 260$
$260 \div 2 = 130$

In her solution, the teacher noted that multiplying 5 by 2 was equivalent to multiplying by 10. She also argued that after multiplying 26 by 10, it was necessary to divide this product by 2 to arrive at the appropriate result since 10 ÷ 2 equals 5. The teacher then proceeded to produce the following equation for the whole class that synthesized the solutions that she had previously presented:

$$5 \times 26 = (10 \times 26) \div 2$$

During the whole class discussion that ensued, I worked to validate and made an in-the-moment decision to build on the teacher's initial solution to demonstrate how she had used the transitive property of equality, namely that if a = b and b = c, then a = c. To do this, I circled the first expression (5 × 2) × 26 and labeled it as expression "a". I continued by circling expression b, 10 × 26, followed by expression c, 260. I concluded by reviewing how a = b, b = c, and finally that a = c.

$$\underbrace{(5 \times 2) \text{ X } 26}_{a} = \underbrace{10 \times 26}_{b} = \underbrace{260}_{c}$$

Next, I asked whether 10 × (26 ÷ 2) = (10 × 26) ÷ 2? The realization that these expressions are equal led a participating teacher to share how she teaches GEMA (grouping symbols, exponential operations, multiplicative operations, and additive operations) rather than PEMDAS (parenthesis, exponents, multiplication and division, addition and subtraction) to support student understanding of the order of operations since with PEMDAS, students may learn to multiply before they divide and to add before they subtract.

Discussion and Implications

In this paper, I have shared a process that I have developed for use in PD sessions in which teachers' mathematical ideas are leveraged to develop their mathematical knowledge and potentially challenge particular beliefs teachers may hold about mathematics instruction and their students. This process involved making in-the-moment decisions about how to teach mathematics in a manner that supports the mathematical learning and pedagogical development of teachers. Using an integrated theoretical framework to make these decisions incorporates noticing, Whiteness and positionality, culturally and linguistically diverse participants' mathematical ideas were continually on display throughout the Institute. In their evaluations of the Institute, teachers wrote that they liked having their ideas highlighted throughout. Some teachers also shared that they needed to provide more space in their classrooms for greater inclusion of their students' mathematical ideas because they have come to believe through our work that their students also have important mathematical ideas to share, just like they do.

In the example provided, I deliberately selected a teacher's solution that incorporated the transitive property of equality; a powerful property that I knew would be beneficial to examine together. Inspired by the teacher's final equation, I also decided that we would collectively examine order of operations. I knew from experience this was worthy of consideration since application of PEMDAS may lead to miscalculations (Dupree, 2016). An intentional strategy here was to take advantage of participating teachers' mathematical ideas to organically push their thinking to consider ideas that most certainly will emerge in their instruction. From participating teachers' responses to their colleagues' mathematical solutions and from information gleaned from evaluations completed, teachers frequently offered how the Institutes helped them learn mathematics and become more confident in their mathematical abilities. As previously mentioned, culturally and linguistically diverse teachers created all three of the solutions shared here. Intentionally having women and culturally and linguistically diverse teachers publicly share their mathematical solutions to problems removed me as the sole mathematical authority in the room. Though I decided whose solutions were on display and what mathematical ideas to develop further, the teachers' ideas were always in the forefront of discussions.

An implication of the research findings of this study is that making diverse teachers' ideas a focal point of instruction is a practical approach to incorporate equity and access to mathematics in one's teaching. Such instruction is equitable for the simple reason that it validates the ideas of members of underrepresented groups in the learning process, erasing the notion that one's competence in mathematics can be predicted by one's race, class, sex, ethnicity, and English proficiency (Gutiérrez, 2002). In the case of the Institutes, teachers' learning from other participants is an example of what Boaler (2008) has referred to as "relational equity."

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THE IMPACT OF HIGH-TOUCH, RESPONSIVE, COLLABORATIVE PROFESSIONAL DEVELOPMENT ON CURRICULUM IMPLEMENTATION AND TEACHER PRACTICE

Karina K. R. Hensberry
University of South Florida
khensberry@mail.usf.edu

George J. Roy
University of South Carolina
roygj@mailbox.sc.edu

Kathe Rasch
Maryville University
krasch@maryville.edu

Vivian Fueyo
Professor Emerita
vfueyo@mail.usf.edu

Philip Vahey
SRI International
philip.vahey@sri.com

Tracie Harwood
School District of Manatee County
harwoodt@manateeschools.net

The SunBay Digital Mathematics Project promotes the use of a curricular activity system (CAS) in which professional development (PD), curricular materials, and dynamically-linked virtual representations help teachers engage in student-centered instruction. Here we describe a district-university partnership that resulted in a unique model for PD focused on supporting teachers to implement technology-based units and adopt effective mathematics teaching practices. We examine teachers' perceptions of the PD model and how it impacted their practice. Data sources were qualitative in nature (e.g., observations, interviews); preliminary analyses suggest the implemented model resulted in relevant, impactful, effective, and empowering PD.

Keywords: Teacher Education, Professional Development, Inservice Teachers, Technology

The initial preparation of mathematics teachers as well as the type of ongoing professional learning provided to them throughout their career has received much attention (Sowder, 2007). Initial preparation is traditionally the responsibility of teacher education programs and as they progress through their careers, the responsibility for their professional development (PD) falls within the purview of school districts. This dichotomous approach brings to light the challenges many districts across the U.S. are facing, where teachers are leaving the profession at alarming rates and at earlier points in their career (CERRA, 2019). Among the factors that predict the attrition and migration of mathematics teachers is useful content-focused PD (Ingersoll & Perda, 2010). One way to share responsibility for ongoing PD is through university-district partnerships (Roschelle, et al., 2010; Roy, Fueyo, & Vahey, 2017; Vahey, Knudsen, Rafanan, & Lara-Meloy, 2013; Vahey, Roy, & Fueyo, 2013).

The SunBay Digital Mathematics Project promotes the use of a curricular activity system (Roschelle et al., 2010) in which focused PD, realistic curricular learning progressions that address state-defined college and career readiness standards, and dynamically-linked representations (Moreno-Armella, Hegedus, & Kaput, 2008) are aligned to assist mathematics teachers as they engage in a student-centered approach to instruction. Curricular materials that incorporate dynamic technology have been shown to support teachers' middle school mathematics content knowledge (Roy, Vanover, Fueyo, & Vahey, 2012). By allowing teachers to initially "do the math" and explore the dynamic representations while participating in the PD as learners they can make mathematical connections and broaden their own content knowledge. These PD experiences can lead to different classroom environments when similar curricular learning units (Vahey, Werner, Roy, Fueyo, & Collier, 2013). Here, we describe a partnership in which university faculty collaborated with district-based leadership and middle school teachers to build on that prior research to explore the intersections among pedagogy, technology, and mathematics to address both the practice of mathematics teachers and student achievement. This paper provides a concrete example of how to successfully support teachers so they will implement effective teaching practices and innovations.

We developed a three-year PD that was research-based, high-touch, responsive, and collaborative (CBMS, 2012). We examine the impact of this four-pronged PD approach on quality implementation and teacher change by investigating the following research questions:

What are middle school teachers' perceptions of this professional development model?
In what ways has the professional development model impacted teacher practice?

Methods

Participants and Setting

During the intervention, middle school teachers across a large, suburban school district (NCES, n.d.) were provided PD to explore the SunBay curricular units as learners, and then later implement those same units with their middle school students. More than half of the students are non-white and two of the ten public middle schools are classified as Title 1 schools. Forty school teachers from the eight middle schools received PD. Teachers were provided with PD on at least two curricular units specific to the grade level they taught. Although students are not the focus of this paper, it is worth noting that approximately 1,700 students were served in year 1; 3,300 students in year 2; and 2,100 students in year 3.

Data Sources and Analysis

Data sources include field notes from classroom observations, audio transcripts from end-of-year focus group interviews, surveys, and notes from team meetings. As data were collected, the first author engaged in open coding (Cresswell & Poth, 2017; Hatch, 2002) to identify themes, which were revised as more data were analyzed. She also engaged in constant comparison to cross-check data for inconsistencies and nonexamples. The codes and associated findings were shared with other authors and participants, and further revisions were made. Illuminating quotes were then selected and presented in this paper.

Intervention

Elements of the curricular activity system include (a) six total curricular units; two units per grade level emphasizing the mathematics content in each respective grade, and each of the units was aligned to the state standards and served as replacement for state adopted textbook lessons; (b) dynamic technology emphasizing multiple mathematical representations; and (c) focused teacher PD exploring the implementation of technology within the unit. Each element of the curricular activity system will be described in the following sections.

The SunBay units. Six units and four web-based apps were developed by curriculum specialists with feedback from district personnel and teachers who would be implementing the units. These were designed as replacement units for 'business as usual' units and covered the same standards. Each of the units: (a) addressed important content identified in both research and the state standards; (b) incorporated web-based dynamically linked representations; (c) incorporated the Predict, Check, and Explain instruction routine (Roy et al., 2016); and (d) emphasized purposeful instructional lesson structures.

Our professional development model. Our framework emphasized teacher's own mathematical content development; attention to students' interaction with the curriculum; teacher beliefs that middle school students can learn deep, rich mathematical concepts; and building a network of teachers who can support each other and attend to their collective learning throughout implementation (Doerr & Lerman, 2010). We also sought to support teachers to implement SunBay units with fidelity and to affect student achievement. We developed a three-year PD model aligned with principles described by the CBMS (2001) by creating a PD that was:

- *High-touch.* The PD was deep and iterative, not the single-session experiences that the teachers reported they regularly receive. We provided multiple-day PD for new and returning teachers prior to the implementation of each SunBay unit, with a minimum of two sessions per academic year. Teachers benefited from individual correspondence with PD facilitators, classroom coaching observations and feedback, and participation in the end-of-year Teacher Forum at which they reflected on successes and challenges (Blank & de las Alas, 2009). This high-touch model allowed the teachers opportunities to reflect on what they learned after each session, implement the units in class, troubleshoot issues that arose with their peers and PD leaders, and make changes for the next implementation (Garet et al., 2010; Demonte, 2013; Tarr et al., 2008).
- *Responsive.* Topics varied depending on teacher and district needs. Initial training always focused on learning both the content and the curricular materials. New-to-SunBay teachers typically received two days of PD that included opportunities to "do the math" in the same manner their students would. Returning and new SunBay teachers also received PD specifically addressing issues of SunBay curriculum implementation identified by teachers and the authors during interviews, observations, surveys, and PD discussions. Sessions focused on Productive Struggle, Student Talk (year 2), and NCTM's (2014) Eight Effective Teaching Practices (year 3).
- *Collaborative.* In planning, the PD developers worked closely with curriculum developers and district personnel, including the Math Curriculum Specialist. This ensured we fulfilled the vision of the developers and met district needs.
- *Research-based.* In addition to the research-based SunBay materials themselves (Vahey et al., 2013), we drew on teacher education research about best practices for teaching mathematics as we made decisions about how to deliver the PD (e.g., NCTM, 2014; Smith & Stein, 2011). We actively engaged the teachers with the content and highlighted sense-making by asking them to "do the math", explain their thinking, and justify mathematical solutions as we hoped they would do with their students.

Findings

When considering teachers' perceptions of the PD model and how that PD impacted their practice, we identified several themes, described below. These findings are preliminary and will be refined after future analysis of questionnaires and interviews of the teachers.

Teachers Believed in the SunBay Technology, Materials, and Pedagogical Approach

Teachers mentioned the use of technology itself, the real-world applications, and the visual nature of the way concepts were displayed as some of the greatest rewards of SunBay. They noted that students took ownership of their own learning and "were encouraged to challenge themselves to think through problems."

Teachers Found the PD Model to Be Relevant, Useful, Responsive, and Impactful

They commented that "this is the best PD we have ever had," and "it's not like other PD where you don't want to go … I wanted to come to this PD." They found the facilitators supportive and responsive, and felt well-prepared to teach each of the units.

Relationships Built with the PD Team Were Critical

The PD facilitators and district personnel involved with SunBay worked hard to build rapport and demonstrate care for participating teachers, and the teachers noticed. They felt that facilitators "were always available and listened to us as they planned PD" and "that if I need anything, it's a phone call away."

The Curriculum and Pedagogy Were Challenging

For most of these teachers, the SunBay approach was a shift in how they were used to teaching. It was challenging to think on their feet, to manage classroom discussions well, as well as to not give away answers to students but to instead allow students time to work things out on their own. Teachers struggled with time management and pacing, especially during the first year of implementation. The curriculum developers responded to this feedback by shortening units and providing better-suited materials for both teachers and students. For instance, the teachers were provided more guidance regarding how to integrate the curriculum materials along with other the mathematics activities (e.g., warm-ups) they use in their class. The teachers were grateful, and reported their timing issues were "much better" due to their exploration regarding how effectively integrate the material without cutting or modifying it.

The PD Was Effective for the Teachers and Their Students

Teachers reported that during their instruction of the SunBay units, their students were more participatory, more actively engaged with content, seemed to enjoy themselves more, and better understood the mathematics as compared to business as usual. Their students "were eager to understand the concepts behind the math." Some teachers used pedagogical strategies learned at the PD when teaching non-SunBay units.

Opportunities for Empowerment Result in Empowered Teachers

Though participation was technically voluntary, many Year 1 teachers expressed being "volun-told" to participate. Over time, this perspective shifted, with participants becoming SunBay advocates and using the SunBay lessons for their formal school-based observations and evaluations. Two teachers presented at their state-level Council for Teachers of Mathematics conference, and four volunteered to serve as Teacher Leaders and carry on SunBay trainings with others in the district after the study was completed.

Concluding Remarks

When adopting new curricula and encouraging teachers to change their thinking and modify their instruction, PD has the potential to make a great impact on teacher practices and student achievement, but many times fails to do so. We offer an alternative. The PD model we implemented and relationships we built with teachers resulted in relevant, impactful, and empowering PD for teachers. This high-touch, responsive, collaborative, and research-based approach may serve as a model for providing PD to mathematics teachers adopting a curriculum and pedagogical approach aligned with best practices for teaching mathematics. We encourage PD providers to offer long-term and responsive PD to teachers, focusing especially on building both content knowledge and relationships with teachers. It is worth noting that when our grant concluded, some of the teacher leaders we trained continue to provide PD to other teachers in their district and the SunBay units continue to be one of the curricular choices for teachers to implement. For these reasons we believe this approach to be sustainable in large districts.

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ONLINE VIDEO COACHING: AN ANALYSIS OF TEACHERS' AND COACHES' NOTICING

Julie M. Amador
University of Idaho
jamador@uidaho.edu

Cynthia Carson
University of Rochester
ccarson3@u.rochester.edu

Ryan Gillespie
University of Idaho
gill6762@vandals.uidaho.edu

Jeffrey Choppion
University of Rochester
jchoppin@warner.rochetser.edu

We discuss our three-part online synchronous and asynchronous professional development model for online video coaching for rural middle grades mathematics teachers and share results of an analysis of teachers' and coaches' noticing. We report on one aspect of this model, an online version of Content-Focused coaching (West & Cameron, 2013). We engaged practicing teachers in repetitive cycles of coaching, which involved co-planning, teaching, annotating video, and debriefing, all through distance technologies. In this study, we focus on the noticing of both teachers and coaches as they watched the recorded lessons and annotated lesson video. Findings indicate that coaches and teachers focused on both the teacher and students in their annotations. Coaches were specific more often in their annotations and made more connections than teachers. Coaches' annotations were more often interpretive whereas teachers were descriptive.

Keywords: Middle School Education, Teacher Education, Technology

Teachers in rural areas face constraints in terms of accessing the expertise and resources required for high-quality professional learning experiences, often because of lack of proximity to such resources as institutions of higher education and critical masses of teachers required to collectively reflect on problems of practice (Howley & Howley, 2005). The innovative online professional development experiences in our project focus on the development of teacher capacity to enact the Five Practices for Orchestrating Mathematics Discussions (Smith & Stein, 2011), with a specific focus on supporting teachers to notice students' mathematical thinking (i.e. van Es & Sherin, 2008; Jacobs, Lamb, & Phillip, 2010). Recognizing the critical need to prepare all teachers to implement rigorous instruction, especially teachers who are not geographically proximate to in-person trainings or coaching resources, we engaged participants in online courses, online lesson labs, and online video coaching over a two year period.

The use of coaching to foster teacher learning and improve students' mathematical understanding has become a popular strategy for schools, districts, and states in the US (Gibbons & Cobb, 2017). Prior studies have shown that coaching can improve certain aspects of teaching (Kraft, Blazar, & Hogan, 2018; Sailors & Price, 2015). In this paper, we focus on one component of our larger model—online coaching—and particularly the use of video annotations made by both coaches and teachers in response to video-recorded lessons. This was completed through the use of an innovative online Content-Focused coaching model (Authors, 2015; West & Cameron, 2013) to stimulate coaching interactions and the development of professional noticing (Jacobs et al., 2010; van Es & Sherin, 2008). The purpose of this study was to better understand the content of the annotations and the interactions of the coach and teacher in the online space. Specifically, we were interested in understanding the subject focus of the annotations (whether the coach noticed student or teacher actions), the specificity of the noticing (whether general or specific), and the analytic stance (how noticing was communicated). Given the dearth of research on coaches' noticing, we consider it valuable to focus on coaches' noticing, in addition to teachers' noticing. We answered the following research question: When annotating lesson video as part of an (synchronous and asynchronous) online coaching model, how is the noticing of coaches and teachers similar and how is it different?

Method

Using a cohort model, we engaged 8 middle-grades teachers in an intensive two-year professional development model focused on supporting teachers to engage in ambitious, responsive instruction. The four coaches on the project were all experienced in coaching teachers using in-person Content-Focused coaching (West & Staub, 2003).

Online Content-Focused Coaching Model

As a component of the overall professional learning model, participants took part in Content-Focused coaching delivered synchronously and asynchronously. To facilitate this online coaching, we used video conferencing software (Zoom), and video capturing/annotating software (Swivl). The online video coaching was purposely designed with features analogous to West and Staub's (2003) in person Content-Focused coaching cycle. First, the teacher and coach met synchronously via Zoom to plan the lesson; second, the teacher video recorded the lesson implementation using Swivl; third, the teacher and coach asynchronously viewed and annotated the video of the enacted lesson; and finally, the coach and teacher met synchronously via Zoom to reflect on the lesson using the annotations to anchor their discussion.

Data Collection

The focal aspect of this study is the lesson annotation component. After teachers taught and recorded their lesson, the coach and teacher each independently watched the video of the lesson and wrote comments (annotated) about the lesson through the online Swivl platform.To enter the comments, the coach or teacher paused the video and typed their comments, which were then synced to the video by a time-stamp. This allowed for the teacher or coach watching the video to connect a comment to a specific moment in the video. The unit of analysis for this study was the annotations from the coach-teacher pairs, with each coach-teacher pair completing two or three coaching cycles over the course of one year.

Data Analysis

We considered each annotation comment a separate data unit for analysis. Given the focus on understanding coach and teacher noticing, we created a codebook with three main categories: subject (who), specificity, meaning the extent to which the comment made connections, and analytic stance (how noticing was communicated) (see Figure 3). We based the subject category on the work of van Es and Sherin (2008) and van Es (2011), with the focus on either the teacher or student(s). We based the specificity component on the work of Author (date) to describe how participants drew connections between teaching and learning in their comments. We based the analytic stance category on the work of van Es and Sherin (2008) and van Es (2011). Categories for analytic stance included tag, describe, evaluate, interpret, suggest, and question. Based on the literature, we considered the codes of *tag* and *describe* to reflect less advanced noticing and *evaluate* and *interpret* to reflect more advanced noticing. We consider *suggest* and *question* to be deliberate types of moves that do not reflect more advanced or less advanced noticing directly, but represent a stance participants took. The code for *tag* was only used in the absence of any other code for analytic stance. Each annotation could receive any number of codes from each of the categories. After several rounds of testing and validating the codebook, coding commenced with two researchers independently coding all annotations from a given coach-teacher coaching cycle. We then calculated Kappa for each coaching pair and the two researchers met to reconcile differences in codes, resulting in final codes for each coach-teacher pair for each coaching cycle. Kappa ranged from 0.63 to 0.70, indicating good to excellent reliability (Landis & Koch, 1977). Following the assignment of codes, we conducted frequency counts related to all codes for the coaches and teachers individually. We then conducted frequency counts for the coaches as a group and the teachers as a group.

Results

In total, 197 annotations were made within the analyzed coaching cycle; of these 60.9% (n=120) of these were from the coaches and 39.1% (n=77) were from the teachers.

Subject (Who)

Coaches focused on the teacher in 73.3% (n=88) of their annotations and students in 68.3% (n=82). The teachers focused on the teacher in 55.8% (n=43) of the annotations and students in 55.8% (n=43). Teachers had fewer annotations that mentioned both the teacher and the student(s) (20%) as compared to the coaches (40.6%). The teachers' annotations commonly focused on only one subject, a teacher or student, rather than two subjects. In contrast, it was more common for the coaches to include a focus on both teacher and students in a given annotation.

Specificity

The coaches were more specific in their annotations than the teachers. The annotations from the coaches were coded as specific 75% of the time and the annotations of the teachers were coded as specific 20.2% of the time. In other words, the teachers were much more likely to make a general annotation when compared to the coaches. For example, one teacher wrote, "Used a fraction bar to start figuring out inches to pounds…" (Hopkins, June 19) A coach wrote, "Good wait time." (Lowrey, June 19) In these examples, the annotations were general because there was no connection drawn between the teacher and student or another person or aspect (see Figure 3). In contrast, the following was a specific annotation from a coach, "Bringing the conversation back to what was important information about the problem provided a bridge between the two days and supported students to re-engage in the problem." (Lowrey, June 19) In this example, there was a connection between the teacher's talk move of "bringing the conversation back to what was important" and student learning by supporting them "to re-engage in the problem" because of this move.

Analytic Stance (How)

In considering the analytic stance of the coaches and teachers, there were differences with respect to the type of stance taken. Table 1 shows the percentage of each type of analytic stance, meaning *how* the annotation was written.

Table 1: Percentages of Annotations by Analytic Stances

	Tag	Describe	Evaluate	Interpret	Suggest	Question
Coaches	3.14	17.28	15.71	24.61	23.04	16.23
Teachers	15.56	25.56	15.56	17.7	16.67	8.89

The frequencies of evaluative annotations was similar for both groups, with interpreting and suggesting being more common for the coaches. In other words, the coaches were more likely to make meaning or make an interpretation through the annotation. The following is an example of an interpretation from a coach:

> "What do you mean by that?" provides opportunity for the student to think through their own process and for the rest of the group to understand the information better. This also provides an opportunity for students to engage in the mathematical practices. (Lowrey, June 19)

In this example, the coach referred to teacher move (describe) and then unpacked the outcome of this move with respect to student opportunity for engagement in the lesson (interpret).

Coaches provided suggestions for the teachers more often than teachers wrote comments to themselves about what they should do differently. The following is an example of a coach providing a suggestion, "After posing the question about 50 people, I wonder if a few minutes of independent think time would have given all of the students a chance to make sense of the question you asked." (Reiss, June 6) The teachers gave themselves pointers in some instances, "Have the student explain how/why." (Sandoval, June 4) In this case, the teacher wrote herself a note that that she should have had the student explain. Approximately 20% of the annotations (irrespective of participant type) contained some type of suggestion for how something could have gone differently in the lesson.

Discussion and Implications

Knowing that the coaches focused on students in their annotations to a greater extent than the teachers raises questions about practices that would better support teachers to notice children's mathematical thinking (e.g. Jacobs et al., 2010; van Es, 2011). To increase teachers' focus on students, coaches may consider how they could encourage teachers to make an intentional shift to focus more heavily on students' mathematical thinking in their annotations.

Another difference in the results concerned the level of specificity of the coaches' annotations as compared to the teachers' annotations. In many instances, the coaches identified student thinking, made interpretations, and made connections to broader principles of teaching and learning, all components of the van Es and Sherin (2008) definition of noticing. This often involved identifying a moment in the lesson and then elaborating on this moment with some interpretation, which is a heightened form of noticing (van Es, 2011). These findings raise questions about how we support teachers to develop more specificity and increase the frequency at which they make interpretations with their own noticing.

An additional question for future investigation would be to connect the coaches and teachers' noticing, evidenced in their annotations, with the content of the post-coaching cycle debrief meetings. As a part of the online Content-Focused coaching model, the coach and teacher met following their completion of the annotations for an hour-long debriefing meeting. Connecting the annotations to the actual conversations the participants have about the lesson would provide further insight into their noticing.

These findings provide an opportunity for professional development providers to understand how teachers notice as they participate in the online learning experience. At the same time, the annotations provide insight about how the teachers focus on students' thinking, which may relate to their ability to monitor student thinking in their teaching (Smith & Stein, 2011). We contend that the annotation analysis is one minute component of the larger project and additional studies are needed to fully understand the noticing practices of the participants.

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IN-SERVICE TEACHERS' NOTICING OF MATHEMATICAL IDEAS IN AN ANONYMOUS PRESCHOOLER'S REPRESENTATION OF COUNTING

Madhuvanti Anantharajan
Stanford University
madhuvan@stanford.edu

This study examines twelve in-service early math teachers' noticing of mathematical ideas in an anonymous preschooler's representation of counting. The findings suggest that stand-alone representations offer a generative opportunity for teacher-learning about student-thinking.

Keywords: Number concepts and operations, Early childhood education, Teacher-noticing, Mathematical representations

Objectives

Literature on the early math activity of Counting Collections suggests that asking children to represent their counting on paper can provide teachers valuable information about children's mathematical strategies and understanding (Franke, 2017; Johnson & Gaxiola, 2018; Schwerdtfeger & Chan, 2007). However, the independent potential of children's representations as evidence of their mathematical thinking has not been studied empirically. The current study examines how, in the absence of other information about the student, teachers who implement Counting Collections make sense of a student's mathematical thinking through a representation of counting. The results can inform professional development and teacher-learning opportunities.

Theoretical Framework

The broad research question that guides this paper is: What do in-service preschool and kindergarten teachers who implement Counting Collections learn about students' mathematical thinking from an anonymous preschool student's representation of counting? The sub-questions are (1) What mathematical ideas do participants notice in the representation, and to what degree is teacher-noticing of mathematical ideas in the representation consistent across participants? (2) How do the constraints of the anonymous representation affect participants' noticing of mathematical ideas? The study is informed by the research in math education discussed below.

Teaching Based on Student-thinking

Researchers and practitioners agree that teaching grounded in student-thinking effectively supports children's mathematical learning (Ball, 1997; Carpenter, Fennema, & Franke, 1996; Lampert, 2003). The current study is based on an activity called Counting Collections, where children count collections of objects and record their counting on paper. Teachers make sense of children's understanding by observing them count, having them explain their thinking, and asking them questions about their counting (Carpenter, Franke, Johnson, Turrou & Wager, 2017; Franke, Kazemi & Turrou, 2018; Schwertfeger & Chan, 2007). Teachers' moves when engaging with students' thinking are conceptualized as teacher-noticing and include attending, interpreting, and responding (Jacobs et al., 2010; Sherin, Jacobs & Philipp, 2011). The current study examines teacher-noticing of a preschooler's representation of counting.

Teacher-learning About Children's Mathematical thinking from Student Work

Research suggests that young children's written work is valuable evidence of students' mathematical strategies, reasoning, and confusions (Brizuela, 2005; Carruthers & Worthington,

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2005, 2006; Johnson & Gaxiola, 2018; Kamii et al., 2001; Schwerdtfeger & Chan, 2007). In the current study teacher-noticing is elicited through an artifact of practice (Ball & Cohen, 1999), namely a representation made by a preschool student during Counting Collections. The study evokes "productive disequilibrium" (Ball & Cohen, 1999, p.15) in participants through a generative piece of student work chosen for rich mathematical details, while limiting external information about the child. The study also examines teachers' restraint in interpreting student work and how they verify their interpretations (Ball & Cohen,1999; Kazemi & Franke, 2003).

Early Math

The study focuses on the early math domain of counting and number sense. Counting involves the principles and strategies of counting (Carpenter et al., 2017; Clements & Sarama, 2009; National Research Council, 2001, 2009). A previous study of Counting Collections indicated that children need distinct understandings to *represent* counting, compared to counting objects (Anantharajan, in press). The current study examines teacher-noticing of all these mathematical ideas in an anonymous representation of counting by a preschooler (See Table 1).

Methods and Data Sources

The participants[1] were 12 in-service teachers based in California, who taught pre-K, transitional kindergarten, or kindergarten. The participants had implemented Counting Collections for between six months and four years. All participants looked at the same representation (Fig.2). They were told the student's grade level and that the student was counting coins. They did not observe the child's counting that preceded the representation and did not know the child. The representation provides the opportunity to notice *multiple mathematical ideas* including number sequence, one-to-one correspondence, cardinality and representing place value in the base ten system. It indicates the student's *strategies* (Schwerdtfeger & Chan, 2007; Franke et al., 2018), such as representing in rows, and associating each object with a numeral. The representation indicates a *problem* the child may be trying to solve (Carpenter et al., 1993), to represent the ordinal value of each object. Finally, the representation indicates a *partial understanding* of a math idea (Carpenter et al., 2017; Franke et al., 2018), representing place-value. Participants looked at the representation during individual semi-structured interviews. Questions focused on their noticing of mathematical ideas in the representation and interpreting what the student may understand or struggle with. Interview transcripts comprise the study data.

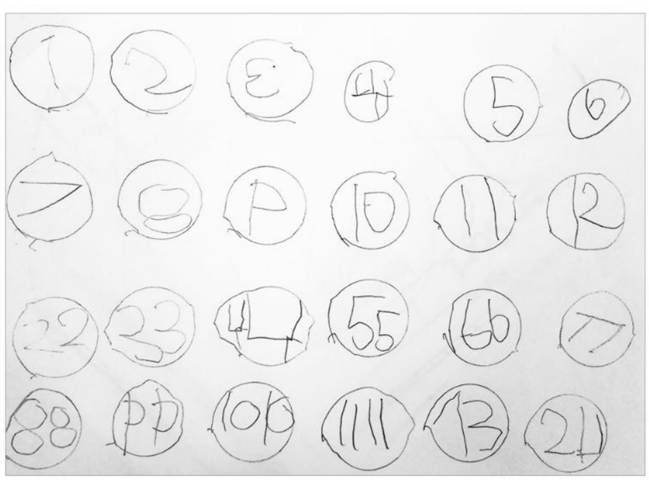

Figure 1: Anonymous Representation of Counting Made by Preschool Student

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

The coding framework consisted of three main codes: specific principles and strategies of counting, and understandings required to represent counting (Anantharajan, in press; Carpenter et al., 2017; NRC 2001). Each code was distinguished to indicate a strong understanding when teachers implied that the child's understanding of an idea is fluent or consistent; a partial understanding when teachers implied that the child is working on or has an inconsistent understanding of an idea; and uncertainty when teachers were unsure of how to interpret particular aspects of the child's thinking. Transcripts were coded line by line applying as many codes as needed to each line. Inter-rater reliability of 90 to 100% was achieved. Nvivo was used to code the transcripts. The analysis indicates the number of participants who noticed a mathematical idea or level of an idea and qualitative patterns across participants' noticing.

Results

Multiple Participants Noticed Certain Mathematical Ideas in the Representation

Table 1 shows that teachers noticed some mathematical ideas consistently in the representation. All twelve teachers noticed one-to-one correspondence and knowing how numerals look and how to write them. Eleven teachers noticed number sequence. Ten teachers noticed the use of visual configurations. Seven teachers noticed an understanding of cardinality, the association of number with object to count, and the symbolic strategies used by the student.

Table 1: **No. of Teachers Who Noticed Mathematical Ideas in an Anonymous Representation**

		No. of teachers
Principles and strategies of counting	One-to-one correspondence	12
	Number sequence	11
	Cardinality	7
	Abstraction principle	0
	Order irrelevance	0
	Other mathematical ideas	9
	Grouping	4
	Visual configurations	10
	Keeping track	1
	Associating number with object	7
	Other mathematical strategies	0
Understandings distinct to representation	Counting can be shown on paper.	2
	Marks on paper can be counted independently	0
	Representations can show what objects are counted	2
	The total number of objects in the collection determines the number in the representation	3
	Using symbolic strategies for representation of counting	7
	Utilizing space on paper	1
	Knowing how numerals look and how to write numerals	12

Participants also commented on the degree to which the student understood the ideas. For instance, eleven of the twelve participants who noticed one-to-one correspondence in the

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representation perceived a strong understanding of it. All eleven participants who noticed an understanding of number sequence, noticed a partial understanding of the principle. Similarly, ten participants noticed a partial understanding of writing numerals and all twelve expressed uncertainty about what the child may know about writing numerals.

Participants Spontaneously Attended to Specifics and Details of the Representation

All participants paid special attention to the third and fourth row of the representation. Participants spontaneously went beyond broad assessments of right or wrong answers to both *specifics* and *details* of what they saw. Consider Betty's observations:

> "They definitely know the counting sequence up here [end of the second row] …[I]t's interesting they counted correctly up to twelve but then started doubling their numbers but then ended with the correct number at the end. That's pretty fascinating." (Betty)

Participants pinpointed the specific place where they saw a shift in the child's understanding – the beginning of row three. They further articulated specifics of what followed – that the student "doubles" the digits, that the last number in the representation seems to reflect the correct total number. The teachers also moved back and forth between what they noticed and their interpretation of what the child may know or struggle with.

Participants Proposed Multiple Explanations for the Same Aspect of the Representation

Each participant proposed between three and eight explanations for the apparent shift from the third row onwards. For example, Amanda used details of the representation to consider multiple possibilities: the student may be struggling with number sequence; trying to invent a system of numerals; represent a counting strategy of grouping; or struggling to remember how to write numerals after twelve. Amanda is one of the participants who relates the representation to a common struggle that young children have with the number sequence, the transition from twelve to the 'teen numbers.' In interpreting such details, seven participants made the distinction between the child knowing number names and sequence and knowing how to write numerals.

Participants Were More Uncertain About Their Interpretations of the Child's Partial Understandings Than About Their Interpretations of the Child's Strong Understandings

In contrast to participants' confidence regarding their perception of strong understandings, more participants expressed uncertainty about their perception of partial understandings in the representation. Eleven out of twelve participants said that the representation indicates a *strong* understanding of one-to-one correspondence. Only three of the eleven indicated that they may need to confirm whether the child has a strong understanding of one-to-one correspondence. Similarly, ten out of twelve teachers felt that the child had a strong understanding of visual configurations to aid and represent counting, but only two expressed uncertainty about their interpretation. In contrast, eight of the eleven teachers who indicated that the child had a *partial* understanding of the number sequence past twelve also indicated that they were uncertain of the child's understanding of the number sequence. Similarly, ten out of twelve teachers felt that based on the representation, the student had a partial understanding of how numerals look and how to write them. All twelve teachers however expressed their own uncertainty about the student's understanding of how numerals look and how to write them.

Significance

The results of this study suggest that the use of representations in professional learning may help teachers develop the habit of attending to specific details in the work; exploring multiple explanations for what students do; perceive a broader range of abilities, struggles and confusions in students; and remain aware of the limitations of what student work alone can reveal about student thinking. These habits and attitudes may help teachers explore alternative explanations for the work of students that they believe they know very well. Teachers can also share their students' work with colleagues and invite fresh insights into their students' thinking.

Endnotes

All participants' names have been changed.

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PUBLIC SENSEMAKING OF NON-REHEARSING TEACHERS DURING DEBRIEFS OF REHEARSALS

Jen Munson
Northwestern University
jmunson@northwestern.edu

Erin E. Baldinger
University of Minnesota
baldi013@umn.edu

Rehearsals have become a common way to support mathematics teacher learning. Primarily, research on rehearsals has focused on the experience of the rehearsing teachers, and we know much less about the experiences of non-rehearsing teachers. In this study, we investigate non-rehearsing teachers' public sensemaking about rehearsals during debrief discussions following rehearsals. We use the lens of professional noticing to capture the ways in which non-rehearsing teachers made sense of the rehearsals and positionality to account for the dual roles of student and teacher that non-rehearsing teachers take on in rehearsals. We found that non-rehearsing teachers primarily attended to and interpreted their experiences from the position of student, and made connections beyond the rehearsal from the position of teacher. These findings have implications for considering the experiences of all rehearsal participants.

Keywords: Teacher Education-Inservice/Professional Development, Instructional Vision, Instructional Activities and Practices, Classroom Discourse

Recent literature in mathematics teacher preparation focuses on how coached rehearsals can be used to support the learning of complex instructional practices (e.g., Ghousseini, 2017; Lampert et al., 2013). In these approximations of practice, one person typically takes on the role of teacher, while the rest of the group takes on the role of student. The rehearsal is usually facilitated by a teacher educator (TE) who takes on the role of coach (Kazemi, Ghousseini, Cunard, & Turrou, 2015). As these approximations of practice have gained prominence in both pre-service and in-service professional development, research has focused on the learning of participants who take on the role of teacher (e.g., Ghousseini, 2017; Lampert et al., 2013). However, we know much less about the experiences and learning opportunities for those who take on the role of student. In this paper, we examine the experiences of secondary mathematics teachers taking on the role of student in rehearsals during a summer professional development institute. In particular, we describe how these non-rehearsing teachers (NRTs) using noticing to make sense of their experiences during the reflective debrief discussions that follow each rehearsal.

Prior Literature

Often, after a rehearsal, all participants (rehearsing teachers [RTs], NRTs, and TEs) engage in a debrief discussion. This type of discussion could take many forms based on the learning goals for the rehearsal. For instance, a debrief could serve as a venue for RTs and NRTs to reflect on and discuss their experiences during the rehearsal (e.g., Baldinger, Selling, & Virmani, 2016). This kind of reflective post-rehearsal discussion, facilitated by the TE, represents an opportunity for public sensemaking, or "collaborat[ing] on sensemaking as a shared group goal" (Ruef, 2016, p. 16). Through public sensemaking, RTs, NRTs, and TEs all have a chance to learn from one another and reflect on the rehearsal experience. Public sensemaking is also valuable from an analytic perspective, because it gives insight into some (but not all) of participants' sensemaking about a given experience. In this kind of reflective debrief structure, we wonder about what sensemaking NRTs share publicly.

One way to describe public sensemaking analytically is through the lens of professional noticing. We draw on the framework for the professional noticing of children's mathematical thinking (Jacobs, Lamb, & Philipp, 2010). When engaging in the work of noticing, teachers *attend* to specific details, *interpret* those details, and finally *decide how to respond*. In the context of a debrief discussion, NRTs might attend to and interpret an event that occurred during the rehearsal. However, in this context, the NRT would not be deciding how to respond to student thinking, but rather might describe implications or connections that extend their thinking beyond the specific context of the rehearsal itself. This noticing framework provides a lens through which we can describe the public contributions made during debrief discussions.

NRTs spend rehearsals acting as students, and then they are asked to apply what they learn from rehearsals to their work as teachers. In other words, NRTs are asked to change their positionality, moving between the positions of teacher and student during the rehearsal and debrief. How a person positioned themselves in an interaction can influence the obligations they feel (Aaron & Herbst, 2012; Herbst & Chazan, 2012). In particular, being positioned as teacher versus being positioned as a student can change how people react to the same contexts (Baldinger & Lai, 2019). Positionality is thus an important consideration that might help explain how NRTs engage in public sensemaking. Given this, we ask the following research questions: (1) How do NRT use debrief discussions to publicly make sense of their experiences during rehearsals? (2) How do NRT position themselves during debrief discussions? (3) In what ways is positioning related to NRT noticing?

Methods

Setting and Participants

This study took place in the context of professional development for early-career (2nd-7th year) secondary mathematics teachers serving lower-income schools. This two-year fellowship included two-week summer institutes and ongoing online coaching during both school years. Our research considers the second summer institute, which focused on facilitating collaborative group work. Participants included 22 high school mathematics teachers from comprehensive public, magnet, and charter schools across the US.

Design

The summer institute culminated in a full day during which all participants had the opportunity to rehearse leading collaborative group work. Participating teachers were randomly assigned to one of two rehearsal rooms. Four rounds of rehearsal were conducted in each room, so that each teacher had the opportunity to rehearse the focal practice once and participated as a NRTs three times. Additional math teachers were recruited to participate as NRTs to increase the class size in each room. After each rehearsal, the facilitating coach prompted the group to discuss the experience, beginning by asking RTs to share "some of the things in your head right now" and then pose any questions they had for the NRTs. Debriefs then pivoted to NRTs addressing any RT reflection questions and segued into general reflection. All debriefs ended with "final thoughts" from the RTs.

Data Sources and Analysis

This study draws on audio and video recordings of the debrief discussions following each of the eight rehearsals at the end of the second summer institute. Debrief discussions ranged in length from 14 to 23 minutes. Each was professionally transcribed for analysis.

Each talk turn (n = 454) was coded by participant type (RT, NRT, TE) and then segmented based on the component of noticing represented in the speech, (i.e., attending, interpreting, implicating), where *attending* and *interpreting* were defined by prior research (Jacobs et al., 2010) and *implicating* was defined as making connections beyond the rehearsal (e.g., to the speaker's own classroom) or considering alternative pathways for the rehearsal itself. Emergent coding was used to classify any remaining utterances. Each talk turn segment was then coded for the positionality the speaker took in the utterance (e.g., teacher, student). If a single utterance took more than one position, it was further segmented such that each utterance could be given a single position and noticing code pair (n = 780). Emergent codes were developed to further classify utterances coded as *attending* based on *to what* the NRTs were attending. Code matrices were developed within and across the eight rehearsals to explore patterns of participation. Initial patterns and descriptive statistics are provided in the findings. A chi-square test was performed to test for independence of positionality and noticing type.

Findings

Through our analysis, we found that *public noticing* was inherent to the sensemaking of NRTs in debrief discussions of rehearsals. Of the 762 talk turn segments in the data, 560 segments (73.5%) were NRTs attending to, interpreting, or implicating the rehearsal experience. The remaining segments were either affirming (n = 132) or connecting to (n = 33) others' contributions. Predominantly, NRTs made sense of rehearsals by attending to and interpreting their experiences. Of the 560 noticing utterances, 34.3% were attending, 45.7% were interpreting, and 20.0% were implicating.

Positionality and Noticing

The positionality of NRTs during debriefs played a role in their noticing. When NRTs reflected on their experiences during rehearsals, they most frequently took the position of students (80.5% of noticing utterances). This pattern was most pronounced when NRTs attended to or interpreted their experience, and NRTs' positionality often switched to teacher when implicating (see Table 1). We used a chi-square test for independence to examine the relationship between the noticing component NRTs engaged in aloud and the position they took when doing so. The relationship between these variables was significant, χ^2 (4, n = 560) = 183.23, $p < 0.0001$. When NRTs were attending to or interpreting their experiences they were more likely to do so as students, but when implicating, they were more likely to do so as a teacher.

Table 1: Contingency Table of Positionality and Noticing Component of NRT Utterances

	Positionality			
Noticing	Student	Teacher	TE	Total
Attending	187	5	0	192
Interpreting	225	31	0	256
Implicating	42	66	4	112
Total	454	102	4	560

Attending To

Looking more closely at NRTs' attending talk turn segments (n = 192), we found that what they attended to during debrief discussions spanned 8 categories: (1) the presence of RTs, (2) the moves RTs used, (3) the outcome of the moves used by RTs, (4) the structure of the activities as planned by RTs, (5) the physical materials used in the rehearsed lesson, (6) NRTs' own mathematical work and decisions, (7) NRTs' mathematical struggles, and (8) the moves TEs used to facilitate the rehearsal. Talk turn segments could be coded for more than one attending category.

NRTs primarily focused their attending on the moves the RTs used with them during collaborative work and the mathematical work and decision-making their own group engaged in (see Figure 1). This split points toward the ways that NRTs juggle attending to themselves and the RT during debriefs, while staying nearly entirely in the stance of student. In the five instances where NRTs engaged in attending from the position of teacher, they primarily attended to RT moves or presence.

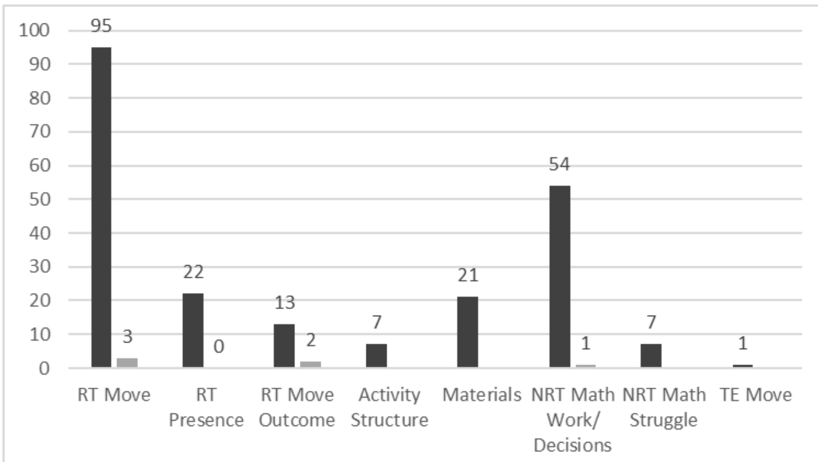

Figure 1: Distribution of What NRTs Attended to During Debrief Discussions When They Positioned Themselves as Student (darker gray) or as Teachers (lighter gray).

Significance

Debriefs of rehearsals created a space for NRTs to publicly make sense of their experiences through noticing, with an emphasis on attending to and interpreting the details of the rehearsal from their position as students. Rather than "deciding how to respond" as the third component of noticing, as it is in the professional noticing of children's mathematical thinking (Jacobs et al., 2010), this study found a third component of noticing in debriefing rehearsals appears to be drawing implications for practice. Responding to student thinking is not the goal of rehearsals as it is in classrooms; rather learning in, from, and for practice is what is called for in the rehearsal setting (Lampert, 2010). This finding contributes to the body of literature on teacher noticing by considering how noticing is used to make sense of learning settings other than the classroom. This study suggests that there is a cognitive shift NRTs make when attending and interpreting from the position of student to implicating in the position of teacher. Further analysis is needed to understand the conditions that foster this shift and how it contributes to the learning goals of rehearsal. In facilitating rehearsals, mathematics teacher educators may need to consider structuring venues for NRTs to make this shift, and debriefs of rehearsals can serve this function.

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NETWORKED PROFESSIONAL DEVELOPMENT: AN ECOLOGICAL PERSPECTIVE ON MATHEMATICS TEACHER LEARNING

Robin Anderson
North Carolina State University
randers6@ncsu.edu

Teacher learning does not happen only during planned traditional professional development programs. In this study, I situate teacher learning within a complex learning network where learning happens amongst and in-between multiple communities of practice. Using survey data from 322 teachers who participate in an online learning community, I argue that teachers draw upon multiple parts of their learning network to respond to the needs of their students. This study seeks to broaden what counts as mathematics teacher professional development.

Keywords: Inservice Teacher Education/Professional Development, Technology

Access to supportive learning communities is critical for the continued professional development of mathematics teachers and the subsequent learning opportunities they provide to their students. With new standards every decade, new students every year, and new technology seemingly adopted every month, teachers must continually learn to keep up with change. Yet, only 25% of teachers report satisfaction with the professional learning that their schools offer (Bill and Melinda Gates Foundation, 2014). Teachers report dissatisfaction with the insufficient time included for professional development during the school year, and the content of professional development often not contextually relevant. To combat inadequate professional development offered by their employers, teachers look for support elsewhere. In this research I examine the complexity of teacher learning networks as they expand beyond traditional forms of professional development.

Framing

Responsive teaching requires teachers to adjust their teaching to meet the needs of their students. Often teachers find themselves modifying or developing new lesson plans overnight in response to their student's needs. Teachers learn in and from practice in multiple ways to become responsive teachers (Ball & Cohen, 1999). Some teachers find value in traditional professional development guided by research based best practices (Desimone 2009; Borko, Jacobs & Koellner, 2010). While others look to informal learning communities, such as online networks, to become responsive teachers (Moore and Chae, 2007; Smith Risser, 2013). But often, teachers are attending to the needs of their students by seeking professional development from multiple learning communities both formal and informal.

In their pursuit to respond to the needs of their students, teachers learn among multiple communities. Often times teachers move between communities to develop a more complex understanding of the teaching and learning of mathematics. Learning among multiple communities is motivated by both necessity and desire. Teachers might find a lack of support in one learning community and therefore turn to another to find the desired knowledge. An ecological perspective on learning allows for a multi-dimensional approach to investigating how teachers learn about teaching and learning within and across multiple communities. Barron (2004) defines "a learning ecology as the accessed set of contexts, comprised of configurations of activities, material resources and relationships, found in co-located physical or virtual spaces that provide opportunities for learning" (p. 6). This framework legitimizes types of teacher learning that are often overlooked within traditional studies on teacher professional development. Within the framework, learning within structured professional development is valued alongside non-traditional forms of teacher learning like blogging or online forums. Researchers have investigated the learning opportunities with an ecological perspective within multiple contexts: youth technology learning (Barron et al., 2009), adult amateur astronomers (Azevedo, 2013), film fans (Sheridan, 2008), and youth summer learning (Akiva, Kehoe, & Schunn, 2016). In this research I seek to add to the examination of mathematics teacher professional development by using an ecological perspective which highlights learning across multiple communities.

Methods

Setting

This report will include initial findings of a survey, which comes from a larger study that examines teacher learning networks. The participants of the study are members of a Facebook group that was started in the summer of 2017 by Youcubed, a mathematics education research group at Stanford University. The group was created to provide a space where teachers, and other individuals interested in mathematics education, could network around reform oriented mathematics education.

The group has new, member-generated, content daily. Members of the group often post questions, share struggles, success stories, and less frequently post links to articles around mathematics education. On average the group has seven original posts a day, 95 comments on existing posts, and 209 reactions. Reactions are the way group members can react to a post, or comment, without writing anything. Members can react by clicking the like, love, haha, sad, or angry emojis at the bottom of the post or comment. In general, activity on the group occurs daily through multiple forms of participation.

Participants

The survey participants are a subset of the Youcubed Facebook group. The survey was posted in the Facebook group and members of the group took it at their convenience. The survey was solicited as a general survey about their professional learning experiences. 322 completed surveys were collected. Because the sampling of the survey was based on convenience, the results are not assumed to be generalizable.

There were 300 female (93.1%), 20 males (6.2%), 1 non-binary/third gender (.3%), and 1 prefer not to state (.3%). The majority of teachers self-identified as being White or Caucasian (86.6%). 3.7% self-identified as Asian, 2.5% self-identified as Hispanic, and .3% identified as Black/African American. The remainder of the respondents self-identified as two or more races.

The majority (97.2%) of respondents have a teaching credential. There was a variety of levels of teaching experience with a majority of teachers having at least 16 years of teaching experience.

Data Sources and Collection

A survey was designed based on a previous learning ecology survey (Barron, 2004) to measure teacher learning networks. The survey was intended to measure, through learning topics and learning partners, the complexity of teacher professional learning networks. The survey included three parts: 1) general demographic questions about the teacher and the school in which they teach, 2) professional development habits through questions around time spent on learning activities, 3) The final part of the survey aims to measure the complexity of the network by collecting data on learning topics and learning partners.

This report will focus on the third part of the survey that intended to measure complexity in the learning networks. The teachers were asked, through a multiple choice question, which topics they learn about when they seek new knowledge about the teaching and learning of mathematics. The topics included: assessment practices, facilitating classroom discussions, scope & sequence of your subject, classroom management, mathematical content knowledge, interpreting student work, unit/lesson design (plans, tasks, activities, etc.), groupwork strategies, math games, strategies for interacting with stakeholders (parents, community, administration, etc.), technology ideas, supporting second language learners, grading practices, supporting students with special needs, classroom routines, questioning, and mindset/brain science. Respondents were also given fill-in-the-blank fields labeled “other” to type in topics that were not covered in the above list.

With the intention of understanding how teachers learn about the above topics, teachers were asked to select learning partners. For each topic that the teacher selected to learn about, a list of 15 learning partners were provided to the teachers to select. The list of learning partners included: colleagues, administrator, coach, in-person learning community (fellowships, conferences, district-sponsored network, etc.), family, university course, websites (blogs, teachers-pay-teachers, Wolfram-Alpha, etc.), online networking community (twitter), online networking community (Facebook), online networking community (other), online webinars, online course, newsletters, books, and journals. The learning partners come from both existing literature on teacher learning networks (Baker-Doyle, 2014) and personal conversation with mathematics education researchers and educators. Thus, for each topic the teachers indicated they wanted to learn about, they also indicated their learning partners. These selections produced a learning network.

Analysis

The survey was constructed to measure the complexity of teacher learning networks. The complexity will be measured based on average number of learning topics, average number of learning partners, and visualization of learning network. While visualizations do not quantify the complexity of the networks, it will help the readers appreciate the interconnected nature of learning within a teacher learning network.

Results

The 322 teachers responding to the survey reported on average they were interested in learning about eight different topics in mathematics education. The top five topics were mindset/brain science (n=254), assessment practices (n=245), unit/lesson design (n=218), facilitating classroom discussions (n=215), and mathematics content knowledge (n=210).

When selecting learning partners, the survey respondents on average elected 7 different learning partners across their desired learning topics. This does not mean that respondents turned to seven different learning partners for each of the topics, but rather between all topics they selected on average seven different learning partners. The top five learning partners were colleagues (n=288), websites (n=273), online networking community/Facebook (n=247), books (n=246), and in-person learning community (n=242).

Complete learning networks of the 322 teacher respondents were built to illustrate the complexity of teacher learning networks. Figure 1 is an example of one respondents learning network. This visualization shows that teachers learn about specific topics within multiple communities. For example, this teacher learns about mindset/brain science from six different learning partners (colleagues, Facebook, websites, coach, university course, and books).

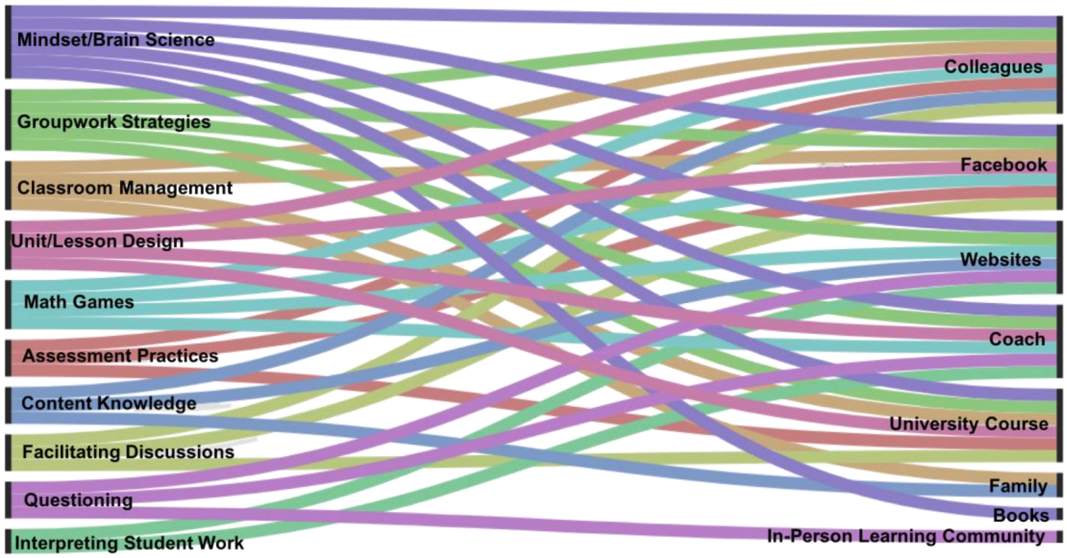

Figure 1: Teacher Learning Network Visualization Example

Discussion

Teachers that responded to this survey desired to learn about multiple different mathematics education topics with multiple different learning partners. As teachers seek out new knowledge to respond to the needs of their students, they are learning about individual topics within multiple learning communities. By moving knowledge between learning communities, mathematics teachers have the potential to develop a more complex understanding of the topic they desired to learn about. With membership in multiple learning communities, teachers experience "boundary encounters" in which they share the practices—new, or augmented, knowledge—between their multiple learning communities (Wenger, 1998). It is during the "boundary encounters" that innovated ideas are shared and teachers have the potential to develop a more complex understanding of their desired learning topic.

An ecological perspective on teacher professional development allows researchers to highlight the complexity of a teacher's learning network. Using a learning ecology survey, data was collected that proved teachers are not just learning from traditional forms of professional development, but rather they report to have multiple learning partners. Further analysis of both the survey data, and also interview data from the larger study, will provide a more complete understanding of how teachers are using their complex networks to make sense of new knowledge of mathematics teaching and learning.

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MATHEMATICS TEACHERS' PERCEPTIONS OF THE INSTRUCTIONAL QUALITY ASSESSMENT AS A REFLECTIVE TOOL

Amber G. Candela
University of Missouri – St. Louis
candelaa@umsl.edu

Melissa Boston
Duquesne University
bostonm@duq.edu

This brief research reports details a professional development using the Instructional Quality Assessment rubrics (Boston, 2012) as the frame for teachers reflecting on their selection and implementation of tasks. The purpose was to find teachers' perspectives of how an observation protocol typically used by researchers could be used as a professional learning tool with teachers. Through case study, we found themes related to teachers finding the rubrics to support their reflection on teaching with cognitively demanding tasks, their ability to provide feedback to colleagues, and their desire for more direct feedback from researchers and instructional leaders.

Keywords: Teacher Education, Inservice/Professional Development

In order to provide students with the opportunity to engage in cognitively challenging mathematical tasks, teachers must regularly select and implement tasks that promote reasoning and problem solving (NCTM, 2014). As such, one of a teacher's most important decisions is choosing the tasks to be used during instruction (Lappan & Briars, 1995). Teachers need knowledge of the characteristics of tasks that have the potential to engage students in high quality mathematics. Once a teacher has selected a cognitively challenging task, they also need to implement that task at a high level to support students' mathematical thinking (Henningsen & Stein, 1997). Teachers can implement cognitively challenging tasks, but not maintain the cognitive demands, depending on how the task is implemented (Son & Kim, 2015; Tekkumru-Kisa & Stein, 2015). Hence, it is not enough to select cognitively challenging tasks, the implementation of the task matters.

A way to provide opportunities for teachers to learn about selecting and implementing cognitively challenging tasks is through professional development (Boston & Smith, 2009, 2011; Stein, Grover, & Henningsen, 1996). The challenge for those designing and providing professional development is to identify a format that adequately supports teachers in enhancing their instruction. This study builds off a previous study on professional development aimed at supporting teachers' selection and implementation of high cognitive demand tasks (Candela, 2017) by using the Instructional Quality Assessment (IQA) rubrics (Boston, 2012) as a frame for supporting teachers' selection and implementation of tasks. In the original professional development, the IQA rubrics were used to rate teachers' implementation of tasks, but teachers were not introduced to and had no interaction with the IQA rubrics. In this pilot study, we intended to use the IQA rubrics as a tool *with* teachers, so teachers could become familiar with the rubrics, use the rubrics to support their selection and implementation of cognitively demanding tasks, and be empowered to direct their own instructional change. In this brief research report, we describe a yearlong professional development utilizing the IQA rubrics as a frame to support teachers' selection and implementation of cognitively demanding tasks and the teachers' perspectives on using the rubrics to support mathematics instruction.

Methodology

In order to understand the teachers' perspectives, we utilized case study (Yin, 2003) to develop thick descriptions and answer questions about teachers' perspectives with respect to the phenomenon of selecting and implementing high cognitive demand tasks, using the IQA as a frame for assessment and reflection. The case was defined as the set of school participants. We collected multiple data points including the individual teacher interviews and final group interview presented in this brief report to gain teachers' voices on the professional development. We analyzed data to find common themes among participants with particular interest in teachers' perceptions of using the IQA rubrics as a tool to support their selection and implementation of tasks, with the following research question driving the research: What aspects of the IQA rubrics and professional development did teachers identify as affecting the implementation of high cognitive demand tasks?

The research took place at a K-8 public charter school in a large Midwestern city. The school population has around 900 students with diverse student body: 44% of the students are White, 26% are Black, 17% are Hispanic, 7% are Asian, and 6% are multi-racial. Sixty-eight percent of students receive free or reduced-price lunches, 22% of students are English Language Learners and 20% receive special education services. The participants (n = 10) included two teachers from grades three through five to provide each with a collaborative partner, each mathematics teacher from grades six through eight, and the mathematics instructional coach. The intention was to create a core of teachers (along with the instructional coach) who could then share and support other teachers with what they learned. Building administrators also attended sessions so they could be familiar with the rubrics and use the rubrics in observations to support mathematics instruction based on cognitively challenging tasks.

The Instructional Quality Assessment and Professional Development

The IQA rubrics to assess students' opportunities to learn mathematics with understanding based on the premise that, in order for students to engage in rigorous mathematics, they need access to tasks that allow and promote such engagement. The Levels of Cognitive Demand (LCD) and the MTF (Stein et al. 1996) inform the IQA's set of Academic Rigor rubrics (e.g., Potential of the Task, Implementation of the Task, Teacher's Questions, and Students' Discussion rubrics). These rubrics measure students' work, the level of the task, and the level of implementation. The work of Michaels, O'Connor, Hall, and Resnick (2010) provides the basis for the IQA's Accountable Talk rubrics (e.g., Teacher and Student Linking, Teacher Press and, Student Providing rubrics). These rubrics capture talk that is accountable to the learners in the classroom community and accountable to mathematics in the classroom. Previously, researchers have used the IQA to observe and rate teachers (e.g., Boston & Wilhelm, 2015) but not in conjunction *with teachers* (Candela, 2017).

As professional development can impact teachers' knowledge in ways that impact their instructional practices (Desimone, 2011), the intent was to increase teachers' knowledge around selecting and implementing cognitively demanding tasks using the IQA rubrics for the professional learning activities. The professional development started with two half-day sessions before school started on using the Potential of the Task rubric to select and modify tasks to engage students in high-level mathematics. Seven times throughout the year, teachers participated in 2.5-hour after-school sessions. Each session was based on a different IQA rubric with the general format of introducing the rubric, watching videos at various score levels, and discussing and rating the videos using the rubric. Between sessions, the teachers then practiced using the rubric to support instruction by reflecting on their own instruction and by offering feedback to colleagues. In the following session, teachers reflected on using the rubric and how it affected their practice and repeated the sequence with a new rubric. A few times throughout the year, teachers videotaped lessons and watched and rated the videos using the IQA rubrics and reflected on their instruction with their peer.

Results

Two themes emerged during analysis from the individual and group teacher interviews. The first theme related to teacher reflection and intentional thought. The teachers claimed the IQA rubrics provided a space for intentional thought in their planning of lessons, reflecting on their instruction, developing a vision of instruction, and providing feedback to peers. The teachers claimed the rubrics supported their planning of cognitively challenging tasks because the rubrics helped teachers think through a lesson in depth, to the point where they felt comfortable implementing a high cognitive task. One teacher commented "I enjoyed thinking about my lessons. In thinking about your task and really knowing the ins and outs before you actually teach something. Really being thoughtful." The teachers also commented on how the IQA rubrics provided a space for reflection. Another teacher said, "When you were reflecting on your own lessons, being able to think back to the rubrics was really helpful." The teachers reflected back on their lessons using the IQA rubrics and then used their reflection to inform instruction.

The teachers thought the instructional activities around using the IQA rubrics to rate lessons, of their peers or of example lessons, provided a good model of instruction. The teachers said these activities helped them visualize maintaining high-level tasks and gained a better idea of what their instruction could look like. One teacher claimed "We have always known it was important for [students] to be discussing, but what I've learned, is how to make that come together. We knew what we needed to do, but now we know how to do it." The teachers also claimed that watching their peers teach helped them see other ways of teaching, and because it was in their school, it was more relatable than some of the videos of outside classrooms. By watching example videos of teachers in mathematics classrooms and watching their peers' videos in conjunction with using the IQA rubrics to rate the lessons, it supported their ideas of what it meant to maintain or lower the cognitive demand of a task. The teachers claimed the peer reflection was also beneficial in hearing others' perspectives on their lesson using the IQA rubrics. The teachers discussed how it is difficult critiquing peers' teaching, but the IQA rubrics helped them by being able to point to specific aspects of the rubrics as evidence.

The second theme that emerged was around the structure of the professional development. The teachers wanted more direct feedback on their teaching and a quicker pace of being introduced to the rubrics. The teachers commented they wanted an expert opinion, as they were unsure if they were implementing tasks well in their own classrooms. One teacher commented:

> I felt like there was a lack of immediate feedback. It doesn't help to come and talk about something I did two months ago when I've still been trying to do it. I didn't just do tasks on the days you asked us to do a task and record it. I was still trying to do them along the way and I kind of felt floundering at times, not knowing am I even doing this right.

While the goal of the professional development was to familiarize teachers with the IQA rubrics and how those rubrics could support practice, we should have provided immediate, ongoing feedback based on the IQA rubrics. This may have supported the teachers with their confidence in implementing tasks and their confidence with the tool. The teachers also commented on the pace of the professional development. In its conception, the plan was to introduce a new rubric each month and give the teachers time between sessions to practice what was learned in the session. The thought was to give teachers smaller bits of information each month in order to not overwhelm them with information and give time each month to practice using a new rubric. The teachers remarked they would have rather learned all the rubrics up front, in the first half of the year, and had the second half of the year to practice using all the components. While the intention was to give smaller sections at a time, it was clear the teachers wanted the rubrics all up front, and then more time to practice.

Discussion

One goal of using an observation instrument, originally designed for research, in a professional development setting with teachers was to provide teachers with a tool that could provide focused feedback and direction to promote growth and improve instruction (Boston, Bostic, Lesseig, & Sherman, 2015). While we looked specifically at the IQA rubrics, there are other observation protocols used in mathematics classrooms to assess instruction. These include The Reformed Teaching Observation Protocol (RTOP; Sawada et al., 2002), The Mathematical Quality of Instruction (MQI; Hill et al., 2008), the Elementary Mathematics Classroom Observation Form (Thompson & Davis, 2014), and the Mathematics-Scan (M-Scan, Walkowiak et al. 2014). For more resources on these observation protocols see Boston and colleagues (2015) or Charalambous and Praetorius (2018). Future research could look at how other observation protocols could be used as tools support teachers' instruction of mathematics.

In recent review of literature, Darling-Hammond and colleagues (2017) found teachers who participated in professional development that involved modeling of the instructional practice enabled those teachers to create the vision they needed to reflect on their own practice. This collaborative aspect aligns with effective elements of professional development (e. g., Darling-Hammond, et al., 2017; McGee, Wang, & Polly, 2013). By providing the teachers time to collaborate and learn from each other with the IQA rubrics, they found the professional development to be more effective in supporting their selection and implementation of high cognitive demand tasks. In regards to providing more feedback, Darling-Hammond and colleagues (2017) claim feedback by an expert and teacher reflection should go hand in hand during professional development to supports the effective implementation of new instructional practices. Desimone and Pak (2017) found integrating coaching into cohesive professional development structures enhance the effectiveness of that professional development. This suggests integrating coaching and feedback into professional development supports teachers and provides an opportunity for teachers to immediately incorporate the feedback into their practice. The professional development in our study had an embedded peer feedback and reflection component, which could be strengthened with an expert feedback and reflection component.

While aspects of the IQA rubrics rate students' engagement with the task, such as students' ability to link their ideas to their peers or justify and explain their mathematical solutions, the IQA does not measure student outcomes related to their performance in mathematics. Future research could collect student data in classrooms where teachers use the IQA rubrics and determine if ratings on the rubrics correlate to student outcomes in mathematics.

In this case study, the goal of using the IQA rubrics with teachers was to support their ability to attend to the cognitive demands of instructional tasks, how tasks are implemented during instruction, and the level of mathematical discussion (teachers' and students' contributions) that occurs around the task. The IQA rubrics provided an accessible way for teachers to reflect on their practice and implementation of high cognitive demand tasks. More generally, this professional development indicates that teachers responded well to begin provided with a platform that allowed for self-assessment and self-reflection and to collaboratively assess and reflect on instruction with colleagues.

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TEACHERS' PERCEPTIONS OF TEACHER-STUDENT RELATIONSHIPS AND THEIR IMPORTANCE FOR STUDENTS' MATHEMATICAL SUCCESS

Dionne Cross Francis
Indiana University
dicross@indiana.edu

Ayfer Eker
Giresun University
ayfer.eker@giresun.edu.tr

Jinqing Liu
Indiana University
jingliu@iu.edu

Kemol Lloyd
Indiana University
kemlloyd@iu.edu

Pavneet Bharaj
Indiana University
pkbharaj@iu.edu

This study investigates teachers' understandings of teacher student relationships (TSRs), and mathematically productive relationships (MPRs). Seven elementary teachers were coached to increase their teaching quality. Semi-structured interviews were conducted at the end of the coaching program to find out how teachers described the core features of TSRs and MPRs, and the factors that influenced students' mathematics achievement. Safe, mistake-friendly environment, trust and respect, establishing and maintaining clear and high expectations, and consistency were considered as core features of TSRs and MPRs across teachers. However, these features did not align with what teachers considered important for deep mathematics learning.

Keywords: Teacher Professional Development; Teacher Attitudes; Teacher-student Relationship

Ensuring that schools are safe, nurturing spaces is critical for children's cognitive and social development (Brophy, 1998; Davis, 2001). In particular, adult-child relationships play a significant role in supporting children's development (Pianta, 1999). In this regard, positive teacher-student relationships (TSRs) are considered essential for quality learning to occur. In this paper, we investigated (i) teachers' perceptions of strong teacher-student relationships (TSRs), (ii) how they should unfold in the mathematics classroom [referred to as mathematically productive relationships (MPR)] and (iii) the extent to which teachers see these relationships as fundamental to students' mathematics learning.

Teacher-Student Relationships (TSRs)

Healthy TSRs promote higher academic achievement and behavioral adjustment and are associated with pleasant student peer relationships (Hughes & Kwok, 2006), better connections with school and engagement in the classroom (Furrer & Skinner, 2003), and higher overall achievement (Crosnoe, Johnson & Elder, 2004). When TSRs are positive, students are more likely to work hard and persevere, to risk making mistakes, and seek help when needed (Murray & Greenberg, 2000) – dispositions essential to being powerful mathematical thinkers. Thus, they tend to have a positive influence on students' achievement, often mitigating the negative effects of socio-economic status (Hattie, 2009). When teachers are unable to develop and sustain these kinds of relationships, students often show lack of focus, low levels of on-task behavior and performance, and engage in disruptive behavior (Marzano, Marzano & Pickering, 2003), thereby negatively influencing learning that occurs in the classroom. As such, how a teacher conceptualizes her teacher-student interactions in the mathematics classroom is important. We refer to these interactions as mathematically productive relationships.

Building Mathematically Productive Relationships (MPRs)

We consider MPRs to be healthy teacher-student interactions that foster mathematical caring, and an emotionally safe environment that encourages intellectual vulnerability, problem solving and academic risk-taking. Similar to TSRs, maintaining healthy MPRs involve communicating positive expectations, engendering feelings of belongingness and pride in the classroom community through warm and responsive interactions, demonstrating caring, and attending to emotional needs that arise personally and for students (Boynton & Boynton 2005; Davis, 2004). However, although there is significant overlap, these features may unfold differently in the math classroom as the primary goal of MPRs is to develop strong mathematical thinkers.

Communicating (and Acting On) Positive Expectations

In the math classroom, acting in ways that help students meet high expectations means selecting worthwhile and rigorous tasks that are developmentally appropriate, then scaffolding students' productive struggles in ways that build ideas. Shaping MPRs may unfold differently than TSRs. Boynton and Boynton (2005) suggest communicating positive expectations by calling on students equitably as a key aspect of promoting positive TSRs. However, two essential practices for supporting mathematical learning is to be selective about the student responses showcased and to sequence these tasks in ways that extend students' thinking (Stein & Smith, 2011).

Relationship Building and Demonstrating Caring

Expressing sincere acts of care towards students will create a positive learning climate, engender trust, and encourage students to achieve the high expectations set by their teacher. Mathematical caring relations "conjoins affective and cognitive realms in the process of aiming for mathematical learning" (Hackenberg, 2010, p. 237). Focusing on productive struggle, where students will experience unpleasant emotions, the mathematically caring teacher may not try to relieve these unpleasant emotions (which is the typical approach of general caring); rather, focus on helping the student channel these emotions in ways that encourage perseverance, and focused and divergent thinking.

Reducing Your Own Frustration and Stress

Teachers' emotional experiences are core aspects of their daily work, particularly in relation to instruction, TSRs, and educational change (Schutz & Zembylas, 2009). In this regard, the continual experience of unpleasant emotions (e.g. anger, frustration) tends to have detrimental effects on teachers' decision-making and instructional practices often leading to attrition (Cross & Hong, 2009). Unless regulated appropriately, heightened unpleasant emotions can interfere with teachers' abilities to develop or maintain an environment that prioritizes the features of productive TSRs and MPRS.

Methods

This study involved seven elementary (K – 5) teachers from three school districts in the Midwest serving high populations of students of color and eligible for free and reduced lunch.

Data Sources and Analysis

The data came from three-hour, semi-structured interviews at the end of the coaching cycles. The questions focused on a range of topics including teacher identity, emotions, beliefs, as well as teachers' definitions of TSRs and their importance, factors that supported positive TSRs, factors that supported mathematical success and teachers' perceptions of productive struggle. The interview transcripts were coded and coded statements were closely examined and categorized to identify overarching themes that captured teachers' perceptions (Saldana, 2005).

Teachers' Descriptions of Factors That Support Positive TSRs And MPRs

Safe, mistake-friendly environment

Several participants foregrounded the importance of trust and building a safe environment where students could be themselves and not be embarrassed to make mistakes. These perspectives were expressed in Wilma's descriptions of the important elements of healthy TSRs. She stated, "It's OK to make mistakes. And we talk about that a lot-- whether it's behavior, whether it's math. yeah, it's OK to make mistakes and using those mistakes as learning platforms." She responded similarly when describing the qualities of MPRs. She emphasized the importance of students feeling comfortable in the environment to share their ideas, ask questions and attempt multiple approaches without ridicule.

Trust and Respect

Two of the teachers emphasized the role of trust and respect in establishing and maintaining TSRs and MPRs. In order to achieve positive TSRs, there needs to be mutual respect between the teacher and student. Linda shared her insight explaining, "[It depends on] trust that is built there. The utmost important thing is that the student respects you and to gain and earn that respect, you have to delve into those relationships - you have to share pieces of yourself." She reiterated later, when asked to describe the core features of MPRs, "Well, I think it's kind of the same thing. I think it's that trust between the student and the teacher. To be able to get down to the nitty-gritty." For Linda trust and respect were fundamental to productive TSRs and establishing this creates and open, safe atmosphere to share.

Establishing and Maintaining Clear and High Expectations

In addition to engendering trust and respect, Julie also underscored the importance of teachers having high expectations "Establishing that safe environment, establishing a personal relationship, a high relationship, where they know their boundaries. They know their expectations. And they know that they are high expectations and that they will be challenged." Julie described the importance of ensuring that the teacher defines the relationship making sure students know the parameters. Both Julie and Anderson were explicit about ensuring that students know what the expectations are, that the expectations are high, and they are there to support them in achieving their goals.

Consistency

There was general sense that these core features of both MPRs and TSRs were to be applied consistently – they should be core pillars of the relationships. Bradley, a third-grade special education teacher, in addition to the attributes mentioned above highlighted the importance of consistency. He captured it well in his statement, "No matter what happens the teacher will most likely react in the same way as they did the day before… consistency is really important for kids…that they know that when you say something that you're going to follow through." Embedded in the teachers' responses was that for both TSRs and MPRs to be healthy, teachers need to be committed to ensuring their classrooms were consistently safe spaces – mistake-friendly, engendering mutual trust and respect with high expectations.

Teachers' Descriptions of Factors That Promoted Mathematical Success

Working with the Student Over Time

All the teachers thought that learning occurred over time through building knowledge of students, how they think and what approaches best support them in advancing that thinking. It's critical that the teacher is "taking time, having the time with them to explore their little mind"

(Sam, post interview). It was important that teachers have opportunities to understand students as learners to best support them mathematically.

Developing a Partnership Around Learning

In addition to time investment, both teacher and student should have similar goals of their interactions - learning. In this regard, teachers found that it was important that this shared goal be enacted in the form a partnership, “like a partnership, like we both learned, [like we] came to this point together.” The invest of time and work should be mutual, Wilma shared that “there's not one particular thing. I think it's a whole body of work on the teacher's part- on the student's part.” For success, it’s important that both parties are committed to doing that work toward a successful end.

Knowing the Basics and Using Manipulatives

Many of the teachers stated that they thought that “a key factor [for] success was knowing the basics” (Anderson, post interview). Similarly, Julie described how she supported a student, highlighting “… just really spending time with her and going back to the basics and using the manipulatives.”

Persistence

In conjunction with the investment of time, teachers also talked about the importance of persevering and being persistent, that is “it was just-- just that constant persistence with them and just trying different ways and they finally got it, but that's what I think of when I think of the success.” Persevering is essential for success because engaging in worthwhile mathematics involves challenge and struggle and sometimes “it takes a long time to get through before you get to this new way” (Bradley, post interview).

Discussion

Alignment Between MPRs, TSRs, And Existing Research

Features of healthy MPRs and TSRs overlapped in teachers’ descriptions suggesting these elements may be grounded in their beliefs about beneficial ways to interact with students. We are cautious here in making connections to their instructional practices as we did not include analysis of the teachers’ instruction. In addition, the relationship between teachers’ thoughts and their actions is not linear (Cross Francis, 2014). The participants expressed that strong TSRs and MPRs involved a safe psychological environment for students, where teachers demonstrated care, one of the characteristics identified in the literature as key to mathematical caring relations (Hackenberg, 2010). Establishing and maintaining clear and high expectations, for both learning and behavior, was important - letting students know they are capable thereby warranting high-quality work was a common thread across the teachers’ responses. The teachers considered maintaining trust and respect to be critical to having a safe space. However, absent from the teachers’ descriptions is the need for the emotional fortitude and skill to consistently provide and sustain opportunities for students to think deeply about mathematical ideas. In order to maintain healthy MPRs teachers must help students navigate through productive struggle, and the emotional fluctuations that accompany this process (Hackenberg, 2010), while maintaining the cognitive demand of the task (Smith, 2000). Teachers did not identify any significant distinctions between healthy TSRs and MPRs, perhaps indicating a lack of awareness of the importance of being able to effectively manage the affective and cognitive realms as students engage in learning (Hackenberg, 2010).

Factors That Contribute to Mathematical Success and Their Descriptions of MPRs

The teachers' descriptions of MPRs did not align with the factors they identified as contributing to mathematical success. This would suggest that they considered the factors that support mathematical success to be direct teacher actions supporting mathematical thinking. Although they did not make this distinction directly, it appeared that the teachers perceived MPRs as more distal to student learning than working with the student over time, developing a partnership around learning, persistence, and knowing the basics and using manipulatives. Although the factors the teachers identified are clearly important, we found this disconnect problematic as it seemed to indicate low awareness of how these elements are integrated to facilitate high quality learning. Let's consider the notion of a mistake-friendly environment - mistakes are fundamental to learning and learning is maximized when students are able to leverage their mistakes to extend thinking. If students do not feel safe to make mistakes and do not appreciate their learning potential, then it's unlikely they will persist in "learning the basics" or be open to working alongside the teacher over time. This disconnect between the elements of MPRs and factors supporting deep learning may explain outcomes of research (e.g., Kane et al., 2013) that show that the nature and quality of both the instructional and emotional/social aspects of TSRs tend to be low generally. Although teachers identify healthy TSRs as important, they may not mentally integrate these features with content-specific instructional practices which may help explain research results that do not show improvement in teachers' instructional practices leading to comparable positive shifts in student outcomes (Kraft, 2019)**.**

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ELEMENTARY TEACHERS' CONFIDENCE IN ENACTING MATHEMATICAL MODELING AND THEIR PERSPECTIVES ON THE NATURE OF MATHEMATICS

Jennifer Suh
George Mason University
jsuh4@gmu.edu

Melissa Gallagher
University of Louisiana at Lafayette
mgallagher@louisiana.edu

Sara Birkhead
George Mason University
sbirkhea@gmu.edu

Padmanabhan Seshaiyer
George Mason University
pseshaiy@gmu.edu

This paper reports on examining teachers' beliefs about the nature of mathematics and their confidence in enacting mathematical modeling in the elementary classrooms after a school-university partnerships institute focused on designing and implementing (MM) tasks. We found high correlation between teachers' Process of Inquiry and their Confidence in MM Competencies. Some of the key findings from the analysis of the lesson studies and teacher design memos revealed that teachers posed MM questions that related to a) student lived experiences drawing on their personal, school and community events that had MM potential; b) situations that connected to their grade level standards; and c) opportunities to use mathematics for decision-making using descriptive, predictive and optimizing models. The study provides promising pathways to work with elementary teachers as designers of MM tasks.

Keywords: Modeling; Affect, Emotion, Beliefs, and Attitudes; Instructional Activities and Practices; Teacher Knowledge

Objectives/Purpose of the Study

Using design research as a method of inquiry, we collaborated with school district leaders to investigate ways to support elementary teachers design and enact mathematical modeling(MM) as a means to promote problem posing and problem solving in the classroom. The district identified a cohort of twenty teachers who were immersed in a weeklong summer institute to experience MM first-hand and improve their content-specific knowledge of MM. In the second phase of the study, in the fall, teacher designers engaged in Lesson Study and collaborated with the research team on designing MM tasks. Through our study we wanted to better understand how teachers learned to problem pose and design early modeling experiences in the elementary grades. As we studied this ambitious practice, we wanted to assess how teaching with an inquiry stance through mathematical modeling impacted teachers' beliefs about mathematics and their confidence in posing real world problems. The following research questions guided this study:

1. Are teachers' beliefs about the nature of mathematics different before and after participating in a PD on MM? Is there a correlation between teachers' beliefs about the nature of mathematics and their confidence in orchestrating MM in the elementary grades?
2. How do elementary teachers develop their ability to pose MM tasks and design early modeling experiences for elementary students?

Theoretical Framework

In the recent years Mathematical Modeling (MM) has gained attention as a comprehensive and dynamic cyclic process that supports the problem posing and the translation of real-world problems into a mathematical language (NGA & CCSSO, 2010). Problem formulation is one of the critical first steps in MM (MM) which includes posing the real-world problem, making assumptions and defining variables, which then leads to developing a mathematical model, analyzing and assessing the solution, and refining the model (COMAP & SIAM, 2016). In examining research on how MM problems are formulated, Galbraith, Stillman, and Brown (2013) reported on turning ideas into modeling problems where “a nucleus of an idea” is developed into a modeling problem and extended by a related problem closer to the “personal experience of adolescents” (p. 133). Zbiek and Connor (2006) investigated the role of contexts and how modeling can “motivate students to study mathematics by showing them the real-world applicability of mathematical ideas” (p. 89). When posing a MM problem, it is important to choose a context familiar to students because “the formulation of the real-world situation itself requires demanding a priori knowledge” (Caron & Bélair, 2007, p. 127). Based on research on professional development around mathematical modeling, we positioned teachers as designers of MM learning experiences for young learners with the support of a six-month professional development design institute.

Methods

Participants

Participants were 19 in-service, K-6 mathematics teachers and one middle school teacher, in a large mid-Atlantic school district, involved in a professional development (PD) initiative focused on mathematical modeling (MM), between August, 2017 and June, 2018.

Data Collection

To assess teachers’ belief about mathematics, we administered teachers’ Beliefs about the Nature of Mathematics (Tatto, et al., 2012) before they participated in the PD and after. We used the Nature of Mathematics to capture teachers’ attitudes about the nature of mathematics, as well as how the subject is best learned and taught. This survey was made up of two scales: (a) mathematics as a set of rules and procedures - “Respondents who score highly on this scale tend to see mathematics as a set of procedures to be learned, with strict rules as to what is correct and what is incorrect” (Tatto et al., 2012, p. 154); and (b) mathematics as a process of inquiry - “Respondents who score highly on this scale see mathematics as a means of answering questions and solving problems” (Tatto et al., 2012, p. 155). Items on this survey are scored on a Likert scale from 1 - Strongly Disagree to 6 - Strongly Agree. We used these scales to see how posing problems in MM would capture attitudes that may be important to the modeling process.

All of our participants completed a survey at the end of the PD and Lesson Study to gauge their level of confidence orchestrating MM with their students. This survey was created by the project team to assess teachers’ confidence in five processes involved in modeling: posing problems, making assumptions and defining variables, building a solution, analyzing and creating a model and finally revising, refining and reporting on the solution. In this survey, teachers self-reported their level of confidence in orchestrating MM in their own elementary classrooms on a 5-point Likert scale. Each Likert scale item was followed by two open-ended questions asking teachers about their successes and struggles with that competency. 19 teachers had complete survey data at all timepoints. Data were also collected from lesson studies and teacher interviews exploring the processes of MM. We focused on better understanding how teachers involve elementary students in the problem formulation process as they co-construct the problem, how they work through making assumptions, building a solution or a model and revising their model as they relate back to their problems.

Data Analysis

For our first research question, we conducted a paired samples t-test to compare teachers' beliefs on the two factors (Rules and Procedures and Process of Inquiry) before and after participating in the PD. In addition, we examined Pearson's correlations between Rules and Procedures and Confidence in Teaching MM, as well as between Process of Inquiry and Confidence in Teaching MM, all at post. For the second research question, we used Dedoose Version 7.5.9 to code our Lesson Study and interview data and dissect the text into text segments. These text segment were analyzed qualitatively for emergent themes and patterns.

Findings

For our first research question, the results of the t-tests indicated statistically significant differences for Rules and Procedures, $p < .05$, and Process of Inquiry, $p < .001$. Means and standard deviations are presented in Table 1. In addition, through the Pearson's correlations. we found that the correlation between Rules and Procedures and Confidence in Teaching MM was non-significant, $p = .59$, but that the correlation between Process of Inquiry and Confidence in MM Competencies was statistically significant, $p < .001$, and also very high, $r = 0.80$.

Table 1: Means and Standard Deviations of Teachers' Beliefs about the Nature of Mathematics and Confidence in MM Competencies

		Pre	Post
Rules and Procedures	M (sd)	3.72 (0.80)	3.26 (0.76)
Process of Inquiry	M (sd)	4.86 (0.73)	5.50 (0.50)
Confidence in MM Competencies	M (sd)	--	3.74 (0.59)

For our second research question, we analyzed responses to open-ended items on the Confidence in MM Competencies survey, lesson artifacts and memos. Qualitative survey data revealed that teachers focused on authentic contexts; finding everyday situations that provided access to familiar contexts and service learning "opportunities [that presented] themselves out of *real need*." In this way, teachers reported that students' engagement was extremely high. Teachers described students as "excited" and "invested in their learning." They described seeing "emotion and empathy" and "ownership investment" from students in math classes engaged in MM. They also described that students "became risk takers" as a result of their participating in MM. These reflections help illustrate why finding local contexts for MM problems is so critical in problem posing. One common challenge with problem posing and content alignment is in finding age appropriate contexts. Teachers found that it could be challenging to find authentic contexts, relevant to their students, that aligned to mathematical content. When contexts were too complex, teachers found that, “students not connecting with the context leads to struggle with assumptions.”

The video analysis of enactment of problem posing for the teachers who reported highest confidence also helped us better understand ways to promote productive problem posing, and the effectiveness of engaging students in problem-posing activities. Some of the key findings from the analysis revealed that teachers tapped into a) student lived experiences drawing on familiar and meaningful contexts from their personal, school and community happenings; b) mathematical situations that related to their grade level standards; c) opportunities to use mathematics for make decisions using descriptive, predictive and optimizing models. For example, one lesson study team focused on getting students to use mathematical modeling to answer the question, “How can we organize and maintain our supplies to last us all year?”. It took the students awhile to get to the point where they decided to count and inventory their supplies. Students then tracked their usage and how many supplies they had over the course of the next couple months. Once they had multiple data points, students then tried to find trends so that they could predict how much supplies they would have by the end of the year. This was a very interesting point in the lesson and students approached this is varying ways. Some groups looked for patterns and did repeated subtraction and division while others used self-invented ways to try to find the average amount used in a month. Some groups found that they predicted they would have supplies left over at the end of the year (pencils), while others found they would run out of supplies before the end of the year (glue sticks). This lead the glue stick group to determining they had to adjust the use of glue sticks to last longer or find a way to raise money to purchase more glue sticks.

Discussion/Conclusion

An important contribution of this study is finding the positive relationship between teachers' beliefs about the Process of Inquiry and their confidence in MM competencies. Respondents who score highly on this scale see mathematics as a means of answering questions and solving problems. They see mathematical procedures as tools of enquiry, as means to an end, but not as ends in themselves. This inquiry stance helps teachers appreciate the nature of mathematical modeling that involves posing problems, allowing students to make assumptions to narrow their investigation to a focused problem and help facilitate the building solution phase without doing the math for the students. They value the need to provide time and space to facilitate discussion with students around the model they developed so that they can revise, refine and report their model so that they learn to critique and revise their models to maximize learning.

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SHIFTS IN SELF-EFFICACY FOR TEACHING ENGLISH LEARNERS: EMERGENT FINDINGS FROM MATHEMATICS TEACHER PROFESSIONAL DEVELOPMENT

Jill Neumayer DePiper	Johannah Nikula	Josephine Louie
Education Development Center	Education Development Center	Education Development Center
jdepiper@edc.org	jnikula@edc.org	jlouie@edc.org

Supports for access and mathematical communication are essential components of mathematics instruction for students who are English learners (ELs), but many mathematics teachers are not adequately prepared to support ELs. The Visual Access to Mathematics project developed and studied teacher professional development (PD) focused on visual representations (VRs) and language strategies in middle grades mathematics. This paper reports on shifts in teachers' self-efficacy for using VRs and for teaching ELs based on participation in the PD. The findings have implications for understanding how PD can affect teacher self-efficacy in these areas.

Keywords: Equity and diversity, Instructional vision, Teacher education, Teacher knowledge.

The Visual Access to Mathematics (VAM) project is engaged in a multi-year effort to develop and study teacher professional development (PD) designed to build teacher knowledge of teaching with visual representations and language access and production strategies to support mathematical reasoning and communication among students who are English Learners (ELs). We recently conducted a cluster randomized trial of the PD to investigate self-efficacy and learning outcomes for participating teachers. Self-efficacy—or one's belief or perceptions about their abilities—can support an individual's incentive to act (Usher & Pajares, 2009) and may lead to change in their instruction. This report presents findings related to participating teachers' shifts in self-efficacy and discusses connections from self-efficacy to specific PD components.

Perspectives and Theoretical Framework

Visual representations (VRs) in mathematics, such as tape diagrams and double number lines, can support student understanding of quantities and relationships (Siegler, Thompson, & Schneider, 2011) and foster students' mathematical problem solving while supporting access to critical mathematical content and processes. Students who are ELs particularly benefit from VRs as VRs support making sense of problems and selecting solution strategies (Ng & Lee, 2008). However, many U.S. teachers are neither experienced nor skilled in understanding and using VRs in mathematics (Orrill & Brown, 2012; Stylianou, 2010), frequently relying on algorithmic thinking, such as the cross-multiplication algorithm in proportional situations, without the ability to demonstrate conceptual understanding or use a VR (Orrill & Brown, 2012).

Additionally, teachers are not consistently provided training for how to support ELs to meet mathematical content standards (Bunch, 2013; Darling-Hammond, Wei, & Adamson, 2010; Samson & Collins, 2012). The integration of language and content during lessons is linked to opportunities for ELs to learn mathematics, and regular participation in "explaining solution processes, describing conjectures, proving conclusions and presenting arguments" both orally and in writing is imperative for ELs (Moschkovich, 1999, p.11). Mathematics teaching practices must "specifically address the language demands of students who are developing skill in listening, speaking, reading, and writing in a second language while learning mathematics"

(Celedón-Pattichis & Ramirez, 2012, p.1). Attention to language is especially important as current mathematics standards emphasize communication and academic language (Bunch, 2013).

The VAM PD was developed to build teachers' knowledge and self-efficacy for planning and implementing mathematics lessons focused on ratio and proportional reasoning content for ELs. The project focuses on ratio and proportional reasoning content because proportionality has been called the "cornerstone of higher mathematics and the capstone of elementary concepts" (Lesh, Post, & Behr, 1988, p. 98). The PD aims to promote teachers' skills in learning and teaching mathematics with VRs because VRs are valuable tools that students developing English skills can use to reason through challenging tasks and to communicate their thinking. The PD also supports teachers in learning how to integrate language access and production strategies into mathematics lessons to support students' mathematical reasoning and communication. The 60-hour VAM PD starts with a four-day face-to-face summer institute, followed during the academic year with eight two-week long asynchronous online sessions, two face-to-face workshops, and four synchronous online meetings. A hybrid face-to-face and online learning approach may promote teacher pedagogical content knowledge and practice as related to student outcomes (Dede, 2006; Treacy, Kleiman & Peterson, 2002). The VAM Theory of Change (Figure 1) identifies key PD activities, hypothesized teacher outcomes, and envisioned classroom and student outcomes. The VAM PD design derives from the research perspective that deep understanding of content is important for teacher practice, and PD that aims to change classroom practice should be embedded in classroom practice and content (e.g., Cohen, 2004). Through insight into students' understandings, teachers can more effectively target math instruction to the concepts, practices, and skills that remain to be learned (Darling-Hammond, et al., 2010).

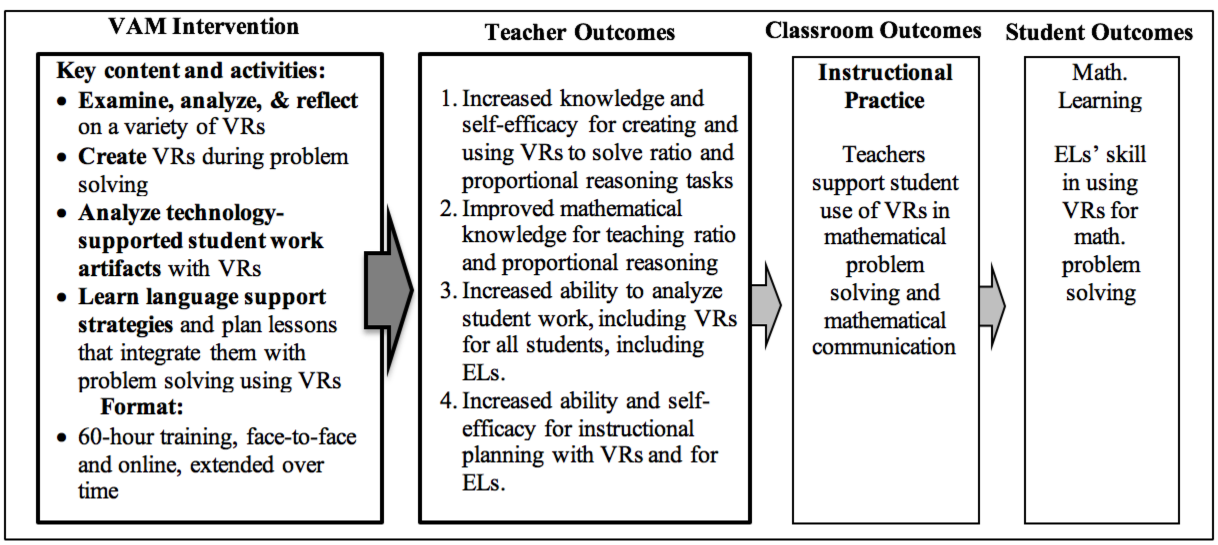

Figure 1: VAM Theory of Change

Methods and Modes of Inquiry

After two years of design and development, a cluster randomized trial involving 101 middle grades teachers across 47 schools in New England was used to examine whether teachers who participate in the VAM PD demonstrate stronger learning outcomes – including higher measures of teacher self-efficacy – than teachers in a control group. This paper addresses the research question: *Compared to control teachers, do VAM PD treatment teachers demonstrate greater* *self-efficacy for planning and implementing instructional activities that integrate support for ELs' language and use of VRs?* A self-efficacy instrument was adapted from items focused on teaching ELs and teaching mathematics (Ross, 2014; Siwatu, 2007; Wright-Malley & Green, 2015). The final instrument included 19 items with a 7-point Likert-type rating scale (Strongly Disagree to Strongly Agree) addressing three areas: using VRs in mathematics (5 items), including, "I am confident I can translate quantitative information into a visual representation to solve a ratio and proportional reasoning task;" teaching with VRs in mathematics (5 items), including "I am confident I can help students learn to create visual representations, such as double number lines and tape diagrams, to solve ratio and proportional reasoning tasks;" and teaching students who are ELs in mathematics (9 items), including, "I can describe strategies that help ELs get started on mathematics tasks in my class."

Treatment and control participants completed the self-efficacy instrument pre and post VAM PD. Based on exploratory factory analysis of the 19 items in our self-efficacy scale, we identified two strong subscales: self-efficacy related to using and teaching with VRs in mathematics, or "self-efficacy-VRs" (10 items, α=0.96; possible score range 7-70), and self-efficacy related to teaching ELs in mathematics, or "self-efficacy-ELs" (9 items, α=0.98; possible score range 7-63). Participants also completed other measures of teacher knowledge that are not reported here.

The effects of VAM PD on teachers' scores on the two self-efficacy subscales were estimated with two-level hierarchical linear models using HLM 7.03 software (Raudenbush, Bryk, & Congdon, 2017). Teachers were clustered within schools (between 1 and 7 teachers per school) with random assignment at the school level. In the final analytic sample after attrition (2% at the school level, 5% at the individual level), pre- and post-test self-efficacy data was collected from 98 teachers (24 treatment schools with 52 teachers; 23 control schools with 46 teachers). Baseline equivalence between our treatment and control groups was established by estimating a two-level model for each subscale with pretest score as the outcome and treatment status entered at Level 2. For self-efficacy-VRs, mean treatment and control group pretest scores differed by an effect size of Hedges $g = |0.10|$; for self-efficacy-ELs, treatment and control group scores differed by $|0.19|$. These statistics suggest baseline equivalence when acceptable statistical adjustments are included in analyses (WWC, 2017.). This study's research question was answered by fitting a sequence of models to predict posttest scores for each self-efficacy subscale; first fitting an unconditional model to estimate the variability at each level, and then a second model that included treatment status at Level 2 and pretest scores (a statistical adjustment, grand-mean centered) at Level 1.

Results

Analysis showed that participation in VAM PD had a positive effect on how teachers scored on the two self-efficacy subscales (Table 1). The mean posttest self-efficacy-VR score was 55.85 points for the control group and 62.34 for the treatment group, or an effect size of g=0.79, controlling for pretest scores. The mean posttest self-efficacy-EL score was 44.74 points for the control group and 54.22 points for the treatment group, or an effect size of g=1.14. Both of these effects are statistically significant at $p<.001$ and may be considered large. The ICC or proportion of variance in teachers' posttest scores that is explained by schools is 0.11 for self-efficacy-VRs and 0.22 for self-efficacy-ELs. The proportion of variance in self-efficacy-VRs posttest scores that is explained by a model that includes both treatment group and pretest scores compared to a model that includes only pretest scores (not shown) is 0.05 at Level 1 and 0.88 at Level 2. The proportion of variance in self-efficacy-ELs posttest scores that is explained by the same model comparison (again not shown) is 0.04 at Level 1 and 0.88 at Level 2.

Table 1: HLM Results for Models of Teacher Self-efficacy in Using VRs and Teaching ELs

	Self-efficacy-VRs		Self-efficacy-ELs	
	Model 1	Model 2	Model 1	Model 2
Fixed effects: Coefficients (SE)				
Model for posttest intercept (β_{0j})				
Intercept (γ_{00})	59.31*** (0.96)	55.85***(1.13)	49.62*** (1.15)	44.74***(1.01)
Treatment Intercept (γ_{01})		6.49***(1.55)		9.48***(1.40)
Model for pretest slope (β_{1j})				
Intercept (γ_{10})		0.28***(0.07)		0.58***(0.07)
Random effects: Variance (χ^2, df)				
Teachers: Var(r_{ij})	66.64	55.59	74.22	39.43
Schools: Var (u_{0j})	8.45 (54.08, 45)	0.95 (36.96, 44)	21.09** (71.55, 45)	2.87 (47.02, 44)
Model fit				
Deviance	699.25	670.40	719.80	641.62
Parameters	3	5	3	5
Effect size: Hedges *g*		0.79		1.14

Discussion

Findings from this study suggest that VAM PD holds promise for improving the self-efficacy of teachers of ELs in mathematics more broadly. A next step is to explore how specific components of VAM PD may have worked to promote greater teacher self-efficacy, and how these components may be scaled. Teacher responses to VAM PD exit surveys indicate course components they found most helpful for their own learning: doing mathematics tasks and analyzing solutions with other teachers; focused work with tape diagrams and double number lines; and interactions with course instructors and teacher peers at face-to-face workshops. In addition, the project team is examining reflections participants wrote in online notebooks to identify possible paths within the PD experience to increased self-efficacy. For example, one teacher wrote: “Doing the tasks myself increase[d] my repertoire of visuals and the value I place on them for both ELs and increasing students’ conceptual understanding.” This comment suggests that under certain conditions, teachers may come to value the use of specific mathematical tools in instruction more highly as they develop a deeper understanding of these tools. Similarly, another participant noted that being “challenged to apply the various [language support] strategies in different ways in our classroom” supported her ability to implement the strategies routinely, even when not part of a course assignment. The positive impact of VAM PD on teachers’ self-efficacy encourages further study of how the components of the PD may have contributed to teacher change as well as the potential for effective scaling of the PD.

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AN EXPLORATORY CASE STUDY OF POSITIONING AND WHITENESS IN A SECONDARY MATHEMATICS TEACHER PROFESSIONAL DEVELOPMENT

Crystal Kalinec-Craig
University of Texas at San Antonio
Crystal.Kalinec-Craig@utsa.edu

Priya V. Prasad
University of Texas at San Antonio
Priya.Prasad@utsa.edu

In this paper, the authors present preliminary findings on an exploratory study about the experiences of three in-service mathematics teachers who were participating in a mathematics professional development (PD). Because the authors believe that mathematics classrooms (and PDs) are racialized spaces that can leave room for teachers to position each other and themselves, the authors use Critical Race Theory and positioning theory as the theoretical frameworks to analyze an interaction among three teachers. The authors briefly describe the context of the PD, the backgrounds of the teachers, and the episode to be analyzed. Preliminary findings show how Wanda, a white woman, and Deirdre, a Black woman, position and reposition themselves and each other as more-experienced teacher and less-experienced student. Implications for future research will be discussed.

Keywords: Teacher Education-Inservice/Professional Development; Equity and diversity; Algebra and Algebraic Thinking

Rationale and Background

Recent investigations into the professional development (PD) of mathematics teachers focuses on ways to improve the effectiveness of teachers' experiences. PD that introduces teachers to new pedagogical strategies must center on the mathematics content because there is evidence that content-oriented PD is impactful for teachers; (Marra et al., 2010). Yet there can be issues during PD that may arise among teachers and facilitators, which are similar to what teachers and students experience within the classroom (Adler, Ball, Krainer, Lin, & Novotna, 2005). For example, some facets of PD reflect the dynamics of a mathematics classroom, such as an instructional triangle consisting of interactions among the nodes representing the PD instructors, the participating teachers, and the mathematics content (Borko, 2004). Teachers in PD also consider their own students as they attend to their own learning, which creates an instructional rhombus with a fourth node representing the teachers' real or hypothetical students (Nipper et al., 2011). Moreover, the dynamics of power and status relationships (a perceived social ranking among members) can play a role among teachers in a PD just as they do in the classroom with their students (Cohen & Lotan, 2014). In fact, in PD, these relationships can become even more fraught, because experienced teachers can be positioned (by themselves, by each other, and by those orchestrating the PD) as less-experienced learners. The power behind each of these identities (e.g., as being the more- or less-experienced learner) can shift quickly during a PD and can affect the mathematical learning of each participant in profound ways. Therefore, it is imperative to investigate the ways mathematical identities, power, and status operate for teachers within in a PD so that more teachers learn to notice and address issues of status in their own practice and with their own students.

This paper described in this proposal presupposes that PD workshops, like K-12 mathematics classrooms and the rest of the world, are racialized spaces (Martin, 2009) where participants' experiences are based on racial, gendered, and experiential identities (whether self-imposed and/or imposed by their fellow teachers and the facilitators). The authors use the theoretical frameworks of Critical Race Theory (Delgado & Stefancic, 2017) and positioning theory (van Langenhove, & Harré, 1990) with the analytical framework of whiteness to analyze how three teachers negotiate their shifting identities while solving a task during a PD. Preliminary findings from this study continues the research about teacher identities in an environment (PD) that needs more attention in the field.

Theoretical Framework

As first described by van Langenhove & Harré (1990), positioning theory "refer[s] to cognitive processes that are instrumental in supporting the actions people undertake particularly by fixing for this moment and this situation what these actions mean" (Harré et al., 2009, p. 6). Wood (2013) describes in her seminal paper an example of students positioning themselves and each other: Rebecca (a white girl), Daren and Jakeel (two Black boys), and how Rebecca positions Jakeel as knowing less mathematics while Jakeel repositions himself (and attempts to adopt a positive mathematical identity) as someone who knows mathematics. When people use language to position each other, there are an associated meaning, storyline, and identities with that language and the interaction(s) (Harré et al., 2009; van Langenhove & Harré, 1990). By understanding the meaning behind the language used and how the language can serve a specific purpose for positioning oneself and another, researchers can understand the ways in which language and meaning inform identity formation within mathematics classrooms (Wood, 2013).

When teachers collaborate in PD, they are forming identities while also negotiating multiple hierarchies of positions. Teachers can position each other by using status-laden language regarding their mathematical knowledge, expertise, and potential to contribute to the PD experience. For example, more experienced teachers might position novice teachers as knowing less mathematics and might take control of a task or react to the novice's contributions with condescension. But there is another layer of positioning that happens in PD: teachers who are participating in PD are positioned by the teacher educators (or facilitators) as knowledgeable learners (yet still learners in their own right). In each situation, the teachers in PD are also negotiating their own identities of themselves and of each other as they position (and seek to position) while making sense of the PD interactions.

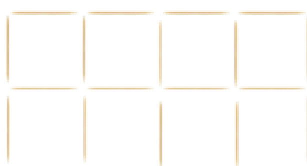
Critical Race Theory (CRT) serves as our second foundational theoretical framework. As summarized by Delgado and Stefancic (2017) and described in more detail by others (e.g., Dixson, Rousseau & Anderson, 2006; Langer-Osuna, 2011; Martin, 2009; Tate, 1997; Taylor, Gillborn & Ladson-Billings, 2015), educational researchers use CRT to examine the ways that race and racism play a role in oppressing and suppressing the voices and success of Black, Brown, and Indigenous people, among others. Martin (2009) argues that mathematics classrooms are racialized spaces in which messages are communicated to students, about students, and for the purpose of positioning students. Martin outlines that:

> Issues of identity—racial, cultural, gender, mathematical—and agency become centrally important when seeking to better understand how students make sense of, and respond to, their mathematical experiences (p. 324).

Similar to Martin, Shah (2017) also argues that "societal narratives about topics like race matter because of how they can become consequential in positioning" (p. 7). Researchers can use CRT to examine the interactions of teachers in PD as they negotiate their identities, especially as related to the notion that PD spaces are also racialized and there are societal narratives in formation. Therefore, given the presumption that learning as a social activity, researchers should examine how race and racism play a role when teachers form their identities within the context of a PD. In this exploratory case study (Merriam, 1994), we explore a single incident from a PD and ask the following research questions: How do teachers in a PD exercise moves of positioning in the context of solving a groupworthy (Lotan, 2003) mathematics task? And how do the teachers' racial identities play out (and become complicated) while solving a task?

Methodology

The authors used critical discourse analysis (Fairclough, 2013) to analyze one episode from a PD workshop facilitated by Prasad. Middle- and high-school mathematics teachers participated in a three-week-long summer institute in which Prasad modeled teaching through problem-solving and using Complex Instruction (Cohen & Lotan, 2014) strategies like assigning group roles. Because all participating teachers planned to teach algebra in the following school year, Prasad presented them with the problem shown in Figure 1 (Driscoll, 1999).



The picture shows a rectangle made up of two rows of four columns and of squares outlined by toothpicks. How many toothpicks would be needed to make a rectangle with R rows and C columns?

Figure 1: Toothpick Task

This episode featured three teachers named Wanda, Deirdre, and Frances (all pseudonyms). All three teachers were experienced, mid-career, self-identified female high-school mathematics teachers of similar ages who taught in the same high-needs, high-density district. Deirdre and Frances were Black, while Wanda was white. The facilitator, Prasad, is Indian-American and is younger than the teachers in the group. Kalinec-Craig is white and the same age as Prasad, and she had worked with the three teachers and Prasad in previous sessions, but was not present for this particular session. Prasad introduced the idea of using group roles to structure equitable participation (Featherstone et al., 2011), but many of the groups in the workshop took up this idea in ways that best suited their own goals. For example, Wanda used Frances' designation as the group's "Presenter" to externalize her struggle to understand Deirdre's solution. Because the interactions and work were recorded using a LiveScribe pen, Deirdre took up her role as the assigned "Recorder." In the preliminary analysis, the authors looked across the transcript of the entire interaction to understand how each participant positioned herself and was positioned by her peers. Preliminary findings focus on the experiences of Wanda.

Preliminary Findings

After the teachers worked individually on the problem, Prasad directed the teachers to discuss their solutions. The following interaction occurred when Deirdre, Frances, and Wanda began talking amongst themselves. Deirdre had developed the expression of "$2CR+C+R$" to express the number of toothpicks in a given rectangle with C columns and R rows. Deirdre attempted to explain her solution (to strategically count the horizontal and vertical toothpicks separately, but in a related way) to Wanda and Frances. For example, each row contained $C+1$

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* St Louis, MO: University of Missouri.

vertical toothpicks, making the total number of vertical toothpicks in the rectangle is $R(C+1)$; similarly, there are $C(R+1)$ horizontal toothpicks in the rectangle.

At one point in the episode, Deirdre attempted to position herself as having a correct solution. However, Wanda, who had not yet developed a viable expression at this point, attempted to position Deirdre as having an incorrect or incomplete solution by stating, "so that's why what [Deirdre]'s talking about might work per se, but I don't understand it the way she's trying to explain it to me." As Wanda tried to replicate Deirdre's solution, she stated, "…so now if I go (oops), I shouldn't have done that, I should have gone over to this one…" to which Deirdre defended the validity of her solution by replying, "…that's why I said one of them [C*R] has to be two." Still unsure of Deirdre's solution, Wanda stated, "...but this isn't going to work for *all of* [the toothpicks] if you're going to multiply by two" (italics added for emphasis). Deirdre persisted and repositioned her solution and herself when she said, "... really? Because what I said was, 'so then my formula became however many rows I have.'" Wanda's insistence that Deirdre (re)justify her solution using Wanda's mathematical constraints and assumptions helped Wanda position herself as the "teacher" and to position Deirdre as a "student" who should mimic the teacher's justification for the solution.

At multiple points in the interaction, Deirdre resisted Wanda's moves to position her as being a learner who is less knowledgeable. Deirdre also positioned herself as a conciliator when Wanda assessed her expression by testing it against a two-column table Wanda generated. Deirdre asked Wanda, "So, do you want to try some more to see if it fits all of them, or do you think it just fits that one? Or are you good with that?" Wanda, in response, positioned herself as being the one who would confirm the group's progress by saying, "Well, I see what you're doing and I think it will work," yet immediately follows by saying, "We just have to get the labeling down so when Frances ... goes up to explain, she can explain [the solution to the class]..." In this moment, Wanda accepted Deirdre's attempt to position herself as a "teacher," and then moved to position Frances (and Frances' assigned role of presenter) as the student who needed to articulate the solution. Wanda implicitly acknowledged Deirdre's solution as valid without confirming the solution herself nor acknowledging that Deirdre displayed a mathematical solution that Wanda was not yet ready to explain in her own words.

Wanda's initial positioning of Deirdre and eventual positioning of Frances seemed to highlight how she exercised the notion of whiteness (Harris, 1993) to maintain her relative high, more-knowledgeable status with respect to the two Black teachers in her group. When Deirdre solved the problem quickly and successfully, Wanda resisted being positioned as a "student" and insisted that Deirdre's explanation conform to her own thinking about the problem. As the group was to present their solution to the whole class, Wanda (serving as teacher) positioned Frances as a student who needed to be knowledgeable about the solution prior to presenting. Wanda's resistance to being positioned as a student was important to note, given that in the context of PD, the teachers were all, explicitly and intentionally, positioned by Prasad as "learners." Prasad's framing of teachers-as-learners was purposeful: to help teachers experience small-group learning through problem-solving, before they unpacked their experience as "teachers", as they had done in the previous problem-solving experience in this PD.

Next Steps

As evident in the brief excerpt of the interaction, Wanda seemingly repositions Deirdre and Frances as students whereas she avoids a learner position in the context of this particular task. Wanda's stance may have subverted the opportunity for her to learn another approach (Deirdre's approach) to the problem because of this insistence. The initial analysis of this interaction offers some insight into how teachers can position each other and themselves in the course of solving a groupworthy task. Moreover, the group's experiences suggest that the notion of racialized identities and whiteness were at play during the task. The next phase of the analysis will use a finer-grain critical discourse analytic approach (Fairclough, 2013) to examine the *micro*positioning moves that highlights each teachers' racialized identities (Wood, 2013).

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AN INVESTIGATION OF TEACHERS' SENSEMAKING AROUND FACILITATING MATHEMATICAL ARGUMENTATION

Hala Ghousseini
University of Wisconsin
Ghousseini@wisc.edu

Elham Kazemi
University of Washington
ekazemi@uw.edu

Sam Prough
University of Wisconsin
sprough@wisc.edu

Alison Fox Resnick
University of Washington
foxla@uw.edu

We report on a study that aims to understand how elementary teachers' thinking and sensemaking may support or constrain their learning to promote disciplinary argumentation. The study context is a professional development model organized around a sequence of learning cycles in which teachers collectively investigate and enact teaching. We share results related to teachers' engagement in the model and their sensemaking around the work of facilitating classroom mathematical argumentation.

Keywords: Classroom discourse, Instructional practices, Practice-based professional development

Whole class discussions that focus on mathematical argumentation are central events of inquiry-oriented environments (Staples & Newton, 2016), and important for promoting conceptual understanding and developing mathematically proficient students (Osborne et al., 2019; Rumsey, 2012). Mathematical argumentation engages students in a collaborative process where they make claims and justify them using reasoning that is based in disciplinary practices and in their existing knowledge and cultural and linguistic resources. Argumentation-based discussions, however, are uncommon in U.S. classrooms (Cazden, 2003). Research has revealed teachers' roles and responsibilities in these socially and intellectually demanding environments, and highlighted teaching practices that support mathematical argumentation (Lampert, 2001; Staples, 2007). However, this research also suggests that implementing these complex practices is not trivial. Teachers must judge how to elicit and respond to student thinking and how to facilitate students' engagement with each other's ideas around disciplinary content. To accomplish these goals, teachers must draw on specialized professional knowledge, which entails taking individual student thinking seriously and learning about the cultural resources and assets that students bring (Bartell et al., 2017) as well as the nature of disciplinary learning in schools for creating a more just society (Gutiérrez, 2008). Understanding how teachers learn to make these judgments and how they reason about when and how to take up these goals and practices is the aim of this study.

Research on teacher learning about facilitating disciplinary argumentation has to date focused for the most part on how teachers socialize students into this process (Forman et al., 1998), teacher noticing of situations for argumentation in the classroom (Ayalon & Hershkowitz, 2018), and design aspects of professional development that support learning about and facilitating argumentation (Osborne et al., 2019). Less is known about teacher learning and sensemaking about disciplinary argumentation in the course of their professional development. In this brief report, we examine teacher learning about facilitating mathematical argumentation within a practice-based professional development model that engages teachers in investigating and enacting practice. We highlight two preliminary findings related to teacher learning: the relationship between descriptive and explanatory talk and supporting student participation.

Theoretical Framework

We frame teacher learning from a sociocultural perspective as a process of changing participation in a community of practice (Rogoff, 1995). This process entails the development of principled goals, commitments, and ways of thinking valued by this community, as well as the continual refinement of a set of principled practices for supporting mathematical argumentation in the classroom. Consonant with this perspective on teacher learning, we use a practice-based design of professional development (PBPD), which centers teacher learning on the activities that characterize their daily work (Ball & Cohen, 1999). In this context, PBPD facilitators support teacher learning using interactive structures that promote teachers' sensemaking with each other and make teacher thinking visible through methods such as video analysis (van Es & Sherin, 2010), problem solving cycles (Koellner et al., 2007), rehearsals (Kazemi, Ghousseini, Cunard, & Turrou, 2016), and teacher time out (Gibbons, Kazemi, Hintz, & Hartmann, 2017). Three key commitments, central to facilitating productive classroom discourse, orient our work with teachers: attending to student thinking (striving to hear students, attending to who they are and the knowledge they bring (Bartell et al., 2017); attending to the integrity of the content (promoting dispositions and practices central to learning disciplinary content); and attending to a classroom community where students learn to articulate their thinking and to listen to and represent others' ideas, even those with whom they disagree (Ball & Bass, 2003).

Context and Professional Development Model

The context of our investigation is a professional development (PD) initiative across two locations in the U.S. We recruited from partner schools three instructional leaders and 13 teachers with varying experience in leading classroom mathematical argumentation. Both schools serve richly diverse student populations with respect to language, race, and ethnicity. We are co-designing and co-facilitating the PD with the instructional coaches from each site. The PD takes place at the school site during the school day, and consists of 12 Learning Labs implemented over two years (6 cycles/year, with a projected cycle every other month). Each Learning Lab begins with new learning during which teachers may engage a reading or analyze artifacts of practice to identify new ideas with respect to students' mathematical thinking as well as elements of teaching that support productive argumentation. In the next phase, teachers collaboratively prepare a lesson to enact in a classroom. Next, teachers rehearse small portions of their lessons in front of other study participants, exhibiting and sharing their understanding and decision making about how to promote classroom communication focused on argumentation and seeking feedback from colleagues. The teachers then collectively enact the lesson in one of the participant's classroom, collect records of their enactment, and debrief what they learned in the last phase of the Learning Lab. The day comes to a close with teachers setting goals for next steps in their ongoing work with argumentation in their own classrooms. New ideas and insights from these trials are then brought back to the subsequent Learning Lab.

Data Collection and Analysis

Currently in our first year of this project, we are collecting data in each cycle in the form of video records of teachers' participation, fieldnotes from each phase of the cycles, copies of teachers' reflections and artifacts of practice (including plans and videos of their teaching), and one-on-one interviews with both teachers and instructional coaches/mentors. For the purpose of this report, we thematically analyzed (Boyatzis,1998) data from fieldnotes and teachers' written reflections to capture (a) what teachers attend to when they debrief the collectively planned and enacted lessons, and (b) the problems of practice that arise for teachers and what knowledge they use to address them. Looking across these dimensions over time allowed us to identify teacher insights that were suggestive of emerging aspects of their sensemaking about facilitating mathematical argumentation.

Results

We focus our results on two themes that emerged from our analysis: student participation and the relationship between descriptive and explanatory talk in argumentation-focused discussions.

Student Participation

Attending to student participation was a key problem of practice for teachers in facilitating argumentation-based discussions, and addressing it leveraged two major understandings about this work for teachers: the way language can function as an enabler and barrier for participation in discussions, and the way classroom culture can invite or constrain participation. Teachers noticed the centrality of language in classroom discussions. In one Lab debrief, for instance, several teachers noted that those not speaking up were the students whose first language is not English. They related this lack of participation to the task structure which appeared to promote quick responses and privilege students who have the mathematical language to verbally communicate their ideas over others. As one teacher elaborated, "you have to take information, process it, translate it . . . I see kids open their mouth to say something and then someone else jumps in." These conversations allowed teachers to consider pedagogical moves that encourage broader participation. Teachers indicated a need for different scaffolds that make language in discussions more accessible. They suggested providing students with sentence stems to support their engagement with the content and using a student-generated word bank based on small group discussions to better support sharing within a large group.

In another respect, teachers stressed how the classroom culture needed to be inviting for participation to occur. In one Lab debrief, the teachers noticed students' hesitation to share until one student contributed a starting point for an idea. This led to a examination of the sort of norms that encourage confidence. One teacher shared that she tries to solicit responses that are not always right, " I like to acknowledge [their] thinking to show the other kids this is what we do, and that usually lowers the risk for other kids to take a guess." These insights led to the identification of several teacher moves that help make the environment more friendly for sharing, including an initial question of "who can get us started?" and stressing that the discussion is aimed at thinking about ideas and not having the right answer.

Relationship Between Descriptive and Explanatory Talk

Teachers engaged in unpacking two aspects of the relationship between descriptive and explanatory talk in mathematical argumentation and considered how understanding this relationship might impact task design and instructional moves. In one respect, teachers recognized that descriptive and explanatory talk are not the same and require different work on the part of students. In one Lab, for instance, teachers considered how students could use talk to describe calculations that prove why a mathematical equation (4x6=2x12) is true. In their discussions, they realized that explaining why the equation was true without calculating produced a more rigorous demand for creating an argument. As a teacher expressed it, "you take [the mathematical work] to the next level by not calculating." This insight led teachers to begin to experiment with how to redesign mathematics discussion tasks to support more explanatory talk. For instance, in relation to the equation task described above, teachers tried removing the need for calculation by stating up front for students that the equation was true and asking students to create models and representations to explain relationships instead of calculating results.

In another respect, teachers realized that descriptive talk sets an important foundation for explanatory talk, especially in generating conjectures or claims. Through experimenting with student discussions, teachers observed that giving students time to notice and describe patterns in a mathematical task supported them in noticing regularity in their descriptions of mathematical reasoning. Students then used their initial descriptive talk to generate and justify claims. This led teachers to experiment with how to facilitate a discussion structure around this realization. In one case, the teachers first had students describe how they would explain that each fraction comparison in a series of statements are true (e.g., $\frac{1}{8} < \frac{3}{8}$; $\frac{3}{5} > \frac{2}{5}$). As students described their thinking, teachers created a public representation of their strategies. The teachers then asked students to look across their thinking for multiple problems and begin to generate a conjecture about how to compare fractions. One teacher explained that the goal was to support students to see "the patterns in the way [they] are thinking about it. Making that obvious to them...that is their conjecture." Hence, teachers' deeper understanding of the relationship between descriptive and explanatory talk in mathematical argumentation influenced their design of instructional tasks and moves.

Conclusion

The preliminary findings we share suggest aspects of the sensemaking that teachers experienced in the context of the PBPD focused on enactment and investigation. These insights related to broadening student participation and the coordination of descriptive and explanatory talk are grounded in the commitments that frame our view of teacher learning: attending to the integrity of the content and to the promotion of a classroom community where students learn to articulate their thinking. The findings suggest the way teachers reason pedagogically about such commitments. Additionally, the emergence of these insights was enabled by the structure of the PBPD model which afforded teachers time to experiment collectively with leading argumentation-based discussions and to reflect and notice aspects they were likely to miss in their own classrooms.

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INFLUENCE OF CERTIFICATION PATHWAY ON MATHEMATICS TEACHER BELIEFS

Jennifer Cribbs
Oklahoma State University
Jennifer.cribbs@okstate.edu

Juliana Utley
Oklahoma State University
Juliana.utley@okstate.edu

John Weaver
Oklahoma State University
john.waver@okstate.edu

Teacher shortages have created an influx of alternatively certified teachers, which raises questions about the effect these teachers are having on classrooms. This study explored potential differences in mathematics teachers' beliefs based on their path to certification. An ANOVA revealed differences in beliefs among the four certification pathways explored, such as teachers certified through a traditional mathematics education program self-reported significantly higher levels of mathematics identity and mathematics teacher identity than teachers certified through a traditional education program that was not mathematics specific. These findings have implications for educational stakeholders and policy makers when considering the certification pathway options available for potential teachers.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Teacher Beliefs

Over the past seven years, there has been a decrease in the percentage of teachers who are entering the teaching profession through a traditional teacher preparation program. During the 2017-2018 school year, less than half of new hires in the state of Oklahoma were holding a standard teaching certificate. Even more striking, between 2012 and 2018 the number of emergency certifications increased from 32 to 2,915 (Lazarte Alcala, 2018). This increase is partly in response to the shortage of teachers in the state across all content and grade levels. With an influx of teachers who are emergency and alternatively certified in Oklahoma, it is important to study the impact these teachers are having on students and the education system in general, particularly with evidence that teachers who are emergency or alternatively certified have lower levels of retention, both in Oklahoma (Lazare Alcala, 2018) and in other states (Zhang & Zeller, 2016). One way to begin to address this concern is to explore potential differences between teachers based on their certification pathway. Other researchers have begun to explore these differences, noting that alternatively certified teachers tend to feel less prepared than teachers certified through a traditional education program (Kee, 2012).

Research indicates that teacher beliefs play an important role on teachers' practice (Stipek, Givvin, Salmon & MacGyvers, 2001; Wilkins, 2008). In particular, a number of specific mathematics related beliefs have been connected to teachers' instructional practices and behaviors including self-efficacy and anxiety (Ghaith & Yaghi, 1997; Trice & Ogden, 1986), mindset (Good, Rattan, & Dweck, 2007), and identity (Jong, 2016). Therefore, the purpose of this study is to explore differences in mathematics teachers' beliefs based on their pathway to certification.

Methods

In late spring of 2018, a survey was sent to all potential mathematics teachers in Oklahoma based on contact information provided by the State Department of Education. 17,444 emails were sent to elementary, middle grades, and secondary teachers; 1,385 of these were bounced emails. Additionally, elementary teachers, some who did not teach math, were included in this initial email, as we had no way of determining which teachers did or did not teach math. After eliminating 155 teachers who elected the option of not being a math teacher and removing 233 surveys that were missing a significant number of responses, we were left with 1,543 completed surveys. As this study focuses on comparing certification pathways and this option is primarily open to secondary teachers, elementary teachers were removed from the sample leaving 217 secondary teachers (6th-12th) that responded to the survey.

The respondents were 85% female and 15% male. In terms of race and ethnicity, respondents were 85.7% White, 6.5% Asian, 3.7% African-American, 3.7% Multicultural, and .5% Other, with only 2.3% of the participants identifying as Hispanic. The age of participants ranged from early 20s to 70 years, with them holding degrees ranging from a Bachelors to a Doctorate. The years of experience reported by participants ranged from 1 year to 49 years with an average of 13.4 years.

The survey included items related to participant demographics (e.g., gender, paths to certification), and mathematics specific beliefs (e.g., mathematics identity, mathematics mindset). In order to determine teachers certification pathway, participants were asked to select one of the following options: (1) traditional undergraduate teacher education program with certification in secondary mathematics, (2) traditional undergraduate teacher education program with certification in a field other than secondary math, (3) master's degree with certification in secondary mathematics, (4) bachelor's degree in mathematics and pursued alternative certification, (5) bachelor's degree in a field other than mathematics and pursued alternative certification, (6) traditional undergraduate teacher education program with certification in elementary, (7) bachelor's degree in field other than education and pursued alternative certification, (8) Troops to Teachers, (9) Teach for America, and (10) other. We removed participants selecting option 8 and 9 as the sample size was small. For the purpose of this study, we grouped similar pathways in the following way:

A. Pathway 1 and 3 - traditional mathematics teacher education pathway (*n* = 49);
B. Pathway 2 and 6 - traditional non-mathematics teacher education pathway (*n* = 58);
C. Pathway 4 - alternative certification with bachelor's degree in math (*n* = 27); and
D. Pathway 5 and 7 - alternative certification with bachelor's degree in field other than math (*n* = 83).

The survey also included measures for mathematics identity (MID, 11 items, 5-point Likert scale) using items developed and tested with adults in prior research (Cribbs, Hazari, Sadler, & Sonnert, 2015). This scale is comprised of three sub-constructs including interest, recognition, and competence/performance. The higher the mean the more a person identifies with being a "math person." Math mindset items (MS, 4 items, 5-point Likert scale) drew from prior research (De Castella & Byrne, 2015; Degol, Wang, Zhang, & Allerton, 2018). Mean scores range from 1 (fixed mindset) to 5 (growth mindset). Mathematics teacher identity (MTID, 13 items, 5-point Likert scale) drew from a study with secondary preservice mathematics and science teachers (Thamotharan & Hazari, 2017) and includes the same three sub-constructs as mathematics identity but focused on mathematics teaching. The higher the mean the more a person identifies as being a mathematics teacher. The 5-point Likert scaled self-efficacy for teaching mathematics instrument (SETMI) was used to measure efficacy for pedagogy in mathematics (EPM, 7 items) and efficacy for teaching mathematics content (ETMC, 15 items) (McGee & Wang, 2014). The higher the mean score the more efficacious the teacher was towards teaching mathematics. The 5-point Likert scaled mathematics anxiety rating scale revised (MARS-R) with two sub-constructs – learning mathematics anxiety (LMA, 8 items) and mathematics evaluation anxiety (MEA, 4 items) - was used in this study (Hopko, 2003). A higher mean score indicates a higher level of anxiety. The 5-point Likert scaled constructivist teaching scale (TP1, 6 items) was used in this study (Woolly & Benjamin, 2004). A higher mean score indicates beliefs that align more with a constructivist teaching philosophy. Finally, a nine-item researcher developed instrument was used to measure the teachers' mathematics teaching philosophy (TP2). This instrument used a semantic differential scale to measure teachers' beliefs about effective mathematics practices as described in the *Principals to Actions* text published by the National Council of Teachers of Mathematics (NCTM, 2014). Mean scores can range from 1 (teacher centered) to 5 (student centered). A higher mean score indicates a more student-centered rather than teacher-centered teaching philosophy. Internal consistency values using Cronbach's alpha ranged from .71 to .95 across all scales. One-way analysis of variance (ANOVA) was performed to determine if there were differences in mathematics and teaching beliefs between the four (A, B, C, and D) certification pathways.

Results

Results for the one-way ANOVA exploring differences in mathematics and teaching beliefs based on certification pathways for mathematics teachers are shown in Table 1.

Table 1: ANOVA Results Comparing Across Four Different Certification Pathways

Measure	A	B	C	D	*F*	*p*
	Mean(SD)	Mean(SD)	Mean(SD)	Mean(SD)		
MID	4.42(.61)	4.05(.74)	4.12(.93)	4.18(.67)	2.473	.063†
MS	4.26(.62)	4.32(.68)	4.39(.56)	4.39(.59)	.553	.646
MTID	4.69(.36)	4.37(.71)	4.23(.60)	4.42(.65)	4.252	.006**
EPM	3.88(.57)	3.76(.70)	3.58(.72)	3.94(.60)	2.548	.057†
ETMC	4.38(.54)	4.24(.67)	4.09(.73)	4.26(.62)	1.280	.282
LMA	1.41(.51)	1.47(.47)	1.58(.72)	1.57(.56)	1.210	.307
MEA	1.72(.83)	1.84(.76)	1.99(.73)	2.07(.82)	2.251	.084†
TP1	3.93(.62)	3.84(.60)	3.88(.59)	3.88(.65)	.199	.897
TP2	3.74(.69)	3.78(.62)	3.88(.69)	3.85(.59)	.507	.678

Note. MID = math identity; MS – mindset; MTID – mathematics teacher identity; EPM - efficacy for pedagogy in mathematics; ETMC - efficacy for teaching mathematics content; LMA - learning mathematics anxiety; MEA - mathematics evaluation anxiety; TP1 – teaching philosophy measure 1; TP2 - teaching philosophy measure 2
†$p < 0.10$. **$p < 0.01$.

Results indicate a marginally significant difference between certification pathway groups, $F(3, 213) = 2.473$, $p = .063$ for Mathematics Identity. A Tukey HSD post hoc test found significant differences ($p = .042$) between groups A and B. A significant difference~~s~~ was found between certification pathway groups, $F(3, 213) = 4.252$, $p = .006$ for Mathematics Teacher Identity. A Tukey HSD post hoc test found significant differences between group A with groups B ($p = .033$) and C ($p = .009$), and a marginally significant difference between group A and D ($p = .062$). Results also indicate a marginally significant difference between certification pathway groups, $F(3, 213) = 2.548$, $p = 0.057$ for Efficacy for Pedagogy in Mathematics. A Tukey HSD post hoc test found significant differences between group C and D ($p = .050$). Finally, results indicate a marginally significant difference between certification pathway groups, $F(3, 213) =$

2.251, $p = 0.084$ for Mathematics Evaluation Anxiety. A Tukey HSD post hoc test found marginally significant differences between group A and D ($p = .063$).

Discussion

Our findings indicate that there are some differences in beliefs between the four different certification pathways explored in this study. Specifically, teachers who were certified through a traditional mathematics education program self-reported significantly higher levels of mathematics identity and mathematics teacher identity than teachers who were certified through a traditional education program that was not mathematics specific. Teachers certified through a traditional mathematics education program reported higher levels of mathematics teacher identity and efficacy for pedagogy in mathematics than teachers who had a mathematics bachelor’s degree and were alternatively certified. Teachers certified through a traditional mathematics education program also reported higher levels of mathematics teacher identity and lower levels of mathematics evaluation anxiety than teachers who had a bachelor’s degree in a field other than mathematics and were alternatively certified. Lastly, teachers who had a bachelor’s degree in a field other than mathematics and were alternatively certified reported higher levels of efficacy for pedagogy in mathematics than teachers who had a mathematics bachelor’s degree and were alternatively certified. It is also worth noting that teachers who were certified through a traditional mathematics education program reported higher means, even if not statistically significant, for all but one factor measured than all other certification pathways. These findings contradict some prior research noting no differences in beliefs between teachers who are alternatively and traditionally certified. For example, Fox and Peters (2013) found no significant differences between teachers’ self-efficacy based on certification pathway. It is also important to note that alternatively certified teachers may not be able to accurately assess their own instruction as compared with traditionally certified teachers (Naugaret, Scruggs, & Mastropieri, 2005). With this in mind, participants self-reporting on their teaching philosophy, where they reflect on their own instructional methods, may be skewed for alternatively certified teachers as they have not observed, reflected, and practiced implementing instruction that is student-centered as is often the focus of traditional education programs. This may account for the lack of significance for the teaching philosophy factors (TP1 and TP2) not having a higher mean for the traditionally certified teachers than the other certification pathways. Given the importance that teachers’ beliefs have on their classroom practice, these results speak to the added benefit a traditional, content-specific education program has in preparing teachers for the classroom. These results could inform policy makers and educational stakeholders as they consider certification pathways for future prospective teachers. Future research should be examined that accounts for differences in years of teaching experience and other demographic differences.

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USING COMPUTER GENERATED ANIMATIONS TO EXAMINE PEDAGOGICAL CONTENT KNOWLEDGE IN STSTISTICS TEACHING

Dhimitraq Duni
Eastern New Mexico University
dhimitraq.duni@enmu.edu

In this study, I examine the nature of pedagogical content knowledge (PCK) (Shulman, 1986) that teachers exhibit while observing computer-generated animations of statistics lessons that bring up pedagogical issues in middle school classrooms. In-service teachers were asked to view different animations and react to different prompts about the animations. Later, teachers were interviewed to stimulate further discussion of pedagogical issues with the intention of clarifying their stance and their knowledge of pedagogy. The computer animations were utilized as a surrogate for classroom teaching, highlighting issues that could be otherwise overlooked during live teaching observation.

Keywords: Data Analysis and Statistics; Computer Generated Animations; Pedagogical Content Knowledge; In-service Teacher Knowledge

In understanding teachers' knowledge of content and pedagogy, researchers have used classroom observations, as well as pen and paper assessment instruments. The next step in this endeavor was the introduction of animations (comics) to help understand teachers' knowledge (Chazan & Herbst, 2012; Herbst, Chazan, Chen, Chieu, & Weiss, 2011; Herbst & Kosko, 2014). These authors have argued that animations present a unique opportunity in understanding teacher's knowledge, because of the inherent concentration on the desired topics afforded by animations, compared to classroom observations.

The purpose of this study was to examine the use of computer-generated animations (CGAs), in lieu of classroom vignettes, for understanding aspects of teachers' pedagogical content knowledge. The topic of "measures of central tendency" was chosen because of the importance of statistics in the curriculum as well as the importance of its first introduction to students in middle school (Franklin et al., 2007; NCTM, 1989, 2000, 2006; NGA & CCSSO, 2010).

The goal of this research report is to present an initial effort in creating several computer-generated animations CGAs that helped assess teachers' knowledge of pedagogy. My research questions concentrated on characterizing teachers' knowledge as it related to the components of PCK put forth by Hill, Ball, & Schilling (2008).

Theoretical Framework

It is a model that has endured and has been used and adapted by other researchers in the study of statistics (Burgess, 2009; Groth, 2013, 2014). In this framework, PCK is partitioned into three categories: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum (KC). This model gives me the ease of understanding how to classify given responses into statements about students, teaching, and curriculum. Within each part, I have introduced components that are advocated for in research about what this knowledge should look like (Duni, 2018). The Expected Statistical Knowledge for Teaching Measures of Center framework (Duni, 2018) is designed and supported by research conducted in the teaching and learning of statistics, as well as the curriculum documents put forth by NCTM (1989, 2000, 2006) as well as CCSSM (2010).

Methodology

Videos as tools to be analyzed have been used since the Third International Mathematics and Science Video Study (1999; Hiebert et al., 2004), which gave video evidence of different teaching styles around the world. Herbst, Chazan, Chen, Chieu, and Weiss (2011), argued that comic-based representations of teaching can be powerful substitutions of videos, in seeing the mathematics taught in the lessons that were observed, allowing PST's to explore decisions made in the classroom while teaching. Furthermore, Herbst et al. (2011) argued that comics could enable a high degree of control over the content as well as the possibility of highlighting issues that might not show up in most videos. As Herbst advocated, I am using animations because it gives me the opportunity to control the narrative and what mathematics teachers pay attention to. These CGAs concentrate on eliciting different aspects of teachers' PCK when introducing the measures of center in middle school statistics.

Development of Computer-Generated Animation Protocol

The first step in designing the CGA's was the creation of the vignettes, the dialogue taking place in each CGA. The vignettes were designed with the idea that the teacher in the animation would pose a question or an activity, and the students would respond to the questions. The prompts that are used after each vignette were specifically designed to assess different aspects of teacher knowledge with regard to the teaching and learning of measures of center. The problems posed, as well as the students' comments, were typical of examples from research on students' knowledge of statistics (Cai, 1998; Russell & Makros, 1990; Strauss & Bichler, 1998).

For brevity's sake; I will give a description of just one vignette. The content of the second vignette was created to discuss measures of center as "typical" instead of the traditional "mean, median, mode". The teacher starts by introduces data gathered from the students (shoe weight) and ask students to find the typical weight. The data are not ordered, and at the end, there is a piece of data that seems like an outlier. The first students, using the algorithm, finds the mean of the data. The second student eliminates the supposed outlier (with no comment on identifying the outlier) and using the algorithm finds the mean. A third student eliminates the smallest and largest data points and then finds the mean. The fourth student, who considers the outliers, chooses to use the median (wrongly identified since the data was not ordered). Lastly, the first student comments on his confusion about always using the mean without consideration of distribution. Student responses and issues with the content were consistent with previous research (Tarr & Shaughnessy, 2003; Zawojewski & Shaughnessy, 2000). The corresponding prompts were:

1. Please identify which student or students seem to be struggling with the idea of measures of center and explain why.
2. How would you help these students improve their conception (please talk about each child individually)?

Table 1 shows which component of the expected framework knowledge was solicited from each vignette. Keeping with vignette 2 as an example, finding typicality from data of weight of shoes, the vignette could have solicited teachers to discuss their students' understanding of the mean as well as which measure of center to choose, depending on characteristics of the distribution. The second vignette could have also prompted teachers to discuss their students understanding of outliers, issues with finding median, and the idea that students tend to ignore the mean when it is not part of the data.

Table 1: Vignettes to Framework Chart

Vignettes	V1	V2	V3	V4	V5	V6	V7
C1 – Median as a central point	✓					✓	✓
C2 – Conceptual understanding		✓		✓	✓	✓	✓
C3 – Representing data with a single number			✓		✓		
C4 – Relate choice to shape & context		✓					✓
C5 – Connect center to variability							
S1 – Min, Max outliers		✓					✓
S2 – Ignore 0			✓				
S3 – Fail to order data		✓					
S4 – Mode to represent typical		✓					✓
S5 – Ignore mean when not part of data		✓	✓				
S6 – Difficulty with Weighted Mean					✓		
S7 – Mean only procedurally		✓	✓		✓	✓	✓
T1 – Investigative process	✓					✓	✓
T1a – Formulate questions							
T1b – Collect data							
T1c – Analyze data		✓	✓				
T1d – Interpret							
T2 – Connect to mathematics			✓		✓		
T3 – Explore real data	✓	✓	✓	✓		✓	✓
T4 – Use technology and assessment							

Note. Tick marks mean that that component of the framework was solicited by the corresponding vignette.

Results

Knowledge of Content and Curriculum

The fourth CGA, which directly asked teachers to connect a concept to a given activity, was the only question that was a source of most of the codes for curriculum. With respect to the concept of mean as fair share or leveling off data, the teachers made comment about it; however, almost all (14 of 15) comments came after the animated teacher (in CGA 4) demonstrated a fair share activity to the classroom.

Thinking of the measures of center as representing the data using a single number, the comments that were offered discussed the idea by mentioning that all measures of center gave different information about the data. As far as relating the choice of the center to the shape and the context of the data, comments were mostly concentrated on singular points (outliers) and did not mention, nor discuss, the shape or the context of the data. Even though the components of the framework were based on curriculum documents and research, participants did not show evidence indicating that their knowledge matched what had been recommended by researchers. Similar findings were observed by Watson (2001), who reported that even though there were a lot of activity-based lesson plans at the primary level, there was almost no evidence a uniformity of curriculum. Watson also found that the major curriculum document, was only used by a quarter of the secondary teachers.

Knowledge of Content and Students

There were numerous prompts that asked teachers to discuss topics that would reveal their knowledge of students; however, only half of the teachers specifically identified a challenge that students have that was also identified by the research. Most comments were not about making a choice about which measure of center to use but were about differences in definition and computational algorithm, between the three measures of center. The other comment about challenges came from one teacher recognizing that students do not see the need to have ordered data when finding median.

Three participants made at least three comments coded as S1 (min max as outliers), two of those teachers took the acknowledging of an outlier by a student as a given and commented on whether the elimination of said outlier was correct or not. One teacher commented that students should not just eliminate outliers and find the mean, whereas the other recognized that the outlier can skew the data. The third teacher acknowledged what the student was doing without passing judgment; interestingly though, this third teacher also recognizes that the students are eliminating what they "thought was an outlier." Watson et al. (2008) reported that, at the low content knowledge level, teachers were not successful at responding to student work or questions. Similarly, Hashweh (1987) found that low content knowledge teachers tended to reinforce preconceptions and incorrectly criticized correct student answers.

Knowledge of Content and Teaching

The one component, about knowledge of content and teaching, that participants gave plenty of evidence about was T3—*relying on exploration of real data*. Of the six teachers who commented on T3, only two tried to recognize the actual context of the real data that they were using, and the rest of the teachers merely mentioned the use of real data. From the comments pertaining to KCT, only 8% corresponded to one of the codes created in the framework; the rest of the comments were attributed to other ideas about general teaching.

It was evident that participants mainly employed direct instruction, a finding corroborated by Ijeh (2013), who reported that most teachers taught the topic in a step-wise and mostly procedural fashion. As with Watson et al. (2008), who reported teachers giving generic suggestions to students work, my participants gave such vague suggestions that they did not even pertain to statistics. Similar, to the comments that a participant made, that she did not feel comfortable teaching because of her missing knowledge, Batanero and Diaz (2010) also reported that their teachers did not feel prepared to help their students. Hashweh (1987) and Carlsen (1993) also reported that teachers with low knowledge of content tended to not deviate from activities presented in books, showing that they were not comfortable teaching the subject.

In submitting this research report, I wanted to add to the conversation about the validity of the data received by conducting research with Computer-Generated Animations being used as an assessment tool. As can be seen from the results introduced in these pages, CGA's are a good way of getting teachers to open up about their knowledge of pedagogy in a comfortable setting. These participants were given a week to go online, at their convenience, and respond to several prompts on seven animations. The animations allowed teachers to concentrate on the statistics discussed in the classroom, and not be distracted by environmental polluters.

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CONVERSATIONAL PATTERNS AND OPPORTUNITIES FOR TEACHER LEARNING IN COLLABORATIVE PLANNING CONVERSATIONS

Grace A. Chen
Vanderbilt University
grace.a.chen@vanderbilt.edu

Samantha A. Marshall
Vanderbilt University
samantha.marshall@vanderbilt.edu

Although many researchers have studied teacher collaboration, few have focused on what and how experienced teachers learn when they co-plan daily lessons. This study examines how mathematics teachers' conversational patterns in co-planning meetings shape their opportunities to learn. Preliminary findings suggest that different features of collaborative conversations, such as their agreements and disagreements, their use of students as conversational resources, and their methods of justifying their suggestions, support teacher inquiry about specific lessons or about teaching more broadly. This has implications for teachers' professional growth.

Keywords: Teacher knowledge; Curriculum; Instructional activities and practices

High-quality teacher collaboration has been linked to improved student outcomes (Ronfeldt, Farmer, McQueen, & Grissom, 2015) and research on one such form of collaboration—co-planning daily lessons—has primarily studied the processes of lesson planning (e.g., Lewis, 2002). Despite evidence suggesting that teacher collaboration is crucial for teachers' professional learning (Horn & Little, 2010), research that links co-planning and teacher learning skews toward individual cognitive processes (e.g., Shavelson & Stern, 1981) or novice teacher learning (e.g., Feiman-Nemser & Beasley, 1999). Because teachers' conversational routines shape their opportunities for examining problems of practice (Horn & Little, 2010), we examine the learning opportunities found in experienced teachers' conversations as they engage in co-planning daily lessons. This is a common and time-consuming activity for many teachers; thus, it is important to understand what and how teachers have opportunities to learn during these conversations.

Theoretical Perspectives

Taking a situated view of teacher learning as being embedded in a community of practice (Lave & Wenger, 1991), we draw on the concept of teachers' opportunities to learn as "affordances [within contexts] for changing participation and practice" (Greeno & Gresalfi, 2008, p. 172). As Grossman, Wineburg, and Woolworth (2001) point out, not all forms of teacher community are generative for improving instructional practice. In pseudocommunities, teachers act as if they agree, but this "illusion of consensus" reduces the likelihood for teachers to experience intellectual renewal, professional growth, and learning (p. 955). By contrast, teachers' opportunities to learn are enabled by processes of inquiry (Ball & Cohen, 1999), which in turn are shaped by, among other things, the norms and expectations that teachers co-construct, the conceptual resources available to them, and the conversational routines in their teacher communities (e.g., Horn, 2010; Horn & Little, 2010; Little, 2003).

Specifically, we focus on "how conversational processes differentially support teachers' professional learning" (Horn, Garner, Kane, and Brasel, 2017, p. 41). We hypothesize that engaging in different forms of inquiry—such as inquiring into a specific lesson versus inquiring into broader problems of practice—opens up different opportunities for teacher teams to learn in ways that would not be possible if they simply agreed on what to teach and how to teach it.

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Therefore, this analysis aims to answer the research question: *How do teachers' conversational patterns in co-planning meetings shape their opportunities to learn?*

Data Collection and Analysis

Study Context

This study analyzes data from a 4-year research-practice partnership between our university and a professional development organization (PDO) in which we collect ethnographic field notes, audiorecordings, and artifacts from teachers' collaborative meetings and classroom teaching, among other data, to study teacher learning. PDO teachers are competitively selected and have between 5 and 20 years' teaching experience. As a result, their conversations often involve sophisticated pedagogical reasoning compared to novice teacher conversations, which are more likely to, of necessity, prioritize classroom and time management.

Participants and Data

For this preliminary analysis, we selected transcripts from two co-planning meetings with teachers teaching the same course at the same pace in the same school. We selected these particular transcripts because PDO leaders recommended these teams as high-functioning and these conversations offered analytically interesting contrasts; they both involved planning for the next day and their structure and goals were entirely up to participants, but conversational patterns were noticeably distinct. At Noether High School, Greg and Abigail planned an Algebra 2 lesson about the fundamental theorem of algebra as it relates to polynomial functions. At Banneker High School, Amanda, Clark, and Franck planned an Algebra 1 lesson on writing algebraic rules for visual patterns representing arithmetic sequences.

Analytic Methods

To answer the research question above, we examined the questions teachers asked, how (if at all) they justified what they said, and the processes of consensus-building in their co-planning conversations. We identified continuous half-hour excerpts from each transcript, beginning when teachers started designing the upcoming lesson, and coded the transcripts using methods from critical discourse analysis (Wodak & Meyer, 2001). First, we looked for the modes expressed in each turn-by-turn utterance, coding for questions, declarative statements, justifications, direct and indirect suggestions, and expressions of agreement and disagreement (Fairclough, 2015). We also considered patterns in turn-taking and the types of justifications that teachers provided, when they provided any at all, to support their suggestions or expressions of dis/agreement.

Preliminary Findings

We found three differences in the two transcripts of co-planning meetings: 1) the nature of suggestions, agreements, and disagreements; 2) the use of students as conversational resources; and 3) the explicitness of justifications. We illustrate these differences using transcript excerpts and then describe the consequences of these differences for teachers' opportunities to learn.

Suggestions and Dis/agreements

Teachers in both teams made both direct and indirect suggestions about the next day's lesson. We coded statements of what "we should" do as direct suggestions, even if they were non-imperative (e.g., "maybe we should..."); most of the suggestions at Noether were direct suggestions. Indirect suggestions, on the other hand, merely implied what the group should do; these were more common at Banneker, where teachers often described their personal practice or preferences as an offering to the other teachers. For example, Clark said that in his previous lesson, he posted questions one at a time rather than all at once, "so that was helpful, I think."

The teachers at Noether regularly agreed vocally with each other's suggestions, responding with utterances like "yes, okay," "I agree," "yes that's true," and "got it." However, they also often disagreed. For example, Abigail disagreed with Greg's suggestion to save fourth-degree polynomials for homework; Greg disagreed with her disagreement.

> Greg: ... Should we just do quadratics and third and for homework do degree four?
> Abigail: Umm—I don't think they're going to be able to make a—ah—an interesting statement between the degrees and the number of zeroes if they don't do degree four. You know? Maybe.
> Greg: Okay. For example. I'm trying to—so if they don't do degree four—
> Abigail: So if we ask the question 'what's the relationship between the degree and the number of zeroes,' I don't know if they'll really have enough information to answer the question. What do you think?
> Greg: I think they already know it, in my mind. Like 50% of the class?

At Banneker, we coded only one instance in which teachers openly disagreed. This disagreement was less direct than disagreements at Noether: Clark had proposed giving students individual whiteboards for a problem. Franck offered, "or paper." Clark responded, "whiteboards I think is helpful for this because it's so easy to make a mistake..." without rejecting Franck's suggestion.

Students as Conversational Resources

At both schools, students became resources for teachers' sensemaking in lesson planning. At Banneker, teachers tended to narrate individual students. For example:

> Amanda: ... This girl is an expert picture-drawer, so she had shaded that. Then the kid next to her had done this, and so he was ... We were talking about, 'Well, why is it minus three?' She was like, 'Because this one got counted three extra times.' So it was like we counted it once, but then we counted again and again and again. So we have to take off the three that got triple-counted. [Clark says "yeah."] Then other kids were like, 'You take off one to get this and multiply it by four and add the extra one back in.'

In the next 13 turns, Amanda and others continued animating students; Franck, for example, noted that a student who often struggled also completed the annotations that Amanda described.

At Noether, on the other hand, teachers referenced students to support their proposals. For example, Abigail said, "Okay, I have another idea? …in my third period… they're not thinking about how the equation connects with the graph." Most references a) were in generalized terms, citing an entire class or "students" more broadly, and b) described what students had found or might find difficult (e.g., "they're still having a hard time graphing quadratics").

The different patterns of invoking students across the two high schools led to qualitatively different learning opportunities for each group of teachers. Banneker teachers' narration of individual students left open a range of interpretive possibilities, letting teachers explore what students might understand about sequences. At Noether, referring to students collectively and in terms of their challenges instead narrowed teachers' learning opportunities to focus on specific ways of responding to specific student difficulties in this specific mathematics lesson.

Justifications

Both teams justified suggestions, but the Noether teachers did so less frequently. Justifications occurred mostly when there was disagreement, and tended to refer to tacit principles of teaching. For example:

> Abigail: …Because what I worry about is if we just go straight into solving with complex they may not make the connections.

Abigail referred to an idea that she seemed to take as shared: that it is good for students to make connections between graphs and algebraic work. She appealed to this idea several times.

Banneker teachers, by contrast, commonly justified their suggestions explicitly, often without being prompted to do so. After Clark proposed giving students individual whiteboards instead of using a shared whiteboard, he added, "because then they could show each other." After Franck disagreed, however, Clark justified his preference with a value statement, declaring that work feels less permanent on whiteboards than on paper (and presumably, that this is a good thing).

Abigail's appeal to tacit principles as justification, occurring mostly when there was disagreement, opened opportunities to consider how these principles could be embedded into tomorrow's lesson and led to more articulated consensus on the Noether team. The Banneker team offered more frequent justifications and did so by appealing to broader values, providing opportunities for teachers to consider less-settled ideas without necessarily arriving at consensus.

Discussion

Although both teams' conversations provide rich learning opportunities by developing teaching concepts directly related to future work (Horn et al., 2017), differences in teachers' suggestions and dis/agreements, use of students as conversational resources, and justifications may nevertheless have implications for teachers' opportunities to learn. Greg and Abigail seemed to share an expectation of reaching consensus but were willing to disagree in order to do so, justifying their reasoning with observations about students' general challenges. Achinstein (2002) stresses the importance of conflict in "creat[ing] the context for learning and thus ongoing renewal of [teacher] communities" (p. 412), so Greg and Abigail's conversational pattern, with clear dis/agreements and few tangents, likely opened up opportunities for them to inquire into and learn specifically about how to integrate connections between algebraic and graphical representations into teaching polynomial functions to Noether High School students.

The teachers at Banneker High School expressed less direct disagreement; they offered more animations of individual students and justifications based on values. In this conversational pattern, teachers played with ideas, allowed room for their colleagues to draw different conclusions, and left questions unanswered, supporting a stance of adaptive expertise (Horn, 2010). We wonder whether this pattern opens up for discussion more ideological commitments, laying the foundation for the type of generative conflict that can challenge the status quo of teaching (Achinstein, 2002). The opportunities for teacher learning afforded by Noether and Banneker's different conversational patterns are both important: learning about teaching specific mathematics content to specific students supports teachers' effectiveness and is efficient, and critical reflection about teachers' philosophies of teaching is also central to professional growth. Ideally, we imagine that high-functioning teacher teams probably engage in both.

To continue this analysis, we will code more co-planning meetings for these and other teacher teams. Given our interview data, we wonder whether the observed differences might be caused by different social histories among teacher teams or whether they might be consequential for outcomes other than teachers' opportunities to learn. We hope that this analysis can shed light on whether the conversational patterns we found in these initial meetings are typical or unusual across teacher teams, school contexts, and situations, contributing to a deeper understanding of how teachers collaboratively plan curriculum and the consequences of such conversations.

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GRADUATE STUDENT INSTRUCTORS' CONCEPTUALIZATIONS OF PEDAGOGICAL EMPATHY

Karina Uhing
University of Nebraska – Lincoln
karina.uhing@huskers.unl.edu

In this study, I explore the phenomenon of pedagogical empathy, which is defined as empathy that influences teaching practices. Specifically, I examine how graduate student instructors (GSIs) conceptualize empathy as it relates to the teaching and learning of mathematics. The findings presented in this paper emerged from interviews with 11 GSIs in which they were asked questions to elicit their thoughts about empathy. The participants provided complex and diverse notions of how empathy relates to teaching mathematics. Conclusions drawn from this paper may help inform professional development efforts targeted at novice and experienced instructors at the post-secondary level.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Post-Secondary Education; Teacher Knowledge, Undergraduate-Level Mathematics

Interpersonal relationships are central in the teaching and learning of mathematics. One way that teachers relate to their students is through experiencing empathy in an educational setting. However, little research has been conducted at the intersection of teacher and student affect, where empathy is often situated. Exploring how empathy influences and shapes teaching interactions can lead to a better understanding of teaching practices and have far reaching implications for the broader education community.

This study aims to explore how mathematics graduate student instructors (GSIs) conceptualize empathy as it relates to the teaching and learning of mathematics. The conclusions drawn from this study will add to the emerging body of literature about math GSIs' knowledge, beliefs, and teaching practices. In addition, this research can be used to inform professional development efforts for both novice and experienced instructors.

Theoretical Perspective

Empathy is a multifaceted construct. Researchers have described empathy in various ways and have developed different characterizations depending on the context in which empathy is referenced (Cuff, Brown, Taylor, & Howat, 2016). Therefore, it is important to provide a working definition of empathy as it relates to the context of teaching. In this paper, I use the term *pedagogical empathy* to refer to the empathy that a teacher experiences for a student or a group of students. This empathy has the potential to influence teaching decisions and directly impact the actions that are taken in the classroom. Thus, pedagogical empathy involves how teachers use their understanding of students' perspectives and experiences to inform their decisions. The terms empathy and pedagogical empathy are used interchangeably in the remainder of the paper.

The findings presented in this paper also build on Zembylas's (2007) framework, which establishes a connection between pedagogical content knowledge and emotional knowledge. Zembylas (2007) contends that "teacher knowledge is a form of knowledge ecology – a system consisting of many sources and forms of knowledge in a symbiotic relationship" including "content knowledge, pedagogical knowledge, curriculum knowledge, knowledge of learners, emotional knowledge, knowledge of educational values and goals and so on" (p. 356). Specifically, Zembylas (2007) narrows in on emotional knowledge and defines the construct of emotional ecology to connect emotional knowledge to PCK. Building on his work, I argue that teachers experience pedagogical empathy when pedagogical content knowledge and emotional knowledge (along with other types of knowledge) are funneled through a filter of empathy, which consequently influences teaching decisions. The findings presented in this paper help demonstrate how pedagogical empathy can be conceptualized as this filter.

Purpose

This paper stems from a larger study that explores how pedagogical empathy might influence the feedback that mathematics graduate student instructors (GSIs) provide to their students. As a first step, it is important to consider how GSIs define empathy within the context of teaching mathematics and how that definition relates to their conceptions and beliefs about empathy. Thus, this paper focuses on the following research question: *How do GSIs conceptualize pedagogical empathy?*

GSIs are an important and unique population to study because they are not only instructors of undergraduate mathematics courses, but also students in graduate mathematics courses. Having these simultaneous roles may make it more likely for GSIs to experience pedagogical empathy for their students as they may have recently had similar or shared experiences as students themselves, an important factor that has been shown to be associated with empathy (Eklund, Andersson-Stråberg, & Hansen, 2009). In addition, many of them are future mathematics faculty members who will continue teaching at institutions of higher education after they graduate. The insights gained from this study can be used to inform GSI professional development efforts and design activities to increase instructors' awareness of pedagogical empathy.

Methods

Eleven mathematics GSIs at a large Midwestern university participated in this study. Each GSI was the instructor of record of a precalculus course (either Intermediate Algebra, College Algebra, Trigonometry, or Precalculus) and had varying levels of teaching experience. None were international GSIs. Participants selected their own pseudonyms, which are used below. Data was collected during the summer and fall of 2018 through multiple interviews and classroom observations of the GSIs. The findings in this paper emerged from the initial interview with each participant, which was designed to explore how GSIs conceptualize empathy as it relates to teaching mathematics.

Interviews were transcribed and analyzed using MAXQDA. Since pedagogical empathy is a novel theoretical construct, open coding was initially used to generate a set of preliminary codes (Strauss & Corbin, 1998). Each interview was coded, and short summaries were then generated for every coded segment. These segments were then categorized and sorted using axial coding (Strauss & Corbin, 1998). During this process, codes were refined and further clarified until a robust understanding of each category was reached. These codes were then organized into three main themes that characterize how GSIs conceptualize pedagogical empathy.

Findings

Three main themes emerged from the interviews: accounting for contextual factors, enacting and experiencing empathy for students, and considering empathy along a trajectory of development. These themes contribute to an understanding of how GSIs conceptualize empathy as it relates to teaching and learning mathematics. Due to space constraints, only the contextual factors are discussed in depth below.

Context

The context in which a class is situated has the potential to influence how teachers conceptualize and experience pedagogical empathy. This context includes a broader disciplinary perspective and a more localized course format that takes into account specific course constraints and affordances to pedagogical empathy. These two components, which are described in detail below, help explain how the content and the structure of a course might mediate empathic interactions between teachers and students.

Mathematics perspective. Several participants discussed mathematics from a disciplinary perspective and referenced how past experiences with math can shape the way that students feel and think about mathematics as a discipline. It was widely acknowledged that teachers and students might have opposing beliefs about mathematics and that it is important to consider these different perspectives as they have the potential to influence how teachers empathize with their students.

A few GSIs discussed how students might feel or think about math and the resulting impact that might have on their experience in the course. Aaron brought up empathizing with a student in his class who was "hesitant" and "scared." He talked about "trying to think of ways to figure out more about how this student is feeling about math" and how he could "bring her more confidence while she approaches problem solving." Another participant, Mark, emphasized the importance of knowing that "students come in from a lot of backgrounds, many in which they're taught to hate math or to think of themselves as bad at math" and remarked that in order to "reach all students" instructors need to consider the perspectives of their students.

In addition to reflecting on how their students might feel about math, a few of the GSIs referenced their own perspectives and experiences with mathematics. Beth talked about how she wants her students to "appreciate math and appreciate that they can do it" because she wants them to feel the way that she feels about math. Mark brought up his history with math and how he had experienced mixed feelings about it: "I did go through a few years in high school where I didn't like my math classes, but even then I didn't really feel like I disliked math. I just disliked the math class because it was boring." He went on to share how his experience with Calculus had changed his perspective and how that influenced his view of the course as an instructor:

> Calculus, for me, had been just such a joyful class. It was sort of like what reignited my interest in math, reminded me that math class was a cool thing. So when I was teaching it the first semester, it didn't really even occur to me that my students could legitimately be worried about this and frightened about this.

Some GSIs openly discussed their negative experiences with math. Aaron explained how he shares a personal story with his students to tell them about his "perspective on the culture of math and how there's a fear-based culture of math in many cases." Aaron's story involves being a student in an undergraduate math class where he was handed a sheet of problems to solve:

> I didn't know how to even start any of them because I hadn't taken the necessary class prerequisites. So I was just sitting there waiting, not really able to do anything, and the professor stood in front of me and just started staring at me, at my blank sheet of paper for like five minutes. He finally broke the silence by asking, "Do you understand?" And it made me feel really bad.

By sharing this story with his students, Aaron hopes that they will be able to recognize and acknowledge their own fears and anxieties about math and can better relate to him as an instructor.

Course format. The format of a course is another contextual factor that can mediate how teachers and students interact with each other. This format influences the ways in which teachers interact with their students, including how often teachers see their students, how many students are in a course, and what types of interactions are possible. The position and level of authority that teachers have within the course structure can also affect how they view their role in the classroom and their perceived instructional responsibilities. These structural elements of courses can either support or deter opportunities for experiencing pedagogical empathy in the classroom.

A few participants discussed how it was easier to experience empathy for students when the format of the course allowed for direct interactions with them. For example, Chris discussed how teachers are more likely to experience empathy "when they're not just lecturing at them [students] the whole time and then telling them to beat it." He went on to describe how empathy can be more easily elicited in courses where "two-way communication is heavily encouraged" and the teacher is not the only person who is allowed to talk. Amanda felt similarly, stating:

> The way that our classes are set up, at least in the precalculus level with active learning, I think that also makes it easier to experience empathy because you have a lot more one-on-one interaction with your students in class.

Similarly, Mark discussed the affordances of being able to "sit down one-on-one" with a student who needs help or is feeling overwhelmed. However, he also mentioned how it is often difficult to "avoid time constraints" and ensure that all students understand a concept before moving on to the next topic and identified the number of students in his class as being a potential barrier to experiencing empathy:

> In an ideal world where I have time to actually make sure each student gets the help they need, I would try to figure out where the disconnect begins and then work from there. But again, I say this in the context of a classroom where I've got time to do this. In a classroom where there's one teacher and 42 students, corners have to be cut unfortunately.

In addition to contextual factors like the structure of a course and the number of students in a class, GSIs' roles and responsibilities might also influence the way that they empathize with their students. In particular, Jane talked about the difference between being a recitation leader and being an instructor of record. As the instructor of record for a precalculus class, Jane felt "a lot more of an emotional burden on my students to do well and for me to teach them the material," whereas leading a recitation comes with less "responsibility for their education and what they learn." She also suggested that GSIs might have more opportunities to feel empathy when they are the instructor of record, as compared to being a recitation leader, because having that responsibility for helping students learn is "another way that you can feel empathy."

Discussion

The findings above dissect one facet of how GSIs conceptualize pedagogical empathy. Specifically, they highlight how perceptions of mathematics as a discipline can influence teachers' experiences of pedagogical empathy and how the format of the course might elicit or suppress this empathy. These findings also help illustrate how pedagogical empathy can be conceptualized as a filter through which different types of knowledge pass: as teachers interact with their students, they draw on their knowledge and use pedagogical empathy as a filter to decide what actions are most appropriate for the situation. After examining this perspective, we can begin to consider how to structure professional development to increase teachers' awareness of this phenomenon and help them become better decision-makers in the classroom.

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INVESTIGATING MATHEMATICS GRADUATE STUDENTS' GROWTH AS TEACHERS: THE UNIQUE CASE OF EMMA

Mary Beisiegel	Claire Gibbons	Alexis Rist
Oregon State University	Pierce College	Oregon State University
mary.beisiegel@oregonstate.edu	cgibbons@pierce.ctc.edu	delonga@oregonstate.edu

The need for research-based professional development programs for mathematics graduate students is significant, yet few studies have investigated their development as teachers. This study aims to fill this gap in the research by studying mathematics graduate students' experiences with teaching as they progress through their graduate programs. Participants responded to surveys and were interviewed semi-annually for two or more years. We analyzed their responses using thematic analysis and a framework that captured their development as teachers. In this report, we describe the developmental stages for teaching and context and methods of the study. We present general findings on the experiences that mathematics graduate students did not move past a certain developmental stage and we present specific findings about one mathematics graduate student who experienced significant growth in her teaching.

Keywords: Professional Development; Post-Secondary Education; Affect, Emotions, Beliefs, Attitudes; Instructional Activities and Practices

At graduate institutions in the United States, mathematics graduate students often receive funding from mathematics departments to support their time working on master's or doctoral degrees. In return, they teach courses as instructors or they work as teaching assistants, supporting faculty instructors of large lecture courses by leading weekly, small-group recitations. Over the course of their graduate programs, mathematics graduate students contribute to the learning experiences of hundreds of undergraduate students. Their contribution to undergraduate students' learning experiences extends beyond their graduate programs with more than 60 percent of new mathematics PhDs finding employment in post-secondary education settings in which teaching makes up a significant portion of their work (Golbeck, Barr, & Rose, 2016). Thus, mathematics graduate students exert a significant impact on undergraduate learners' trajectories in STEM fields (Belnap & Allred, 2009; Ellis, 2014).

Despite the amount time they interact with undergraduate learners, mathematics graduate students receive little professional development (PD) for their teaching (Deshler, Hauk, & Speer, 2015; Miller et al., 2018). The PD they do receive varies greatly across institutions – programs range from a few hours, to an intensive week of PD, to a seminar that spans a full academic year (Deshler et al., 2015). The field of mathematics graduate student PD is relatively new and experts in the field have not reached consensus on the depth and breadth of PD programs that prepare mathematics graduate students to teach. This lack of consensus might be due to the fact that mathematics graduate students' growth "is a largely unexamined practice" (Miller et al., 2018, p. 2). Miller and colleagues (2018) suggest that this field of study would benefit from longitudinal studies of mathematics graduate students' growth as teachers.

Our primary long-term goal is to develop an extended PD program for mathematics graduate students. Prior to developing and implementing such a program, our research team sought to understand how mathematics graduate students think about teaching, what they learn about teaching, and what type of PD might be relevant at different stages of their development. We wondered – how do mathematics graduate students develop as teachers? What are salient, transformative experiences and what are experiences that might hinder mathematics graduate students' growth as teachers? In this report, we describe a longitudinal study of mathematics graduate students growth. We focus on one specific participant who illustrated significant growth in her teaching.

Theoretical Framework

Because research has not yet addressed MGTAs growth as teachers (Miller et al., 2018), we looked to the K-12 literature, where researchers have studied schoolteachers' experiences in order to gain an understanding of their growth over time. Katz (1972) described four developmental stages, which include: (1) *survival* of the first year of teaching, with particular focus on classroom management and the routines of classrooms and schools; (2) *consolidation*, in which teachers begin to understand which skills they have mastered, and what tasks they still need to master; (3) a period of *renewal*, when teachers become tired of their routines and start to think of how things might happen differently; and (4) reaching *maturity*, where teachers think more broadly about the contexts of schools and students' learning (p. 52-53). Based on the literature, we posit that mathematics graduate students' teaching practices in the survival and consolidation stages will be lecture-based, and in the renewal and maturity stages, more focused on teaching practices that support student engagement in mathematical work during class time (e.g., active learning, inquiry).

Context and Methods of the Study

Our study takes place in a mathematics department in which mathematics graduate students receive two to three days of professional development when they first arrive, as well as a year-long, introduction-to-graduate-studies-in-mathematics seminar part of which is focused on teaching. Mathematics graduate students are most often assigned to teach recitations for large lecture courses, but they also have the opportunity to teach their own courses. Sometimes they work as a grader for upper-division or graduate courses. Mathematics graduate students were recruited for the study in 2015, 2016, and 2017. They are asked to complete a beginning-of-the-year survey, a mid-year interview, and an end-of-year interview. Three mathematics graduate students have participated in the study for its duration (currently in its fourth year), five have participated for three years, and other mathematics graduate students in their second year of participation in the study. We are using the six steps of thematic analysis (Braun & Clarke, 2006) as a method for analyzing surveys and transcripts of interviews.

Findings

We have found Katz's framework (1972) to be a useful lens to study mathematics graduate students' growth. We have observed that mathematics graduate students do not pass through Katz's (1972) developmental stages linearly. They sometimes returned to the *survival* stage if a new teaching assignment varied significantly from their prior teaching assignments. Most mathematics graduate students appeared to be stuck in the *consolidation* stage and their descriptions of teaching were remarkably unchanged year after year. In addition, out of the nine mathematics graduate students who have participated in the study for three or more years, only two of the participants spoke of teaching in ways that Katz (1972) would categorize as *maturity* – and none so profoundly as one participant, Emma (pseudonym).

Focusing on Emma's Growth as a Teacher

Emma began participating in the study in 2015 as a second year graduate student. Emma has had several different teaching assignments over the years of the study. She has worked as a teaching assistant in both in-person and on-line courses, as a grader for graduate courses, and as an instructor for a course. These positions were determined by the department based on Emma's schedule. Early in the study, Emma expressed her belief that a goal of teaching mathematics is to impart knowledge – not only mathematical knowledge but also "the knowledge of how to gain more knowledge, how to be a student, how to be a mathematician, how to think like a mathematician." Whether and how she felt she had an impact on students (in their learning and what they thought about mathematics) was very important to Emma. Her role as a teaching assistant made her feel that she had very little impact on students.

Difficulties of Being a Teaching Assistant Instead of Instructor (Survival, Consolidation)

Like many participants in the study, Emma found the role of teaching assistant to be "less than ideal" because she did not have control over the classroom dynamic or over the work the students were assigned. This feeling continued into the third year of the study, where Emma described the instructor of the course as the decision-maker who has the biggest impact on student learning. Emma explained that the instructor chooses the content for recitation, which meant that she had "No power over what happens in the recitation hour" which made her feel ineffective: "It's the instructor's course. That's fine. But it means I'm not choosing any of the problems, or anything like that. So what can I do?" Because she saw the instructor as having total control over both the lecture and her own work with students, Emma felt like she had "such little power to actually impact [students'] learning."

Eventually, her role as teaching assistant began to have a negative impact on how she saw herself and what she felt she could contribute to students' learning:

> For the impact on their learning, I think I'm interchangeable with all the other graduate students. Every once in a while, there'll be a problem that everyone's asking questions on, and I'll be like, "Okay, let's review something that you all might enjoy having a small review of." And that's fine, but they could get that from going to anybody's office hour.

Teaching Her Own Course for The First Time (Consolidation With Hints of Renewal)

When Emma had the opportunity to teach her own course, she very much wanted to have an impact on students and aimed to do so through detailed, planned lectures. She spoke in detail about how she planned for her lectures:

> So I worked so hard on that class. Oh, my God. It was a full-time job for me. I was preparing lectures, and I supplemented the text in all these ways, and I spent boatloads of time coming up with awesome examples and activities. So I spent all this time and energy into the activities, and I do think that I had a big impact on people's enjoyment of it, and I got personal notes from students saying as much.

Even though Emma emphasized lecturing, we observed evidence that the message of active learning was having an impact on Emma. She described "full on lecture" as "easier" than leading recitations. But she also remarked that she wanted to include both group work and direct instruction. She vocalized some uncertainty of how successful she would be: "Having never done this before, I don't know how effective it's going to be."

In the interview following her first experience teaching her own class, Emma reported using strategies that supported student engagement in mathematical work. She lectured during most class periods, but had students working on problems three or four times per class. Her goal was to talk for no more than several minutes before letting students work on an example:

> I would introduce a concept, try to put it in the big picture. This was my whole thing, was like, "Okay, fundamental theorem of calculus. Let's put it in the big picture. Why am I even talking about this? 'Why is it called fundamental'? What makes this worth studying?" And then instead of doing example, I would do a simple example on the board, and then I would make them do an example on their own.

Unfortunately, Emma saw students' exam performance as the most important measure of her impact as a teacher and she reflected on her lecturing:

> But my students got the exact same average on the midterm as every other instructor who was teaching this course over the summer. That made me think, "Maybe I'm not having as big an impact as I thought." […] I would like it to be, "I just nailed all of my lectures," and I want that to have a measurable impact on student performance. I think [lecturing] doesn't have as big an impact as I used to think it did.

In a later interview, Emma remarked: "I have started to see the in-class lecture as a colossal waste of time, and not as big an impact."

Emma's Move Away from Lecturing (Renewal, Maturity)

Recently, we observed Emma reach the renewal and maturity stages through a unique experience in which she had the opportunity to design and teach a course. With some advice from the department, she chose the curriculum, lesson formats, and assignments. She incorporated many things that were important to her about mathematics, including a focus on deep mathematical reasoning (e.g., *Why* would a mathematician think about a problem in a particular way?). She made the choice to not lecture and instead had students engage in mathematical activities. When asked about her transition from lecturing to active learning and whether it was difficult, Emma admitted her love of "performing" and stated, "I like being center of attention and look like I know everything." Reflecting on her choice to not lecture, Emma said, "It bums me out because I like lecturing. I think I'm good at it. But now, I'm convinced that people don't learn by lecturing." She later said, "So I did feel a sense of loss at first. And then by the end of that class, it totally changed me. I maybe got more out of the class than the students did because I got to the end of that class and I said, 'I am never ever lecturing again, ever.'"

Conclusion

What can we learn from Emma's experiences and her growth as a teacher? What prompted Emma to make the changes to her teaching practices? Emma noted that some events in the mathematics department influenced her. For example, the department that had selected a few courses to be more focused on active learning and Emma had exposure to an active learning classroom in her first year as a graduate student, a visiting colloquium speaker provided examples of active learning and some of her peers were incorporating active learning into their classes. However, we acknowledge that not all mathematics graduate students will have the same exposure to active learning and opportunities to grow in their teaching practices. Thus, we see as Emma's desire to feel that she had an impact on students and how she reflected on her own learning experiences for more guidance critical to her growth as a teacher. We suggest that PD programs be designed such that mathematics graduate students understand the contributions they make to students' learning. We also propose that PD programs include a reflective component that prompts mathematics graduate students to consider their most meaningful learning experiences. Our study will continue and it remains to be seen whether Emma will continue to develop as a teacher after this unique experience. What might happen to her views about active learning should she be assigned to an instructor who wants her to lecture mathematics to students and/or if she feels that her role does not allow her to contribute to student learning?

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ITERATIVE DESIGNS OF MODELING TOOLS FOR INSTRUCTION

Rebekah Elliott
Oregon State University
elliottr@oregonstate.edu

Elyssa Stoddard
Oregon State University
stoddare@oregonstate.edu

Megan Brunner
Oregon State University
brunnerm@oregonstate.edu

The modeling literature offers insights on student learning but remains scarce on teachers' instruction of mathematical modeling. We investigate the efforts of a team of secondary teachers, teacher leaders, and mathematics and science teacher educators to co-design and use instructional tools for mathematical modeling. We adopt formative interventions (Engestrom, 2011) to understand how the team coalesced around a shared problem rooted in modeling and uncover the ways that the team respond to contradictions in the activity system. This study contributes a view of professional learning that deeply connects research, design, and enactment fostered by the team's agentic actions.

Keywords: Professional Development, Modeling, Instructional activities and practices

Professional development (PD) designed by educational researchers to change instruction is often developed outside of the educational setting for teachers and districts inside the educational setting. With the aim to impact knowledge, instruction, and student achievement, researchers often cite the complexity to affect change within the time span of typical research (Koellner & Jacobs, 2015). This brief report investigates efforts of secondary teachers, teacher leaders, and mathematics and science teacher educators (hereto called the TLE team) to co-design and use instructional tools for mathematical modeling to impact teachers' knowledge, instruction, and student achievement.

Modeling researchers provide insights on student learning, but there are limited details on teachers' mathematical modeling practice associated with student learning. Even though modeling is a part of many state's K-12 standard (NGA, 2010), research on modeling instruction has not kept pace (Kaizer, 2017). The questions driving this study are: How are modeling instructional tools co-designed by the TLE team in ways that support co-production of learning? How does tool use uncover further complications of modeling enactments and foster agentic action? To consider the co-production of learning, we examine the TLE team's efforts in design summits, two-day events in which the TLE team learned about modeling and designed and refined instructional tools. This study contributes a view of professional learning that deeply connects research, design, and enactment, to tool-design and use for mathematical modeling.

Framework

Literature on professional learning, with an eye toward reimagining practitioner/ researcher roles and redesigning systems of working in partnerships of research, design, and enactment are theoretical rooted of design-based research (Bevan & Penuel, 2018). This growing scholarship offers insight on the typical divisions between researchers' and practitioners to re-negotiate the research hierarchy and share roles in research, design, and enactment.

We adopt Engestrom's (2011) formative interventions as a starting point for design and research with a focus on shared problems in a system. Modeling, as a mathematical activity, is demanding and merges STEM disciplines and real-world questions, in novel ways of reasoning with no pre-designed pathway to a solution (Barquero, Bosch & Romo, 2018). In this study, modeling instruction was identified as a problem of practice by the TLE team due to the lack of instructional supports and the complex layering of human activity in which school mathematics and modeling coexist. Engstrom's construct allowed us to examine the contradictory motives that emerged and subsequent responses both unpredictable and agentic.

Mathematical modeling is a cycle of activity iterating across real world situations, promoting problem formation, mathematizing problems, and coordinating a mathematical model and justification to make meaning within the real world (Bliss, Fowler, & Galluzo, 2014). Researchers posit to build students' modeling competence they need wholistic opportunities to engage in the modeling process *and* specific opportunities to build new skills with particularly demanding aspects of modeling (Schukajlow, Kaiser, & Stillman, 2018). This atomized approach was catalyst for instructional tool designs that establish classroom routines for modeling reasoning. Across three design summits, TLE team members advanced two instructional tools to resolve shared problems of enactment.

Context and Methods

Eleven math and science teachers and four teacher leaders working in rural, suburban, and urban districts in the western United States and five math and science teacher educators from two campus of the University comprised the TLE team. Data were collected from the TLE team activities in three design summits taking place across seven months. The summit goals were two-fold, to advance the TLE team's capacity to enact modeling with learners and to identify or design tools in support of enactments.

Many in the TLE team had teaching experience using routines as instructional tools. Although the genesis of the summits was initiated by teacher educators, teachers were seen *and became* co-developers in considering issues and designing solutions. Across the summits, responsibility for facilitating shifted to teachers and teacher leaders. In the first summit, the TLE team designed two instructional routines focused on key elements of the modeling cycle, defining assumptions and variables (called Donkey Ume) and iterating, refining and defending a model (called Is it Good Enough). In the second summit, enactments of the routines were discussed and further refined. In summit three, the routines were rehearsed to further the revision of the two instructional tools. The agentic actions of the team are the focus of our analysis to further understand how PD can link research, design, and enactment.

Methodology

Data were collected to trace meaning making that took place in design summits. Data reduction was accomplished by identifying *design/re-design* and *enactment discussions* in each summit. Data included: (1) teacher created design cycle documents in summit one, (2) post-summit one enactment reflections and classroom artifacts, (3 & 4) fieldnotes and design cycle documents from summit two and three.

Our analysis drew upon Engestrom's (2011) formative interventions to pinpoint starting points, processes and outcomes. Starting points were the ways in which the TLE team identified problematic and potentially contradictory goals essential to teaching mathematical modeling. Claims were inventoried and themed (Saldana, 2012). After an exhaustive list of themes were identified, we clumped themes into larger categories that captured the initial starting points, catalysts for action, and mediating artifacts. Documenting the claims chronologically, we could parse how issues emerged and shifted activity post modeling routine-design. We discuss how themes resulted in the layered character of a formative intervention, tracing the starting point, process, and outcome.

Findings & Discussion

Our analysis identified themes we conjecture are the main contradictions that the TLE team worked to overcome.

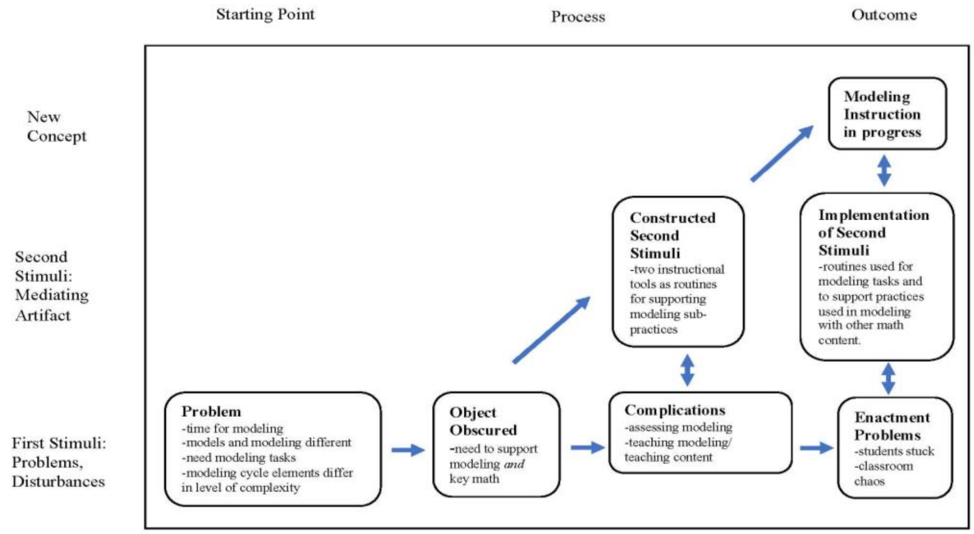

Figure 1: Formative Intervention with TLE Team (adopted from Engestrom, 2011)

In Figure 1, we share our conjectures on learning. Our aim was to document how instructional tools for modeling were co-designed in ways that supported co-production of learning across stakeholders. Due to space limitations we limit the discussion to highlights of the analysis.

The TLE Team saw a contradiction between the modeling practice-standard and the act of modeling and were aware that not all features of the modeling cycle were easily navigated. The six elements of the modeling cycle became a tool to talk about where instructional "chaos" erupted because of "cognitively demanding" mathematics. We saw efforts here similar to calls for atomizing modeling to support students' reasoning (Schukajlow, et al., 2018). Team members readily agreed that it wasn't because students weren't capable of doing the mathematics, but rather their instruction hadn't adequately supported students.

These enactment problems didn't hampered enthusiasm for modeling, rather theTLE team honed in on the elements of the modeling cycle that were more demanding. Via two tools for defining assumptions and variables and iterating, refining and defending a model, they conjectured that students would be better equipped to engage in all the elements of the modeling. After designing the tools, members gathered in two subsequent design summits to examine and refine the tools.

Summit data analysis identified two central agentic actions of TLE members. The first was an assessment question of capturing students' improvement on modeling and the second was a tension between teaching modeling verses teaching content. Assessment questions emerged from almost all members in enactment discussions relating the tools' role in learning of math content, math modeling, and specific focal practices. The assessment ideas coalesced into an action plan for an "experiment" alternating "full modeling cycle" tasks and "hard stop" tasks. They thought that strategically alternating types of tasks would allow team members to know if student were able to "transfer" thinking.

The second theme was a tension between modeling and content that the TLE team felt compelled to reconcile. One member's comment summarizes this idea when she said, "does it have to be modeling OR content?" noting the tension between the two and pushing back. TLE team members discussions lingered on "what was *not* being taught" by focusing on modeling. We marked this tension between content and modeling and saw it transform in interesting ways.

A number of teachers had shaped tasks and lessons to use the routines in ways that were supportive of the focal practice and were doing so on a semi-regular basis. They conjectured that this allowed students more agency to engage in learning and supported building norms for reasoning. As the third summit approached, teachers shared online video, tasks, and student work. From these data we saw team members use the routines with non-modeling tasks to leverage justification of a different representation and consider assumptions and variables when working with functions.

In the third summit, discussions of enactments led to TLE team members advocating for rehearsing with the tools to resolve instructional complications they had encountered. The collective agency for rehearsal resulted in iterating the tools to refine the instructional sequence, develop clarity of mathematical aims, and to generate two tools for each routine, one for teachers to use in planning and one as a public record for students. Also notable was that TLE members were thinking about how other teachers could use the tools modifications for non-TLE members.

Conclusion

Often in research on PD, the researcher looks to intervene on practice to improve it. As researchers, we did not have a full-proof design and as Engestrom (2011) suggests, formative interventions do not have one end point in mind. Rather, we are interested in the agentic actions of the TLE team and how they resolve contradictions. This report documents the TLE team's commitment to resolving contradictions that emerge by bringing the real world of modeling into the world of school mathematics. Unlike other studies of PD aimed at documenting impact of a predetermined treatment, this research attempts to document the agentic actions of professionals and the unfolding responses to contradictions that are part and parcel of human experience.

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TEACHER LEADER LEARNING THROUGH PARTICIPATION IN AND FACILITATION OF PROFESSIONAL DEVELOPMENT ADDRESSING PROBLEMS OF PRACTICE

Anthony Muro Villa III
Stanford University
anthony1@stanford.edu

Kelly L. Boles
Stanford University
kboles@stanford.edu

Hilda Borko
Stanford University
hildab@stanford.edu

Research on facilitation of professional development (PD) has been limited. Further limited is our knowledge of how learning to facilitate PD takes place and how PD gets adopted or adapted, once disseminated to different school contexts. As part of a district and research university research-practice partnership (RPP), a PD model was developed to help teachers implement a new mathematics curriculum and center discussions around addressing problems of practice. This study examined how teacher leaders from different school sites learned to facilitate site-based PD focused on discussions about mathematics group tasks and video clips from their own classrooms. Teacher leader learning and enactment of facilitation was analyzed throughout this ongoing and adaptive RPP PD model.

Keywords: Teacher Education-Inservice/Professional Development, Teacher Knowledge

Introduction and Research-Practice Context

This study examined teacher leader learning while participating in and then subsequently facilitating professional development (PD) aimed at addressing persistent problems of practice within their school contexts. The professional development was part of a larger ongoing research-practice partnership (RPP) between an Urban Unified School District (UUSD) and a research university. UUSD re-designed their curriculum and aimed to implement *group-worthy tasks* (Lotan, 2003) to address the CCSS-M. In addition, the district developed interdisciplinary Dimensions of Teaching and Learning, inspired by the Teaching for Robust Understanding (TRU) Framework (Schoenfeld, 2013, 2014). These dimensions include visions of equitable teaching and learning around three main components: assessment; access to content; and authority, agency, and identity. A central goal of the RPP is to support UUSD in implementing the new curriculum and the vision of teaching and learning represented by these dimensions.

Teacher leaders (TLs) from middle school sites within UUSD participated in a program led initially by the university research team and district mathematics personnel. The program incorporates the Problem-Solving Cycle (PSC) model of PD and Teacher Leadership Preparation (TLP) model of PD leader preparation (Borko et al., 2015), as well as adaptations to these models developed within the RPP (cf. Borko et al., 2017).

The Teacher Leadership Preparation model prepares teacher leaders to plan and lead Problem Solving Cycle workshops with the math teachers at their school sites (see Figure 1). The central component of the TLP is a series of Leader Support Meetings (LSMs) that provide ongoing, structured guidance for the TLs as they prepare to lead the PSC workshops. The first LSM activity (LSM: Modeling and Debriefing Discussions) is the modeling of mathematics task-based or video-based discussions (VBDs). During these discussions, teacher leaders participate as teacher learners within the project-team-facilitated PD. For the debrief, the partnership team invites TLs to ask questions during the "pulling back the curtain" component, where intentional planning, facilitation, and focal question choices are made explicit.

The final activity of each Leader Support Meeting is the LSM: Rehearsal and Planning portion. Mirroring the math discussions or VBDs from earlier in the day, TLs plan and rehearse facilitating discussions to be enacted at their school site. TLs engage in these rehearsals as an opportunity to develop discrete components of complex practice in settings of reduced complexity (Grossman, 2009; Grossman et al., 2005). Grounding the PD in artifacts such as videos of practice makes practice available to peer review, opens opportunities to learn new practices and pedagogical strategies, and allows deep analysis of the complexities and richness of teaching and student learning (Ball & Cohen, 1999; Borko et al., 2008; Sherin, 2004; Sherin & Han, 2004; Sherin & van Es, 2009).

PSC workshops that follow the LSMs are facilitated by the TLs, and are enacted at their respective school sites. Non-TL colleague mathematics and support services teachers attend as teacher learners. When a VBD is the focus of the PSC, the video clip is taken from the classroom of an attending TL or colleague teacher.

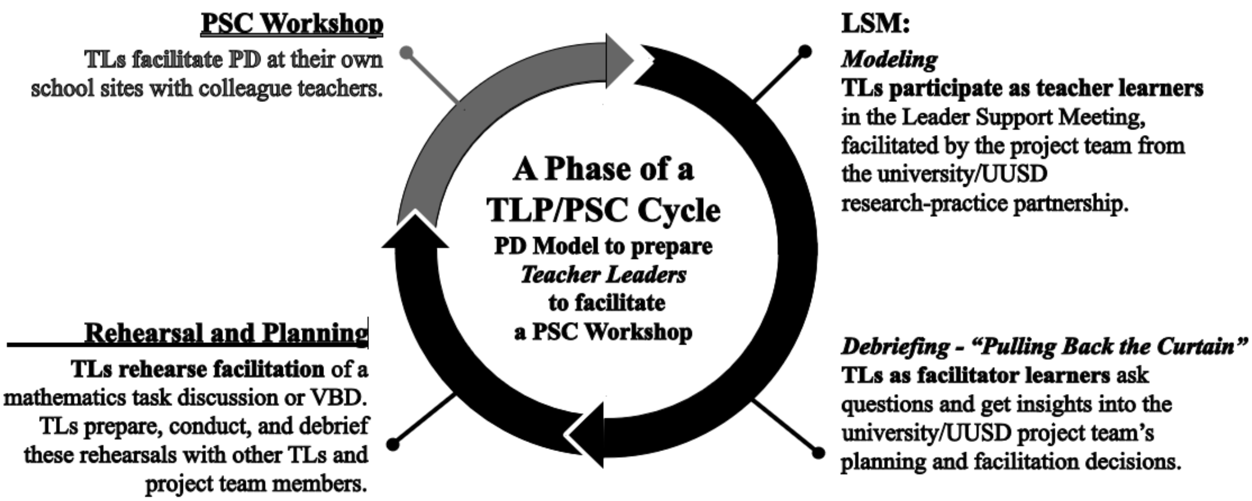

Figure 1: Teacher Leader Roles During Each Phase of a TLP/PSC Cycle. Three Interconnected Phases Occur Per Semester, Which Comprises One TLP/PSC Cycle.

Purposes of Study

This study examined the impact of TLs' participation in TLPs on their facilitation moves when conducting corresponding PSC workshops at their respective school sites. Little attention has been given to how leaders learn to facilitate PD, and even less research considers how teacher leaders learn from the facilitation of other PD providers (Elliott et al., 2009; Goldsmith & Seago, 2008; Roesken-Winter, Schüler, Stahnke, & Blömeke, 2015). As such, we investigated each layer of the teacher leaders' participation in the LSM and PSC -- as teacher learners in the LSM: Modeling and Debriefing Discussions, as facilitator learners during LSM: Rehearsal and Planning, and as TLs during PSC PD workshops enactment at their respective school sites. This investigation was particularly focused on how teachers are learning and enacting facilitation of VBDs focused on problems of practice.

A central goal of the partnership was to support implementation of the new curriculum and incorporation of the dimensions of equitable teaching and learning into classroom practice. Thus, it was important to consider the ways in which TLs were supported in developing facilitation skills to conduct site-based PD that addressed these partnership goals. With these aims in mind, this study sought to analyze the relationships between and within the following: (1) the activities, practices, and facilitation moves that were adopted or adapted during LSM: Rehearsal and Planning, and (2) the TLs' facilitation of PSC workshops at their own school sites.

Theoretical Perspective

We adopt a situative perspective to study teacher learning supported by participation in and facilitation of discussions addressing problems of practice (Putnam & Borko, 2000). A situative perspective asserts that a purposeful progression of activities provides opportunities for learning, and that all learning occurs in a situation (Greeno, 1998). This perspective emphasizes the social interactive development of understanding and individual identity development during participants' engagement in activities. It views knowledge as distributed among participants and their context, including the artifacts, tools, materials, and their communities (Greeno, Collins, & Resnick, 1996). In addition, this perspective highlights the activities and problems of practice that arise from participating in various learning communities and interrogates how those activities and problems can change with changes in context.

For this study, we observed the TLs in multiple spaces of learning and facilitation. During LSMs, TLs engaged with problems of practice as well as learning activities to promote inquiry and sense-making among their colleagues. The school site contexts varied across TLs, presenting situations where TLs first considered the individual and collective goals of their colleagues and departments, and then examined how the focus of VBDs and their facilitation could be adapted to address these differentiated goals. A situative perspective acknowledges the possibility that different school settings may give rise to different kinds of learning. Therefore, we analyzed TL facilitation moves during LSM: Rehearsal and Planning as well as during their own school-site PSC workshops, considering how TLs adopted or adapted facilitation moves and activities as they moved between LSM and PSC contexts.

Methods

Data Sources

This study focuses on the LSMs and PSC workshops conducted during the 2016-2017 school year, which was the first year of full implementation of the four-year university/UUSD research-practice partnership. Sixteen TLs from seven different middle schools within UUSD were involved in the RPP during the 2016-2017 school year. The data sources for the analyses reported in this study include video and audio recordings from the six LSMs and corresponding TL-facilitated PSC workshops conducted that year. Records represent four different school sites. Three sites were omitted based on missing video recordings from either the LSM: Rehearsal and Planning or corresponding PSC workshop. Thus, we examined video records from 24 rehearsals (one rehearsal by TLs from each school during each LSM) and 24 PSC workshops.

Analysis

Video and audio data sources were coded using a video-analysis software called Angles (fulcrumtec.com). Initial codes from frameworks for facilitation and orchestration of video-based discussions by Borko and colleagues (2014) and van Es and colleagues (2014) were adapted by the first and second authors. These frameworks include a total of seven main practices: orienting group to the video analysis task; eliciting teacher's thinking about the lesson segment; probing for evidence of their claims; sustaining an inquiry stance; maintaining a focus on the video and the mathematics; supporting group collaboration; and helping the group to connect their analyses to key mathematical and pedagogical ideas. Additionally, moves and activities that did not precisely fit in this framework were identified, discussed and categorized by a larger research team of mathematics educators. This process resulted in the addition of two practices: debriefing the facilitators, and TL adaptation for own context.

Results

The following results are preliminary in nature and offer broad themes found across each TL facilitation context. TL participation in the debriefs of VBDs modeled during Leader Support Meetings increased over the course of the school year. Early debriefs consisted primarily of one project team member asking questions of the LSM: Modeling facilitators to "pull back the curtain" on their thinking. Later sessions included more questions by TLs and fewer by LSM leaders. During LSM rehearsals, most TLs practiced using focal questions to launch their video-based discussions. We observed a shift in the nature of TL participation during rehearsals from "walking through their presentations" more toward approximation of practice.

Not all the facilitation moves rehearsed transferred to the PSC workshops. Shortened time for VBDs seemed to limit TLs' enactment of facilitation moves that the foster high-quality discussion (e.g. connecting ideas, distributing participation). Time constraints affected each site in our sample; every site had at least one PSC workshop limited to 30 minutes because of various context-related constraints such as shortened department meeting times, mandated testing discussions, and other school/district goals for the department. Differences were also found in the nature of facilitation across PSC workshops. Regardless of context-based constraints, some PSCs included more facilitator-directed whole-group discussions during the VBDs, limiting the time that participating teachers had to respond or share insights with their colleagues, and using pair-share or small group participation in a limited manner. The TLs in these situations also had varying patterns of facilitation participation; some TLs would take turns facilitating rather than co-facilitating, while other TLs participated as a colleague teacher rather than facilitator. Other PSC workshops had more participant-centered VBDs and used a variety of participation structures. TLs at these sites typically co-facilitated more; for example, one TL facilitated a discussion while another TL recorded responses and occasionally asked clarifying or follow-up questions. Some of the PSCs also included intentional connections to problems of practice in their VBDs to elicit colleague teachers' ideas that could address the needs of their students.

Discussion

This study contributes to the growing number of research projects that are focused on what PD leaders should know and be able to do, and how to best prepare and support them (Elliott et al., 2009; Even, 2008; Koellner et al., 2011). More specifically, it builds on the call to researchers to develop Mathematical Knowledge for Professional Development (MKPD) proposed by Borko, Koellner, and Jacobs (2014) and the framework for organizing and categorizing facilitation moves during video-based discussions developed by van Es and colleagues (2014). We extend this work by examining the nature of novice facilitators' activities, practices, and moves enacted during rehearsals and then incorporated into their PD workshops. Our findings support using rehearsals as an approximation of practice (Grossman et al., 2005) for novice facilitators. They also highlight the importance of considering how facilitation can be affected by contexts such as school-based constraints when preparing novice facilitators. Future research should address both adapting to school-based constraints and facilitation that fosters discussions of equitable teaching practices.

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A LONGITUDINAL STUDY OF INSERVICE TEACHERS' VISION FOR TEACHING WITH TECHNOLOGY

Suzanne R. Harper
Miami University
HarperSR@MiamiOH.edu

Dana C. Cox
Miami University
Dana.Cox@MiamiOH.edu

Keywords: Instructional Vision; Teacher Beliefs; Technology; High School Education

For almost two decades, Miami University has prepared preservice secondary mathematics teachers (PSMTs) to teach with technology. PSMTs take a required course where they revisit their own learning of secondary mathematics and investigate concepts by way of problem solving with various technological tools. In previous work, we sought to understand the impact of this course on future teaching practice. We defined *vision of teaching with technology* as "an imagined state wherein PSMTs are able to translate their technological beliefs into principles on which they will base future instructional decisions and practice," (Cox & Harper, 2016, p. 169). We found that as a result of participation in this course, PSMTs develop a vision of teaching with technology that is better aligned with that expressed in modern policy documents (i.e., National Council of Teachers of Mathematics, 2000) (Cox & Harper, 2016). We also found that PSMTs draw heavily on their index of personal experiences to illustrate their visions and that descriptions of curricular experiences were central of what PSMTs referred to as "responsible use of technology" (Cox & Harper, 2016).

The current study is a longitudinal examination of the development of a vision of teaching with technology. Since beliefs and practice are dialectic (Thompson, 1992) it is likely that an individual's vision has been impacted by actual teaching experience. Given the importance of indexing personal experiences when articulating a vision, we will likely be able to discern key professional events and activities that point to both continued evolution and devolution of beliefs. No longer limited by imagined states, we have an opportunity to answer the questions: (1) What are seminal teaching experiences that impact an individual's vision of teaching with technology? (2) What is the longitudinal value of collegiate curricular experiences in shaping teachers' vision of teaching with technology?

Participants in this study are recent graduates of Miami University who have been teaching secondary (6-12) mathematics since graduation. Participants were asked to complete an online survey. The survey includes three lines of inquiry. First, we invite participants to narrate their experiences with teaching mathematics with technology. Second, we provide participants with their original vision statements written at the conclusion of the technology course and invite them to identify passages that still represent their thinking as well as passages about which they now think differently. Lastly, we ask participants to identify experiences such as graduate study, professional development, or classroom episodes as well as the impact (or lack thereof) they have had on their developing vision of teaching with technology. A complete analysis of data with findings will be included on this poster.

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A PROFESSIONAL DEVELOPMENT FRAMEWORK TO SUPPORT INCLUSIVE ELEMENTARY MATHEMATICS INSTRUCTION

Stefanie D. Livers
Missouri State University
stefanielivers@missouristate.edu

Kristin E. Harbour
University of South Carolina
kharbour@mailbox.sc.edu

Sara C. McDaniel
University of Alabama
scmcdaniel@ua.edu

Jim Gleason
University of Alabama
jgleason@ua.edu

Nicolette Nalu
AMSTI
nnalu@montevallo.edu

While meaningful mathematics for all students is the expectation stipulated by leading organizations, the noted opportunity and achievement gaps between students with and without disabilities remains. Given the current climate based on accountability, rigorous standards, and increased student diversity, professional development is a necessity for teachers. We present a practical, three component framework design to in-service professional development for general and special educators for improving inclusive mathematics instructional practices. The framework hinges on the use of high-quality mathematics tasks, differentiation strategies, and effective co-teaching to create a change in both instructional and collaborative practices.

Keywords: Co-teaching, Differentiation, High-quality mathematics, Professional development

The growing number of students with identified disabilities served in general education classes (Mayrowetz, 2009) necessitates the need for lessons to be differentiated to provide meaningful learning experiences for all students (National Council of Teachers of Mathematics [NCTM], 2014). To do this, teachers need support as they work to meet these high expectations for a diverse student population. We present a practical professional development framework that was developed across a three-year service delivery project.

Conceptual Framework

The components of the conceptual framework included opportunities for growth with (a) co-teaching strategies, (b) differentiated instructional strategies, and (c) implementing these high-quality mathematics tasks. The initial emphasis of our conceptual framework centered on increasing the rigor and engagement through the use of high-quality tasks (Stein, Smith, Henningsen, & Silver, 2001). This was followed by the inclusion of differentiation strategies to meet the diverse student needs (Tomlinson, 2017), and then effective and efficient co-teaching models (Friend & Cook, 2010) were of focus to ensure a collaborative atmosphere was in place and that the expertise of both general and special education teachers were capitalized. The relationship of the components of the presented framework were critical for successful teaching and learning of mathematics. The use of high-quality tasks provided the basis of meaningful mathematics instruction. However, without differentiation strategies for the task, some students may not have access to the mathematics. Coupled with the support of both the general and special educators, all students can be supported in their problem-solving approaches, meaning-making, and solutions of the high-quality task.

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ACTIVIDADES DIDÁCTICAS DEL LÍMITE DE UNA FUNCIÓN PARA PROFESORES DE EDUCACIÓN MEDIA SUPERIOR USANDO TEORÍA APOE

José Javier Guerrero Maldonado
Benemérita Univ. Autónoma de Puebla
jjguerre@unet.edu.ve

Lidia Aurora Hernández Rebollar
Benemérita Univ. Autónoma de Puebla
lhernan@fcfm.buap.mx

Múltiples investigaciones enfocadas en el concepto de límite permiten concluir que los estudiantes presentan dificultades que retrasan su proceso de aprendizaje. La investigación presentada se enfoca en docentes de preparatoria como integrantes del triángulo didáctico. Basado en la Teoría APOE se plantea el diseño de una secuencia de actividades didácticas que contribuyan a enfrentar de manera óptima situaciones problemáticas donde dicho concepto se encuentre involucrado. Las actividades se organizaron en dos bloques, el primero a manera de diagnóstico y el segundo de desarrollo de las estructuras y mecanismos mentales necesarios para la construcción del concepto de límite de una función.

Palabras claves: Pensamiento matemático avanzado, Conocimiento del profesor, Actividades y prácticas de enseñanza.

Con este trabajo se busca contribuir, a través del diseño de una secuencia de actividades didácticas basadas en la Teoría APOE (Acción, Proceso, Objeto y Esquema), a la promoción del nivel estructural de un concepto fundamental del cálculo diferencial como lo es el del límite de una función en un grupo de profesores. El método a seguir está basado en la Teoría APOE (Arnon, Cottrill, Dubinsky, Oktaç, Fuentes, Trigueros, & Weller, 2014) en cuanto al análisis de las descomposiciones genéticas existentes (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas & Vidakovic, 1996) y el diseño de las actividades, tomando como inspiración las propuestas por Tomás (2014).

Estas actividades utilizan las representaciones analítica, tabular y gráfica del concepto de función y se aplicaron a doce profesores asistentes a un curso de 20 horas con la finalidad de favorecer en ellos la concepción métrica y de aproximación del límite de una función. En algunos casos los docentes observaron la aproximación de las imágenes de una función mientras los valores del dominio se aproximan a un número dado en una tabla y, en otros casos, observaron este proceso gráficamente. Teóricamente, la coordinación de estos procesos lleva al docente a la concepción de aproximación. Otras actividades semejantes se proponen para efectos de la concepción métrica. Se espera que la coordinación de los procesos de estas dos concepciones favorezca la concepción Proceso del límite de una función en el docente. A continuación, se presenta el contenido de una de las actividades con parte de su análisis:

Dada $f(x) = \begin{cases} 2x+1 & x<0 \\ -2x-3 & x \geq 0 \end{cases}$ Se pide completar la tabla siguiente:

x	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1
$f(x)$									

Para realizar esta actividad el docente debe realizar la acción de evaluar una función definida por partes en algunos valores del dominio que se aproximan a cero. También se solicita describir el comportamiento de la variable dependiente respecto a la independiente, para determinar si hay

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coordinación o no en el proceso de aproximación en el dominio cuando *x* tiende a cero por la izquierda con la aproximación en el rango cuando *f(x)* tiende a 1 o cuando tiende por la derecha, con el proceso de aproximación en el rango cuando *f(x)* tiende a -3.

En este trabajo se muestra cómo se corresponden dos bloques de actividades con la teoría y, en un estudio posterior, se reportará el impacto de su aplicación en los participantes. Por lo que concluimos que la actividad de enseñanza del concepto de límite puede ser modificada al aplicar la Teoría APOE para promover el desarrollo de estructuras mentales más elaboradas en los profesores.

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ACTIVIDADES DIDÁCTICAS DEL LÍMITE DE UNA FUNCIÓN PARA PROFESORES DE EDUCACIÓN MEDIA SUPERIOR USANDO TEORÍA APOE

José Javier Guerrero Maldonado
Benemérita Univ. Autónoma de Puebla
jjguerre@unet.edu.ve

Lidia Aurora Hernández Rebollar
Benemérita Univ. Autónoma de Puebla
lhernan@fcfm.buap.mx

Multiple investigations focused on the concept of limit allow us to conclude that students present difficulties that delay their learning process. The presented research focuses on high school teachers as members of the didactic triangle. Based on the APOS Theory, is considered the design of a sequence of didactic activities that contribute to optimally deal with problematic situations where this concept is involved. The activities were organized in two blocks, the first in a diagnostic manner and the second in the development of the structures and mental mechanisms necessary for the construction of the concept of the limit of a function.

SEEING MATHEMATICAL LEARNING: TEACHER NOTICING AND GESTURE

Caro Williams-Pierce
University of Maryland, College Park
caro.williams@gmail.com

Janet Walkoe
University of Maryland, College Park
jwalkoe@umd.edu

Keywords: Teacher Knowledge, Communication, Technology

Students gesture while learning mathematics, and will often gesture new concepts before they can verbally communicate their budding understanding (Alibali & Nathan, 2012). In addition, gestures can complement spoken or written language, communicating additional and unique mathematical information (Pier et al., 2019; Williams et al., 2012; Williams-Pierce et al., 2016). Given that gestures play such a crucial communicative role, an equally crucial question is: how do teachers notice gestures when evaluating student learning?

Teacher noticing has been identified as an important construct of teacher knowledge (Sherin, Jacobs, Philipp, 2011). There is consensus that teachers' attention to the substance of students' ideas has important implications for instruction (e.g., Ball, Lubienski & Mewborn, 2001; Fennema et al., 1996). Oftentimes, however, the focus has been on teacher noticing of students' verbal or written expression of ideas (e.g., Sherin & van Es, 2009). Our motivation for examining teacher noticing of gesture is two-fold: 1) teachers may be missing crucial mathematical information when not attending to gesture that could otherwise greatly improve student mathematical learning; and 2) attending to spoken and written language means privileging those who are adept with English mathematical language – by supporting teachers in noticing gesture, we lessen that privileging. To that end, this poster will report on a pilot study that has been designed and will be conducted with up to 30 in-service teachers in April, 2019.

In Phase 1 of the study, participants will view three videos of students talking and gesturing about mathematics from the second author's previous research, which we chose specifically because these students communicate unique mathematical information in gesture. These videos will be viewed on LessonSketch, an online platform that allows viewers to drop timestamped "pins." They can drop pins of two different colors: red when they see powerful evidence of mathematical learning; and blue when they wish they could step into the video and ask the student a question. After each video, participants will be presented with text boxes that ask them to describe the mathematics learning that they noticed in the video, and to elucidate what questions they still have about the student's learning.

In Phase 2, the participants will read a short paper by the first author (Williams-Pierce et al., 2012), which specifically addresses the role that gestures can play in communicating mathematical learning. They will follow this by Phase 3: similar to Phase 1, but with an additional question about what they noticed differently this second time watching the videos. We hypothesize that participants will be unlikely to notice – or be explicitly aware of what they are noticing about – what the gestures communicate about student learning during Phase 1, but that the brief article will serve as a small intervention that will increase their ability to notice relevant mathematical gestures in Phase 3. This pilot study will serve to confirm or disconfirm these intuitions. Alongside reporting on the results of this study, we plan to share our plans for a larger follow-up study, and look forward to gaining additional advice and insights from the poster session attendees.

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REFLECTION, CASE STUDIES, AND DISCOURSE: COMPONENTS FOR TRAINING NEW UNDERGRADUATE MATHEMATICS TUTORS

Joseph Antonides
Ohio State University
antonides.4@osu.edu

Carolyn Johns
Ohio State University
johns.125@osu.edu

Keywords: Undergraduate-level; Instructional activities and practices

Introduction and Rationale

Almost all colleges and universities in the United States house a mathematics tutoring center, most often staffed by undergraduate students majoring in STEM disciplines (Johnson & Hanson, 2015). Altogether, these tutoring centers cater to nearly 20 million undergraduate students annually (National Science Board, 2018). These tutors, while typically prepared mathematically, often have had little to no pedagogical training or teaching experience. This trend raises important concerns regarding the preparation of tutors, an issue rarely studied rigorously in the literature.

In this study, we analyzed four 55-minute tutor training sessions to capture specific ways that "apprentice" mathematics tutors' pedagogical capacities were developed. Particular focus was on the teacher's decisions in facilitating the training sessions and the ways in which the teacher attempted to raise mathematical knowledge for teaching among tutors. Our study was guided by the following research question: what are the major approaches used to develop apprentice undergraduate mathematics tutors' pedagogical capacities during these training sessions?

Context, Methodology, and Framework

This study was conducted at a large, public university in the Midwest region of the U.S. The researcher observed four 55-minute sessions during which newly-hired "apprentice" tutors were trained to perform as undergraduate mathematics tutors. The range of courses tutors were expected to support included: remedial mathematics, college algebra, trigonometry, Calculus I, II, and III, differential equations, as well as mathematics courses for business and biological sciences. Audio recordings were collected and analyzed through a sociocultural framework (Smagorinsky, 2007, 2018).

Findings and Implications

The sessions promoted reflection and discussions regarding what it might mean to know mathematics and how student learning might be traced. The most prominent approach in successfully gauging discussion and reflections on effective pedagogical moves included a reliance on an examination of case studies of student-tutor interactions that depicted both effective and problematic tutoring practices. This approach promoted not only the greatest level of engagement by the tutors, but also allowed the facilitator to identify effective pedagogical moves that could be utilized by the tutors in their future work.

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INFLUENCES ON THE USE OF INSTRUCTIONAL FRAMEWORKS IN MATHEMATICS CLASSROOMS

William S. Walker, III
Purdue University
wswalker@purdue.edu

Signe E. Kastberg
Purdue University
skastber@purdue.edu

Keywords: Instructional Frameworks, Professional Development, Practical Knowledge

Instructional frameworks can provide teachers with a vision for research-based instruction and a guide to support changes (e.g. National Council of Teachers of Mathematics, 2014). Three such frameworks were used in a mathematics professional development program. The frameworks included *Launch, Explore, Summarize* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002); *Math Talk* (generated within the program) and *Positive Norms for Math Class* (Boaler, 2017). These frameworks addressed needs identified by the teachers. This research investigated the question, how do teacher practical knowledge and the situated context of the school relate to the use of instructional frameworks in participants' classrooms?

The concepts of practical knowledge and the situated nature of the school were central considerations in the design of the professional development program for this research. Practical knowledge includes the knowledge teachers have of instructional techniques, students' learning styles, classroom management skills, and the appropriate place and time to use this knowledge (Chapman, 2011; Elbaz, 1983). A situated perspective recognizes knowledge as the product of the context and culture where it is used (Brown, Collins, & Duguid, 1989) and allows researchers to view actions as part of the larger system of social practices (Cobb & Bowers, 1999).

Subjects for this study were eight teachers at a Midwestern middle school who completed a 14-month mathematics professional development experience. Teachers were interviewed to learn how practical knowledge and school context influenced the use of instructional frameworks. Data consisted of field notes from the interviews with researcher reflections and artifacts from the professional development. An inductive approach of comparative pattern analysis was used to create a category system for the data (Merriam, 2009). For example, the statement "I feel free to use or not use different activities" was categorized as "autonomous decision-making."

Findings

Elements of teacher's practical knowledge and the situated context of the school supported and constrained the use of the instructional frameworks. *Math Talk* and *Positive Norms* were widely used because they were consistent with ways the teachers understood their classroom learning environment, like including group work and having multiple solution methods. At the same time, short class periods (45 minutes) and planning time needed to create lessons aligned with the frameworks limited their use. Similarly, the situated context of the school influenced the use of the instructional frameworks. The teachers stated that support from school leadership allowed them to make autonomous decisions about instructional strategies. Additionally, teachers noted a lack of a school-wide curriculum, which made it difficult to integrate aspects of the instructional frameworks. The professional development providers worked with participating teachers to select the instructional frameworks, yet constraints on the use of the instructional frameworks could not be anticipated. This finding suggests that having prior long-term relationships with the teachers and spending extended time as a participant in the school may be necessary conditions for the development or selection of productive instructional frameworks.

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THE MANY FACES OF "NOT ENOUGH TIME"

Victoria R. Jacobs
University of North Carolina at Greensboro
vrjacobs@uncg.edu

Susan B. Empson
University of Missouri
empsons@missouri.edu

Naomi Jessup
Georgia State University
njessup@gsu.edu

Gladys Krause
William and Mary
ghkrause@wm.edu

Keywords: Teacher Education-Inservice/Professional Development, Instructional Activities and Practices, Elementary School Education

When teachers work to improve their instruction following professional development (PD), the tensions they encounter often undermine the positive effects of PD (Kennedy, 2005). To begin to explore this phenomenon, we documented and categorized the tensions articulated by 92 teachers who participated in our design study focused on the teaching and learning of fractions in grades 3–5. The study included 3 years of PD that used research-based frameworks of children's thinking and instructional practices to encourage a vision of instruction that is responsive to children's mathematical thinking (Jacobs et al., in press; Richards & Robertson, 2016).

Based on a thematic analysis of 72 focus groups across the 3 years of PD, we identified "not enough time" as a significant tension teachers encountered, but this tension had many faces, as illustrated by the following two categories. First, teachers identified time tensions linked to their *instructional contexts*. Some of these tensions were general in nature (e.g., school or district policies that reduced instructional time) while others were mathematics-specific (e.g., pacing guides that controlled use of instructional time). Second, teachers identified time tensions linked to the *instructional vision* in the PD. Some of these tensions focused on time-consuming instructional practices that centered children's ideas (e.g., whole-class discussions based on children's strategies) while others reflected teachers' understanding that supporting children's mathematical sensemaking often takes longer than showing procedures. (Figure 1 exemplifies the two categories.) Our overarching goal is to better understand the tensions teachers encounter when working to implement what they learn in PD so that we can help them view these tensions as manageable, even necessary, rather than as roadblocks (Braaten & Sheth, 2017).

Tensions Linked to Instructional Contexts	Tensions Linked to Instructional Vision
"We are getting more district mandated things. We lose 40 minutes a day for intervention and then we lose 40 minutes a day for [a language program] and so the amount of time that we have to teach is a lot shorter. So I mean, like the district is saying you have got all these opportunities and everything, but as far as how much teaching we get to do, it is less and less each year."	"This whole philosophy of students sharing strategies and things like that. A typical math lesson—those things take time. Just fitting it all in and still having time for reading and writing and everything else because this type of math is wonderful but it does take a lot longer than the traditional "turn to page 51 and do the"— you know."

Figure 1: Sample Teacher Comments Illustrating Two Categories of Time Tensions

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SECONDARY MATHEMATICS TEACHERS OVERCOMING DIFFICULTIES IN AN ONLINE REAL ANALYSIS COURSE

Kanita DuCloux
Western Kentucky University
kanita.ducloux@wku.edu

Keywords: Teacher Education - Inservice/Professional Development, Advanced Mathematical Thinking, Instructional Activities and Practices

Topics in real analysis such as understanding and constructing proofs and creating counterexamples have a history of being important yet difficult topics for many students (Doruk and Kaplan, 2015; Weber, 2001). The current ways, however, in which these topics are taught, may not be helpful to prospective secondary mathematics teachers when they begin teaching. Thus, a team of researchers (Wasserman, Fukawa-Connelly, Villanueva, Mieja-Ramos, & Weber, 2017) proposed an alternative model to teach an analysis content course for secondary teachers. The model contains 12 modules that are closely aligned with Abbott's *Understanding Analysis* (2nd edition). The goal of this exploratory study was to examine the difficulties that secondary mathematics teachers experienced while engaging in real analysis teaching modules. Any of the 12 modules could be used to examine the teachers' difficulties but given the online setting and time constraints only nine modules were used. Chapters 1 and 6 each had one module aligned with the content, thus both modules were selected. Chapters 2 and 4 each had three modules, so two were chosen from each chapter. There were no modules for Chapter 3. Chapters 5 and 7 each had two modules. Only one was selected from chapter 5 (both addressed differentiation) and both from chapter 7 (Riemann Sums and the Fundamental Theorem of Calculus).

Four secondary mathematics teachers enrolled in the researcher's analysis course for an online master's program participated in the study. During nine weeks, the teachers completed the modules and textbook problems and participated in required asynchronous discussions. They also completed reflection questions where they discussed important concepts, difficulties encountered, and the relevance of the modules. The reflection question concerning difficulties, from each module, was coded using Doruk & Kaplan's list of seven types of student difficulties. The findings revealed that all four teachers struggled with writing convergence proofs, integrating the bell curve, as well as distinguishing among various concepts and definitions. The teachers reported relying heavily on the textbook, online discussion boards, and videos to overcome difficulties encountered while engaging in the modules. This study has implications for the teaching and learning analysis to increase its relevance for teachers.

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MIDDLE LEVEL TEACHERS' ENGAGEMENT IN THE STATISTICAL PROBLEM-SOLVING PROCESS

Christopher Engledowl
New Mexico State University
chriseng@nmsu.edu

Tracey Gorham Blanco
New Mexico State University
tgblanco@nmsu.edu

Mohammad Al-Younes
New Mexico State University
mmigdadi@nmsu.edu

Keywords: Data Analysis and Statistics; Teacher Education–Inservice/Professional Development

An important component of professional learning of statistics content is engagement in the statistical problem-solving process (Burgess, 2011; Franklin et al., 2007)—also referred to as the practice of statistics (Watson et al., 2018). This cycle involves formulating a question, collecting data, analyzing data, and interpreting results (Franklin et al., 2007). The research base is relatively weak for studies that involve the entire cycle, with very few including the component of formulating questions (Watson et al., 2018), and even fewer that document this process among practicing teachers (Burgess, 2011). Burgess and Watson et al. both call for more research in this area, with Burgess (2011) proclaiming that "part of the core work of teachers involves responding to students' questions, and evaluating students' responses to questions or tasks. Without appropriate teacher knowledge, such work can be compromised" (p. 267). This study aims to contribute to this research base by documenting middle level teachers' engagement in the entire statistical problem-solving process within a professional learning context.

This qualitative study involved 7 middle school mathematics teachers of statistics, who all taught in diverse settings in 3 districts in the Southwestern United States. Teachers exhibited a range of teaching experience (1–40+ years) and were observed to be at both beginner (Level A) and intermediate (Level B) levels of statistical literacy level, according to the levels in the GAISE report, as measured by the Levels of Conceptual Understanding in Statistics (LOCUS) assessment (Jacobbe, 2016)—administered prior to the start of the professional learning sessions.

This study is part of a larger project examining these teachers' engagement in the statistical problem-solving process across all three days of the professional learning session, which was facilitated by the authors. This study focuses on teachers' whole class discussion as they engaged in all components of the process during the Raisins task which took place across 4 hours during the first day of professional learning. During the Raisins task, teachers were provided with small boxes of raisins. They then engaged in formulating statistical questions about the boxes of raisins, making predictions about the answer to their questions, collecting data, analyzing data, and interpreting results. Data comes from a video-recording of the session and teachers' written work. Video data was analyzed by first identifying the moments when each component of the problem solving process, as described in the GAISE report (Franklin et al., 2007), was occurring. Then, teachers' talk was coded for which statistical literacy level (A, B, or C) it most fit within.

Analysis is currently on-going and full results will be shared on the poster. However, preliminary results indicate that, across all components, teachers mostly were observed to be within Level B. Moreover, it appears that the formulating questions component of the task enabled teachers to participate in higher levels of statistical literacy in later components. For instance, discussion about specific words to include in the statistical question allowed teachers to analyze data drawing on multiple statistical summaries of the data—such as using a broader term like *typical* rather than *mean*. Results of this study respond to calls for more research on teachers' engagement in the statistical problem-solving process (Burgess, 2011; Watson et al., 2018).

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ANALYZING DESIGNS FOR TEACHER ONLINE DISCOURSE AROUND VIDEOS OF PRIMARY STUDENTS' MATHEMATICAL THINKING

Sarah Larison
Northwestern University
slarison@u.northwestern.edu

Jennifer Richards
Northwestern University
jrichards@northwestern.edu

Miriam Sherin
Northwestern University
msherin@northwestern.edu

Keywords: Teacher Education-Inservice/Professional Development

Discussing classroom video, with a focus on students' mathematical thinking, can be a generative forum for teacher learning (e.g., Cohen, 2004; van Es & Sherin, 2008, 2010). However, in online settings, opportunities and challenges arise in supporting productive discourse (Falk & Drayton, 2009). This study explored primary mathematics teachers' discourse around video from each other's classrooms during a six-week online course, asking: How might different designs engender productive teacher conversation around video, particularly about student thinking?

We analyzed teachers' contributions within two different designs: comments teachers made about peers' videos and reflections on a discussion board (N=39) and tags teachers made directly on peers' videos (N=112). We drew on conceptions of substantive discourse from prior work (van Es & Sherin, 2008, 2010) coding contributions where teachers described and/or interpreted specific students' mathematical thinking and provided evidence for their statements as "substantive." We also coded for two emergent dimensions of productivity: whether contributions were "connected" (meaning explicitly tied to preceding contributions) and "expansive" (went beyond the bounds of the video).

Table 1 compares teachers' contributions across the two designs.

Table 1: Percent of Productive Dimensions of Teacher Contributions Across Designs

	Comments	Tags
% Substantive	36% (15/39)	50% (56/112)
% Connected	36% (14/39)	11% (12/112)
% Expansive	69% (27/39)	10% (11/112)

More tags were coded as "substantive" than comments. In contrast, comments extended beyond the video more frequently, often to teachers' own classroom settings. While few contributions met criteria for productivity across all dimensions (5% of comments, 0% of tags), 26% of comments (10/39) were coded as *both* substantive and expansive, representing occasions when teachers noticed specific moments of students' mathematical thinking in a peer's artifact and compared to what they saw from their own students — which did not occur when tagging.

Thus, both designs for teacher discourse used in this study supported some productive teacher discourse around student thinking, but in different ways. We envision each design serving a different niche in online teacher learning using video as a tool. Moving forward, we will explore relationships between productive discourse, the nature of classroom mathematical tasks, and the substance of the videos and reflections shared.

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SCHOOL-BASED PROFESSIONAL DEVELOPMENT TO ADDRESS EQUITY IN MATHEMATICS INSTRUCTION

Sara Birkhead
George Mason University
sbirkhea@gmu.edu

Keywords: Equity and Justice, Instructional activities and practices, Teacher Education-Inservice/Professional Development

High-quality professional development (PD) is often provided through university-based partnerships and requires teachers to attend sessions in the summer or in evenings after school. While this PD does have positive impact on teacher practice (Desimone, Porter, Garet, Yoon, & Birman, 2002), it is not available to all teachers, nor does it take into account the contextual differences of teachers' work sites. This poster will present a PD model utilizing teacher inquiry as well as a video coaching platform to provide school-based PD that is responsive to teacher contexts and can be implemented with minimal input of both human and monetary resources.

Professional Development to Address Issues of Equity

The distribution of teacher quality in the United States is notably inequitable. Students who are most in need have the least qualified teachers (Goldhaber, Quince, & Theobald, 2018). Impacting this distribution requires that all teachers, especially those in under-resourced schools, have access to high-quality PD. School-based PD models are more accessible to teachers and thus offer a greater potential for instructional change than traditional off-site centers of expertise.

School-based PD Using Facilitated Teacher Inquiry

Teacher research has been found to have a positive impact on teacher practice (Cochran-Smith & Lytle, 2009). The PD model shared here uses a teacher inquiry structure and a video coaching platform to facilitate teacher reflection as well as interaction with video of teachers' practice as data (see Figure 1). The model described is centered in individual classrooms, making it responsive to all elements of context.

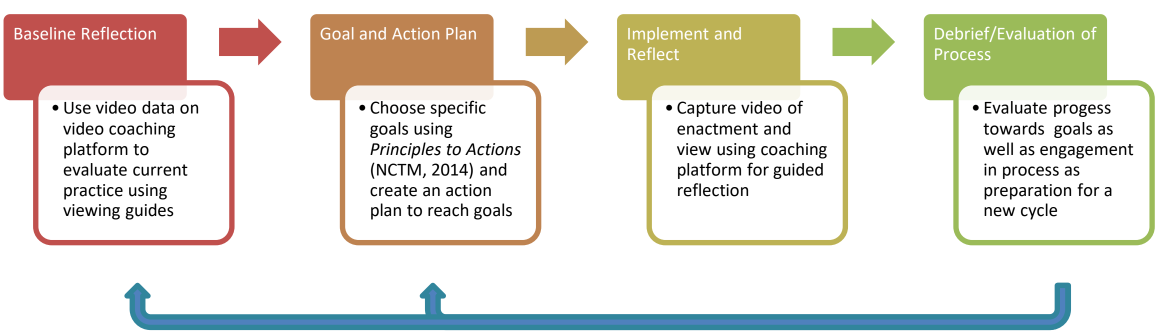

Figure 1: Teacher Inquiry Cycle Supported by Use of a Video Coaching Platform

The goal of this model is both to provide effective PD with minimal investment of resources and also to create a sustainable focus on instructional improvement in the classrooms of

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

participating teachers. By participating in the teacher inquiry, it is hoped that teachers will continue the cycle, setting new goals and reflecting on their practice.

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CONCEPTUALIZATIONS OF AND ACCESS TO RESOURCES FOR RURAL MATHEMATICS TEACHERS' MINDSETS AND IDENTITIES

Shande King
The University of Tennessee
nking5@vols.utk.edu

Nick Kim
The University of Tennessee
nkim2@vols.utk.edu

Dr. Lynn L. Hodge
The University of Tennessee
lhodge4@utk.edu

Michael Lawson
The University of Tennessee
mlawso13@vols.utk.edu

Keywords: Teacher Education-Inservice/Professional Development; Teacher Beliefs; Affect, Emotion, Beliefs, and Attitudes

Mathematics teacher education is emphasizing a greater need to understand the relationship between mathematics teachers and the resources they use in their instruction (Lee, 2012). Particularly, more impoverished schools within rural contexts are recognized as lacking in resources and needing to close the gap to retain mathematics teachers in these areas (Roberts, 2004; Yarrow, Ballantye, Hansford, Herschell, & Millwater, 1999), but further research is needed to understand current teachers' perceptions of themselves and their mindsets in approaching mathematics instruction in their rural contexts.

In a multiple summer professional development for mathematics teachers in a rural county, K-8 mathematics teachers participate in three two-week summer mathematics institutes that emphasized the learning of math pedagogy and content in the context of their rural setting. This professional development institute recognized the needs of these rural teachers and provided both pedagogical and physical resource support on current research-based mathematical methods in K-8 classrooms, including STEM literature and other manipulatives. Data was collected via surveys, daily reflections, and interviews concerning the teachers' pedagogical methods before and after the institute and their mindsets about themselves as successful math teachers and math people. Through a situated learning lens (Lave & Wenger, 1991), this study examines the development of these rural math teachers' conceptualizations of resources and their view on access to resources in the data analysis, particularly as they considered their own identities as math teachers and math people. The professional development institute itself adopted an approach of focusing the teachers' views of resources on the students learning so that they could consider the modification of certain tasks when resources were not accessible to them in their rural contexts, as well as the conceptualization of what a resource is to be successful in mathematics teaching in these rural contexts. Thus, research questions in this study were:

How do rural teachers conceptualize resources in the mathematics classroom?
How does access to and mindsets about access to resources affect rural mathematics teachers' identity as mathematics teachers?

Findings indicate that teachers in rural contexts began to conceptualize resources as necessary characters in the mathematics classroom to successfully implement research-based mathematics methods. Further, their qualitative responses indicate a direct connection between these teachers' conceptualizations and attempts for implementation of these teaching methods.

These findings have implications for future research to understand how to support rural teachers' access to resources, as well as how to assist rural teachers in reconceptualizing different resources to change their views and identities of themselves as math people and math teachers.

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DESIGNING FLEXIBLE, WEB-BASED FORMS OF PROFESSIONAL DEVELOPMENT FOR SUPPORTING MATHEMATICS TEACHER LEARNING

Angela Knotts
WestEd
aknotts@wested.org

Elizabeth B. Dyer
WestEd
edyer@wested.org

Nanette Seago
WestEd
nseago@wested.org

Keywords: Teacher Education-Inservice/Professional Development; Technology

Incorporating video in the professional development (PD) environment offers great potential for teachers to unpack the relationships among pedagogical decisions and practices, students' work, and the disciplinary content (Borko, Koellner, Jacobs, & Seago, 2011). However, research on video-based PD focuses on face-to-face formats, which do not capitalize on the benefits of digital platforms for professional learning (e.g., scalability, flexibility to tailor to individual needs, financial/logistical accessibility). Research to date in online professional development has shown positive effects for teachers, even when compared to face-to-face (O'Dwyer, Masters, Dash, De Kramer, Humez, & Russell, 2010). For example, teachers who are reluctant to share ideas in face-to-face settings can be more comfortable doing so in digitally-mediated interaction (Dede, Jass Ketelhut, Whitehouse, Breit, & McCloskey, 2009). However, significant questions still remain for how to adapt professional learning programs from face-to-face to digital formats, especially in terms of the social dimensions of professional learning.

This study adapts the Learning and Teaching Linear Functions face-to-face PD materials (Seago, Mumme, & Branca, 2004), which support improvement in teacher and student knowledge of functions. The digital adaptation includes multiple self-contained modules that can be experienced in two-hour time frames, varied sequences, and in flexible collaboration formats. The modules place a video clip at the center, or "in the middle," of a professional learning experience as teachers take part in an online experience of mathematical problem solving, video analysis of classroom practice, and pedagogical reflection. Usability research, conducted in two phases, was completed with 15 mathematics teachers and teacher educators. The participants completed the activities in one digital module, and were surveyed about their experiences and the potential for the modules to support teacher learning. Participants' comments from the surveys were qualitatively analyzed to identify emergent themes (Glaser & Strauss, 1967) about positive experiences with design elements and individual suggestions for improvement.

Preliminary findings indicate that the adapted digital materials show promise for engaging teachers in similar learning as the face-to-face PD. Participants found two new design elements most useful: the lesson graph and comparing solution methods. The lesson graph gives context for the video clip by showing a sequential overview of the lesson the video comes from, which is chunked by the main activities. Comparing solution methods shows teachers a variety of solution methods uploaded by other participating teachers, similar to what they might experience when solving the math task in a face-to-face format. The results from the upcoming feasibility study, which will be shared in the poster, will allow us to compare teachers' experiences in three different settings: independent, locally-based facilitated, and project-facilitated.

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HOW TEACHERS REASON WITH THEIR MATHEMATICAL MEANINGS WHEN MAKING CURRICULAR DECISIONS

Porter Peterson Nielsen
Brigham Young University
porternielsen@gmail.com

Dawn Teuscher
Brigham Young University
dawn.teuscher@byu.edu

Shannon Dingman
University of Arkansas
sdingman@uark.edu

Travis A. Olsen
University of Nevada, Las Vegas
travis.olson@unvl.edu

Lisa Kasmer
Grand Valley State University
lisa.kasmer@gmail.com

Keywords: Geometric Reflections, Mathematical Meanings, Curricular Reasoning

Teachers' curricular decisions are important in mathematics classrooms because they determine the learning opportunities for all students. Students do not learn that which they are not given the opportunities to learn (Hiebert, 1999). Researchers have identified a number of influences on teachers' curricular decisions. For example, we know professional development, the written curriculum in the form of the district-adopted textbook or supplementary materials, and teachers' content knowledge all influence curricular decisions (Ball, & Cohen, 1996; Ball, Thames, & Phelps, 2018; Desimone, Porter, Garet, Yoon, & Birman, 2002). Despite our knowledge of influences on curricular decisions, few researchers have investigated *why* teachers make these decisions. In this poster I provide evidence addressing the following research question: *In what ways do middle grades teachers reason with their mathematical meanings when making curricular decisions about the definition for geometric reflections?*

Teachers' mathematical meanings (Thompson, 2016) provide insight into their reasoning about curricular decisions. Two teachers who make different curricular decision may have more in common than not. They may have graduated from similar universities or colleges and received a similar education. Both teachers may have similar mathematical knowledge, based on completing similar course work and scoring similarly on required state content assessments. However, these similarities may obfuscate the genesis for teachers' curricular decisions. By studying teachers' mathematical meanings we gain insights into teachers' reasoning and can identify similarities and differences across teachers as we study their curricular decisions.

Using data collected through our National Science Foundation study (NSF #1561542, #1561554, #1561569, #1561617), I created models of two middle grades mathematics teachers' mathematical meanings. By using these models I predicted concepts teachers would emphasize while teaching geometric reflections and I could accurately explain the teachers' reasons for their curricular decisions. Some of the curricular decisions that I investigated and explicate in this presentation concerned whether the teachers would emphasize the orientation of a figure in relation to reflections, if they would use perpendicular bisectors in defining reflections or merely rely on the concept of equidistance, if they would emphasize reflections on the coordinate grid or off the coordinate grid, and if they would seek to have the students learn coordinate rules for reflections.

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TEACHERS' VIEWS OF STUDENTS WITH DISABILITIES AS MATHEMATICALLY CAPABLE

Erica N. Mason
University of Missouri
ericamason@mail.missouri.edu

Keywords: beliefs, instructional vision

The majority of students with disabilities receive the majority of their instruction in general education settings, which positions general education teachers as the daily instructional decision-makers for these students. However, general education mathematics teachers report feeling instructionally unprepared to support students with disabilities in inclusive settings (DeSimone & Parmar, 2006), or view instruction as ineffective for these students (Jordan, Schwartz, & McGhie-Richmond, 2009). Two factors that may influence teachers' instructional decision making are their visions of high-quality mathematics instruction (VHQMI; Munter, 2014; Wilhelm, 2014) and their views of students as mathematically capable (VSMC; Jackson, Gibbons, & Sharpe, 2017). What then is the relation between these constructs and a student's disability status?

Data for this study come from interviews conducted with grades 3-8 general education mathematics teachers whose classes included students with disabilities, in one urban school district (n = 15) and one suburban district (n = 15), both in the Midwest U.S. Using a semi-structured interview protocol, I asked teachers about their instructional vision and then asked them to order students in one of their classes from "most struggling" to "least struggling" based on their own conception of "struggling". Teachers articulated what they considered the underlying reason for a student's learning struggle and described any instructional adjustments they made for that student including their rationale for doing so. All interviews were audio recorded and transcribed. I used Munter's (2014) VHQMI rubrics to characterize the level of sophistication of each teacher's instructional vision. I also used the VSMC coding scheme (Jackson et al., 2017) to characterize teachers' views of their individual students as mathematically capable and to characterize the instructional adjustments teachers reported making for those individuals. Responses about individual students were analyzed in relation to the student's disability status, and patterns within and across teachers were identified.

This poster will share findings and implications for in-service teacher development and learning.

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ANALYZING THE MEDIATED EFFECT OF ELEMENTARY MATHEMATICS SPECIALIST CERTIFICATION ON JUSTIFICATION OPPORTUNITIES

Christopher Austin
University of Southern Maine
christopher.austin@maine.edu

Keywords: Reasoning and Proof, Elementary School Education, Teacher Beliefs, Teacher Knowledge

It is critical that elementary teachers be prepared to support their students in developing the mathematical practice of justification. In a classroom, shared understandings of both the need to justify and what counts as a justification are important as members of the classroom community communicate mathematical ideas (Cobb, Wood, Yackel, & McNeal, 1992). This demands that elementary teachers possess knowledge of the mathematics involved, the practice of justification, and their students' understandings (e.g. Ball, Thames, & Phelps, 2008). Furthermore, teachers' beliefs about teaching and learning mathematics (e.g. Hill, 2011) and the nature of mathematics itself (Raymond, 1997) influence their instructional decisions. Given historical concerns about elementary teachers' preparedness, multiple states have developed elementary mathematics specialist (EMS) graduate certificate programs to improve elementary teachers' ability to provide high-quality mathematics instruction. Calls to study EMS graduates as classroom teachers (e.g., National Mathematics Advisory Panel, 2008) are only recently beginning to be answered. The purpose of this study was to investigate the relationship between elementary teachers' EMS status (i.e. certified, not certified), knowledge, beliefs, and attitudes, and the opportunities they provided students to engage in justification during typical classroom instruction.

Data for participating EMS (n = 28) and non-EMS (n = 33) teachers in Grades 3-5 included four subscales of the Learning Mathematics for Teaching instrument (Hill, 2011) as a measure of mathematical knowledge for teaching, and surveys utilizing 29 beliefs items and 30 attitudes items from existing instruments (Banilower et al., 2013; White, Way, Perry, & Southwell, 2006). Three teaching observations were also conducted, resulting in scores for eight elements of the classroom environment based on rubrics adapted from an existing observation protocol (e.g., Tarr & Soria, 2015). For analyses here, the average of the three scores for the element "Students created and defended mathematical justifications" is used as measure of teaching practice.

Quantitative analyses were informed by an existing framework for relationships between teachers' background characteristics, knowledge, beliefs, attitudes, and teaching practice (Ernest, 1989; Wilkins, 2008). Exploratory factor analyses resulted in belief and attitude factors which were included in a path analysis with directly measured variables for EMS status, knowledge, and teaching practice. Multiple mediation analysis revealed a significant indirect effect of EMS status on teachers' practice, mediated by teachers' knowledge and beliefs. Furthermore, the only significant direct effects on teaching practice were from two belief factors: *mathematics should be a meaningful challenge* (β = .47, p < .01), and *mathematics lessons should be highly structured experiences* (β = -0.31, p < .01). Implications of the role of these factors in mediating effects of EMS status and knowledge on teaching practice will be discussed.

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Chapter 7:
Instructional Leadership, Policy, and Institutions/Systems

ACQUIRING A STANCE ON EQUITY AND SOCIAL JUSTICE WITHIN AN ONLINE MATHEMATICS SPECIALIST PROGRAM

Courtney K. Baker
George Mason University
cbaker@gmu.edu

Terrie M. Galanti
George Mason University
tgalanti@gmu.edu

Laura Bitto
George Mason University
lbitto@gmu.edu

Theresa Wills
George Mason University
twills@gmu.edu

Cassandra Cook Eatmon
George Mason University
ccook19@gmu.edu

This research report will discuss mathematics specialists and the way in which they communicate ideas of equity and social justice within a master's program that leads to a state endorsement in mathematics education leadership. Many mathematics specialist programs focus on developing their mathematical content knowledge for teaching, but these specialized leaders also need to learn about curriculum, assessment, pedagogy, and leadership for school change. This research describes one mathematics specialist program's initial efforts in developing mathematics teacher leaders who take action within their K-12 school contexts. Preliminary findings suggest that the mathematics specialists within this program are aligning their views of equitable mathematics instruction with Principles to Actions (National Council of Teachers of Mathematics, 2014) and the Teaching for Robust Understanding *Framework (Schoenfeld, 2015). Implications for the program are that we are still exploring how to incorporate these ideas and facilitate critical conversations as teacher educators within our classrooms.*

Keywords: Mathematics specialists, Online learning, Equity, Social justice

Mathematics education researchers are called to view equity as a collective professional responsibility (Aguirre et al., 2017). While recent research publications related to our online mathematics specialist program have contributed to the field in the arenas of self-study (Baker & Bitto, under review), online teaching (Hjalmarson, 2015; Baker & Hjalmarson, 2019), and field experiences (Baker, Bitto, Wills, Galanti, & Eatmon, 2018), we recognize that we have more to do to improve our professional practice and to turn our research lens toward inequities in education. As reflective mathematics teacher educators, we strive to advance our master's students' understanding of equity and their readiness to advocate for more socially just educational structures in their contexts.

Our research journey began in Summer 2017 as we purposefully reflected on national equity conversations in our field within position papers and conference strands. Our emerging knowledge informed our interactions with one another and with our students. We questioned whether our program substantively engaged our master's students in ideas of equity and social justice. We further recognized that we needed to be intentional in creating learning experiences which would challenge their thinking and build their awareness as they worked toward becoming agents of change in their school communities.

This study captures our theorized entry points for making equity more explicit in our courses and our analysis of our students' evolving descriptions of equity. Our work is motivated by the tenets of the National Council of Supervisors of Mathematics (NCSM)/TODOS position statement (2016) titled *Mathematics Education Through the Lens of Social Justice:*

Acknowledgment, Actions, and Accountability. We place our work in the "acknowledgement" phase, and the results of this study drive our movement toward the "action" phase in our own practice and in our candidates' practice as emerging mathematics teacher leaders. The implications of this work are especially important as our we move toward a 100% synchronous online program design which extends our sphere of influence beyond in the state of Virginia and builds an increasingly diverse community of learners. We seek to build a program that both respects their current thinking yet challenges their perspectives of equity and social justice in ways that make all of us more accountable.

Research Question

How do candidates in a synchronous online K-8 mathematics specialist master's program describe equity and social justice?

Theoretical Framework

Equity and Social Justice in Mathematics Education

As White female mathematics teacher educators in a highly-resourced suburban mid-Atlantic community with diverse public school populations, we acknowledge both our positionality and our own pursuit of deeper understandings of equity and social justice. We have collaboratively contemplated entry points into broader conversations about equity in our master's level courses for aspiring mathematics teacher leaders. Gutiérrez (2007) placed access and achievement together on a *dominant axis* in her framework for equity in mathematics education. This dominant axis reflects "the status quo" and privileges a "static formalism" which equates resource allocation and access for historically marginalized students. It further creates an achievement-driven pipeline which defines societal economic participation (p. 39). Gutiérrez also theorized a *critical axis* of equity in which "mathematics… squarely acknowledges the position of students as members of society rife with issues of power and domination" (p. 40). She offered a window/mirror metaphor for identity as a precursor to building power through mathematics to challenge marginalization in society. "Students should have the opportunity to see themselves in the curriculum as well as to have a broader view of the world" (Gutiérrez, 2012, p. 20).

The tensions between these two axes (Gutiérrez, 2012) orient our research community toward a broader agenda of social justice in which mathematics education evolves to effect change. This tension is deeply embedded in our own instructional decisions as we facilitate conversations about equity which both value and challenge the contexts that our experienced teachers bring to their graduate work. Rubel (2017) categorized standards-based teaching and complex instruction as "equity-directed instructional practices" (p. 69) within the dominant axis and culturally-responsive teaching and mathematics for social justice within the critical axis. Yet, we wonder if situated instructional moves which are intended to build mathematical competence (Berry, 2018; Gresalfi, Martin, Hand, & Greeno, 2009) are so clearly categorized. We believe that instructional practices can be highly contextualized and actually place our teachers at the uncertain intersection of achievement as the dominant axis and identity as the critical axis of equity in mathematics education.

We are asking our aspiring teacher leaders to operate in this tenuous space as we seek to understand how their views of equity evolve with the use of reflective tools designed to foster student identities as productive learners and doers of mathematics (Aguirre, Mayfield-Ingram, & Martin, 2013, Aguirre et al., 2017) and change student and teacher beliefs about mathematics

(Berlin & Berry, 2018; Horn, 2007). In our mathematics specialist courses, we engaged our students in the National Council of Teachers of Mathematics (NCTM) (2014) *Principles to Actions* effective teaching practices and the Teaching for Robust Understanding (TRU) framework (Schoenfeld, 2015) pedagogical moves as tools which promote equity in classrooms. We also facilitated conversations around the calls to take a social justice stance in the joint position paper from NCSM and TODOS. These tools and papers were a starting point for broader conversations among our faculty as we prepare our mathematics specialist candidates to understand all four dimensions of equity (Gutiérrez, 2007) and to become agents of change.

Mathematics Specialists

Mathematics specialists, also called mathematics coaches or mathematics teacher leaders, are recommended to, "enhance the teaching, learning, and assessing of mathematics in order to improve student achievement" (Association of Mathematics Teacher Educators (AMTE), Association of State Supervisors of Mathematics, NCSM, & NCTM, 2010, p. 1). To be prepared for varied contexts and responsibilities, specialists need expertise that includes deep knowledge of mathematical content and pedagogy (Ball, Thames, & Phelps, 2008) as well as the ability to develop teaching peers' mathematical knowledge for teaching (AMTE, 2013). It is recommended that credentialing and certificate programs develop content knowledge for teaching mathematics, which includes deep understanding of mathematics for grades K-8 and further specialized mathematics knowledge for teaching; pedagogical knowledge for teaching mathematics, which includes learners and learning, teaching, and curriculum and assessment; and leadership knowledge and skills (AMTE, 2013). NCTM *Principles to Actions* (2014) acknowledges the critical role of mathematics specialists in enhancing teacher capacity to enact research-based teaching practices.

Potential agents of change. Mathematics specialists can act as agents of change to implement policy initiatives related to standards, curriculum, assessment, and professional development within their K-12 school contexts (AMTE, 2013; Fennell, 2017). However, mathematics specialists face many challenges, including meeting the expectations of multiple stakeholders in the school system (e.g., teachers, administrators, and parents) (Chval et al., 2010), while teaching the learning of mathematics to those involved (Felux & Snowdy, 2006). Therefore, they require guidance and support to become experts (Borko, Koellner, & Jacobs, 2014; Chval et al., 2010) as experience and time are needed for both developing their leadership abilities and changing the mathematics practice within their schools (Campbell, 2012).

The mathematics specialist master's program at George Mason University is developing specific teacher leader skills and dispositions (Baker et al., 2018) which prepare our specialist candidates to become agents of change. But as we, as faculty, reflect upon the candidates' ability to take action to reform and advance their educational settings to be equitable and accessible, we wonder: How do candidates describe equity and social justice in a mathematics specialist advanced program?

Methods

Context

The George Mason University K-8 Mathematics Specialist Leader program, established in 2005, is comprised of ten 3-credit graduate courses (five content, four leadership, and one capstone internship course) based on current coaching leadership frameworks (Bitto, 2015) and national standards for mathematics specialists (NCTM, 2012; AMTE, 2013). Over the past decade the program has transitioned from traditional face-to-face to an online format. In Fall

2017, the first iteration of a 100% synchronous online program began. All coursework was delivered using a synchronous online format that utilized Blackboard Collaborate Ultra for communication (audio, video, and text) along with interactive Google slides in which candidates had editing rights allowing for simultaneous engagement with their instructors and peers on slides.

Participants

There were 16 mathematics specialist candidates in the first synchronous online cohort, including female (87%) and male (13%); a combination of White (69%), African American (19%), and Indian American (6%) educators. All of the candidates were employed in K-12 schools, with the majority (81%) in elementary schools (K-5) and 19% who taught at the secondary level (6-12). The candidates represented varied educational settings, including 81% who resided in districts in Virginia, which does not use Common Core State Standards; others were outside of Virginia in Common Core states (13%) or in international schools (6%). Prior to the beginning of the cohort, 25% of the candidates were already working in mathematics specialist positions without formal preparation coursework.

Data Collection and Analysis

Data was collected via identical surveys that were given at the beginning of the program and midway through coursework. Consisting of 23 questions total, initial questions centered on participant demographics such as job title and K-12 setting. Eight questions utilized a Likert scale (1-5), were taken from the EMC survey (Yopp, Burroughs, & Sutton,, 2010), and delved into one’s comfort facilitating leadership activities such as modeling instructional practices, collaborating with stakeholders, providing feedback, and identifying gaps in their peers’ knowledge. Six questions aimed at understanding participants’ perceptions of high-quality mathematics instruction and collaborating with various K-12 stakeholders (e.g. teachers, administrators). Five questions were open-ended and centered on each of the dimensions of the TRU Framework (Schoenfeld, 2015).

Responses were analyzed qualitatively. An in vivo coding (Saldaña, 2016) scheme was used in order to capture the specific language of the participants. Categories shared by the participants were aligned and the reduction of these categories led to the emergence of a few prominent themes (Maxwell, 2005). These themes will be elaborated on and discussed in the following sections.

Results

Preliminary findings indicate our mathematics specialist candidates’ descriptions of equity within mathematics education evolved during their program in the following ways: 1) changing their descriptive language; 2) increasing understanding of a teacher’s responsibility; and 3) developing knowledge of facilitating rich mathematics experiences.

Advancing Their Perspectives of Equity

At the beginning of the program, mathematics specialist candidates characterized equity as differentiation. In the pre-program survey, they spoke about “meeting students’ needs” 27 times and used the term “differentiation” 10 times. More specifically, they described “differentiation” and “meeting the needs” as a function of resource allocation. That is, if teachers and mathematics specialists provided the right combination of tools, access, and resources, they were being equitable in their instructional approaches. Participants articulated a critical role for differentiation in achieving equitable mathematics instruction. They also recognized that many factors contribute to ensuring each student has access to meaningful learning.

The math specialist candidates discussed how differentiation in mathematics instruction was a way to "level the playing field" for low-performing students. One candidate said, "Each student enters a mathematics classroom with a different skill set. Collectively, they possess a variety and multitude of levels and deficiencies." They also communicated a commitment to access for all students by "making content available to all and treating all students with dignity regardless of their level is important." Furthermore, they described equity as teachers utilizing multiple formats to meet the needs of students, either collectively or individually, as demanded by the differences in student readiness.

This focus on providing access for the purpose of higher achievement was aligned with the dominant axis (Gutiérrez, 2007) of equity. They viewed equity as addressing mathematical weaknesses instead of fostering the identities of students of varied cultural backgrounds as productive doers of mathematics. The mathematics specialist candidates talked about the (lack of) prior knowledge as a deficit rather than communicating a mindset that views potential in each student.

However, in the mid-program survey the language the specialist candidates used to discuss students and their understandings shifted. There were fewer references to differentiation and meeting students' needs and a deeper recognition of the value all students bring to classroom learning experiences. Their language also shifted from viewing (lack of) prior knowledge as a deficit to an opportunity for meaningful learning. For example, one candidate wrote:

> Equitable math instruction means that all students have access to high quality math instruction. All students need to have a voice in the classroom with lots of opportunities to share their thinking. Teachers need to hold all students to high standards and provide support to them. By using best practices, all students can have a deeper level of math understanding and be problem solvers and think critically.

Mathematics specialist candidates recognized complexities that they had not previously identified. One student considered socioeconomic status as she described equity, "All students should be provided with the same resources to learn mathematics. However, oftentimes, this is a challenge for students who are disadvantaged socioeconomically." This subtle shift indicates progress towards the critical axis (Gutiérrez, 2007) since the candidate acknowledges the position of the student in terms of societal power. Another candidate alluded to the institutional structures beyond the classroom which affect opportunities to learn. This candidate wrote, "Access is huge. If students are in a school or setting where they do not have access to quality instruction and safe learning environments, they do not have an equitable education." Again, this shift in language from the pre-program survey -- phrases such as "leveling the playing field" and "differentiating" -- to acknowledging systemic inequities in the mid-program survey indicates a deeper understanding of equity as the candidates shift their thinking from the dominant to the critical axis (Gutiérrez, 2007).

Increasing Understanding of a Teacher's Responsibility to Ensure Equitable Experiences

At the beginning of the program, mathematics specialist candidates communicated that students could achieve with sufficient access to resources. After class discussions about equity in the context of *Principles to Actions* and the TRU framework, they recognized that students brought their own mathematical understandings into K-12 classrooms. However, perceptions of equity emphasized resource allocation and achievement goals. For example, one candidate said, "We all know students work at different levels and all learn in different ways so providing equitable mathematics instruction would be providing our students with all the resources and tools they would need to achieve the same goal." When sharing potential additional supports, mathematics specialist candidates identified students as a means to provide support to one another "via peers and one-on-one tutoring." These solutions to addressing student understandings placed the responsibility of learning on the students and minimized a teacher's role in facilitating equitable mathematics instruction.

The majority of mathematics specialist candidates recognized that there was a need to take more action in their instructional practices but were unable to articulate specific actions beyond differentiation. For example, a candidate said, "The teacher knows their students best and will be able to differentiate the instruction to meet their students' needs." This candidate indicated some responsibility to the teacher but was unable to verbalize specifics.

However, in the mid-program survey mathematics specialist candidates identified specific ways in which a teacher could begin to facilitate equitable mathematics instruction by addressing issues of identity. Responsibilities of learning were still placed on students, but in an appropriate way that indicated students would construct their own meaning based on instructional strategies. Now, candidates were communicating about individual student participation in constructing meaning:

> Students need to feel that they have a voice and role in the classroom. Rather than just a few students doing that majority of the talking and sharing, all students need to have ways to contribute in class. The teacher's role needs to be a facilitator so that the students are doing the heavy lifting of the thinking and learning.

Similarly, another candidate shifted the responsibility of learning towards the teacher, viewing the teacher as a leader who empowered her students to take ownership of their own learning. Furthermore, this candidate recognized the identity of each individual student and the students' societal positionality.

> Students will take ownership when they feel empowered to think and reason through math problems. When they are successful, this contributes to a positive math identity. Another thing about identity is students may not see themselves as a "math person" because of gender, race, or socio-economic status. We need to empower all students to become math people.

As candidates' views towards learning responsibilities shifted, questions simultaneously arose in how a teacher might move forward. "Equitable instruction ensures that teachers are focusing on the specific needs of students. But, how can each student be successful? What do they each need?" Such questions acknowledge the complexities of equitable instruction and indicate that the candidates are engaging in critical reflection about equity.

Developing Knowledge of Facilitating Rich Mathematics Experiences to Promote Equity

Mathematics specialist candidates acknowledged their role in creating equitable classrooms as facilitators of rich mathematical experiences. Their knowledge of this role is defined through initial surveys as a teacher's ability to teach mathematics content equally to all students. They recognized that while they do not have control of grade-level content requirements, they do have control of the facilitation of rich mathematics experiences that are delivered with this content. They stated this limitation, however, in the initial survey. Candidates described the ways that they overcame this limitation with descriptions of fairness and equality as definitions of equity. One candidate responded that, "While there is a belief that not one math program has the content necessary for all learners to be successful, I do believe that every school should have a common textbook program to which teachers can add supplementary tasks and activities." Another candidate described equity through a teacher's decisions when delivering the content by stating that, "Content is set by the state. However, how the teacher structures the content experiences for students allows for equitable instruction." Further, a candidate stated that math content is inherently equitable because it can be delivered through various teaching practices.

> Math content contributes to equitable math instruction in focusing on the big ideas of math and conceptual understanding like equivalence, multiple representations, patterns, form, relationships, etc. that are also cross disciplines other than math. These are the enduring understandings that we want students to understand in the math content.

In each of these examples, the candidates are describing equity as equalizing access to content and/or the teacher's decisions in delivery of the content.

By the mid-program survey, the candidates shifted to describe equity as their ability to teach content through differentiation and student-centered tasks that value the uniqueness and individuality of each student. They began to value differences as an opportunity for rich mathematical experiences. One candidate reported a shift in their teaching by stating that s/he was "recognizing cultural and socio-economical differences and allowing students' experiences to guide task creation." As these candidates engage students in math tasks, they are not only recognizing differences, but valuing them by highlighting those differences through the mathematics content.

Also, candidates transitioned from defining equity as equality to equity as a teacher's ability to differentiate the mathematics experiences. For example, "If you teach the same content but differentiate it based on the varied abilities it is equitable." This differentiation is described as individualized by a candidate who stated that "When problems are presented it should be with content that students can make a connection to. If you know your students have never been to the beach, do not give them problems related to the ocean without first providing background knowledge" and "math is relatable to all students and there are multiple entry points to the problems." In differentiating the math experiences based on students' prior knowledge, they are facilitating student-centered math experiences, transitioning to individualization. Another candidate described rich content as individualizing content understanding and application:

> Content is content. Rich content goes beyond the surface and provides experiences with math that give pathways to a higher understanding of content. This is what we want for all children. Not a show and tell mathematics, but an environment where students can develop their own understanding, work collaboratively with others, and see the beauty in math.

Candidates began to focus on "all students" and how "students should be provided with ways to engage in the content." One candidate wrote, "All students need to be able to contribute to the classroom and have a voice. High quality tasks with multiple solutions provide ways for all students to have a voice and share their thinking." However, while the specialist candidates identified the need for students to take ownership of their own learning, no one discussed how teachers could develop opportunities for students to see themselves as doers of mathematics.

Discussion

We, as a program, reiterate that we are continuing to learn how to better prepare mathematics specialist candidates to engage in issues of equity and social justice. We embrace the call to change the structures which perpetuate inequities and recognize that we, as faculty, need to take a stronger stance. We acknowledge that the content and leadership experiences we provide to our candidates are valuable, but they are not aggressively moving the conversation forward to empowering our master's students as agents of change. This analysis of the candidates' descriptive language of equity and social justice can inform our future programmatic actions. *Principles to Actions* (NCTM, 2014) and the TRU framework (Schoenfeld, 2015) resonated with our students' conceptions of effective K-12 mathematics teaching and was a comfortable way to begin conversations that center on designing high-quality classroom experiences. The conversations that arose from using the NCSM/TODOS (2016) position statement on equity and social justice were more critical in nature and allowed our candidates to see the differences between good mathematics teaching and equitable mathematics teaching. However, our candidates need more opportunities to envision ways in which they can challenge inequitable structures.

We, as a program, also need to look at how we address issues of equity and social justice across our courses and develop a common vision that will promote the systematic change we desire. Our candidates need more experiences to grapple with ideas of equity and social justice. We are reflecting on critical questions: How do we begin these critical conversations with mathematics specialist candidates so that we have evidence of their growth? How does one develop excellent teaching practices and advocacy skills in parallel?

Although we are developing classroom teachers and leaders who are embracing *Principles to Actions* (NCTM, 2014) and the TRU framework (Schoenfeld, 2015), we need additional tools which promote critical awareness and prepare our candidates to act. We also need to re-examine our data collection tools, specifically the survey. We need to capture how our candidates not only describe equity, but we also need to gather data that illuminates what equity looks like in their daily practice. We anticipate some disconnect between beliefs and actions as our mathematics specialist candidates, like pre-service teachers, are challenged by the reconciliation of exploring what they are 'supposed to' say with what they perceive possible in their school contexts. We hope that our candidates will feel empowered to change the systems within which they work. Without taking any action or modeling equitable teaching practices in our own graduate classrooms, our practices will continue to center on instruction that does not contribute to systemic change.

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WHAT DOES IT TAKE TO BE A FOX? NEW HORIZONS FOR MATHEMATICS COMMUNITIES OF PRACTICE

Nicole M. Wessman-Enzinger
George Fox University
nicole.enzinger@georgefox.edu

Joshua T. Hertel
University of Wisconsin-La Crosse
jhertel@uwlax.edu

Justin K. Dimmel
University of Maine
justin.dimmel@maine.edu

In this theoretical research report we reflect on the challenges of becoming more fox-like in mathematics education work. Using a communities of practice motivating theoretical lens, we compare and discuss the differences in defining, creating, and accessing knowledge between virtual and scholarly communities of practice in mathematics education. We present four claims that virtual communities of practice in mathematics education are inherently foxy work. As part of our claims, we discuss how scholarly communities of practices are inherently hedgehog work. We conclude with a list of recommendations of those within the scholarly communities of practice in mathematics education. These recommendations include looking toward the successful fox-like attributes of the virtual communities in mathematics education.

Keywords: Instructional Leadership; Systemic Change

Drawing on Berlin's (1953) essay, *The Hedgehog and the Fox*, Boaler, Selling, and Sun (2013), ask the question to the mathematics education field: "Where are the Foxes in Mathematics Education?" Berlin uses the metaphor of hedgehogs and foxes for illustrating two types of thinkers and writers: hedgehogs as those focused on a singular idea and foxes as those who draw across ideas and make changes. Boaler and colleagues challenge those in the academy to be more "foxy" in their collaborations and research with intentional outreach to other communities. They note "although the production of research ideas is extremely worthy, if research ideas in mathematics education do not get taken up and used, by teachers, parents, and other educators, then their worth diminishes significantly" (p. 191).

In this theoretical paper, we explore questions around this challenge including: What does it mean to be a fox in mathematics education? To what degree does the field support members in becoming foxes? How might foxes within mathematics education use online communities in their work? Moreover, we consider how newly formed virtual communities in mathematics education might serve as new horizons for traditional scholarly communities.

An Array and a Skulk

Berlin's (1953) essay developed the conceit of the hedgehog and the fox as a lens for assessing the life and work of Leo Tolstoy. What endures from the essay is the idea that there are hedgehog-like thinkers and writers and fox-like thinkers and writers. Kilpatrick (2013) captures the difference between the two as a matter of scale and humility: the hedgehog knows all there is to know about one big idea, uses this information to understand the world in its entirety, and fails to recognize that there aspects of the world that cannot be known or explained. The fox, on the other hand, knows just enough about many things and also knows that some things are unknowable. Berlin (1953), Kilpatrick (2013), and Boaler and colleagues (2013) use the hedgehog/fox dichotomy for describing individual or cooperative activities (e.g., the work of Tolstoy, specific mathematics education research projects). Our interest in discussing this distinction is not for focus on individual hedgehogs or foxes, but rather to consider *collections* of each and the collective behavior of these different groups. Moreover, we use the hedgehog/fox distinction as lens for examining the kinds of activities that take place by different collections of math educators: the arrays of hedgehogs in academia and the skulks of foxes in what we describe below as the network of virtual mathematics education communities of practice and scholarly mathematics education communities of practice.

Motivating Theoretical Lens: Communities of Practice

Our motivating theoretical lens is formed from perspectives of communities of practice (Lave & Wenger, 1991). We apply this lens to virtual mathematics education communities of practice (vMEC) and also scholarly mathematics education communities of practice (sMEC). Although we make the distinction of these two different types of communities, virtual and scholarly, we recognize that some members of the sMEC are also engaged participants in the vMEC. Because the overlap between vMEC and sMEC is so limited, we treat these communities as distinct in our subsequent discussion and claims. These communities, the vMEC and sMEC, each have the three essential characteristics discussed by Wenger-Trayner and Wenger Trayner (2015): (1) a shared domain; (2) an engaged community of participants; and (3) a focus on practice (see, e.g., Hertel, Wessman-Enzinger, and Dimmel, in press, for further discussion on virtual mathematics education communities of practice).

Virtual Mathematics Education Communities

As social media platforms (e.g., Facebook, Instagram, Twitter) have become integral to people's daily lives over the last decade, virtual communities have flourished. We draw a distinction between platforms (websites that facilitate interactions among users) and the multitude of virtual communities that reside on such platforms. Moreover, although we are focused on virtual communities, the interaction between virtual and physical communities is complex (Ellis, Oldridge, & Vasconcelos, 2004). As Ellis, Oldridge, and Vasconcelos (2004) note, virtual communities have different features than traditional social networks:

> Virtual communities are both narrow and specialized, in terms of the information posted, but at the same time broadly social and supportive. Consistent evidence suggests that many individuals go to virtual communities because of these social and supportive characteristics: the many weak ties supported by virtual communities provide access to a much wider network of people than conventional, social networks. The potential for invisibility regarding normal social cues such as gender, race, class, and age opens up the potential for networking and interaction that may be inhibited elsewhere. (p. 148)

We conceptualize these virtual communities as types of *communities of practice* (Lave & Wenger, 1991; Wenger, 1998; Wenger-Trayner & Wenger-Trayner, 2015) acknowledging that there are inherent differences between physical and virtual communities of practice. Moreover, within this theoretical paper we concentrate on vMECs that are focused on the teaching and learning of mathematics.

The shared domain of a virtual community of practice is independent of platform. Members of a particular vMEC might use several platforms concurrently (e.g., Pinterest, Facebook) or a single platform might host multiple vMECs. The shared domain of any particular vMEC is visible through the shared focus on teaching and learning of mathematics in the discussions, interactions, resources, etc. Additionally, smaller communities of practice each having a narrow focus may also be part of a larger vMEC (e.g., mathematics content at the elementary level, the professional development of mathematics teachers). Collectively vMECs make up a group that is different from other virtual communities of practice and, at the same, time they form smaller communities that are distinct from one another.

The individuals who constitute a vMEC share information and help each other. Through this interaction, members build relationships, collaborate, support one another, and engage in the shared domain. The methods and kinds of engagement are influenced by the structure of the platform on which the vMEC resides. Some platforms provide robust tools for interaction and knowledge building (e.g., Mathematics Stack Exchange) while others provide limited tools for structuring interaction. The presence or lack of tools influences that degree to which community members can shape discourse.

Practice is always social (Wenger, 1998) and involves performing an activity within a social as well as historical context. Practice includes:

> What is said and what is left unsaid; what is represented and what is assumed. It includes the language, tools, documents, images, symbols, well-defined roles, specified criteria, codified procedures, regulations, and contracts that various practices make explicit for a variety of purposes. But it also includes all the implicit relations, tacit conventions, subtle cues, untold rules of thumb, recognizable intuitions, specific perceptions, well-tuned sensitivities, embodied understandings, underlying assumptions, and shared worldviews. Most of these may never be articulated, yet they are unmistakable signs of membership in communities of practice and are crucial to the success of their enterprises. (Wenger, 1998, p. 47)

The explicit and visible practices of an vMEC may be quickly recognized and adopted by a newcomer. The implicit, untold, underlying assumptions, on the other hand, may go unnoticed. Moreover, the space in which a vMEC resides can also create room for miscommunication since, unlike physical communities, participating in a vMEC typically requires no more than visiting a publicly viewable website or registering a free account on a platform. Consequently, although newcomers can easily participate in a vMEC, they may do so without understanding the practices of the community and the assumed knowledge base.

Scholarly Mathematics Education Communities

The shared domain of sMECs is ultimately the academy. Although participants of the sMEC belong to different communities themselves, all participatns of the sMEC have an affiliation with a university in some capacity. The engaged participants are often graduate students, researchers, research assistants, tenured professors, or untenured professors. Becoming an engaged participant within the sMEC requires a vetting—this usually entails participation in thesis or dissertation work at some point (e.g., Golde & Walker, 2006; Reys, 2000). The expectations in the sMEC for vetting through theses and dissertations mimic research processes described in common mathematics education journals (e.g., Williams & Leatham, 2017) and conferences, such as the Psychology of Mathematics Education-North America (PME-NA). Although some members of the sMEC are participants of the vMECs, the two groups operate largely independently with participants of the sMEC knowing little about vMEC active members of the vMEC (e.g., classroom teachers, math coaches, consultants) outside of the sMEC.

Claims

We now discuss claims that we believe are critical issues in the discussion of hedgehogs and foxes within the field of mathematics education. As noted, our focus with these claims is on general behavior of collections of foxes and hedgehogs. This shift to focusing on collections rather than individuals allows for consideration of how sMEC and vMEC support different kinds of engagement with issues of teaching and learning in mathematics. We see these four claims as supporting the following position:

> The field is currently focused on generating hedgehogs (Claim 1) for work within sMECs and work within hedgehog systems. As a result, few within sMECs are positioned to obtain fox status (Claim 4). In contrast, vMECs offer a vision of mathematics education communities where there is more access and opportunity for becoming a fox (Claims 2 & 3).

Claim 1: The Scholarly Mathematics Education Communities are Designed to Produce Hedgehogs.

The very act of working on or obtaining a terminal degree (PhD, EdD, etc.) in mathematics education or a related field is a hedgehog idea. The academic journey members of sMEC undergo in graduate work is that of a hedgehog with narrow focus around developing robust knowledge of the field (pedagogy, research, etc.) before posing and researching a circumscribed, theoretically framed question whose investigation contributes to this knowledge base. This work is disseminated in hedgehog ways through a dissertation and (sometimes closed) defense. The structure of the dissertation itself is a hedgehog production built upon accepted norms of academic discourse—hundreds pages filled with complex, academic jargon that only insiders in that area read.

After graduate school, those in the field seek positions in the academy. Many in the sMEC obtain tenure-track positions at universities, which are built upon hedgehog systems. For example, tenure, whether at a teaching college or research university, is achieved through a combination of demonstrated proficiency in teaching, scholarship, and service. Related to scholarship, peer-reviewed publications are highly valued. Although some publications may be written for a more general audience, most are focused more narrowly on a subset of the sMEC community. Moreover, the articles themselves are produced to adhere to the norms of academic discourse and written for members of the sMEC community, rather the mathematics education community at large.

Claim 2: Hedgehogs Can Flourish in a Closed System; Foxes Require Access to Flourish.

Members of the sMEC are inherently part of systems where production of knowledge is more important than the access to knowledge. These mathematics education hedgehogs produce knowledge through research that is focused in a particular domain in the field (e.g., children's thinking about operations with fractions, student positioning in number talks). Likewise, the systems in which graduate students are trained to work (i.e., higher education) value the production of knowledge and the systems in which we work require the production of knowledge (e.g., dissertations, publications). This knowledge is disseminated through peer-reviewed articles, conference presentations, or other publications. Most of these publication venues are creations of the sMEC community itself although some may be owned by other entities. As such, these venues are maintained by members of the sMEC and access to them challenging for those in the vMEC who lack institutional financial supports. Likewise, an individual who is trained as a hedgehog, but resides outside of the academic structure, will struggle to access most of the newly created knowledge.

The monetization of knowledge generated by members of the sMEC showcases how access to knowledge is less important than the production of knowledge. This monetization occurs in various ways even beyond the required memberships to organizations or journals stated above. Only a few of the sMEC organizations have open access to the knowledge they curate or promote the freely sharing of knowledge to those outside the community. In fact, open access is not given a particular value by the field or the sMEC as a whole, which can lead to inconsistent policies within groups. For example, although the proceedings for PME-NA are freely available online, the proceedings of the larger parent organization, the International Group for the Psychology of Mathematics Education, are not free to download. Some scholars seeking to work around the prohibitive structures of academic publication make prepublication versions of their work (papers, book chapters, etc.) available for download. Although this shows the desire of many authors to work around the access-limiting structures, it is also evidence of the degree to which the field ignores the importance and challenge of access.

The very nature of the expectations in the academy, or the sMEC, limits what those in the academy consider transformational. Reaching out in transformational work beyond the research required for typical peer-reviewed journals is an afterthought or an "if we have time" mantra. Time constraints notwithstanding, there is also the risk, when reaching out to the general public, that one's scholarly work will become popular, at which time one's academic reputation may be impugned for doing unserious work. Or, one may spend time working with the general public and lose credibility within the academy for doing unscholarly work. This is evidenced by the academy's emphasis on scholarly production and de-emphasis on service.

Conversely, within vMECs the sharing of knowledge and the creation of knowledge are equally important. Reaching out to those beyond a niche community is the purpose of any post or work; this starkly contrasts many situations within sMEC. Even if those with large-scale funding (e.g., NSF) engage in transformational work, the access to this work is often lacking. This presents hedgehog findings making their way into thousands of teachers' classrooms in the same ways available from MEC. Likewise, although an individual in sMEC may spend years studying children's thinking about integer addition and subtraction deeply, sharing this work in an open and accessible way if the individual wishes to have an impact within the vMEC.

Members of the vMEC need to make sense of student's thinking about many domains of mathematical content and also the accompanying pedagogical work that leverages this thinking forward. Members of the vMEC come together and use social media platforms as a ways to share knowledge about making transformative work in mathematics education. Simply put, those in the vMEC are trained mathematics practitioners and inherently engage in foxy work (e.g., instructional coaches for a range of grades, teachers who teach many different topics). Despite the varying roles of vMEC (e.g., math coaches, teachers, math consultants), there is a common goal of connecting larger mathematical ideas and transforming education for students. Although those in the sMEC may value transformational education for students, the field is submerged in a hedgehog system that perpetuates closed structures (e.g., publications in peer-review journals, tenure).

Claim 3: The vMEC Presents Opportunities for Developing Foxes More than sMEC.

The vMECs have more opportunities for developing foxes than the sMECs. Moreover, we claim that vMECs are inherently foxy communities of practice. In the shared virtual spaces of the vMECs (e.g., Twitter chats like #mathchat, Facebook groups), the engaged participants wish to evoke change. This constitutes sharing things in public, free, open-source ways. Because the work is inherently foxy, there are more opportunities for the members of the vMEC to flourish when compared to those within the sMEC. Participants in the sMEC have less opportunities to engage in discussion of knowledge and shared knowledge. Most participants in the sMEC discuss their work through conferences or journal articles. Although they may get to speak with members of the sMEC about their work at a conference, the members of the sMEC have limited opportunity to discuss their work published in articles beyond receiving feedback from reviewers and editors. Members of the sMEC rarely know if their work is being taken up or are able to share it freely. Conversely, the very nature of knowledge expectations of the vMEC requires interactions with others publicly and freely shared.

There is clear evidence of the influence these communities have had on mathematics education. An example of the inherent foxy work of vMEC are 3-Act Tasks. 3-Act Tasks are mathematical cultural phenomenon that constitutes an ideal example of highlighting transformative, foxy work part of the vMEC. A 3-Act Task is a task presented as a way of mathematical storytelling in three specific parts or acts (e.g., http://blog.mrmeyer.com/2011/the-three-acts-of-a-mathematical-story/). In the first act, the story is visually told. This act entails beginning with a video of a contextual situation, like picking up eggs out of a cartoon or packing meatballs in a pot of water. The videos do not contain words, symbols, or questions. It is essential the videos are visual with a change happening and open-ended. The second act, the protagonist/student/learner has a need for overcoming an obstacle. This may entail mathematizing the situation, looking for resources, or creating tools. For example, in a video where the protagonist is packing meatballs in a pot of water, one may wonder how many meatballs can be packed without overflowing the water in the pot. Learners may mathematize this situation by starting to unpack ideas of volume and displacement. In the last act, act three, there is resolution to the story and conflict. This is the space where mathematical problem solving occurs. Due to the open nature of the video and conflict, it is likely that there are multiple strategies, solutions, and robust discussion about the conflict resolution.

Countless teachers and other practitioners know what a 3-Act Task is and use them routinely. Although 3-Act Tasks have recently found themselves in traditional, scholarly peer-review spaces (e.g., Lomax, Alfonzo, Dietz, Kleyman, & Kazemi, 2017), the inaugural roots of 3-Act Tasks began on a blog by Dan Meyer. Through the Mathematics-Twitter-Blogosphere (MTBoS) and vMECs, numerous educators began using 3-Act Tasks. Graham Fletcher categorized a vast array of 3-Act Tasks for elementary school; these tasks with videos are freely available (e.g., https://gfletchy.com/3-act-lessons/). Using the hashtag #3ActMath on Twitter platform reveals quick insight into the power of the MEC in these the virtual communities and transformational impact on mathematics pedagogy. In a way, the 3-Act Tasks have been “researched” by the engaged participants in the shared domains of the MEC. There are many examples of this within the vMEC. In fact, all of the examples in the vMEC are fox-like work. Yet, very little of the work in the vMEC is picked up or used within the sMEC.

Claim 4: Few Within the sMEC Are Positioned to Be a Fox Whereas Anyone Within the vMEC Has Potential to Become a Fox.

Our final claim is that only a handful of individuals within the sMEC are positioned to become foxes. This is because the path to becoming a fox from within the sMEC requires first demonstrating one’s competence and ability as a hedgehog. The practices and norms of the sMEC create narrow avenues for becoming a fox. For example, some members become foxes through their successes as hedgehogs (e.g., articles, books, grants). Others attain fox status through their work as leaders within professional organizations. The sMEC celebrates and rewards the achievements of hedgehogs through a variety of structures including awards, honors, and guest lectures. Foxy work that lies outside of typical hedgehog activities is not valued in the same way. For example, non-peer reviewed work (e.g., blog posts, Youtube Channel) that is open to the public is not seen in the same light as scholarly publications.

In contrast, foxes in the vMEC are typically individuals who exist outside of the traditional academy. These bloggers, writers, and teachers have built their credentials upon popularity.

Moreover, the popularity itself is made possible by open access to the created content and serves as a proxy for an individual's position within the vMEC. Thus, becoming a fox within the vMEC relies upon the actions of the individual rather than their position within a system.

One's position as a fox within either the sMEC or vMEC determines the amount of influence one is able to yield. Members of sMEC, for example, may listen to those who have earned prominence in a plenary speech with assurance that the structure of the academy has guaranteed the speaker's qualifications as a hedgehog. In contrast, within vMEC prominence is earned through popularity of content. As such, prominent foxes are those whose content creations have become well known. Furthermore, whereas foxes with the sMEC largely have doctorates, a long list of publications, and a history of groundbreaking research, those prominent in the vMEC, in contrast, do not necessarily have they characteristics. What vMEC foxes do have, however, are curated free, open-source content that is connected to many mathematical topics.

Encouraging a Skulk of Foxes from Within Academia

As others have identified (Boaler, Selling, & Sun, 2013; Kilpatrick, 2013) there is a need for more foxes within the sMEC. But what are the attributes of being a fox? How can those outside of privileged positions (e.g., tenure, prominence in field) do foxy work? As we argue above, the field is currently focused on generating hedgehogs (Claim 1) to work within a system that is closed to most of the outside world (Claim 2). This structure limits opportunities to become a fox (Claim 3) and positions few within sMEC to obtain fox status (Claim 4).

Calls for more foxes in mathematics education are important, but what we need are more *skulks of foxes*—groups of foxes—in mathematics education. This is not an individualistic endeavor; it is a group or systemic change that must take place. We believe that the field must begin to address some of the structural barriers toward becoming a fox. Most importantly, the limited access to content needs be addressed so that innovations within research and teaching from within sMEC can have a broader influence with vMEC. This is more than a call for increased open access journal articles. Rather, it is a call for structural changes and community expectations so that those in the sMEC can engage in work that integrates both the scholarly rigor our field requires alongside the outreach needed to reach the practitioners who reside outside the closed system.

Those within the sMEC have made initial steps toward structural changes and issues of access. For example, the journals *Teaching Children Mathematics*, *Mathematics Teaching in the Middle School*, and the *Mathematics Teacher* have monthly free-preview articles with accompanying Twitter chats. The *Mathematics Teacher Educator* (MTE) journal also started a podcast that is open access about select articles in the journal.

This is challenging work that needs to be done by the sMEC community—individuals alone cannot make systemic changes. Those in the sMEC need to think about collective ways they make changes to *access* of knowledge—particularly since views on creations of knowledge cannot be changed (e.g., dissertations, research). We challenge readers who belong in the sMEC to reflect and take-up the following questions:

How can we focus on access within the field?
How can we change those journals within the control of the sMEC (e.g., Journal for Research in Mathematics Education, MTE) to focus on open access?
How can we open up meetings at conferences (e.g., PME, PME-NA), where knowledge is being shared so that more practitioners benefit?

How can we integrate the work and tangible artifacts of the vMEC into the sMEC?
How can we value open access to a level similar to academic rigor?
How can we change the systems to focus on access and research, particularly for communities beyond the sMEC?

In some ways, being a fox counters the very nature of the academy, which supports and rewards hedgehog work. In this sense, the only way those in the sMEC can do foxy works is to be a rebel. A member of the sMEC may not feel it financially or personally worth it be a rebel; but, ultimately to engage in fox-like work, a member of the sMEC needs to push back on the closed systems that dictate hedgehog work.

Concluding Remarks

We affirm the need for more foxes in mathematics education (Boaler et al., 2013; Kilpatrick, 2013); we highlight that the calls are not individualistic, but collective. We emphasize structural concerns of the closed, hedgehog system that we reside in within sMECs. And, we hope that our field can begin the challenging, deep work of systemic change that encourages a skulk of foxes.

We recognize that there are challenges in using a strict dichotomy to describe mathematics education communities of practice and their members as either hedgehoges or foxes. Within a community, members might position themselves and their work along a continuum from all-fox on one end to all-hedgehog on the other. However, we contend that examining communities (vMECs, sMECs) through the lens of foxes and hedgehogs offers a way to re-examine and re-frame scholarly spaces. In doing so, this perspective incorporates a social justice stance that allows for examining institutional structures, practices of community engagement, and teaching and learning environments (NCSM & TODOS, 2016). Thus, by adopting this lens we can consider individual actions and identity the institutional and contextual factors that may support or constrain one's work along various points in the hedgehog-fox continuum.

The vMEC and the sMEC have differing expectations about knowledge, knowledge creations, and access to knowledge within their communities of practice. These differences dictate the type of work, hedgehog or fox, which occurs. The sMEC, with marginal involvement in virtual spaces, has different ideals of what constitutes transformative work. Boaler and colleagues (2013) challenge scholars for intentionality in reaching out to differing communities. The very nature of the expectations in the sMEC limits what those in the academy consider and value as transformational. Those in the vMECs preference open access are inherently foxy communities of practice (Berlin, 1953; Boaler et al., 2013). In the shared virtual spaces—i.e., Twitter, Facebook groups, Instagram—communities formed around shared domains with engaged participants who wish to evoke change. This constitutes sharing things in public, free, open-source ways. Even if members of the sMEC remain dedicated hedgehogs, we believe they should spend more time within social media platforms and begin to participate with active members of the vMEC. In this way, members of the sMEC may play a more active role in transforming these virtual communities and shaping the field in the future.

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MATHEMATICS EDUCATION RESEARCHERS' INTERDISCIPLINARY COLLABORATION PRACTICES

Elizabeth Suazo-Flores
Purdue University
esuazo@purdue.edu

William S. Walker, III
Purdue University
wswalker@purdue.edu

Hanan Alyami
Purdue University
alyamih@purdue.edu

Mahtob Aqazade
Purdue University
maqazade@purdue.edu

Signe E. Kastberg
Purdue University
skastber@purdue.edu

Susan Hahn
Purdue University
hahn31@purdue.edu

Mathematics Education Researchers (MERs) working in interdisciplinary research groups face challenges as the group members expose themselves to new ways of knowing and operating. To explore collaboration practices containing at least one MER, we conducted an examination of peer reviewed papers. Three common practices were identified: (1) partnering on similar research interests, (2) cultivating relationships that promote trust and open-mindedness, and (3) receiving institutional support. We contend that interdisciplinary research groups that involve MERs need to give consideration to these practices to set themselves up for success.

Keywords: Interdisciplinary Collaborations, Research Practice, Policy

Interdisciplinary collaborations have become common in the work of mathematics education researchers (MERs) since the integration of science, technology, engineering, and mathematics (STEM) was identified as a tool for solving large-scale problems and enhancing education (e.g., National Governors Association, 2007). To understand the workings of MERs involved in such groups, we conducted a pilot study to review research articles that explored interdisciplinary research collaborations (Walker et al., 2018). We found challenges that researchers experience in interdisciplinary groups derive from exposure to others' ways of knowing and operating. Following that study, we investigated the origins of disciplines (Schön, 1983; Williams et al., 2016) and frameworks to conceptualize interdisciplinary collaborations (e.g., Akkerman & Bakker, 2011). Historically, disciplines developed under a positivist epistemology of practice (Schön, 1983) which positioned researchers as problem-solvers with little attention given to problem settings. We argue that practices are needed to guide MERs in interdisciplinary research groups. We examine: what practices do MERs use in interdisciplinary groups?

Conceptual Framework

At the 13th International Congress on Mathematical Education, MERs recommended positioning mathematics as interrelated with other disciplines (Williams et al., 2016). Disciplinarity is described as a "phenomenon" involving "specialization" of work and discourse (Williams et al., 2016, p. 4). From this view, disciplines include specialized identities, practices, and knowledge bases that develop and are transmitted. As a discipline develops, inquiry activities take up practices and discourse from other disciplines, leading to levels of integration including mono-, multi-, inter-, trans-, and meta- disciplinarity (Williams et al., 2016).

A defining characteristic of disciplines are the practices developed and used by its members. Practices are methods or techniques that become tacit, spontaneous, and automatic ways of operating or knowing within a discipline (Schön, 1983). Differences in practices and knowledge bases among researchers bring challenges related to communication, work processes (e.g., research methodologies), institutional support, and translating findings into viable results of existing domains (Bruce et al., 2017; Walker et al., 2018). This makes MERs' interactions within interdisciplinary groups a phenomenon to be studied.

Researchers who are members of a discipline have a reflexive relationship with the discipline. They contribute to the development of the discipline and the discipline enables and constrains their activities (Walshaw, 2016). When MERs join an interdisciplinary group, they are people with personal qualities and experiences, familiar with mathematics education research and practices that influence the particular ways they interact within the group. For MERs to conduct successful interdisciplinary projects, the interactions and negotiations of goals among researchers should be given consideration.

Successful interdisciplinary collaborations provide opportunities for researchers to learn new practices and develop knowledge. Yet, MERs' participation in interdisciplinary work exposes them to challenges, which can discourage them from joining others. In our previous search, we found that there are a few published articles that present these issues and consider them from the position of the researcher working with others in an interdisciplinary group (Walker et al., 2018). We assert this is due to the development of disciplines from a positivist epistemology of practice (Schön, 1983), which has contributed to deemphasize the reflexive relationship between individuals and their discipline.

Schön (1983) described how the positivist epistemology of practice in the nineteenth century promoted science and technology and their achievements to improve humans' lives. This movement with industrialism and the technological program became a dominant horizon in the Western society and contributed to a common understanding that the primary way to generate knowledge was through empirical science (Schön, 1983). Under this horizon, disciplines are seen as a problem solving process (i.e., researchers from different disciplines came together to solve a problem). With "this emphasis on problem solving, we ignore problem setting, the process by which we define the decision to be made, the ends to be achieved, the means which may be chosen" (p. 40). Uncertainty, uniqueness, instability, and value conflict are troublesome for practitioners of a discipline because dealing with these is not seen as productive practice. Thus, we argue that not acknowledging MERs' practices could limit the ability of an interdisciplinary group to conduct research.

Methods

We conducted a case study (Flyvbjerg, 2011) to explore what practices existed or emerged when MERs work with researchers from other disciplines. We identified peer reviewed papers in mathematics education journals, written in English, as the data for this study. The papers included at least two researchers from different disciplines, one of them a MER. For this research we considered mathematics and mathematics education as two different disciplines. To find research articles that involved at least one MER, we performed a search using Ebsco and Scopus databases. The keywords used for the search in titles, abstracts, and keywords were: interdisc* AND mathematics AND education. These searches resulted in 45 articles. Afterwards, each member of our research group reviewed the abstracts according to these criteria: (1) Is it a research article (i.e., we did not include commentaries, essays, book review, or theoretical papers)? (2) Are there at least two authors (researchers)? (3) Are the authors coming from at least two different disciplines or does the topic of the article explore an idea using at least two different disciplines, where one involves mathematics education? Five of the 45 articles fit these criteria and comprise the data for this study. We describe each article here. Bruce et al. (2017) explored spatial reasoning across mathematics education, mathematics, psychology, and neuroscience. Goos and Bennison (2018) investigated faculty collaboration in mathematics content and methods courses. Krummheuer, Leuzinger-Bohleber, Müller-Kirchof, Münz, and Vogel (2013) used a socio-constructivist approach from mathematics education and psychoanalysis to explain a child's mathematical creativity. Ramful and Narod (2014) explored proportional reasoning in stoichiometry problems. Thompson et al. (2013) described revising a curriculum for college biology.

We examined the five articles using Schön's (1983) definition of practice. Each article was analyzed by each member of our research group to identify the purpose of the study, disciplines involved, and practices used by the authors. Practices were identified using phrases from the research articles. For example, the following quote was identified as practices for partnering researchers: "our Spatial Reasoning Study Group (SRSG) first gathered in 2012 to explore possible research synergies" (Bruce et al., 2017, p. 144). Practices that were similar to each other were grouped and given a descriptive title. The quote above was grouped with similar quotes using the title *partnering on similar research interests.* Using this method, we identified three groups of practices that comprise the findings of this paper.

Findings

Our analysis of the practices MERs used within the five articles resulted in the following groups of practices: *partnering on similar research interests, cultivating relationships that promote trust and open-mindedness*, and *receiving institutional support*.

Partnering on Similar Research Interests

Researchers in the five studies had a common research interest that aligned with their personal and discipline-based research paths. For example, the faculty in Thompson et al. (2013) were committed to the common goal of infusing quantitative approaches to the biological sciences curriculum and faculty partnered according to their research agendas. In Bruce et al. (2017), researchers formed a study group to explore the "role of spatial reasoning in mathematics teaching and learning" (p. 144). In Ramful and Narod (2014), Ramful, a MER, wanted to explore the role of proportional reasoning in stoichiometry problems. "I guess my intention in doing cross disciplinary research is to apply the theoretical understanding in mathematics education in the sciences" (A. Ramful, personal communication, December 20, 2018).

Cultivating Relationships that Promote Trust and Open-mindedness

Researchers described the need for trust and open-mindedness among the group to understand each other's points of view. Goos and Bennison (2018) described how MERs and mathematicians recognized the importance of trust to improve the mathematics content and methods courses. Bruce et al. (2017) explained that working with people from different disciplines "entails careful study of discipline-specific vocabularies and methodologies, alongside the articulation of problems in ways that permit all potential research collaborators to identify and situate themselves. It might also require patience and perseverance" (p. 158). In Thompson et al. (2013), a core set of faculty and staff participated in nearly all efforts to revise the curriculum, which contributed to strengthening interpersonal relationships in the project.

Receiving Institutional Support

Personal and common research interests are not sufficient to promote interdisciplinary research collaborations. Institutional support was also identified as a key practice among the articles. Bruce et al. (2017) described, "[t]ransdisciplinary will not always be a comfortable fit in institutions that have been traditionally defined by rigid boundaries, vertical hierarchies, and fixed agendas" (p. 158). Thompson et al. (2013) noted that building a shared vision, college-level support for the work (e.g., release time, funding, alignment to college goals), and open conversations about course outcomes allowed the interdisciplinary curricula to be built by the group. Noting a lack of institutional support, Goos and Bennison (2018) described struggles that faculty faced due to distant campus locations, or because their work was not seen as part of their workload, but as they were seeing as "doing extra stuff" (p. 266).

Discussion and Conclusion

We explored the practices MERs used when working in interdisciplinary groups. In our data, MERs were studying concepts using multidisciplinary or interdisciplinary lenses or designing interdisciplinary curriculum. These articles included descriptions of practices that were important features of their collaborations. These practices are described in this research as *partnering on similar research interests, cultivating relationships that promote trust and open-mindedness,* and *receiving institutional support*.

Our work followed Schön's (1983) view of attending to problem settings when working in interdisciplinary groups. Our study brings attention to limitations of seeing interdisciplinary research groups as solvers of problems, which neglects group members that want to be trusted and heard (Bruce et al., 2017; Goos & Bennison, 2018; Thompson et al., 2013). It is important for researchers to acknowledge the individuals involved in the groups to successfully conduct their studies.

While the focus of the articles was not to highlight the practices researchers used while conducting their studies, almost all of them referred to practices. Goos and Bennison (2018) is an exception to this because the purpose of their research was to explore practices and learning mechanisms experienced by mathematicians and MERs during their collaboration. This article in particular, prompted us to consider how practices influenced the outcomes of research collaborations.

Being exposed to other methods brings opportunities for MERs to develop new understandings of a topic. In our data, the authors describe how collaborating with people from other disciplines brought them to know concepts in a new way. In Krummheuer et al. (2013), using a socio-constructivist perspective of mathematics learning with a psychoanalytic attachment theory helped the research team describe a child's mathematical creativity. Ramful and Narod (2014) learned that stoichiometry problems had different levels of difficulty for students when the problems involved proportional reasoning. Bruce et al. (2017) concluded that using a transdisciplinary approach could address "complex issues related to teaching and learning mathematics" (p. 157). As the field of mathematics education explores issues that are of interest to other disciplines, Bruce et al. (2017) called for MERs to "combine insights and the interdisciplinarian's practice of conducting parallel analysis" (p. 157).

We acknowledge the challenges of defining boundaries in a literature review search (Kennedy, 2007). Our findings were restricted to mathematics education journals identified by Scopus and Ebsco databases. We recognize that some journals, chapters, and commentaries are left out of these databases. Given that our search for data resulted in only five articles, we plan to interview interdisciplinary groups containing at least one MER to explore their practices and compare them with the ones identified in this paper.

Although interdisciplinary research is often portrayed as researchers coming together to solve a problem, we illustrate how consideration of problem setting helped identify three practices that were important for success of an interdisciplinary research group that included a MER. We hope these findings contribute to the construction of a new horizon where MERs feel fulfilled when engaging in multi-, inter-, trans-, and meta- disciplinary research projects.

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MORE THAN "DOING A GOOD JOB": COACHING FOR MEANINGFUL ENGAGEMENT WITH MODEL-ELICITING ACTIVITIES

Courtney K. Baker
George Mason University
cbaker@gmu.edu

Tammy S. Kraft
George Mason University
tkraft2@gmu.edu

Terrie M. Galanti
George Mason University
tgalanti@gmu.edu

Mathematics specialists are often challenged with shifting instructional practice in school settings. This qualitative study examines two mathematics specialists' facilitation of The Box Turtle model-eliciting activity (MEA) in their schools. While the mathematics specialists had received multi-year professional development on the potential affordances of MEAs, the initial implementations in K-6 classrooms challenged their presumptions about how mathematics needed to be taught and learned in their schools. Preliminary analysis identified two important findings about the mathematics specialists' positioning in bringing innovative STEM curricula to their schools. They witnessed the potential for MEAs to equitably engage students in mathematics process standards, and they were ready to advocate for more meaningful change as they began to design future MEA implementations to connect to content standards.

Keywords: Mathematics specialists, Model-eliciting activities, Equity, Instructional vision

Purpose of Study

This project drew upon educational research expertise across the STEM disciplines to develop and implement a model-eliciting activity (MEA) using data from the Smithsonian Conservation Biology Institute. University researchers collaborated with mathematics specialists, teachers and administrators in Port City Public Schools (PCPS) to develop and pilot the Box Turtle MEA. This paper explores how mathematics specialists supported the early adoption of this innovative curricular resource in K-6 settings. It will contribute to the base of research on mathematics specialist-teacher facilitation of modeling experiences and interpretation of student mathematical thinking. This research team aims to advance the relevance and affordances of MEAs across multiple fields. Specifically, we theorize the mathematics specialist as an agent of sustained curricular change within K-6 settings. We explored the following research question: In what ways does the positioning of mathematics specialists influence the outcome of MEA implementations in K-6 classrooms?

Theoretical Framework

School divisions and policymakers across the United States have "embraced" STEM (Bybee, 2010), yet mathematics may be set up to play only a supporting role in STEM integration (Fitzallen, 2015) when it is characterized as the computations or the data representations in science classrooms and technology labs. MEAs encourage ideation, testing, and revision (Zawojewski, Hjalmarson, Bowman, & Lesh, 2008) and engage elementary school students with mathematics in novel ways. MEAs also promote greater equity in mathematics assessment by shifting the focus from memorization of procedures toward "spurring, nurturing, and supporting mathematical contributions from a larger pool of students" (Lesh, Hoover, & Kelly, 1992, p. 1). Prior research has described mathematics specialists' beliefs in the power of MEAs to elicit a new vision of STEM integration for their teachers through open-ended mathematics problems with client-driven, real-life contexts (Baker & Galanti, 2017). However, there is limited research on how mathematics specialists can build this capacity in their schools.

McGatha and Rigelman (2017) define a mathematics specialist as “a professional with an advanced certification as a mathematics instructional leader or who works in such a leadership role” (p. xiv). The National Council of Teachers of Mathematics (NCTM, 2014) acknowledges the critical role of mathematics specialists in enhancing teacher capacity to enact research-based teaching practices. Mathematics specialists face many challenges, including meeting the expectations of multiple stakeholders in the school system (Chval, et al., 2010), while teaching the learning of mathematics to those involved (Felux & Snowdy, 2006). The impact of these unique challenges and environments highlights the varying roles and responsibilities mathematics specialists take on and the stakeholders they must support (Chval, et al., 2010). However, changing the mathematics practice within a school takes time, and it is essential to examine both how mathematics specialists position themselves as change agents in addition to the results of their actions.

Method

Design-based implementation research (DBIR) is a methodology in which stakeholders are committed to iteratively developing an educational innovation with a goal of broader and sustainable impact (Penuel, Fishman, Cheng, & Sabelli, 2011). This collaborative PD connected an emerging body of research on MEAs which “engage learners in productive mathematical thinking” (Hamilton, Lesh, Lester, & Brilleslyper, 2008, p. 5) to STEM integration efforts within one school division and built upon a multi-year school-university partnership (Baker & Galanti, 2017).

Setting and Participants

PCPS is located approximately 30 miles from a major city and has 7,605 students; (60.6% Hispanic, 18.4% white, 12.1% black). Within PCPS, 42.4% of students have limited English proficiency (STATE, 2015a), 73% are traditionally underserved and 53% are economically disadvantaged (STATE, 2015b). All seven elementary schools in this division hold Title I designation. Four have subgroups with proficiency gaps in mathematics of 15% or higher in either the black or Hispanic subgroups (STATE, 2015a). Specifically, this qualitative research examines the positioning of two full-time mathematics specialists: Mary, a veteran mathematics specialist situated in one school (grades K-4) full-time, and Katherine, a novice mathematics specialist equally split between two intermediate schools (grades 5-6).

Data Collection

The university-school division partnership drew upon research on specific enactments of MEAs as tools in mathematics education and explicitly focused on mathematics coaching as an in-situ resource to advance mathematics content in K-6 classrooms. Purposeful sampling was utilized as the nature of the stakeholders’ interactions warranted further exploration of these groups (Patton, 2002). Qualitative data sources such as participant surveys, observations, individual and focus group interviews, and MEA work samples were used to characterize the enactment as a vehicle for STEM integration.

The Box Turtle MEA

The Box Turtle MEA drew upon data from the Smithsonian Conservation Biology Institute and was aligned with state standards in mathematics, literacy and science so that the K-6 school stakeholders could flexibly implement the MEA to meet their instructional needs. Triads of mathematics specialists, classroom teachers and administrators were came to the Smithsonian-

University School of Conservation to experience and reflect on this novel MEA firsthand prior to implementing at their schools. The initial exploration allowed school stakeholders an opportunity to not only analyze their own collaborative engagement with this particular MEA as learners but also to reflect on transferring these ideas to their practice. The focus of the Box Turtle MEA centered on determining individual turtle's ages and creating a generalizable procedure for the client, a county park ranger. MEA materials included images of eight box turtle plastrons (bottoms), a map of where each turtle was found, and a description of the specific terrain.

Data Analysis

We identified relevant information through the use of open coding (Merriam, 2009) and collaborated to categorize codes after examining documents. Upon completion of the analyses, categories were grouped and the reduction of these categories led to the emergence of a select few themes. Internal validity was developed by searching for alternative explanations. Incorporating multiple sources into the research design was a purposeful decision that provided validation for our findings and helped to achieve construct validity (Maxwell, 2005).

Results

Mary positioned herself as a co-teacher. She identified a first-year teacher to work with in order to "build capacity" with "a willing participant." Katherine positioned herself as a provider of resources. She identified twelve teachers across two grade levels who collaboratively implemented the MEA. Findings indicated that there were two primary outcomes of the mathematics specialist positioning as they responded to their contextual needs: the potential for MEAs to equitably engage students and productive reflection to inform and advocate for future implementations.

Potential for MEAs to Equitably Engage Students

Mathematics specialists and the classroom teachers they collaborated with cited their enjoyment in the MEA training as the motivating factor for implementing the Box Turtle MEA within their schools. Prior to the implementation, mathematics specialists expressed concerns about school buy-in, teacher beliefs that MEAs were only for advanced learners, and MEA mathematical content not aligning with content standards and pacing. After implementation, mathematics specialists and teachers found that their initial concerns were unwarranted as MEAs afforded them and their students' opportunities to access new understanding of content and pedagogy.

> I've learned that it's important for teachers to see that all students can access MEAs and have entry points into the lesson. It's important to give teachers opportunities where they can see that their students can do things they think they can't. (Katherine)

Specifically addressing the "our students can't do this" concern, teachers noticed that "their" students found multiple entry points, were engaged and willingly took on the roles of collaborators, communicators, problem-solvers, and citizen scientists - making hypotheses and exchanging ideas. In this manner, students took ownership of their learning and the products of their thinking. "The equity for all kids to be able to jump in at any level was built in so that every single child felt as if they could participate at their level" (Mary).

Over the course of the implementation, teacher views evolved from MEAs as "fun" activities to MEAs as rich mathematics tasks with the potential to engage learners in productive mathematical thinking. Mathematics specialists played a major role in fostering these evolving views by facilitating discussions prior to and after the implementation.

> I don't think (the teacher) saw the richness of the math until we had the discussion about the process goals. She and I have talked about the process goals and how those overarching pieces are so much more important to be able to facilitate math long term and that way she sees the math application in the project. (Mary)

Reflection on MEAs to Inform and Advocate for Future Implementations

At the conclusion of each MEA implementation, the mathematics specialists reflected on specific actions required to continue the relationships with the identified teachers and generate additional buy-in from other school stakeholders. Katherine spoke of the need to ensure everyone involved understood the embedded mathematics content within the MEA. "All teachers need to experience the MEA before facilitating with their students - coaches need to do the math (with their teachers?) before they collaboratively design the lesson." Mary identified ways in which she could connect future MEAs to other school initiatives and even the district pacing guide. "I would like to use MEAs to kick off units. I would love for students to delve into an open-ended problem and use that as background information [that is applied] as they go through a unit of study." Katherine also spoke of the essential work required to connect MEAs to teachers' current instructional practices in order for MEAs to be sustained within the school district. "A long-term goal for me with MEAs is to incorporate them into a unit of study to really deepen students and teachers understanding of the units and to build their connections" (Katherine).

Discussion

While MEAs have been an instrument for research on modeling in K-16 mathematics education for nearly 20 years, recent studies in elementary classrooms describe MEAs as rich mathematics tasks with problem solving potential (Bostic, 2012; English, 2017). The mathematics specialists' experiences with the Box Turtle MEA, first as learners and then as teacher leaders, allowed then to see the curricular potential of MEAs as more than fun, supplementary activities. The Box Turtle MEA connected their personal commitments to making mathematics learning more equitable in their schools and their pedagogical goals of engaging students in more meaningful STEM content experiences. Each mathematics specialist described the multiple entry points as the key to engaging all learners in problem solving and communicating. Thus, the Box Turtle MEA provided a vehicle for mathematics specialists to challenge the status quo of mathematics instruction and to promote more equitable instructional experiences. MEAs more generally became a window on the realizable potential of promoting positive math identities for students who may not have previously experienced success.

> One of our students who just really has a hard time focusing on school and is always a behavior concern, was doing the best I've ever seen him do. He was so engaged which was very interesting to me. I [shared with his teacher] that she needed to document that, needed to write that down. He's working with a group. He's doing a really good job. What more could you ask for? (Emma, intermediate classroom teacher)

However, it is important to purposefully reflect on what it means to do a "good job" in mathematics classrooms. As we continue to collaborate, design, and facilitate future MEAs as equitable curricular elements, the following question needs to be at the forefront of our thinking: How do we make sure that doing a 'good job' translates to learning? Identifying ways in which mathematics specialists not only highlight these critical content connections but also relate particular MEAs to their mathematical objectives will define the next steps in our research.

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CONTROLLING FOR RACE IN MODELING MATHEMATICS TEACHERS' "VALUE-ADDED"

Phi Nguyen
University of Missouri
phinguyen@mail.missouir.edu

Cara Haines
University of Missouri
carahaines@mail.missouri.edu

We analyzed district leaders' and mathematics teachers' interpretations of race as a control variable in value-added measures of teaching effectiveness and found three distinct interpretations. Participants argued that controlling for race: (1) produces more precise statistical models; (2) removes dis-incentives from teaching particular student populations; and/or (3) creates fairer teacher evaluations. Implications for equity and policy are discussed.

Keywords: Value-added, Teacher assessment, Policy, Equity

Using value-added modeling (VAM) to assess teacher effectiveness has been a controversial issue. Though some have embraced VAM, many researchers have criticized the approach for its statistical limitations (Darling-Hammond, Amrein-Beardsley, Haertel, & Rothstein, 2012; American Statistical Association, 2014) and its role in high-stakes evaluation systems that determine compensation or termination of employment (Harris, 2009; Konstantopoulos, 2014). One consistent point of contention has been the relation between student background factors, such as race and socioeconomic status, and measures of achievement. In investigating this issue, the Value-Added Research Consortium compared multiple value-added models, with and without race and other demographic factors, and found that teachers' contributions to student achievement were often sensitive to these variables (University of Florida, 2000). In contrast, Ballou, Sanders, and Wright (2004) reported that the introduction of student background controls to the Tennessee Value-Added Assessment System (TVAAS) had only a negligible impact on predicted teachers' contributions. Prior, TVAAS had only implicitly controlled for student demographic factors to the extent that their influence on post-achievement data was already reflected in pre-data. Though much of the discourse around VAM, particularly the influence of student-level factors, has been concerned with statistical validity and reliability, some scholars have also considered issues of equity. For example, Rivers and Sanders (2002) asserted that VAM data could aid school leaders in problematizing and addressing existing policies that disproportionately assign less-effective teachers to schools with low-income, marginalized student populations. In contrast, Johnson (2015) argued that evaluation based on value-added measures may discourage effective teachers from working in high-need schools.

As one of the two most tested subjects (U.S. Department of Education, 2002; 2015), mathematics is especially vulnerable to implications of student achievement-based teacher assessment policies such as VAM. Differences in test scores have also attracted considerable attention to racial "gaps" in mathematics opportunity and achievement (Flores, 2007). If value-added policies are to improve teaching and learning, particularly for historically marginalized populations, it is important to attend to how district leaders and mathematics teachers frame race as a control variable, as their discourse can help to either maintain or transform institutional culture (Bertrand, Perez, & Rogers, 2015). For example, the ways teachers frame VAM as being unfair to those who work in more challenging schools are rooted in deficit assumptions about African American, Latinx, and Native American students (Jiang, Sporte, & Luppescu, 2015).

Consequently, discourses around value-added policies contribute to the social devaluation of these students and reinforce the racial hierarchy of mathematical ability narrative (Martin, 2009). In this paper, we draw on critical race theory to pursue the question of how district leaders and mathematics teachers interpret race as a control variable in teacher value-added measures.

Theoretical Perspectives

Originating in legal scholarship (Bell, 1992) and introduced into education by Ladson-Billings and Tate (1995), critical race theory (CRT) challenges the view of racism as a problem of individual beliefs and prejudices and instead interrogates how structures and discourse reproduce and normalize racism. Aligning with perspectives that view race as socially and politically constructed, we see district leaders and teachers as consumers and producers of racial ideologies and discourses. In examining their interpretations of race as a control variable, CRT's theoretical arguments, including the view that racism is endemic and skepticism towards claims of objectivity and meritocracy, guided us in interpreting the race-related structures of inequity that are reproduced through the formulation and enactment of value-added policies. As an enterprise, it is important for mathematics educators and policy makers to understand how value-added policies might allow racism to function in the teaching and learning of mathematics and contribute to broader narratives about who is (and is not) good at mathematics (Martin, 2009).

Methods

Our investigation originated from a multi-year, multi-institutional partnership with an urban school district in the northeast US focused on addressing racial (in)equity in secondary (6-12) mathematics. At the time, the district was focused on launching a new teacher evaluation system that included observation, student surveys, and value-added modeling of standardized state test scores. Unlike the state's value-added model that used a longer history of achievement, the district's model incorporated demographic control variables including race. The district was also pursuing equity-focused efforts such as establishing a central equity office and attending to equity in district initiatives, including curriculum and professional development, and personnel evaluations. The student population was approximately 55% African American and 33% White, with 62% identified as economically disadvantaged.

A variety of data was collected for the partnership's primary efforts, including annual interviews with district leaders and mathematics teachers. All interviews were audio-recorded and transcribed. For the investigation discussed in this paper, we focused only on interviews in which participants addressed the district's teacher VAM. Within these interviews, our main focus was district leaders' and teachers' discussions of race as a control variable, but we also considered general responses about teacher VAM to understand the broader district policy context. The data set for this analysis included 20 interviews with 19 individuals. The individuals held a range of positions at the district and school levels. Nine participants were district leaders (e.g., assistant superintendents; directors of district departments) and the other 10 were teachers from seven different middle or high schools. We analyzed interview transcripts with the purpose of identifying different interpretations that emerged from the data for controlling race in teacher value-added models. Our analysis was informed from the literature previously reviewed, including arguments for statistical precision (e.g, University of Florida, 2000), not dis-incentivizing teachers to work in high-need schools (e.g., Johnson, 2015), and fairness for teachers (e.g., Jiang, Sporte, & Luppescu, 2015). We analyzed the interviews and identified a single response category in some, and multiple in others, and then revisited our initial pass to refine our characterizations of the categories. Peering through a CRT lens, we also considered whether and how district leaders and teachers reinforced deficit assumptions of historically marginalized student populations and perpetuated claims of objectivity and meritocracy.

Results

Our analysis resulted in the identification of three distinct interpretations among district leaders and teachers for why and/or what it entails to control for race in teacher VAM. We present each of the explanations accompanied by relevant interview data and brief discussions.

The Statistical Precision Interpretation

The first rationale is that controlling for race and other demographic factors results in more precise statistical models. For example, Taylor, a district leader, discussed:

> Our value-added measure only looks back at up to 2 years of previous assessment results, and then pulls in student demographic information like race. [The state model] pulls as many years as back as they have available in terms of assessments and so essentially, those 2 approaches end up being very very similar in terms of the extent to which, um, the results are controlled for things are outside of teachers' control, right? So, for what we lack in terms of only pulling two years of data back, we kind of gain with the demographics. And then, essentially, [the state model] doesn't need to include the demographics because going all the back to all the assessments that are possible, they kind of build a full picture from that lens… we've actually done analyses looking at teacher level [state model] and value added, and for the vast majority of teachers, they're getting pretty similar signals from both of the systems.

In both the district and state VAMs, the focus was on producing precise estimates of what an individual teacher contributes to a student's success as compared to what a history of achievement (or demographic factors) would suggest. In comparing the two, Taylor suggested that controlling for demographic factors is a statistically valid substitute for a longer history of achievement. By invoking statistical precision as basis for comparison, both models claim objectivity, yet fail to account for why the correlation that supports such a substitution exists. In particular, neither model explicitly attended to the influence of institutional racism on achievement. As such, both value-added models contribute to color-blind ideologies in which race (beyond its framing as a categorical variable) is portrayed as irrelevant to understanding students' learning and school experiences.

The Removing Dis-incentives Interpretation

The second rationale is that VAMs include demographic control variables so that teachers are not dis-incentivized to work in schools with particular student populations. In explaining the decision for incorporating race as a control variable, Morgan, a district leader, said:

> It came down to the idea that we want to incentivize our best teachers to work with our most challenged students. And if we are not recognizing the additional challenge that these students present and don't give them essentially, you know credit for that then, we would be dis-incentivizing that. And you know we felt really comfortable about using this as a control when we're looking at teachers because this is not something that's within the teacher's control.…And we're actually encouraging schools to work towards evidencing significantly higher contributions for African American students than all their other students in the school, because it's only through achievement of that type of pattern that we will actually be unpacking and eliminating any sort of racial disparities.

Morgan suggested that including race as a control variable in the district's value-added model might incentivize teachers to work in predominantly African American schools, as doing so would account for the "additional challenge that these students present." Though Morgan acknowledged the role of structural inequities (e.g., distribution of high-quality teachers) in student achievement differences, the cause of those differences was attributed to student characteristics. That is, rather than recognizing inequitable social structures that often affect schools that are predominantly African American (e.g., a lack of resources), Morgan suggested that teachers are unmotivated to teach at such schools due to the inherent qualities of its students. Moreover, Morgan's framing of students' racial identities as an "additional challenge" communicates (and reinforces) deficit narratives of African American children.

The Fairness Interpretation

The third interpretation we found is that, in controlling for race, teachers are more fairly evaluated. This argument stems from the notion that student characteristics (e.g., race) and circumstances (e.g., economic status) influence their achievement, and therefore mediate teachers' contribution to student achievement. For example, Mason, a teacher, explained, "I do like [the district's] value added measures as far as they take in more factors based on kids' individual issues… Our population is much different than our suburbia populations. I don't think it's fair that we're under the same microscope as teachers at other districts." Like Morgan, Mason engaged in deficit discourse by attributing differences in achievement to student qualities rather than structural disparities such as the inequitable distribution of resources and quality teachers. In doing so, this interpretation perpetuates the myth of meritocracy in which students' academic achievements depend on their individual merits rather than the (lack of) structural resources and opportunities they are afforded. Mason's interpretation also centers teachers and their evaluation rather than students and their learning opportunities.

Discussion & Conclusion

Our study contributes to the research base on value-added measures by exposing issues of equity that are often overshadowed by statistical concerns. In their interpretations of race as a control variable, district leaders and teachers employed deficit-oriented talk in attending to student qualities – in some cases, even as they were describing the district's VAM policy as part of a broader set of equity-focused initiatives. Though, in failing to account for the reality of institutional racism, they reinforce the "discourse of deficiency" that views these communities as lacking the resources to be academically successful, alongside a mathematics education discourse that mathematics is a discipline in which those who are successful are White, middle-class males (Martin et al., 2017; p. 623). Using CRT to analyze one district's framing of race in VAM uncovered the ways deficit discourses reproduced the inequitable status quo.

Our purpose for this paper was not to argue if/how race should be controlled in teacher value-added measures, nor to uncover all the possible interpretations of controlling for race. Rather, in revealing the ways racial discourses were operating and being perpetuated in district leaders and teachers' interpretations, we provide evidence for why researchers and policy makers should move beyond methodological considerations in the formulation and enactment of value-added policies. If such policies are to address racial (in)equity, our findings suggest a need for a more critical conceptualization of race that centers student learning and school experiences rather than teacher evaluation. This includes not only an explicit attention to the influence of institutional racism on achievement, but also a consideration of the ways in which the enactment of and discourses around value-added policies shape students' learning opportunities.

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NUMBER TALKS AS A HIGH-LEVERAGE METHOD TO FOSTER EQUITY IN DIVERSE CLASSROOMS

Cassandra Cook Eatmon
George Mason University
ccook19@gmu.edu

Courtney K. Baker
George Mason University
cbaker@gmu.edu

Kimberly Morrow-Leong
George Mason University
kleong@gmu.edu

Theresa Wills
George Mason University
twills@gmu.edu

This research report discusses a study of mathematics specialist candidates' response to and reflection on equity supporting practices in a K-12 setting. In this study, mathematics specialist candidates learned a structure that encourages mathematics discourse and sense-making. They received English learner-specific professional development in implementing number routines and reflected on their practice as it relates to their learning, student learning, and the learning of the professional community. Preliminary findings suggest that EL-specific professional development increases understanding of equitable instructional practices and supports culturally responsive instruction.

Keywords: Instructional leadership, Equity, Teacher educators, Teacher knowledge

Mathematics education is compelled to address issues of equity in mathematics classrooms by preparing teachers and school leaders for working with diverse populations. One of the challenges faced by school districts is that the number of mathematics teachers who are adept at differentiating for the needs of ELs is not keeping pace with the demand for these highly-qualified teachers. While there has been a consistent increase of ELs in the schools, professional development for teachers on EL-specific knowledge and skills has not increased at the same rate (Lucas & Villegas, 2013; Quintero & Hansen, 2017; Ross, 2014).

Providing appropriate professional learning for in-service mathematics teachers to work with ELs in the mathematics classroom is a complicated task. Given the large number of mathematics teachers, a systematic way to provide accessible, ongoing professional development for in-service teachers is required to develop the specific linguistic and cultural knowledge and skills for working with ELs (de Araujo, Roberts, Willey, & Zahner, 2018; Lucas & Villegas, 2013; NCSM/TODOS, 2016). We began this study to consider a method for fostering equity of student voice through professional learning around number sense routines for mathematics specialist candidates in a master's program at George Mason University (MASON). This targeted professional development experience offered candidates a structure to share with teachers in order to promote productive language use in mathematics classrooms.

Research Question

How do math specialist candidates respond to and reflect on professional learning around supporting ELs in classroom number routines?

Theoretical Framework

Equity

Understanding that our context matters (Gutiérrez, 2012), and given that we are all White, female educators from a highly-resourced area that is linguistically and culturally diverse, and our mathematics specialist candidates are also predominantly White and female, it is important that we intentionally focus on equity as part of the mathematics specialist program. We must recognize our positionality in relation to access to high-quality education and our responsibility for making high-quality education accessible for the culturally and linguistically diverse learners in the settings where our candidates work. Gutiérrez (2012) identified dimensions of equity supporting a *dominant axis*, the traditional role of students and education in society, and a *critical axis* that reflects the position of students in society. This study was designed to support equity on both axes by providing professional learning for the mathematics specialist candidates around understanding the assets and needs of ELs in a classroom discourse activity. Candidates learned to provide opportunities for access (*dominant axis*) and supportive practices for empowering students as classroom discourse participants (*critical axis*) (Gutiérrez, 2012).

The Roles of Language in Learning

There exists an assumption that good teaching for native English-speaking students is adequate for teaching ELs (de Jong & Harper, 2005), so teachers are not given specific training for working with linguistically diverse learners. De Jong and Harper (2005) offer a framework for the consideration of the role of language in education that makes explicit what mainstream teachers need in terms of knowledge, skills, and dispositions to successfully meet the needs of English learners. Language allows teachers and students to tap their individual and common funds of knowledge to anchor new learning. As such, teachers need an understanding of the language acquisition process to be able to support student development in language and literacy through the content (Bunch, 2013; de Jong and Harper, 2005; Faltis & Valdés, 2016). They also need the ability to identify the language demands of a mathematical task (Lucas & Villegas, 2013) and a collection of scaffolding strategies for ELs. Providing EL-specific professional learning for teachers around the roles of language and culture supports increasing teachers' sense of self-efficacy for working with ELs (Ramos, 2017; Ross, 2014) and promotes the positive dispositions of teachers for the roles of language and culture instruction, in addition to the content (de Jong & Harper, 2005).

Number Routines

A number routine is a brief, planned classroom activity that enhances students' ideas of magnitude, supporting calculation strategies and skills in manipulating numbers. As an oral and aural activity, a number routine promotes language use. Mathematics education is called to move beyond rote memorization of procedures and focus on conceptual understanding and mathematical reasoning (CCSSI, 2010; NCTM, 2000). Number routines may help educators make the shift to fostering reasoning and sense-making through oral routines that also provide access for ELs who may be stronger in oral language skills (Chen & Bofferding, 2017; Humphreys & Parker, 2015) by having students communicate their thinking and develop "more accurate, efficient, and flexible strategies" (Parrish, 2011, p. 199). Teachers' roles as facilitators (Chen & Bofferding, 2017) of the discussion are reliant on their ability to anticipate student responses and have strategies to respond and move the conversation along with appropriate instructional moves that support all students' participation (Anderson, Chapin, & O'Connor, 2011).

Method

Context

Participants were candidates in a Mathematics Education Leadership Master's Degree Program at MASON. The program consists of 10 courses, five content-related and five leadership-related. The mathematics specialist master's program at MASON develops teacher leader skills and dispositions (Baker, Bitto, Wills, Galanti, & Eatmon, 2018) that support our specialist candidates to provide ongoing support for teachers with whom they work and to advocate for best practices that foster equitable access for all students. The candidates were enrolled in two sections of a mathematics content course in Fall 2018, Number Systems and Number Theory for K-8 Teachers: one a traditional face-to-face format; the other a 100% synchronous online format. The course is designed to meet NCTM content and process standards for mathematics specialists (NCTM, 2012) and uses the *Principles to Actions* (NCTM, 2014) as reference for equity practices.

As part of the course, candidates learned about using oral number routines in mathematics classrooms to develop conceptual understanding and fluency while also supporting mathematics discourse. The assignments contained two opportunities to research, design, and implement a number routine. After the first number routine assignment was completed, and before the second number routine was planned and presented, a short intervention was provided to each class as a mini-lesson on understanding the role of culture and language for ELs during number routines and possible scaffolds and supports that could be used to support EL participation. The WIDA English language development standards (WIDA, 2012) provided the framework, including examples of appropriate questioning and responses to be expected at each level of English language development.

Participants

There was a total of 31 mathematics specialist candidates across the two sections; the majority (90%) were female. They were 77% White, 16% African American, and 7% Other. The candidates represented various public school districts in Common Core and non-Common Core states, and one was overseas in a Department of Defense Education Activity (DoDEA) school.

Data Collection and Analysis

Reflection assignments for each shared number routine submitted by the candidates were collected and analyzed. One experience preceded a brief lesson on EL assets and strategies for supporting ELs during a number routine, and one followed. In addition, researcher observations and shared slides from each experience were also considered in the analysis.

The student work was analyzed qualitatively using a phenomenological approach (Creswell & Poth, 2017), looking for patterns in participant behavior or expression. Specifically, any mention of consideration for EL needs or supports to facilitate participation by ELs or specific groups of students was noted. In addition, in vivo coding (Saldaña, 2016) was used to capture the participants' words as they expressed examples of the reflections discussed below.

Results

Reflection Prior to Intervention

Prior to the explicit instruction on language and cultural understandings needed to support ELs in mathematics and, specifically, during a number routine, the reflections of the mathematics specialist candidates were focused solely on the procedures for the number routine and their role as teacher more than the students' role as participants in learning. None of the reflections noted differentiation for ELs or students who require additional support. While there was little reflection on supporting diverse learners, there was a good amount of reflection on the learning of the practice of number routines and some potential challenges. One candidate noted, “There were a lot of people sharing their thinking, but there were also many people who didn’t get a chance to share at all. Those students were only able to practice their receptive communication skills.”

Reflection After Intervention

In the second round of number routines several groups tried “Which One Doesn’t Belong” (WODB, 2013) and similar routines noting that the multiple possible responses, as well as multiple strategies for answering, was supportive of increasing participation among the students, “The visual nature of the WODB routine was the first layer of support we selected to give English language learners (ELLs) an entry point in discourse,” using sentence frames as scaffolds.

Candidates also reflected on a significantly increased number of strategies they employed to support participation of students, and ELs in particular, in the routine based on English language proficiency levels (WIDA, 2012). These included such supports as leveled sentence frames, word banks, pictorial representations, visualization strategies, and anchor charts. Several candidates noted the increased confidence of their students in addition to their own increased confidence with the routines.

Discussion

We recognize that the intervention provided on knowledge and skills for working with ELs was not the only factor that supported the growth of the mathematics specialist candidates in this study. However, the significant increase in their consideration and implementation of EL-specific strategies and their increased reflection on participation of all students is notable in the second round of reflection papers.

For mathematics specialist candidates, this experience with classroom number routines is one that can be easily replicated for supporting teachers of diverse learners with little impact on instructional time; teachers can see it as a routine that is easy to try and implement. One participant reflected, “I plan on using some of these resources with new teachers at my school during a planning meeting. I want them to have another option of things they can do with their students to activate prior knowledge.”

For future consideration of these methods and outcomes, we may consider changing the assignment to include a requirement for considering differentiation strategies in the candidates’ preparation for the initial number routines to provide consistency in assignment expectations for both experiences. Additionally, having similar opportunities in subsequent courses in the cohort may provide a more longitudinal look at the effects of EL-specific professional learning.

By providing dedicated professional development on support strategies for EL students, mathematics specialist candidates, and by extension, the classroom teachers in their practice, directly promote the language use and contributions of EL students. The number routine shows evidence of being a high-leverage practice given the short time commitment and focus on oral language. They have the potential to support sense making around numbers and also to support a wide variety of oral language practices appropriate for ELs. To meet the growing need for mathematics teachers who are knowledgeable of language and cultural factors, scalability of this professional development is an area for future study.

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HOW AN IMMERSIVE SCHOOL-BASED PARTNERSHIP SURFACED CRITICAL TENSIONS IN PRACTICE FOR PRESERVICE TEACHERS

Corey Webel
University of Missouri
webelcm@missouri.edu

Stephanie Hopkins
University of Missouri
sand54@mail.missouri.edu

Field experiences present an opportunity for preservice teachers to practice teaching skills in an authentic educational context. In this paper, we describe the case of a partnership between a teacher preparation program and an elementary school that included daily one-on-one sessions pairing junior-year preservice teachers with individual students to discuss mathematics (the "Math Buddies" program). The partnership brought significant tensions to the surface, including a conflicting vision of mathematics instruction between the partner school and the preparation program, which led to the eventual abandonment of the partnership.

Keywords: Field experiences, School partnerships, Joint work, Boundary practices

Research in teacher education has long emphasized the importance of clinical sites as an authentic opportunity to practice teaching skills (e.g., Darling-Hammond, Hammerness, Grossman, Rust, & Shulman, 2005). However, clinical work for preservice teachers (PSTs) occurs in spaces where there are a variety of programs, stakeholders, and potentially conflicting goals at play, and we argue that our field would benefit from detailed descriptions of the ways that the particulars of such clinical spaces get negotiated.

In this paper, we describe an example of a partnership between a teacher preparation program and "Briar Elementary School," which we frame in terms of "joint work" (Penuel, Allen, Coburn, & Farrell, 2015). Using interview and survey data, we describe the initial goals of the partnership from multiple perspectives, the ways in which those goals came into conflict as the partnership was implemented, the negotiation of specific features and logistics of the partnership over the course of the program, and the concerns that were and were not able to be addressed. We conclude our analysis with potential explanations for why the program was eventually abandoned, including implications of what might be seen as a failed attempt at joint work.

Rationale and Literature Review

Recently, there has been a shift in focus from developing knowledge *for* teaching to developing knowledge *through* teaching (Ball & Forzani, 2009). This could include a focus on specific teaching particular skills, like noticing or responding to children's thinking (Jacobs & Empson, 2016), or particular instructional activities that draw on children's thinking, like enacting a sorting task (Baldinger, Selling, & Virmani, 2016). These practices and activities are often developed through cycles of enactment to explore, rehearse, enact, and debrief (Lampert et al., 2013). However, *enacting* teaching practices requires contexts in which teaching is occurring; for preservice teachers, this usually involves field experiences.

In early field experiences, PSTs typically spend time in a classroom with a cooperating teacher while they take courses at the university. These experiences can be important sites of learning for preservice teachers (e.g., Borko et al., 2000), but the quality of such experiences can be highly variable (Darling Hammond et al., 2005). In many cases, field placement schools have only loose relationships with teacher education programs, resulting in the relegation of the school to a source of children to “practice on” rather than a more democratic partnership where all stakeholders contribute to the learning experiences (Zeichner, Payne, & Brayko, 2015).

Penuel and colleagues (2015), in describing partnerships between researchers and schools, contrasted “translation,” in which schools are expected to translate research findings into practice, with “joint work,” in which “both district goals for improvement and aspects of the research are defined and evolve through interaction, rather than being planned fully ahead of time or defined by either researchers or practitioners independently of one another” (p. 183). They describe “boundary practices,” which are “routines that only partially resemble the professional practices of researchers and practitioners” (p. 187) and include the crossing of typical boundaries in order to increase shared understanding of problems.

The boundary practice we focus on in this paper consists of structured one-on-one mathematical interactions between PSTs and elementary students (“Math Buddies”) that was negotiated as part of the partnership between the teacher preparation program and Briar Elementary. PSTs worked at the school from 8:00 am to 2:00 pm every weekday. Elementary children were selected from Grades 3 – 5 to be Math Buddies based on teacher recommendations. PSTs met with their Math Buddies for 30 minutes each morning and would spend the rest of the day either in methods classes or in their host teacher’s class. The partnership lasted two semesters, after which the program was discontinued, primarily based on the judgment of the director of elementary education at the teacher preparation institution.

Our goal in this paper is to document the tensions and conflicts that arose as stakeholders negotiated the details of how the Math Buddies boundary practice would be structured and supported. Our research questions are: 1) *What are the tensions between the goals articulated by major stakeholders in the proposed partnership?* 2) *How do these tensions play out in the negotiations of the details of the joint activities?*

Methods

Data Collection

Multiple interviews and meetings were conducted with the major stakeholders in the partnership, including the principal of Briar Elementary School (Kyle), the university instructors who held their classes at the school, and the director of elementary teacher education (Brittany). All names are pseudonyms except for Corey, the first author and instructor for the mathematics methods course. A meeting with teachers comprising the “math team” at Briar School was also recorded. In addition, we also analyzed data from PSTs enrolled in the elementary teacher preparation program, including three reflection papers spaced out across the semester, an anonymous survey collected mid-way through the semester, and anonymous course evaluations.

Data Analysis

Recordings of interviews and meetings were analyzed using V-Note, a software program that can be used to assign codes directly to audio/video data. We initially coded the data according to the goals each participant articulated about the partnership, as well as the strengths and concerns that were articulated as the semester progressed. On a second pass through the data, we identified tensions between how different stakeholders expressed their goals, strengths and concerns regarding the partnership. In the results section, we share these tensions in relation to specific components of the partnership (the schedule and the vision for mathematics teaching), highlighting the ways in which the tensions were undiscussed and unresolved at the administrative level, while becoming sources of increasing conflict for the PSTs.

Results

Shared Goals

Broad goals discussed by multiple stakeholders included supporting students' mathematical learning and providing opportunities for PSTs to practice teaching. The school principal, Kyle, described a need for additional resources to help with struggling students: "I wanted to have additional tutoring for our kids… I was worried that we would have students who were struggling and no one to help. I was trying to figure out, how can we get help?" Kyle met with university administrators to develop the idea of Briar Elementary hosting methods classes, with the rationale that PSTs would get more and better access to elementary children, and in turn would provide support for struggling students via one-on-one interactions. As Kyle described,

> The university would benefit in that their students will be exposed to lots of different teaching methods and be around the kids more, and we would benefit through the extra tutoring that we could get the students to do. It seemed like a win-win to me.

In Spring 2017, Brittany met with faculty who typically teach in the program to discuss the possibility of partnering with Briar Elementary. Corey, the instructor for the math methods course, was particularly drawn to the possibility of increasing the quantity and quality of opportunities for methods student to interact with children and their mathematical ideas: "our [PSTs] would get so much more time to talk to kids about math than they currently get." He also hoped that Math Buddies would constitute a more equitable partnership: "[The district] often doesn't feel like they are benefitting from this relationship and this is a way they can." Brittany, the director of elementary education at the university, saw the partnership as an opportunity to more authentically position the PSTs as professional educators and to help them understand the daily work of teaching: "being in this environment all the time, we are going to treat them like teachers." In summary, each stakeholder believed the partnership could be mutually beneficial, providing new learning opportunities for both elementary students and PSTs.

Tensions Over Specific Goals

The Math Buddies schedule. Over the course of the partnership, one substantial tension arose around the weekly schedule. Kyle wanted Math Buddy sessions every day of the week in order to provide as much support as possible to children, but Corey expressed concern about the time that would be needed for PSTs to debrief and make sense of children's thinking. At the same time, he wrestled with an obligation to ensure that the Math Buddies program was meeting a "felt need" for the Briar community: "we have been trying to be responsive to what is happening in [the Briar] class and what we are talking about in [our methods] class."

The schedule remained a source of tension throughout the semester. According to Brittany, PSTs voiced concerns amount of time they spent preparing for Math Buddies and their lack of time in host teachers' classes. Eventually a new schedule was negotiated for the Spring semester in which PSTs did not come to Briar Elementary on Fridays.

Vision for mathematics instruction. While Kyle did not fully articulate a vision for mathematics instruction, he provided a specific "Daily Five" routine in which groups of students (usually determined by "similar ability") rotated through stations. The teacher worked with each group for about 15 minutes, while other groups worked on other activities (games, fluency practice, a mathematics software program) that might not be related to the main lesson.

In contrast, the vision of mathematics instruction promoted in the methods course, drawing on recommendations set forth by the National Council of Teachers of Mathematics (2000; 2014), focused on posing problems, encouraging sensemaking, facilitating discussion between students, and developing understanding through a "launch, explore, summarize" lesson format in which all students work on the same task at the same time. While Corey hoped that Math Buddies would provide PSTs opportunities to practice skills like eliciting and responding to student thinking, he also acknowledged that the vision of teaching promoted in the course would likely be difficult to enact within the "Daily Five" structure expected at Briar.

These differences in vision for mathematics instruction created tension for the PSTs, who were asked to work on the practices being encouraged in the methods class and also respond to obligations to their Math Buddies. For example, in their reflections and evaluations, they conveyed many sentiments such as: "I am unsure about how meeting every day will help my math buddy in her math class," and "This experience probably helped [my Math Buddy] with word problems but not much else. Also, I wasn't allowed to tell her if she was correct or not when she answered each word problem which honestly seemed to be confusing for her…. While it is important to learn how to observe and ask guiding questions, I think it is also valuable to learn how to actually teach math too."

Our data showed that several PSTs perceived a clear distinction between the practices emphasized the methods class ("observe and ask guiding questions") and the needs of their Math Buddies. While both Corey and Kyle agreed that the program should help children and PSTs, they did not discuss the details of *how* it would do so, and did not connect the different contexts that children would experience to a coherent vision for mathematics teaching and learning.

The dissolution of the program. Though changes were made the second semester that seemed to improve Math Buddies, the change to a four-day schedule resulted in fewer opportunities for PSTs to see classroom teaching. This led them to question the benefits of being at Briar Elementary School, and their end-of-program evaluations were overwhelmingly negative. Based on these, Brittany decided not to commit to the program for the next year.

Discussion and Conclusion

From our analysis, we draw the possibility that the program did not create new tensions between the methods course and the field placement so much as it brought existing tensions to the surface in ways that PSTs could not ignore. Unlike typical arrangements, where PSTs fulfill obligations to the school context and university context in largely separate settings, in our project, PSTs had to respond to the pressures of the methods class and the pressures of the field placement *simultaneously*—that is, during their interactions with their Math Buddies (see Figure 2). On the one hand, they were obligated to demonstrate particular types of moves to elicit, support, and extend students thinking (Jacobs & Empson, 2016), which some seemed to interpret as being forbidden from confirming or critiquing student solutions. On the other hand, because they knew the Math Buddies and their teachers well, they felt obligated to support student learning, which many did not see as consistent with the use of supporting and extending moves. We conjecture that such tensions are always at play when PSTs move between their university courses and field placements, but the Math Buddies partnership made these particularly salient because PSTs were expected to respond to both sets of obligations in the same setting.

Other lessons learned from the partnership include a lack of attention to the needs and perspectives of the host teachers. At the December math team meeting, the teachers described a lack of clarity about what the Math Buddies were doing, and seemed to experience little agency in deciding what the program entailed or how it would meet their needs. In future such partnerships, we suggest a more substantive and inclusive process to develop and strengthen a shared vision across contexts, one that addresses the details of instruction and draws on a common conception of what it means to learn and do mathematics.

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THE DISTANCE BETWEEN US: MATHEMATICS AND SPECIAL EDUCATION TEACHER EDUCATORS' EVALUATION OF PURPOSEFUL QUESTIONS

Rob Wieman
Rowan University
gomathman@yahoo.com

Mollie Sheppard
Saint Joseph's University
msheppar@sju.edu

Keywords: (Dis)ability Studies; Teacher Beliefs; Instructional Vision

Background

Just as special education and mathematics teachers are increasingly called upon to collaborate, special education teacher educators and math teacher educators increasingly share responsibility for training *all* teachers to work together to teach mathematics to a diverse population of students. However, research in math education and special education has led to different descriptions of "best practice," stemming from different ideas about content, students, learning and teaching (Boyd & Bargerhuff, 2009). Despite the apparent lack of alignment between special education and mathematics education *research*, there have been few attempts to examine whether special education and mathematics education faculty differ in their *practice*.

Our Study

In designing our study, we chose a problem of practice that would be very familiar to both mathematics teachers and special education teachers – posing purposeful questions (NCTM, 2014) to an elementary student with learning disabilities who asked for help with a word problem. We then asked a group of special education and mathematics teacher educators to rank a set of possible purposeful questions from most effective to least effective. We wanted to see how their rankings compared.

Findings

Our findings indicate that special education and math education teacher educators do have some general areas of agreement. All groups of teachers gave high rankings to initial assessment questions that probed the student's understanding and prior knowledge. Questions that focused on operations or directed students to a particular solution method were ranked low and described as overly leading by all groups. One interesting difference was in how the two groups ranked a question asking the student to underline important words in the problem. We hypothesize that mathematics teacher educators may see it as advocating a key-word strategy, while special education teacher educators may see it as an essential literacy and language support.

We see this work as pushing mathematics and special education teacher educators to move beyond the horizons that define their fields towards a shared understanding of mathematics teaching and learning for diverse groups of students. While we cannot say how different groups of teachers would use the information that they gain from their initial questions, our findings indicate that there may be fertile ground for future work in this area.

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DEFINING STEM AND MATHEMATICS IN A STEM STANDARDS DOCUMENT

Miriam Gates
Boston College
gatesmd@bc.edu

Min Jung Kim
Boston College
kimbzw@bc.edu

Even though mathematics is part of STEM education, how the subject should be conceptualized is uncertain due to a lack of common definition and goals of STEM education. This paper addresses how STEM and mathematics are conceptualized in a state standards document by using a frame analysis and Ernest (1989)'s categories of mathematics.

Keywords: STEM, Mathematics, Policy analysis

As of 2016, forty-one U.S. States provide K-12 standards contain references to STEM education (Carmichael, 2017); however, there is no agreement in field about the definition of STEM (e.g., Breiner, Harkness, Johnson, Koehler, 2012; Sanders, 2009). Further, educators have expressed the concern that STEM experiences cannot replace courses within individual disciplines. In light of this concern and the explosion of policy documents around STEM, the purpose of this study, is to examine policy documents to understand how policymakers conceptualize mathematics in the context of STEM education policies. For this study, we have focused on the Maryland STEM Standards. The policy documents clearly lay out an integrated view of STEM, whereby the disciplines are supportive of each other (Carmichael, 2017). Both English (2015) and Fitzallen (2015) argue that in integrated applications of STEM education, mathematics has the tendency to be underrepresented or represented as only an applied science.

We undertook a frame analysis (Reese, 2001) to understand how STEM and mathematics were conceptualized in the Maryland STEM standard policy documents. To understand how mathematics was defined we analysed the policy documents using Ernest's (1989) categories: instrumentalist, Platonist, and problem-solving. The instrumentalist conception suggests mathematics is a collection of skills for application to another context; the Platonist conception identifies mathematics as a coherent static body of knowledge; and the problem-solving conception centers mathematics is a constructive, creative, human endeavor (Ernest, 1989).

As a result, we discovered that the instrumentalist view of the subject has the highest frequency, both in mathematics and STEM standards. This implies that mathematics in this document is generally conceptualized as a tool or set of tools that can be applied in contexts outside of mathematics itself. After the instrumentalist view follows the Platonist perspective. Finally, the view that considers mathematics as a creative human endeavor came in last. Similar to this pattern, we found that 62% of the standards are designed to enhance students' career-related skills. With a heavy emphasis on how STEM education prepares students for future careers, mathematics is considered a set of knowledge that can be used to seek complicated solutions in interdisciplinary contexts. We believe that the findings raise a very important question - what is the purpose of STEM education as separate or the same as mathematics education? We suggest that the way that mathematics will be portrayed in future STEM projects depends on the purpose of those projects.

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INTEGRATING 360 VIDEO INTO A MATHEMATICS METHODS COURSE

Karl W. Kosko
Kent State University
kkosko1@kent.edu

Keywords: Instructional activities and practice; Technology; Teacher Educators

Use of videos, and other representations of practice, to prepare preservice teachers (PSTs) in methods coursework is commonplace (Grossman et al., 2009). The effectiveness of such representations is often tied to various technological scaffolds that help to better approximate the practices of educators (Herbst et al., 2016; Stockero et al., 2017; van Es & Sherin, 2002). Although evaluation of both representations of practice, and related technology, has shown how their use affects PSTs, there is relatively little work on examining mathematics teacher educators' (MTEs) experiences integrating such representations and technology.

The purpose of this study is to examine my own experience as an MTE integrating a novel video medium: 360 video. 360 video differs from traditional video by recording in a spherical direction and allowing the viewer to adjust and choose what is viewed in this recorded sphere. Experiences included creating three videos of math instruction (multiplication/division & fractions), and using VR headsets and tablet technology for PSTs to view these videos.

To evaluate my experiences, I engaged in a narrative-based self-study (Loughran, 2007; Tzur, 2001). Data included written reflections, emails with colleagues, and a yearlong text message based conversation between myself and a colleague in educational technology. These written sources were evaluated for themes in how I integrated 360 videos for teaching PSTs. Analysis of written records revealed three themes that influenced my pedagogical decision making: accessibility, perceptual capacity, and attending to mathematics. *Accessibility* of the 360 media emerged in my evaluation of various platforms for disseminating 360 videos (YouTube, Kaltura, Vimeo, etc.). This theme also was influential in identifying virtual reality (VR) headsets to use with PSTs, as some allowed for casting (sharing on a viewable screen) what a headset wearer sees, but may not function in certain wifi network conditions. *Perceptual capacity* refers to what aspects of a scenario are available to view (Ferdig & Kosko, in review). This theme emerged regarding how video resolution differed by recording device (brand of 360 camera), platform, and VR headset. Additionally, I began to experiment with extending perceptual capacity to allow for multiple vantage points by recording with multiple 360 video cameras. *Attending to mathematics* focused on my own efforts to scaffold PSTs' professional noticing. Often, this theme interacted with the prior two. PSTs attended to specific mathematics when viewing 360 videos on their own, but this was less effective when I adapted the viewing for small groups. Within small groups, one PST watched a 360 video with a headset and cast their view to tablet shared by others. When a group size was too large (> 4), some PSTs lacked accessibility to the video. Additionally, when too many PSTs attempted to direct where the headset wearer should look, it served to limit what was viewable in the scenario. This self-study is ongoing. Preliminary findings provide potential themes to inform MTEs in implementation of representations of practice in preparing PSTs.

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HOW COACHES DEVELOP EFFECTIVE ONLINE COACHING PRACTICES

Cynthia D. Carson
Warner School of Education - University of Rochester
ccarson3@u.rochester.edu

Keywords: Technology, Instructional Leadership, Professional Development

Online professional development is growing in popularity (Bates, Phalen, Moran, 2016). Online programs can be adapted to teachers' schedules, provide avenues to draw on resources that may not be available locally, and provide work-embedded training with sustained support (Dede et al., 2009). Online environments remove geographic barriers, giving access to high quality experts regardless of physical location (Butler, Whiteman, & Crow, 2013; Prouty, 2009).

We designed an online content-focused coaching model (West & Staub, 2003) that included typical coaching components such as pre- and post-lesson conferences, and atypical components, such as video-recorded lessons that could be viewed and annotated. We hypothesized that online coaching might lead to the development of new coaching practices due to the online platform. Although none of the coaches (n=9) in our project were new to coaching, seven of these coaches were new to online coaching, which led us to develop a mentor-apprentice coaching model to acculturate them to the online environment. We implemented this mentor-apprentice coach model (see Figure 1) in one coaching cycle for 2 years.

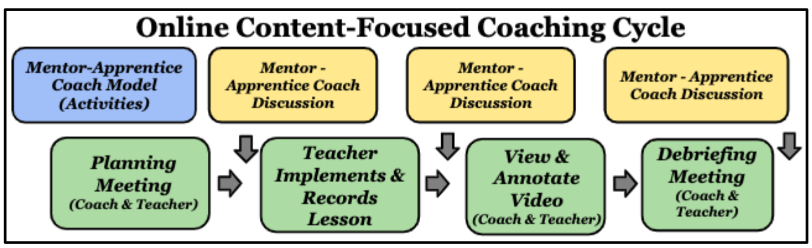

Figure 1: Mentor-Apprentice Coaching Model

We gathered data from semi-structured interviews, surveys, mentor and apprentice coaching interactions, and apprentice coach and teacher discussions from coaching cycles. We grounded the analysis of the development of the apprentice coaches in Lave and Wenger's (1991) theory of legitimate peripheral participation and Rogoff's (1994) theory of transformation of participation within a community of practice. We analyzed the coaches' development to understand changes in their coaching practices as a result of interacting with the more experienced coaches and the online platform.

Preliminary findings suggest that mentor coaches engaged the apprentice coaches as thinking partners by sharing coaching challenges, asking for new ideas and feedback on their coaching practices. For example, Alvarez, a mentor coach, characterizes her goal for the work with her apprentice coach: "It's not really that we're doing a mentorship relationship here. I figure we're just both in here figuring this out together" (all names are pseudonyms). These findings also suggested that apprentice coaches who helped the mentors reflect on their own practice; simultaneously developed their own coaching practices in ways that showed they could be effective coaches after the support of the mentors concluded. These findings and future research could inform training programs for supporting coaches and developing more effective coaching practices.

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MATHEMATICS EDUCATORS' PERCEPTIONS OF PASSING THROUGH THE GATEWAY OF OCCUPATIONAL POLITICIZATION

Daniel L. Clark
Western Kentucky University
daniel.clark@wku.edu

Keywords: Policy Matters, Standards (broadly defined), Teacher Education-Inservice/Professional Development, Elementary School Education

Studies concerning the implementation of the Common Core State Standards for Mathematics (CCSSM) have been requested by the mathematics education research community (Heck, Weiss, & Pasley, 2011). This study sought to document educational stakeholders' views as CCSSM implementation turned into a political controversy. More specifically, this study sought to answer the following research question: In what ways were the job performances of State Department of Education (SDE) officials, regional professional development providers, and elementary teachers affected by the emerging political nature of CCSSM implementation?

SDE officials, regional professional development providers, and elementary teachers were surveyed and interviewed as part of this mixed methods study. Responses were analyzed for commonalities and differences in the perceptions of participants at the varying levels of the state's education system.

SDE Officials' Perceptions

At the state level, many of the participants from the SDE felt micromanaged by the legislature. The legislature passed laws revoking and later reinstating funding for CCSSM implementation and also threatened to remove responsibility for CCSSM-aligned assessments from the SDE entirely. In short, this caused morale at SDE to suffer. This contentiousness followed the pattern of what McDonnell and Weatherford (2016) referred to as the politics of enactment versus the politics of implementation.

Regional Professional Development Providers' Perceptions

Professional development providers were affected in a variety of ways as well. First, despite being long-time advocates for quality mathematics education, they found being called to testify at legislative hearings in front of legislators who had already seemed to have made up their minds on the issue to be frustrating. This sentiment has been corroborated by Polikoff, Hardaway, Marsh, and Plank (2016). Furthermore, the professional development providers found that engaging teachers at inservice training sessions was more difficult. This was partially due to the uncertainty around CCSSM-aligned assessments caused by the legislature.

Elementary Teachers' Perceptions

Elementary teacher participants in the study generally identified themselves as being apolitical with respect to the CCSSM political controversy. While they said the controversy had affected their work with everyone from students' parents to their own family members asking them pointed questions about the CCSSM, they said they would teach their students the best way they knew how regardless of what the standards were.

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Chapter 8:
Mathematical Knowledge for Teaching

PRESERVICE TEACHERS' UNDERSTANDING OF TWO-VARIABLE INEQUALITIES: APOS THEORY

Kyunghee Moon
University of West Georgia
kmoon@westga.edu

This study investigated preservice secondary mathematics teachers' difficulties with and understanding of graphs of two-variable inequalities with the lens of the APOS theory. The analysis showed that most teachers understood inequality graphs as actions, at best, and used informal rules to construct graphs. They also had difficulty justifying graphs, such as overgeneralization of linear inequality graphs, lack of understanding of the meaning of solutions and the Cartesian Connection, and inability to transfer algebraic/verbal to geometric rays. This study suggests two methods for the visualization of inequality graphs, which uses the concepts of variable and parameter.

Keywords: Inequality, Graph, Cartesian Connection, APOS theory

Inequalities is an important topic in mathematics due to its relations to equations and functions as well as its applications to real-life situations. Research on inequalities has been, however, under-represented to date despite calls for more research from the mathematics education community (Boero & Bazzini, 2004; Vaiyavutjamai & Clements, 2006). In the case of one-variable inequalities, studies have suggested that many of students' difficulties are associated with the properties of inequalities or the meaning of inequalities. For instance, students multiply a negative number to an inequality without changing the direction of the inequality sign (Kroll, 1986), overgeneralize the Zero Product Property to inequalities (Kroll, 1986), solve inequality problems by solving parallel equations (Almog & Ilany, 2012; Blanco & Garrote, 2007; Sfard, 1994; Tall, 2004; Vaiyavutjamai & Clements, 2006), and fail to interpret "solve" as finding values that make inequalities true (Blanco & Garrote, 2007; Frost, 2015).

In the case of two-variable inequalities, an article by Switzer (2014) describes how a guided discovery approach helps students understand the meaning/graph of a linear inequality in a specific form. Yet due to a lack of in-depth research on the topic, not much is known about what issues are involved in understanding various kinds of two-variable inequalities. In this paper, I study this under-examined area with the research question: What are the difficulties and understandings in constructing and justifying graphs of two-variable inequalities?

Theoretical Framework

This study utilizes the notion of the Cartesian Connection (CC) in equations and inequalities. The CC in equations is a relationship between an equation and its graph, which stipulates that 'a point is on the graph of a mathematical relation $R(x,y) = 0$ if and only if its coordinates satisfy the equation.' Past studies have reported that this seemingly trivial idea is rather complex, especially when individuals encounter the idea in non-typical tasks (Moon, Brenner, Jacob & Okamoto, 2013; Moschkovich, Schoenfeld, & Arcavi, 1993). 'CC in inequalities' is a newly minted terminology I created for this study. It means either 'a point is on the graph of the mathematical inequality $R(x,y) < 0$ if and only if its coordinates satisfy the inequality' or 'a point is on the graph of the $R(x,y) > 0$ if and only if its coordinates satisfy the inequality.'

On the other hand, as a graph of $R(x,y) = 0$ is a locus of points whose coordinates satisfy the equation, a graph of $R(x,y) < 0$ is a locus of points whose coordinates satisfy the inequality. As such, an understanding of the graph of $R(x,y) < 0$ may involve both localization and globalization: localization to attend to a pair of x and y values that satisfies $R(x,y) < 0$ and its representational transfer to a point in a plane, and globalization to attend to *all* such pairs and the graph of their corresponding points as a whole (see Even, 1998 for equations).

To transition from localization to globalization, one may need a mental mechanism that elevates actions and thoughts. Examples of such transitions include: a point-wise view to a global view (Monk, 1992), an operational view to a structural view (Sfard, 1991), and an action conception to a process conception in the APOS theory (Dubinsky & Harel, 1992). In this study, I use the APOS theory, which seems to best explain the phenomena involved in the visualization and understanding of graphs of two-variable inequalities.

An *action* is a mental or physical manipulation of objects that can transform one object into another. It can be a single-step action performed explicitly, including recalling memorized facts, or a multi-step action involving a number of steps without conscious control (Arnon et al., 2013; Cottrill et al., 1996). When actions are repeated and interiorized, the actions collectively become a mental *process* (Breidenbach et al., 1992). As such, an individual with a process conception may execute actions without explicitly running through each action.

Research has suggested that many individuals conceive graphs of one-variable functions or two-variable relations only as actions. They may plot and trace points to represent function equation graphs, but do not understand the existence of infinitude points or the continuity of the graphs (Kerslake, 1981; Leinhardt, Zaslavsky & Stein, 1990). They also overgeneralize linear function graphs to other graphs, even with their abilities to perform actions of transformations, from a table of x and y values to points in the plane (Presmeg & Nenduradu, 2005). Many calculus students also have, at best, an action view of two-variable functions. They can neither make connections between algebraic and geometric objects other than points nor explain cross sections of $y = x+z+3$ at fixed values of x (Trigueros & Martinez-Planell, 2010).

On the other hand, it has been shown that covariational reasoning (Weber & Thompson, 2014) is essential in the visualization of two-variable function graphs. A student who understands the graph of $y = x^2–2x$ as a collection of points on a plane (a process in one-variable functions) can visualize the graph of $y = z(x^2–2x)$ as a sweeping out of the $y = a(x^2–2x)$ graph, with a acting as a parameter and then as a variable (a process in two-variable functions). In comparison, another student who sees the $y = x^2–2x$ graph only as a shape is unable to visualize the graph of $y = z(x^2–2x)$.

Methodology

The participants were 15 preservice secondary teachers (referred as "teachers") at a mid-sized state university in the Southeast region of the United States. Their mathematical backgrounds were varied, with four taking Precalculus, one taking Calculus I, and the other ten taking Calculus II or above. The participants were individually interviewed twice, for about one-and-a-half hours each time, in a form of semi-structured clinical interview. The interviews were recorded with a video camera and were transcribed. Their written responses were collected.

The interview items used for this study are the two questions included in the first interview.

Q1: (a) Find a solution of an inequality, $x+2y–32 < 0$.
(b) Represent all the solutions of the inequality above in the Cartesian plane.

Q2: (a) Find a solution of a system of inequalities, $y < x^2+1$ and $x^2+y^2 > 1$.
(b) Represent all the solutions of the system of inequalities above in the Cartesian plane.

During the interview, I asked the teachers why they represented the solutions in certain ways or why the x and y coordinates of all the points in their shaded regions satisfied the inequalities. I also asked the teachers to represent mathematical statements, in words or in algebraic forms, graphically in the plane when opportunities arose.

Genetic Decomposition

The genetic decomposition in APOS theory is a hypothetical model describing learners' mental structures and mechanisms for a certain concept (Arnon et al., 2013). For inequality graphs, I created a genetic decomposition by incorporating the action and process conceptions in the APOS theory and covariational reasoning (Weber & Thompson, 2014) to make up for the deficits in the current methods of teaching inequality graphs: the solution method and the ray method. The details follow.

In general, the solution method (ST) for the $y < f(x)$ graph involves two steps:

ST-1: A graph of $y = f(x)$ is drawn in a Cartesian plane.
ST-2: One or a few points are chosen from the plane, followed by the testing of the truth values of the points on $y < f(x)$. The graph is then determined as one of the two regions divided by the $y = f(x)$ graph in which points with true truth-values lie—normally shaded in gray or color (see, for example, David et al., 2011).

The first step of the ray method—see, for example, the CME Project, Algebra I (Cuoco et al., 2013)—is identical to ST-1. In the second step, however, the ray method uses verbal descriptions of rays to determine and justify inequality graphs.

> Try $x = 0$. The point on the line with x-coordinate 0 would be (0, 5). Any point on the vertical line with equation $x = 0$ with y-coordinate less than 5 is part of the solution set. Next try $x = 5$. … It would take forever to write out the situation for every possible value of x, but you can see that any point that is below the line is part of the solution set. (p. 756).

The $y < 3x+5$ graph is then represented as the region under the $y = 3x+5$ graph, shaded in pink.

Both methods provide tools to construct and/or justify inequality graphs, yet they fall short of emphasizing the variable concept as a critical idea in inequality graphs, logically and/or graphically. To explain, the solution method yields inequality graphs by generalizing a few points with true test values to the entire region, with no justification provided; the ray method does not incorporate their verbal descriptions of rays in geometric representations, and inequality graphs are represented as static objects. The genetic decomposition, which I created and named the "concept of variable" (COV) method, addresses these issues. In COV, one simultaneously constructs and justifies graphs with the actions and process below:

- COV-1: One converts $R(x,y) = 0$ to its graph—a probable action as one generally depends on one's memory from the previous learning of equation graphs in this context.
- COV-2: One performs an action of assigning a value for the x variable in $R(x,y) < 0$, such as $x = k$, and performs algebraic treatments (Duval, 2006) on $R(k,y) < 0$ to solve for y. (It is possible that one fixes y as constant k and performs treatments on $R(x,k) < 0$)

- COV-3: One performs a conversion from the algebraic—$x = k$ and $R(k,y) < 0$—to a geometric object—line segment(s), ray(s), or line—depending on the nature of $R(k,y) < 0$.
- COV-4: With similar actions repeated for other values of x, one interiorizes the actions into a process and visualizes the $R(x,y) < 0$ graph as a sweeping out of segments, rays, and lines—similar to the two-variable function graphs through covariational reasoning (Weber & Thompson, 2014).

It is possible that one uses a combination of ST and COV to construct and justify inequality graphs: ST for construction and COV-related reasoning for justification. Although it is inefficient, some individuals may use this method, especially if they justify their graphs after they have constructed inequality graphs using ST.

For analysis, in addition to the genetic decomposition, I used an open coding strategy (Strauss, 1987) to examine teachers' difficulties with and understanding of inequalities, by using a priori codes such as action/process, algebraic/verbal/geometric, treatment/conversion, and equation/inequality. The initial coding showed patterns, and thus yielded categories and subcategories. I then performed the second stage of coding: reexamining and revising the prior codes, and at the same time performing an axial coding (Strauss, 1987) to focus on the categories and subcategories from the previous coding.

The results are presented in the next two sections. The first section focuses on teachers' understanding of inequalities with the lens of genetic decomposition, and the second their difficulties with inequalities in general.

Understanding Inequalities: Genetic Decomposition

Linear Inequality

Eleven teachers attempted to construct the $x+2y–32 < 0$ graph, with seven successfully doing so. All seven teachers with a correct inequality graph drew the line graph of $y = –x/2+16$, converted from $x+2y–32 = 0$ (ST-1, or equivalently COV-1), but none used the rest of ST or COV when constructing the inequality graph. Instead, five of them applied a rule, such as "less than is lower," and two simply shaded the lower part of the line with no explanation.

When justifying their graphs, two of the seven teachers with a correct inequality graph performed actions of checking the truth values of a few points under the line to $y<–x/2+16$, and then generalized the few points to the entire region. When asked why they included the entire region below the line, however, they could not articulate their reasoning. Another two teachers followed COV-2, making comments such as, "my y-values are never going to be bigger than 16 when $x = 0$," which were verbal representations of rays. However, when they were asked to represent their words in a plane, they claimed the entire region under the $y = –x/2+16$ graph as the geometric representation of their words, failing COV-3. The other three also failed to provide correct reasoning with various issues, including a shaky understanding of the Cartesian Connection, which I will explain in more detail in the next section.

Among the four teachers with an incorrect inequality graph, three teachers incorrectly drew a $y = 16–x/2$ graph (ST-1), with two using the "less is under" rule and one claiming the line graph itself as the inequality graph. One remaining teacher did not provide any explanation.

Parabolic Inequality

Seven teachers attempted to construct the $y < x^2+1$ graph, with six correctly representing the $y = x^2+1$ graph and three the $y < x^2+1$ graph. All three teachers with a correct $y < x^2+1$ graph first drew $y = x^2+1$ (ST-1) and then either performed ST-2 or used the rule, "less is under," to construct a graph. Among the three teachers with an incorrect inequality graph, but with a correct $y = x^2+1$ graph (ST-1), one claimed that the upper part of the parabola was the inequality graph by guessing, and two claimed that the lower part of the $y = 1$ graph was the inequality graph, using the rule, "less is under." The remaining teacher represented $y = x^2+1$ as a curve similar to the $y = x^2$ graph (ST-1) and claimed that the curve itself was the inequality graph.

When justifying their graph, two of the six teachers who had a correct $y = x^2+1$ graph did not provide any explanation, and one teacher used the "less is under" rule. The other three teachers followed COV-2 by fixing x values and by saying such things as "I will make x equals 0, then you will get y is less than 1." However, all three incorrectly represented their words in a plane, failing COV-3. One represented his words as the half plane under the $y = 1$ graph, and the other two represented them as the entire region under the $y=x^2+1$ graph.

Circular Inequality

Seven teachers attempted to construct the $x^2+y^2 > 1$ graph, with five providing a correct graph and two an incorrect one. All five teachers with a correct inequality graph drew a $x^2+y^2 = 1$ graph (ST-1) and used a "greater is outside" rule to determine the inequality graph. The other two with an incorrect inequality treated $x^2+y^2 = 1$ into $y = \sqrt{1 - x^2}$ and sketched the latter as a curve similar to the $y = \sqrt{x}$ graph (ST-1). One teacher then shaded the upper part of the curve using the rule, "greater is upper," and the other claimed that the curve itself was the inequality graph.

When justifying the graph, one of the five teachers with a correct inequality graph used ST-2 as a justification; another teacher used a verbal description of a ray, "at $x = 2$, anything plugged in for y could be greater than 1" (COV-2), without performing the rest of COV; and another teacher did not provide any explanation at all. The other two teachers used an idea of $x^2+y^2 = k$ as a part of $x^2+y^2 > 1$—more generally, $R(x,y) = k$ as a part of $R(x,y) > 0$, implicitly utilizing the concept of parameter (COP), which I will explain in more detail in the Discussion and Conclusions section.

As for the two teachers who had a $y = \sqrt{x}$-like curve, one teacher provided reasoning by using a verbal description of a ray (COV-2), but represented his words as the entire region above the curve, failing COV-3. The other teacher was not asked to justify as she did not provide any justification for the linear and parabolic inequalities.

Difficulties with Inequalities

The teachers in general struggled to construct and justify inequality graphs, with fewer than half of them providing a correct graph—seven for the linear, five for the circular, and three for the parabolic inequality. The sources of their difficulties were various and intermingled, with some implicitly shown in the previous section. Below are the four most noteworthy characteristics of their difficulties.

Misunderstanding the Meaning of Solution

Some teachers did not understand the meaning of "solution" of an inequality and/or of a system of inequalities. Four teachers claimed that $y < –x/2+16$ (or $y <$ something) was a solution of $x+2y–32 < 0$. There were also three teachers who correctly interpreted the meaning of solution in a single inequality, but not in the system of inequalities. Instead of finding solutions of the system of inequalities, they found a solution for each inequality separately as if they had been given two independent inequality problems. Considering the fact that they had successfully found a solution of a system of equations—$y = x^2+2$ and $y = –2x$—in a different question, they

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did not seem to understand the role of the *and* connective in the system of equations and inequalities.

Lacking Understanding of the Cartesian Connection

Four teachers successfully found the solutions of inequalities; however, they either falsely claimed the graphs of parallel equations themselves as the graphs of inequalities, or could not determine the inequality graphs due to their lack of understanding of CC. In one case, my questioning helped a teacher revise his thinking, reflect on the meaning of solution, and change his answer to a correct inequality graph. However, for the other three teachers, my questioning did not help them reorient their thinking.

> I: Why do you think everything on the line is a solution [of the inequality]?
> S: I see 1, 16 [in x+2y-32<0]. Oh no, that will be greater than 0. It is 0, 16. No, it wouldn't be because 0 is not less than 0. So, I guess when x is 1, y is 15.5, so 1+31 is 32 minus 32 is 0.
> I: Would everything on the line make [x+2y-32] zero?
> S: Well, we will see. 2 gives me 15. It seems like everything on this line will give me 0.
> I: So how does that help you determine the answer?
> S: It doesn't. I am confused.
> I: You are saying everything on the line is 0. But you can't determine what values will make it less than 0.
> S: Right.

As shown above, when asked whether the x and y coordinates of any point on the line would satisfy $x+2y–32 = 0$, the teacher was compelled to find another solution of $x+2y–32 = 0$, after having already found two such pairs meeting the condition, showing a lack of understanding of CC in equations. Further, when he had a pair of x and y values, $x = 1$ and $y = 16$, which made $x+2y–32$ greater than 0, he was unable to use the pair to determine the graph of $x+2y–32 < 0$; instead, he adjusted the x value to 0 to have a pair that satisfied $x+2y–32 = 0$, showing a lack of understanding of CC in inequalities.

Overgeneralization of Linear Inequality Graphs

The teachers' overgeneralizing behavior of linear inequality graphs appeared when two teachers represented the parabolic inequality graph. Despite having a correct $y = x^2+1$ graph, they shaded the half plane under the $y = 1$ graph instead of the region under the $y = x^2+1$ graph. One teacher provided valid reasoning, but an incorrect graph, by inserting the $y = 1$ graph. The other teacher simply said, "I don't know how to do this one," and shaded under the $y = 1$ graph. It was noteworthy that this behavior came from two higher performing teachers and, in particular, from a teacher whose reasoning was otherwise in the right direction.

Figure 1: Teachers' Representations of Rays in $y<x^2+1$

Problems with Ray Conversions

Teachers also struggled with ray conversions. Of the five teachers who were asked to do the conversions, with two of them being asked twice, four teachers could not convert from algebraic/verbal rays to geometric rays. For the linear inequality, four teachers verbally described a ray, saying such things as “when x = 31, y is less than 0,” but three of them falsely claimed the entire region under the y = 16-x/2 graph as the graph of their verbal ray. For the parabolic inequality, one teacher represented the statement, “$y < x^2+1$ when $x = 0$,” as the entire region below the y = l graph instead of a ray on the y axis (Figure 1a). Another teacher represented the statement, “$y < x^2+1$ when $x = 3$,” as a ray on the y-axis instead of a ray on the line, x = 3 (Figure 1b).

Discussion and Conclusions

This study examined preservice secondary teachers’ abilities to construct and justify inequality graphs by applying the framework of the APOS theory (Dubinsky & Harel, 1992). The results showed that the vast majority of the teachers had, at best, an action conception of inequality graphs. They used rules, such as “less is lower” or “greater is upper,” when constructing or justifying graphs, rather than reasoning relevant to the concept of variable or the Cartesian Connection. Even when they were able to implicitly use the concept of variable in their argument, they could not utilize the concept to construct or justify graphs, due partly to their lack of ability to convert their arguments of verbal/algebraic rays to geometric rays.

Their difficulties with inequalities were comparable to individuals’ difficulties with inequalities and functions shown in other studies. Some teachers incorrectly found solutions of two-variable inequalities from parallel equations, similar to those who struggled with one-variable inequalities in earlier studies (Almog & Ilany, 2012; Blanco & Garrote, 2007; Frost, 2015). Some teachers falsely claimed that graphs of inequalities were the graphs of parallel equations, despite knowing the meaning of solutions in algebraic forms—similar to the individuals who could not make the Cartesian Connection in two-variable equations in other studies (Moon et al., 2013; Moschkovich et al., 1993). Some teachers also falsely represented the graph of a parabolic inequality with a linear inequality graph, similar to the teacher who represented the table of a non-linear function with a line graph in Presmeg and Nenduradu (2005).

Teachers also showed struggles that were comparable to those of college students in the case of two-variable functions. Many teachers knew the shapes of inequality graphs by memory, or by using rules, but did not understand inequality graphs at fixed values of x, similar to students who could not visualize the cross-sections of two-variable functions in other studies (Trigueros & Martinez-Planell, 2010; Weber & Thompson, 2014). As such, visualizing an inequality graph as a sweeping out of rays, line segments, or lines— similar to the way that a student visualized a two-variable function graph as a sweeping out of curves in Weber and Thompson (2015)—was out-of-reach for the teachers.

On the other hand, two teachers’ reasoning shown in the circular inequality graph, which implicitly utilized the concept of parameter, inspired me to generate a new method for inequality graphs. In this concept of parameter (COP) method, inequality graphs can be constructed and justified as a sweeping out of curves or lines. To use the COP method, one first converts $R(x,y) = 0$ into a graph (COP-1, identical to COV-1). One then performs an action of assigning a value k to $R(x, y)$ by seeing $R(x,y) = k$, with $k > 0$, as part of $R(x,y) > 0$ (COP-2). One then converts

$R(x,y) = k$ to its graph, which is a curve (including lines) (COP-3). After repeating these actions for various k, one interiorizes these actions into a process and visualizes the inequality graph as a sweeping out of all curves (COP-4).

The results of the COV and COP methods are quite different when the graphs are shown as dynamic processes. While COV represents a graph as a collection of vertical or horizontal rays, lines, and line segments (Figure 2a & 2b, respectively), COP represents it as a collection of lines or curves (Figure 2c). As shown in the work by Moon (2019), the COV and COP methods have their own merits. As such, both methods deserve to be considered in the instruction of inequality graphs.

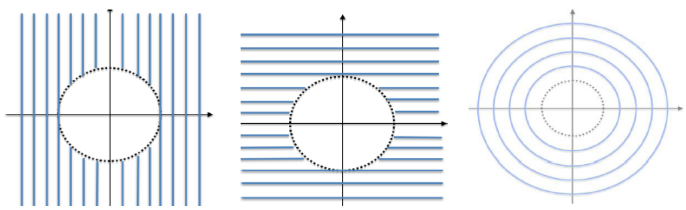

Figure 2: Graphs of $x^2+y^2>1$ through COV and COP Methods

I conclude this paper with a suggestion for those who contemplate implementation of the concepts of the variable and parameter methods. I believe that instructors should not use the methods as rote means to draw graphs of inequalities. Rather, they should focus on the concepts and ideas embedded in the methods: the meaning of inequalities, variable and parameter, the meaning of solutions, and the connection between algebraic and graphical inequalities, including the Cartesian Connection and ray connection.

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HORIZON CONTENT KNOWLEDGE IN PRESERVICE TEACHER TEXTBOOKS: AN APPLICATION OF NETWORK ANALYSIS

Joanna G. Jauchen
George Mason University
jjauchen@gmu.edu

Horizon Content Knowledge (HCK, Ball, Thames & Phelps, 2008), knowledge that teachers hold about the interconnectedness of mathematics, has been recognized as an integral component of Mathematics Knowledge for Teaching, yet little is known about how or where it is taught in the preservice teacher (PST) curriculum or how researchers should study HCK. This study used network analysis to examine HCK in the context of undergraduate PST textbooks. I describe my approach to the study and report on the analytical questions I encountered. I then describe the affordances and constraints of network analysis for understanding HCK and reflect on the power of this tool to understand HCK in a variety of contexts.

Keywords: Curriculum Analysis, Data Analysis and Statistics, Mathematical Knowledge for Teaching, Teacher Education-Preservice

Ball, Thames, and Phelps (2008) developed a framework for the Mathematical Knowledge for Teaching (MKT) which divides MKT into six subdomains, three of which are oriented around subject matter knowledge (SMK) and three around pedagogical content knowledge (PCK). SMK is further divided into three dimensions: Common Content Knowledge (CCK), Specialized Content Knowledge (SCK) and Horizon Content Knowledge (HCK). CCK is mathematical content that is generally known outside of the world of teaching; SCK is mathematical content that teachers rely on to teach; and HCK is often noted as knowledge teachers hold about the interconnectedness of mathematics. One example of HCK is the knowledge that teachers possess regarding how addition and multiplication are connected mathematical domains.

Little work has been done to examine what HCK is being taught to PSTs. One place to look for answers to these questions is in PST textbooks. Because of the varying definitions of HCK in the literature, and because the interconnectedness of mathematics may be more difficult to define and analyze than CCK or SCK, the study of HCK presents unique analytical challenges that network analysis may address. For this study, I examined three texts for a mathematical content course for elementary PSTs using network analysis to examine HCK in the context of addition of whole numbers. I present my methodological approach to this study, along with reflections on the use of network analysis for examining HCK. I discuss the analytical choices that I encountered and the affordances and constraints of network analysis for understanding HCK in PST textbooks. Finally, I outline how network analysis may be used in future work on HCK.

Theoretical Framework and Purpose of Study

PSTs come to understand disciplinary knowledge within a socio-cultural environment. As Lave and Wenger (2007) note: “This world is socially constituted; objective forms and systems of activity, on the one hand, and agents’ subjective and intersubjective understandings of them on the other, mutually constitute both the world and its experienced forms” (p. 51). Thus, the HCK of individual teachers is the result of a process of socially reproducing culturally and socially situated knowledge. In that process of social reproduction, PST textbooks, as artifacts, transmit cultural tools the mathematics community values in the education of PSTs (Wertsch, 1998). As such textbooks play an important role in cultivating the HCK of PSTs.

This study, situated in a socio-cultural framing, makes three major assumptions. First that PST textbooks, as cultural artifacts, describe some of the cultural tools PSTs are exposed to during a math content course. Second, that the Common Core State Standards for Mathematics (CCSSM; National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) as a community-generated artifact, represents the domains of CCK that teachers are expected to be familiar with. The final assumption is that I, as a full participant in the mathematical community and the community of teacher educators, understand the cultural tools and norms of this community. This positions me to create a reflection of the HCK that I recognize in PST textbooks. A result of this framing is that the HCK I have identified is one of many representations of HCK that may facilitate PST understanding of the interconnectedness of mathematics. Results about HCK from this study are bounded by the cultural artifacts included in the analysis and by my own knowledge. This is a socially, culturally, and historically situated analysis of HCK from three textbooks.

Though the MKT framework is well-established, there is not full agreement in the literature on what constitutes HCK (Ball & Bass, 2009; Ball, Thames & Phelps, 2008; Figueiras, Ribeiro, Carrilo, Fernández, and Deulofeu, 2011; Zazkis & Mamolo, 2011). Ball, Thames, and Phelps (2008) defined HCK as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). The following year, Ball and Bass (2009) expanded on this idea and outlined four aspects of HCK: (1) "A sense of the mathematical environment surrounding the current ‘location’ in instruction"; (2) Major disciplinary ideas and structures"; (3) Key mathematical practices; and (4) Core mathematical values and sensibilities (p. 6). For this study, I operationalize HCK using Ball and Bass’s (2009) first and second aspects of HCK and also integrate a similar definition proposed by Figueiras et al. (2011) which defines HCK as "connections between mathematical concepts and ideas, grounded in the coherence of mathematics, in which all concepts and ideas are precisely defined and logically interwoven" (p. 28). Based on these two definitions, I identify eighteen broad mathematical concepts (major disciplinary ideas) present in four cultural artifacts (the CCSSM and three PST texts). I consider HCK present when two or more of those eighteen mathematical concepts are connected by the textbook author in a coherent way (within a single paragraph).

Method

Sampling and Coding

Three textbooks were chosen for this study on the HCK based on a single criterion: the needed to be reasonable to use in a mathematical content course for elementary undergraduate PSTs. The three books selected were: Manes and Holmes (2018); Sowder, Sowder and Nickerson (2017); and Caldwell, Karp, and Bay-Williams (2011). Manes and Holmes (2018) authored an Open Educational Resource written to help students "think like a mathematician," (p. 1) and which emphasized the Common Core Standards for Mathematical Practice (SMP). Similarly centering the SMP, Sowder et al. (2017) aimed to support classrooms where "students take an active part in learning" (p. xi) and provide students insight into "children’s mathematical thinking" (p. xi). Caldwell et al. (2011) is a text written solely about addition and subtraction as part of a larger series of texts published by the National Council of Teachers of Mathematics. I consider this to be a high-level text for an undergraduate PST content course and a possible exemplar of HCK. After texts were chosen, I selected a single topic for study: addition of whole numbers. Sections of the book that explicitly referenced addition of whole numbers were coded and analyzed. Data were chunked into paragraphs, using one paragraph as the base level of analysis. Relying on the definitions by Ball and Bass (2009) and Figueiras et al. (2011), paragraphs were coded as containing HCK if they connected two different mathematical concepts in a coherent picture of mathematics. I did not code paragraphs as containing horizon content knowledge if the text only implicitly relied on previous concepts, as many new mathematical concepts rely on existing mathematical knowledge. For this study, I was interested in the information explicitly presented to students, not what they might intimate from the text. I also chose, because of the exploratory nature of this study, to code two concepts as “connected” without being more specific about the relationship of that connection (Carley, 1993). For this analysis, I make no assumptions about the linearity in mathematical domains. For example, I do not assume counting precedes addition or that addition precedes subtraction. I only document that the textbook describes a “connection” between the two concepts by discussing them in the same paragraph. This choice impacts the resulting analysis and I discuss those limitations below.

The study relied on the CCSSM to identify disciplinary ideas that Ball and Bass’s conceptualize. When applied, the CCSSM was problematic for uncovering HCK in the textbook content because of differences in the level of content detail between the CCSSM and the PST textbooks. Once I identified this analytical issue, I developed emerging categories from the PST texts, in light of CCSSM standards. I formalized these emerging categories into new conceptual codes (Saldaña, 2016) and these were used to code the disciplinary ideas that defined HCK. These conceptual codes, developed in the first round of coding, more faithfully described the sort of concepts that were connected in the textbooks. The final eighteen codes arose from the CCSSM in the context of the PST textbooks, and described eighteen different disciplinary ideas that I found to be present in the textbooks and standards. The codes are addition (A), subtraction (S), multiplication (M), division (D), word problems (WP), concrete models (C), decomposing (DEC), properties (P), place value (PV), unknown addend (or the relationship between addition and subtraction; UA), counting (CNT), mental math (MM), equations (EQN), algorithms (ALG), definition of addition (DEF), measurement (MEAS), estimation (EST), and functions (FCN). For consistency, all paragraphs were coded a second time using the conceptual codes. Finally, the conceptual codes were checked against the initial CCSSM codes for alignment.

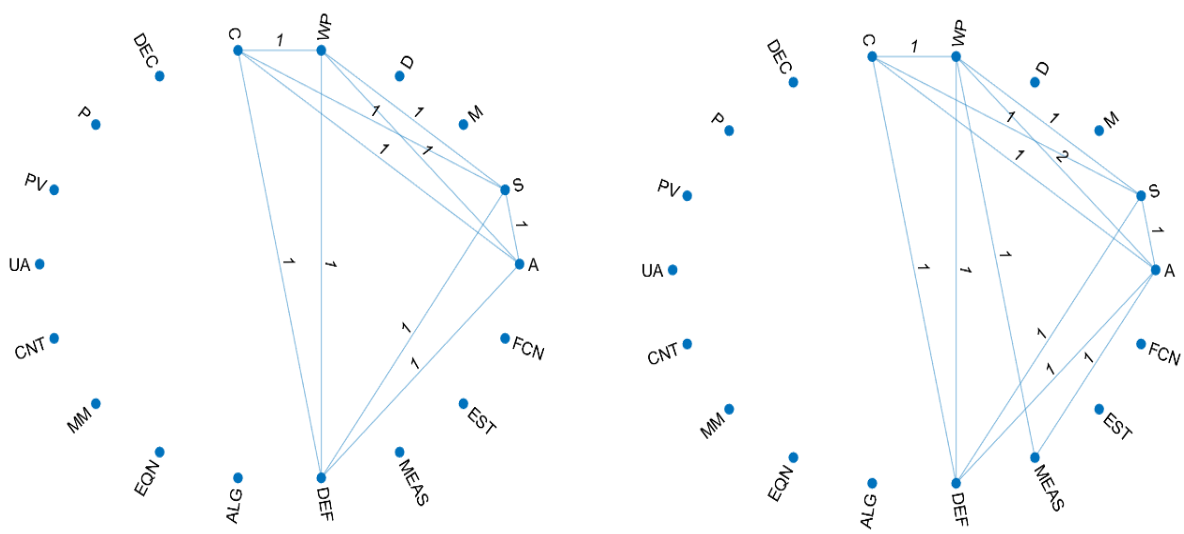
Analysis

Once data was coded, I used network analysis to examine the connections (Borgatti, Mehra, Brass, & Labianca, 2009). Network analysis relies on nodal graphs to visualize connections between objects (in this case, mathematical disciplinary ideas). Borgatti et al. (2009) list several types of analysis that can be conducted using network theory, including an analysis of types of connections, and an analysis of the underlying structure of connected systems. Network analysis has been used in education in a variety of contexts (Carolan, 2013) and network analysis is widely used outside of education to understand connected or relational data (Scott, 2013).

Since network analysis has not been previously used to analyze HCK, I briefly describe how the resulting visualizations originate from the initial coding. Here is an example paragraph from Sowder, Sowder, and Nickerson (2017). This paragraph was coded as containing addition, subtraction, concrete representations (the paragraph references a picture of addition which I omit for space considerations), word problems, and definitions. Codes are noted in brackets. The paragraph mentions “situations” which, in context, is about word problems. A network graph of this paragraph is given on the left side of Figure 1.

> Although the above distinction between meanings for addition [A; DEF] may be subtle, the distinction between meanings for subtraction [S] is quite stark. In Take-from situations [WP], part of a quantity is removed and another part is left over. Such situations [WP] are consistent with the familiar take-away meaning of subtraction [S]. This meaning tends to be emphasized in instruction in the primary grades and tends to be more familiar to students. In Compare situations, by contrast, there is no removal action. Two quantities are simply being compared additively. Children who associate subtraction with take-away may have difficulty solving Compare problems (p. 48).

Each of the eighteen conceptual codes is represented by a node on the graph. The entire network (all nodes) are displayed, even if there are no connections to them. Lines are drawn between nodes that are connected in the paragraph (these are called edges). The five codes from the example paragraph will lead to ten total connections. All codes in the paragraph are connected with edges, so addition is connected to each of the other four topics, subtraction is connected to each of the other four topics, word problems is connected to each of the other four topics, and similarly for concrete representations and definitions. In my analysis, these connections were considered bidirectional, though unidirectional analysis is possible. The number 1 written along the edge is called the edge weight, and it indicates that one paragraph connected these two topics.



Example paragraph Example and second paragraphs

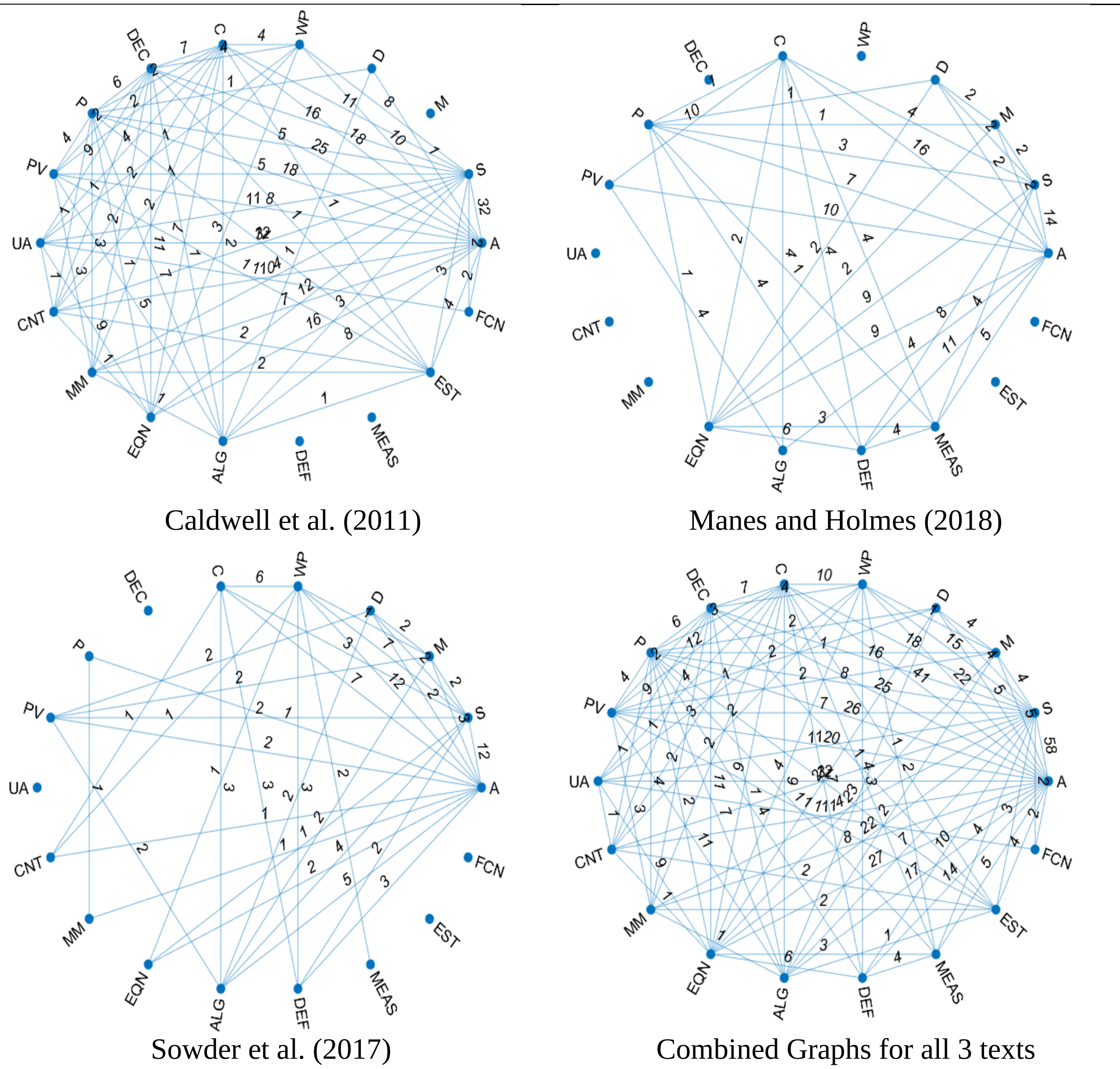
Note. Codes begin with A on the west pole of each graph and progress counter-clockwise
Addition (A), Subtraction (S), Multiplication (M), Division (D), Word Problems (W), Concrete (C), Decomposing (DEC), Properties (P), Place Value (PV), Unknown Addend (UA), Counting (CNT), Mental Math (MM), Equations (EQN), Algorithms (ALG), Definition of Addition (DEF), Measurement (MEAS), Estimation (EST), and Functions (FCN).

Figure 1*: Example Network Analysis Graphs of HCK*

If I add another paragraph coded with addition, word problems, and measurement, the graph changes to look like the right side of Figure 1. The edge weight for addition to word problems now increases to 2 (meaning 2 paragraphs contain this connection) and new edges are added linking addition to measurement and word problems to measurement. Each new paragraph adds edges or adds to the edge weight in the visualization.

Results and Discussion

For HCK, I present a nodal network for each textbook and discuss the results. I consider these results the affordances of using network analysis, namely, the ability to structure and visualize the sorts of HCK textbook authors have discussed. These results are best analyzed as a comparison, so in Figure 2, I provide the three individual graphs for Caldwell et al. (2011), Manes and Holmes (2018) and Sowder et al. (2017), along with a graph that combines their HCK across texts. Looking at Figure 2, Addition and Subtraction are connected in 32 paragraphs in Caldwell et al. (2011). Addition and division are only connected in eight paragraphs, and surprisingly, addition and multiplication are never connected (there is no line between addition



Note. Codes begin with A on the west pole of each graph and progress counter-clockwise
Addition (A), Subtraction (S), Multiplication (M), Division (D), Word Problems (W), Concrete (C), Decomposing (DEC), Properties (P), Place Value (PV), Unknown Addend (UA), Counting (CNT), Mental Math (MM), Equations

(EQN), Algorithms (ALG), Definition of Addition (DEF), Measurement (MEAS), Estimation (EST), and Functions (FCN).

Figure 2: Network Analysis Graphs of HCK

and multiplication). Because this analysis centered on addition of whole numbers, addition is well connected to many topics with significant edge weights, indicating its importance in this network. Subtraction is similarly connected, again with significant weights, as is decomposing indicating that Caldwell et al. routinely linked these concepts. However, division is only linked to three other topics (properties, algorithms and addition) and all with a weight of 1, indicating that Caldwell et al. did not strongly connect division to other topics in the addition section. Topics that were not connected by Caldwell et al. (2011) include multiplication, measurement, and the definition of addition. Caldwell et al. is the only text that connected to functions.

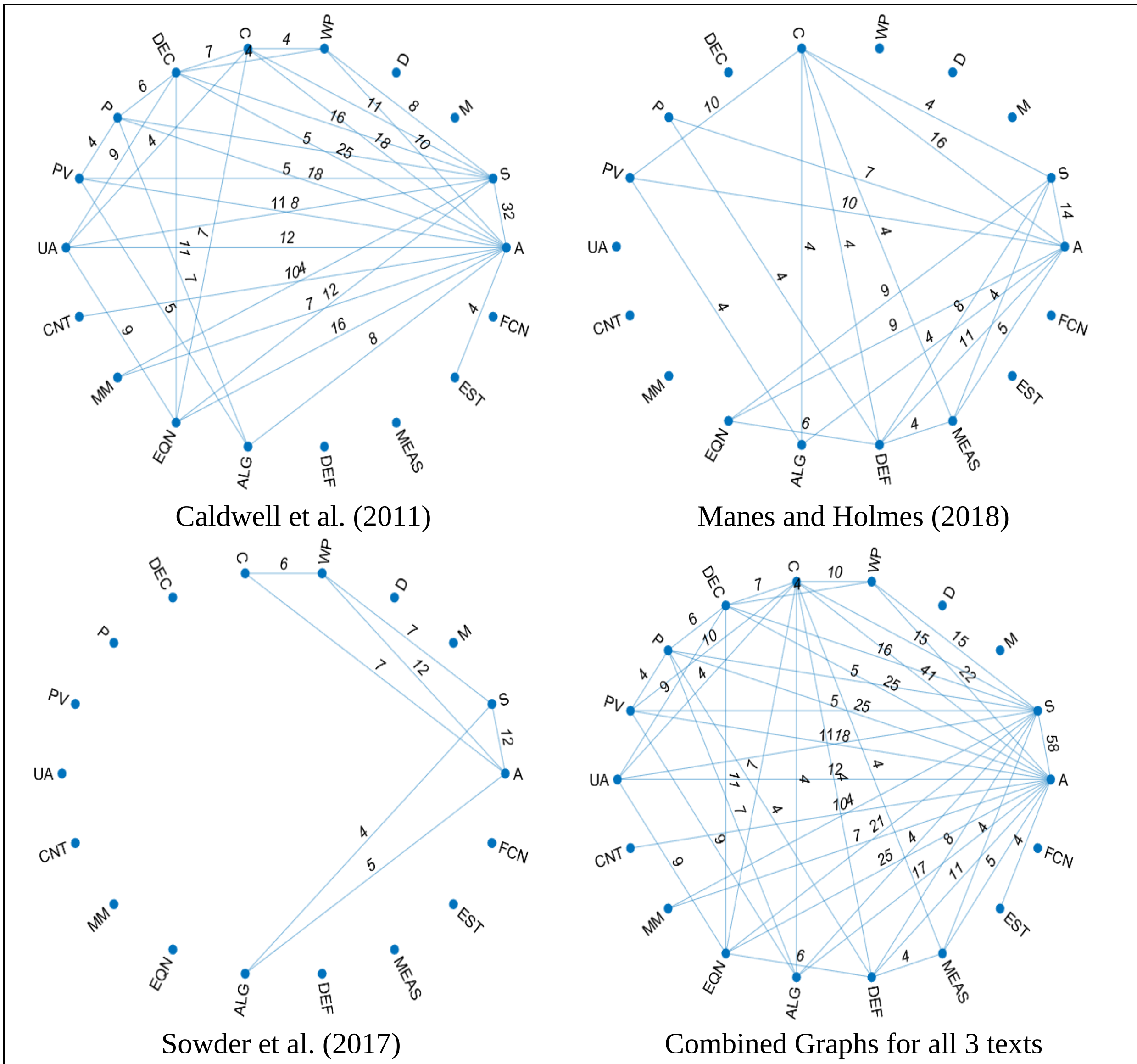
The graph for Manes and Holmes (2018) is less dense than the one for Caldwell indicating fewer paragraphs were coded as containing connections. The graph shows seven disconnected nodes: word problems, decomposing, unknown addend, counting, mental math, estimation and functions. Nodes that connect heavily with addition in Manes and Holmes (2018) are concrete models (16), place value (10), equations (9), and the definition of addition (11). Concrete models are also heavily connected to place value. The graph is less dense for Manes and Holmes than for Caldwell et al., but it is difficult to tell if that is the result of a real difference or because of the disproportionate number of paragraphs coded for Caldwell et al. (2011). Finally, the graph for Sowder et al. (2017) shows even fewer connections. Addition is still heavily connected. Nodes that are completely disconnected are decomposing, unknown addend, estimation, and functions. Word problems connect to eight other topics, which confirms that word problems were heavily emphasized in this text. Thirteen is the highest edge weight here (between word problems and addition) with weights decreasing significantly away from the addition subtraction or word problem nodes. Counting is only related to concrete models, word problems and addition, all with a weight of 1. Finally, the combined graph that visualizes the connections present across texts is useful to analyze what sort of HCK textbook authors were emphasizing across texts. So, if the research question were not comparative, but interested in the sort of HCK knowledge that the community of textbooks authors valued, the combined graph would be the tool to answer that question. I consider decisions to make that knowledge more accessible in the next section where I reflect on my analytical techniques and suggest future work.

Analytical Questions and Constraints

Operationalizing HCK and Descriptive Validity

This paper presents a fundamentally new approach to analyzing HCK. As such, there are still ongoing analytical questions and decisions that need to be made before utilizing network analysis. I operationalized HCK by claiming that HCK was present if two concepts were present in a single paragraph in a cohesive way, even if the book did not make the exact nature of the connection explicit. For example, my results claim that addition to subtraction HCK is present if addition and subtraction were present in the same paragraph. This operationalization and the resulting analysis raised questions about the descriptive validity of my operationalized variable as compared to the definitions by Ball and Bass (2009) and Figueiras et al. (2011). First, is addition to subtraction HCK “present” in a textbook, if a single paragraph mentions addition and subtraction together? This answer must be no. Single mentions of two topics in one paragraph do not present a "sense of the mathematical environment surrounding the current ‘location’ in instruction" (Ball & Bass, 2009, p. 6). However, if a textbook contains 34 paragraphs that all mention “addition and subtraction,” does that present a more cohesive picture of mathematical terrain? I believe the answer here is yes.

Given these observations, I propose it may be useful to set a bottom required edge weight. Consider the graphs in Figure 3, which represent the same network as Figure 2, but assume, for discussion, a required bottom edge weight of 4 (meaning, four separate paragraphs need to mention the connection in order to be included in the graph). This bottom edge weight privileges concepts that are mentioned at least four times in the text, which may help to more precisely operationalize HCK in texts. Setting a bottom edge weight also lowers the visual complexity of the figure, allowing the most prominent connections to be easily understood. It highlights both the interconnectedness of the Caldwell text and that most of the connections in Sowder et al. had an edge weight of 1, 2 or 3.



Caldwell et al. (2011)

Manes and Holmes (2018)

Sowder et al. (2017)

Combined Graphs for all 3 texts

Note. Codes begin with A on the west pole of each graph and progress counter-clockwise

Addition (A), Subtraction (S), Multiplication (M), Division (D), Word Problems (W), Concrete (C), Decomposing (DEC), Properties (P), Place Value (PV), Unknown Addend (UA), Counting (CNT), Mental Math (MM), Equations (EQN), Algorithms (ALG), Definition of Addition (DEF), Measurement (MEAS), Estimation (EST), and Functions (FCN).

Figure 3: Network Analysis Graphs of HCK with Bottom Edge Weight of 4

But other questions arise about how to define and operationalize connections between content. Other types of connectedness in the context of HCK may be present. For example, HCK can refer to both knowledge across grade bands as well as knowledge across domains. It is possible that the four types of HCK (Ball & Bass, 2009) should be coded independently. Other ways to possibly categorize HCK may be: explicit connection made in the text, implicit connection made in the text, across grade band connection, or within grade band connection. It may also be possible to conceptualize an analysis that does rely on a linear conception of these mathematical ideas, for example, using the knowledge from learning trajectories to map the unidirectional connections present (Clements & Sarama, 2004; Confrey et al., 2011). Regardless of the refinement, refining the operationalization of HCK is necessary for more nuanced analysis and a fuller understanding of the content present in texts.

Sampling and Bounding the Analysis

As Scott (2013) noted, inorganic bounding of relational data can lead to false conclusions about the full network. In the context of textbook analysis, bounding is a necessary pragmatic decision. In this study, the sample was bounded to paragraphs that were explicitly focused on addition. Because of this, my network does not faithfully represent the entire network of connections authors made *even for addition* because I did not code every paragraph in the text that mentioned addition. In other words, sampling decisions in network analysis not only impact the edges present in the network, they impact the network nodes as well. For example, perhaps an author does not mention the connection between addition and multiplication in the sections on addition but mentions that connection in the section on multiplication. In that circumstance, and because of the way I have bounded my analysis here, the network I create would obfuscate the actual HCK presentation in the text. If a researcher wanted to create a full network for a text, it is reasonable to expect significant difficulties in generating the network (i.e. the coding scheme) for all content knowledge that PSTs are exposed to across textbooks. Understanding that network analysis of a bounded sample leads to a partial network is essential for interpreting these results.

Conclusion

I have presented what PST texts present as HCK as defined by Ball and Bass (2009), outlining the affordances and constraints of network analysis for HCK and posing some of the analytical questions that require further work and reflection to optimize network analysis for HCK. If these questions can be answered, network analysis can be a significant tool to examine the connectedness of HCK in a variety of situations, not just PST textbooks. A logical next step after textbook analysis is a study of the impact of HCK heavy textbooks on PST knowledge.

The amount and density of information presented in a text for PSTs should be a conscious and strategic choice by textbook authors and teacher educators. There are real differences across populations of students; different teaching environments require different texts. Teacher educators need to be aware of the choices they are making in order to select an appropriate text for their own classes. Further, the teacher education community needs to reach a consensus around what HCK should be included in PST content courses. Taking a careful inventory of what HCK may currently be presented in PST textbooks is one step in that process. An understanding of HCK is essential to the teacher education community so we can understand what textbooks emphasize, what they leave out and develop consensus in the community about what should be taught in an undergraduate content course for PSTs. Because of the vital role of content courses in PST education, careful attention to HCK in these courses is warranted.

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AN EXPLORATION OF TEACHERS' ABILITIES TO IDENTIFY PROPORTIONAL SITUATIONS AND MAKE SENSE OF STUDENTS' THINKING

Rachael Eriksen Brown
Penn State Abington
reb37@psu.edu

Chandra Hawley Orrill
UMass Dartmouth
corrill@umassd.edu

In this paper we extend our previous work acknowledging teachers, like students, struggle to make sense of a pseudoproportional situation. For this study, we further analyzed to determine the significance of this problem in terms of teachers' abilities to support a range of students' thinking. We found teachers who were able to reason appropriately about the original situation were also able to make sense of students' varied approaches to solving the task, whereas teachers who struggled to reason about the situation were less able to make sense of students' thinking. Implications for teaching and professional development will be discussed.

Keywords: Teacher Knowledge, Mathematical Knowledge for Teaching, Number Concepts and Operations

Purpose

Research on students' persistent use of proportional reasoning in non-proportional situations suggests that this is a concept with which students struggle. In one study, De Bock, Van Dooren, Janssens, and Verschaffel (2002) examined the persistence of proportional thinking using an item we call the Santa Task (a version modified for our study is shown in Figure 1). In their study, the researchers posed the Santa Task to high school students, then provided five scaffolds to help the students work toward attending to the area relationship. De Bock et al. found participants struggled to recognize the Santa task focused on area not height. Initially only two students were able to answer the task correctly. After examining the five scaffolds, 32 of the 40 participants were able to determine a correct answer.

Building from the De Bock et al. (2002) study, we modified the task to both Americanize it (e.g., changing "Father Christmas" to "Santa") and to make it appropriate for teachers. Thus, we removed some scaffolds and situated scaffolds as student approaches. Our version of the Santa Task for teachers included these scaffolds:

1. Ms. Yarbrough's class had two favorite answers. About 40% of her class chose 18 ml and about 40% chose 54 ml. What might the students who were wrong been thinking about?
2. One of Ms. Yarbrough's students drew rectangles around the images. Do you think this is a helpful strategy for a student? Why or why not?
3. One of Ms. Yarbrough's students tried to solve the problem with easier numbers. She thought of the smaller picture as taking 1 tube of red paint and tried to figure out how much red paint would be needed for the larger picture. She decided that the right amount would be 9 tubes. Does this method work? Is the student's answer reasonable?

In previous analysis (Orrill & Brown, 2018), we found several teachers were misled by the task. Of the 32 participants, only nine (28%) initially used an area-interpretation. Two teachers corrected their error in Scaffold 1, one more corrected their approach in Scaffold 2, and Scaffold 3 led to no changes. One additional teacher corrected her response, but was not clear about when she realized her error as she updated her answers to all parts of the task. By the end of the task, only 14 of the 32 (44%) teachers had correctly applied the area interpretation.

In this paper, we extend the analysis of teachers' interactions with the scaffolds to determine (a) whether they engaged with students' reasoning and (b) whether they could determine the reasonableness of a student's approach. Our purpose was to explore whether there is a connection between teachers' demonstrated content knowledge and their ability to make sense of students' reasoning.

Painting Santa Task

Part 1

Bart is publicity painter. In the last few days, he had to paint Christmas decorations on several store windows. Yesterday, he painted a 56 cm high Santa on the door of a bakery. He needed 6 ml of paint. Now he is asked to make an enlarged version of the same painting on a supermarket window using the same paint. This copy should be 168 cm high. How much paint will Bart need to do this?

Ms. Yarborough gave her students the question above. Please take a moment to work the problem so you're comfortable with it, then we will look at sample responses.

Figure 1: Santa Task

Perspective

Teaching mathematics requires facilitating students' interactions with mathematics in ways that allow them to develop meaning. A variety of resources exist that provide guidance for such engagement. As Kilpatrick, Swafford, and Findell (2001) argue, teachers decide when to ask questions, provide guidance, and allow students to struggle. They also shape classroom conversations. These moves require teachers to engage with students' mathematical reasoning. *Principles to Actions* (NCTM, 2014) suggested mathematics teachers be able to "elicit and use evidence of student thinking" (p. 10), including being able to assess students' understanding as a means for making instructional decisions.

While scholars have included making sense of students' understanding as an aspect of the knowledge teachers need (e.g., Shulman, 1986; Ball, Thames, & Phelps, 2008), little research has been done linking teachers' understandings of mathematics to their understandings of students' ideas about mathematics. This study seeks to address that opening in the literature.

Methods

For this study, we analyzed the data from the same 32 teachers as in our previous work. The participants were a convenience sample of middle school teachers from four states. They ranged from one to 26 years of experience. Twenty-four participants identified as female.

Each participant completed a think-aloud interview that included the Santa Task. They were asked to solve the Santa task, then respond to each scaffold. For this study, the authors considered in the verbatim transcript (a) whether the participant's answer to the prompt was correct; (b) whether the participant was able to make sense of each approach; and (c) whether the participant identified the reasonableness of the approach. For the participants who changed their solution, we analyzed only those responses that were given after the switch to area reasoning. (Note: no participants switched to linear reasoning.) Our aim was to determine whether there were relationships between participants' own mathematical thinking and their engagement with making sense of the students' thinking. Transcripts were coded by each author independently and then discussed to reach 100% agreement.

To determine whether participants made sense of each scaffold, we looked for particular evidence. In Scaffold 1, the evidence was an explanation about how students arrived at the answer the participant thought was incorrect. In Scaffold 2, the evidence was understanding the ability to compare the bigger Santa to the smaller one using tiling or similar approaches. In Scaffold 3, the evidence was understanding every aspect of the image could be thought of as the bigger Santa being nine times larger.

To determine whether participants identified the reasonableness of the answer, we considered whether a rationale for the approach was provided. In Scaffold 1, were participants able to explain why students might have chosen that scale factor? In Scaffold 2, were they able to see the benefit of the rectangle? In Scaffold 3, were they able to understand this was a mathematically viable way to reason about the paint?

Findings

Overall, there was a connection between participants' engagement with students' ideas and their abilities to solve the task. For Scaffold 1, participants who answered the task correctly (n=12) were able to both make sense of the approach and provide a reasonable explanation for that approach. For example, Charlotte (all names are pseudonyms) noted:

> Well, the students who were wrong were just thinking that the Santa is three times as high. So, they were thinking that…just means three times as much paint. So, what's interesting is what they were not thinking about. They were not thinking about that the Santa is not expanding in just one direction. The Santa is expanding in two directions.

In contrast, of the participants who answered the task incorrectly (n=20), only 11 were able to make sense of the students' approach and none were able to determine the reasonableness. For example, Matt was able to determine how students got the "wrong" answer, but offered no explanation about why: "…instead of multiplying both by three, they multiplied the milliliters times nine and ended up with fifty-four."

For Scaffold 2, we found similar results. Of the 13 participants who answered correctly, all were able to make sense of the student's approach and to determine reasonableness. For example, Alan said, "So, yes, I think this is a helpful strategy because the rectangles that they drew around the images would perhaps lead them to think that this problem is about the area being proportional to the amount of paint as opposed to the height alone being proportional to the amount of paint." In contrast, only 13 of the 19 participants who were incorrect on the task were able to make sense of using rectangles and only one was able to determine why. One kind of answer these participants offered focused on only one attribute of Santa (e.g., height) similar to their initial solutions. For example, David was coded as making sense of the student's thinking but not being able to determine a reasonable way to use the representation when he explained, "this is a good way because if you look at the height of the rectangle the big one is three times bigger than the little one. Three lines to nine lines from top to bottom so that means they can multiply the first ratio by three thirds is equal to one whole." Participants who were unable to make sense of the student's thinking often introduced information into the task that was unnecessary, for example questioning whether the images were drawn to scale or answering vaguely. Brenda provided one such response, "I think it will show them that the two Santas are proportioned... proportional, but... but they would still have to figure out where their numbers go, where… I'm sorry where the numbers would be placed to find the solution."

Finally, for Scaffold 3, of the 13 participants who attended to area, 12 were able to make sense of the student's approach and 10 were also able to articulate the reasonableness. For example, in a particularly insightful response, Georgia commented:

> There's so much left out of this explanation… I wouldn't understand how she got the 9 if this is all that I would know about it. But I would assume this method works because the factor is the main thing. And if she understands how to use the factor once she knows what the factor is, then I would accept this as a reasonable answer. I would need to know more about this answer or approach. But I wouldn't mind my students using this type of logic. Because it shows understanding of the problem, even though they may lack the number sense to use those numbers. It may be a good way to get the same level of critical thinking...

This was an important response because it shows that participants who were able to make sense of student thinking may not have initially understood that thinking. Of the 19 linear-focused participants, only seven were able to make sense of the approach and none were able to see how it was reasonable. One common approach for these participants was to link the 1:9 ratio to the 6:56 ratio the participants believed was correct. This led them to misinterpret the approach. One example of this was offered by Eileen who said, "So if this student is going to use these easier numbers and they're going to say that 1 tube of red paint will paint the entire 56 centimeter Santa, then following that 2 tubes should paint 112 centimeters and 3 tubes should paint 168 centimeters. So, I'm getting 3 tubes. I don't know where the 9 tubes came from."

Significance

This study provides insight into teachers' mathematical knowledge and their ability to make sense of students' thinking. The participants who answered the task using an area-focused approach provided more insight into how students were thinking. Despite unfamiliarity with an approach, as shown in Georgia's example from Scaffold 3, correct participants were still more often able to understand students' thinking and determine reasonableness.

We lack the data to know whether the incorrect participants' lack of sensemaking around the students' work was caused by teacher disposition or ability; however, if a mathematics teacher's job includes sensemaking, as suggested above, then this study uncovers an area for serious consideration. Teachers cannot be correct 100% of the time and need to be able to engage with students' ideas even when they (teachers) are not correct. More research is needed to understand why teachers in this study with a correct mathematical approach were more able to do this: Is it a disposition toward students? Is it a disposition toward mathematics? or Is it that more mathematics knowledge better prepares one to make sense of ideas? These are important questions as the field continues to refine our understanding of what knowledge matters for the work that teachers do.

Acknowledgments

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CONOCIMIENTO PEDAGÓGICO DEL FORMADOR PARA ENSEÑAR LA ADICIÓN A FUTUROS PROFESORES

Ana Rosa Arceo Luna
Universidad Autónoma de Aguascalientes
aanarceo@gmail.com

David Alfonso Páez
CONACyT- UAA
dapaez@correo.uaa.mx

Enseñar la adición a futuros profesores requiere del formador un conocimiento matemático y didáctico en el marco de las demandas curriculares y contextuales. El objetivo del presente estudio es identificar el conocimiento pedagógico inmerso en la práctica del formador al enseñar el algoritmo de la adición y su didáctica. Para ello, se observaron cuatro sesiones de un formador de profesoras de educación primaria en México al desarrollar este contenido. El análisis de los datos fue de tipo descriptivo e interpretativo a través de categorización. Los resultados muestran que en este formador destaca el subdominio del Conocimiento del Contenido y la Enseñanza, ya que instruye a sus alumnas principalmente sobre la didáctica de la matemática en función de cómo los niños de educación primaria se acercan al algoritmo de la adición.

Keywords: Conocimiento del profesor, Preparación de Maestros en Formación, Conceptos de Números y Operaciones

Diversas investigaciones han dado cuenta que el conocimiento didáctico y matemático del profesor es una característica clave para la mejora y la calidad en el aprendizaje (Ball, Thames, & Phelps, 2008; Jaworski, 2008). El docente que imparte matemáticas en instituciones formadoras de maestros tiene como tarea principal ayudar a los profesores en su formación inicial a desarrollar y mejorar las tareas docentes de modo que construyan el conocimiento profesional necesario para enseñar matemáticas (Jaworski, 2008; Rojas, & Deulofeu, 2015). Ello significa que este actor, también conocido como formador, requiere un mayor dominio de conocimientos pedagógicos sobre actividades de aprendizaje, aspectos referentes al currículum y evaluación, además del contexto social, características personales, conocimientos previos y procesos de aprendizaje de los estudiantes (Ball et al., 2008; Jaworski, 2008). Al considerar la práctica del formador de educación primaria (6 a 12 años de edad) como una preocupación a nivel nacional e internacional (Llinares, 2013), se ha enfatizado el interés por estudiar su conocimiento didáctico y matemático, con la intención de hacer propuestas de intervención donde tenga la posibilidad de discutir sus conocimientos inmersos en la práctica para la mejora de la enseñanza (Ball et al., 2008), en particular, aquellos que emplea para desarrollar contenidos que, por las habilidades que ponen en juego, constituyen una herramienta valiosa para la construcción de algoritmos matemáticos, como lo es la adición (Godino, Font, & Wilhelmi, 2006). El estudio aquí reportado tiene como objetivo: Identificar características del Conocimiento Pedagógico del formador inmerso en su práctica al enseñar el algoritmo de la adición.

Marco teórico

Para estudiar el Conocimiento Pedagógico del formador se retoma el modelo Mathematical Knowledge for Teaching (MKT) propuesto por Ball, Thames y Phelps (2008), desde la perspectiva de quien enseña a los futuros profesores; es decir, que tenga un dominio mayor del objeto matemático y cómo este debe enseñarse a los niños (Zopf, 2010).

El MKT expresa una relación recíproca de dos dominios del conocimiento para la enseñanza de las matemáticas: del Contenido y de la Pedagogía del Contenido. La relación entre ambos constituye una composición que "implica una idea o procedimiento matemático particular y la familiaridad con los principios pedagógicos para enseñar ese contenido" (Ball et al., 2008, p. 402), por lo que en conjunto forman un insumo para analizar múltiples aspectos inmersos en la práctica de quien enseña matemáticas. El primer dominio constituye un conocimiento particular y característico de quienes se dedican a la enseñanza de las matemáticas (Hill, Ball, & Schilling, 2008) y se compone, a su vez, de tres subdominios: Conocimiento del Contenido y de los Estudiantes (KCS), Conocimiento del Contenido y de la Enseñanza (KCT), y Conocimiento del Contenido y el Currículo (KCC). Respecto a la adición, el formador debe poseer una comprensión sobre este contenido matemático, sus significados (Godino, Font & Wilhelmi, 2006), representaciones y modos en que el niño construye el algoritmo convencional (Vergnaud, 1985/2013) con la intención de mostrar la didáctica.

Metodología

Se trata de un estudio de caso en el que, desde un enfoque cualitativo, se percibe la realidad como un sistema complejo que se encuentra integrado y sujeto a un análisis (Cohen, Manion, & Morrison, 2007). El participante es un formador que labora en una escuela mexicana para profesoras e imparte la materia *Aritmética. Números Naturales,* en la que se enseña la adición y su didáctica en la Licenciatura en educación Primaria 2018 (Secretaría de Educación Pública [SEP], 2018). Cuenta con una formación inicial como docente de educación básica,y tiene doce años de experiencia en las escuelas para maestros, de los cuales en nueve ocasiones ha enseñado las materias en las que se muestra la adición.

Para la obtención de los datos se utilizó la observación no participante centrada en captar explicaciones, ejemplos e interacciones que dan cuenta del Conocimiento Pedagógico del formador (Ball et al., 2008). El formador fue observado (videograbado) en cuatro sesiones de clases (duración: dos horas cada una) en las tenía prevista el tratamiento de la adición según su planeación didáctica y el programa de la materia.

Se llevó a cabo un análisis interpretativo descriptivo, por lo que las videograbaciones fueron vaciadas en una matriz de acuerdo a las categorías del Conocimiento Pedagógico para Enseñar Matemáticas (Ball et al., 2008) y los indicadores propuestos por Sosa y Carrillo (2010). En este sentido, se determinaron tres categorías deductivas conformadas por subcategorías: KCS, *conocimiento del estudiante de primaria, y conocimiento del estudiante para profesor*; KCT, *didáctica del profesor de primaria, uso de ejemplos, preguntas y, respuestas de los estudiantes*; KCC, c*urrículum oficial y currículum real*.

Además, otras tres subcategorías emergieron de los datos: *Conocimiento del estudiante de primaria,* que se refiere al reconocimiento que hace el formador acerca de los modos de pensamiento de los niños con quienes trabajaran las futuras docentes; *Didáctica del profesor de primaria,* que se refiere a las estrategias de enseñanza que se sugieren para mostrar la adición a los niños; *Currículum real* (Casarini, 2015), alude a los saberes que el profesor ha aprendido en su experiencia previa como docente y, sin indicarse oficialmente, emplea en su práctica.

Resultados

1. KCS

En la práctica del formador se observó un conocimiento acerca de dos tipos de alumnos: el niño de educación primaria y las futuras profesoras. El primer tipo de conocimiento fue mostrado al hacer alusión a los modos de pensamiento del niño que el docente debe considerar para abordar el contenido de la adición; por ejemplo, el formador menciona que "antes de plantear situaciones convencionales, primero hay que poner al niño a que haga conteo de situaciones en las que hay que agregar o quitar, de ahí se trabaja la composición y la descomposición". Además el formador sugirió a sus estudiantes considerar las dificultades de los niños al enseñar la suma, en particular la comprensión de los problemas matemáticos donde hay dificultad para discriminar los datos necesarios, identificar el tipo de operación -como el restar en lugar de sumar- y al uso del algoritmo escrito; por ejemplo, comentó a sus estudiantes: "los niños no mantienen un orden en el sentido de las unidades, decenas y centenas, por lo que llegan a cometer errores al ubicar el valor posicional".

El conocimiento de las estudiantes para profesoras se observó cuando el formador reconoció que tienen procesos de pensamiento distintos a los niños, en particular en la resolución de ejercicios matemáticos: "ya ustedes son expertas en este procedimiento, por eso lo resuelven de una manera más rápida. Pero pensemos en el trámite de los niños de apropiarse de un procedimiento que ellos desconocían". En este sentido, a las futuras profesoras les solicitó que además de la participación en los juegos sugeridos para trabajar con los niños y la resolución de problemas, diseñaran actividades, considerando el conocimiento previo de la resolución de problemas y el empleo de la suma y la resta: "diseñen una actividad donde comiencen a introducir los signos de sumar y restar". En general, predomina el conocimiento del estudiante de primaria, por lo que las actividades y sugerencias estuvieron centradas en el proceso de los niños y cómo éstos se acercan al contenido de la adición, por ejemplo ante resoluciones equivocadas, el formador hizo expresiones como: "sucede con los niños", enfatizando que es el estudiante central.

2.KCT

El subdominio del KCT fue mostrado por el formador a través de cuatro subcategorías: a) Didáctica del profesor de primaria, b) Uso de ejemplos, c) Preguntas y d) Respuestas de los estudiantes. En lo que refiere al inciso *a* se observaron dos aspectos principales: el primero respecto a las consideraciones acerca de los procesos que debe seguir el profesor para guiar al niño desde la fase manipulativa al algoritmo convencional de la adición, pues dijo que el docente de primaria "tiene que ponerlos a jugar, a contar, a que iluminen, a que comparen, que pongan los signos, primero, y ya al final el procedimiento usual" ya que considera que esto hace que para el niño sea más "fácil y divertido, al niño le interesan más este tipo de actividades y hacemos algo distinto". El segundo aspecto es la delimitación de roles o tareas que debe desempeñar el niño y el profesor; el formador expresó que las nociones acerca del signo y los elementos que conforman la suma –sumandos, resultado y signo- "deben ser enseñados por el profesor, pues el niño no los descubre solo", mientras que en la resolución de problemas la función del profesor es "darle libertad al niño para que represente las colecciones […] ya al final el maestro lo representa de manera horizontal y vertical, pero no le dice al niño que lo resuelva así, el niño lo resuelve como puede ".

Respecto al inciso *b,* algunos ejemplos consistían en la introducción al tema de forma expositiva, por ejemplo para presentar nociones y conceptos relacionados con la adición se utilizó una proyección en el pizarrón, otros ejemplos servían para presentar problemas en el pizarrón en los que se involucran situaciones que el formador denomina de "agregar o quitar". Estas situaciones tuvieron dos intenciones: que las estudiantes ejercitaran el proceso de resolución de problemas en los que utilizan el algoritmo escrito de la adición, y mostrar la didáctica del contenido enfatizando cómo se integran los conceptos matemáticos revisados en la clase de formación para profesoras –como cambiar la expresión "sobra" por el término "transformaciones", y hacer sugerencias sobre el uso de material manipulativo.

En lo que refiere al inciso *c*, las preguntas se plantearon principalmente con cuatro intenciones: para enlazar contenidos de la clase con la experiencia previa de las estudiantes; "¿Cómo les enseñaron a ustedes a sumar y a restar cuando eran niñas?" "¿Les ponían problemas o les ponían ya el algoritmo?" y contrastarlo en cómo deben enseñarlo a sus futuros estudiantes de primaria. Para valorar la didáctica y estrategias propuestas por el formador: "¿Por qué es necesario trabajar el tablero y en general actividades lúdicas antes del algoritmo usual?". En las operaciones matemáticas, para comprobar procedimientos y hacer propuestas de mejora a los recursos; "¿qué necesitaría este problema para hacerlo más comprensible para el niño?", y para construir reflexiones acerca de la enseñanza.

Por su parte, las respuestas de las estudiantes fueron consideradas por el formador para profundizar en los contenidos así como para hacer correcciones a los procedimientos y a los conceptos, por ejemplo, al pasar una estudiante al pizarrón y utilizar expresiones como "sobrantes", el formador la corrige y le señala que "se transforma en una decena" haciendo alusión al empleo de un concepto más de tipo matemático que de sentido común. Las respuestas dadas por las estudiantes son utilizadas también para generar discusión acerca de la práctica futura de las docentes y los procedimientos que los niños siguen al utilizar el algoritmo de la adición; por ejemplo, cuando las estudiantes explican cómo han resuelto un problema, el formador comenta "tú lo hiciste con tu mente y con ayuda de los dedos, tú lo hiciste en la libreta, el cálculo mental, tú usarías el ábaco […] así como ustedes, también los niños utilizan diferentes materiales para poder contar" y a partir de ello, el formador explica cómo los niños resuelven la suma.

3. KCC

En el KCC se observaron dos tipos de currículum en el aula; el oficial y real. Respecto al primero, el programa de estudios (SEP, 2018) señala que en este curso se espera estudiar la didáctica y la matemática, que es una intención observada en la práctica del formador, además utilizó la metodología sugerida por este programa, donde se da prioridad a la resolución de problemas. Sin embargo, la secuencia de actividades es distinta a la propuesta del programa, lo que hace suponer que el formador ha utilizado referencias que surgen de su experiencia previa como profesor de primaria, por lo que se observa un currículum real en el que el formador hace propuestas didácticas y matemáticas que surgen de su experiencia y son distintas a los planteamientos del currículum oficial.

Conclusiones

Este trabajo permitió plantear el dominio del Conocimiento Pedagógico del MTK en la práctica de los formadores y con ello dar cuenta que éstos actores manifiestan características distintas a las de los docentes que tienen como objetivo de su práctica el enseñar la adición y no la didáctica. Los ejemplos –KCT- constituyeron un eje central para el desarrollo del contenido y sirvieron para enseñar o reaprender la matemática en las futuras profesoras (Rojas & Deulofeu, 2015), en especial, los conceptos relacionados con la adición. Los subdominios que se retoman de la práctica del formador presentaron subcategorías diferentes a las del estudio de Sosa y Carrillo (2010) relacionadas con la enseñanza de futuros profesores; por ejemplo, en el KCS las actividades y exposiciones fueron diseñadas atendiendo al nivel cognitivo y los procesos de los niños para adquirir el algoritmo, priorizando este estudiante, mientras que en el KCC el formador lleva a cabo un currículum real que utiliza su propia experiencia como herramienta principal. En síntesis, se pudo observar que el Conocimiento Pedagógico al enseñar la adición a futuras profesoras se presenta con un énfasis particular en las tareas de instrucción, las cuales se relacionan con los procesos que el niño lleva a cabo. Conjuntamente, se mostró la relevancia del KCT en la práctica del formador (Zopf, 2010) y además de reconocer los conocimientos inmersos en la práctica, sería necesario desarrollar estudios encaminados a estudiar la reflexión que el formador puede realizar para mejorar su práctica.

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PEDAGOGICAL KNOWLEDGE OF THE TEACHER EDUCATOR TO TEACH THE SUM TO FUTURE TEACHERS

Teach the sum to future teachers requires from the teacher educator a mathematical and didactic knowledge within the framework of curricular and contextual demands. The objective of the present study is to identify the pedagogical knowledge in the teacher educator's practice when teaching the algorithm of the sum and its didactics. For this, four sessions of a teacher educator of primary education in Mexico were observed when developing this content. The analysis of data was descriptive and interpretative through categorization. The results show that in this teacher educator highlights the subdomain of Knowledge of Content and Teaching, he mainly instructs his students mainly on the didactics of mathematics according to how primary school children approach the algorithm of the sum.

CONNECTING ADVANCED AND SECONDARY MATHEMATICS: INSIGHTS FROM MATHEMATICIANS

Xiaoheng Yan
Simon Fraser University
xkyan@sfu.ca

Ofer Marmur
Simon Fraser University
omarmur@sfu.ca

Rina Zazkis
Simon Fraser University
zazkis@sfu.ca

This paper reports a preliminary analysis of mathematicians' detailed accounts on why and how advanced mathematical knowledge matters for secondary mathematics teaching, and what specific connections can be made between advanced and secondary mathematics. Drawn from interviews with 18 mathematicians from 9 universities, the findings indicate that having advanced mathematical knowledge would allow secondary mathematics teachers to 1) develop an appropriate framework of mathematics; 2) gain mathematical experiences that would enhance their students' experience with mathematics; and 3) grow awareness of the extent of one's mathematical knowledge. Specific examples that connect linear algebra to secondary mathematics are presented and discussed.

Keywords: Advanced mathematical thinking, High school mathematics, Teacher knowledge, Mathematicians.

Introduction

How important is advanced mathematics for good teaching? While Monk's (1994) study shows that teachers' mathematical background, as measured by the number of mathematics courses a teacher took in university, is positively related to how much mathematics students learn at the secondary level, it is not always clear how and why advanced mathematics contributes to teaching. In fact, teachers often struggle to find specific examples that connect advanced and secondary mathematics (e.g., Zazkis & Leikin, 2010). Previous studies sought the relevance and contribution of academic mathematics to teaching by taking into account mathematicians' views on curriculum planning and course design (e.g., Goos, 2013; Hoffmann & Even, 2018; Leikin, Zazkis & Meller, 2018). However, many questions remain unanswered regarding which connections between advanced and school mathematics are important, and how they might lead to the improvement of teachers' practice (Murray et al., 2017). This paper reports a preliminary analysis of mathematicians' view on why advanced mathematical knowledge matters for secondary mathematics teaching, and what specific connections can be made between linear algebra and secondary mathematics.

Theoretical Background

Our paper builds upon 1) the notion of advanced mathematical thinking (AMT) and advanced mathematical knowledge (AMK); and 2) previous studies on mathematicians' perspectives on the importance of advanced mathematical knowledge for teaching.

Tall (1991) examined differences between elementary and advanced mathematical thinking as transitions from *describing* to *defining*, from *convincing* to *proving* based on abstract entities. These transitions, considered challenging processes for most prospective teachers, must be addressed and overcome during their undergraduate education. In exploring secondary mathematics teachers' conceptions of the role and usage of AMK in their teaching practice, previous studies have shown that while some teachers acknowledged the importance of AMK, others are unaware of the connections between secondary and advanced mathematics, and are often dismissive of their upper-level training (e.g., Goulding, Hatch, & Rodd, 2003; Zazkis & Leikin, 2010). To help motivate practicing and prospective teachers in deepening their mathematical understanding, previous studies also investigated how abstract algebra might support teachers' efforts to unpack a secondary mathematics topic, and how an understanding of advanced mathematics might shape pedagogy in the secondary classroom (Christy & Sparks, 2015; Murray et al., 2017).

Considering mathematicians' perspectives on teaching, Leikin, Zazkis, and Meller (2018) noted that, "mathematicians act as teacher educators de facto, without explicitly identifying themselves in this role" (p. 452). Indeed, mathematicians and teachers typically have different approaches to the knowing of mathematics: while mathematicians focus on the epistemic nature of the subject, teachers often focus on cognition and pedagogy without rigorous justification (Cooper & Arcavi, 2018). Despite these differences, Bass (2005) argued that mathematicians' knowledge, practices, and habits of mind are relevant to school mathematics education because their mathematical sensibility and perspective are essential for maintaining the mathematical balance and integrity of the educational process. This preliminary research report aims to further investigate mathematicians' perspectives on possible connections between advanced and secondary mathematics.

Method

The participants were 18 mathematicians from 9 universities: 15 were actively engaged in mathematics research and 3 were focused on teaching only. Of the 18 participants, 15 were teaching undergraduate or graduate courses in mathematics at Canadian research universities; two were professor emeriti; and one was a visiting professor from outside of Canada.

The main corpus of data was comprised of individual semi-structured interviews with the participating mathematicians. The aim was to gain insights into their views on advanced mathematical knowledge and its relevance to secondary school mathematics teaching. The interviews included two main guiding questions:

1. How do you think advanced mathematical knowledge might contribute to secondary school mathematics teaching?
2. What are specific examples that reveal connections between advanced and school mathematics?

All the interviews were audio-recorded and transcribed. Additional written artifacts generated by the interviewees were collected for the qualitative analysis. Recurring themes were identified and categorized using an iterative and comparative process. In this paper, all participants were coded starting with M, followed by a number assigned to each participant.

Findings

Why Advanced Mathematical Knowledge Matters

Three major themes emerged from the mathematicians' responses to "how do you think advanced mathematical knowledge might contribute to secondary school mathematics teaching?" The results of the analysis show that advanced mathematical knowledge would allow teachers to 1) develop an appropriate framework of mathematics; 2) gain rich mathematical experience that would enhance their students' experience with mathematics; and 3) grow awareness of the extent of one's mathematical knowledge.

Developing the framework of mathematics. Some of the mathematicians claimed that a clear vision of what mathematics is and robust understanding of a mathematical concept comes from advanced mathematical knowledge. For M5, "the conceptual framework that underpins complex procedures and mechanical applications" is a component of AMK. Along the same line, M10 explained the framework of mathematics with an analogy:

> If you are knitting something, you must get some kind of overarching concept of what it is that you are doing. You are working locally but there is some framework in your mind that how your local thing is sort of fitting in.

In this sense, the importance of advanced mathematical knowledge lies in the heart of developing or changing the framework of mathematics. Having advanced mathematical knowledge would allow a teacher to see the fundamental structure and to reach "a kind of gestalt," as M10 stated, "I mean it is the same idea but somehow you are looking at a piece of mathematics on completely different terms."

The framework of mathematics was also elaborated in the sense of the scope of the subject. For example, "What comes next in the material? And how is it used later on?" (M14). As M16 put it, "This is what you teach now, but maybe this will appear again in a slightly different form; because definitions and problems are adjusted to students' comprehension level. How will they appear later on? That is the key." Concerning teaching at the secondary level, M14 stated that, "Having more knowledge means more tools, more examples, and a deeper understanding. […] It allows you to bring the material to life via a compelling narrative."

Gaining mathematical experience. Mathematics is not a spectator sport (Phillips, 2005). The mathematicians interpreted advanced mathematics not only as a form of knowledge, but also as "the process and experience of gaining that knowledge and thinking about how that knowledge impacts what they [teachers] are going to try to put forth to their students" (M11). As M10 suggested, "Roll up your sleeves and do it […] because it embodies structure and you have to get into it and understand the structure." M9 also explained:

> It [mathematics] is the study of complex structure and especially when there are simple elegant aspects to the structure. So they [teachers] need to understand how to work with such a structure. How to take it apart so they can understand how the different pieces interact and then put it back together? The only way to get that understanding and experience is to spend time doing it.

One of the benefits from doing a lot of mathematics, for M13, is that learners would be able to attack a mathematics problem using a "power drill" rather than some very simple "screwdrivers" that by definition take longer (see example 3 in Table 1).

Growing awareness of the extent of one's mathematical knowledge. Another aspect of the importance of AMK lies in the growth of awareness that guides and enables actions. M15 regarded the importance of AMK as a reflective process that allows teachers to identify their strengths and weaknesses in mathematics:

> If people are going to be teaching, they should be able to tell they actually know something well or more importantly whether they are still confused. Really getting that distinction down

> is one of the goals of undergraduate training at the level of a major. I think that only if you study a certain amount of mathematics [you] can do that [make that distinction].

Other mathematicians also shared this view. For example, M4 pointed out that one of the expectations in the undergraduate mathematics program at his university was to learn the limits of both students' knowledge and mathematicians' knowledge. This suggests that being aware of the extent of one's mathematical knowledge would consequently enable prospective teachers to mediate between elementary and advanced mathematical thinking.

Connections Between Advanced and Secondary Mathematics

The examples provided by the mathematicians span a wide range of topics in mathematics. In this paper, we focus on three examples in linear algebra to illustrate the connections drawn between advanced and secondary mathematics.

Table 1: Examples in Linear Algebra

A High School Problem	AMK	Advantages of AMK for Teachers
1. Solve systems with 3 or more equations.	- *Gauss-Jordan elimination*: "A systematic way to simplify a set of equations" (M5). - *Matrices and the idea of inverse*: "Use the inverse matrix if it is invertible, then you know you get a unique solution" (M8).	- "'Read off' the solution" (M5); - Identify whether a system has unique, many, or no solution.
2. Solving $\begin{cases}\mathbf{x}+\mathbf{y}=\mathbf{2}\\ \mathbf{x}-\mathbf{y}=\mathbf{0}\end{cases}$ A student's incorrect method $\mathbf{x}+\mathbf{y}-\mathbf{2}=\mathbf{x}-\boldsymbol{y}\Rightarrow\boldsymbol{y}=\mathbf{1}$, thus the solution set is y=1, x any number.	- *Solution to a system of equations*: "In the background of [the] Gauss-Jordan elimination is the abstract idea of what algebraic operations are permitted on the system of equations (manipulating them) that don't change the solutions set" (M5).	- Understanding that "equating two or more equations together is NOT an allowed operation; it could alter the solution set" (M5).
3. Find the equation of a line through the points (2, 7) and (8, 10).	- *Use of the determinant of a 3 by 3 matrix*. (M13) e.g. (2,7) and (8,10). $0 = \det\cdot\begin{bmatrix} x & y & 1\\ 2 & 7 & 1\\ 8 & 10 & 1\end{bmatrix} = x\cdot\det\begin{bmatrix}7 & 1\\ 10 & 1\end{bmatrix} - y\det\begin{bmatrix}2 & 1\\ 8 & 1\end{bmatrix} + 1\cdot\det\begin{bmatrix}2 & 7\\ 8 & 10\end{bmatrix}$ $= -3x - y\cdot(-6) - 36 = 0$ Then solve to get $6y = +3x+36$ $\boxed{y = \frac{x}{2}+6}$	- Offer an alternative solution; - Create a need for a justification of why the method works, which may lead to a formal proof.

Discussion

In response to why advanced mathematical knowledge matters for good teaching, the mathematicians placed emphasis on the understanding of the breadth and depth of mathematical knowledge in relation to mathematical structure. Two of their ideas – to establish the framework of mathematics and to gain rich mathematical experience resonate with Mamona-Downs and Downs' (2008) elaboration on the importance of the process of mathematical thinking rather than the product of mathematical thought. This is also in line with Burton's (1999) suggestion that mathematicians often see connections set within a global image of mathematics as an important part of knowing mathematics. The third idea regards the importance of teachers' awareness of the extent of their own knowledge, and their ability to distinguish between what they know and what requires further mathematical investigation. Additionally, by providing examples in linear algebra that may require AMK for an explanation, an alternative method, or simply a quick solution, the mathematicians provided insightful connections between advanced and secondary mathematics. Nonetheless, the analysis of the data set is still at its early stages. Further investigation and a more in-depth analysis would help us better understand the nature of advanced mathematical knowledge, and its relationship to school mathematics.

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VALIDATION OF A U.S. INSTRUMENT FOR DIAGNOSING MATHEMATICS TEACHER KNOWLEDGE FOR USE IN CHINA

Xingfeng Huang
Shanghai Normal University
xfhuang@shnu.edu.cn

Rui Kang
Georgia College & State University
rui.kang@gcsu.edu

Yawei Shen
University of Georgia
yawei.shen51@uga.edu

Andrew Izsák
Tufts University
Andrew.Izsák@tufts.edu

We report the results of a validity study of a multidimensional fraction arithmetic test with a sample of 371 Chinese middle school mathematics teachers. The test was originally developed and validated in the U. S. using diagnostic classification models (DCMs). It contains four attributes related to fraction multiplication and division reasoning. The results provide cross-cultural support for DCMs to reliably measure multidimensional constructs. The response patterns from the Chinese sample were compared to those of a US sample of 990 middle school teachers and provide a strong rationale for international collaboration.

Keywords: Mathematics Knowledge for Teaching, Rational Numbers, Measurement, Middle School Education

The field of mathematics education has seen a growing interest in cross-national studies of teaching and learning. Cross-national comparisons of teacher knowledge are afforded by measures such as those developed in the contexts of (a) the Teacher Education and Development Study: Learning to Teach Mathematics (TED-M) and (b) Learning Mathematics for Teaching (LMT). In 2012, the journal *ZDM: Mathematics Education*, devoted an entire issue, *44*(3), to *Assessment of Teacher Knowledge across Countries*, which served as a platform for international scholars to engage in rich discussions about the various methodological challenges surrounding cross-cultural measures of teacher knowledge (e.g. Delaney, Ball, Hill, Schiling, & Zopf, 2008; Stylianides & Delaney, 2011). Delaney et al., in particular, recommend that researchers, when adapting measures to use internationally, first employ psychometric and interview-based methods to determine a correspondence between the constructs being measured.

We believe that studies of Eastern Asian and Chinese learners and teachers in the past (e.g., An, Kulm, & Wu, 2004; Ma, 1999; Stigler & Perry, 1988) have not only helped mathematics teachers better understand the practices in their own societies and cultures, but also enhanced their awareness of the various alternative models, methods, and possibilities, and will continue to have a profound impact on the development of practices and policies in mathematics education.

Theoretical Perspectives

Mathematical Perspective

The instrument we use for this validity study was developed by an interdisciplinary team of psychometricians and mathematics education researchers in the context of a National Science Foundation-funded project, Diagnosing Teachers' Multiplicative Reasoning (DTMR). The DTMR fractions survey was designed to measure knowledge of middle grades mathematics teachers (grades 5-7) in multiplicative reasoning about fraction arithmetic. It emphasizes standards-based practices (CCSSO, 2010; NCTM, 2000) in which numbers are embedded in problem situations as measures of quantities (e.g., lengths and areas) and are accompanied with drawn models (e.g., number lines, rectangular areas). For a detailed discussion of the test construction process and sample test items, please see Izsák, Jacobson, and Bradshaw (2019). The final version of the instrument includes four attributes or components.

In brief, the first attribute is called *referent unit* (RU), which is closely related to Lamon's (1996) notion of unitizing, i.e., assigning a given quantity a unit of measurement. The second attribute, *iterating and partitioning* (PI), combines two closely related aspects of reasoning about fractions into a single construct, similar to Steffe's (2001) splitting construction scheme. *Partitioning* refers to subdividing a given quantity into equal-sized pieces. *Iterating* refers to the process of concatenating a unit of length, which is particularly important for developing meaning for improper fractions (e.g., Tzur, 1999, 2004). The third attribute, *appropriateness* (APP), is tied to the difficulties that teachers tend to experience when making distinctions among situations that call for dividing by a fraction, dividing by a whole number, and multiplying by a fraction (e.g., Ball 1990; Ma, 1999; Sowder, Philipp, Armstrong, & Schappelle, 1998). The fourth and last attribute, *reversibility* (REV), usually deals with returning to a starting point after some process (Iszák et al., 2019), for example, after partitioning a non-unit fraction (n/m) into n parts to produce the unit fraction ($1/m$), and then producing the original whole (m/m) from which the unit fraction is a part (also see Tzur, 2004).

Psychometric Perspective

In the majority of the large-scale studies on mathematics knowledge for teaching (e.g., Hill, Sleep, Lewis, & Ball, 2007; Center for Research in Mathematics & Education, 2010), researchers have developed measures for use with unidimensional item response theory models. However, measures based on unidimensional item response theory are not suitable for measuring multiple attributes of reasoning simultaneously.

Diagnostic Classification Models (DCMs) is a family of recently developed psychometric tools that can be used to construct practical, multidimensional tests. For example, DCM analyses provide specific feedback to individual teachers in terms of their strengths and weaknesses on each component of reasoning. In addition, these attributes are determined based on research literature and theory prior to analyses. Compared to multidimensional IRT (MIRT), DCMs need far fewer items per dimension to yield reliable examinee estimates (Templin & Bradshaw, 2013). Bradshaw, Izsák, Templin, and Jacobson (2014) were the first to report an instrument that successfully measures a multidimensional construct using a generalized form of DCM, *log-linear cognitive diagnostic model* (LCDM, see Henson, Templin, & Willse, 2009) with the support of empirical data. Prior studies involving DCMs relied on retrofitting DCMs to existing test data that was originally designed for unidimensional IRT. We know of no cross-cultural validity studies of any DCM-based instrument for mathematical knowledge for teaching.

In this study, we ask two research questions. First, can a multidimensional U.S. diagnostic instrument for mathematics teacher knowledge in a specific domain (i.e., fraction arithmetic) be used to reliably measure the same attributes with Chinese teachers? Second, what is the distribution of proficiency with these attributes among Chinese mathematics teachers?

Methods

We distributed our instrument to a convenience sample of middle grades Chinese teachers (Grades 6-9) from two districts in Shanghai. Among the 450 surveys we sent out, 414 surveys were returned with a response rate of about 92%. Among the 414 returned surveys, 43 surveys were blank, and subsequently were removed from further analysis, resulting in 371 usable surveys. All the Chinese teachers in our sample have at least a Bachelor's degree in mathematics. Some have a Master's degree. Their average years of teaching is about 16 years. About 1/3 of our sample are male teachers. We used forward and backward translation to prepare the Chinese version of the instrument. The two English versions were deliberated and compared among a panel of mathematics educators and doctoral students. Conceptual equivalence, accuracy, and appropriateness were established by the panel. A generalized form of DCM, *log-linear cognitive diagnosis model* (LCDM; see Henson, Templin, & Willse, 2009) was used to fit the test data. All the analysis below were performed in *R* i386 3.5.1 using the *cdm* package.

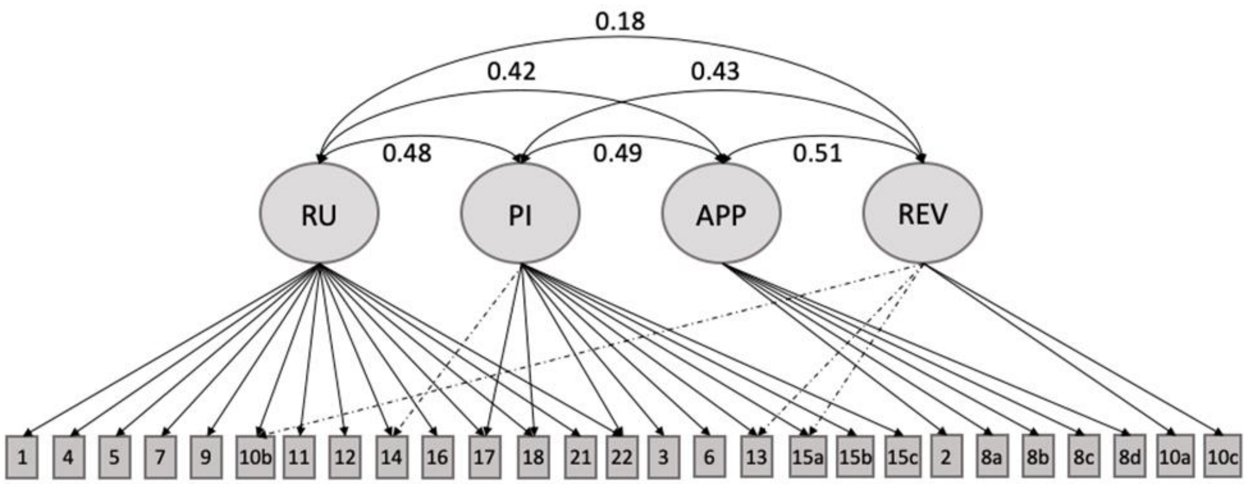

Figure 1: Path Diagram for DTMR Fractions Survey

Results and Discussions

Model Specifications

Figure 1 depicts our final model, which is highly comparable to the final model in the earlier study with 990 U.S. middle school teachers (Bradshaw et al., 2014). A few paths in our study did not show reasonable effect sizes and were signified with dotted lines. The results show that these items only had significant contribution to measure one of the two attributes they were intended to measure. For example, Item 14 significantly contributed to the measures of RU and PI in the U.S. study, but in our study, Item 14 only significantly contributed to the measure of RU but not PI. We also modeled the bivariate relationships between attributes using tetrachoric correlations as shown in Figure 1. The correlation coefficients ranged from .18 to .51. These slightly-to-moderately positive correlations provide evidence that the four attributes are distinct, although related. This result supports the conclusion from the earlier study for using a multidimensional model rather than a unidimensional model. If the underlying trait was unidimensional, we would expect that these correlations to be close to 1.

Item Parameter Estimates and Diagnostic Utility

Odds ratios were determined to indicate the effect sizes for the final 28 items. Conventional criteria for evaluating effect size (Chin, 2000) are small (between 1.44 and 2.47), medium (between 2.47 and 4.25) and large (above 4.25). Two of the items had extremely small effect sizes (Item 2 and Item 10c), which may be interpreted as these items contributed little to the measure of the specified attributes. Among the rest of the 25 items, 9 yielded small effect sizes; 4

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

medium effect sizes, and 12 large effect sizes. These 25 items demonstrated reliable diagnostic utility.

Item Reliability

All the above results focused on item-attribute validity. To quantify reliability of the attribute classifications, we used the DCM measures of reliability from Templin and Bradshaw (2013). Classification reliabilities are extremely high based on the item responses from the Chinese teachers, even for attributes that are measured by only a small set of items. These reliability scores are 0.902, 0.995, 0.994, and 0.999 for RU, PI, APP, and REV respectively.

Attribute Classifications

Because the DTMR fractions survey measures four binary attributes, teachers are classified into 2^4 or 16 possible patterns of attribute mastery or latent classes. The LCDM estimates the probability that each teacher is a member of each latent class. These estimates were aggregated across examinees and are provided in Figure 2. The most likely attribute profiles for our sample of Chinese middle school teachers were (a) all four attributes were mastered (40%) followed by (b) mastery of three attributes except for RU (25%). The earlier U.S.-based study showed that the most likely attributes for U.S. middle school teachers were all attributes were mastered (25.5%), followed by none of the four attributes were mastered (21.2%), and then followed by all attributes but RU were mastered (13.5%). These results demonstrate that RU is a common challenge for teachers in both countries.

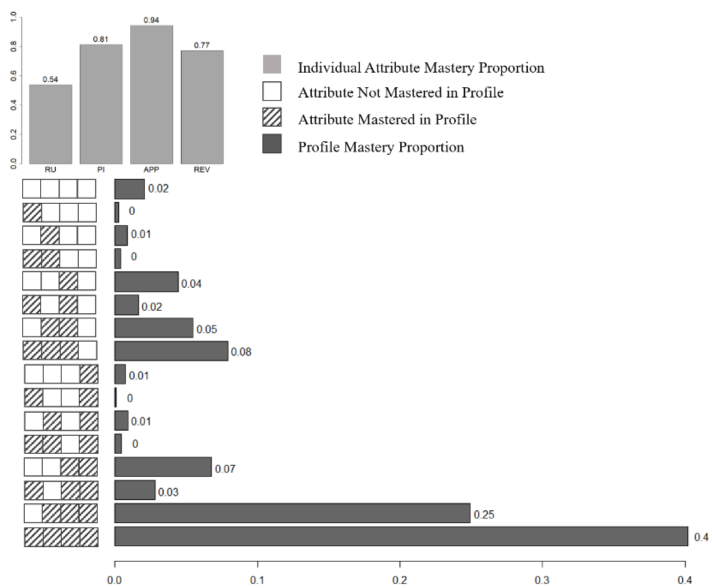

Figure 2: Attribute Profile Mastery Proportions. Profiles Use 0 or 1 to Indicate Nonmastery or Mastery in the Following Order: RU, PI, APP, and REV.

Conclusions and Implications

In terms of psychometric development, this study provides empirical support for the capacity of DCMs to reliably model and measure multidimensional constructs in a new cultural context. The overall responses from the Chinese and U.S. teachers to the DTMR fractions survey have revealed some similar patterns. For example, the lack of proficiency in identifying and choosing appropriate referent units is a common challenge for practicing teachers in both countries. We recommend that international mathematics educators continue to harness the recent advances in psychometric models such as DCMs to build common theoretical grounds for rich and deeper dialogues and for collaboration in developing teacher education and professional development programs.

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CORRELATIONS BETWEEN PROFESSIONAL NOTICING OF STUDENTS' MATHEMATICAL THINKING AND SPECIALIZED CONTENT KNOWLEDGE

Raymond LaRochelle
University of Delaware
rlaroche@udel.edu

Behailu Mammo
Hofstra University
Behailu.Mammo@hofstra.edu

Professional noticing of students' mathematical thinking and mathematical knowledge for teaching are two constructs that appear to have much overlap. However, empirical studies investigating connections between the two constructs have not found much. In this study, we narrowed our focus to specialized content knowledge (SCK), which is a sub-domain of mathematical knowledge for teaching, and found a connection between SCK and professional noticing expertise.

Keywords: Professional noticing of students' mathematical thinking, mathematical knowledge for teaching

In instructional styles where teachers regularly elicit and use student thinking throughout the lesson, it is crucial for teachers to be able to make sense of their students' mathematical thinking in-the-moment as well as productively decide how to respond to students' emergent ideas (Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010; Stein, Engle, Smith, & Hughes, 2008). This crucial practice is called *professional noticing of students' mathematical thinking*, and studies have shown that it is not always an easy practice for teachers to develop (Casey, Lesseig, Monson, & Krupa, 2018; Jacobs, Lamb, & Philipp, 2010; LaRochelle, Nickerson, Lamb, Hawthorne, Philipp, & Ross, in press). One way to think about helping teachers develop their professional noticing expertise is by investigating what knowledge teachers need in order to effectively "professionally notice" a student's work. Some researchers have hypothesized that teachers' *mathematical knowledge for teaching* (MKT; Ball, Thames, & Phelps, 2008) plays an important role in teachers' professional noticing of students' mathematical thinking, but empirical connections between the two constructs have been tenuous (Fisher, Thomas, Schack, Jong, & Tassel, 2018; Skultety, 2018). In this brief research report, we share preliminary findings that show a connection between teachers' *specialized content knowledge* (SCK; which is a subset of MKT) and teachers' professional noticing of students' mathematical thinking, a relationship that has been eluding studies.

Literature Review

Professional noticing of students' mathematical thinking is a particular noticing expertise that happens during instruction, when a teacher notices a student's mathematical idea (Jacobs et al., 2010). It consists of three component-skills: the teacher (a) *attends* to the details of the student's strategy, (b) *interprets* the student's mathematical understanding, and (c) *decides* how to respond to the student based on the student's mathematical understanding. Jacobs and her colleagues defined these three component-skills as happening simultaneously, as three interrelated and interdependent processes, that occur before the teacher responds.

Mathematical knowledge for teaching is defined to be the mathematical knowledge used in the profession of teaching. Ball, Thames, and Phelps (2008) defined six domains of mathematical knowledge for teaching, such as common content knowledge (CCK), specialized content

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knowledge (SCK), and knowledge of content and students (KCS). In this study, we focus on SCK, which is defined as “the mathematical knowledge and skill unique to teaching” (p. 400). In particular, this includes mathematical knowledge that is not typically needed in areas outside of teaching. For example, this includes knowing the different meanings of division and being able to deduce when a student’s non-standard method works.

Thomas, Jong, Fisher, and Schack (2017) highlighted many connections between MKT and professional noticing of students’ mathematical thinking. For example, an important component of enacting MKT is being able to *interpret* the mathematical thinking of each learner. Conversely, robust professional noticing expertise involves consistency with *the research on children’s mathematical development*. Citing numerous other examples of overlap between the two, they posited that effective professional noticing occurs at the intersection of well-developed MKT and a high level of responsiveness to the mathematical activities of students.

However, empirical connections remained elusive. Thomas et al. (2017) and Skultety (2018) looked for connections between teachers’ performance on a professional noticing assessment (e.g., Jacobs et al., 2010), and the Learning Mathematics for Teaching assessment (LMT; Hill, Schilling, and Ball, 2004). In both studies, the authors chose a content domain of the LMT closest to the content domain of the professional noticing assessment. Using a Spearman’s correlation test, Thomas et al. (2017) found no significant correlations between MKT and any component-skill of professional noticing in the pre-test, and only one in the post-test between MKT and attending. Skultety (2018) used a linear regression model to see if MKT might predict changes in professional noticing, and did not find evidence for this hypothesis. As Thomas et al. (2017) remarked, “Given the strong theoretical connections between MKT and professional noticing, such findings are… surprising” (p. 19).

In their conclusion, Thomas et al. (2017) questioned whether the mathematical content in the LMT assessment was too broad in comparison to the content domain of their professional noticing assessment. In this preliminary study, we followed this line of reasoning and wondered what would happen if we significantly narrowed our attention to teachers’ SCK surrounding a single type of problem, and teachers’ professional noticing expertise related to that problem. In particular, we ask the following research question: Are there correlations between teachers’ professional noticing of students’ mathematical thinking skills and teachers’ specialized content knowledge, when assessed in the domain of missing-value proportional reasoning tasks?

Methods

In this study, 15 practicing middle and high school mathematics teachers from the northeast region of the United States completed a survey that captured data about their professional noticing expertise and SCK (see LaRochelle, 2018, for the math tasks and solution strategies). Participants had on average 4.6 years of teaching experience, with a range of 0 to 8 years.

Data Collection: Survey

The survey consisted of three parts, but only the first two parts collected data about teachers’ professional noticing expertise and SCK, and so we only describe those two parts. In each of the first two parts, participants first solved a missing-value proportional reasoning task (Kaput & West, 1994). Participants were invited to solve the task in two ways, and all 15 participants provided two strategies for both tasks. The mathematical knowledge needed to solve a task in multiple ways is considered SCK because teachers are expected to know how to solve mathematical tasks in multiple ways, while those in other professions are not.

Next, participants considered three students' strategies solving that same task, and responded to the following professional noticing prompts (adapted from Jacobs et al., 2010):

1. Please describe in detail what you think each student did in response to the problem.
2. Describe the mathematical understandings of these three students.
3. Pretend you are the teacher of these students. What problems might you pose next, and why?

Each of these prompts related to a different component-skill of professional noticing; the first related to attending, the second related to interpreting, and the third related to deciding how to respond. Parts one and two were identical in structure. The difference between the two parts was in the mathematical task (in part 1 the task exhibited integer ratios, while in part 2 the task exhibited non-integer ratios) and the subsequent solution strategies.

Data Analysis

In order to investigate correlations between teachers' SCK and professional noticing expertise, we rated SCK on a three-point scale, and then rated each component-skill on a three-point scale. We then calculated three Spearman's rank correlations, and tested each for statistical significance: between SCK and attending, SCK and interpreting, and SCK and deciding. In the next two sub-sections we describe how we scored teachers' SCK and professional noticing skills.

Rating teachers' SCK. Each teacher provided two strategies for solving two missing-value tasks, for a total of four strategies. Carney, Hughes, Brendefur, Crawford, and Totorica (2015) claimed that an important goal with missing-value tasks is to develop flexibility using both the *scalar* multiplicative relationship (e.g., 24 caterpillars is four times as big as 6 caterpillars; notice the relationship is between caterpillars and caterpillars) and the *functional* multiplicative relationship (e.g., each caterpillar eats three leaves; notice the relationship is between caterpillars and leaves). The first author chose numbers that equally lent themselves to either multiplicative relationship, so that teachers might be primed to use both when solving the tasks (Carney, Smith, Hughes, Brendefur, & Crawford, 2016). Hence, we coded each strategy for whether the teacher utilized a scalar or functional multiplicative relationship. Cross-multiplication strategies did not count as using either relationship because many students use this strategy without understanding proportional reasoning (Cramer & Post, 1993). Teachers received a score of 2 if they used both multiplicative relationships, 1 if they only used one of the multiplicative relationships, and 0 if they used neither relationship (i.e. they used cross multiplication for all four strategies).

Rating teachers' professional noticing skills. For each response, we followed Jacobs et al. (2010) and coded for the *amount of evidence* exhibited in the response that teachers were consistent with the students' strategies and were consistent with the literature on students' proportional reasoning (e.g., Carney et al., 2015; Lobato & Ellis, 2010). In the attending prompts, this included completely describing important mathematical details of each strategy. For the interpreting prompts, this included differentiating among the students and identifying specific understandings related to multiplicative reasoning. For the deciding prompts, this included selecting problems tailored to each of the three students (sometimes teachers selected multiple problems; this was not a requirement though), and clearly specifying how the problems related to the students' understandings of multiplicative relationships. (For a complete description of these analyses, see LaRochelle, 2018.) For each component-skill, teachers received either a 0 (lack of evidence), 1 (limited evidence), or a 2 (robust evidence).

Results

Results can be seen in Table 1. We did not find significant correlations between attending and SCK, or between interpreting and SCK. However, we did find significant correlation between deciding how to respond and SCK. This meant that higher scores in SCK occurred with higher scores in deciding how to respond based on the students' mathematical understandings.

Table 1: Correlations Between Professional Noticing Component-Skills and SCK

N = 15	**Correlation**	**T-statistic**	**p-value**
Attending vs. SCK	0.2484	0.9244	0.1861
Interpreting vs. SCK	0.2875	1.0822	0.1494
Deciding vs. SCK	0.6989	3.5227	**0.0019***

*Significant at the $\alpha = 0.0166$ level, after adjusting using a Bonferroni correction.

Discussion

The correlation between deciding how to respond and SCK was about 0.7, which is considered to be a strong correlation. Our measure of SCK implicitly measured teachers' understandings of the various multiplicative relationships embedded in missing-value proportional reasoning tasks (Carney et al., 2015). The strong correlation indicated that there was a relationship between teachers' deciding skills and teachers' knowledge of multiple strategies for solving these proportional reasoning tasks. This may indicate that the ability to find (or knowledge of) multiple strategies to a particular task is related to the ability to decide how to respond to students based on their understandings, as exhibited in that task. Hence, this study implies that knowing multiple solution strategies is an important knowledge that teachers use when deciding how to respond to students based on students' mathematical understandings.

We did not find statistically significant correlations between attending and SCK nor interpreting and SCK. However, we wondered if a more rigorous study could shed light on these relationships. We recognize that our sample size was small, and this limited our potential for uncovering relationships. Additionally, we only invited teachers to share two solutions to each proportional reasoning task. It is also possible that attending and interpreting might relate to different domains of MKT. For example, other studies have found relationships between teachers' MKT and teachers' analyses of classroom situations (Kersting, Givvin, Thompson, Santagata, & Stigler, 2012; Kersting, Givvin, Sotelo, & Stigler, 2010). Although these researchers did not specifically investigate teacher noticing, their construct of "useable knowledge" and its related methodology is similar to studies of teacher noticing. Overall, more work must be done to understand the relationship between MKT and professional noticing.

With this study, we question the appropriateness of using MKT assessments such as the LMT for finding relationships between MKT and professional noticing expertise. These assessments collect data on teachers' knowledge with respect to multiple aspects of mathematics, teaching, and learning, and may be too broad for researchers to find direct relationships between MKT and professional noticing. We conjecture that assessing our teachers' MKT with respect to one problem type allowed us to uncover the correlation we found.

While searching for connections is important, we also believe it is important to recognize theoretical differences between the two constructs. For example, noticing is a teaching practice and MKT is a teaching knowledge. As a practice, researchers investigate teacher noticing by documenting what teachers notice; there are fewer inferences to be made because what the researcher documents is what the teacher noticed. However, the scope of MKT is much broader, and allows researchers to consider multiple teaching actions simultaneously. These theoretical differences afford researchers with different opportunities for studying teacher learning.

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REFUTING A FRACTION MISCONCEPTION: A BRIEF INTERVENTION PROMOTES TEACHERS' CONCEPTUAL CHANGE

Jessica Rodrigues
University of Southern California
jessica.rodrigues@usc.edu

Ian Thacker
University of Southern California
ithacker@usc.edu

Research shows that some teachers overgeneralize the whole number rule of "multiplication always makes bigger" to multiplication with fractions. The present study explores this misconception not only with fractions with unfamiliar denominators (e.g., thirty-fifths) but also fractions with familiar denominators (e.g., halves) and evaluates an intervention for remediating the misconception. In- and pre-service teachers (N = 100) completed a direction of effects task prior to and after random assignment to either read a control text or a refutation text that directly refuted the misconception. The misunderstanding was present among items with unfamiliar and familiar denominators, demonstrating the pervasiveness of the misconception. Overall, the intervention was found to be effective, showing promise of refutation texts for promoting teachers' mathematics conceptual change.

Keywords: Rational Numbers, Teacher Knowledge, Number Concepts and Operations, Elementary School Education

Studies of teachers' mathematics knowledge have shown that many teachers possess a limited conceptual and procedural understanding of fractions (e.g., Ma, 1999; Moseley, Okamoto, & Ishida, 2007), limiting their preparedness to support their students' fraction learning. Preservice teachers struggle to comprehend the most foundational conceptual understanding of fraction arithmetic, which is the *direction of effects* that an operation produces (e.g., whether multiplying by a given value makes a number smaller or larger; Siegler & Lortie-Forgues, 2015).

Research on direction of effects errors has concentrated on multiplication of fractions (Siegler & Lortie-Forgues, 2015). A common misconception that individuals hold is to overgeneralize the whole number rule that "multiplication always makes bigger" to multiplication with fractions, when in fact, multiplying a positive number by a fraction that is less than one does *not* make that positive number bigger. In a study conducted by Siegler and Lortie-Forgues (2015), 41 preservice teachers completed a direction of effects task for which they were asked to predict without calculating whether the answer to an inequality would be larger or smaller than the larger fraction in the multiplication problem. The fractions used in the task were selected to be too large and uncommon so that the items could not be solved easily with mental arithmetic (e.g., "True or False: 41/66 x 19/35 > 41/66"; Siegler & Lortie-Forgues, 2015). The preservice teachers answered correctly less often than chance on multiplication items designed intentionally to capture the misconception (i.e., multiplication items with fractions between zero and one).

The present study adds to the literature in two important ways by (a) examining the presence of the direction of effects misconception for items involving unfamiliar denominators and items involving familiar denominators and (b) evaluating a brief intervention for remediating the misconception.

Theoretical Framework

Individuals who transfer the whole number rule that "multiplication makes bigger" to all multiplication problems with fractions may need to undergo conceptual change to overcome this misconception. Conceptual change is a process in which individuals restructure conceptual knowledge and shift misconceptions prompted by discrepant information (Carey, 2009). We draw from the Cognitive Reconstruction of Knowledge Model (CRKM) model of conceptual change that discerns between characteristics of the learner (i.e., their prior conceptions and motivation) and of the message (i.e., whether the learning material is comprehensible, coherent, compelling, and plausible; Dole & Sinatra, 1998). This model is particularly useful for our research because it explains why individuals might shift their misconceptions about fraction multiplication based on novel information presented in a refutation text.

Refutation texts have been shown to be an effective tool for promoting conceptual change in science education (e.g., Broughton, Sinatra, & Nussbaum, 2012; Tippet, 2010). These texts are designed to state a misconception, refute those ideas, and then present the accurate explanations as plausible and fruitful alternatives (Hynd, 2001). Whereas positive effects of refutation texts for the remediation of misconceptions have been demonstrated in science education for years (e.g., Diakidoy, Mouskounti, & Ioannides, 2011), the texts have not yet been explored extensively for addressing mathematics misconceptions. A noted exception is research conducted by Lem and colleagues (Lem, Onghena, Vershaffel, & Van Dooren, 2017) that explored the use of refutation texts for correcting students' misconceptions about box-and-whisker plots.

Present Study

To our knowledge, there are currently no studies investigating the presence of the "multiplication always makes bigger" misconception when fractions being multiplied have common denominators (e.g., halves, fourths) and no work investigating an instructional intervention for remediating this specific misconception. In response, the present study was designed with the following two objectives: (a) to assess if pre- and in-service teachers correctly answer less often than chance on easy, medium and hard direction of effects problems involving multiplication of fractions between zero and one and (b) to evaluate the effectiveness of a refutation text that targets the misconception for promoting conceptual change.

Method

Participants

Participants were pre-service and in-service elementary school teachers (N = 100), 38% self-reported as male, 42% White, 45% Asian American/Pacific Islander, 1% Hispanic, 3% Black/African American, 8% Mixed and 5% Other. The mean age was reported as 21.1 years (SD = 3.7). All participants were recruited via a Qualtrics survey panel that exclusively targeted in- and pre-service elementary teachers.

Measures

Texts. Two texts were created, a refutation text and a control text. The refutation text consisted of 364 words, had an 11th grade Flesch-Kincaid reading level, and stated and directly refuted the "multiplication always makes bigger" misconception. For example, an excerpt reads: "When multiplying numbers, you may think that the product will always be greater than the original number. This is incorrect! It is only when you multiply a positive number by a number greater than 1 that the product will be greater than the original number." This direct refutation was followed by a description of when multiplication makes bigger, and does not. The control text was a 100-word expository text paired with an illustration that was adapted from Hake (2007) that describes a strategy for multiplying fractions: "We often translate the word *of* into a multiplication symbol. We find 1/2 of 1/2 by multiplying: 1/2 of 1/2 becomes $1/2 \times 1/2 = 1/4$."

Direction of effects task. The direction of effects task assessed participants' understanding of the direction of effects of arithmetic, which was used to measure participants' conceptual understanding of fraction addition and multiplication (Siegler & Lortie-Forgues, 2015). Participants were asked to evaluate the accuracy of addition inequalities in the form of $a/b + c/d > c/d$ and multiplication inequalities in the form of $a/b \times c/d > c/d$. Participants answered four addition inequalities and four multiplication inequalities within three categories of varying difficulty, resulting in 24 total items. The "easy" items all included the fraction 1/2 paired with a fraction with a familiar denominator such as fourths (e.g., $3/4 \times 1/2 > 1/2$); "medium" items included familiar denominators but did not present any unit fraction such as 1/2 (e.g., $9/10 \times 2/5 > 2/5$); and "hard" items included the items used in a previous study (Siegler & Lortie-Forgues, 2015) with unfamiliar denominators (e.g., $19/35 \times 41/66 > 41/66$). For each inequality, participants were instructed to decide, without calculating, whether the answer would be greater than the answer indicated in the inequality.

Procedure

All materials were presented via an online Qualtrics survey. After agreeing to participate, teachers completed the pretest direction of effects task. Then, teachers were randomly assigned to either the refutation text or the control text group. After reading the text, teachers were asked to complete the direction of effects task again which served as a posttest assessment. The total survey required an average of 18 minutes ($SD = 19$).

Results

We conducted a simple one sided test of proportions to determine if teachers correctly answered multiplication problems with fractions between zero and one less than half of the time. Analysis of teachers' pretest performance on the easy and medium items with familiar denominators and hard items with unfamiliar denominators revealed that teachers chose the correct item about 49% of the time, which was not significantly different than chance (50%; chi-squared = .375, $p = .27$). When considering all 24 pretest items, 31% of teachers were correct on 100% of the 18 addition and multiplication items for which the greater-than inequality was true and 0% on the 6 multiplication items for which the greater than inequality statement was false. In other words, 31% of teachers at pretest showed the true misconception that "multiplication always makes bigger."

Of the 57 participants in the refutation text group, 9 participants continued to show the true misconception (16%) at posttest. We then looked more closely at participants' performance on the six multiplication items that were designed to capture the misconception. We focused attention on the refutation group participants who answered at least one of the misconception items incorrectly at posttest ($n = 28$). Of this subgroup, participants correctly answered at a rate similar to chance for five of the six items ($p > .05$) and less often than chance on one of the six items with unfamiliar denominators: $19/35 \times 41/66 > 41/66$ (71.4%; $p = .01$).

To assess the effectiveness of the refutation text intervention, we ran a repeated measures ANOVA with Arithmetic Operation (addition or multiplication) as a within-subject factor and text condition (refutation text or control) as between-subject factor and number of correct judgments as the dependent variable yielded significant main-effects of Arithmetic Operation ($F(1,98) = 64$, $p < .001$, *partial-eta squared* = .395) and main effects of condition that are

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approaching significance ($F(1,98) = 2.24$, $p = 0.14$, *partial-eta squared* = 0.022). These were qualified by a significant Operation x Condition interaction ($F(1,98) = 4.462$, $p < .001$, *partial-eta-squared* = 0.043). As expected, post hoc comparisons with Bonferroni correction showed no significant differences between ref-text and control groups in addition knowledge (89% vs 89%, $t(42) = 0.03$, $p = .98$, *Cohen's d* = 0.00). However, we found significant differences between refutation text and control groups in multiplication knowledge (74% vs 63%, $t(42) = 2.45$, $p = .02$, Hedge's $g = 0.56$).

Discussion

We assessed the effectiveness of a refutation text intervention for the remediation of a misconception regarding fraction arithmetic. Though there is some research on the use of refutation texts to shift students' misconceptions about mathematical topics (Lem et al., 2017), our study investigates the use of a refutation text to shift teachers' misconceptions about fraction multiplication. We also explore whether the misconception is specific to multiplication of unfamiliar fractions or common fractions more generally. Overall, we found that teachers answered easy, medium and hard fraction multiplication problems correctly at about the same rate as guessing, suggesting that the misconception is even more widespread and problematic than demonstrated in prior research (Siegler & Lortie-Forgues, 2015). Furthermore, we found that teachers who read a refutation text had fewer misconceptions about fraction multiplication compared with those who read a control text. That is, refutation texts have the potential to substantially impact teachers' mathematics misconceptions. This finding has important implications for research on conceptual change in mathematics and refutation texts.

Teachers Hold the Multiplication Misconception

Our results showed that a sample of pre- and in-service teachers answered fraction multiplication problems with fractions between zero and one at about the same rate as by chance. This finding is consistent with prior research showing that pre-service teachers were not successful in answering items regarding directions of effects in multiplication at a rate better than chance (Siegler & Lortie-Forgues, 2015). This replication of the finding suggests that many pre- and in-service teachers are in need of remediating misconceptions that may be limiting their readiness to advance their students' fraction understanding.

Refutation Texts Support Mathematics Conceptual Change

Teachers who were randomly assigned to read a refutation text that directly refuted the "multiplication always makes bigger" misconception had fewer misunderstandings at posttest than those assigned to read a control text, with modest effect sizes (Hedge's $g = 0.56$). Refutation texts targeting specific misconceptions thus hold potential as a relatively quick method for promoting teachers' conceptual change.

Future Directions

Although the present study shows promise of a refutation text serving as an instructional approach for targeting a specific fraction misconception, replication studies are required before widespread recommendation of refutation texts for supporting mathematics conceptual change. In light of the limited studies thus far exploring refutation texts for mathematics learning, there are multiple research curiosities that future work can explore, such as refutation texts for remediating not only other fraction misconceptions but also misconceptions in other mathematics content areas. Researchers should also explore the effectiveness of refutation interventions among not only teachers but also students, as such texts may hold promise for professional development opportunities and classroom instruction alike.

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IDENTIFYING ACTIVATED KNOWLEDGE FOR TEACHING

James Smith
University of Arizona
Jsmith62@email.arizona.edu

Yalcin Udun
University of Arizona
yudun@email.arizona.edu

Beay Vezino
University of Arizona
beauvezino@email.arizona.edu

Nicole Kersting
University of Arizona
nickik@email.arizona.edu

In this brief report, we describe preliminary results for a study in which we identify and analyze the types of knowledge activated by 101 elementary math teachers in response to a question from about short video clips of mathematics instruction (classroom video assessment; Kersting, Givvin, Sotelo, & Stigler, 2010; Kersting, Givvin, Thompson, Santagata, & Stigler, 2012). Conceptualizing knowledge representations as (re)structured into domain- and context-specific usable knowledge (Kersting et al., 2012), we operationalize the grades 3-5 Common Core State Standards (CCSS; Council of Chief State School Officers & National Governors Association Center for Best Practices, 2010) to capture the mathematical knowledges activated in teachers' written responses. Preliminary results suggest the CCSS might provide a valuable classifier for analyzing teachers' mathematical knowledge bases and offer insights into patterns of knowledge activation.

Keywords: Mathematical Knowledge for Teaching, Teacher Knowledge, Cognition

Since Shulman's (1986, 1987) conception of pedagogical content knowledge (PCK) as knowledge that integrates content and pedagogy, PCK has become the central knowledge domain for teaching. Ball and colleagues (Ball, Thames, & Phelps, 2008) elaborated on Shulman's notion of PCK in the area of mathematics, identifying the Mathematics Knowledge for Teaching (MKT) construct consisting of 6 subdomains. Little progress has been made, however, in capturing the knowledge as it is activated in classroom teaching. One possible reason might be that in order to capture the knowledge teachers activate and use in real classroom situations, teachers' need to engage in assessment tasks that approximate the perceptual-interpretative processes teachers engage in in the process of teaching (Blömeke, Gustafsson, & Shavelson, 2015).

To study teachers' usable knowledge, we developed an assessment approach that uses video clips of authentic mathematics instruction that teachers view along with various prompts to which they respond. Under the CVA-M approach we hypothesize that the teaching episodes shown in the video clips (i.e., the visual auditory input) are sufficiently proximal to an actual classroom situation, tapping into teachers' usable knowledge. We further proposed that for knowledge to become usable it must be restructured to form an interconnected, domain- and situation-specific knowledge teachers can access in a classroom situation (Kersting et al., 2012).

Here we present preliminary results from an attempt to further understand usable knowledge as it is activated. We used the CCSS grades 3-5 math standards as an ontology (c.f. Giaretta & Guarino, 1995) to classify the different knowledge categories activated in a teachers' responses. This approach affords the possibility to capture the variability of knowledge activation across teachers and teaching situations. In this study we investigate three questions:

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1. How applicable are the CCSS as an ontology for capturing mathematical knowledge activation?
2. Do teachers' responses reflect the activation of multiple knowledges?
3. Are the most frequently activated knowledges those associated with students struggle in the clip?

Methods

Data were collected as part of an on-going study using the classroom video assessment (CVA; (Kersting et al., 2010, 2012) administered online to a sample of grades 4 and 5 teachers from one of the largest districts in the US (N = 101). The teachers' classroom experience ranged from 1 (n = 4) to 33 (n = 1) years with a mean of 14 and median of 15. About half of the teachers held a master's (n = 50), the other half a bachelor's (n = 47) degree; 2 teachers reported earning a Ph.D. and 2 teachers reported earning an education specialist degree.

Video Clip

Each teacher watched to the same 3-minute video clip in which the teacher poses the following word problem to the whole class: *If 132 cupcakes are distributed evenly among 6 containers, how many cupcakes would end up in each container?* The clip begins with the students noticeably confused by the problem, so the teacher asks for a student to offer a hint without giving away the answer. A student responds by suggesting dividing 132 by 6. The teacher asks the class if the hint is helpful, but when students confirm they are still unsure, the teacher takes a different approach by asking students what $\frac{1}{6}$ of 132 means and instructing them to talk about this question at their tables.

After a moment for group discussions, the teacher asks one student (who she noticed had arrived at 22 in her group) to explain how she arrived at the answer at the overhead projector. The student fills each box of the tape diagram displayed on the projector with 22. The teacher then follows up with asking her what $\frac{1}{6}$ of 132 means. The student responds that $\frac{1}{6}$ is 22 cupcakes because there are 6 boxes, so $\frac{1}{6}$ would be 22. The clip finishes with the teacher asking the class for any questions they have for the student.

After watching the clip, teachers were prompted: "If you were the teacher in this situation, what mathematical question might you pose to the class, and how would your question help improve the students' mathematical understanding?"

Data Analysis

First, we coded all teacher responses (N = 101) for response type. (a) Responses that included a mathematically specific question that directly addressed the students' struggle were coded as TQP (n = 40); (b) responses that included mathematically specific questions not primary to the students' struggle were coded as TQS (n = 28). Responses that did not include a mathematically specific question or that repeated questions already asked by the teacher in the clip were coded as PS (n = 29); responses containing incorrect math were coded as IM (n = 4). Second, we identified in each response the underlying activated knowledge using the grades 3-5 CCSS-M standards (see Table 1). To capture non-mathematically specific responses, we created three categories: Representational strategies (e.g., Can you draw a picture or write a number sentence that shows us or helps us answer?*)*, simplification (decreasing cognitive demand of the task), or other (e.g. commenting on the teacher in the clip).

Responses were coded exhaustively, but categories were not mutually exclusive. For instance, a statement could be coded as both a representation strategy and a mathematical knowledge if a statement contained both category types. Based on pairwise comparisons, interrater reliability measured by direct agreement ranged from 85% to 100% across all dimensions with an average of 93%.

Results and Discussion

CCSS as an Ontology

We found that the CCSS were useful to capture the mathematical knowledge activated by teachers in response to the video clip. The standards provided enough sensitivity for the different mathematical knowledges reflected by teachers' responses as well as sufficient coverage (i.e. all mathematical responses could be categorized). Teacher responses contained other usable knowledge, such as general mathematical and pedagogical strategies, which indicates the need for additional frameworks.

Analysis of Knowledge Activation

Across response types, most teachers demonstrated more than one knowledge activated, with an average of two per response (TQP: M = 2.23; TQS: M = 2.04). Interestingly, even responses coded as IM had a similar average of knowledge activation. Responses revealing activation of non-mathematical knowledge tended to reflect mostly a single knowledge base (M = 1.69). The results provide preliminary evidence that carrying out common teaching tasks, such as generating teacher questions, require the integration of different knowledges.

Which Knowledge is Activated?

Further investigation of TQP-coded responses illustrates the most frequent mathematical categories were those more relevant to the students' struggles (Table 1). Predictably, responses mostly reflected basic fractions understanding, fractions-as-division, and the partitioning relationship of a quantity and a fraction. We were surprised, however, by the variety of other knowledges referenced in support. Each teacher watched the same video clip and answered the same prompt, but not all teachers demonstrated the same knowledge activation. The results provide evidence on the complexity and idiosyncrasies of teachers' usable knowledge.

Table 1: Knowledge Activation by Response Type

Knowledge Type (*CCSS code*)	Frequency				Knowledge Type (*CCSS code*)	Frequency			
	TQP	TQS	PS	IM		TQP	TQS	PS	IM
Equivalent fractions (*4.NF.A.1*)	3	0	0	0	Representation *(NA)*	7	12	10	2
Addition of fractions (*4.NF.B.3.A*)	4	0	0	0	Simplification *(NA)*	3	1	5	2
Decomposing fractions (*4.NF.B.4.B*)	1	2	0	0	Other *(NA)*	14	11	18	3
Multiplication of fractions (*4.NF.B.4.C*)	4	1	1	0	Basics of fractions (*3.NF.A.1*)	19	9	0	0
Fractions-as-division (*5.NF.B.3*)	13	2	2	1	Long division (*4.NBT.B.6*)	4	8	4	1
$\frac{a}{b} \times q = q \times a \div b$ (*5.NF.B.4.A*)	16	5	5	1	Multiples/Factors (*4.OA.B.4*)	1	0	1	1

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Interestingly, the most frequently activated knowledges across response types were non-mathematical categories: representation strategies and other. While the pedagogical other category might be artificially inflated by being a broad category for non-mathematical statements, the representation category is a defined category. The relatively high frequency of these types of statements suggests the most frequently activated knowledges for teachers in our study for this prompt are non-mathematical knowledges; surprising given the explicit focus on the mathematics in the prompt. The results might provide some evidence for what has been identified in the literature as one of the challenges for improving teaching: Teachers' ability to generate targeted questions that can help students improve or further their mathematical understanding. While our findings could be a consequence of our measure (analyzing video clips as opposed to being in a real classroom), we hypothesize that they are indicative of the knowledge that gets activated in their classroom. Teachers whose go-to response to the video clip reflects a non-mathematical strategy, might rely on that same knowledge also in their classroom. In other words, some teachers' usable knowledge might be heavily influenced by or weighted towards general strategies.

Implications and Conclusion

In this brief report, we identified and analyzed the different underlying knowledges evident in teachers' response to the CVA task, using CCSS as a classifier. For the given clip, most teacher responses demonstrated the activation of more than one knowledge category with the most frequently activated knowledges being knowledges of pedagogical strategies. However, for responses that contained a mathematical question, most responses leveraged relevant mathematical knowledge and a variety of different knowledges activated in support of the most relevant knowledge for the clip.

This work is still in an exploratory phase. In future work, we plan to incorporate this type of feature scaling to create a knowledge base for automated scoring of CVA responses. We hypothesize by better accounting for the complexity of knowledge activated in teachers' responses, we improve measurement of usable knowledge. Additionally, our approach affords mapping knowledge activation to further support more complex computational modeling of usable knowledge involved in in-the-moment decision making, which can then lead to more focused professional development to overcome knowledge "blind spots".

Acknowledgments

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TOWARDS A CONCEPTUALIZATION OF THE DENSITY PROPERTY OF DECIMAL NUMBERS: A STUDY WITH TEACHERS IN TRAINING

Mayra Suárez-Rodríguez Olimpia Figueras
Centro de Investigación y de Estudios Avanzados del IPN
mayra.suarez@cinvestav.mx figuerao@cinvestav.mx

Understanding the density property of rational numbers involves a process of conceptual change from 'the discrete' to 'the dense'. With mathematics teachers in training in Mexico City, a didactic sequence on this property in the decimal number system was tried out. Evidence that this change can be promoted is shown in this document. Some middle-school teachers in training were able to visualize a six-digit expansion to prove that there are more decimals between two numbers. However, others continued with the idea that a decimal number has a successor.

Keywords: Concepts of numbers and operations, Rational numbers, Teacher knowledge, Mathematics knowledge for teaching

Difficulties to understand decimal numbers have been of interest to researchers worldwide. In Indonesia, Widjaja, Stacey, and Steinle (2008) conducted a study with teachers in training and found evidence that they only consider numbers with a limited number of decimals. They argue that such situation reflects contents contained in the textbooks. Ávila and García (2008) found that elementary-school teachers in Mexico often claim that there are no more numbers between two decimal numbers due to the belief that decimals have at most two decimal places. This way of thinking affects the understanding of the density property of decimal numbers. Tirosh, Fischbein, Graeber and Wilson (1999) in a research with future teachers in the United States, found difficulties of a high percentage of the participants because they think that there is a finite number of intermediate numbers between two rational numbers and that a rational number has a successor. Vamvakoussi and Vosniadou (2004), in studies with Greek students of 15 years of age, typified those ideas as an image of 'false consecutives'; that is, between 1/4 and 1/5 or between 0.05 and 0.06 there are no other numbers. Their evidence showed the students' thinking is closely linked to 'the discrete', a property of natural numbers. Valencia and Ávila (2015) and Ni and Zhou (2005) have also emphasized this issue. Vosniadou and Vamvakoussi (2004, 2010, and 2012) hold that a process of conceptual change is required: a re-conceptualization of the properties of natural numbers those of rational numbers.

Aims of the Research

The main purpose of the study carried out is to promote, using a didactic sequence, meta-conceptual awareness among teachers in training to promote a conceptual change regarding the density property of decimal numbers (see Suárez-Rodríguez, 2017). Meta-conceptual awareness is a recognition process in which a person tests his beliefs and presuppositions (Vosniadou, 1994). To describe explanatory frameworks —different ways to express students' interpretations (Vamvakoussi, Vosniadou & Van Dooren, 2013)— of future mathematics teachers about the density property of decimal numbers, the following research aims were proposed: 1. To describe conceptions on the density property of decimal numbers, and 2. To identify performances while solving activities that indicate elements associated with a conceptual change.

Theoretical Foundation

Researchers such as Vamvakoussi and Vosniadou have used the cognitive-development approach of conceptual change proposed by Carey (1987) to study mathematics learning. The human being begins the process of knowledge acquisition through the development of a 'naive theory' (Vosniadou, 2013). The term theory is used to refer to a relational and explanatory structure based on daily experience and information from culture and surrounding situations regarding events or concepts, particularly before entering school (Vosniadou, 2002). Learning science requires a fundamental restructuring of the naive theory that can be named as a theoretical change. Specifically, Vosniadou (2013) defines conceptual change as the result of a complex cognitive and social process of the naive theory.

From the Conception of the Discrete to the Density Property

Based on the results of their study with 15-year-old students, Vamvakoussi and Vosniadou (2004) characterized the thinking of a subject regarding finite or infinite quantities of numbers in an interval (see Table 1). The results obtained support the hypothesis that understanding density is a slow and gradual process limited by the presupposition of 'the discrete.'

Table 1: Thinking Characterization Regarding the Quantity of Numbers in an Interval

Naive thinking about 'the discrete'	It is thought there is no other number between two consecutive false rational numbers; that is, it is thought there is a successor of a rational number.
Advanced thinking about 'the discrete'	It is thought there is a finite quantity of numbers between two consecutive false rational numbers.
Mixed thinking between discrete and dense	In some cases, it is thought that between two rational numbers there is an infinite quantity of numbers; and in other cases, it is thought there is a finite quantity of intermediate numbers.
Naive thinking about 'the dense'	It is understood that there is an infinite quantity of numbers in an interval, but the fact is not justified using the density property. The symbolic representation of the extremes of an interval influences the way of thinking; it is believed there can only be an infinite quantity of decimal numbers between decimals and an infinity of fractions between fractions, but not an infinity of fractions between decimals or otherwise.
Advanced thinking on the dense	There is a sophisticated understanding of the density property; that is, it is evidently understood that there is an infinite quantity of numbers between two rational numbers, regardless of their symbolic representation and this is justified through the use of the density property.

Methodology

The experimentation of a didactic sequence was carried out with 10 students from an institution in Mexico City that trains secondary school teachers. The sequence was structured in 2 stages: one stage to identify initial explanatory frameworks of participants through a paper-and-pencil test and individual interviews. The other stage included collective activities mediated with instruction. This part of the sequence was carried out in 4 sessions. The sessions are about: 1. The perception of the dense in concrete materials, 2. Addition and subtraction, 3. Location in intervals, and 4. Comparison.

Results

This report includes evidence of the trainee teachers' performance in 3 different activities, which are elements of the conceptual change from the discrete to the dense.

Performance Linked to Addition and Subtraction with Decimal Numbers

For the second session of the teaching sequence, 2 activities inspired by those created by Broitman, Itzcovich and Quaranta (2003) were designed in order to identify skills for writing numbers with decimal expansions. In one of the activities, one of the participants chooses a number between 0 and 5 and another student chooses one between 5 and 10 and write them down in a table. The first one adds a positive number to the chosen number in such a way that its sum is strictly less than the number written by the other participant. The latter subtracts a positive number from the selected number so that the result is strictly grater than that written by the other member, and so on.

In Karen and Karina's performances, an underlying notion of a 'infinite' process through adding digits was observed; it was also seen they introduce zeros in the middle of the decimal part of a number (see Figure 1). In this activity, it was also observed that Karen, who in the test showed 'discrete thinking' and believed that a decimal number only had tenths, has extended the number of decimal places. This fact evidences the beginning of a process of conceptual change, since Karen is expanding her knowledge system by adding digits, especially zeros in the subtrahends (see right side in the box, Figure 1). Likewise, both managed to visualize the existence of several decimals between 5.5 and 5.8 (see the numbers written outside the box, Figure 1). The other participants had a similar performance to that of Karen and Karina.

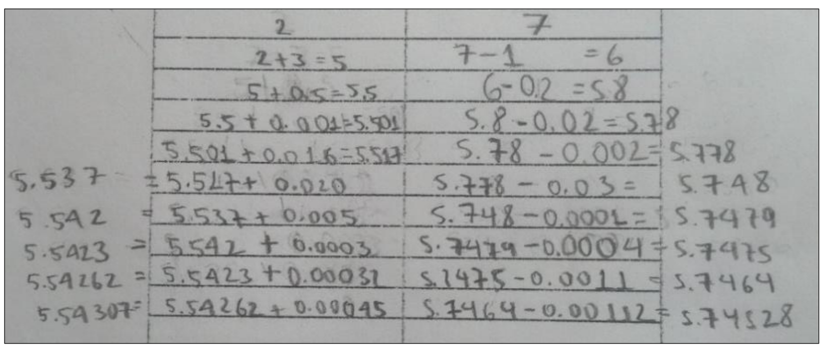

Figure 1: Karen and Karina's Records on the Addition and Subtraction Activity

Performances Linked to the Location of Decimal Numbers in Intervals

The third session of the sequence contains activities related to the location of a number in an interval, which were inspired by an item elaborated by Brousseau (1981). In one of these activities, the student has to find the interval in which the number 'hidden' by a classmate is located; the extremes must be numbers with consecutive decimal digits. Figure 3 has part of a couple' dialogue. Fabiola wrote down the number *13.415* in a piece of paper and her classmate Amanda tries to find the interval where the number hidden by Fabiola is located. At the beginning, Amanda had recorded several intervals before finding one whose extremes were natural consecutive numbers. Then she starts to consider the extremes with tenths and finally with hundredths. For Amanda, this is the beginning of a process of conceptual change since she exhibited a naive thinking about the discrete in her answers to the questions in the test.

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Amanda: -13.4 and 13.5?
Fabiola: -Yes, but now with hundredths.
(Fabiola means that the extremes of the Interval must have two decimals).
Amanda: -13.42 and 13.48?
Fabiola: -No.
Amanda: -Is it between 13.45 and 13.47?
Fabiola: -No … [silence], this interval is in the other one you mentioned.
Amanda: -Ahhh … the number is between13.42 and 13.43?
Fabiola: -Mmm … yes, ah, no!
Amanda: -I'm confused again. I meant, is it between13.41 and 13.42?
Fabiola: -Yes! It is the Interval.
Amanda: -Which one is it?
Fabiola: -13.415.

Figure 2: Interaction Between Amanda and Fabiola in the Activity of the Hidden Number

Performances Identified in the Activities on Comparison of Decimal Numbers

Comparison activities were based on an activity proposed by a future teacher who took part in the research done by Castillo (2015) in Mexico. The aim of the task was to understand the density property of decimal numbers through comparison. In one of the activities of the fourth session, each couple took a piece of paper with an interval written on it. The teacher-researcher took out from an envelope cards that had numbers with up to six decimals written on them. Those who had the interval in which the number drawn was located, would ask for the card. At the end, the cards were pasted on the board (see Figure 3). With the intention that teachers in training acquire meta-conceptual awareness that the density property helps to visualize the non-existence of consecutive decimal numbers, it was shown that between pairs of false consecutive there is at least one decimal number, consequently, an infinity. For example, in the third row (listed with 3 on the right in Figure 3), 21.001 and 21.002 are false consecutives. It was shown that between this pair, one could find at least four decimal numbers greater than the first one and lower than the second, namely 21.0011, 21.0017, 21.00175, and 21.0018.

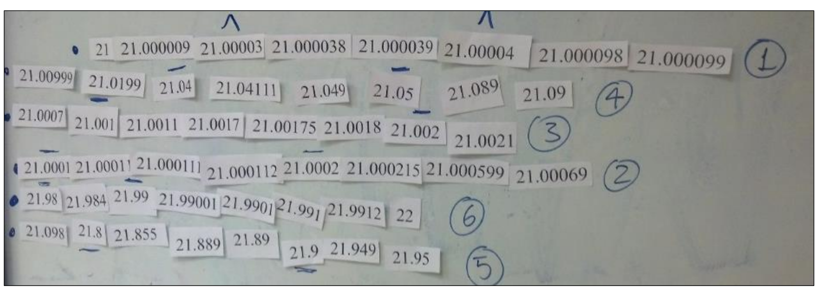

Figure 3: Record of Six Sequences in the Activity of Locating Numbers at Intervals

Conclusions

The didactic sequence promotes performances that can be linked to components of an initial process of conceptual change regarding the explanatory frameworks that teachers in training have about the density property of decimal numbers. The 10 participants showed naive or advanced thinking about the discrete at the beginning of the study. During the application and development of the didactic sequence, most of the future teachers extended the decimal expansion: they wrote numbers of up to six decimal digits and, in some cases, they included interleaved zeros in the decimal part of a number. This strategy helped the participants to visualize that numbers in an interval can always be found. The didactic sequence constitutes an innovative teaching model that can be interesting for mathematics teachers. It can be also used to initiate the study of density in basic education.

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HACIA UNA CONCEPTUALIZACIÓN DE LA PROPIEDAD DE DENSIDAD DE LOS NÚMEROS DECIMALES: UN ESTUDIO CON PROFESORES EN FORMACIÓN

La comprensión de la propiedad de densidad de los números racionales conlleva un proceso de cambio conceptual de 'lo discreto' a 'lo denso'. Con profesores de matemáticas en formación, de la Ciudad de México, se puso en marcha una secuencia didáctica acerca de esa propiedad en el sistema de los números decimales. En este documento se muestran evidencias de que se puede promover ese cambio. Algunos estudiantes para docentes de secundaria lograron visualizar una expansión de hasta seis cifras decimales para mostrar que hay más decimales entre dos números; empero, otros continuaron con la idea de que un número decimal tiene un sucesor.

TEACHERS' UNDERSTANDING OF THE CONNECTION BETWEEN CONGRUENCE AND TRANSFORMATION IN CONGRUENCE PROOFS

Julia St. Goar	Yvonne Lai	Rachel Funk
Merrimack College	University of Nebraska-Lincoln	University of Nebraska-Lincoln
stgoarj@merrimack.edu	yvonnexlai@unl.edu	rachel.funk@unl.edu

We are called by recent standards to promote teachers' learning of a transformation approach in geometry and its proofs. Teachers who learned geometry from a perspective based on Euclid's Elements are now called to teach geometry using different proofs and axioms from what they learned. Moreover, there is little literature on teachers' learning of geometry from a transformation perspective. To begin to address this problem, we analyze the teachers' use of the conceptual link between congruence and transformation in the context of constructing proofs. We identify possible key developmental understandings involved in using the definition of congruence and in the construction of transformation proofs, pointing to concepts that may need to be specifically addressed in teacher education.

Keywords: Teacher Knowledge, Geometry and Geometrical and Spatial Thinking, Reasoning and Proof

Teacher preparation programs face a transition in geometry instruction. In the past several decades, geometry has been taught primarily from a perspective based on Euclid's *Elements* (Sinclair, 2008); in recent years, *geometry from a transformation perspective* has come to the fore (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010; NCTM, 2018).

These changes in geometry standards have implications both mathematically and pedagogically. For instance, consider the well-known triangle congruence criterion "Angle-Side-Angle (ASA)": *If* ΔABC *and* ΔDEF *are triangles such that* $\overline{AB} \cong \overline{ED}$, $\measuredangle BAC \cong \measuredangle EDF$, *and* $\measuredangle ABC \cong \measuredangle DEF$, *then* $\Delta ABC \cong \Delta DEF$. In secondary and college geometry texts using an *Elements* approach, this criterion is often taken as a postulate: it is intended to be accepted as mathematical truth without proof (e.g., Education Development Center, 2009; Musser, Trimpe, & Maurer, 2008; Serra, 2008; Boyd, Cummins, Mallow, Carter, & Flores, 2005). These and other texts help teachers establish conviction in ASA through empirical exploration – a scheme for conviction, that taken by itself, can be unproductive when the objective is to construct a deductive proof (Harel & Sowder, 2007). In contrast, from a transformation approach, if a teacher is to show that two triangles ΔABC and ΔDEF in a plane are congruent, they must show that no matter the triangles' locations, there exists a sequence of translations, rotations, and/or rotations that map ΔABC to ΔDEF. (See Wu (2013) for a schematic for such a proof.) In the transformation approach, even if empirical exploration is beneficial, the goal is to establish a deductive proof of ASA. In the *Elements* approach, a proof would be mathematically impossible.

Consequently, teachers, who may have learned geometry from one perspective but are called on to teach geometry from another perspective, may not be familiar with what can be proven, what cannot be proven, or how particular proofs operate. It is critical for teachers to understand not only the abstract notion that different axiom systems result in different proof approaches (Van Hiele-Geldof, 1957), but also that they may be teaching students from an axiomatic system different from the one they learned first. Our study addresses this problem from the perspective of developing knowledge for teacher educators, including understanding how teachers learn. We investigated: *What concepts are entailed in teachers' construction of congruence proofs?*

We focused this study on constructing congruence proofs, as.this topic is fundamental to the study of geometry, and it is one where differences between *Elements* and transformation approaches are salient.

Conceptual Perspective

Following Usiskin and Coxford (1972), we take a *transformation approach* to geometry as one that features:

- Reflections, rotations, and translations which are assumed without proof to preserve geometric properties such as length and angles; and
- Definition of congruence in terms of transformations: two subsets X and Y of the plane (e.g., two triangles) are said to be congruent if there exists a reflection, rotation, or translation, or sequence of these transformations, that maps X and Y.

The details of these features may differ across texts. For instance, different statements of postulates of transformations may be taken. However, they have in common that the postulates are about transformations, rather than congruence criteria for particular objects such as triangles.

Hence, from a transformation perspective:

- Two objects in the plane, such as two triangles, are congruent if and only if there exists a single one of or a sequence of reflections, rotations, or translations that maps one object to the other.

We take an *Elements approach* to be one that features the postulation of at least one triangle congruence criterion (e.g., SSS, ASA, or SAS), and definition of congruence in terms of individual geometric objects (e.g., congruence for triangles is defined separately from congruence of circles).

As Jones and Tzekaki (2016) reviewed, there is "limited research explicitly on the topics of congruency and similarity, and little on transformation geometry" (p. 139). To our knowledge, there have been few studies on teachers' conceptions of congruence *proofs* from a transformation perspective. One exception is Hegg, Papadopoulos, Katz, and Fukawa-Connelly (2018), who examined how teachers managed their prior knowledge of congruence criteria when showing the congruence of two triangles. They found that teachers preferred to use triangle congruence criteria rather than transformations, but could, when asked, successfully complete proofs using transformations. However, their study did not examine the case of proving congruence of figures that are not triangles.

Data and Method

Data

A post-hoc analysis was conducted of 20 prospective secondary teachers' responses to two congruence proof tasks, the Line Point Task and the Boomerang Task (below). The tasks were distributed as part of an in-class midterm examination in a mathematics course taught by one of the authors in Fall 2017.

Tasks Used

The tasks for this study were:

- **Line Point Task.** Let ℓ, m be lines. Among all the points that are a unit distance from ℓ, choose one point P. Among all the points that are a unit distance from m, choose one point Q. Prove that no matter what points P and Q you chose, it is always true that $\ell \cup P \cong m \cup Q$.
- **Boomerang Task.** Let ΔABC and ΔDEF with congruences marked as shown. Let O be a point on the inside of ΔABC and P be a point on the inside of ΔDEF so that the angle measures $\alpha = \gamma$ and $\beta = \delta$ as shown. Given the all the above, prove that $\Delta AOB \cup \Delta ABC \cong \Delta DPE \cup \Delta DEF$ (Figure 7).

These tasks allowed us to examine teachers' capabilities for writing congruence proofs beyond standard triangle congruence proofs. The tasks used were two in a sequence of tasks intended for developing teachers' understanding of using definition of congruence from a transformation perspective to prove or disprove the congruence of given figures. This sequence was designed using variation theory (Lo, 2012) to support not only finding sequences of transformations between familiar objects, but showing that a sequence could simultaneously map the objects in a *union* of these objects to another *union.* Furthermore, these tasks included working with lines and points—objects which, though familiar—are not often discussed in the context of congruence proofs.

Analysis

The analysis focused on identifying potential *key developmental understandings* (KDU: Simon, 2006) used in constructing congruence proofs. A full conceptualization of KDU is beyond the scope of this brief report, but we emphasize that a KDU affords a learner a different way of thinking about mathematical relationships (Simon, 2006). For our purposes, this meant that to determine whether something may be a KDU, we must be able to identify how having or not having the KDU could make a difference in teachers' capacity to construct congruence proofs. We proceeded by coming to consensus about the logic of each teacher's response to the tasks, then generating potential descriptions of ways of thinking about congruence and proof that account for differences among responses. These descriptions became provisional codes. We consolidated or distinguished codes based on how and whether the use of the definition of congruence changed what was possible mathematically later in the argument.

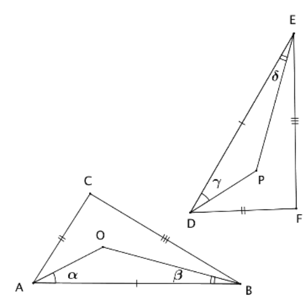

Figure 7: A figure Distributed with the Boomerang Task

Results

Potential KDU 1: Understanding that applying the definition of congruence to prove congruence of two figures means establishing a sequence of rigid motions mapping one <u>entire</u> figure to the other <u>entire</u> figure.

Teachers without this KDU may know that rigid motions are involved in congruence proof, but they may not understand that figures remain fundamentally un-altered with every motion. For instance, we found responses that established rigid motions and thus congruence between parts that compose a whole (such as between ℓ and m as well as P and Q, or ΔAOB and ΔDPE as well as ΔABC and ΔDEF) but that did not necessarily establish congruence of entire wholes ($\ell \cup P$ and $m \cup$ Q, or $\Delta AOB \cup \Delta ABC$ and $\Delta DPE \cup \Delta DEF$).

To illustrate, in the Boomerang Task, some responses used the premise that $\overline{AB} \cong \overline{DE}$ to claim abstractly the existence of a transformation mapping $\overline{AB}$ to $\overline{DE}$, but then the responses concluded that $\Delta AOB \cup \Delta ABC \cong \Delta DPE \cup \Delta DEF$ because $\Delta AOB \cong \Delta DPE$ and $\Delta ABC \cong \Delta DEF$ – and not because the transformations could extend to the unions.

Additionally, some teachers' responses described rigid motions that mapped some or all corresponding parts of the first figure to the second, but the rigid motions constructed did not extend to the entire figures – in this case, the responses exhibited different rigid motions for different components that could not extend. Other responses constructed rigid motions that did extend to the entire figure, but this extension was not recognized explicitly in the responses. Furthermore, some teachers defined a transformation that did "double-duty", that is the teacher noted that two parts of the figures are congruent and therefore claimed the existence of a single transformation that mapped both pieces to their corresponding parts at the same time.

Potential KDU 2: Understanding that using a sequence of transformations to prove that two figures are congruent means justifying deductively that the image of one figure under the sequence of transformations is exactly the other figure.

To understand the necessity of proving that two figures need to be superimposed, one must conceive of the possibility that they may *not* be superimposed. Being able to conceive of this possibility allows for a learner to realize that there is more to show than identifying a candidate sequence of transformations.

Teachers without this KDU may declare the proof complete after defining the transformations or providing minimal justification. For instance, on the Line Point Task, some teachers defined a sequence of rigid motions and claimed that $\ell \cup P$ had been mapped to $m \cup Q$ without further justification. Several other teachers minimally attempted to justify superposition by stating that rigid motions preserve distance. We note that in this case, teachers showed evidence of potential KDU 1 but not potential KDU 2.

Discussion/Conclusion

In this study, we analyzed teachers' responses to tasks, designed using variation theory, for underlying understandings that support constructing congruence proofs. Based on this analysis, we proposed two potential KDUs for the use of the definition of congruence in congruence proofs.

Applications of this work may include the construction of lessons, assignments, and assessments that directly address each above potential KDUs and conceptual links. Such materials may help instructors as they attempt to help teachers learn the subtle concepts listed above in addition to those involved in notation. Future work is needed to interrogate the accuracy of these KDUs.

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COMPARACIÓN DE ARGUMENTOS DE PROFESORES DE PRIMARIA AMERICANOS Y MEXICANOS ANTE UNA FIGURA GEOMÉTRICA INEXISTENTE

Lidia Aurora Hernández Rebollar
Benemérita Universidad Autónoma de Puebla, México
lhernan@fcfm.buap.mx

Chepina Rumsey
University of Northern Iowa
chepina.rumsey@uni.edu

José Antonio Juárez López
Benemérita Universidad Autónoma de Puebla, México
jajul@fcfm.buap.mx

Josip Slisko
BUAP, México
jslisko@fcfm.buap.mx

En este trabajo se presenta un análisis de los argumentos de dos grupos de profesores de primaria, uno de México y otro de Estados Unidos. A ambos grupos se les presentó un problema que apareció en un libro de texto mexicano de quinto de primaria el cual involucraba un polígono irregular imposible de existir con las medidas de los lados que exhibía. Los dos grupos de profesores encuestados proporcionaron diversos argumentos para sostener la existencia o no del polígono. Aquí se presentan las diferentes garantías o apoyos utilizados por las profesoras de Estados Unidos y se comparan con los de los profesores mexicanos reportados en una investigación anterior. La intención de este trabajo ha sido aportar al conocimiento de los recursos argumentativos de los docentes del nivel básico.

Keywords: Teacher Knowledge, Geometry and Geometrical and Spatial Thinking, Reasoning and Proof

Propósito del estudio

En Juárez, Hernández y Slisko (2014) se reportaron los argumentos proporcionados por un grupo de 69 profesores de primaria para validar la existencia o no de una figura geométrica que, con las medidas que presentaba, no era posible que existiera. La mayoría de los argumentos de estos profesores fueron para aceptar que la figura existía y las garantías que aportaron eran pragmáticas. "En la variedad de argumentos que presentan los docentes, observamos que la gran mayoría hace un análisis superficial, se centra en la forma del terreno e ignora las medidas de los lados indicadas numéricamente" (Juárez et al, 2014).

En Alatorre, Flores y Mendiola (2012) los autores observaron un divorcio entre los trazos y sus longitudes cuando cuestionaron a profesores de primaria mexicanos sobre la posibilidad de construir triángulos con medidas arbitrarias. Ellos concluyeron en su estudio que la desigualdad triangular no formaba parte de la práctica profesional de los participantes. Estos trabajos nos motivó a preguntarnos: ¿Qué argumentos presentarían profesores de primaria de Estados Unidos para validar la existencia o no de una figura geométrica inexistente?

La argumentación es una de las habilidades que se plantea desarrollar en los estudiantes a lo largo de la vida escolar. En los planes de estudio de México y Estados Unidos se busca la formación de ciudadanos críticos, reflexivos y con capacidad de análisis y de argumentación. Aunque en el currículo no se plantean actividades explícitas para desarrollar esta habilidad, es usual que se recomiende que los estudiantes validen sus procedimientos y resultados. Es claro que, a nivel institucional y social, se espera que los docentes de todos los niveles educativos posean dicha habilidad, así como suficientes recursos para promoverla en sus alumnos.

Por lo anterior, el objetivo de la investigación que presentamos aquí fue: obtener una categorización de los argumentos que proporcionaron un grupo de profesoras de primaria de Estados Unidos ante el cuestionamiento de la existencia o no de un polígono irregular con medidas imposibles y comparar dichos argumentos con los de un grupo de profesores de primaria mexicanos.

Marco teórico

En esta investigación se considera que un argumento es un enunciado que utilizan las personas para validar algo y convencer a otros. Se utiliza el modelo de Toulmin (1958) en el que un argumento está constituido por datos, justificaciones o garantías y una conclusión. Dependiendo del campo de la argumentación, existen diferentes tipos de garantías y éstas le dan diferentes grados de fuerza a la conclusión. Para clasificar los argumentos se ha recurrido a Balacheff (2000) quien distingue los tipos de pruebas o demostraciones entre pragmáticas e intelectuales, dependiendo del tipo de argumentos que se utilicen. En las primeras, los enunciados se validan con acciones concretas o mentales, e incluyen dos niveles, empirismo ingenuo y experimentos cruciales. Las segundas basan sus argumentos en conceptos y en el lenguaje e incluyen dos tipos: ejemplos genéricos y experimentos mentales.

Metodología

Este es un estudio cualitativo de tipo exploratorio interpretativo. Se comparan las respuestas dadas por dos grupos de profesores de primaria, uno de Estados Unidos y otro de México, a un cuestionario común.

Participantes

Grupo de Estados Unidos. 26 maestras de educación primaria que participaron en un programa de desarrollo profesional de dos semanas en el verano. Las maestras eran de tres escuelas en el medio oeste de los EE. UU. y enseñaban a niños de jardín de infantes a quinto grado de primaria.

Grupo de México. 69 docentes de educación primaria en servicio que se encontraban terminando un curso de capacitación en matemáticas al momento de contestar el cuestionario. Su preparación era de educación normalista con 15 años de servicio en promedio.

Instrumento

Se elaboró una hoja de trabajo que contenía un polígono irregular que representaba a un terreno y que formaba parte de una actividad de un libro de texto de matemáticas mexicano del quinto grado de primaria (SEP, 2010). La actividad pedía calcular el número de rollos de malla necesarios para cercar el terreno. Este instrumento se aplicó a los profesores de México y fue traducido al inglés para encuestar a los maestros estadounidenses. El terreno es el de la figura 1.

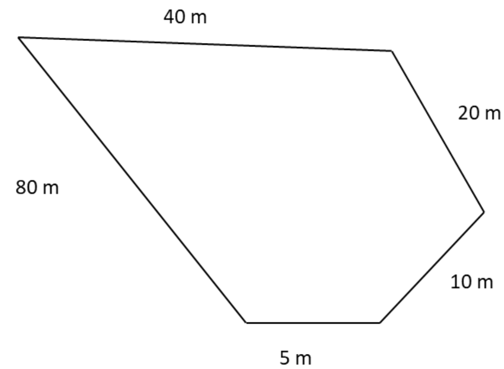

Figura 1: Polígono irregular incluido en la hoja de trabajo

En el instrumento se dice explícitamente que la tarea de los docentes no es resolver el problema planteado en la actividad, sino responder la pregunta acerca de la existencia o inexistencia del terreno y dar los argumentos que apoyan su respuesta. La instrucción en inglés fue:

Your task is not to solve the problema, but rather to respond to the question: Can the piece of land described in the problem above exist in reality?

a) It can exist b) It cannot exist c) I cannot decide

Justify your answer with as many details as you can to support your response.

Respuesta esperada

Se esperaba que los profesores eligieran el inciso b) It cannot exist y que en su justificación utilizaran conceptos matemáticos como los siguientes:

1. Las longitudes de los lados no corresponden con los números o medidas proporcionadas. El lado más largo tiene la etiqueta de 80 m, el segundo en longitud se dice que mide 40 m. Sin embargo, se aprecia en el dibujo que el segundo no cabe dos veces en el primero. El resto de los lados están marcados con 20, 10 y 5 metros, pero igual que los dos primeros el lado que "mide" 5 m no cabe dos veces en el que "mide" 10 m y así sucesivamente.
2. Este terreno tiene un lado cuya medida es mayor que la suma de los demás lados, relación que no puede ocurrir en los polígonos. El lado que tiene la etiqueta de 80 m mide más que la suma del resto de los lados, que es de 75 m, lo cual es imposible. Tal imposibilidad se demuestra aplicando la desigualdad del triángulo, sucesivamente, en los triángulos cuyo vértice común es el punto que une a los lados de 80 y 40 metros.

Resultados

Los porcentajes de respuestas dadas por las profesoras de primaria de Estados Unidos se muestran en la tabla 1, y debajo de ellas las de los profesores de México (Juárez et al, 2014).

Tabla 1: Porcentajes de Respuestas en Cada Inciso del Instrumento

Profesores	It can exist	It can not exist	I cannot decide	No response

Estados Unidos	38.5%	46.1%	15.4%	0%
México	89.9%	4.3%	1.4%	4.3%

Los porcentajes de las profesoras que afirmaron que sí podía existir el terreno en la realidad y las que afirmaron que no, son muy parecidos, alrededor del 40 por ciento cada uno. Resulta mayor la cantidad de profesoras que afirmaron que el terreno no puede existir en la realidad con una diferencia de casi siete puntos porcentuales. Hubo cuatro profesoras que no supieron cómo contestar y ninguna dejó en blanco su respuesta.

En contraste con este resultado, la gran mayoría de profesores mexicanos afirmó que el terreno sí podía existir (casi 90%), y el porcentaje de los que afirmaron que esto no era posible fue solo del 4.3%, sorprendentemente bajo.

Dos profesoras de Estados Unidos que respondieron que el terreno no podía existir en la realidad se dieron cuenta que el lado de 80 m es mayor que la suma del resto de los lados y que de esta forma la figura no podría cerrar. Este es un argumento que alude a una propiedad matemática, la desigualdad triangular (argumento esperado número 2). Siete profesoras argumentaron que los lados no eran proporcionales, por ejemplo, que el de 80 m no era cuatro veces el de 20 m. Estas profesoras apreciaron visualmente que las medidas de los lados no corresponden con la longitud trazada y que no cumplen con ser proporcionales cuando las comparan (argumento esperado número 1). Una profesora dijo que con esas medidas no se formarían esos ángulos. Otra profesora anotó, “simplemente, no se ve bien”. Así, la mayoría de las que respondieron que no puede existir en la realidad, se dio cuenta que las medidas no corresponden con la longitud real de la figura. La propiedad matemática que identificaron con mayor frecuencia fue la de proporcionalidad. De esta forma, un 40% de los argumentos presentados en este grupo fue de tipo intelectual.

Las profesoras que afirmaron, -sí puede existir -, argumentaron en su mayoría, que “los terrenos pueden tomar cualquier forma”. Algunas apoyaron su argumento en que los terrenos toman cualquier forma por los límites naturales como ríos o caminos, o porque “no todos son rectangulares”. Dos profesoras no dudaron en la existencia del terreno ya que “las medidas son realistas” o porque “es una figura cerrada”. Una más contestó que “todo es posible”. A dos profesoras de este grupo les pareció extraña la pregunta y cuestionaron la intención de cercar un terreno tan pequeño. Así, encontramos que las profesoras que respondieron que sí puede existir, fue porque observaron la forma de la figura superficialmente y no detectaron ningún problema con las medidas. Por lo tanto los argumentos de quienes sostuvieron la existencia del terreno fueron pragmáticos y sus garantías fueron similares a las de los profesores mexicanos.

Al comparar los argumentos de las profesoras de Estados Unidos con los de México detectamos claramente que las estadounidenses tuvieron una mayor capacidad de análisis al responder la pregunta sobre la existencia. Los argumentos de las profesoras de Estados Unidos contenían propiedades matemáticas como garantía o apoyo, para justificar la imposibilidad de la existencia del terreno, como proporcionalidad y la desigualdad triangular. Mientras que en los argumentos de los mexicanos no aparecieron estos recursos, aun entre los que respondieron que la figura no podía existir en la realidad.

En ambos grupos se observó que, quienes respondieron que el terreno sí podía existir en la realidad, se basaron únicamente en la forma de la figura. Lo visual e inmediato les fue suficiente para responder y para justificar su selección. Sin embargo, esto ocurrió en mayor proporción entre los profesores mexicanos que entre las estadounidenses.

Discusión y Conclusión

Aunque las profesoras de Estados Unidos fueron capaces de detectar una inconsistencia entre las medidas de los lados que aparecían en la figura y las longitudes reales de la misma, y lograron establecer argumentos de tipo intelectual, en un porcentaje mucho mayor que el de los mexicanos, observamos que tal porcentaje es menor al 50%. Por lo que es posible hablar de una tendencia, en los profesores de primaria, a reponder después de un análisis superficial y a proporcionar argumentos pragmáticos. En estos dos estudios, los resultados coinciden con lo reportado por Alatorre, Flores y Mendiola (2012) en la que observaron un divorcio entre los trazos y sus longitudes, y en la cual concluyen que “el razonamiento y la argumentación no son parte de la práctica profesional de muchos docentes de primaria”.

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A COMPARISON OF AMERICAN AND MEXICAN PRIMARY TEACHERS’ MATHEMATICAL ARGUMENTS FOR A NONEXISTENT GEOMETRIC FIGURE

This paper presents an analysis of the arguments of two groups of primary teachers, one from Mexico and the other from the United States. Both groups were presented with a geometry problem that appeared in a fifth-grade Mexican textbook. The problem involves an irregular polygon that can not exist. The two groups of teachers provided various arguments to support the existence or not of the polygon. An initial analysis of different arguments used by the teachers in the United States are presented and compared with those of the Mexican teachers, which were reported in a previous study. The intention of this work is to contribute to the knowledge of the argumentative resources of elementary school teachers.

CROSS-CASE EXAMINATION OF MIDDLE SCHOOL MATHEMATICS TEACHERS' TOPIC-SPECIFIC CONTENT KNOWLEDGE IN THE U.S. AND RUSSIA

Mourat Tchoshanov
University of Texas at El Paso
mouratt@utep.edu

Maria Cruz Quinones
Universidad Autónoma de Ciudad Juárez
titacq@hotmail.com

Gulshat Shakirova
Kazan Federal University
gkharis@gmail.com

Liliana Shakirova
Kazan Federal University
liliana008@mail.ru

This cross-case study examined the U.S. and Russian teachers' topic-specific knowledge. Teachers (8 – from the U.S., and 8 – from Russia) were selected using a non-probability purposive sampling technique based on scores on the teacher content knowledge survey. Teachers were interviewed on the topic of fraction division using questions addressing their content and pedagogical content knowledge. The study revealed that there are explicit similarities and differences in teachers' content knowledge as well as its cognitive types. The findings also suggest that in the cross-national context teachers' knowledge could vary depending on curricular and socio-cultural priorities placed on teaching and learning of mathematics.

Keywords: teacher knowledge, cross-national study, number concepts and operations, middle school mathematics

Focus of the Study

Cross-national studies allow understanding of how teacher education is contextualized in different countries which requires "a range of analytical methods that draw out conflicting views, contested areas and shared beliefs" (LeTendre, 2002). In last decade, a number of studies on teacher education were focusing on unpacking "culturally contextualized and semantically decontextualized dimensions" in order to create "a more balanced comparative perspective" (Ewha, Ham, Paine, 2011).

However, few comparative studies addressed in-service teachers' content knowledge. Moreover, the field lacks research that provides an in-depth analysis of teacher knowledge at a topic-specific level. Considering the importance of teachers' topic-specific knowledge, the study focused on the following *research question*: to what extent is the U.S. and Russian lower secondary mathematics teachers' content knowledge similar and/or different in the topic-specific context?

Framework

Existing studies vary in a scope addressing different issues including general aspects in teacher education, teacher knowledge, different types of teacher knowledge, connections between teacher knowledge and student performance, to name a few. Studies suggest that international comparisons in teacher education should be sensitive to the social, historical, and cultural contexts. Few studies aimed at the international comparison of the effects teacher mathematics knowledge and pedagogy have on student achievement (Baumert, Kunter, Blum, Brunner, Voss, Jordan, Klusmann, Krauss, Neubrand, & Tsai, 2010; Marshall & Sorto, 2012).

Analysis of a body of literature in cross-national research in teacher education and teacher knowledge demonstrates that the field is short of studies that provide a close evaluation of teacher knowledge at a topic-specific level. Addressing this deficiency, the proposed study attempts to qualitatively examine the U.S. and Russian lower secondary school mathematics teachers' content knowledge in the topic-specific context – the division of fractions.

Methodology

We selected the interpretive cross-case study design (Merriam, 1998) to examine the U.S. and Russian teachers' topic-specific knowledge of one of the important themes in the lower secondary mathematics curriculum in both countries - division of fractions.

The Teacher Content Knowledge Survey (TCKS) instrument employed at the initial stage of the study consisted of 33 multiple-choice items addressing main topics of lower secondary mathematics curriculum as well as different cognitive types of content knowledge with its' reliability as measured by the Cronbach alpha coefficient at .839 (Tchoshanov, 2011). The TCKS was administered to the initial sample of lower secondary mathematics teachers in the USA (grades 6-9, N=102) and Russia (grades 5-9, N=97) (Tchoshanov, 2011; Tchoshanov, Cruz, Huereca, Shakirova, & Ibragimova, 2017). A non-probability purposive sampling required that selected the U.S. and Russian teachers represent different quartiles of the total scores on the TCKS measure. It was also required that selected teachers teach at similar school settings (e.g. urban public schools). Addtitionally, both the U.S. and Russian participants have similar teaching assignments – lower secondary school mathematics. Thus, 8 teachers from the U.S., and 8 from Russia were selected for the study.

The study used structured teacher interviews on the topic of the division of fractions. Teachers were interviewed using the following five questions related to the topic: (1) When you teach fraction division, what are important terms, facts, procedures, concepts, and reasoning strategies your students should learn? (2) What is the fraction division rule? (3) Apply the rule to the following fraction division problem: $1\frac{3}{4} \div \frac{1}{2} =$. (4) Construct a word problem for the given fraction division: $1\frac{3}{4} \div \frac{1}{2} =$. (5) Is the following statement $\frac{a}{b} \div \frac{c}{d} = \frac{ac}{bd}$ (a, b, c, and d are positive integers) ever true?

Taking into account the categorical nature of the quantitative data (e.g., frequency counts) collected in the study, we used a non-parametric Chi-square technique to compare two groups on a response variable (Huck, 2004, p. 463). Considering a small sample size, in the 2x2 contingency case of the chi-square test, for expected frequencies less than 5 Yates' correction is employed. In order to analyze qualitative data, we conducted meaning coding and linguistic analysis of teacher narratives as a primary method of analysis (Kvale & Brinkmann, 2009). To increase the credibility of the qualitative data analysis, the meaning coding and linguistic analysis were performed and cross-checked by two independent raters.

Findings

We selected the interpretive cross-case study design (Merriam, 1998) to examine the U.S. and Russian teachers' topic-specific knowledge. The data analysis clearly demonstrated that the participants had similarities and differences while responding to the interview questions. The analysis below unpacks teacher responses to each question.

Teacher responses to question 1 were coded using the following categories: 1) vocabulary, 2) facts and procedures, 3) concepts and connections, and 4) reasoning.

Table 1: Frequency of the U.S. and Russian Teachers' Responses to Question 1

Category	US teachers	Russian teachers	χ^2
Vocabulary	33	38	*2.003*
Facts and Procedures	27	32	*3.471*
Concepts and Connections	20	23	*0.893*
Reasoning	0	6	*6.667***

*p<.05, **p<.01

As reported in Table 1 with chi-square statistic and p-values, the most frequently used category in response to question 1 was "vocabulary" with no significance observed between the groups ($\chi^2=2.003$). With regard to categories "facts and procedures" and "concepts and connections", we also didn't detect any significant differences ($\chi^2=3.471$ and $\chi^2=0.893$ accordingly). The only category where the significance was detected is the category of "reasoning" ($\chi^2=6.667$).

In Table 2, we present the frequency of terms used by the U.S. and Russian teachers while explaining the rule for fraction division in response to question 2 along with chi-square values for each reported term. All U.S. and Russian teachers correctly responded to this question.

Table 2: Frequency of Terms Used By the U.S. and Russian Teachers

Terms used by teachers	US teachers	Russian teachers	χ^2
Flip	7	1	6.250*
Reciprocal	7	8	1.067
Dividend	0	6	6.667**
Divisor	0	6	6.667**
First fraction	6	2	2.250
Second fraction	6	2	2.250
Quotient	0	1	1.067

*p<.05, **p<.01

As expected, teachers' responses to the procedural question 3 (divide two given fractions) were the least insightful. Most of the teachers in both groups silently performed the division on a scratch paper that was provided to every participant. All participating teachers correctly solved the given fraction division task. Slight differences were observed in the representation of the answer. One of the U.S. teachers illustrated the division by a pictorial model.

There are several distinct meanings of the division of fractions discussed by scholars. Ma (1999, p. 72) claimed that there are three main models and corresponding meanings to represent the division of fractions: measurement, partitive, and product and factors. Question 4 was challenging to the U.S. teachers - only five teachers were able to construct a correct word problem compared to eight Russian teachers.

Table 3: Frequency of Meanings Used By the U.S. and Russian Teachers

Meanings of fraction division	US teachers	Russian teachers	χ^2
Part-to-whole (partitive)	0	2	*0.571*
Measurement (quotitive)	5	2	*2.286*
Rectangular area model (product and factors)	0	4	*5.333**
Incorrect	3	0	*3.692*

**p<.05, **p<.01*

In Table 3, chi-square analysis showed statistically significant difference not only for the rectangular area model but also an overall difference in performance of the U.S. and Russian teachers on this particular task ($\chi2 = 10.286$, $p<0.05$).

Question 5 was challenging to both the U.S. and Russian teachers. As depicted in Table 4, we were not able to observe any significant differences between groups in a number of correct responses (only one correct and one partially correct solution proposed from the U.S. teachers compared to three correct and one partially correct solutions provided by Russian teachers).

Table 4: Frequency of Solutions Proposed By the U.S. and Russian Teachers

Teacher responses	US teachers	Russian teachers	χ^2
Never true	5	4	*0.254*
True if a=b=c=d	1	1	*0*
True if c=d	1	3	*1.333*
No solution provided	1	0	*1.067*
Using numerical values to prove	4	0	*5.333**

**p<.05, **p<.01*

Discussion and Conclusion

Most insightful finding in teachers' responses to question 1 was the fact that both U.S. and Russian teachers quite similarly define learning objectives for the division of fraction. Both groups clearly outlined the main vocabulary students should learn, facts and procedures students should master, and concepts students should understand. The revealing difference was observed in the teachers' response to the reasoning category.

Despite the fact that question 1 explicitly asked to articulate "what are important … reasoning strategies your students should learn?", none of the U.S. teachers responded to this part of the question compared to six Russian teachers who highlighted the importance of "developing logical reasoning" (4 teachers) as well as "checking for reasonableness" (2 teachers). This finding may suggest that the U.S. teachers do not see a "reasoning" potential in the topic of the division of fractions whereas their Russian counterparts emphasize the development of students' critical thinking as one of the important learning objectives for the topic of fraction division.

As mentioned earlier, both the U.S. and Russian teachers emphasized the importance of developing students' mathematical vocabulary related to the topic of the division of fractions. Between two groups of teachers, there were 13 terms recorded in response to the "vocabulary" category of question 1. Most frequently used term among the U.S. teachers was "division" (6 frequency counts) whereas "reciprocal" (7 counts) and "multiplicative inverse" (6 counts) were the most frequently used terms by Russian teachers. This result may suggest that the U.S.

teachers focused on the operation in general (e.g. division) whereas Russian teachers emphasized the operation specific to the division of fractions (e.g. reciprocal, multiplicative inverse).

Next observation is concerned with the use of accurate mathematical terminology: "dividend" vs. "first fraction" and "divisor" vs. "second fraction" which was statistically significant in both cases as depicted in table 2. Our observation also revealed a strong tendency on the part of the U.S. teachers to use the term "flip" as a sub-language for reciprocal/ multiplicative inverse with a reported $\chi 2=6.250$ at $p<0.05$.

Following on the models used by teachers for fraction division, we found that the measurement model was the most popular one (5 frequency counts as presented in table 3) and the only one model used by the U.S. teachers in response to question 4 asking to construct a word problem for the given problem. In contrast, Russian teachers applied three different models for the fraction division meaning with the rectangular area model being the statistically significant ($\chi 2=5.333$, $p<0.05$).

Analysis of teacher narratives to question 5 did not show significant differences between groups in a number of correct responses. Whereas the U.S. teachers proposed only one correct (c=d) and one partially correct solution (a=b=c=d), their Russian counterparts provided three correct and one partially correct solutions. A statistically significant difference ($\chi 2=5.333$, $p<0.05$) was reported with regard to a method of proof used by teachers. None of the Russian teachers attempted to prove the statement numerically compared to four U.S. teachers who tried to plug in different numbers to check if the statement works.

Synthesizing the main findings of the study, we report that the topic-specific level of analysis helped us to unpack hidden insights in terms of differences and similarities in teacher knowledge among participants. Considering the qualitative nature of the study, we are cognizant of its limitations and, congruently, we are sensitive do not overgeneralize its results. The granualized methodology used in the study to unpack and analyze teacher topic-specific knowledge could be considered as a potential contribution to the field of cross-national studies on teacher knowledge. The study suggests close comparison and learning about issues related to teacher knowledge in the U.S. and Russia with a potential focus on re-examining practices in teacher preparation and professional development.

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TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING FUNCTION REPRESENATIONS: AN EXPLORATORY STUDY

Stacey C. Zimmerman
University of North Carolina Greensboro
sczimmer@uncg.edu

P. Holt Wilson
University of North Carolina Greensboro
phwilson@uncg.edu

This exploratory study attempts to learn more about mathematical knowledge for teaching by examining the knowledge teachers use to teach in secondary mathematics classrooms. Drawing upon Ball and colleagues' (2008) mathematical knowledge for teaching framework and the literature, we construct an initial model of teacher knowledge for function representations. Using written responses and interview data, findings indicate that the nature of secondary mathematics teachers' knowledge can be broadly categorized as subject matter knowledge and pedagogical content knowledge.

Keywords: Mathematical Knowledge for Teaching, Teacher Knowledge

For the past three decades, researchers have worked to develop detailed descriptions and categorizations of teacher knowledge. Of note in mathematics is the work of Ball and colleagues (Ball, Thames, & Phelps, 2008) who expanded upon Shulman's (1986) notion of pedagogical content knowledge to develop the mathematical knowledge for teaching (MKT) framework. This framework has been instrumental in detailing ways of knowing elementary grades mathematics useful for teaching, and studies have demonstrated a strong association between MKT and student achievement (Hill, Rowan, & Ball, 2005) and MKT and the quality of instruction (Hill, Ball, & Schilling, 2008).

Unlike MKT for elementary grades, there are far fewer studies examining MKT for secondary mathematics. While some secondary researchers use Ball et al.'s (2008) MKT framework, others question the transferability of the frameworks' constructs, specifically common content knowledge and specialized content knowledge, to the secondary level (Speer, King, & Howell, 2015). However, considering the usefulness of the MKT framework in elementary mathematics, the dearth of secondary mathematics MKT research, and the disagreement about the nature of secondary MKT, this two-phase exploratory study investigates the extent to which the MKT framework can aid in the understanding of secondary MKT by answering the question: *What is the nature of secondary mathematics teachers' MKT related to the multiple representations of functions?*

Background

To investigate how well the MKT framework models secondary MKT, this study focuses on a significant topic in secondary mathematics, function representation. Functions are a "unifying theme in United States mathematics curricula" (Steele, Hillen, & Smith, 2013, p. 454). Instruction on functions can begin as early as elementary school and continue to grow in complexity throughout middle school, high school and college. For students to be successful in mathematics, they must develop a rich conceptual understanding of functions (Cooney, Beckmann, & Lloyd, 2010), grounded in the ability to move flexibly among representations (Even, 1998; Kaput, 1992; National Council of Teachers of Mathematics, 2000; Thompson, 1994). Key aspects of understanding the concept of function include an awareness of the limitations and strengths of a particular representation and knowing when one representation is more useful than another. Considering the prominence and importance of functions in the secondary mathematics curriculum in the United States, insights into secondary teacher knowledge related to the multiple representations can add to our understanding of MKT at the secondary level.

Ball's MKT framework is composed of two overarching domains, subject matter knowledge (SMK) and pedagogical content knowledge (PCK). When considering the SMK domain and knowledge related to the multiple representations of functions, the National Council of Teachers of Mathematics (NCTM) has identified several understandings, termed "essential understandings" (Cooney et al., 2010, p. 10). Essential understandings are a vital component of teacher knowledge. Such knowledge enables teachers to support students in developing deep conceptual understandings. Three of the essential understandings (denoted *EU* in Table 1) serve as an initial conjecture as to what constitutes SMK related to the multiple representations of functions.

Unlike SMK, knowledge-specific statements like NCTM's essential understandings are not readily available or lack the detail needed to describe PCK. While there are few studies focused on both PCK and multiple representations of functions, research examining student learning and multiple representations offer a starting point for conjecturing PCK related to multiple representations of functions. A review of the existing literature suggests that students typically focus on local rather than global aspects of a function (Leinhardt, Zaslavsky, & Stein, 1990), may prefer algebraic representations over other representations (Dubinsky & Wilson, 2013), may not make the necessary switch between representations to assist in solving problems (Even, 1998), and lack the conceptual understandings of functions needed to perform accurate algebraic calculations (Kalchman & Koedinger, 2005). Knowledge of students' tendencies connects what teachers teach to how they teach it, an underlying concept of PCK.

To build our initial framework for secondary MKT for representing functions, we considered NCTM's essential understandings as key aspects of SMK and selected literature focused on student learning and multiple representations as PCK. The two domains of this conceptual mapping (see Table 1, categories column) represented an initial, conjectured framework for secondary MKT for which we sought empirical support with an exploratory study.

Methods

This two-phased exploratory study used mathematical tasks to investigate secondary teachers' mathematical knowledge related to multiple representations of functions. Guided by the Mathematical Work of Teaching Framework (Selling, Garcia, & Ball, 2016), six open ended-tasks were specifically designed to elicit teachers' understanding and use of multiple function representations when teaching. Four tasks were adapted from tasks previously used in studies examining student learning (Eraslan, Aspinwall, Knott, & Evitts, 2007; Even, 1998; Kalchman & Koedinger, 2005; Leinhardt et al., 1990) and two were based on our personal experiences as mathematics instructors.

Participants and Data Collection

In the first phase, an invitation to complete the online survey was distributed to secondary mathematics educators statewide via social media, an email listserv, and word of mouth. Those electing to participate were randomly assigned two of the six tasks to complete. The survey was open for two weeks and task responses served as the primary data for phase one. Using the initial framework, we coded these responses for evidence of SMK and PCK and used these preliminary results to focus our investigation in the second phase of the study on MKT domains not well represented in phase one**.**

In phase two, we used semi-structured interviews based on the survey tasks to elicit additional evidence of MKT for representing functions. Ten teachers involved in a yearlong professional development focused on mathematics teaching practices completed the entire survey (all six tasks) as a part of the professional development activities. Four teachers volunteered to participate in a follow-up interview to discuss, modify, provide additional details to their initial survey responses, and discuss their instructional uses and perceptions of function representations.

Data from phase one and two were analyzed using a hypothesis coding approach (Miles, Huberman, & Saldana, 2014). In this approach, codes are predetermined based on the framework for the study. For data not mapping to the codes, four additional codes representing some "other" type of SMK, PCK, or the lack of evidence of SMK or PCK were added for a total of 11 codes. Survey and interview responses could receive multiple codes. After hypothesis coding, we then focused on the "other" codes in an attempt to elaborate the initial framework.

Findings

Table 1: Categorizations of Responses

	Categories	**Survey**	**Interview**
SMK	*EU1*: Functions can be represented in various ways	129 (52%)	32 (80%)
	EU2: Changing a function's representation does not change the function, although different representations highlight different characteristics, and some may show only part of the function	89 (39%)	29 (73%)
	EU3: Some representations of a function may be more useful than others, depending on the context.	11 (9%)	25 (63%)
	Other SMK demonstrated	98 (43%)	8 (20%)
	No SMK demonstrated	2 (<1%)	0 (0%)
	Total Demonstrating SMK	**227 (99%)**	**40 (100%)**
PCK	Focus on local rather than global aspects of a function	19 (8%)	3 (7%)
	Students may prefer algebraic representations	2 (<1%)	0 (0%)
	Students may not make the necessary switch between representations to assist in solving problems	16 (7%)	6 (15%)
	Weak conceptual understandings of functions inhibit accurate algebraic calculations	31 (14%)	8 (20%)
	Other PCK demonstrated	96 (42%)	22 (55%)
	No PCK demonstrated	77 (34%)	6 (15%)
	Total Demonstrating PCK	**152 (66%)**	**34 (85%)**

As shown in Table 1, all but two responses (99%) demonstrated some category of SMK. Although essential understanding EU1, *functions can be represented in various ways*, was evident in over 52% of the survey responses, NCTM's essential understandings did not fully categorize all the responses that demonstrate SMK. Consequently, 43% of the survey responses were evaluated as demonstrating some other type of SMK. As compared to the survey responses, the interview responses mapped somewhat better to the essential understandings; 80% to essential understanding EU1, 73% to essential understanding EU2, and 63% to essential understanding EU3. In the interviews, participants tended to elaborate more on their survey responses which often highlighted several categories of SMK. However, with 43% of the survey responses and 20% of the interview responses being categorized as “other SMK” it is evident that the SMK domain in our initial framework is moderately characterized by the three essential understandings.

Our findings in Table 1 also suggests the need to further develop the PCK domain in the initial framework. Whereas 66% of the survey responses and 85% of the interview responses demonstrate PCK, many of these responses did not map to either of the four PCK categories selected from the literature. As a result, of the before mentioned 152 survey responses, 42% demonstrate some type of PCK not included in the initial framework. Likewise, 55% of the interview responses demonstrate some *other* type of PCK. Only one PCK category in the initial framework, *weak conceptual understandings of functions inhibit accurate algebraic calculations,* exceeds 10% in both survey phases and the interview. Agreeably, responses that illustrate the knowledge of students is an aspect of PCK, but with the high number of responses categorized as “other PCK”, evidence does not confirm that the current subdomains of the initial framework are adequate to describe the nature of teachers’ knowledge.

Discussion

These findings provide empirical support for aspects of the initial framework, implying that Ball’s MKT framework can aid in the exploration of secondary level MKT. The SMK domain, along with the essential understandings as SMK subdomains, is effective in describing the nature of teachers’ MKT related to multiple function representations. Although evidence for the PCK subdomains is not as compelling, findings did support the overarching domain of PCK as being a part of the initial framework. While the initial framework provides a foundation for this initial exploration, the considerable number of responses classified as “other” in both the SMK and PCK domains suggests that additional or revised subdomains are needed for a thorough representation of the nature of secondary teachers’ MKT related to teaching multiple representations of functions.

Gaining insights into teachers’ MKT is important to the field as we constantly seek to improve mathematics education. The Conference Board of Mathematical Sciences clearly states that “all teachers need continuing opportunities to deepen and strengthen their *mathematical knowledge for teaching*” (2012, p. 68). In order to provide *all* teachers, including secondary mathematics teachers, the opportunity to expand their MKT, we must first identify what MKT is and possibly could be. This study is a step in that direction.

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HIGH SCHOOL MATHEMATICS TEACHERS' DISCERNMENT OF VARIANCE AND INVARIANCE IN A DYNAMIC GEOMETRY ENVIRONMENT

Gili Gal Nagar
UMass Dartmouth
gnagar@umassd.edu

Chandra Hawley Orrill
UMass Dartmouth
corrill@umassd.edu

Stephen Hegedus
Southern Connecticut State University
hegeduss1@southernct.edu

In this study we investigated the ways high school teachers deal with the task of looking for invariant properties in a Dynamic Geometry Environment (DGE). We also examined if they even attend to invariant properties; and what invariant properties they discern, describe and discuss. We found that the teachers tend to discern and discuss invariant properties using a DGE, but this occurred mostly when they were probed to consider invariance. We also found four categories of invariant properties that seem to be important for a rich and deep understanding of geometrical objects in the context of invariance and DGE. It appears also that the use of DGE allowed teachers to see and interact with invariant properties, thus suggesting that accessing geometry dynamically may have structural affordances especially when seeking for invariance. Possible future research directions and implications to teacher education are discussed.

Keywords: Teacher Knowledge, Geometry and Geometrical and Spatial Thinking, Technology

Purpose and Background

In conceptualization of and reasoning about geometrical objects, the literature suggests that variance plays a crucial role in that it invites the individual to look for invariant properties (Baccaglini-Frank, Mariotti, & Antonini., 2009; Sinclair, Pimm, & Skelin, 2012). Discerning invariant properties helps to explain and organize experiences (Gruber & Voneche, 2005), and such discernment is an essential part in understanding objects in the context of invariance (Sinclair et al. 2012). Yet, this can be a complex task because the variability of geometric objects is often invisible, especially, when the geometric property most of the time is expressed as dealing with a single static object (Laborde, 2005). Sinclair et al. (2012) explained that because invariance is part of any geometric theorem, teachers should develop skills and understandings to seek for and discern certain invariant properties and relationships in a given geometric situation.

What is not clear yet is how teachers themselves deal with the task of looking for invariant properties in Dynamic Geometry Environment (DGE) settings; do they even attend to invariant properties; and what invariant properties they might discern, describe and discuss. Thus, the current exploratory study investigates the following research questions: (a) Do six high school mathematics teachers tend to discern invariant properties in a Dynamic Geometry Environment? And (b) what types of invariant properties these teachers discern and discuss? Such an examination is important because what teachers know is associated with what they are likely to be able to do when they teach in their classrooms (Thompson, Carlson & Silverman, 2007).

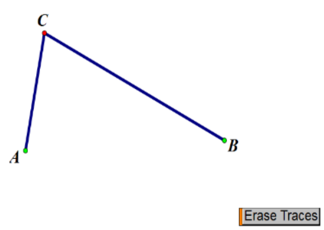
Theoretical Framework

We define conceptualization and reasoning in a DGE as a symbiosis between three key elements: (a) a DGE Object, (b) the individual formal (definitions, axioms, theorems) and non-formal knowledge of the object, and (c) the operations and actions that a person enacts. In this work, conceptualization and reasoning in a DGE, in terms of variance and invariance, deals with dynamic objects that can be perceived, conceived, understood, and reasoned about, by attending to specific aspect of variance and invariance - *discernment of invariant properties*. The prevalent way scholars define invariance in a DGE is through referring to certain invariant geometrical properties (e.g., Baccaglini-Frank et al., 2009; Hadas, Hershkowitz, & Schwarz, 2000; Laborde, 2005) that remain unaltered by transformations.

Methods

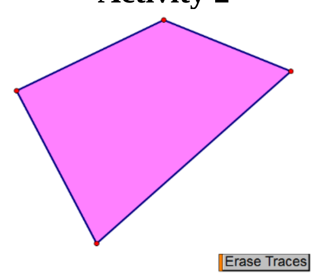
Six high school mathematics teachers were recruited for this study. The participants included four females - Amanda, Lisa, Diana and Susan, and two males - Andy and Mark (pseudonyms). Four participants had 2-4 years of mathematics teaching experience, Andy had 10 years and Susan had 30 years. The primary data for this study came from a set of two 45-minutes videotaped interviews with each teacher. As shown in Figure 1, we designed four activities using Sketchpad® Explorer (Jackiw, 1991, 2009) to provide teachers with opportunities to explore variance and invariance as they examined tasks appropriate for high school geometry. Two activities were based on similar ideas that other scholars who investigate variance and invariance in a DGE used in their work (the ideas in Activity 1 were presented by Leung (2003); and the ideas in Activity 3 were used by Baccaglini-Frank & Mariotti (2009)). Every activity has two parts: In Part 1 (called *Noticing*), participants were asked to describe what they notice when they drag one or more points in the space; and in Part II (called *Maintaining*), participants were asked to think about ways to drag one or more points in a way that maintains an invariant property. All draggable aspects left a colored trace when dragged. We focused our data analysis on descriptions in which participants discerned and described invariant properties. The analysis process included two cycles of coding (Miles, Huberman, & Saldaña, 2014) that laid the ground for a further analysis of the data in looking for themes and patterns. We used *descriptive coding* (Miles et al., 2014) to look at what participants said (verbally) and did (through actions).

Activity 1



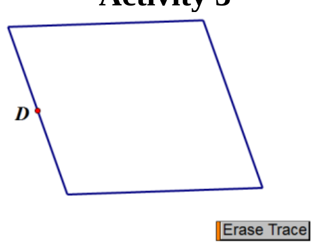
Participants could manipulate a draggable point (point C) connected to two fixed points. In Part II, participants were asked to think about ways to drag C so that there exists a circle passing through A and B with C as its center; and then to maintain a circle.

Activity 2



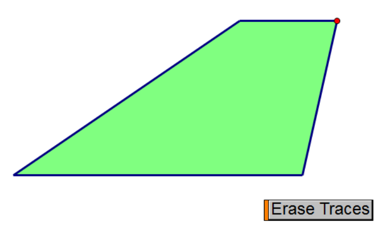
Participants were able to manipulate a dynamic polygon, by dragging one or more vertices. In Part II, participants were asked to try and drag one or more points, so the area of the figure stays the same.

Activity 3



Participants were able to manipulate a dynamic parallelogram by dragging D (a midpoint). In Part II, participants were asked to drag point D to create a rectangle and then to try and maintain the rectangle.

Activity 4



Participants could manipulate a dynamic trapezoid by dragging the red vertex. In Part II, the task was to think about ways (if possible) to drag the red point, so the area of a trapezoid is maintained.

Figure 8: Screenshot of the Four Activities Used in the Study

Results

We found 152 descriptions referring to invariant properties. Participants discussed invariant properties in 49 descriptions in the first part of the activities (*Noticing*) versus 103 descriptions in the second part of the activities (*Maintaining*).

Teacher's Tendency to Discern Invariant Properties

In all activities, we found many more descriptions referring to invariant properties in the second part of the activities than in the first part. In the first part, participants were mainly asked to describe what they notice (we found 39 descriptions of invariant properties), and if participants did not discuss about invariant properties, they were asked to describe what is changing and not changing (we found 10 descriptions of invariant properties). On the other hand, in the second part, the exploration was directed by asking participants to think about how it is possible to maintain an area of a figure the same, how to maintain a circle/rectangle, etc.

Types of Invariant Properties Discerned and Discussed by Teachers

We found four categories of invariant properties: (a) shape, (b) measurement, (c) location, and (d) calculation. In all four activities, descriptions were mostly focused on an invariant property of the visible visual dimensions of the shape of the figure (n=65), i.e., instances in which participants explained that the shape remains unchanged under a transformation and this suggested that participants might have identified the shape based on visual features that are maintained (the shape remains: an angle, triangle, parallelogram, etc.).

(a) (b)

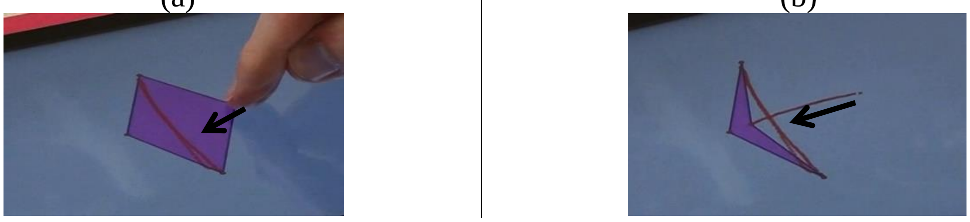


Figure 9: Mark Drags a Vertex When the Polygon is: (a) Still Convex and (b) Not Convex

For example, in Activity 2, Mark explained that by "dragging one of the corners" the shape stays convex, but "if you go far enough then it sort of starts to self-intersect and, it's no longer convex". Mark emphasized this by dragging the right corner horizontally to the left and showing how the polygon stays convex (see Figure 2a) until it is no longer convex (Figure 2b).

(a) (b)

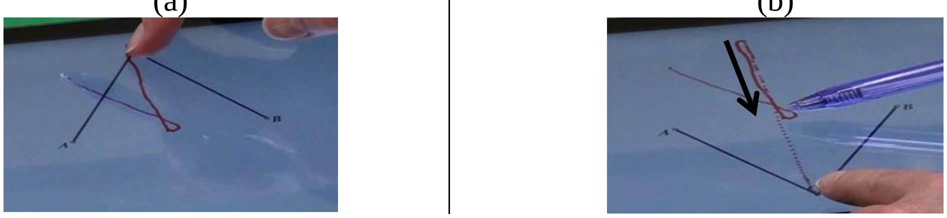


Figure 10: Susan's Reflection of Point C

We also found that many descriptions were related to properties of measurement of attributes (n=53) as being preserved under different transformations. This is interesting given that the teachers were not provided with access to the Sketchpad measurement tool. For instance, Figure 3 shows how in Activity 1, Susan reflected point C to maintain the distance of the midpoint of

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the segment AB and point C the same. Susan used her fingers and a pen to emphasize and explain that the measure of the length of the distance is the same before and after the reflection.

Across all activities, we found descriptions of a location of a geometrical object as being preserved under transformations (*n*=25). Participants discussed these properties in two ways: (a) Invariant property of a location of an attribute as staying fixed in terms of not moving (e.g., points A and B in Activity 1); versus (b) an invariant property of location of an aspect that moves but stays fixed with respect to other objects, e.g., Andy translated point D in Activity 3 to maintain a rectangle and explained that he "sees them [three rectangles that he created] being different lengths and widths, but with point D always being the midpoint of one side".

Several descriptions (*n*=9) referred to a property of a calculation between attributes of a figure as being maintained under a transformation (e.g., the product 'height times base'). We also found that participants were also able to describe invariant properties (*n*=13) that were not visually presented on screen, e.g., a *Circle* in Activity 1; *Height Times Base* in Activity 2, etc.

Conclusions

One of the skills necessary to develop rich and deep understanding of geometrical objects, in terms of variance and invariance, is to look for and discern invariant properties (Sinclair et al., 2012). Related to our first research question, teachers in this study were able to discern and discuss invariant properties when asked about them but did not seem to naturally offer discussions of invariance without prompting. This finding suggests that careful consideration should be given to activities to ensure learners are engaging in identifying both the things that stay the same and the things that change – noting that invariance is often not as readily apparent.

In answering our second research question, we found four categories of invariant properties that seemed to be important for these participants in discussing variance and invariance in a DGE. Based on our interviews, we argue that discerning invariant properties of a shape of the figure might be an important starting point in making connections to valid domains, in unpacking a situation, and in looking for different ways to approach a problem. Leung and Lee (2008) showed that the measurement tool in DGE can be powerful in creating manipulations that can be guided and directed by the user. Moreover, the use of calculations and measurements is an integral part of problem solving in mathematics. Thus, attending to invariant properties of measurement and calculations might be an important component of rich understanding of objects and problem solving in DGE. As evidenced in our data, attending to invariant properties of location may help in sensemaking in geometric situations. Thus, teachers might benefit from learning opportunities that include explicit reference to the pursuit of these four different types of invariant properties and for developing skills to use such invariant properties in unpacking a geometric situation, problem solving, argumentation, etc.

Laborde (2005) explained that the invisible nature of invariant properties makes the task of discerning them a complex one. However, in this study, using the DGE allowed participants to see and interact with those properties, thus suggesting that accessing geometry dynamically may have structural affordances especially when considering invariance. The findings also show that it is possible to discuss and describe invariant properties that are not visible on the screen. This ability to talk and think about invisible attributes that have invariant properties is an important skill worth developing among teachers and should be further examined.

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CONNECTING ADVANCED AND SECONDARY MATHEMATICS: INSIGHTS FROM MATHEMATICIANS

Xiaoheng Yan
Simon Fraser University
xkyan@sfu.ca

Ofer Marmur
Simon Fraser University
omarmur@sfu.ca

Rina Zazkis
Simon Fraser University
zazkis@sfu.ca

This paper reports a preliminary analysis of mathematicians' detailed accounts on why and how advanced mathematical knowledge matters for secondary mathematics teaching, and what specific connections can be made between advanced and secondary mathematics. Drawn from interviews with 18 mathematicians from 9 universities, the findings indicate that having advanced mathematical knowledge would allow secondary mathematics teachers to 1) develop an appropriate framework of mathematics; 2) gain mathematical experiences that would enhance their students' experience with mathematics; and 3) grow awareness of the extent of one's mathematical knowledge. Specific examples that connect linear algebra to secondary mathematics are presented and discussed.

Keywords: Advanced mathematical thinking, High school mathematics, Teacher knowledge, Mathematicians.

Introduction

How important is advanced mathematics for good teaching? While Monk's (1994) study shows that teachers' mathematical background, as measured by the number of mathematics courses a teacher took in university, is positively related to how much mathematics students learn at the secondary level, it is not always clear how and why advanced mathematics contributes to teaching. In fact, teachers often struggle to find specific examples that connect advanced and secondary mathematics (e.g., Zazkis & Leikin, 2010). Previous studies sought the relevance and contribution of academic mathematics to teaching by taking into account mathematicians' views on curriculum planning and course design (e.g., Goos, 2013; Hoffmann & Even, 2018; Leikin, Zazkis & Meller, 2018). However, many questions remain unanswered regarding which connections between advanced and school mathematics are important, and how they might lead to the improvement of teachers' practice (Murray et al., 2017). This paper reports a preliminary analysis of mathematicians' view on why advanced mathematical knowledge matters for secondary mathematics teaching, and what specific connections can be made between linear algebra and secondary mathematics.

Theoretical Background

Our paper builds upon 1) the notion of advanced mathematical thinking (AMT) and advanced mathematical knowledge (AMK); and 2) previous studies on mathematicians' perspectives on the importance of advanced mathematical knowledge for teaching.

Tall (1991) examined differences between elementary and advanced mathematical thinking as transitions from *describing* to *defining*, from *convincing* to *proving* based on abstract entities. These transitions, considered challenging processes for most prospective teachers, must be addressed and overcome during their undergraduate education. In exploring secondary mathematics teachers' conceptions of the role and usage of AMK in their teaching practice, previous studies have shown that while some teachers acknowledged the importance of AMK, others are unaware of the connections between secondary and advanced mathematics, and are often dismissive of their upper-level training (e.g., Goulding, Hatch, & Rodd, 2003; Zazkis & Leikin, 2010). To help motivate practicing and prospective teachers in deepening their mathematical understanding, previous studies also investigated how abstract algebra might support teachers' efforts to unpack a secondary mathematics topic, and how an understanding of advanced mathematics might shape pedagogy in the secondary classroom (Christy & Sparks, 2015; Murray et al., 2017).

Considering mathematicians' perspectives on teaching, Leikin, Zazkis, and Meller (2018) noted that, "mathematicians act as teacher educators de facto, without explicitly identifying themselves in this role" (p. 452). Indeed, mathematicians and teachers typically have different approaches to the knowing of mathematics: while mathematicians focus on the epistemic nature of the subject, teachers often focus on cognition and pedagogy without rigorous justification (Cooper & Arcavi, 2018). Despite these differences, Bass (2005) argued that mathematicians' knowledge, practices, and habits of mind are relevant to school mathematics education because their mathematical sensibility and perspective are essential for maintaining the mathematical balance and integrity of the educational process. This preliminary research report aims to further investigate mathematicians' perspectives on possible connections between advanced and secondary mathematics.

Method

The participants were 18 mathematicians from 9 universities: 15 were actively engaged in mathematics research and 3 were focused on teaching only. Of the 18 participants, 15 were teaching undergraduate or graduate courses in mathematics at Canadian research universities; two were professor emeriti; and one was a visiting professor from outside of Canada.

The main corpus of data was comprised of individual semi-structured interviews with the participating mathematicians. The aim was to gain insights into their views on advanced mathematical knowledge and its relevance to secondary school mathematics teaching. The interviews included two main guiding questions:

1. How do you think advanced mathematical knowledge might contribute to secondary school mathematics teaching?
2. What are specific examples that reveal connections between advanced and school mathematics?

All the interviews were audio-recorded and transcribed. Additional written artifacts generated by the interviewees were collected for the qualitative analysis. Recurring themes were identified and categorized using an iterative and comparative process. In this paper, all participants were coded starting with M, followed by a number assigned to each participant.

Findings

Why Advanced Mathematical Knowledge Matters?

Three major themes emerged from the mathematicians' responses to "how do you think advanced mathematical knowledge might contribute to secondary school mathematics teaching?" The results of the analysis show that advanced mathematical knowledge would allow teachers to 1) develop an appropriate framework of mathematics; 2) gain rich mathematical experience that would enhance their students' experience with mathematics; and 3) grow awareness of the extent of one's mathematical knowledge.

Developing the framework of mathematics. Some of the mathematicians claimed that a clear vision of what mathematics is and robust understanding of a mathematical concept comes from advanced mathematical knowledge. For M5, "the conceptual framework that underpins complex procedures and mechanical applications" is a component of AMK. Along the same line, M10 explained the framework of mathematics with an analogy:

> If you are knitting something, you must get some kind of overarching concept of what it is that you are doing. You are working locally but there is some framework in your mind that how your local thing is sort of fitting in.

In this sense, the importance of advanced mathematical knowledge lies in the heart of developing or changing the framework of mathematics. Having advanced mathematical knowledge would allow a teacher to see the fundamental structure and to reach "a kind of gestalt," as M10 stated, "I mean it is the same idea but somehow you are looking at a piece of mathematics on completely different terms."

The framework of mathematics was also elaborated in the sense of the scope of the subject. For example, "What comes next in the material? And how is it used later on?" (M14). As M16 put it, "This is what you teach now, but maybe this will appear again in a slightly different form; because definitions and problems are adjusted to students' comprehension level. How will they appear later on? That is the key." Concerning teaching at the secondary level, M14 stated that, "Having more knowledge means more tools, more examples, and a deeper understanding. […] It allows you to bring the material to life via a compelling narrative."

Gaining mathematical experience. Mathematics is not a spectator sport (Phillips, 2005). The mathematicians interpreted advanced mathematics not only as a form of knowledge, but also as "the process and experience of gaining that knowledge and thinking about how that knowledge impacts what they [teachers] are going to try to put forth to their students" (M11). As M10 suggested, "Roll up your sleeves and do it […] because it embodies structure and you have to get into it and understand the structure." M9 also explained:

> It [mathematics] is the study of complex structure and especially when there are simple elegant aspects to the structure. So they [teachers] need to understand how to work with such a structure. How to take it apart so they can understand how the different pieces interact and then put it back together? The only way to get that understanding and experience is to spend time doing it.

One of the benefits from doing a lot of mathematics, for M13, is that learners would be able to attack a mathematics problem using a "power drill" rather than some very simple "screwdrivers" that by definition take longer (see example 3 in Table 1).

Growing awareness of the extent of one's mathematical knowledge. Another aspect of the importance of AMK lies in the growth of awareness that guides and enables actions. M15 regarded the importance of AMK as a reflective process that allows teachers to identify their strengths and weaknesses in mathematics:

> If people are going to be teaching, they should be able to tell they actually know something well or more importantly whether they are still confused. Really getting that distinction down is one of the goals of undergraduate training at the level of a major. I think that only if you study a certain amount of mathematics [you] can do that [make that distinction].

Other mathematicians also shared this view. For example, M4 pointed out that one of the expectations in the undergraduate mathematics program at his university was to learn the limits of both students' knowledge and mathematicians' knowledge. This suggests that being aware of the extent of one's mathematical knowledge would consequently enable prospective teachers to mediate between elementary and advanced mathematical thinking.

Connections Between Advanced and Secondary Mathematics

The examples provided by the mathematicians span a wide range of topics in mathematics. In this paper, we focus on three examples in linear algebra to illustrate the connections drawn between advanced and secondary mathematics.

Table 1: Examples in Linear Algebra

A High School Problem	AMK	Advantages of AMK for Teachers
1. Solve systems with 3 or more equations.	- *Gauss-Jordan elimination*: "A systematic way to simplify a set of equations" (M5). - *Matrices and the idea of inverse*: "Use the inverse matrix if it is invertible, then you know you get a unique solution" (M8).	- "'Read off' the solution" (M5); - Identify whether a system has unique, many, or no solution.
2. Solving $\begin{cases} \mathbf{x+y=2} \\ \mathbf{x-y=0} \end{cases}$ A student's incorrect method $\mathbf{x+y-2=x-}\boldsymbol{y} \Rightarrow \boldsymbol{y}=\mathbf{1}$, thus the solution set is y=1, x any number.	- *Solution to a system of equations*: "In the background of [the] Gauss-Jordan elimination is the abstract idea of what algebraic operations are permitted on the system of equations (manipulating them) that don't change the solutions set" (M5).	- Understanding that "equating two or more equations together is NOT an allowed operation; it could alter the solution set" (M5).
3. Find the equation of a line through the points (2, 7) and (8, 10).	- *Use of the determinant of a 3 by 3 matrix*. (M13) e.g. (2,7) and (8,10). $0 = \det \cdot \begin{bmatrix} x & y & 1 \\ 2 & 7 & 1 \\ 8 & 10 & 1 \end{bmatrix} = x \cdot \det\begin{bmatrix} 7 & 1 \\ 10 & 1 \end{bmatrix} - y \det\begin{bmatrix} 2 & 1 \\ 8 & 1 \end{bmatrix} + 1 \cdot \det\begin{bmatrix} 2 & 7 \\ 8 & 10 \end{bmatrix}$ $= -3x - y \cdot (-6) - 36 = 0$ Then solve to get $6y = +3x + 36$ $\boxed{y = \frac{x}{2} + 6}$	- Offer an alternative solution; - Create a need for a justification of why the method works, which may lead to a formal proof.

Discussion

In response to why advanced mathematical knowledge matters for good teaching, the mathematicians placed emphasis on the understanding of the breadth and depth of mathematical knowledge in relation to mathematical structure. Two of their ideas – to establish the framework of mathematics and to gain rich mathematical experience resonate with Mamona-Downs and Downs' (2008) elaboration on the importance of the process of mathematical thinking rather than the product of mathematical thought. This is also in line with Burton's (1999) suggestion that mathematicians often see connections set within a global image of mathematics as an important part of knowing mathematics. The third idea regards the importance of teachers' awareness of the extent of their own knowledge, and their ability to distinguish between what they know and what requires further mathematical investigation. Additionally, by providing examples in linear algebra that may require AMK for an explanation, an alternative method, or simply a quick solution, the mathematicians provided insightful connections between advanced and secondary mathematics. Nonetheless, the analysis of the data set is still at its early stages. Further investigation and a more in-depth analysis would help us better understand the nature of advanced mathematical knowledge, and its relationship to school mathematics.

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EXAMINING PRESERVICE TEACHERS' MATHEMATICAL CONTENT KNOWLEDGE THROUGH PROBLEM SOLVING

Monte Meyerink
Montana State University
montemeyerink@montana.edu

Fenqjen Luo
Montana State University
fluo@montana.edu

Prior studies have highlighted the importance of teachers' mathematical content knowledge and its impact on teaching practices and students' learning. Applying the mathematical knowledge for teaching framework and problem-solving strategies framework, this small-scale study sought to examine preservice teachers' mathematical content knowledge and problem-solving tendencies. Participants' examples of various problem-solving strategies are discussed, as well as recommendations for future research.

Keywords: teacher knowledge, number concepts and operations, algebra and algebraic thinking

Problem solving has become an area of emphasis in mathematics education (Chapman, 2015; Lesh & Zawojewski, 2007; Schoenfeld, 2013). This emphasis can be observed on the Smarter Balanced mathematics assessment – one of the four domains tested is problem solving (SBAC, 2018). An important aspect of problem solving is the development of algebraic thinking. Emphasizing algebraic thinking at an early age has been shown to better prepare students for future applications of algebraic thinking (Carraher & Schliemann, 2018), and the Common Core State Standards for mathematics address algebraic thinking beginning at the kindergarten level (CCSSI, 2010). However, studies have shown that teachers may be insufficient in algebraic thinking, as "preservice elementary teachers' conceptions of algebra as subject matter are rather narrow" (Stephens, 2008, p. 33). Researching potential causes of this lack in algebraic thinking may prove beneficial for teacher education programs. This study's research question is the following: What problem-solving strategies do PSTs tend to use most frequently?

Theoretical Framework

The mathematical knowledge for teaching framework proposed by Ball et al. (2008) highlights six separate domains of teaching mathematics. One of these domains, common content knowledge (CCK), refers to teachers' abilities to solve mathematics problems correctly. Knowledge of problem-solving strategies is an essential aspect of CCK – one must first identify potential problem-solving strategies in order to solve a problem correctly. Problem solving is highlighted by the various strategies that can be implemented to reach a solution. While the comprehensive list of these strategies has evolved over time, Van de Walle, Karp, Lovin, and Bay-Williams (2018) presented a list of problem-solving strategies that were well-aligned with the research of both Polya (1957) and Schoenfeld (1985). The problem-solving strategies used in this study were: (a) visualize; (b) look for patterns; (c) predict and check for reasonableness; (d) formulate conjectures and justify claims; (e) create a list, table, or chart; (f) simplify or change the problem; and (g) write an equation (Van de Walle et al., 2018, p. 22). This study focused on investigating PSTs' CCK related to problem solving while understanding that CCK is just one aspect of mathematical knowledge for teaching.

Methodology

Participants in this study were PSTs (n = 25) enrolled in K-8 Mathematics Methods during the spring 2019 semester at a large university in the Northwest US. Participants were asked to

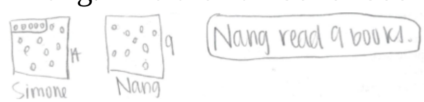
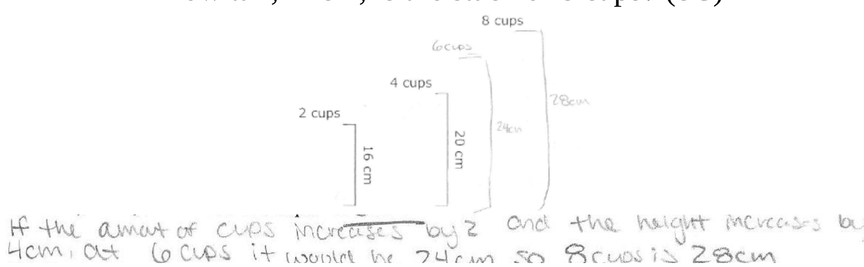
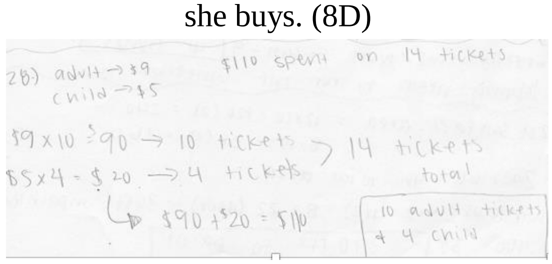
Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

independently complete Smarter Balanced mathematics problem solving sample items for sixth (n = 2), seventh (n = 2), and eighth (n = 4) grade. These specific sample items were selected due to their potential to be solved using either number operations strategies (NOS) or algebraic thinking strategies (ATS). Problem-solving strategies were coded according to the categories highlighted by Van de Walle et al. (2018) and two additional categories: (a) multiple strategies demonstrated and (b) no strategies demonstrated. Additionally, strategies were coded according to whether NOS or ATS strategies were employed. Correctness was not considered when coding.

Results

Problem-solving strategies were coded in regard to two factors: (1) whether NOS or ATS strategies were demonstrated and (2) which strategy(ies) presented by Van de Walle et al. (2018) best described the strategy(ies) demonstrated by PSTs. Examples of each coding category observed in this study are presented in Figure 1. Participants' names have been replaced with aliases to protect participants' identities.

Figure 1: Examples of Problem-Solving Strategies

<table>
<tr><th>Strategy</th><th>Example</th></tr>
<tr><td>Number Operations, Visualize (NV) - Jasmine</td><td>Simone and Nang read a total of 23 books over the summer. Simone read 5 more books than Nang. Find the number of books Nang read. (8B)

</td></tr>
<tr><td>Number Operations, Look for Patterns (NP) - Cindy</td><td>How tall, in cm, is the stack of 8 cups? (8C)

</td></tr>
<tr><td>Number Operations, Predict and Check for Reasonableness (NG) - Suzi</td><td>Mary is buying tickets for a movie. Each adult ticket costs $9. Each child ticket costs $5. Mary spends $110 on tickets. Mary buys 14 total tickets. Find the total number of adult tickets and total number of child tickets she buys. (8D)

</td></tr>
<tr><td>Number Operations; Create a List, Table, or Chart (NT) - Taylor</td><td>Sample Item 8D
<table>
<tr><th>Adult ($9)</th><th>Child ($5)</th><th>Total $</th><th>Adult ($9)</th><th>Child ($5)</th><th>Total $</th></tr>
<tr><td>1</td><td>13</td><td>74</td><td>7</td><td>7</td><td>98</td></tr>
<tr><td>2</td><td>12</td><td>78</td><td>8</td><td>6</td><td>102</td></tr>
<tr><td>3</td><td>11</td><td>82</td><td>9</td><td>5</td><td>106</td></tr>
<tr><td>4</td><td>10</td><td>86</td><td>10</td><td>4</td><td>110</td></tr>
<tr><td>5</td><td>9</td><td>90</td><td>11</td><td>3</td><td>114</td></tr>
<tr><td>6</td><td>8</td><td>94</td><td>12</td><td>2</td><td>118</td></tr>
</table>
Mary bought 10 adult tickets and 4 children's tickets.</td></tr>
</table>

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

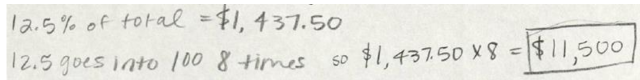
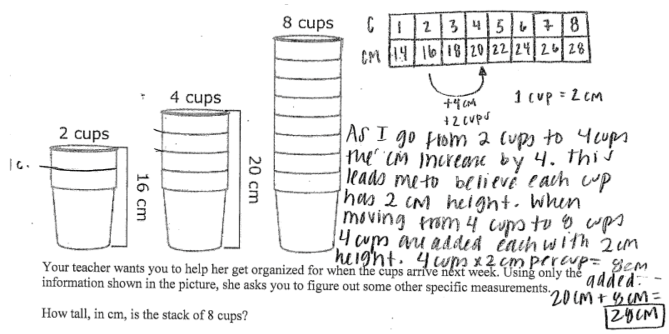
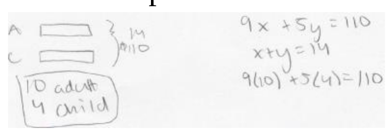
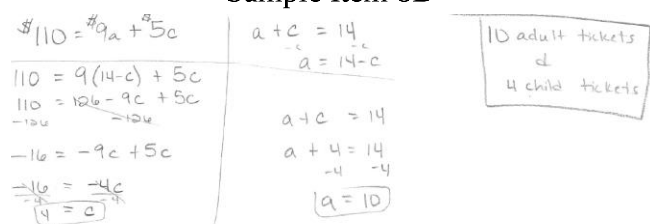
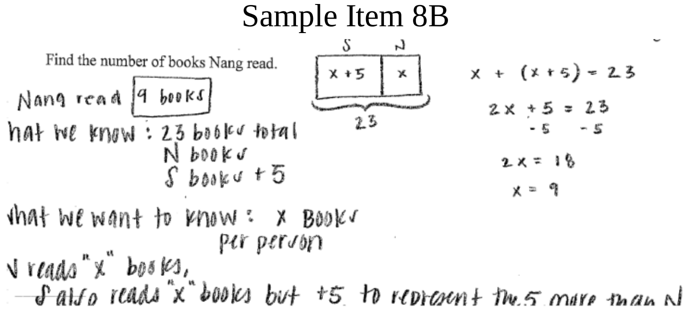
Number Operations, Write an Equation (NE) - Nancy	Lenny bought a motorcycle. He paid 12.5% in tax. The tax added $1,437.50 to the price of the motorcycle. What was the price of the motorcycle, not including tax? (7B) 
Number Operations, Multiple Strategies (NM) - Karlie	Sample Item 8C 
Algebraic Thinking, Predict and Check for Reasonableness (AG) - Mike	Sample Item 8D 
Algebraic Thinking, Write an Equation (AE) - Cindy	Sample Item 8D 
Algebraic Thinking, Multiple Strategies (AM) - Karlie	Sample Item 8B 

Table 2 displays the frequencies of problem-solving strategies demonstrated by PSTs. Overall, it appears that PSTs tended to use NOS (n = 167) more frequently than ATS (n = 28). When examining the data more closely, it appears that PSTs tended to write equations that rely on NOS (e.g., solving a series of simple arithmetic operations) on the sample items designed for lower grades, while the tendency to write equations that rely on ATS (e.g., solving a system of equations) appears to have increased as the grade level increased. It also seems that PSTs' tendency to employ "guess and check" strategies (i.e., predict and check for reasonableness) increased as the demand for algebraic thinking increased. Particularly, ATS are arguably most appropriate for sample items 8A, 8B, 8C, and 8D. However, "guess and check" strategies relying on NOS were not utilized on sixth- or seventh-grade sample items but were demonstrated on eighth-grade sample items (n = 19). Additionally, NOS were more frequently demonstrated (n = 77) on eighth-grade sample items, rather than employing ATS (n = 23). Possible explanations of this finding might be a lack of PSTs' knowledge in regard to ATS or a greater comfort or confidence level in solving problems with NOS.

Table 2: PSTs' Problem-Solving Strategies

	NV	NP	NG	NT	NE	NM	AG	AE	AM	None	Total
6A	–	–	–	–	24	1	–	–	–	–	25
6B	–	–	–	1	22	1	–	1	–	–	25
7A	–	–	–	1	21	1	–	2	–	–	25
7B	–	–	–	–	18	–	–	6	1	–	25
8A	–	–	–	–	20	–	–	3	–	2	25
8B	1	–	5	–	9	4	–	4	1	1	25
8C	–	8	–	–	9	6	–	–	–	2	25
8D	–	–	14	1	–	–	1	9	–	–	25
PSQ	1	8	19	3	123	13	1	25	2	5	200

Discussion and Implications

Based on the findings of this study, PSTs tended to rely on NOS rather than ATS when working on problem-solving items. While the collected data revealed PSTs' tendencies related to the use of NOS or ATS and the problem-solving strategies highlighted by Van de Walle et al. (2018), little insight was provided regarding PSTs' thought process while working on the PSQ. For example, consider Jasmine's solution to sample item 8B (Figure 1). One possibility might be that she initially placed five books in Simone's square. Then, her next step could have been to place a book in Nang's square, then Simone's, etc., until she reached the total number of books, 23, read by Simone and Nang. Another possibility is that she solved the sample item mentally, using a "guess and check" strategy, and simply used the visual representation to explain or check her solution. Solutions of this type leave uncertainly as to how PSTs solved sample items.

Additionally, the collected data on problem-solving strategies did not provide information regarding PSTs' reasons for utilizing or avoiding certain problem-solving strategies. For example, consider Suzi's solution to sample item 8D (Figure 1). Suzi demonstrated a "guess and check" strategy that relied on NOS. However, why she chose this strategy is unclear. One possibility is that Suzi does not feel comfortable applying ATS, so she decided to try out several combinations of adult and child tickets until she discovered a pair that met the criteria for the sample item. Another possibility is that Suzi discovered how to solve the sample item more efficiently. She may have noticed that the total price of tickets, $120, has a 0 in the ones place, and the price of multiple child tickets must end in a 0 or a 5, since a child ticket is $5. Using this information, Suzi could have recognized that the price of adult tickets must have a ones place that matches the ones place of the price of child tickets. Thus, the number of adult tickets must be 5 or 10. Discovering this "shortcut" creates a scenario where she would only need to use "guess and check" a maximum of two times. Therefore, Suzi's reason for not applying ATS is unclear. Qualitative data will be included moving forward, as individual follow-up interviews may enhance this study by: (a) providing additional information regarding how PSTs solved sample items, (b) improving criteria for coding problem-solving strategies, and (c) providing information regarding PSTs' rationale for using or avoiding certain problem-solving strategies.

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QUANTITY: IT MAY NOT BE AS EASY AS IT APPEARS

Chandra Hawley Orrill
UMass Dartmouth
corrill@umassd.edu

Rachael Eriksen Brown
Penn State - Abington
reb32@psu.edu

James P. Burke
UMass Dartmouth
jburke@umassd.edu

Martha Epstein
UMass Dartmouth
mepstein1@umassd.edu

Akira Harper
UMass Dartmouth
aharper@umassd.edu

In this case study, we uncover that a key taken-as-shared concept, quantity, may not have the shared understanding assumed in the literature. We discuss implications for research and practice if this teacher is typical.

Keywords: Teacher knowledge, Number concepts, Mathematical knowledge for teaching

Purpose and Perspective

Across the literature, the term *quantity* is used to describe something important about numbers and their relationships. Lamon (2007), Lobato and Ellis (2011), and Thompson, Carlson, Byerly, and Hatfield (2014) all provide different facets of a definition of quantity. In our own work, we previously took quantity as having a shared meaning and focused on more complex proportional-reasoning ideas. In this study, we describe one teacher's experience in one session of a professional development (PD) course as her understanding of *quantity* evolved.

Methods

The six-hour pilot PD program focused on proportional relationships. Twenty-one teachers met three days after school for two hours each day. In the first two days, teachers engaged with mathematics through dynamic applications designed to support conjectures and arguments about proportional relationships. The third day focused on classroom applications. Each class session was recorded using multiple video cameras to capture whole class and small group conversation.

We analyzed video from the third day when a rare moment in PD was captured in which a participant, Diane (pseudonym), clearly experiences and explains a pivotal "aha!" moment regarding her understanding of quantity and how she has been teaching this concept.

Results and Discussion

Participants began discussing the importance of language and how difficult it can be to engage students in using proper mathematical language. Through this conversation, Diane began to ask questions about quantity. She indicated she had been focusing on labels with ratios. The facilitator reinforced that ratios are comprised of two quantities and quantities are measurable. Diane exclaimed this was an "aha" moment for her and would impact her teaching. This brief case suggests more research is needed about how teachers understand, communicate, and teach the topic of quantity.

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MATHEMATICAL KNOWLEDGE FOR TEACHING: A COMBINATORIAL UNDERSTANDING OF ALGEBRAIC IDENTITIES

Erik S Tillema
Indiana University
etillema@indiana.edu

Desiree Ippolito
Indiana University
drippoli@iu.edu

Keywords: Mathematical Knowledge for Teaching, Algebraic Identities, Combinatorics

Mathematical knowledge for teaching (MKT) has received a significant amount of attention over the last three decades (e.g., Ball, 1990). At the secondary level, researchers have used two broad approaches to specifying secondary teachers' MKT. One approach has focused on specifying this knowledge as it relates to particular topical areas (e.g., geometry, fractions, complex numbers) (e.g., Heid, Wilson & Blume, 2015; Herbst & Kosko, 2014; Izsak, Jacobson, de Araujo, Orill, 2012). A second approach has focused on specifying this knowledge as it relates to key developmental understandings (KDUs) (Simon, 2002) that cut across multiple topical areas (e.g., co-varational reasoning) (e.g., Thompson, 2016).

For the purposes of this poster we adopt the second approach. Specifically, we report on a year-long design experiment conducted with four PSSTs whose focus was on helping the PSSTs to: a. establish an increasingly general understanding of the role that combinatorial reasoning can play in understanding common algebraic identities in 8^{th}-12^{th} grade; and b. support them to design and teach a sequence of lessons in their student teaching placement that was connected to what they had learned. The following research question guides this poster: What were the central components of PSSTs establishment of a combinatorial understanding for algebraic identities?

The PSSTs worked to develop a combinatorial understanding of algebraic identities using problems like the Card Problem:

> *The Card Problem.* You have a 2, 3, and King of Diamonds. You draw a card, replace it, draw another card, replace it, and draw a third. How many possible three card hands could you make? How many three card hands have no kings? Exactly one king? Exactly two kings? Exactly three kings?

The aim of this problem was for PSSTs to develop the equivalence that $3^3 = (2 + 1)^3 = 2^3 + 3\cdot(2^2 \cdot 1) + 3 \cdot (2 \cdot 1^2) + 1^3$. We conjectured that this equivalence could grow out of reasoning that there were a total of 3^3 possible three-card hands, that this total could be quantified as $(2 + 1)^3$ because each person had 2 non-face cards and 1 face card, and also that this total could be quantified as $2^3 + 3 \cdot (2^2 \cdot 1) + 3 \cdot (2 \cdot 1^2) + 1^3$ because: the number of three card hands with no face cards was 2^3; there were 3 ways to have one face card with each way having $(2^2 \cdot 1)$ three card hands, etc. The students were then asked to generalize this situation to one in which they drew from an unknown number of face and non-face cards, and to expand this generalization to a range of related situations.

For the poster, we highlight two results from the study: a. important points along the path in the PSSTs development of a combinatorial understanding of algebraic identities; and b. differences in the quality of the MKT among the four PSSTs. One key point along the path for the PSSTs was the transition they made from thinking about an algebraic identity as tied to a particular combinatorial situation (with whole number possibilities known or unknown) to seeing the underlying structure of the partial products that could be applied to fraction and integer values. Two of the four PSSTs developed a combinatorial understanding that enabled this transition. We will outline the reasons for these differences among the PSSTs on our poster.

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REFUTING A FRACTION MISCONCEPTION: A BRIEF INTERVENTION PROMOTES TEACHERS' CONCEPTUAL CHANGE

Jessica Rodrigues
University of Southern California
jessica.rodrigues@usc.edu

Ian Thacker
University of Southern California
ithacker@usc.edu

Research shows that some teachers overgeneralize the whole number rule of "multiplication always makes bigger" to multiplication with fractions. The present study explores this misconception not only with fractions with unfamiliar denominators (e.g., thirty-fifths) but also fractions with familiar denominators (e.g., halves) and evaluates an intervention for remediating the misconception. In- and pre-service teachers (N = 100) completed a direction of effects task prior to and after random assignment to either read a control text or a refutation text that directly refuted the misconception. The misunderstanding was present among items with unfamiliar and familiar denominators, demonstrating the pervasiveness of the misconception. Overall, the intervention was found to be effective, showing promise of refutation texts for promoting teachers' mathematics conceptual change.

Keywords: Rational Numbers, Teacher Knowledge, Number Concepts and Operations, Elementary School Education

Studies of teachers' mathematics knowledge have shown that many teachers possess a limited conceptual and procedural understanding of fractions (e.g., Ma, 1999; Moseley, Okamoto, & Ishida, 2007), limiting their preparedness to support their students' fraction learning. Preservice teachers struggle to comprehend the most foundational conceptual understanding of fraction arithmetic, which is the *direction of effects* that an operation produces (e.g., whether multiplying by a given value makes a number smaller or larger; Siegler & Lortie-Forgues, 2015).

Research on direction of effects errors has concentrated on multiplication of fractions (Siegler & Lortie-Forgues, 2015). A common misconception that individuals hold is to overgeneralize the whole number rule that "multiplication always makes bigger" to multiplication with fractions, when in fact, multiplying a positive number by a fraction that is less than one does *not* make that positive number bigger. In a study conducted by Siegler and Lortie-Forgues (2015), 41 preservice teachers completed a direction of effects task for which they were asked to predict without calculating whether the answer to an inequality would be larger or smaller than the larger fraction in the multiplication problem. The fractions used in the task were selected to be too large and uncommon so that the items could not be solved easily with mental arithmetic (e.g., "True or False: 41/66 x 19/35 > 41/66"; Siegler & Lortie-Forgues, 2015). The preservice teachers answered correctly less often than chance on multiplication items designed intentionally to capture the misconception (i.e., multiplication items with fractions between zero and one).

The present study adds to the literature in two important ways by (a) examining the presence of the direction of effects misconception for items involving unfamiliar denominators and items involving familiar denominators and (b) evaluating a brief intervention for remediating the misconception.

Theoretical Framework

Individuals who transfer the whole number rule that "multiplication makes bigger" to all multiplication problems with fractions may need to undergo conceptual change to overcome this misconception. Conceptual change is a process in which individuals restructure conceptual knowledge and shift misconceptions prompted by discrepant information (Carey, 2009). We draw from the Cognitive Reconstruction of Knowledge Model (CRKM) model of conceptual change that discerns between characteristics of the learner (i.e., their prior conceptions and motivation) and of the message (i.e., whether the learning material is comprehensible, coherent, compelling, and plausible; Dole & Sinatra, 1998). This model is particularly useful for our research because it explains why individuals might shift their misconceptions about fraction multiplication based on novel information presented in a refutation text.

Refutation texts have been shown to be an effective tool for promoting conceptual change in science education (e.g., Broughton, Sinatra, & Nussbaum, 2012; Tippet, 2010). These texts are designed to state a misconception, refute those ideas, and then present the accurate explanations as plausible and fruitful alternatives (Hynd, 2001). Whereas positive effects of refutation texts for the remediation of misconceptions have been demonstrated in science education for years (e.g., Diakidoy, Mouskounti, & Ioannides, 2011), the texts have not yet been explored extensively for addressing mathematics misconceptions. A noted exception is research conducted by Lem and colleagues (Lem, Onghena, Vershaffel, & Van Dooren, 2017) that explored the use of refutation texts for correcting students' misconceptions about box-and-whisker plots.

Present Study

To our knowledge, there are currently no studies investigating the presence of the "multiplication always makes bigger" misconception when fractions being multiplied have common denominators (e.g., halves, fourths) and no work investigating an instructional intervention for remediating this specific misconception. In response, the present study was designed with the following two objectives: (a) to assess if pre- and in-service teachers correctly answer less often than chance on easy, medium and hard direction of effects problems involving multiplication of fractions between zero and one and (b) to evaluate the effectiveness of a refutation text that targets the misconception for promoting conceptual change.

Method

Participants

Participants were pre-service and in-service elementary school teachers (N = 100), 38% self-reported as male, 42% White, 45% Asian American/Pacific Islander, 1% Hispanic, 3% Black/African American, 8% Mixed and 5% Other. The mean age was reported as 21.1 years (SD = 3.7). All participants were recruited via a Qualtrics survey panel that exclusively targeted in- and pre-service elementary teachers.

Measures

Texts. Two texts were created, a refutation text and a control text. The refutation text consisted of 364 words, had an 11th grade Flesch-Kincaid reading level, and stated and directly refuted the "multiplication always makes bigger" misconception. For example, an excerpt reads: "When multiplying numbers, you may think that the product will always be greater than the original number. This is incorrect! It is only when you multiply a positive number by a number greater than 1 that the product will be greater than the original number." This direct refutation was followed by a description of when multiplication makes bigger, and does not. The control text was a 100-word expository text paired with an illustration that was adapted from Hake (2007) that describes a strategy for multiplying fractions: “We often translate the word *of* into a multiplication symbol. We find 1/2 of 1/2 by multiplying: 1/2 of 1/2 becomes $1/2 \times 1/2 = 1/4$.”

Direction of effects task. The direction of effects task assessed participants’ understanding of the direction of effects of arithmetic, which was used to measure participants’ conceptual understanding of fraction addition and multiplication (Siegler & Lortie-Forgues, 2015). Participants were asked to evaluate the accuracy of addition inequalities in the form of $a/b + c/d > c/d$ and multiplication inequalities in the form of $a/b \times c/d > c/d$. Participants answered four addition inequalities and four multiplication inequalities within three categories of varying difficulty, resulting in 24 total items. The “easy” items all included the fraction 1/2 paired with a fraction with a familiar denominator such as fourths (e.g., $3/4 \times 1/2 > 1/2$); “medium” items included familiar denominators but did not present any unit fraction such as 1/2 (e.g., $9/10 \times 2/5 > 2/5$); and “hard” items included the items used in a previous study (Siegler & Lortie-Forgues, 2015) with unfamiliar denominators (e.g., $19/35 \times 41/66 > 41/66$). For each inequality, participants were instructed to decide, without calculating, whether the answer would be greater than the answer indicated in the inequality.

Procedure

All materials were presented via an online Qualtrics survey. After agreeing to participate, teachers completed the pretest direction of effects task. Then, teachers were randomly assigned to either the refutation text or the control text group. After reading the text, teachers were asked to complete the direction of effects task again which served as a posttest assessment. The total survey required an average of 18 minutes ($SD = 19$).

Results

We conducted a simple one sided test of proportions to determine if teachers correctly answered multiplication problems with fractions between zero and one less than half of the time. Analysis of teachers’ pretest performance on the easy and medium items with familiar denominators and hard items with unfamiliar denominators revealed that teachers chose the correct item about 49% of the time, which was not significantly different than chance (50%; chi-squared = .375, $p = .27$). When considering all 24 pretest items, 31% of teachers were correct on 100% of the 18 addition and multiplication items for which the greater-than inequality was true and 0% on the 6 multiplication items for which the greater than inequality statement was false. In other words, 31% of teachers at pretest showed the true misconception that “multiplication always makes bigger.”

Of the 57 participants in the refutation text group, 9 participants continued to show the true misconception (16%) at posttest. We then looked more closely at participants’ performance on the six multiplication items that were designed to capture the misconception. We focused attention on the refutation group participants who answered at least one of the misconception items incorrectly at posttest ($n = 28$). Of this subgroup, participants correctly answered at a rate similar to chance for five of the six items ($p > .05$) and less often than chance on one of the six items with unfamiliar denominators: $19/35 \times 41/66 > 41/66$ (71.4%; $p = .01$).

To assess the effectiveness of the refutation text intervention, we ran a repeated measures ANOVA with Arithmetic Operation (addition or multiplication) as a within-subject factor and text condition (refutation text or control) as between-subject factor and number of correct judgments as the dependent variable yielded significant main-effects of Arithmetic Operation ($F(1,98) = 64$, $p < .001$, *partial-eta squared* = .395) and main effects of condition that are approaching significance ($F(1,98) = 2.24$, $p = 0.14$, *partial-eta squared* = 0.022). These were qualified by a significant Operation x Condition interaction ($F(1,98) = 4.462$, $p < .001$, *partial-eta-squared* = 0.043). As expected, post hoc comparisons with Bonferroni correction showed no significant differences between ref-text and control groups in addition knowledge (89% vs 89%, $t(42) = 0.03$, $p = .98$, *Cohen's d* = 0.00). However, we found significant differences between refutation text and control groups in multiplication knowledge (74% vs 63%, $t(42) = 2.45$, $p = .02$, Hedge's $g = 0.56$).

Discussion

We assessed the effectiveness of a refutation text intervention for the remediation of a misconception regarding fraction arithmetic. Though there is some research on the use of refutation texts to shift students' misconceptions about mathematical topics (Lem et al., 2017), our study investigates the use of a refutation text to shift teachers' misconceptions about fraction multiplication. We also explore whether the misconception is specific to multiplication of unfamiliar fractions or common fractions more generally. Overall, we found that teachers answered easy, medium and hard fraction multiplication problems correctly at about the same rate as guessing, suggesting that the misconception is even more widespread and problematic than demonstrated in prior research (Siegler & Lortie-Forgues, 2015). Furthermore, we found that teachers who read a refutation text had fewer misconceptions about fraction multiplication compared with those who read a control text. That is, refutation texts have the potential to substantially impact teachers' mathematics misconceptions. This finding has important implications for research on conceptual change in mathematics and refutation texts.

Teachers Hold the Multiplication Misconception

Our results showed that a sample of pre- and in-service teachers answered fraction multiplication problems with fractions between zero and one at about the same rate as by chance. This finding is consistent with prior research showing that pre-service teachers were not successful in answering items regarding directions of effects in multiplication at a rate better than chance (Siegler & Lortie-Forgues, 2015). This replication of the finding suggests that many pre- and in-service teachers are in need of remediating misconceptions that may be limiting their readiness to advance their students' fraction understanding.

Refutation Texts Support Mathematics Conceptual Change

Teachers who were randomly assigned to read a refutation text that directly refuted the "multiplication always makes bigger" misconception had fewer misunderstandings at posttest than those assigned to read a control text, with modest effect sizes (Hedge's $g = 0.56$). Refutation texts targeting specific misconceptions thus hold potential as a relatively quick method for promoting teachers' conceptual change.

Future Directions

Although the present study shows promise of a refutation text serving as an instructional approach for targeting a specific fraction misconception, replication studies are required before widespread recommendation of refutation texts for supporting mathematics conceptual change. In light of the limited studies thus far exploring refutation texts for mathematics learning, there are multiple research curiosities that future work can explore, such as refutation texts for remediating not only other fraction misconceptions but also misconceptions in other mathematics content areas. Researchers should also explore the effectiveness of refutation interventions among not only teachers but also students, as such texts may hold promise for professional development opportunities and classroom instruction alike.

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COMPARING SECONDARY TEACHERS' MKT FOR GEOMETRY IN VIDEO AND WRITTEN REPRESENTATIONS OF PRACTICE

Alyson E. Lischka
Middle Tennessee State University
Alyson.Lischka@mtsu.edu

Yvonne Lai
University of Nebraska-Lincoln
yvonnexlai@unl.edu

Jeremy F. Strayer
Middle Tennessee State University
Jeremy.Strayer@mtsu.edu

Lindsay Czap
Middle Tennessee State University
Lnc4a@mtmail.mtsu.edu

Keywords: Teacher Knowledge, Geometry and Geometrical and Spatial Thinking

A persistent challenge facing any instructor is to "continually find ways to expose [learners'] thinking" (Kennedy, 2016, p. 12). When the learners are prospective mathematics teachers (PTs), this challenge can take the form of exposing mathematical thinking, especially related to mathematical knowledge in and for teaching (MKT; Ball, Thames, & Phelps, 2008; Rowland, Thwaites, & Jared, 2016; Thompson & Thompson, 1996). However, as a recent review of research in MKT suggests, it is not clear how task design impacts the MKT that is visible (Hoover, Mosvold, Lai, & Ball, 2016). This issue is additionally complicated by the potential for differently designed assessments, such as submitted written or video assignments, to expose different aspects of PTs' thinking or MKT.

To investigate this issue, we analyzed 36 representations of secondary geometry teaching practice (Grossman et al., 2009) generated by 6 PTs in a college geometry course. We asked: *In what ways do video and written representations of practice expose different aspects of PTs' MKT?* We argue that attending to potential differences in video and written representations is critical to developing PTs' MKT from an asset-based perspective.

We take the theoretical perspective that development of MKT involves key developmental understandings of mathematical topics (Silverman & Thompson, 2008) and decentering in teaching practices (Ader & Carlson, 2018). Our analytic framework draws from this theoretical perspective and is elaborated in Lai, Strayer, and Lischka (2018). The study design involved six representations of practice per PT. In three of these representations, PTs were asked to write a plan for a whole class discussion. In the other three, PTs prospective teachers were asked to video-record themselves giving a response to a student. Written and video tasks were designed in pairs to address the same particular content standards. In all tasks, secondary student thinking on definitions in geometry (e.g., reflection) were provided to teachers.

To illustrate differences represented across the data, we use the case of one PT's (Heidi) responses. In Heidi's video responses, she demonstrated fluent MKT. She built on student thinking presented, used mathematical language in precise ways, and posed questions that supported advancing the learner's key developmental understandings. In contrast, Heidi's written responses to an assignment addressing the same mathematical ideas included few mathematical statements, and did not use the student thinking. Heidi's responses make a compelling case for the importance of examining the impact of different formats of approximations of practice on instructors' perceptions of PTs' knowledge and a commitment to asset-based perspectives on what teachers know.

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PROFESSIONAL EXPERIENCES OF RURAL TANZANIAN MATHEMATCS TEACHERS

Michael Lolkus
Purdue University
mlolkus@purdue.edu

Jill Newton
Purdue University
janewton@purdue.edu

Keywords: Teacher Knowledge, Instructional Activities and Practices, International Education, Social Justice

Aligning with the redistribution component of Cazden's (2012) education-oriented framework for social justice, many education reform efforts have been aimed at improving infrastructure as opposed to instructional development (Pryor, Akyeampong, Westbrook, & Lussier, 2012). Many of the professional experiences of rural Tanzanian mathematics teachers (RTMT) are currently centered on issues in the school, and less on curriculum and effective pedagogical practices (Kafyulilo, 2013). Conversations with RTMTs and anecdotal evidence from one author's experiences have revealed a need for professional experiences that will create opportunities for RTMTs to reflect on their pedagogical practices, collaborate with their peers, and enhance their prestige and mobility in their instructional careers. We utilize collective case study methods (Stake, 2003; Yin, 2017) to examine how RTMTs develop pedagogical content knowledge (PCK) (Ball, Thames, & Phelps, 2008 2008; Shulman, 1986) through both formal and informal collaborative experiences.

Methods

This study explores current professional experiences with and for RTMTs that address the redistribution of teacher development resources through investigation of (a) how RTMTs develop PCK through formal and informal collaborative experiences and (b) how RTMTs are supported in developing their PCK. We use a collective case study with an embedded multiple case design (Yin, 2017) to investigate RTMTs professional experiences and the development of their PCK. Participating RTMTs serve as embedded units within each of the cases, the partner schools. Data were collected from RTMTs through questionnaires as well as individual and group interviews. We used descriptive and provisional coding methods for the first cycle of data analysis (Saldaña, 2016).

Summary of Findings

Preliminary findings suggest that RTMTs' develop their PCK through frequent and informal collaboration with their school-based colleagues resulting in a replication of pedagogical practices (Pryor et al., 2012). RTMTs are participating in formal collaborative conversations in bi-annual panel discussions composed of RTMTs in the district in order to overcome challenges in their mathematics classrooms. While national trends in Tanzania's teacher credential levels reveal that RTMTs generally hold "lower-grade" instructional certifications (Komba & Nkumbi, 2008), all of the teachers we interviewed held at least a diploma from a two-year teacher's college, with over half of all interviewed RTMTs holding a bachelor's or master's degree in education. This project highlights openings to work with RTMTs to develop ongoing professional experiences that can continue to improve upon instructional practices and further develop PCK amongst RTMTs.

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INVESTIGATING PRESERVICE TEACHERS' CONCEPTUALIZATIONS OF MATHEMATICAL KNOWLEDGE FOR TEACHING THROUGH ANALYSIS OF VIDEO LESSONS

Bima Sapkota
Purdue University
bsapkota@purdue.edu

Keywords: Instructional Activities and Practices, Mathematical Knowledge for Teaching, Teacher Education-Preservice, Video Analysis Activities

It has been suggested that mathematics pre-service teachers (M-PSTs) conceptualize their mathematical knowledge for teaching (MKT) (Ball, Thames, & Phelps, 2008) through various instructional activities, including video-lesson analysis; however, particular dimensions of video analysis activities are underexplored. Previous studies (e.g., Nilsson, 2008; Santagata, Zannoni, & Stigler, 2007) have indicated that discussion and reflection on the video-lessons were major instructional strategies used during the activities. In this study, I adopted a collective case study approach (Stake, 2003; Yin, 2017) to investigate how four M-PSTs, enrolled in a secondary mathematics methods course, conceptualized MKT when they analyzed a video lesson using the MKT framework, and discussed and reflected on the lesson. On the first and third days of the activities, the participants watched a video lesson from an eighth-grade classroom and offered their written reflections/analyses on the lesson. They analyzed their Day 1 reflection papers using the MKT framework on Day 2. After each day's written reflections/analyses, the participants discussed teacher and student actions from the video that seemed productive/unproductive for mathematics teaching and learning. In addition, the participants evaluated those actions using the MKT framework during the Day 2 and Day 3 discussions. Data analysis involved bottom-up and top-down interactive modes of analysis (Chi, 1997), which included identifying codes using descriptive and simultaneous coding methods (Saldaña, 2016), categorizing the codes using the MKT framework, and identifying themes.

The findings suggested that, in their initial (i.e., Day 1 and Day 2) analyses of the video lesson, the participants considered "content" and "pedagogy" as two separate and distinct bodies of teacher knowledge, resulting in their evaluation of teacher knowledge either as pedagogical or content knowledge. As such, M-PSTs divided classroom events into two categories: "what teachers do" vs "how teachers do." They then evaluated those events as *subject matter knowledge (SMK)* and *pedagogical content knowledge (PCK)* respectively. Furthermore, the participants evaluated SMK in a way that generalized it as *common content knowledge (CCK)*. They seemed to think that teachers' instructional decisions were not explicitly associated with their content knowledge. In their Day 3 analyses, however, the participants distinguished the notion of MKT from generic pedagogy and/or CCK and began to consider how teachers' professional content knowledge influences their instructional decisions. Since the M-PSTs reflected on and discussed the MKT-framework domains between their initial and second analyses, I argue that the changes in the participants' conceptions were due to these discussions and reflections. However, some participants still tended to evaluate SMK as CCK and PCK as general pedagogy after their participation in the discussions. Given the complexity of the framework, the short discussion sessions were not enough to reasonably conceptualize the MKT framework domains. I propose that had the participants engaged in additional discussion and reflection sessions, their understanding of MKT would have been further enhanced.

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Chapter 9:
Mathematical Processes

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

STEM WAYS OF THINKING

David Slavit	Elizabeth Grace	Kristin Lesseig
Wash. St. U. Vancouver	Wash. St. U. Vancouver	Wash. St. U. Vancouver
dslavit@wsu.edu	elizabeth.grace@wsu.edu	kristin.lesseig@wsu.edu

We explore the epistemological issues that arise when considering STEM as a curricular and instructional construct. Our approach is somewhat unique in that we are not focused on the curricular or instructional boundaries of STEM education, but consider the nature of the cognitive activity at play during STEM-focused activity, with an emphasis on mathematical thinking. We focus specifically on the epistemological underpinnings of mathematics and other STEM disciplines, and the possibility of an epistemology of STEM as a curricular construct. The implications on students' STEM ways of thinking (SWoT) are discussed in detail from a theoretical and empirical lens. Future research directions are identified.

Keywords: Advanced mathematical thinking; Learning theories; Reasoning and proof

Objectives

This paper explores various theoretical perspectives on what we are terming *STEM ways of thinking* (SWoT). We provide an extensive review of the literature on the perspectives taken on this topic, followed by a discussion of our own empirical work in this area. We are not focused on the nature of STEM itself, but on the ways in which students think about STEM in interdisciplinary curricular contexts. We focus specifically on SWoT as **argumentation in STEM contexts (argumentation)** and **thinking about STEM interdisciplinary concepts (explanation).** The roles of epistemology, content, schooling, and cognition are considered.

Theoretical Perspectives

Teachers' facilitation of classroom conversations that both model and provide opportunities for students to use reasoning and conceptual understanding is an ambitious practice receiving significant attention in teacher education (Ball, Sleep, Boerst, & Bass, 2009; Lampert & Graziani, 2009; Franke, Borko, & Whitcomb, 2008) as well as within STEM education policy and standards (CCSSI, 2010; NRC, 2001; NGSS, 2013). Argumentation or explanation, as well as other valued mathematical practices such as perseverance, modeling, and attending to precision (CCSSI 2010), can develop from conceptually-based conversations embedded in problem-solving contexts. In mathematics, problem-solving processes utilize representations (e.g., numbers, polygons, graphs) in support of arguments and explanations, but different representations and processes are often foregrounded in other STEM areas. For example, in science, student arguments usually derive from experimental evidence linked to scientific principles. In engineering, they usually derive from the testing, data analysis, and redesign efforts central to design-based project activities. The claims, evidence, reasoning (CER) framework is a widely-used perspective on scientific thinking related to argumentation (McNeill & Krajcik, 2012) that can be generalized to both mathematics and engineering (Figure 1).

In addition to differences in representations, processes, and expectations for the certainty of evidentiary claims, each STEM discipline also has its own theoretical frameworks for discussing content-specific ways of thinking in relation to the development of conceptual understanding (Wasserman & Rossi, 2015). For example, there are specific ways that children think about fractions, including as part-whole relations, ratios, and numbers (Lamon, 2012). Middle and high school students often initially conceive of functions as an output-producing action before developing a more object-oriented understanding grounded in domain-range correspondences and the properties possessed by function classes, such as lines and parabolas (Sfard, 1991; Slavit, 1997). There are numerous other frameworks in mathematics, science, and engineering that help describe students' ways of thinking about disciplinary concepts.

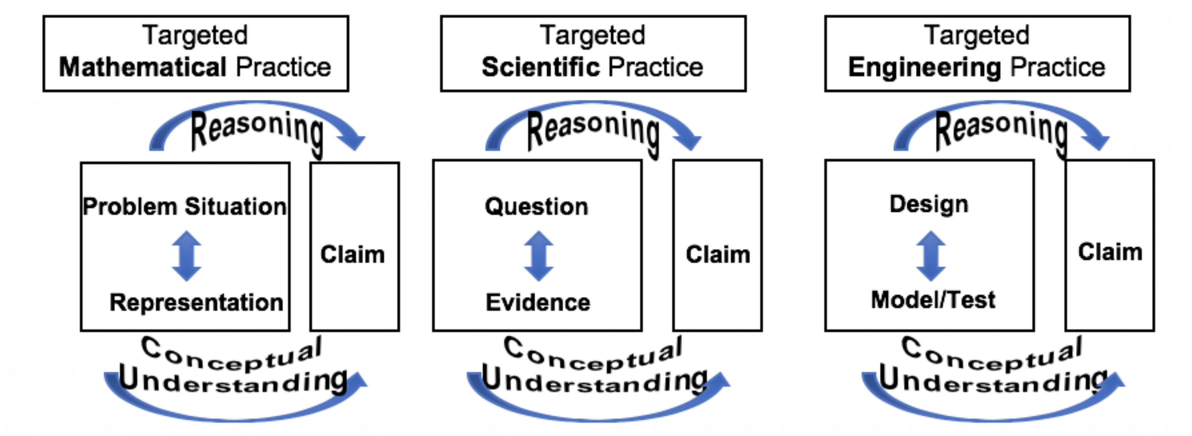

Figure 1: Disciplinary Ways of Thinking; Using Reasoning to Make Claims in Mathematics, Science, and Engineering

We are interested in the theoretical notion of SWoT and its particular relationship to mathematical ways of thinking. This theoretical endeavor has three immediate challenges. First, there are virtually no current frameworks on which to draw. Second, there are differences in the epistemologies of each individual STEM discipline, making the construction of an epistemology of STEM difficult to conceive. Herschbach (2011), one of the few scholars who have addressed this issue, states the following:

> The four STEM fields, in sum, have epistemological characteristics that differ markedly. These characteristics must be fully recognized and accommodated in programming in order to preserve the intellectual integrity of each field. Otherwise a very limited understanding results that undervalues specific intellectual contributions or ignores the collective value of each. (p. 110)

Third, STEM is not a well-defined term, making the construction of a theory related to SWoT problematic. For example, Holmlund, Lesseig, and Slavit (2018) illustrated the variety of perceptions of STEM education held by content teachers, administrators, and policy makers. However, we hypothesize that STEM-focused instruction could support the development of cognitive processes consistent with our view of SWoT, delineated further in this paper. This hypothesis includes the development of the discipline-focused ways of thinking discussed above, including mathematical thinking. However, it might also include a way of thinking specific to STEM when manifested in interdisciplinary contexts.

Our theoretical approach to SWoT consists of two related but different cognitive processes: **argumentation in STEM contexts**, and **thinking about STEM interdisciplinary concepts**. These types of thinking are not independent, but not necessarily interdependent.

Because science, mathematics, and engineering are grounded in their own epistemologies, ontologies, and practices, it is an open question as to whether such cross-disciplinary SWoT can actually exist. Can STEM be a discipline in and of itself with its own ways of thinking? If so, what are the ways of thinking that define it? This paper will explore the theoretical issues related to the above questions. We begin with a discussion of five distinct ways of considering SWoT found in the current literature. These include relationships between SWoT and 1) learning theory, 2) 21st-century skills, 3) disciplinary lenses, 4) curricular foci, and 5) epistemology.

Learning Theories Related to SWoT

Some researchers have explored the notion of SWoT by relating it to specific theories of learning. Asunda (2014) presented a conceptual framework for attaining STEM integration based on principles of pragmatism that drew from four different theoretical underpinnings: systems thinking, situated learning theory, constructivism, and goal orientation theory. Denick and colleagues (2013) emphasized social learning theory and discourse, as SWoT and integrated thinking require understanding concepts from multiple perspectives. Kelley and Knowles (2016) presented a conceptual framework grounded in situated cognition that foregrounded SWoT in the context of engineering design. They suggest that both students and practitioners engage in communities of practice to integrate and develop interdisciplinary thinking, including engagement with mathematicians and other STEM experts. Because of the common use of interdisciplinary, real-world projects in STEM education, it seems natural that the theories of learning most commonly related to SWoT involve discourse and draw from a situated or community of practice perspective.

SWoT as 21st-century Skills Applied in Real-world Contexts

From this perspective, SWoT is more than thinking across the four disciplines of STEM. Instead, primary importance is given to the presence of 21st-century skills, such as inquiry processes, problem-solving, critical thinking, creativity, and innovation (English, 2016). While content-based thinking and knowledge are important, these are secondary in focus to the above-mentioned process skills. Hence, SWoT can bridge disciplinary thinking by deemphasizing content-isolating topics, and emphasizing more general thinking processes. Chalmers et al. (2017) suggest that STEM-encompassing endeavors (or “grand challenges”) promote the exploration and transfer of “big ideas” across disciplines, using these broader ways of thinking. The encompassing big ideas are important problems that can invoke a particular SWoT, which students need to solve these problems. From this perspective, STEM is more than the sum of its parts and fostering 21st-century skills is fundamental.

While also emphasizing the influence of curriculum on SWoT, Chalmers et al. (2017) proposed that students need to be inducted into STEM “habits of mind” that will promote application of STEM ideas. They presented three types of “big ideas” that can facilitate in-depth STEM learning: within-discipline ideas, cross-discipline ideas, and encompassing big ideas. Within-discipline ideas are those primarily found in one STEM discipline that have application in another (e.g., scale, ratio, proportion, energy). Cross-discipline ideas are represented by content or processes found in two or more STEM disciplines (e.g., variables, patterns, models, computational thinking, reasoning and argument, etc.). The final type, encompassing big ideas, can also be manifested by either concepts or content. Chalmers et al. claim that conceptual encompassing big ideas (e.g. representations, conservation, systems, coding, change) create interdisciplinary lenses across the STEM disciplines to be utilized on important, thematically-based problems that relate to global challenges.

SWoT Through a Disciplinary Lens

Some researchers argue a specific content area should be the framework for analyzing STEM thinking. Bennett and Ruchti (2014) suggested using the Standards for Mathematical Practices (SMPs) from the CCSSM to provide links across the four disciplines of STEM. This approach could serve two purposes; the SMPs would make the mathematical connections across STEM contexts clear and more accessible, and it would also help emphasize the role of practices in the development of content knowledge. Bennett and Ruchti mention that other standards (e.g., the NGSS Science and Engineering Practices) could also serve this role, but the mathematical connections would not be as clear and connected in such an approach. While English (2016) highlighted the need for a more balanced integration of STEM subjects, she suggested mathematics should be foregrounded to more explicitly emphasize this discipline.

A focus on other STEM disciplines is also present in the literature. Asunda (2014) suggested that the *Standards for Technological Literacy* provide an approach to integrate STEM, and that Career and Technical Education (CTE) be the platform for STEM integration. Sengupta, Dickes, and Farris (2018) argued that computational thinking serves as a disciplinary entrée into STEM due to its uncertain and complex nature. Many researchers, either implicitly or explicitly, assign engineering as the framework for SWoT, with mathematics and science as supporting roles (English & King, 2015). Denick et al. (2013) suggested that engineering design through model-eliciting activities supports an “integrated thinking” in STEM, specifically in informal environments. Kelley and Knowles’ (2016) conceptual framework, grounded in situated cognition, suggests that engineering is the context in which all four STEM disciplines can exist on an equal platform, and thus where STEM thinking best takes place. While focused on “learning by design,” Purzer et al. (2015) considered SWoT as “making knowledge-based decisions” using a combination of scientific inquiry and engineering design. They highlighted the similarities between engineering design and scientific inquiry that support this framework, including an emphasis on reasoning and the role of uncertainty as a starting point for SWoT.

SWoT as a Product of a Curricular Focus

Some researchers claim that interdisciplinary student thinking in STEM is dependent on how the teacher frames the STEM task, and the type of integration emphasized by the curriculum. There are several examples of this in the literature. In an engineering design context, English and King (2015) found that students were able to identify mathematics and science concepts, particularly in the latter stages of the design process (including redesign), but only 1/3 of the students did so, and after the teacher intervened and explicitly highlighted this content. Kelley and Knowles (2016) argued that students might not naturally know how to integrate content and thinking across disciplines. Like English and King, this suggests that SWoT are not natural cognitive processes for students, or perhaps the complexities of SWoT do not lend themselves to easy development. Mathematics can be more difficult to integrate than other STEM disciplines, as there is evidence that effect sizes of STEM integration on student achievement are lower when M is integrated than other integrated curricular models (e.g., S-T and E-S-T) (Becker and Park, 2011) Kelley and Knowles claim that students have difficulty knowing which ideas are relevant across disciplines when engaged in interdisciplinary content. For example, students need support to elicit relevant science and mathematics ideas in an engineering design task and to reorganize their thinking to be interdisciplinary, rather than a composite of different kinds of content-based thinking. However, current instructional norms do not always lend themselves to this way of thinking. Chalmers et al. (2017) applied their disciplinary framework, discussed above, to discuss how to introduce topics and develop contexts for SWoT. They suggest a sequence that begins with a *worthwhile task*, then an exploration of *big ideas*, followed by *a synthesizing activity* to provide closure and attempt to make the big ideas explicit knowledge objects of thought through reflection on and refinement of the students' understandings and SWoT.

Purzer et. al (2015) found that while engaging in design work, the focus of two students' thinking was primarily on engineering processes. However, near the end of the project, when the students were asked to examine the different affordances and constraints of their designs, the students began to engage in scientific inquiry, particularly when working within the explicit constraints of the problem. This suggests that the intentional introduction of specific constraints in an engineering design task could elicit more integration of science content knowledge and inquiry, which aligns with the claim of English and King (2015) that interdisciplinary thinking tends to take place in the latter stages of engineering design. The lack of presence of mathematical thinking is notable, and English (2016) called for more research on whether sequenced and structured mathematics instruction hinders in-depth STEM learning. It seems that mathematics is not normally perceived as the central content focus of STEM, and that mathematical thinking can be difficult to include in SWoT or need explicit scaffolding. This somewhat nebulous role of mathematics in the above findings surfaces several questions about the role of mathematics in SWoT. For example, what are the implications of Bennett and Ruchti's (2014) recommendations to make mathematics the overarching framework of STEM? Does the above evidence support the need to foreground mathematics when designing curriculum, or does it discourage educators from including mathematics in integration?

SWoT Through an Epistemological Lens

There are very few treatments of STEM in the context of its underlying epistemology that exist in the current literature base. Sengupta et al. (2018) argue that an "epistemology of computational thinking" can foreground the uncertainty and complexity that should exist in STEM classrooms. Their discussion of epistemology highlights the role of abstraction and representation in thinking, as well as the contexts that ground this thinking in use. They suggest that computational thinking can bring in other disciplinary ways of thinking and highlight modeling as a key potential tool for integration. Herschbach (2011) developed an "organization of knowledge" framework to argue for the incompatibility of epistemologies across the STEM content areas, and the resulting challenge this yields for STEM education. Treating STEM as a curricular concept best represented through activities, he distinguishes formal and applied knowledge, constructs he claims are inherent in STEM, to further argue for the difficulty in a coherent STEM epistemology.

Limitations of the Current Literature

While useful in a variety of theoretical and practical ways, the above theoretical perspectives and empirical findings have collective limitations. Perhaps most notably, not much research exists on STEM ways of thinking, with only a few researchers broaching the role of epistemology. Variations in defining STEM education can also create differences in STEM education models, and subsequent differences in conceptualizations of SWoT. This can lead to a variety of methodological issues related to how SWoT might be measured, and a lack of consistency across studies. The many different models for how STEM content should be integrated, or which content should be foregrounded, makes analyses of SWoT not applicable to all STEM scenarios. Current methods also have limitations; for example, English and King (2015) used annotations in students' notebooks and verbal explanations in their empirical analysis, but these might not accurately represent the students' actual SWoT. The existing research also tends to report small samples and effects (e.g., English and King, 2015; Becker and Park, 2011), and more research is needed on students from diverse groups (Purzer et al., 2015).

Many of the existing empirical studies target the effectiveness of STEM in increasing student learning outcomes, but do not address student thinking. Becker and Park (2011), in a meta-analysis of 28 studies that related the impact of STEM education on student achievement, found no study that explicitly highlighted student thinking. English (2016) has argued the need to better highlight connections among STEM disciplines, and make them more transparent to both teachers and students. Finally, we note that our literature search did not reveal interdisciplinary ways of thinking as a model of SWoT, with perhaps the best approximation found in the set of studies related to SWoT as 21st-century skills in interdisciplinary contexts. Thus, we see a distinct theoretical need for such a perspective.

Modes of Inquiry

Our basic research question is: What SWoT are evident in student activity and discourse during classroom activities involving STEM content and practices, and what role does mathematical thinking play in these cognitive processes? Our analysis is exploratory and qualitative in nature. Our primary data source is 69 videotaped segments of second grade (24), middle (35), and high school (10) classrooms during interdisciplinary, project-based instruction. The videos ranged from a few minutes to over one hour, with the majority lasting approximately 15 minutes. Our sampling was opportunistic, as we leveraged our relationships with local schools and teachers doing project-based instruction. Most episodes were of design-based engineering activity and included examples of interdisciplinary thinking grounded in argumentation. Our observations targeted the nature of the SWoT enacted by students during student-student interactions, although instructional segments and whole-group discussion were also included in the longer videotaped segments.

Two researchers (the first two authors) individually coded all videotaped segments, and met regularly to discuss coding decisions and results. Due to the exploratory nature of the analysis, open coding (Merriam & Grenier, 2019) was used. Three levels of analysis were progressively initiated, with codes based partly on the five ways of describing SWoT discussed above. First, we conducted a “meta” analysis to identify and describe the target content and the nature of collaboration that was occurring amongst the students. For example, a video segment might be coded as a design-based project that highlighted mathematical functions, with brief lecture followed by student group investigation. Codes related to the nature of the SWoT were then added, which were later refined through the use of analytic memos to include specific attention to the epistemological stances taken by the students, particularly when using reasoning to make claims (see Figure 1). Finally, codes related to interdisciplinary thinking were constructed to more fully articulate the SWoT being observed. For example, students who used geometric ideas and explanations to make a claim about the design of a lunchbox, and perhaps how this might help keep items inside cool, were noted as interdisciplinary across S-E-M. Students who modeled a buoyancy situation with a quadratic function to find its maximum were noted as superficially interdisciplinary, with an emphasis on mathematical skills.

Results

Although the observed content revolved mostly around engineering ideas, the instructional context shifted this focus on several occasions. For example, an exploration of water flow by a group of middle school students shifted from engineering principles to mathematical formulas related to force when the teacher inserted this focus into the discussion. A noted shift in the epistemological stance taken by the students was revealed in this episode, and the nature of claims and reasoning shifted from conceptual-based claims on aspects of force in a boating context to procedural-based claims related to force using abstract, mathematical formulas.

Our collective observations suggest that integration in the realm of SWoT consists of shifting disciplinary practices across STEM contexts. Student thinking that explored real-world problems and drew on 21st-century skills possessed such fluidity, but when the disciplinary content and practices became more explicit (we refer to this as being “hardened”), the different tools, practices, representations, and epistemologies within the STEM disciplines provided barriers for interdisciplinary thought. When the interdisciplinary ideas that were the subject of SWoT were not hardened, the individual epistemologies mattered less.

Several student groups were seen exploring STEM content that was accessible and perceived as “fun,” with a focus on successful task accomplishment. This led to a variety of ways of thinking about the STEM activity. Other times, the presence of a specific content objective, such as solving algebraic equations, grounded thinking and limited the nature of SWoT present. For example, a group of second-grade students exhibited interdisciplinary thinking when designing a lunch container with a cold bottle attached to the top. The SWoT observed in this action had aspects of mathematics (solids and volume, area and measurement), science (heat flow, energy transfer), and engineering design. But the S, E, and M emerged very informally, and the epistemologies of each discipline were not in conflict. We assert that the notion of SWoT is highly dependent on the nature of each discipline, the nature and intent of the activity, the instructional oversight, and the (internal and external) epistemological forces that drive student thought. We hypothesize that the existence of shifting practices across the STEM disciplines allows for more fluid SWoT. It also appeared that shifting practices occurred more frequently during exploratory phases of STEM activity, and less frequently when seeking solutions, as occurred in the case above when a mathematical formula was inserted into the SWoT.

Discussion

STEM as a curricular construct is usually considered through a disciplinary lens (single or interdisciplinary), or as a forum for the application of 21st-century skills in real-world problem contexts (Holmlund et al., 2018). STEM education consistently balances student curiosity and subsequent “natural” ideas with the need to explore predetermined academic content, a situation complicated by the competing epistemologies of the STEM disciplines. From a cognitive perspective, claims and reasoning tend to drive the activity (Figure 1), but the practices and ways of thinking often shift depending on the content and nature of the activity. Mathematical problem solving might be bypassed because of the curricular or instructional focus, or because of the current epistemology at play during student exploration. Prior research, such as English and King (2015), suggests that teacher scaffolding of mathematics content through the design process can allow for a more integrated way of thinking that aims to balance the representation of all STEM subjects. Specifically, students should be given sufficient time to complete the engineering design process so that the latter stages can be highlighted. One possible implication of this work is that students do not have natural SWoT, but rather that SWoT need to be facilitated in explicit ways. We claim the place of any content-specific cognitive activity, including mathematical thinking, is ill-defined in many STEM education contexts, and needs explicit nurturing.

Our observations suggest that when content becomes hardened, epistemological stances change and interdisciplinary thought becomes more difficult. We do not claim that there is a universal epistemology around STEM, or that the individual STEM disciplines are either universally compatible or incompatible. Our observations suggest that the nature of certain STEM activity, particularly that which does not involve hardened content and facilitates shifting practices across STEM disciplines, lends itself to more conceptual and fluid interdisciplinary thought, as epistemological barriers matter less.

This paper yields more questions than answers. When exploring STEM contexts, do students shift between disciplines, or think about them collectively? What is the implication of siloed content-based instruction, or interdisciplinary STEM instruction, on SWoT and/or mathematical thinking? What curricular and instructional adjustments might be necessary to promote SWoT or mathematical thinking in STEM contexts? Our limited observations indicate curriculum and instruction that limits the hardening of content can facilitate interdisciplinary SWoT. More research is needed to identify the nature, and even presence, of a SWoT and its relationship to mathematical thinking. Further delineation of SWoT can aid researchers' and mathematics educators' efforts to engage students in conceptually-based argumentation and explanation.

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MEASUREMENT AND DECOMPOSITION: MAKING SENSE OF THE AREA OF A CIRCLE

Dana C. Cox
Miami University
Dana.Cox@MiamiOH.edu

Jane-Jane Lo
Western Michigan University
Jane-Jane.Lo@wmich.edu

As a component of a course on geometry for preservice elementary teachers (PSTs), we derive area formulas for a variety of polygons including triangles, quadrilaterals, and both regular and irregular shapes whose areas can be measured empirically using decomposition. Decomposing a circle to justify why its area can be measured using the standard formula is more challenging as it requires both empirical and deductive reasoning involving limits. In spite of the challenge, we expected decomposition strategies to transcend work with polygons and support PSTs when thinking about the area of circles. Results show that few PSTs utilized decomposition and instead focused on finding meaning in the symbolism of the formula. Concept images related to area will be discussed.

Keywords: Geometry and Geometrical and Spatial Thinking; Mathematical Knowledge for Teaching; Teacher Education-Preservice; Reasoning and Proof

Background

As a component of a course on geometry for preservice elementary teachers (PSTs), we derive area formulas for a variety of polygons including triangles, quadrilaterals, and both regular and irregular shapes. Generalized formulas for finding the area of these types of shapes can be justified using the actions of composition and decomposition. For example, one way to derive a general formula for the area of a trapezoid is to decompose it into two triangles (with altitude equal to that of the original trapezoid and each taking one of the parallel sides as a base) and then summing the two areas.

Being able to give an informal derivation between the circumference and the area of a circle is a common middle-grades standard (CCSSO, 2010), yet the curved boundary lends challenge to students used to working with polygons. There are multiple justifications possible. Tent (2001) describes one method of figuring out the area of a circle through the action of composition and decomposition that were commonly used in the sixteenth and seventeenth centuries. A circle with radius, *r* is sliced into sectors, which are then rearranged to approximate a parallelogram with height equal to the radius and width equal to half of the original circumference. When the sectors are sliced infinitely thin, the parallelogram becomes a rectangle. The area of which, we deduce, can be measured as πr^2 *(See Fig. 1).*

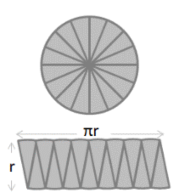


Figure 1: Decomposing a Circle to Reason About Its Area

We want to better understand how PSTs might transfer strategies for conceptualizing area in the context of polygons to the circle. At the end of our instructional unit on area of polygons, and before any instruction on circles, we gave a pre-assessment where we asked 69 PSTs to justify the standard formula for the area of a circle. Through the analysis of this set of written work, we sought to gain a better understanding of the claims and supports PSTs utilized when justifying why the standard formula for area of a circle, Area = $\pi(radius)^2$. This study sought to answer the question: how can we characterize PST's justifications of the general formula for finding the area of a circle?

Theoretical Perspective

Mathematics education research community knows very little about students' (PSTs included) conceptions of the area formula of the circle beyond memorizing and applying the formula to simple routine problems. In a recent review of research on the teaching and learning of measure (Smith and Barrett, 2017), discussion focused on area in the context of rectangular regions alone. Other than the general notion that area is an attribute that measures the amount of space inside the boundary of a 2-D shape and can be quantified by counting the number of area units, research studies focusing on rectangular regions or polygons provide limited insights into the challenges students face when trying to make sense of the area formula of the circle.

First of all, the boundary of a circle is curved instead of straight which makes the idea of tiling the space inside with square units seem impossible. Second, a justification of why the area can be measured using the standard formula, Area=$\pi(radius)^2$, requires an infinite number of subdivisions. The challenge emerges in our inability to physically decompose the shape (using common tools such as grid paper or scissors) into an infinite number of subshapes that can be used to quantify the amount of surface inside a circle. To do so conceptually requires a good understanding of the concept of limit through approximation metaphor (Oehrtman, 2004). Lastly, researchers have found that the idea of a non-repeating, non-ending number is very challenging for both preservice teachers and students (Fischbein, Jehiam, & Cohen, 1995; Güven, Çekmez & Karataş, 2011). So the presence of π in the area formula can pose an additional challenge for sense-making.

In addition to this analysis of the challenges in making sense of area formula of the circle, the design of this study was guided by prior work on proof and justification. Based on Stylianides's (2007) conception of proof and proving in elementary school, we take justification to mean a viable argument that supports a new claim with previously established definitions and facts as well as a connected sequence of assertions backed by by sound reasoning. We had worked with our PSTs' previously on justifying area formulas for various special polygons and we expected that some of them might even use diagrams to augment their argument.

Theoretical Framework

Harel and Sowder's (1998) proof schema were initially helpful in framing categories for the written responses. Specifically, we used *externally-based*, *empirical* and *analytical* proof schemes to organize the data. A written response is classified as an externally-based proof scheme when either 1) the argument is based on a textbook or other authoritative figure (authoritarian proof scheme), or 2) manipulation of symbols without meaning (symbolic proof scheme). The empirical proof scheme consists of arguments based on 1) multiple numerical examples (inductive proof scheme) or 2) rudimentary mental images (perceptual proof scheme), that is, 'images that consist of perceptions and a coordination of perceptions but lack the ability to transform or to anticipate the results of a transformation," (Harel & Sowder, 1998, p. 255). When an argument is based on logical deduction, it is said to use an analytical scheme.

Methods

Setting and Participants

This study was conducted in a Midwestern university. All participants (n=69) were enrolled in one of two sections of a course on geometry for preservice elementary teachers taught by the first author. The data used in this study were collected at the end of a unit on measurement in which significant work had been done toward a deep understanding of linear and area measurement. This work included activity related to the meaning of perimeter and area in the context of simple and composed polygons. Participants in this study had previously been asked to justify the area formulas for a variety of polygons including triangles, rectangles, parallelograms, and trapezoids using decomposition as a technique with tools such as dot and grid paper, tracing paper, and geoboards. At the time of data collection, participants had not engaged in course activity involving circles, including the measurements of radii, diameters, or circumferences nor the ratio of π.

Data Collection and Analysis

Students were asked to justify individually why the area of a circle could be found using the formula $Area = \pi r^2$. This formula was explicitly given, though accompanied by no additional guidelines or specifics. PSTs had been asked to justify or derive formulas for other shapes prior to this work, though they had not had any instruction regarding circles or the meaning of π. Their work was done by hand on blank white paper.

We used the proof schemes by Harel and Sowder (1998) to sort all written justifications into three piles: those used external authority based, empirical based and analytical based proof schemes. While some data fit clearly within the framework, we did not find it was useful in all cases. We needed the flexibility to group responses that lacked the coherence of an intentionally written proof or who blended schema from across the framework; also, the writing was informal and spontaneously generated. We then turned to grounded theory (Corbin & Strauss, 1990) to categorize responses in order to bring out themes that had been visible to us, but were not emerging through the framework. Specifically, we wanted to understand how PSTs were using what they had been learning about area and decomposition to justify the area of a circle.

Our process of categorization was to first read through a subset of the data to look for similarities and differences. Once an original set of categories were identified, we used an iterative process of reading through the data and refining the categories. In all cases, the entire response was used to garner meaning. In other words, we used the figures that were drawn as a means to interpret what was written and vice versa. Through this process, it became apparent to us that the drawn figures were being used as tools for different purposes and we will speak more to that in the discussion. When a response seemed to span two categories, we refined the categories when possible. However, when the category did not warrant subdivision, we placed each response according to the most sophisticated reasoning provided. For example, if a PST used a worked example, but also provided a justification for decomposing the circle into sectors, we chose to place it within the decomposition category.

Findings

We present five different categories of responses, grounded in the work of Harel and Sowder (1998). Second, we look across our categories to identify key conceptual challenges indicated by the responses. Reasoning about π and conflicting area concept images are visible within attempts at justification.

Characterizing PSTs' Written Justifications

Non-justification. We begin by noting that about 27.5% (n=19) of PSTs in this study made no attempt at justification. In a few cases PSTs defined *r* as the radius of a circle or noted that π was a number, but were not able to make use of that information relative to the formula. In other cases, PSTs simply wrote explicitly that they did not know or provided a description of a circle (i.e. infinite or round) that was not related to the formula, such as Malana, who wrote, *"the circle is shaped like a pie, so the formula works!"* These responses did not supply a viable argument to support new claim based on these definitions, so we did not categorize them as a justification (Stylianides, 2007).

Externally-based and empirical schema. We found 7 cases that fit the proof scheme framework by Harel & Sowder (1988). Six of them possessed the characteristics of an externally-based proof scheme by invoking authorities such as "mathematicians who discovered it" or simply reciting the formula from memory: "*multiply radius x itself and then multiply by π*". The remaining student submitted a true "proof by example" within the induction scheme.

The framework proved limited in analyzing the remaining 43 responses. While the majority of them (n=30) were motivated by a study of symbolism, there were clear attempts to go beyond and give meaning to those symbols. So, they did not quite fit under the big umbrella of externally-based schema. We created a new category for these which we called "Dimensional Analysis" (DA). This helped us capture one way PSTs attempted to give meaning to the area formula of a circle. The remaining 13 responses were classified either as Approximation Strategies or Decomposition Strategies. We will go on to describe each of our new categories illustrated by the responses that fit within them. Table 3 provides an orienting view of the entire data set according to these categories.

Table 1: Justifications by Category

Strategy	**Frequency**
Non-Justification	19
Externally-based	6
Empirical	1
Dimensional Analysis	
Diameter is r^2 *(only)*	10
Circumference (π) times diameter (r^2)	8
Other	12
Approximation	6
Decomposition	
Summation of Linear Measures	2
Summation of Equal-sized Sectors	5
Total	**69**

Dimensional analysis. In our study, 43.5% (n=30) of PSTs justified the area formula through a process of symbolic deconstruction aimed at associating meaning with the symbols and operations within the general formula. In most cases, this meant finding meaning in the isolated terms r, r^2, and π. While r was always defined as the radius of the circle (and often accompanied by a drawn and labeled figure), there was more variation in the ways that PSTs found meaning in r^2 and π.

In 10 cases, PSTs associated the square of the radius with measuring the diameter of the circle and then ended their justification without connecting that meaning back to the area of the circle. Within that group, some justified the squaring action on the basis of the relationship between the radius and the diameter, "*You square it because the radius is half of the diameter*" while others explicitly said that squaring was a doubling action, or indicated that r^2 was equivalent to the measurement of the diameter. Rhoda was the opposite, finding no meaning behind r^2, except to say that "*in order to go around the circle, it takes the r^2 3.14 times.*" Rhoda, like the others, was finding a way to associate the symbolism with measurement.

In 8 more cases, PSTs went one step further to include π in their sense-making, claiming that the formula worked because it multiplied that diameter (r^2) by the circumference (π). While that group of 8 is the largest cluster of similar meaning we could find, there were 12 other PSTs who also created symbolic combinations that focused on identifying measurements, but were either less specific in their meaning or unique in their interpretation. We present three examples of justifications from that category to highlight both the level of specificity and uniqueness of the responses:

> Brandon: "Because radius is half of a circle in order to get the full circle, you must square the radius. You then multiply by π because π is a measurement used in circles,"
>
> Bree: "Squaring the radius will get you the diameter of the circle going up and down and π will get you a measurement of the insides."
>
> Katie: "We do πr to find the distance around the entire circle. Then, you must multiply that number r again to account for all the area from the edge of the circle to the center of the circle."

Seneca, whose response is also included in this category, is the only one in our sample who explicitly mentioned that area was measured in square units, although her attention to those units was limited to a justification of why we square the radius and was diluted by a comparison to finding the area of a rectangle by multiplying length times width "*but just squaring the one length.*"

Approximation strategies. Six PSTs justified the area formula for a circle by comparing it to the area of a square made up by r^2 or $(2r)^2$. Four of them supported their arguments with a diagram of either a square with an inscribed circle or circumscribed square (Figure 2 shows both). The analysis of their diagrams indicated the struggle they had with the relationship between the square they had imagined and its relationship to the radius. This tension can be seen in Heather's written below (Figure 2).

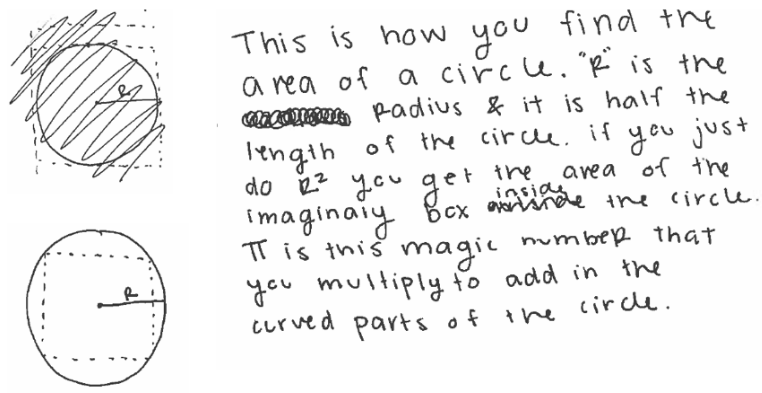


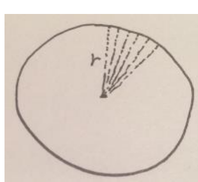
Figure 2: A PST's Attempted to Approximate the Area of a Circle with a Square

Five of these PSTs, including Heather, used π to justify the difference between the area of the circumscribed square and the area of the circle. Only one PST was able to successfully label the square with the circle inscribed (similar to the one crossed out by Heather) as $(2r)^2$, and acknowledged that the true area of the circle lied in-between those two approximations.

Decomposing strategies. Seven PSTs tried to make sense of the area formula for a circle by decomposing the circle in some fashion. Two of them imagined sweeping the region inside of the circle with a line segment (either *r* or *2r*) through *π*. For example, Chase wrote "*The radius is half the diameter of the circle, so you have to square the radius or r.*" (Note: On Hir paper, ze drew a diagram to support this). Ze went on to write, "*A circle is a symmetrical shape that is never ending. You multiply it by π to incorporate all of the degrees and angles of the circle."* At first glance, this justification seemed to have some validity, but further examination reveals a common misconception that area can be found by summing linear instead of area measures.

Another 5 PSTs decomposed the circle into equal-sized sectors. They made different associations between the decomposed shape and the formula as our class had done with polygons. Four struggled, but based reasoning on the belief that area was calculated by multiplying two length measures. For example, two PSTs (including a student named Audrey) made a distinction between the multiplied radii: they thought of one *r* as the length of the radius and the other as the number of sections, describing π as "*each of the sections in between the radii*", in another words, the arclength of the pie shape.

Only one PST's written justification (Figure 3) showed some glimpse of the idea of finding the area of circle through the concept of limit. The drawings and the accompanying explanation, though too brief to be classified as having an analytical proof scheme, showed evidence of finding the area of the circle by first decomposing a circle into many tiny sections and then summing the area of these tiny sections, which would resemble triangles.



I will cut a circle into a lot of pieces of triangles and get the area of one triangle and then add them together.

Figure 3: One PST's Decomposition Strategy to Justify the Area Formula of the Circle.

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

Discussion

This data was collected after instruction on justifying the area formulas for polygons, but prior to any instruction related to circles or π. As such, we should assume that these justifications are just quick snapshots or rough drafts and that PSTs ability to reason about the general formula and area in general would evolve and change with exposure to these concepts in class, or if this assignment had first been given to small groups to discuss.

While it was useful to utilize proof-based framework to categorize some responses, there were a number of responses that existed somewhere in between empirical and analytic schema. Those that we categorized into the new Dimensional Analysis schema were applying more analytic reasoning than empirical, though the resulting justification sought more to find meaning in this particular arrangement of symbols than in measurement concepts or imagery. In each of these 30 justifications, PSTs are searching among the symbols for cues as to which two dimensions would yield a 2-dimensional measurement. Becca sums it up when she says, *"This works because the radius squared is similar to bxh which is area of a parallelogram."*

Those we categorized within an approximating schema utilize empirical methods that are more solidly grounded by measurement concepts. However, as their arguments include some generality, they cannot be categorized under the empirical umbrella. Lastly, those that we categorized within the decomposition schema represent what we think of as pre-analytic arguments. While it can be useful to create sectors within the circle, finding the areas of individual sectors for the final sum creates a circular argument as these areas are often expressed as fractions of the whole circle. In order to make full use of the individual sectors, it's easier to rearrange them to form a new whole.

Conceptual Challenges

In the remaining space, we will identify two more conceptual challenges that were apparent when looking across the data. We will talk about concept imagery and the contradictions that created struggle. Then we explore in some depth the role π might have played in that struggle.

Concept imagery. While we did not expect PSTs to be able to fully carry out similar arguments, we were surprised to see only 7 of them even made an attempt to use the decompose/compose strategies they had learned when trying to justifying the area formula for polygons. Given that all of the participants had learned geometry in classrooms supported by the CCSS for mathematics, certainly they had previous knowledge of the geometry of circles. However, the curved boundary appeared to be a challenge for the PSTs in this study. Furthermore, only 3 PSTs acknowledged that the area of the circle could be calculated by summing up the area of an infinite number of smaller pieces in a way similar to the one by Chase discussed above. Looking across our data, we can see evidence of conflicting area concept images. In Seneca, we see her association with square units acting in contradiction to her reasoning about the curved boundary. She, like almost half of her colleagues, struggled to assimilate the formula with their image of area as the product of two one-dimensional measurements. We will share more of this in our presentation.

Meaning of π. Our analysis of PSTs' uses of π in their written justification also indicated their limited conception of this number. Just over half of our PSTs mentioned π in their written justifications. Of those that did mention it, some simply referred to it as a number attributed with important, yet indescribable power; "*an important ingredient of a circle*" or as Pat said, "*π is this magic number that you multiply to add in the curved parts of the circle.*"

Measurement was another theme in the way PSTs addressed π, whether in terms of length, area, or angle. It was common for PSTs to connect π with a measure of circumference. For example, Lynn wrote, "*π is the standard circumference of a circle with a radius of 1 unit*." Often PSTs who had this conception of π drew a diagram (like the one in Figure 4) to support their justification. Rose considered it an area measure when she said, "*π gives you area because the number is exactly how many times that radius can fit in the circle."* Just five PSTs thought of π as a ratio, either as a ratio between the circumference and radius or the circumference and diameter.

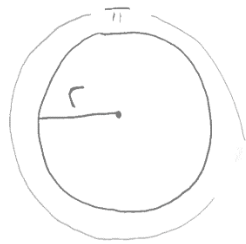


Figure 4: A PST's Diagram to Illustrate the Meaning of π and Radius

Implications

The fact that decomposition/recomposition did not factor more heavily into PSTs responses indicates that it is not enough to exclusively explore area concepts in the context of polygons. While concept imagery related to area such as *covering, iteration* and *decomposition* might be well-developed in the context of polygons, it seems necessary to carry over conceptual development to figures with curved boundaries including measuring circles with square units.

The variation that we saw in PSTs struggle to carve out dimensional measure from within the formula is of heightened interest to us. Jumping to an application-based instructional model (one that emphasizes plugging in values for π and r) earlier in their education might have increased our PSTs willingness to analyze each symbol separately and to overlook a more cohesive or conceptual justification based on existing images. It was clear that most were not cognizant of the relationship between circumference and area formulas, even if they might have recalled them on a different type of assessment. Moving away from plug-and-chug instruction and practice would better serve students of all ages.

There are some interesting concept images present in some of the approximation and decomposition justifications that might point to ways to extend the study of circles. First, approximating the area of the sectors using the radius and arc length might be an interesting activity. Second, using this context to make a connection to exponents and algebra might be a way to help students return to this concept to make additional sense outside the scope of an algebra lesson. At the very least, making the difference between doubling and squaring explicit in this context would raise awareness among PSTs about language and mathematical precision. Last, we were really compelled by those arguments that are based on accumulation. Specifically, the accumulation of length to create a measure of area. This reasoning could be developed further, specifically related to defining angle as a turn (Keiser, 2000) or as an early representation of radian measure.

One activity that models the sector strategy (Tent, 2001) is to have students use scissors to physically decompose and recompose the shape. However, Or (2012) suggests that a digital applet that uses sliders to guide the process (and incorporates far more precision) might help students bridge the epistemic gap between approximation and an exact value.

Setting Up Future Study

While the written justifications were the target for analysis in this study, we suspect that the nature of and ways in which PSTs use images or drawn figures will change after exposure to the types of activities mentioned in the previous section and that this would be an interesting line of inquiry for future studies.

A second area for future work will be to revisit and expand the proof schema to make room for the types of reasoning shown here. In particular, we think there is work to be done to expand the ways in which we reason about and with symbols. We believe that categorizing the ways in which people enact proof is something that merits more study and validation.

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COVARIATIONAL REASONING SUPPORTING PRESERVICE TEACHERS' MATHEMATIZATION OF AN ENERGY BALANCE MODEL FOR GLOBAL WARMING

Dario A. Gonzalez
Center for Mathematical Modeling, Universidad de Chile
dario.gonzalez@fulbrightmail.org

I discuss how three preservice mathematics teachers' (PSTs') covariational reasoning supported the mathematization of a simple energy balance model (EBM) for global warming, and how such mathematization shaped PSTs' understanding of the link CO2 pollution and global warming. I use Thompson & Carlson's (2017) levels of covariational reasoning and Thompson, Carlson, Byerley, and Hatfield (2014) descriptions of understanding and meaning to inform the discussion of results. The PST completed the EBM Task during an individual, task-based interview. The analysis revealed that Chunky Continuous Covariation level supports the mathematization of the EBM in terms of a covariation's rapidity of change. The analysis also revealed that particular mathematizations resulted in particular meanings for radiative equilibrium, which in turn have implications for understanding the link between CO2 pollution and global warming.

Keywords: Modeling, Advanced Mathematical Thinking, and Teacher Knowledge

Introduction and Purpose

Global warming refers to an increase in the mean global surface temperature caused by human emissions of greenhouse gases. The planetary scale of this phenomenon makes it difficult for a single person to experience it in its entirety. Mathematical modeling can make global warming visible for people (Barwell & Suurtamm, 2011; Barwell, 2013a, 2013b; Gonzalez, 2016, 2018; Mackenzie, 2007), thus helping them understand it and take action against this new horizon. I consider preservice mathematics teachers an important group to be informed about global warming because they will educate the members and future leaders of this democratic society. Thus, there is a need for studies examining how preservice mathematics teachers can learn the mathematics behind global warming.

Lambert and Bleicher (2013) have found that there are two key concepts from climate sciences that preservice science teachers need to learn about in order to understand global warming: (a) the Earth's energy balance, and (b) the link between carbon dioxide (CO2) pollution and global warming. Extending this premise to mathematics education, preservice mathematics teachers (PSTs) can model global warming by reasoning about these two concepts as dynamic situations involving covariation between quantities. In this paper, I discuss how three PSTs' covariational reasoning supported the mathematization of a simple energy balance model (EBM) for global warming, and how such mathematization shaped PSTs' understanding of the link CO2 pollution and global warming.

Conceptual Framework

Covariational reasoning refers to "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, p. 354). Thompson and Carlson (2017) have identified six distinctive levels of covariational reasoning (Table 1). Thompson and Carson suggested that "a researcher could use [the levels] to describe a class of behaviors, or she could use it as a characteristic of a person's capacity to reason … covariationally." For the current study, I used Thompson and Carlson's levels as a framework to characterize PSTs' covariational reasoning as they mathematize the EBM. Mathematization refers to the process of translating a real-life, non-mathematical situation into a mathematical problem, and then using mathematical tools and processes to solve it (Freudenthal, 1991).

I also made use of Thompson and colleagues' (Thompson, Carlson, Byerley, & Hatfield, 2014) descriptions of understanding, meaning, and way of thinking, in order to characterize PSTs' understandings of the EBM and meanings for radiative equilibrium. Thompson et al. suggests that a person's understanding is an in-the-moment state of equilibrium resulting from assimilating sensory information to the person's current schemes, or from accommodating those schemes to assimilate the new information. A person's meaning, then, is the space of implications that emerges from the assimilation to or accommodation of the person's current schemes. A person's way of thinking is that person's "pattern of utilizing specific meanings or ways of thinking in reasoning about particular situations" (Thompson et al., 2014, p. 12).

Table 1: Major Levels of Covariational Reasoning

Level	Description
Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables changing by intervals of a fixed size (not necessarily of the same size). The person imagines, for example, the variable's value varying from 0 to 1, from 1 to 2, from 2 to 3 (and so on), like laying a ruler. Values between 0 and 1, between 1 and 2, between 2 and 3, and so on, "come along" by virtue of each being part of a chunk—like numbers on a ruler—but the person does not envision that the quantity has these values in the same way it has 0, 1, 2, and so on, as values.
Coordination of values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.
Preccordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

Methodology

This paper is part of a larger study that investigated how PSTs make sense of introductory mathematical models for global warming. Three secondary PSTs —hereafter Jodi, Pam, and Kris— enrolled in a mathematics education program at a large Southeastern university participated in the larger study. These PSTs had completed Calculus I and II and an Intro to Higher Mathematics course and were completing a Math Modeling for Teachers course by the time the larger study took place. The PSTs were asked to complete an original sequence of mathematical tasks while participating in individual, task-based interviews (Goldin, 2000). In this paper, I focus on the PSTs' responses to the EBM task.

The Energy Balance Model (EBM) Task

An *energy balance model* (EBM) describes the continuous heat exchange between the sun, the planet's surface, and the atmosphere (Figure 1a). The sun warms up the planet's surface at an approximately constant rate S. As the surface heats up, it radiates heat to the atmosphere (R), the majority of which (B) is absorbed by *greenhouse gases* (GHG). The atmosphere then re-radiates a fraction of the absorbed heat back to the surface (A), further increasing its temperature. The continuous heat exchange between the surface and the atmosphere is known as the *greenhouse effect*, which is responsible for enhancing the *planet's mean surface temperature* T(t). Changes in the greenhouse effect result in changes in T(t). Let $N(t) = [S + A(t)] - R(t)$ be the *net planetary energy imbalance*, then it is said that the energy balance is in radiative equilibrium when $N(t) = 0$, which implies that T(t) remains constant. There are *forcing agents* that can push the energy balance out of radiative equilibrium, resulting in $N(t) \neq 0$. My study focuses on modeling the impact that an increase in the atmospheric CO_2 concentration has over the Earth's energy balance, and how such impact affects the planet's mean surface temperature. CO_2 pollution is one of the main drivers of global warming (Intergovernmental Panel on Climate Change [IPCC], 2013).

The Energy Balance Model (EBM) Task (Figure 1b) describes a simplified scenario with a unique, instantaneous increase in the *Atmospheric CO_2 Concentration Function*, C(t), at time t = 0. This increase results in an initial positive heat imbalance $N(0) = [S + A(0)] - R(0) > 0$. This initial imbalance is known as *positive forcing by CO_2* and is denoted by $F = N(0)$. The positive forcing results in a surface absorbing heat at a higher rate than that at which it is releasing it. Thus, the surface warms up as time passes, which causes it to radiate heat at an increasing rate R(t). The atmosphere then absorbs more heat from the surface, which causes it to radiate heat back to the surface at an increasing rate A(t). This feedback process continues until radiative equilibrium is restored so that $N(t) \to 0$ and $T(t) \to T_{NE}$ as time increases, where T_{NE} represents a new and higher equilibrium temperature.

The EBM Task thus required PSTs to reasoning about the above process and draw the graphs of the functions N(t) and T(t). The EBM Task has two prompts: (a) Determine how N vary over time t (in years) and sketch the graph of N(t) and (b) Determine how T vary over time t (in years) and sketch the graph of T(t).

Data Collection

PSTs watched a 7-minute long video introducing the concept of EBM, followed by a Q&A session with me. I next gave them the EBM task. PSTs were also given a diagram of the EBM showing initial values for S, R, B, and A. They were expected to sketch the graphs of N(t) and T(t) assisted by the recursive rules $B_i = 0.794 \cdot R_i$, $A_i = \frac{1}{2} \cdot B_i$, $R_{i+1} = S + A_i$, and $N_i = [S + A_i] - R_i$ (for *i* = 0, 1, 2, …). The rules were meant to give PSTs a sense of how the heat flows change after a positive forcing. Each PST completed the task in a 60-minute, semi-structured, task-based interview (Goldin, 2000). The interview was video recorded and transcribed for analysis. PSTs' work on paper was also collected for analysis.

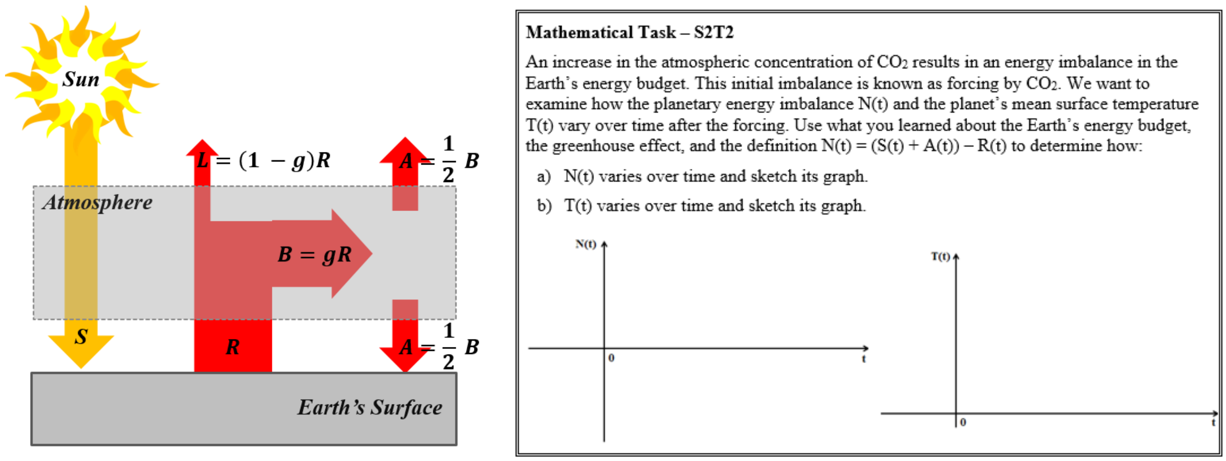

Figure 1: (a) The Earth's Energy Balance (Left) and (b) The EBM Task (Right)

Data Analysis

Videos and transcripts were analyzed through Framework Analysis (FA) method (Ward, Furber, Tierney, & Swallow, 2013). FA has five stages of data analysis: the familiarization with the data, the development of the analytic framework, indexing and pilot charting of the data, summarizing data into the analytic matrix, and the synthetization of the data by mapping and interpreting. Through these stages the research develops an analytic framework and uses it to analyze, reduce, and index data into analytic matrices, FA's distinctive feature.

I watched all interview videos and took notes while doing so. The videos were separated into shorter, more manageable *episodes*. An *episode* showed evidence of PSTs' ways of understanding the EBM or ways of reasoning about covariation. The notes informed my first round of coding. Then, the episodes were sorted according to particular ways of understanding the Earth's energy balance. Looking for patterns in participants' responses, I developed six *energy balance (EB) codes*. I repeated the process with PSTs' ways of envisioning covariation, which resulted in four *covariational reasoning (CR) codes*. These codes represented the *Analytic Framework* for the study.

Using the analytic framework, I indexed all episodes into three analytic matrices, one per participant. I looked for patterns in the distribution of CR codes in relation to EB codes across all three matrices, examining ways in which covariational reasoning supported PSTs' mathematization of the EBM. Then, I compared EB codes across participants in order to identify particular ways of thinking about the EBM. The analysis of such patterns provided the information needed to meet the research goals.

Results

The analysis of PSTs' responses revealed that coordination of values and coordination of change are key to mathematize the EBM for global warming. The analysis also revealed that particular mathematizations resulted in particular meanings for radiative equilibrium.

Covariational Reasoning and Mathematizing the EBM

Coordinating values represented a key step in mathematizing the notion that the energy balance restores radiative equilibrium over time. Coordinating values allowed PSTs to translate radiative equilibrium into a quantity, N(t), that decreases as time increases (direction of change). For instance, Jodi's initial understanding of the EBM did not include an energy balance restoring radiative equilibrium after a positive forcing. For her, the increase in C (atmospheric CO_2 concentration) *pushed* the energy balance out of its *normal* state, and it would remain out of that *normal* state unless C decreased to its original value, which is reflected in the way she described change in N(t) ("[N] wouldn't increase or decrease if CO_2 is kept stable"). Coordinating t-values and N-values using the given recursive rules allowed her to create and plot a collection of pairs (t, N) with which she drew a accurate graph for N(t) (Figure 2a). She interpreted the graph as follows "[the graph means] that we are going back to an equilibrium, or we are not as far from equilibrium as we were." While looking at her graph, she added "each time we are increasing t, we are decreasing N by smaller amounts." Jodi's understanding of the EBM extended to include: (a) the idea of radiative equilibrium being restored over time, and (b) a mathematized representation of such process in terms of a covariation, N(t), that decreases (direction of change) by decreasing amounts of change (rapidity of change) as t increases. Describing rapidity of change of N(t), I would argue, reveals a more sophisticated mathematization of radiative equilibrium than describing direction of change alone; it represents a higher degree of complexity in understanding and describing a covariation.

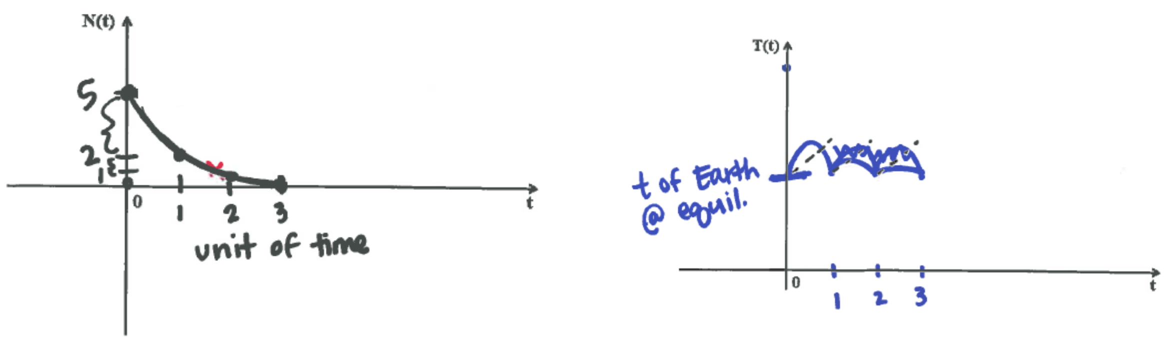

Figure 2: (a) Jodi's Graph of N(t) (Left) and (b) Jodi's Graph of T(t) (Right)

Pam's initial understanding of the EBM included the idea of radiative equilibrium being restored over time; she correctly anticipated N(t) to be a decreasing function of time. Her mathematization of radiative equilibrium, however, was limited to indicating direction of change alone ("[N] was five, and then it would decrease to be zero again") and did not support drawing a graph for N(t). After coordinating t-values and N-values using the given recursive rules, Pam noticed that:

> [N] decreased pretty quickly, like 3 units of J/s/m^2. Then, it decreased by about 1 J/s/m^2 … I am assuming [N] is going to decrease by a little bit, and a little bit, and a little bit, until it reaches zero again.

Coordinating values allowed Pam to think about changes in t in relation to changes in N. This, in turn, helped her draw an accurate graph for N(t) (Figure 3a). Pam's mathematization of radiative equilibrium extended from describing *direction of change* alone ("[N] was 5, and then it would decrease to be zero again") to describing *rapidity of change* ("[N] is going to decrease by a little bit, and a little bit, and a little bit, until it reaches zero again"). Pam reasoned about rapidity of change by coordinating two sequences indicating change in N: the sequence of values of N and the sequence of amounts of change $\Delta_i N$. The coordination of these two sequences allowed her to mathematize radiative equilibrium as a covariation, N(t), that decreases by decreasing amounts of change as t increases.

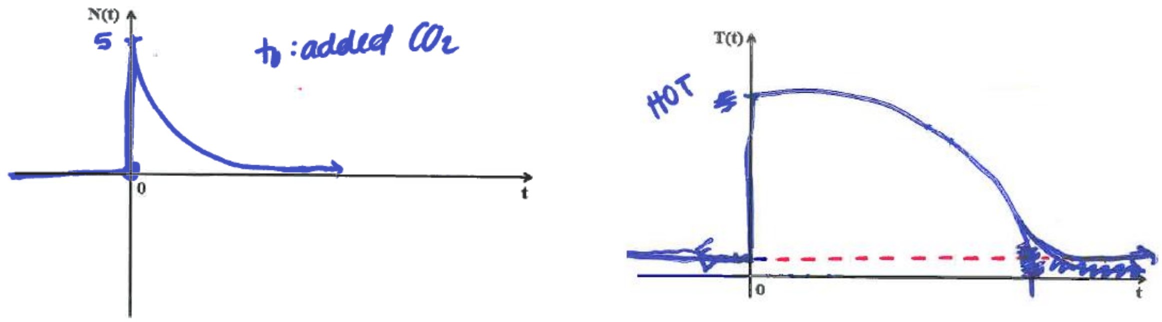

Figure 3. (a) Pam's Graph of N(t) (Left) and (b) Pam's Graph of T(t) (Right)

Kris's initial understanding of the EBM included the idea of radiative equilibrium being restored over time. She, however, struggled to see that idea reflected in the covariation N(t). Since the heat flows R and A were increasing as t increased, Kris thought that N(t) would be increasing too given that $N(t) = [S + A(t)] - R(t)$. She, however, expressed surprise about her conclusion "So, as R increases, A increases [*points at R and A in* $N = (S + A) - R$] … [N] can't just keep increasing!" Coordinating values by using the given recursive rules helped Kris clarify her confusion. She noticed that the heat flow B(t) was increasing by decreasing amounts of change

> Well, this difference right here, between 320 and 328 ($\Delta_1 B = 328 - 320 = 8$), is greater than the difference between these two values, the 328 and 331.5 ($\Delta_2 B = 331.5 - 328 = 3.5$), and the difference between these two ($\Delta_3 B = 332.9 - 331.5 = 1.4$) is less than those [*points at 328 and 331.5*], which is less than that [*points at 320 and 328*]. That tells me that there is eventually going to be a limit … Yeah, it is going to reach a new equilibrium point somewhere

The coordination of values helped Kris reconcile (and mathematize) radiative equilibrium with (and in terms of) the covariation N(t). Her analysis of the rapidity of change of B(t) supported drawing an accurate graph for N(t) (Figure 4a). Like Pam, Kris's extended her mathematization of radiative equilibrium from direction of change to rapidity of change by coordinating two sequences indicating change in B: the sequence of values of B and the sequence of amounts of change $\Delta_i B$. The coordination of these two sequences allowed her to mathematize radiative equilibrium as a covariation, N(t), that decreases by decreasing amounts of change as t increases.

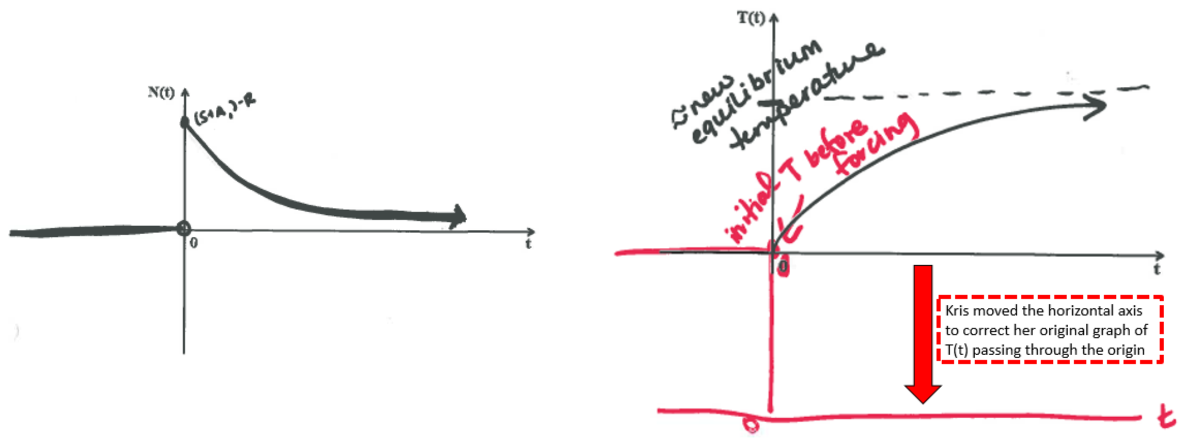

Figure 4. (a) Kris's Graph of N(t) (Left) and (b) Kris's Graph of T(t) (Right)

PSTs use the recursive rules to coordinate values of an independent variable *t* and a dependent variable *y*. Coordination allowed them to mathematize radiative equilibrium in terms of the direction of change of a covariation N(t). Coordination also made sequences of values and graphs available to PSTs. They used such objects to coordinate changes in *t* with changes in *y*. The coordination of changes, in addition to the coordination of values, helped them extend their mathematization of radiative equilibrium from direction of change (N(t) decreases as t increases) to rapidity of change (N(t) decreases by decreasing amounts of change as t increases). The later represents a higher level of complexity in describing covariation.

Mathematization and Understanding Global Warming

PSTs' ways of thinking about their mathematization of EBM resulted in two different meanings for radiative equilibrium: *Single Equilibrium Meaning* (SEM) and *Multiple Equilibrium Meaning* (MEM). The distinctive feature between the two meanings for radiative equilibrium was the PSTs' conception of *what is measured* by N(t) in the EBM.

I asked PSTs to interpret their graphs of N(t) in terms of whether the energy balance was losing heat (cooling down) or gaining heat (warming up) as time increased. Jodi and Pam concluded that the energy balance was losing heat or cooling down as time increased. Jodi stated that "the line (the graph of N(t)) is going in the negative direction, and we know that as N decreases, the surface is losing energy," while Pam stated that "we are losing because if we have gained energy [the surface] would get hotter, but it is not getting hotter because N is smaller so it's cooling off." It seems that Jodi and Pam arrived to such conclusion because they conceived N(t) as a measure of *how much heat* needs to be lost for the energy balance to *return* to radiative equilibrium (e.g., Jodi: "when the input is greater than the output, then we need to decrease N [*writes* $-N$ *on the right side of* $(S + A) = R$] so we can get back to equilibrium"). Jodi and Pam saw N(t) as an amount of excess heat in the energy balance. Therefore, if N(t) is decreasing, then the energy balance must be losing heat. Notice that seeing N(t) as excess heat also involves envisioning the energy balance *returning* to radiative equilibrium. I use the word *returning* to indicate that Jodi and Pam thought of radiative equilibrium as a unique state, the original state before the forcing by CO_2. For her, the energy balance is *returning* to its original equilibrium because the excess heat, caused by the initial forcing by CO_2, is decreasing. I called this meaning of radiative equilibrium Single Equilibrium Meaning (SEM).

SEM led Jodi and Pam to draw graphs of T(t) showing an overall decline in temperature as time increased (Figure 2b and Figure 3b, respectively). The particular shape of Jodi and Pam's graphs for T(t) were rooted in the way they reasoned about graphs and covariation, particularly in terms of the rapidity of change. A discussion of Jodi's case is presented elsewhere (Author, 2018). As for Pam, a discussion of the particular shape of her graph is beyond the scope of this paper. For now, I am drawing attention to the fact that both, Jodi and Pam, drew a graph of T(t) showing an overall decline in temperature. Their graphs indicate that the planet's surface warms up *solely* when C increases. Thus, if C stops changing, then the energy balance returns to its original radiative equilibrium and the planet's surface cools down as time increases.

When I asked Kris to interpret her graph of N(t) in terms of whether the energy balance was losing heat or gaining heat, she concluded that the energy balance was gaining heat.

> Well, [the surface] keeps in taking. I think it is warming up because once we added more CO_2, that is less of the emitted energy that is getting just like shut out passed the atmosphere, leaks from it. So then, more [radiation] is going to be absorbed by the atmosphere … Whatever is absorbed by the atmosphere [*points at B*] is going to be absorbed back into the [*points at the planet's surface*] … which is going to keep increasing.

Kris did not demonstrate any conflict between the energy balance gaining heat and N(t) decreasing as t increased. This suggests that Kris saw N(t) as another representation of heat gain in the EBM. A possible explanation is that Kris conceived N(t) as measuring *how much heat* needed to be gained for the energy balance to reach a *new* radiative equilibrium. When Kris drew the graph of the Planet's Mean Surface Temperature T(t) (Figure 4b), she wanted to show that T(t) was increasing but stabilizing at a certain value. Such graph implied that: (a) the energy balance gains heat to reach radiative equilibrium, and (b) radiative equilibrium is not unique and can occur at higher levels of heat. These are distinctive characteristics of a *Multiple Equilibrium Meaning* (MEM) for radiative equilibrium. This meaning includes attention to the increasing values of R, B, and A, and their implications in the context of the EBM. It also includes understanding the feedback of heat between the atmosphere and the surface: a fraction of the heat released by the surface is reabsorbed by it, enhancing its temperature.

Conclusion

I use Thompson and Carlson's (2017) levels of covariation to characterize PSTs' covariational reasoning. PSTs needed to reason about covariation at the Coordination of Values Level in order to mathematize radiative equilibrium in terms of *direction of change* of the Net Planetary Energy Imbalance N(t). Through coordinating values, PSTs can notice that N(t) decreases as time increases, which indicates that the Earth's energy balance is restoring radiative equilibrium. When PSTs coordinated changes $\Delta_i y$ with equal changes Δx, in addition to coordination of *y*-values with *x*-values, they were able to mathematize radiative equilibrium in terms of the *rapidity of change* of N(t). In other words, PSTs envisioned changes in t and N as occurring simultaneously and by intervals of fixed size. This suggests that covariational reasoning at the Chunky Continuous Level supports the mathematization of radiative equilibrium in terms of rapidity of change.

PSTs' mathematizations of the EBM resulted in two different meanings for radiative equilibrium; these meanings have different implications for understanding the link between CO_2 pollution and global warming. In particular, PSTs' conception of what is measured by N(t) led to two meanings for radiative equilibrium. SEM includes conceiving N(t) as an amount of *excess heat* that must be lost for the energy balance to *return* to radiative equilibrium. SEM also includes understanding radiative equilibrium as a unique state, the original equilibrium. SEM leads to the idea that the planet's surface cools down after a positive forcing by CO_2. This way of thinking about global warming is unproductive because CO_2 pollution has long-term and long-lasting effects on the planets mean surface temperature (IPCC, 2013). SEM may also open the door for misconceptions regarding global warming (e.g., "we can stop global warming at the moment we stop CO_2 emissions" or "as long as we maintain our current level of CO_2 emissions, the planet wouldn't get any hotter"). MEM includes conceiving N(t) as an amount of *heat to be gained* for the energy balance to *reach* radiative equilibrium. MEM also includes understanding that there are radiative equilibriums at higher levels of heat. MEM leads to the idea that the planet's surface warms up after a positive forcing by CO_2, even after the atmospheric CO_2 concentration is stabilized. This way of thinking about the link between CO_2 pollution and global warming is a productive one. It supports an understanding of the real impact of CO_2 emission in the climate, as well as their long-lasting effect in the planet's mean surface temperature.

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ACERCAMIENTOS GEOMÉTRICOS A PROBLEMAS VERBALES EN UN AMBIENTE DE RESOLUCIÓN DE PROBLEMAS CON GEOGEBRA

Adrián Gómez-Arciga
Cinvestav-IPN
agomeza@cinvestav.mx

Isaid Reyes-Martínez
CCH Vallejo, UNAM
isaid.reyes@chh.unam.mx

En este estudio se analizan los acercamientos que muestran estudiantes de bachillerato al resolver problemas verbales aritmético-algebraicos con el uso de un Sistema de Geometría Dinámica (SGD). Se analizan los recursos, representaciones, estrategias y formas de razonamiento matemático que exhiben cuando utilizan GeoGebra durante el proceso de resolución de los problemas. Los resultados muestran que el SGD favorece en los estudiantes diferentes formas de razonamiento y da la posibilidad de transitar de una representación geométrica a una representación algebraica.

Palabras clave: Resolución de Problemas, Álgebra y Pensamiento Algebraico, Tecnología

Introducción

La imagen tradicional que ha caracterizado el estudio del álgebra en el bachillerato enfatiza la manipulación de expresiones algebraicas, la resolución de ecuaciones lineales y cuadráticas, y la modelación algebraica de problemas verbales que contienen variables e incógnitas (Kaput, 2000; Kieran, 2014). Sin embargo, Kieran (2014) considera que el álgebra escolar no debe limitarse a estas actividades, sino también debe incluir procesos que permitan al estudiante tener la capacidad de describir relaciones y resolver problemas que involucren la construcción y uso de relaciones funcionales. En otras palabras, no es suficiente estudiar aspectos simbólicos en la actividad algebraica, sino también resulta importante que los estudiantes construyan formas de razonar asociadas con el pensamiento relacional que subyace en el razonamiento algebraico.

En este sentido, la agenda de investigación en esta área se ha enfocado hacia estudios donde los estudiantes construyan significados o le den sentido a los objetos y procesos algebraicos, de tal manera que se favorezca la comprensión conceptual y la resolución de problemas (Kieran, 2014). Al respecto, la NCTM (2010) hizo una propuesta, que incluye el uso de tecnología, que buscó darle sentido a los procesos algebraicos que llevan a cabo los estudiantes de bachillerato. Además, señaló algunas áreas que presentan retos: expresar generalidad con notación algebraica y con la notación de función, razonar sobre la pendiente, construir y usar funciones algebraicas, plantear ecuaciones apropiadas para resolver problemas verbales.

En un estudio realizado por Gómez-Arciga, Olvera-Martínez, Aguilar-Magallón, y Poveda (2018) con futuros profesores de matemáticas se observó, mediante la interpretación geométrica de problemas verbales con el uso de GeoGebra, cómo la herramienta permitió analizar el comportamiento de los objetos matemáticos involucrados, desarrollar diferentes formas de razonamiento, implementar nuevas estrategias y transitar de la representación geométrica a la algebraica. Se seleccionaron los problemas verbales porque, a pesar de estar incluidos en los planes de estudio desde nivel básico hasta superior, no se ha discutido lo suficiente sobre qué aspectos del razonamiento matemático construyen realmente los estudiantes durante su proceso de aprendizaje al trabajar este tipo de problemas (Verschaffel, Depaepe, & Van Dooren, 2014).

La pregunta de investigación que se planteó para este estudio fue: ¿cómo el uso de un Sistema de Geometría Dinámica (SGD), específicamente GeoGebra, favorece el desarrollo de recursos, estrategias y formas de razonamiento en los estudiantes de bachillerato cuando se enfrentan a resolver problemas verbales aritmético-algebraicos? Se documentan los diferentes acercamientos dinámicos que desarrollaron los estudiantes durante la resolución de problemas verbales y se identifican sus recursos, estrategias y formas de razonamiento.

Marco Conceptual

Álgebra escolar, resolución de problemas verbales y uso de tecnología digital

Uno de los objetivos que se busca en el álgebra escolar es que los estudiantes perciban al álgebra, entre otras cosas, como el estudio de las relaciones entre las cantidades (NCTM, 2010). Por ejemplo, si un estudiante se enfrenta a un problema verbal que involucre una ecuación algebraica del tipo $a = b/x$ en el proceso de su resolución, entonces el análisis puede orientarse hacia la comprensión y análisis de qué sucede con a si el valor de x se acerca a cero o, viceversa, si aumenta.

Las competencias que se han identificado como necesarias para resolver problemas matemáticos, en particular, problemas verbales, involucran: dominio de conocimiento conceptual y procedimental, métodos heurísticos o estrategias cognitivas, estrategias metacognitivas y actitudes positivas ante las tareas que se desarrollan (Schoenfeld, 1985, 1992; Verschaffel, Depaepe, & Van Dooren, 2014).

Las actividades o problemas se desarrollaron alrededor de episodios que incluían las fases de comprensión, exploración del problema, búsqueda de diversas maneras de resolver la tarea, y la visión retrospectiva del proceso de solución y extensión de la tarea (Santos-Trigo & Camacho-Machín, 2013). Los episodios son elementos esenciales de un marco que sustenta y promueve el uso de diversas tecnologías digitales en la resolución de problemas. La principal característica del marco es el uso de un SGD para construir modelos dinámicos de los problemas para explorar y buscar relaciones, así como generar conjeturas que puedan establecer las bases para una posterior búsqueda de los argumentos formales que las respaldan. Otro elemento del marco propuesto por Santos-Trigo y Camacho Machín fue la importancia de plantear preguntas durante todas las fases de resolución de problemas.

Metodología

Este estudio se llevó a cabo con un grupo de 25 estudiantes en un curso de álgebra de primer semestre de nivel bachillerato. Se trabajó durante siete semanas, dos sesiones de dos horas por semana. El desarrollo de las sesiones se realizó en un aula de cómputo donde cada estudiante tuvo acceso a una computadora con el SGD, GeoGebra, instalado. En las primeras cuatro sesiones, con la finalidad de que los estudiantes comenzaran a familiarizarse con GeoGebra, se trabajó en la construcción de una representación geométrica de un problema verbal (Problema 1). Cabe señalar, que dicha tarea fue guiada por el profesor y que el problema que se seleccionó era de nivel básico para darle énfasis al uso de la herramienta. Posteriormente, los estudiantes abordaron diversos problemas verbales aritmético-algebraicos como, por ejemplo, de porcentajes, de velocidades, de trabajo, de edades, que presentaban mayor grado de complejidad en el proceso de comprensión y donde el profesor se limitó a resolver dudas asociadas con la herramienta.

En este reporte se analizan los acercamientos que tuvieron los estudiantes a dos problemas verbales con el uso de GeoGebra. El primero fungió como problema introductorio y permitió sentar las bases para la resolución del segundo problema, el cual se implementó en las últimas sesiones.

Problema 1: Un barco navega a la velocidad de 45 km/h, ¿en cuánto tiempo recorrerá 180 km?

Problema 2: Un estanque se llena por una de dos llaves en 4 horas y la segunda lo llena en 6 horas, ¿cuánto tiempo tardarán en llenar el estanque vacío si se abren ambas llaves al mismo tiempo?

Cada problema se representó geométricamente en el sistema cartesiano para contrastar y analizar cómo se relacionan los conceptos y datos descritos en cada enunciado. Después de la exploración y análisis de cada modelo, se determinó la solución analítica y la respectiva ecuación algebraica.

Los datos se recolectaron a través de los reportes escritos, videograbaciones de las sesiones, archivos de GeoGebra con las construcciones dinámicas que elaboraron los estudiantes y las notas de campo de los investigadores.

Resultados

En esta sección se muestran los modelos geométricos planteados por los estudiantes y se discute sobre los recursos y estrategias que utilizaron. Se exhiben y analizan los resultados del Problema 1 (problema introductorio) y, posteriormente, se discute cómo estos resultados impactaron o influyeron en el desarrollo de un problema con mayor grado de complejidad. En el caso del Problema 1 se divide en dos momentos: el primero contempla la solución con el uso de papel y lápiz y el segundo la exploración hecha con GeoGebra. El primer momento permitió observar los recursos con que contaban los estudiantes, así como las estrategias que utilizaban. En el segundo momento, además de apropiarse de la herramienta, se mostró la forma en la que se puede transformar una actividad rutinaria en una no rutinaria con el uso de un SGD.

Problema 1

Solución con el uso de papel y lápiz

Antes de explorar el problema con el SGD, se les pidió a los estudiantes que resolvieran el Problema 1 con el uso de papel y lápiz con la intención de evaluar la comprensión del enunciado. Se observaron diferentes procedimientos que surgieron de identificar una relación de proporción directa en los datos proporcionados en el problema, en la Figura 1 se muestran los tres acercamientos que surgieron en esta etapa: plantear una regla de tres, construir una tabla y plantear una ecuación, cabe resaltar que todos los estudiantes lograron resolver el problema.

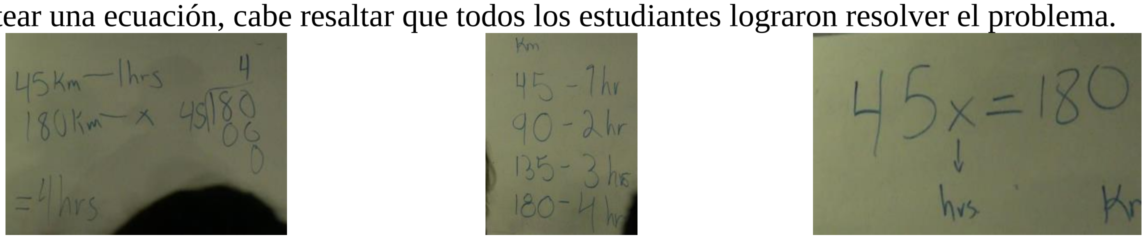


Figura 1: Resultados del problema 1 con el uso de papel y lápiz.

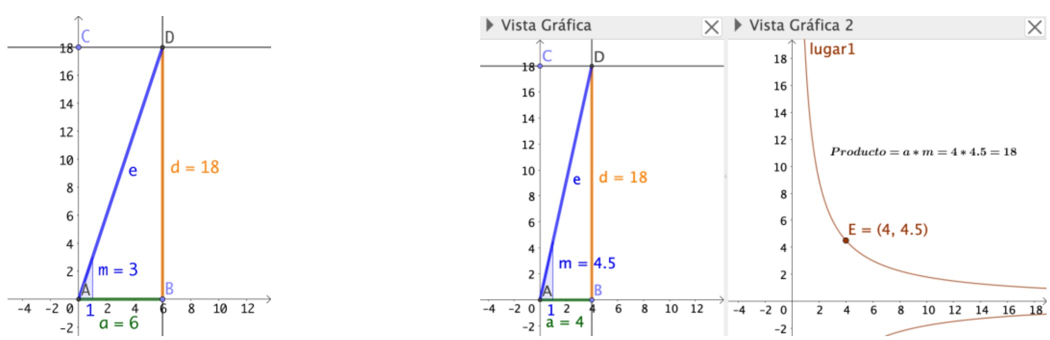
Solución con el uso de GeoGebra

A continuación, se muestra cómo se desarrollan los cuatro episodios propuestos por Santos-Trigo y Camacho-Machín (2013) cuando se incorpora un SGD en la resolución de problemas matemáticos destacando los recursos, estrategias y formas de razonamiento que surgen en este proceso.

Episodio de comprensión. ¿Cómo representar la información descrita en la situación problemática con el uso de un SGD? Buscar formas de representar el problema en un ambiente dinámico implica identificar los conceptos involucrados y asociarlos a objetos geométricos. En este sentido, se utilizó el plano cartesiano no solo para representar el problema geométricamente, sino también para explorar e identificar posibles relaciones entre los datos proporcionados.

La Figura 2a muestra el modelo dinámico que se construyó para explorar el problema. El *eje x* representa tiempo (medido en horas) y el *eje y* la distancia (medida en kilómetros) en escala de 1:10. Así, el *segmento a* representa el tiempo que viaja el barco, el *segmento d* la distancia total que recorrió (la cual se mantiene constante) y la pendiente del *segmento e* la velocidad a la que viaja el barco.

Episodio de exploración. ¿cómo se resuelve el problema utilizando el modelo dinámico? En el modelo es posible explorar la variación de la velocidad del barco en función de la posición del punto *B*, de esta manera, el problema se reduce a encontrar la posición de *B* que haga que la velocidad del barco sea de 45 km/h (Figura 2b). ¿Cómo cambia o varía la velocidad del barco al mover el punto *B*? En la exploración del modelo se puede observar que al mover el punto *B*, la velocidad del barco asociada a la *pendiente m*, cambia (al observar directamente el valor de *m* cuando *B* se mueve). Por lo tanto, tiene sentido investigar la variación de la *pendiente m*. Con la ayuda de GeoGebra se puede generar una representación gráfica de la variación de la pendiente. Es decir, se establece una relación entre el *segmento a* (tiempo del recorrido) y el valor de la *pendiente m* (velocidad del barco). Para explorar esta relación, se define el punto *E* con coordenadas *(longitud del segmento AB, valor de la pendiente correspondiente)*, es decir, *E* = (*a*, *m*) (Figura 2b). El punto *E* describe la relación entre el tiempo y la velocidad del barco y, por lo tanto, su análisis se hizo en una segunda vista gráfica, en un sistema cartesiano donde el eje *x* representa tiempo y el eje *y*, velocidad, ya que, el modelo dinámico del problema y la relación descrita por el punto *E* no pueden coexistir en el mismo plano. Se observa que cuando el punto *B* se mueve sobre el eje *x*, la posición del punto *E* cambia. ¿Cuál es el lugar geométrico descrito por el punto *E* cuando el punto *B* se mueve sobre el eje *x* y cómo se interpreta en el contexto del problema? La Figura 2b muestra que el lugar geométrico descrito por el punto *E* es una curva que indica que, si el tiempo aumenta, la velocidad disminuye o viceversa. Además, se observa que el producto de las coordenadas del punto *E* es constante e igual a 18. Esto permite concluir que la relación que hay entre el tiempo y la velocidad es inversamente proporcional. Pero esta interpretación solo es posible cuando el punto *B* se mueve en el lado positivo del eje *x* debido a las características del contexto del problema. La representación gráfica de la variación de la velocidad se obtiene sin tener de manera explícita un modelo algebraico.



a) En esta configuración se considera un viaje de 6 h con una velocidad de 30 km/h.

b) Modelo dinámico y visualización de la variación de la velocidad.

Figura 2: Representación dinámica de los conceptos involucrados en el problema

Episodio de búsqueda de distintos acercamientos a la solución. ¿cómo proponer una solución geométrica del problema a partir de la exploración del modelo dinámico? En la búsqueda de una solución geométrica del problema resulta importante analizar el lugar geométrico que describe el punto *E* de forma analítica. Es decir, se busca construir el modelo algebraico del lugar geométrico y encontrar su intersección con $y = 4.5$ (Figura 3). La solución del problema en el modelo se obtiene cuando $m = 4.5$. Por esa razón, la solución geométrica se encuentra en la intersección entre la recta $y = 4.5$ y la curva $y = 18/x$ (punto F).

Partiendo del punto $E = (a, m)$ se tiene que: $x= a$ y $y = m$.
Escribiendo a m en términos de a: $y = d/a$
El valor de d es constante e igual a 18, también se sustituye a a por x que es la incógnita del problema: $y = 18/x$
Por lo tanto $y = 18/x$ es el modelo algebraico del lugar geométrico que describe la variación de la velocidad.

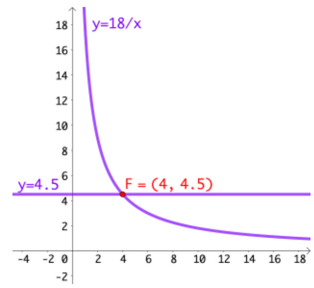


Figura 3: Modelo algebraico del lugar geométrico y solución geométrica del problema

Finalmente, la representación algebraica del problema, es decir, la ecuación que permite resolverlo resulta de igualar las funciones graficadas. Esto es, $4.5 = 18/x$.

Episodio de integración y reflexiones. ¿cuáles fueron las ideas principales en este proceso? Tanto el profesor como los estudiantes discutieron sobre el significado o la interpretación geométrica de los conceptos involucrados en el enunciado del problema. Por ejemplo, algunas preguntas que surgieron de este resultado fueron: ¿qué valores puede tener la pendiente *m* en relación con el segmento *a* y cómo se interpreta eso en el problema?, ¿cuál es el valor de *m* cuando la longitud del segmento *a* se reduce a 0 y cuando la longitud de *a* tiende a infinito?, ¿en dónde debe estar situado el punto B para que el modelo tenga sentido según el contexto del problema?

Conceptos como la velocidad y la proporcionalidad están conectados al concepto de pendiente y fueron esenciales para representar el problema mediante un modelo dinámico. El movimiento controlado del punto *B* permitió a los estudiantes identificar la relación entre el tiempo y la velocidad. El hecho de que GeoGebra no permita intersectar lugares geométricos, necesario para obtener la solución mostrada en la Figura 3, motivó la búsqueda de un modelo algebraico.

Problema 2

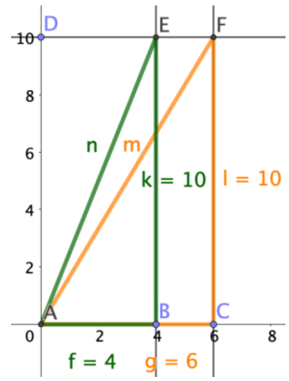
Solución con el uso de GeoGebra

En la Tabla 1 se presenta el desarrollo de los episodios en la resolución del problema 2. En este proceso, los estudiantes exhibieron dos modelos que exploran diferentes propiedades o conceptos y, en consecuencia, distintos acercamientos a la solución.

Tabla 1: Modelos dinámicos del problema 2

Episodio de comprensión

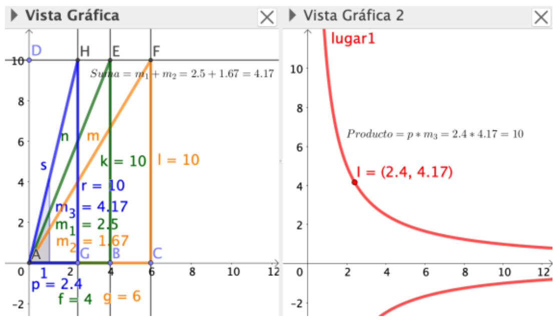
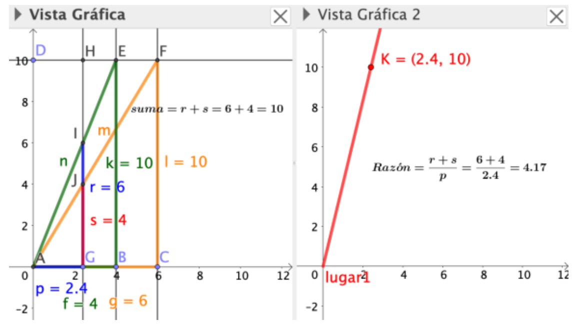
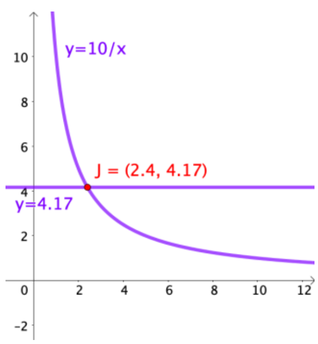
En ambos modelos se consideró el eje *x* como tiempo (medido en horas) y el eje *y* como volumen (medido en litros). ¿Cómo representar dinámicamente la relación entre la cantidad de agua acumulada en el estanque con el tiempo de llenado de cada llave? Los segmentos *f* y *g* representan el tiempo en el cual la primera y la segunda llave llenan el estanque y los segmentos *k* y *l* el volumen total del estanque, que está asociado a la posición del punto *D* (*D* es cualquier punto del eje *y*). Finalmente, se trazaron los segmentos *m* y *n*. En términos del problema, las pendientes de los segmentos *m* y *n* representan una razón de cambio constante entre el volumen y el tiempo, es decir, la velocidad de llenado. Asimismo, los puntos que se encuentran sobre estos segmentos proveen información sobre la cantidad de litros que ha llenado cada llave en un tiempo específico.

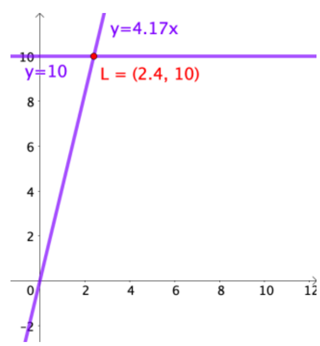


Ambos acercamientos fueron explorados por los estudiantes y se describen a continuación.

Episodio de exploración

Primer modelo: Pendientes de *m* y *n*	**Segundo modelo: Puntos sobre *m* y *n***
Para la exploración se construyó una tercera llave. El segmento *p* representa las horas que se ha abierto la llave, el segmento *r* la cantidad total de litros del estanque y la pendiente m_3 la velocidad con la que se llena el tanque en *p* horas. Esta construcción depende del punto *G* (punto con movimiento ordenado sobre el eje *x*). Cuando el punto *G* se mueve el valor de la pendiente m_3 cambia. La solución se alcanza cuando $m_3 = m_1 + m_2$ (m_1 y m_2 son las pendientes de *n* y *m*, respectivamente). En el contexto del problema, esto es, observar en qué momento la velocidad con la que llena la tercera llave es igual a la suma de las velocidades de las dos llaves descritas en el problema. En la imagen se observa que la solución se obtiene cuandoes $p = 2.4$, es decir, se necesitan 2.4 horas para llenar el estanque a una velocidad de 4.17 litros por hora. Para analizar la relación entre la longitud del segmento *p* (tiempo) y la pendiente m_3 (velocidad), se definió el punto I = (p, m_3) y se obtuvo su lugar geométrico en la vista gráfica 2. Nuevamente, se puede concluir	Si las dos llaves se abren al mismo tiempo ¿cuántos litros llenan del estanque en *x* horas? Para responder a esta pregunta, se define un punto *G* sobre el eje *x* y se traza una perpendicular al mismo eje que pase por *G*. Con esto, se definen el segmento *p* (tiempo), el segmento $GI=r$ (litros llenados por la primera llave en *p* horas) y el segmento $GJ=s$ (litros llenados por la segunda llave en *p* horas). Cuando el punto *G* se mueve sobre el eje *x* las longitudes de los segmentos *r* y *s* cambian. ¿Cómo se interpretan estos trazos en términos del problema? La suma de las longitudes de los segmentos *r* y *s* determina la cantidad de litros que han llenado las dos llaves en *p* horas. La solución se obtiene cuando $r+s = k= l$. En la imagen se observa que la solución es cuando $p = 2.4$, esto es, las dos llaves abiertas al mismo tiempo llenan el estanque en 2.4 horas. Se definió el punto $K = (a, r+s)$ en la vista gráfica 2 para analizar la relación entre la longitud del segmento *p* y la suma de las longitudes de los segmentos *r* y *s*. Se identificó que hay una proporción directa porque la razón de $(r+s)/p$ es constante.

<table>
<tr><td>que hay una relación de proporción inversa como en el problema 1.</td><td></td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td colspan="2">Episosdio de búsqueda de distintos acercamientos a la solución</td></tr>
<tr><td>Algebrización del lugar geométrico:
$I = (p, m_3)$ entonces $y = m_3$;
$y = r/p$; m_3 se escribió en términos de p;
$y = 10/x$; p es la incógnita y $r = 10$ (puede tomar cualquier valor según la posición del punto D).
Por lo tanto, se graficó $y = 10/x$.
La solución analítica se obtiene al intersectar la función $y = 10/x$ con la recta $y = m_1 + m_2$, que para la posición del punto D se obtiene la recta $y = 4.17$.


Solución algebraica:
Se obtiene de igualar las funciones $y = m_3$ y $y = m_1 + m_2$. Entonces,
$m_3 = m_1 + m_2$;
$r/x = k/4+l/6$ pero $r = k = l$, así
$r/x = r/4+r/6$ se divide todo entre r
la ecuación es:
$1/x = 1/4+1/6$</td><td>Algebrazación del lugar geométrico:
$K = (a, r+s)$ entonces $y = r+s$;
$y = (k/4)p+(l/6)p$; $r+s$ se escribió en términos de p;
$y = (10/4)x+(10/6)x$; p es la incógnita y $r = l = 10$ (puede tomar cualquier valor según la posición del punto D).
Por lo tanto, se graficó $y = (10/4)x+(10/6)x$.
La solución analítica se obtiene al intersectar la función $y = (10/4)x+(10/6)x$ con la recta $y = k$, que para la posición del punto D se obtiene la recta $y = 10$.


Solución algebraica:
Se obtiene de igualar las funciones
$y = (10/4)x+(10/6)x$ y $y = 10$. Entonces,
$(10/4)x+(10/6)x = 10$.
La ecuación simplificada es:
$(1/4)x+(1/6)x = 1$</td></tr>
<tr><td colspan="2">Episodio de integración y reflexiones</td></tr>
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Proporcionalidad, velocidad y razón de cambio, son conceptos que están conectados con el concepto de pendiente. Esto permitió representar el problema geométricamente y explorar dos modelos partiendo de analizar diferentes propiedades. La estrategia de visualizar el comportamiento de dos objetos del modelo a partir de un punto con movimiento controlado fue esencial para identificar las relaciones entre estos.

Discusión de los resultados

El Problema 1 es una tarea rutinaria que se resolvió aplicando operaciones aritméticas básicas cuando se trabajó en un ambiente de lápiz y papel. Introducir un SGD para resolver la misma tarea, permitió transformarla en una serie de actividades no rutinarias características del proceso de resolución de problemas. Fue decisivo seleccionar las herramientas que ofrece el SGD adecuadamente para que los estudiantes pudieran enfrentarse a la resolución de problemas verbales aritmético-algebraicos con el uso de éste. Así, el objetivo inicial fue representar los conceptos y la información descrita en el enunciado del problema de forma geométrica. Las estrategias que implican mover puntos de forma ordenada, cuantificar o medir atributos, definir relaciones de objetos mediante un punto y trazar su lugar geométrico, fueron importantes para explorar el modelo y obtener distintos acercamientos a la solución del problema. Un recurso importante además del concepto de pendiente, que permitió conectar con los conceptos de velocidad, razón de cambio y proporcionalidad, fue el de dominio del problema. Es decir, en qué intervalo del eje x tuvo sentido explorar las relaciones u objetos involucrados en el modelo del problema.

En el Problema 2 se observó que con el nivel de apropiación del SGD que tenían los estudiantes, así como los recursos y las estrategias con las que contaban, les permitió generar dos modelos dinámicos en donde se exploraron y analizaron conceptos matemáticos diferentes. En ambos se logró llegar a la solución del problema sin necesidad de plantear un acercamiento algebraico de manera inicial. También, en el episodio de exploración los estudiantes identificaron si la relación que se analizaba era una proporción directa o inversa a partir de visualizar el lugar geométrico. Sin embargo, esto exigió que argumentaran las conjeturas planteadas.

La construcción y exploración de los modelos dinámicos permitió que los estudiantes lograran representar algebraicamente el problema al dar seguimiento a los pasos de la construcción. De esta manera, los estudiantes pudieron interpretar y contextualizar las expresiones algebraicas, y dar sentido a diferentes objetos y conceptos matemáticos como la pendiente.

Conclusiones

El uso de un SGD en la resolución de problemas verbales aritmético-algebraicos ofrece la posibilidad de que los estudiantes busquen nuevas rutas para resolverlos. Por ejemplo, un enfoque novedoso fue representar los problemas con modelos dinámicos en GeoGebra ya que estas configuraciones se construyen con base en el análisis de las propiedades matemáticas de los objetos asociados a la tarea y exige que los estudiantes les asignen un sentido geométrico. Estos modelos permiten al estudiante identificar y explorar relaciones y patrones. Además, resulta fácil cuantificar o medir atributos como longitudes de segmentos, pendientes y observar cómo estos atributos cambian cuando ciertos objetos de la configuración se mueven. Un aspecto relevante es que las representaciones gráficas funcionales que se construyeron, en la vista gráfica 2, no requieren de una representación algebraica inicialmente y que con la ayuda del SGD se puede obtener y representar gráficamente en otro sistema de referencia independiente al asignado al modelo dinámico.

Buscar distintos acercamientos a la solución fue clave en el desarrollo de diferentes formas de razonamiento y favoreció el transitar de la representación geométrica a la representación algebraica y permitió a los estudiantes, transformar o extender los problemas verbales, dándoles la posibilidad de generar y conectar ideas matemáticas en un contexto geométrico dinámico.

De manera general, los rasgos que se muestran en los distintos episodios en la resolución de los problemas esbozan elementos de un marco inquisitivo donde se resalta la importancia de que los estudiantes problematicen su aprendizaje. Es decir, que formulen preguntas, que utilicen distintas representaciones, busquen conjeturas y relaciones y donde el uso de un SGD les ayude a explorar y sustentar con argumentos que van desde aspectos visuales y empíricos hasta acercamientos formales.

Agradecimientos

Se agradece el apoyo recibido del proyecto SEP-Cinvestav-12 durante el desarrollo de esta investigación.

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GEOMETRIC APPROACHES TO VERBAL PROBLEMS IN A GEOGEBRA PROBLEM SOLVING ENVIRONMENT

In this study we analyze the approaches that high school students show when solving arithmetic-algebraic verbal problems with the use of a Dynamic Geometry System (DGS). The resources, representations, strategies and forms of mathematical reasoning that they exhibit when they use GeoGebra during the process of problem solving are analyzed. The results show that the DGS favors in the student's different forms of reasoning and gives the possibility of transiting from a geometric representation to an algebraic representation.

LEARNING THE FOURIER TRANSFORM AND THE Z-TRANSFORM: UNDERSTANDING INSTRUMENTAL AND UNDERSTANDING RELATIONAL

Antonio Rivera-Figueroa
CINVESTAV, IPN, México
arivera@cinvestav.mx

Allan T. De la Cruz-Oliva
CINVESTAV, IPN, México
a_sternova@hotmail.com

The Fourier and Z-transforms are important topics in communications and electronics engineering careers. Given the difficulties implied by the mathematics involved in these transforms, we conducted a research with students from a Mexican university about their understanding they achieved once they had completed a course on these transforms. As a theoretical framework we take Richard Skemp's ideas on instrumental understanding and relational understanding. For the research a questionnaire was used for whose design a conceptual map was elaborated according to the ideas indicated and unstructured interviews were made to deepen on the conceptions of the students. The results suggest the need to attend in classroom the understanding of basic mathematical concepts as well as the nature of the mathematical objects involved in these transforms.

Keywords: Understanding Instrumental, Understanding Relational, Fourier Transform, Z-Transform.

Introduction

In this paper we present the results found on the students comprehension into the mathematics involved in the Fourier transform (FT) and Z-transform (ZT), quite fundamental in communications and electronics engineering career, whose construction is based on diverse concepts and mathematical objects and their correlations. These, when translated into a conceptual map with the mathematical contents as a whole, play an important role in the relational understanding of the FT and ZT, in the sense of how Skemp (1976) conceives. A characteristic of the relational understanding in a mathematical topic is the understanding of meaning and of the nature of mathematical objects involved. Following the Richard Skemp's ideas we focuse first on the construction of a conceptual map about FT and ZT, to then build the instrument used in this research and carry out the results analysis.

Problem Statement and Research Question

The Fourier series (FS), FT and ZT, are usually taught in university engineering careers in communications and electronics. This is the case of the careers offered by the Escuela Superior de Ingeniería Mecánica y Eléctrica (ESIME) of Instituto Politécnico Nacional in México city, among others. The FS, FT and ZT, have a certain degree of intrinsic mathematical complexity so, it explains somehow the learning difficulty, as shows the evaluations in the transforms subjects in ESIME. For the above, we decided to find out the level of understanding of the students in themes and basic mathematical concepts involved in FT and ZT. Here, the general question that we ask is:

- To what extent the students understand the fundamental concepts and the nature of the mathematical objects involved in the Fourier transform and Z-transform?

At this point, we formulated a second question pursuing to specify the mathematical contents:

- Which are the contents and the mathematical concepts involved in the Fourier transform and Z-transform?

Some of these contents are prerequisites for the student to be able to study the FT and ZT and some others are he learns during the study of these transformations, so acquiring new knowledge. After reviewing exhaustively, the related textbooks used by students and teachers, such as Hwei (1970) and Glyn (2002), we highlight the contents and concepts involved in the FT and ZT that we consider relevant, which we present in a conceptual map later on.

Instrumental Understanding and Relational Understanding

Skemp (1976) describes, in a simple and didactic way, what is meant by instrumental understanding and relational understanding, as follows. Suppose a person arrives at a town for the first time, and somebody teaches him how to move from a point X, where he lives, to a point Y in the town (for instance, his new job), with several indications such as ' you turn right in that corner', 'after the post office, you walk two streets and then turn left'. He repeats the round paths so many times that at the end he is certain of the routes. Similarly, he learns how to move from point X to point Z (as could be a branch of his job) and from Z to X, and in all of them he knows how to make the round trip and so on for another routes, all of them with initial and final fixed points. Later on, he explores other places by himself, with no a fixed destination, just to find an interesting place, but mainly to try to construct a cognitive city map. These two ways of routes are quite different, because in the first case the trial is to go from fixed initial point to a fixed final point with well-defined instructions that he learns properly, and he gets them easily. For the second case the objective is to expand and consolidate a mental city map, which is a stage of Skemp´s knowledge (1976).

A characteristic of the realization in this way of a route is the knowledge of what to do at each step. But, if any mistake, the person will surely be lost and will be in troubles to return to the right path. Also, a mental city map makes one better to create different starting and end points routes without becoming lost and, if a mistake, the right path will be recovered. From these ideas, we draw a conceptual map of either mathematical concepts, objects or subjects as a description of the fundamental elements and objects to their construction, including their relationships or connections between them, for the FT and ZT.

Methodological Considerations

First, from the answers to the second question, we analyze the mathematics involved just to create a conceptual map with the Skemp's ideas, in order to identify the associated relevant parts.

Secondly, that map is used to designed 10 questions with the main concepts, etc., for FT and ZT, and then asked to 20 students just finishing their courses on Transformations in the Communications and Electronics Engineering career at ESIME, with previous one semesters courses on differential and integral calculus, on elements of Linear Algebra and on Differential Equations. For the third semester they followed courses on Transformations and on elements of Complex Variable Functions.

The obtained information was supplemented by unstructured interviews to deep on the arguments of the students on which they support their answers. The analysis of four selected questions-answers is given in Results and Conclusions.

Conceptual Map of the Fourier and Z Transforms

Some of the mathematical knowledge that FT and ZT requires are a standard course of university calculation, which includes concept of function, elementary functions, limits, continuity, derivative and integral and properties. Further, an important concept here is the improper integral from 0 or $-\infty$ to $+\infty$, as well as a basic knowledge of complex variable functions, such as: derived from complex variable functions, contour integrals and developments in power series and Laurent series.

For the conceptual map on FT and ZT, the Fourier series is the starting point to construct the transforms. Figure 1 shows the most relevant aspects of the map, where boxes connected by arrows display objects, definitions, deductions or demonstrations. Intrinsic difficulties of some mathematical objects or relationships must be noted, which deserve special teaching attention, as for the integral Fourier formula and the Dirac delta function.

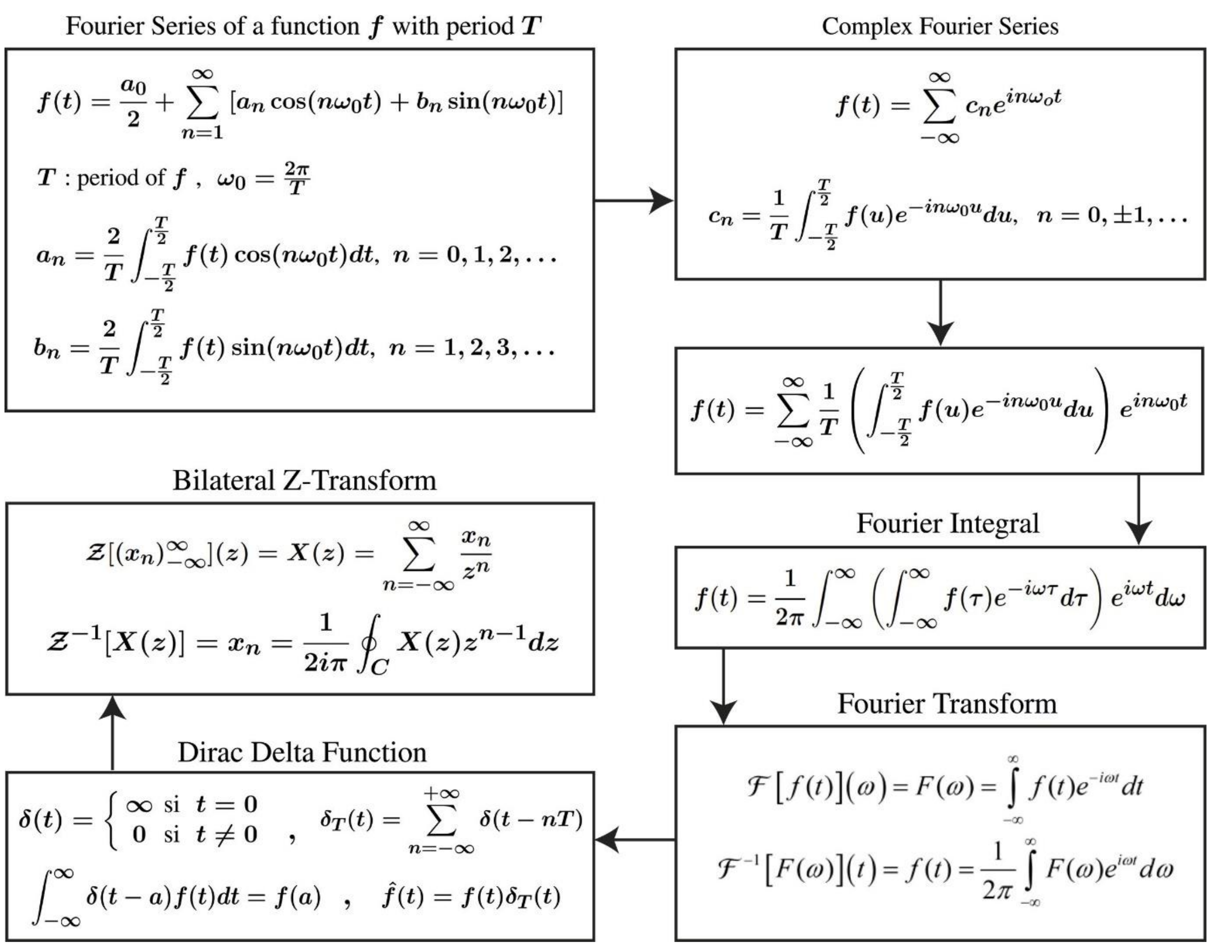

Figure 1: Diagram that Synthesizes the Conceptual Map of FT and ZT

Sometimes, as often happens in the ESIME, the formulas for the coefficients of the FS:

$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u)\cos(k\tfrac{2\pi}{T}u)du \quad \text{and} \quad b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u)\sin(k\tfrac{2\pi}{T}u)du$$

are given with no explanation and then applied to specific functions, being useful its deduction and explanation to the students that the partial sums

$$s_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n}\left[a_k\cos(k\tfrac{2\pi}{T}t) + b_k\sin(k\tfrac{2\pi}{T}t)\right]$$

play a very important role in FS theory, since from here one gets the infinite sum. FT is usually explained and established through the limit

$$f(t) = \lim_{T\to\infty}\sum_{n=-\infty}^{\infty}\left(\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u)e^{-in\omega_0 u}du\right)e^{in\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du\right)e^{i\omega t}d\omega \ . \quad (1)$$

This is a crucial moment on teaching of FT. The calculation of the previous limit lacks mathematical rigor, even though it is quite common in engineering mathematics books, see for example Hwei (1970, pp.72-73), its full demonstration demands knowledge of advanced mathematical analysis, see for example Rudin (1987, pp.180-181) and Beerends et al. (2003, pp. 164-186).

The following functional symbology strengthens the relational understanding of the students

$$f(t) \overset{\mathcal{F}}{\mapsto} F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt \ , \quad F(\omega) \overset{\mathcal{F}^{-1}}{\mapsto} f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\,d\omega \ .$$

In the reviewed textbooks, an approach to the ZT is to apply the FT to a particular function, here the Dirac delta function, introduced and used by Paul Dirac (1926, pp. 621-641) and Oliver Heaviside (1899, pp. 54-57), plays a very important role. Actually, the δ function is not a function as seen in classical mathematical analysis, where it is like a sophisticated object with a formal definition, but very useful in engineering (Lathi, 1998) and physics, where it works properly. Its definition through its properties allows us a proficient developing of the theory while teaching. The theory of Distributions created by Laurent Schwarz (1950) gives a rigorous treatment for this function, here a generalized function. Then the impulse train function is defined for any positive real T, $\delta_T(x) = \sum_{n=-\infty}^{+\infty}\delta(x - nT)$ (Hwei, 1970; Glyn, 2002), so $\delta_T(x) = 0$ outside of the points of the form $nT(n \in Z)$ and $\delta_T(x) = +\infty$ in these points. A weighted impulses train $\hat{f}$ is defined as $\hat{f}(x) = f(x)\delta_T(x)$, for $f: \mathbb{R} \to \mathbb{R}$ an arbitrary function, being a paradox that the functions $\hat{f}$ and δ_T take the same values, but the formal FT calculations give different results, which follows from the same condition of the integral for δ, which does not correspond to any rigorous definition of classical mathematical analysis. It must be recognized in the classroom that the calculations are carried out in a formal way, and thus avoid in the student a feeling of low self-esteem due to his lack of understanding of these mathematical objects.

The FT of $\hat{f}$ is defined by

$$\hat{F}(\omega) = \int_{-\infty}^{+\infty}\left(\sum_{n=-\infty}^{+\infty} f(t)\delta(t-nT)\right)e^{-i\omega t}dt = \sum_{n=-\infty}^{+\infty}\int_{-\infty}^{\infty} f(t)\delta(t-nT)e^{-i\omega t}dt = \sum_{n=-\infty}^{+\infty} f(nT)e^{-i\omega nT}.$$

or

$$\sum_{-\infty}^{+\infty} f(nT)e^{-i\omega nT} = \sum_{-\infty}^{+\infty} f(nT)z^{-n}\,, \qquad (2)$$

with $z = e^{i\omega T}$. If we think of $\hat{F}$ as a function of z and not of ω, the formula (2) is written as:

$$\hat{F}(z) = \sum_{n=-\infty}^{+\infty} f(nT)z^{-n}\,. \qquad (3)$$

Some authors of engineering mathematics books, after an independent variable change for a function of one or more variables, usually do not change notation for the new function, see for example Roberts (2004, pp. 762-763), Pipes (2014, pp.956-957), Kreyszig (2011, pp.392-394), Wylie (1966, pp. 550-551), Sokolnikoff (1958, pp.230-235), but they are however, conceptually different functions. For instance, the functions $\hat{F}$ of (2) and (3) are different, because they have different domain. The function $\hat{F}$ of (3) is the ZT of the succession $\left(f(nT)\right)_{n=-\infty}^{+\infty}$ and we write:

$$\mathcal{Z}\left[\left(f(nT)\right)_{n=-\infty}^{+\infty}\right](z) = \hat{F}(z) = \sum_{n=-\infty}^{+\infty} f(nT)z^{-n}\,.$$

In general, the ZT of a succession $(x_n)_{n=-\infty}^{+\infty}$ it is defined as the Laurent series

$$\mathcal{Z}\left[\left(x_n\right)_{n=-\infty}^{+\infty}\right](z) = X(z) = \sum_{n=-\infty}^{+\infty} x_n z^{-n} = \sum_{n=0}^{+\infty} x_{-n} z^{n} + \sum_{n=1}^{+\infty} x_n \frac{1}{z^n}\,. \qquad (4)$$

The inverse of the ZT assigns to each function $X(z)$ that allows a representation as in (4), the succession of coefficients x_n, and it is obtained by means of a contour integral on a circle C with center z = 0 and with adequate radius r. After multiplying both members of $X(z) = \sum_{-\infty}^{+\infty} x_n z^{-n}$ by z^{k-1}, and the line integration, we get

$$x_k = \frac{1}{2\pi i}\oint_C X(z)z^{k-1}\,dz \quad \text{for all } k \in \mathbb{Z}.$$

This formula is found in a first course of functions of complex variable when calculating the coefficients of a Laurent series. As for FT, the following notation helps to understand the nature of the ZT and its inverse

$$\left(x_n\right)_{-\infty}^{+\infty} \overset{\mathcal{Z}}{\mapsto} \mathcal{Z}\left[\left(x_n\right)\right](z) = \sum_{n=-\infty}^{+\infty} \frac{x_n}{z^n}, \quad X(z) \overset{\mathcal{Z}^{-1}}{\mapsto} x = \left(x_n\right)_{-\infty}^{+\infty}, \text{ where } x_n = \frac{1}{2\pi i}\oint_C X(z)z^{n-1}\,dz.$$

Questionnaire

This has 10 questions on the meaning or nature of the mathematical objects involved in FT and ZT. Here we present four of the questions. As noted earlier, to understand (1) it is required an advanced mathematical analysis tool, so the teacher must recognize and explain that what he presents is an empirical deduction; with the question three we try to realize if the teacher clarified this situation.

1. Choose the options that describe the convergence region of a Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n}\,.$$

They are the values of z for which:

a) $\sum_{n=0}^{\infty} a_n z^n$ converges together with the values of where $\sum_{n=1}^{\infty} a_{-n}\frac{1}{z^n}$ converges.

b) One of the two series $\sum_{n=0}^{\infty} a_n z^n$ or $\sum_{n=1}^{\infty} a_{-n}\frac{1}{z^n}$ converges.

c) $\sum_{n=0}^{\infty} a_n z^n$ converges, due to that the series $\sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n}$ always converges.

d) Both series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n}$ converge.

e) $\sum_{n=0}^{\infty} a_n z^n$ converges except for the value of z where $\sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n}$ is divergent.

2. Choose one or more of the options that are true. A Fourier series is:
   a) Any series of the form $\sum_{k=0}^{+\infty} \left[X_k \cos(k \frac{2\pi}{T} t) + Y_k \sin(k \frac{2\pi}{T} t) \right]$.
   b) Any finite sum $\frac{a_0}{2} + \sum_{k=1}^{n} \left[a_k \cos(k \frac{2\pi}{T} t) + b_k \sin(k \frac{2\pi}{T} t) \right]$ or any infinite sum $\frac{a_0}{2} + \sum_{k=1}^{+\infty} \left[a_k \cos(k \frac{2\pi}{T} t) + b_k \sin(k \frac{2\pi}{T} t) \right]$.
   c) Any series of the form $\frac{a_0}{2} + \sum_{k=1}^{+\infty} \left[a_k \cos(k \frac{2\pi}{T_1} t) + b_k \sin(k \frac{2\pi}{T_2} t) \right]$.
   d) Any series of the form $\frac{a_0}{2} + \sum_{k=1}^{+\infty} \left[a_k \cot(k \frac{2\pi}{T} t) + b_k \tan(k \frac{2\pi}{T} t) \right]$.
   e) Any series of the form $\frac{a_0}{2} + \sum_{k=1}^{+\infty} \left[a_k \cos(k \frac{2\pi}{T} t) + b_k k \sin(\frac{2\pi}{T} t) \right]$.
3. The Fourier integral formula is usually obtained as the following limit:
$$f(t) = \lim_{T \to +\infty} \sum_{-\infty}^{\infty} \frac{1}{T} \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u) e^{-in\omega_0 u} du \right) e^{in\omega_0 t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(u) e^{-i\omega u} du \right) e^{i\omega t} d\omega .$$
   Choose one of the items that corresponds to your learning of the integral formula.
   a) The professor never made such a deduction and only established the integral formula without explanation.
   b) The professor calculated with mathematical rigor the limit, but I never understood it.
   c) The professor calculated with mathematical rigor the limit, at that moment I understood his explanation, but now I do not remember it.
   d) The professor obtained the limit, but he commented that the calculation he made was heuristic, therefore it was not rigorously mathematical.
   e) The professor established the integral formula, but he said that his demonstration was done in advanced books of Mathematical Analysis.
4. Indicate which of the following statements are false and which are true. The bilateral Z transform:
   a) Transforms a function into an impulse train function.
   b) Transforms an impulse train function into a complex variable function.
   c) It is equivalent to the Fourier transform due to that Z-transform is deduced from Fourier transform.
   d) Transforms a sequence into a complex function that is generally defined in an annular region $R < |z| < r$.

Results and Conclusions

Although the 20 students, who completed their course of transforms, show an instrumental learning of FT and ZT, however in general they do not achieve a relational understanding of the involved objects and concepts. For example, 15 of the 20 students were unable to discriminate the definition of convergence region of a Laurent series, three of them chose only one of the two true options that they could choose but they did it in combination with other incorrect options.

On question 2, only four of the 20 answered correctly. The remaining 16 students even though were able to work with the Fourier series, they could not identify them of a list of similar trigonometric series.

Regarding question 3 on the Fourier formula, three students answered that the professor mentioned that the demonstration of Fourier formula was lies in advanced books of mathematical analysis and that they would only enunciate it there, another three students said that the professor indicated that the calculation they were doing was heuristic, so it lacked mathematical rigor. Six students said that their teacher rigorously calculated the limit, but that they either did not understand it or had already forgotten it. Their answers were not entirely reliable because none of the texts used by teachers justify such a limit in a rigorous way.

In question 4, which refers to the mathematical nature of the ZT, seven students answered correctly that the statement in the option (a) was false, but they answered incorrectly that the false statement in the option (b) was true, their answers were not consistent. In the option (c) which refers to the false assertion that FT and ZT are equivalent, six students answered that the statement was true, they knew the connection between FT and ZT, which was revealed in the interview, but the students did not understand that these transformations cannot be considered equivalent. The affirmation of the option (d), which is the only true one, was correctly chosen by only five students.

In summary, we can say that the students were not able to discriminate, from a list of statements that differed subtly from a correct statement, the concept of the radius of convergence of a power series, region of convergence of a Laurent series, nor were they able to recognize the form of an FS from a list of trigonometric series. On the other hand, it was not clear to students what kind of mathematical object is the domain over which the FT operates, the majority of the participants responded that FT applies to periodic functions, its confusion comes from the fact that the FT is deduced from the FS, which are calculated for periodic functions. The students showed incomprehension about the meaning of the improper integrals of Riemann and the contour integrals for complex functions. Finally, in general the students are not clear that ZT transforms a succession of real numbers into a complex Laurent series, nor do they understand the meaning of the inverse ZT.

These results reveal that, in the best case, students learn FS, FT and ZT instrumentally but do not get a relational understanding.

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APRENDIZAJE DE LA TRANSFORMADA DE FOURIER Y LA TRANSFORMADA Z. COMPRENSIÓN INSTRUMENTAL Y COMPRENSIÓN RELACIONAL

Antonio Rivera-Figueroa
CINVESTAV, IPN, México
arivera@cinvestav.mx

Allan T. De la Cruz-Oliva
CINVESTAV, IPN, México
a_sternova@hotmail.com

Las transformadas de Fourier y Z son temas importantes en las carreras de ingeniería en comunicaciones y electrónica. Dadas las dificultades que entraña la matemática involucrada en estas transformadas, llevamos a cabo una investigación con estudiantes de una universidad mexicana acerca de la comprensión lograda una vez culminado su curso sobre estas transformadas. Como marco teórico tomamos las ideas de Richard Skemp sobre la comprensión instrumental y comprensión relacional. Para la investigación se utilizó un cuestionario para cuyo diseño se elaboró un mapa conceptual de acuerdo con las ideas señaladas, además se hicieron entrevistas no estructuradas para profundizar sobre las concepciones de los estudiantes. Los resultados sugieren atender en el aula la comprensión de conceptos y objetos matemáticos básicos involucrados en estas transformadas.

Palabras clave: Comprensión Instrumental, Comprensión Relacional, Transformada de Fourier, Transformada Z.

Introducción

En este artículo presentamos los resultados de una investigación acerca de la comprensión de los estudiantes de la matemática involucrada en la transformada de Fourier (TF) y la transformada Z (TZ) que son fundamentales en las carreras de ingeniería en comunicaciones y electrónica. La construcción de estas transformadas está sustentada en diversos conceptos y objetos matemáticos y relaciones entre ellos. Estas relaciones, traducidas a un mapa conceptual de este contenido matemático como un todo, juega un papel crucial en la comprensión relacional de las TF y TZ, en el sentido de como lo concibe Skemp (1976). Una de las características de la comprensión relacional de un tema matemático es la comprensión del significado y naturaleza de los objetos matemáticos involucrados. Hemos adoptado las ideas de Skemp como marco de referencia, enfocándonos primero en la construcción de un mapa conceptual sobre la TF y TZ, para después construir el instrumento empleado en esta investigación y llevar a cabo el análisis de los resultados.

Planteamiento del problema y preguntas de investigación

Las series de Fourier (SF), TF y TZ suelen enseñarse en las carreras universitarias de ingeniería en comunicaciones y electrónica. Este es el caso de las carreras que ofrece la Escuela Superior de Ingeniería Mecánica y Eléctrica (ESIME) del Instituto Politécnico Nacional en México, entre otras. Las SF, TF y TZ tienen un cierto grado de complejidad matemática intrínseca por lo que, de alguna manera se explica que los estudiantes tengan dificultades en su aprendizaje, como lo revelan los resultados en las evaluaciones en las asignaturas sobre transformadas en la ESIME. Por lo anterior decidimos averiguar el nivel de comprensión de los estudiantes, de temas y conceptos básicos matemáticos involucrados en la TF y la TZ. La pregunta general que planteamos es

¿En qué medida los estudiantes han comprendido los conceptos fundamentales y la naturaleza de los objetos matemáticos involucrados en la transformada de Fourier y la transformada Z?

Para tratar de responder la pregunta anterior nos formulamos una segunda pregunta con el propósito de precisar los contenidos matemáticos:

¿Cuáles son los contenidos y conceptos de la matemática involucrados en la transformada de Fourier y la transformada Z?

Algunos de estos contenidos son prerrequisitos para que el alumno esté en condiciones de estudiar la TF y la TZ y, otros son contenidos y conceptos que el alumno aprende durante el estudio de estas transformadas, son nuevos conocimientos que el estudiante adquiere. Después de revisar de manera exhaustiva los libros de texto sobre la TF y TZ utilizados por alumnos y profesores, por ejemplo, Hwei (1970) y Glyn (2002), destacamos los contenidos y conceptos involucrados en la TF y TZ que consideramos relevantes, los cuales presentamos en un mapa conceptual más adelante.

Comprensión instrumental y comprensión relacional

Skemp (1976), en su artículo describe, de manera simple y didáctica, lo que significa comprensión instrumental y comprensión relacional lo cual ahora explicamos. Supóngase que una persona llega por primera vez a una gran ciudad. Alguien le enseña a esa persona a trasladarse caminando de un punto X, lugar donde reside, a un punto Y de la ciudad (por ejemplo, el lugar donde va a trabajar). Las instrucciones que recibe consisten en varias indicaciones como ‘en la esquina de la farmacia das vuelta a la derecha’, ‘cuando llegues a la oficina de correos camina dos calles y das vuelta a la izquierda’. La persona aprende ese camino de ida y vuelta y lo hace tantas veces que con el paso de los días adquiere confianza para realizar los trayectos. Después le enseñan a trasladarse del punto X al punto Z (por ejemplo, una sucursal de la empresa donde trabaja) y del punto Z al lugar de su residencia X, todos estos trayectos aprende a realizarlos de ida y vuelta. Con indicaciones similares, aprende otro número determinado de trayectos, todos ellos con puntos de partida y puntos destino, fijos, de manera automática. Después, la persona en sus ratos libres decide explorar por su propia cuenta otros lugares de la ciudad, pero ahora sin tener un destino fijo, lo hace sólo por el placer de conocer la ciudad y averiguar si hay algún punto interesante, pero sobre todo por tratar de construir un mapa cognitivo de la ciudad. Estas dos maneras de recorrer trayectos son bastante diferentes. En el primer caso se trata de una caminata en donde el objetivo es ir de un punto de partida fijo a un destino fijo con instrucciones bien determinadas que aprende al pie de la letra y llega a realizarlas sin pensar mucho en ellas. En el segundo caso el objetivo es ampliar y consolidar un mapa mental de la ciudad, que es un estado de conocimiento Skemp (1976).

Una característica de la realización de un trayecto basado en una serie de instrucciones es que se le indica a la persona qué hacer en cada punto del trayecto. Pero, si en algún momento la persona comete un error, probablemente se perderá y no podrá retomar el camino correcto. Por el contrario, una persona con un mapa mental de la ciudad está en mejores posibilidades de crear trayectos con diversos puntos de partida y puntos finales sin perderse y, cuando cometa un error le será más fácil retomar o rehacer un camino de manera correcta. Basados en estas ideas concebimos un mapa conceptual sobre un concepto, objeto o tema de matemáticas como la descripción de los elementos y objetos fundamentales que permite construirlo (concepto, objeto o tema), la cual incluye sus relaciones o conexiones entre ellos. Esto es lo que llevamos a cabo para la TF y TZ.

Consideraciones metodológicas

La primera parte de la metodología utilizada en esta investigación para responder las preguntas antes planteadas, consistió en un análisis de la matemática involucrada en el estudio de las TF y TZ con el propósito de crear un mapa conceptual, en el sentido de las ideas de Skemp, que nos sirviese para identificar las partes relevantes de la matemática cuya comprensión es fundamental en el estudio de las TF y TZ.

Basados en este mapa conceptual, como segunda parte de la metodología, se diseñó un cuestionario de 10 preguntas en donde predominaron las ideas, conceptos y objetos matemáticos básicos sobre los que se desarrolla la teoría sobre las TF y TZ. El cuestionario se aplicó a un grupo de 20 estudiantes de la ESIME que en ese momento recién completaban su curso sobre transformadas en la carrera de ingeniería en Comunicaciones y Electrónica. La preparación matemática previa de los estudiantes consistía en un curso semestral de Cálculo diferencial e integral, uno de elementos de Algebra Lineal y otro de Ecuaciones Diferenciales. Durante el tercer semestre ellos estudian de manera simultánea su curso de Transformadas y otro de elementos de Funciones de Variable compleja.

La información obtenida del análisis de las respuestas a las preguntas del cuestionario se complementó mediante entrevistas no estructuradas para profundizar en los argumentos de los estudiantes en los que sustentaron sus respuestas. Para este reporte hemos seleccionado cuatro preguntas del cuestionario cuyo análisis de las respuestas de los estudiantes presentamos en la sección de resultados y conclusiones.

Mapa conceptual de la transformada de Fourier y la transformada Z

Algunos de los conocimientos matemáticos que el alumno requiere para estudiar la TF y la TZ son los de un curso anual de cálculo universitario. Esto incluye, el concepto de función, el conocimiento de funciones elementales, los conceptos de límite, continuidad, derivada e integral y sus propiedades básicas. Un concepto que juega un papel importante en el tema de transformadas es el de integral impropia con límite inferior 0 o $-\infty$ y límite superior $+\infty$. También se requiere que el estudiante tenga conocimientos básicos sobre funciones de variable compleja, como son: derivada de funciones de variable compleja, integrales de contorno y desarrollos de funciones en series de potencias y series de Laurent.

Para construir el mapa conceptual sobre las TF y TZ, consideramos como punto de partida las series de Fourier, a partir de las cuales podemos comenzar a construir las trasformadas. En el diagrama de la Figura 1 mostramos los aspectos que consideramos más relevantes del mapa conceptual. En cada uno de los recuadros aparecen los objetos o definiciones importantes, las flechas muestran las conexiones entre ellos, que en algunos casos conllevan deducciones o demostraciones. Al respecto hacemos notar sobre las dificultades intrínsecas de algunos objetos matemáticos o relaciones entre ellos, que merecen especial consideración en la enseñanza. Este es el caso, por ejemplo, de la fórmula integral de Fourier y de la función delta de Dirac.

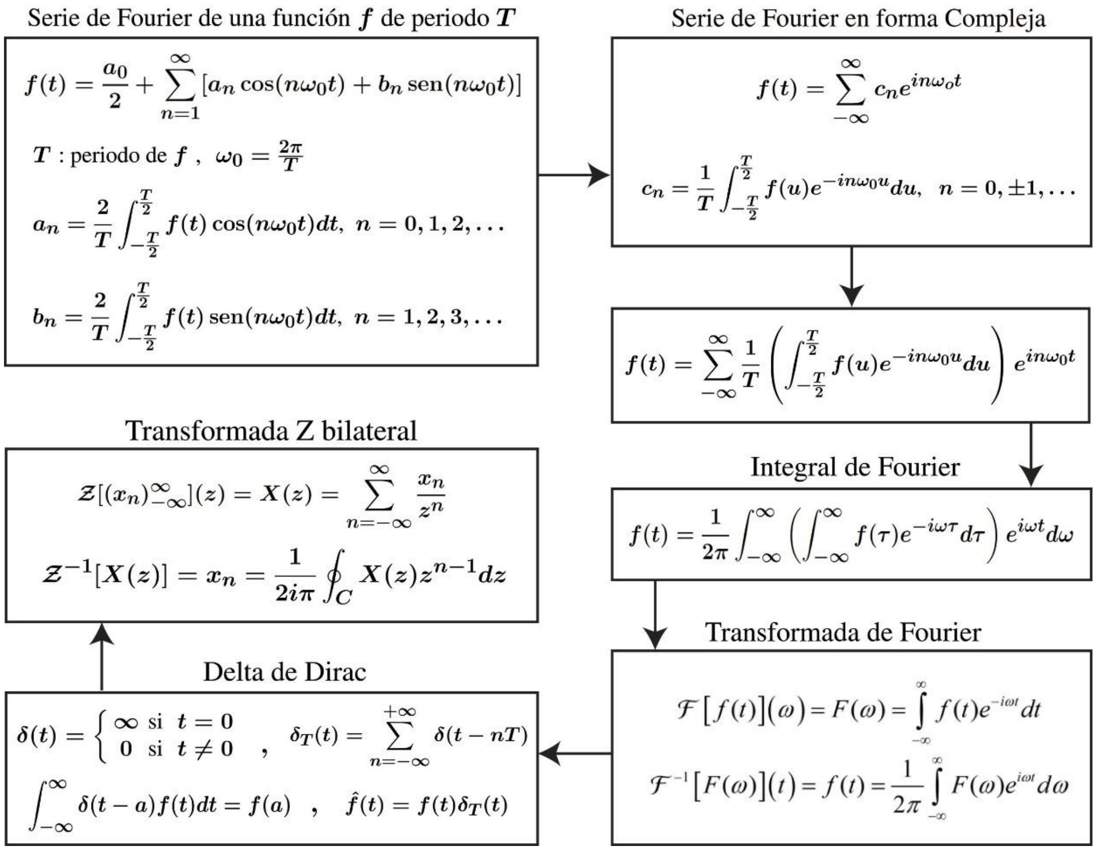

Figura 1: Diagrama que sintetiza el mapa conceptual de TF y TZ

En ocasiones, como suele ocurrir en la ESIME, las fórmulas para los coeficientes de la SF

$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u)\cos(k\tfrac{2\pi}{T}u)du \quad \text{y} \quad b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u)\operatorname{sen}(k\tfrac{2\pi}{T}u)du$$

se presentan en el aula sin explicación alguna y se procede a aplicarlas a funciones específicas. Sería conveniente deducirlas, también es importante explicar a los alumnos que las sumas parciales

$$s_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n}\left[a_k \cos(k\tfrac{2\pi}{T}t) + b_k \operatorname{sen}(k\tfrac{2\pi}{T}t)\right]$$

juegan un papel muy importante en la teoría de las SF, pues son las que permiten darle sentido a la suma infinita. La TF suele explicarse y establecerse mediante la obtención del límite

$$f(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty}\left(\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u)e^{-in\omega_0 u}du\right)e^{in\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du\right)e^{i\omega t}d\omega\ . \quad (1)$$

Este es un momento crucial en la enseñanza de la TF. El cálculo del límite anterior carece de rigor, pero es un acercamiento común en los textos de ingeniería o de matemáticas para ingeniería, véase por ejemplo Hwei (1970, pp.72-73), lo cual es explicable pues la demostración rigurosa de la fórmula (1) demanda conocimientos de análisis matemático avanzado, véase por ejemplo Rudin (1987, pp.180-181) y Beerends et al. (2003, pp. 164-186).

La siguiente simbología funcional fortalece la comprensión relacional de los estudiantes

$$f(t) \overset{\mathcal{F}}{\mapsto} F(\omega)=\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt \ , \quad F(\omega) \overset{\mathcal{F}^{-1}}{\mapsto} f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}d\omega \ .$$

Un acercamiento a la TZ, que se observa en los libros de texto revisados consiste en aplicar la TF a una función particular. En este acercamiento, la función delta de Dirac δ, introducida y usada por Paul Dirac (1926, pp. 621-641) y Oliver Heaviside (1899, pp. 54-57), juega un papel muy importante. En realidad, δ no es una función en el sentido del análisis matemático clásico, pero es un objeto matemático de suma utilidad en ingeniería (Lathi, 1998). Se trata de un objeto sofisticado que en el análisis matemático clásico no tiene una definición rigurosa, pero que en ingeniería y física trabaja muy bien. En la enseñanza ha de destacarse que son las propiedades de δ, las establecidas en su definición, las que nos permiten desarrollar con éxito la teoría. El área de la matemática que le da un tratamiento riguroso a δ, en la que recibe el nombre de función generalizada, es la teoría de Distribuciones creada por Laurent Schwarz (1950). Con base en la función generalizada δ se define, para cualquier real positivo T, la función tren de impulsos $\delta_T(x)=\sum_{n=-\infty}^{+\infty}\delta(x-nT)$ (Hwei, 1970; Glyn, 2002). Esta función toma el valor cero fuera de los puntos de la forma $nT(n\in Z)$ y en ellos toma el valor $+\infty$. Si $f:\mathbb{R}\to\mathbb{R}$ es una función arbitraria, la función $\hat{f}$ definida como $\hat{f}(x)=f(x)\delta_T(x)$ es llamada tren de impulsos ponderado. Aquí resulta paradójico que las funciones $\hat{f}$ y δ_T toman los mismos valores, sin embargo, los cálculos formales de la TF de cada una de ellas conducirán a resultados diferentes, esto es algo que causa conflicto a estudiantes y profesores. Todo esto se desprende de la misma condición de la integral para δ, que no corresponde a definición rigurosa alguna del análisis matemático clásico. Debe reconocerse en el aula que los cálculos se llevan a cabo de manera formal para no provocar en el alumno un sentimiento de baja autoestima por su incomprensión de estos objetos matemáticos.

Por definición, la TF de $\hat{f}$ está dada por

$$\hat{F}(\omega)=\int_{-\infty}^{+\infty}\left(\sum_{n=-\infty}^{+\infty} f(t)\delta(t-nT)\right)e^{-i\omega t}dt=\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{\infty} f(t)\delta(t-nT)e^{-i\omega t}dt=\sum_{n=-\infty}^{+\infty} f(nT)e^{-i\omega nT}.$$

Si el número complejo $e^{i\omega T}$ lo denotamos por z, $e^{-i\omega nT}$se escribe como z^{-n} y entonces

$$\sum_{-\infty}^{+\infty} f(nT)e^{-i\omega nT}=\sum_{-\infty}^{+\infty} f(nT)z^{-n} \ . \qquad (2)$$

Si concebimos $\hat{F}$ como función de z y no de ω, la fórmula (2) se escribe como

$$\hat{F}(z)=\sum_{n=-\infty}^{+\infty} f(nT)z^{-n} \ . \qquad (3)$$

Algunos autores de libros de texto de ingeniería o de matemáticas para ingeniería, al hacer un cambio de variable independiente para una función de una o varias variables suelen conservar la misma letra para la nueva función con la nueva variable, vea por ejemplo Roberts (2004, pp. 762-763), Pipes (2014, pp.956-957), Kreyszig (2011, pp.392-394), Wylie (1966, pp. 550-551), Sokolnikoff (1958, pp.230-235), sin embargo, conceptualmente se tienen funciones diferentes. En nuestro caso, aunque hemos usado la misma letra, la función $\hat{F}$ de (2) es diferente de la función $\hat{F}$ de (3), pues tienen dominios diferentes, entonces son diferentes. La función $\hat{F}$ de (3) es la TZ de f relativa a T. En realidad, la TZ se aplica a sucesiones, en este caso a la sucesión $\left(f(nT)\right)_{n=-\infty}^{+\infty}$ y no a la función f. Entonces es mejor escribir

$$\mathrm{Z}\left[\left(f(nT)\right)_{n=-\infty}^{+\infty}\right](z)=\hat{F}(z)=\sum_{n=-\infty}^{+\infty} f(nT)z^{-n} \ .$$

En general, la TZ de una sucesión $(x_n)_{n=-\infty}^{+\infty}$ se define como la serie de Laurent

$$\mathrm{Z}\left[\left(x_n\right)_{n=-\infty}^{+\infty}\right](z)=X(z)=\sum_{n=-\infty}^{+\infty} x_n z^{-n}=\sum_{n=0}^{+\infty} x_{-n} z^n+\sum_{n=1}^{+\infty} x_n \frac{1}{z^n} \quad . \qquad (4)$$

La inversa de la TZ asigna a cada función $X(z)$ que admite una representación como en (4), la sucesión de coeficientes x_n. El cálculo de la transformada inversa se obtiene mediante una integral de contorno sobre un círculo C con centro $z=0$ y, por ejemplo, radio 1. Si después de multiplicar ambos miembros de $X(z)=\sum_{-\infty}^{+\infty} x_n z^{-n}$ por z^{k-1} integramos sobre un círculo C apropiado obtenemos

$$x_k=\frac{1}{2\pi i}\oint_C X(z) z^{k-1}\, dz \text{ para toda } k\in\mathbb{Z}.$$

La fórmula anterior no es otra cosa que la estudiada en un primer curso de funciones de variable compleja, para calcular los coeficientes de una serie de Laurent. Como en el caso de la TF, la siguiente simbología ayuda a comprender la naturaleza de la TZ y su inversa

$$\left(x_n\right)_{-\infty}^{+\infty} \overset{\mathrm{Z}}{\mapsto} \mathrm{Z}\left[\left(x_n\right)\right](z)=\sum_{n=-\infty}^{+\infty} \frac{x_n}{z^n}, \quad X(z) \overset{\mathrm{Z}^{-1}}{\mapsto} x=\left(x_n\right)_{-\infty}^{+\infty}, \text{ donde } x_n=\frac{1}{2\pi i}\oint_C X(z) z^{n-1} dz .$$

El cuestionario

Constó de 10 preguntas que se refieren a la comprensión de significados o de la naturaleza de los objetos matemáticos involucrados en la TF y TZ. Aquí presentamos cuatro de las preguntas. Como se señaló antes, una de las dificultades en la comprensión de la fórmula integral de Fourier (1), se debe a que para establecerla se requiere herramienta del análisis matemático avanzado, por lo que resulta importante que el profesor reconozca y explique al alumno que se trata de una deducción totalmente empírica; con la pregunta tres pretendemos averiguar si el profesor hizo una tal explicación o intentó hacer una demostración de (1).

1. Elija las opciones que describan la región de convergencia de una serie de Laurent

$$\sum_{n=-\infty}^{\infty} a_n z^n=\sum_{n=0}^{\infty} a_n z^n+\sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n} \quad .$$

Son los valores de z para los cuales:

a) $\sum_{n=0}^{\infty} a_n z^n$ converge junto con los valores de z donde $\sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n}$ converge.

b) Alguna de las dos series $\sum_{n=0}^{\infty} a_n z^n$ o $\sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n}$ converge.

c) $\sum_{n=0}^{\infty} a_n z^n$ converge, pues la serie $\sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n}$ siempre converge.

d) Ambas series $\sum_{n=0}^{\infty} a_n z^n$ y $\sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n}$ convergen.

e) $\sum_{n=0}^{\infty} a_n z^n$ converge exceptuando los valores de z donde $\sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n}$ diverge.

2. Elija una o varias de las opciones que sean ciertas. Una serie de Fourier es:

a) Cualquier serie de la forma $\sum_{k=0}^{+\infty}\left[X_k \cos(k \tfrac{2\pi}{T} t)+Y_k \operatorname{sen}(k \tfrac{2\pi}{T} t)\right]$.

b) Cualquier suma finita $\frac{a_0}{2}+\sum_{k=1}^{n}\left[a_k\cos(k\,\tfrac{2\pi}{T}t)+b_k\,\mathrm{sen}(k\,\tfrac{2\pi}{T}t)\right]$ o cualquier suma infinita $\frac{a_0}{2}+\sum_{k=1}^{+\infty}\left[a_k\cos(k\,\tfrac{2\pi}{T}t)+b_k\,\mathrm{sen}(k\,\tfrac{2\pi}{T}t)\right]$.

c) Cualquier serie de la forma $\frac{a_0}{2}+\sum_{k=1}^{+\infty}\left[a_k\cos(k\,\tfrac{2\pi}{T_1}t)+b_k\,\mathrm{sen}(k\,\tfrac{2\pi}{T_2}t)\right]$.

d) Cualquier serie de la forma $\frac{a_0}{2}+\sum_{k=1}^{+\infty}\left[a_k\cot(k\,\tfrac{2\pi}{T}t)+b_k\tan(k\,\tfrac{2\pi}{T}t)\right]$.

e) Cualquier serie de la forma $\frac{a_0}{2}+\sum_{k=1}^{+\infty}\left[a_k\cos(k\,\tfrac{2\pi}{T}t)+b_k k\,\mathrm{sen}(\tfrac{2\pi}{T}t)\right]$.

3. La fórmula integral de Fourier suele obtenerse como el siguiente límite:

$$f(t)=\lim_{T\to+\infty}\sum_{-\infty}^{\infty}\frac{1}{T}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}}f(u)e^{-in\omega_0 u}du\right)e^{in\omega_0 t}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(u)e^{-i\omega u}du\right)e^{i\omega t}d\omega\ .$$

Elija uno de los incisos que corresponda a tu aprendizaje de la formula integral.

a) El profesor nunca hizo tal deducción y sólo estableció la fórmula integral sin explicación.
b) El profesor calculó con rigor matemático tal límite, pero yo nunca lo comprendí.
c) El profesor calculó con rigor matemático tal límite, en ese momento comprendí su explicación, pero ahora no lo recuerdo.
d) El profesor obtuvo el límite, pero comentó que el cálculo que hacía era heurístico por lo tanto no era riguroso matemáticamente.
e) El profesor estableció la formula integral, pero dijo que su demostración se hacía en textos avanzados de Análisis Matemático.

4. Indica cuáles de las siguientes afirmaciones son falsas y cuáles son verdaderas. La transformada bilateral Z:
a) Transforma una función en una función tren de impulsos.
b) Transforma una función tren de impulsos en una función de variable compleja.
c) Es equivalente a la transformada de Fourier pues se deduce de ésta.
d) Transforma una sucesión en una función compleja que en general está definida en lo que se llama una región anular $R<|z|<r$.

Resultados y Conclusiones

Si bien 20 alumnos, que completaron su curso de trasformadas, muestran un aprendizaje instrumental de las TF y TZ, en general ellos no logran una comprensión relacional de los objetos y conceptos involucrados en estas transformadas. Por ejemplo, 15 alumnos no pudieron discriminar la definición de región de convergencia de una serie de Laurent. Tres de los alumnos que erraron eligieron una de las opciones correctas, de las dos que podían elegir, pero lo hicieron en combinación con otras opciones incorrectas.

Respecto a la pregunta 2, solo cuatro estudiantes respondieron correctamente a la pregunta. Los restantes 16 alumnos, aunque fueron capaces de trabajar con las series de Fourier, como lo demostraron en su curso sobre transformadas, no lograron identificarlas de una lista de series trigonométricas similares.

Respecto a la pregunta 3, la cual se refiere a la fórmula de Fourier, tres alumnos indicaron que su profesor les mencionó que la demostración de la fórmula de Fourier se hacía en textos avanzados de análisis matemático y que ahí solo la enunciarían, otros tres dijeron que el profesor indicó que el cálculo que estaban haciendo era heurístico, por lo que carecía de rigor matemático. Seis estudiantes afirmaron que su profesor calculó rigurosamente el límite, pero que ellos o no lo comprendieron o ya lo habían olvidado. Sus respuestas no fueron del todo confiables debido a que ninguno de los textos empleados por los profesores justifican tal límite de forma rigurosa.

En la pregunta 4, que se refiere a la naturaleza matemática de la TZ, siete estudiantes acertaron al responder que era falsa la afirmación del inciso (a), pero respondieron incorrectamente que era verdadera la afirmación falsa del inciso (b), sus respuestas no fueron consistentes. El inciso (c) que se refiere a la afirmación falsa de que TF y TZ son equivalentes, seis estudiantes respondieron que la afirmación era verdadera, ellos conocían la conexión que existe entre TF y TZ, y así fue revelado en la entrevista, pero no tenían claro que estas transformadas no pueden considerarse equivalentes. La afirmación del inciso (d), que es la única verdadera, fue elegida correctamente sólo por cinco estudiantes.

En resumen, podemos decir que los estudiantes no fueron capaces de discriminar, de una lista de enunciados que diferían de manera sutil de un enunciado correcto, el concepto de radio de convergencia de una serie de potencias, región de convergencia de una serie de Laurent, ni fueron capaces de reconocer la forma de una SF de una lista de series trigonométricas. Por otra parte, los estudiantes no tienen claro qué tipo de objeto matemático es el dominio sobre el que opera la TF, la mayoría afirmó que la TF se aplica a funciones periódicas, su confusión proviene del hecho de que la TF se deduce de la SF la cual puede ser calculada para funciones periódicas. Los estudiantes mostraron incomprensión sobre el significado de las integrales impropias de Riemann y de las integrales de contorno para funciones complejas. Finalmente, no es claro para una mayoría de los estudiantes que la TZ transforma una sucesión de números reales en una serie de Laurent compleja, tampoco comprenden el significado de la TZ inversa. Estos resultados revelan que, en el mejor de los casos, los estudiantes aprenden las SF, TF y TZ de manera instrumental pero no logran una comprensión relacional.

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LEARNING THE FOURIER TRANSFORM AND THE Z-TRANSFORM: INSTRUMENTAL UNDERSTANDING AND RELATIONAL UNDERSTANDING

The Fourier and Z transforms are important topics in communications and electronics engineering careers. Given the difficulties involved in the mathematics involved in these transformations, we conducted an investigation with students of a Mexican university about their understanding that they achieved once they had completed a course on these transformations. As a theoretical framework, we take Richard Skemp's ideas about instrumental understanding and relational understanding. For the investigation, a questionnaire was used for whose design a concept map was drawn up according to Skemp's ideas and unstructured interviews were conducted to deepen the students' conceptions. The results suggest attending in the classroom the understanding of basic mathematical concepts and objects involved in these transformed

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EXAMINING THE DEVELOPMENT OF A FOURTH GRADE STUDENT AS SHE ENGAGES IN REASONING AND PROVING

Jenna R. O'Dell
Bemidji State University
Jenna.odell@bemidjistate.edu

Todd Frauenholtz
Bemidji State University
Todd.frauenholtz@bemidjistate.edu

In this study, we investigated one student as she engaged in the exploration of a graph theory problem, the Graceful Tree Conjecture. The study took place in a classroom in the Midwestern United States which contained eighteen students. We report on a descriptive case study of one student, Heidi, as she works her way through different classes of tree graphs and advances through different types of reasoning, justifications, and proving a select class of tree graphs can be labeled gracefully. We found that with repeated experiences, Heidi, was able to advance her level of reasoning and creations of justifications.

Keywords: Problem Solving, Reasoning and Proof, Elementary School Education

The current horizon in mathematics education for elementary school focuses on procedural and conceptual understanding of mathematical content (Bieda, Ji, Drwencke, & Picard, 2013). What would happen if we went against the horizon and instead focused on the mathematical processes of reasoning and proving?

Students engaging in mathematical proof is not characteristically presented until middle school (Lin & Tsia, 2016) or even high school (Stylianides, 2007). Students at the elementary level are typically focused on finding the correct answer but justifying their solutions is not included (Kieran, 2004). Carpenter, Franke, and Levi (2003) have identified three ways students tend to give an argument or mathematical justification: (a) appeal to authority, (b) justification by example, and (c) generalizable argument. A student appealing to authority would give a reason for their answer by stating a rule or procedure their teacher has shared. A justification through example would be where a student makes a case about something through sharing a specific case, such as four times two is eight therefore an even multiplied by an even is always even. For a generalized argument students would give an argument that would be applicable to all cases in the given conjecture.

Carpenter et al. (2003) furthered their team's argument by stating that elementary students do not typically complete generalizable arguments but as they advance in grade level they should be encouraged to develop more generalized arguments and understand how just giving an example restricts their argument. Past researchers (Ball, 1993; Keith, 2006; Lin & Tsai 2016) have documented that elementary students are able to make conjectures and develop justifications. This has been done through students in second and third grade exploring the idea of the sums of even and odd numbers through using blocks and definitions. Many researchers have recommended all elementary students engage in argumentation, proof, and making mathematical justifications in all mathematics content areas (e.g., Ball & Bass, 2003; Carpenter et al., 2003; NCTM 2014; Stylianides, 2007). Bieda, Ji, Drwencke, and Picard (2013) argued students' difficulties with learning to construct formal proofs in geometry high school classes and college level mathematics could be because of the lack of experiences elementary and middle schools students have had with justifying, reasoning and proving. Further, Beida et al. (2013) stated that tasks where elementary students could engage in reasoning and proving is an area of needed research. For this study, we examined a fourth grade student's development in reasoning and proving as she worked through a graph theory problem.

Research Questions

The following questions guided our research study:

1. What ways does a fourth grade student attempt to construct an argument or mathematical justification?
2. How does a fourth grade student progress through repeated opportunities with justifying and proof?

Theoretical Framework

Throughout this study, we wanted students to experience mathematics more similar to how a mathematician might experience it than what they routinely see in a traditional elementary school setting. Thus, we tried to create a community of practice where students had the freedom to understand a mathematical topic through exploration and focused on the topic of reasoning and proving. Because of this, we used the idea of communities of practice (Wenger, 1998) as an overarching theoretical framework for this study and attempted to turn the fourth grade classroom into such an environment.

We used communities of practice as a way to create an experience among the students that would be similar to the ways a mathematician might engage in mathematics. In communities of practice the two major components are practices and identities. This idea develops from the thought that we are "active participants in the *practices* of social communities and constructing *identities* in relation to these communities" (Wenger, 1998, p. 4). The main idea of practice is how we experience the world and create social engagement. There are three components of the relationship between practice and the created community, which are mutual engagement, joint enterprise, and shared repertoire. We attempted to create mutual engagement by allowing the students opportunities to work together; joint enterprise by allowing the students to create a similar goal of finding a solution to the graph theory problem; and shared repertoire by teaching the students new words such as node, edge, conjecture, and the idea of a graceful labeling to use while working in a community of mathematicians. The other main component of communities of practice is identity. According to Wenger (2008), identity is the relationship between the personal and social and thus it allows it to shape a person's belonging in the community.

Methods

In this qualitative study, we investigated fourth grade students while they engaged in the exploration of a graph theory problem, the Graceful Tree Conjecture. The study took place in a classroom containing eighteen students at a private school in the Midwestern United States. All eighteen students participated in the study but because of the length of the paper, we will report on only one student, who we will call Heidi. The students participated in three teaching experiments (Steffe & Thompson, 2000), each lasting 75 minutes. Both of the authors were teacher-researchers in this teaching experiment but not the regular classroom teacher.

During the teaching experiment, the students engaged in parts of the Graceful Tree Conjecture. The used tasks come from a previous research study (O'Dell, 2017). This study looks to extend that research by integrating the ideas of reasoning and proof into a classroom setting.

Graceful Tree Conjecture

The Graceful Tree Conjecture is currently an unsolved problem from graph theory that is accessible to elementary school students. The problem has children explore different types of tree graphs, graphs that are connected with no cycles (see Figure 1). A tree graph must be acyclic which means if you follow along the edges from node to node, you will never cycle back to the same node without retracing an edge. All tree graphs contain one more node than edge.

To create a graceful label for a tree graph, you assign numbers then create a labeling for the edges. For a tree graph of order *m*, every node is labeled distinctly from 1 through m and the edges are then labeled with the absolute value of the difference of the labels on their attached nodes. A tree graph is labeled gracefully if the edges are labeled 1 through *m*-1 distinctly (see Figure 1).

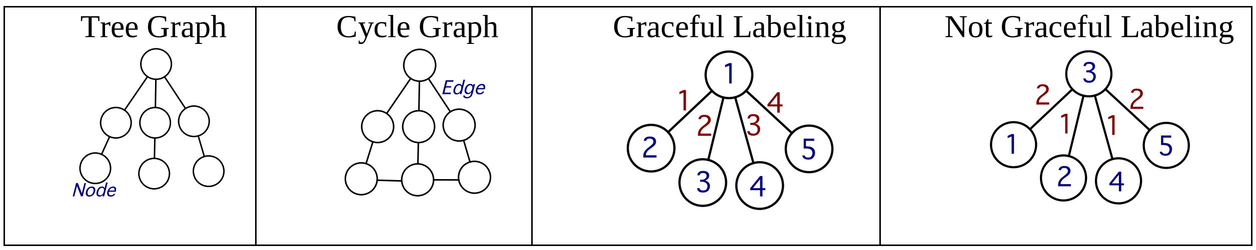

Figure 1: Graceful Tree Concepts

Overview of Teaching Experiment

During the three teaching experiment sessions, students explored different classifications of tree graphs in increasing sophistication (see Figure 2). We tasked students to not just create a graceful labeling for each graph but to find patterns, create justifications, and prove that any graph in the given category could be labeled gracefully. During the exploration of each graph classification, students were given a page that contained the first four distinct tree graphs in a given class, enlarged copies of each of the graphs, and numbered circle and square chips (see Figure 3). The use of the enlarged graphs and chips allowed the students to try multiple configurations without having to erase in hopes of easing frustration.

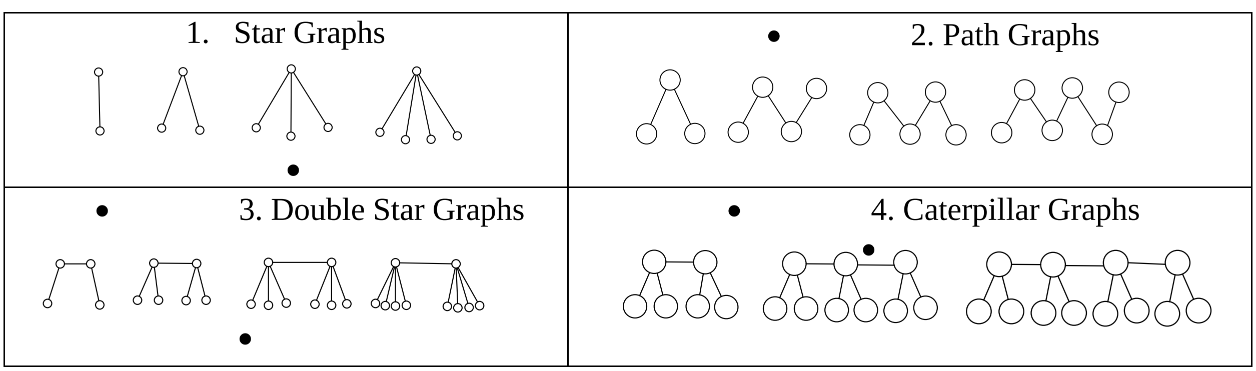

Figure 2: Different Classes of Tree Graphs in Increasing Sophistication

During the first class period students were taught what a conjecture was, each subsequent session started with a reminder that we are looking for a conjecture and what that means. Students were introduced to graph theory and tree graphs, explored graceful labelings, and were

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

tasked to create a graceful labeling of the Star class. Throughout the second class period, students worked to develop a pattern or justification for path graphs. In the third session, students explored and developed justifications for graceful labelings for the group of graphs in the Double Star class and Caterpillar class. At the end of each class, we would have a whole group discussion about the different patterns, justifications, and ideas for proving that a certain class of graphs could be labeled gracefully. For every classification of tree graph students were able to generate at least one generalizable solution. During all three sessions, we encouraged the students to work together, giving them only enough hands-on materials for every two people which meant they would have to work with a partner but record solutions on their own page.

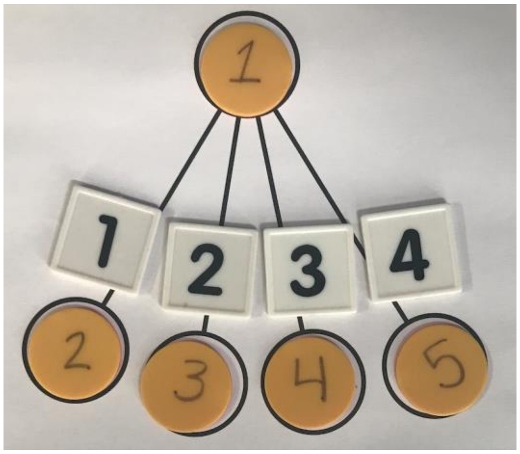

Figure 3: Enlarged Star Graph and Circle and Square Numbered Chips

Results and Discussion

All of the eighteen fourth grade students were able to make progress on each class of tree graphs. They were able to find graceful labelings for the first four distinct graphs in each of the first three classes of tree graphs, Stars, Path, and Double Star Graphs. However, not all of the fourth grade students were able to find patterns and create justifications for each of the graphs. Heidi was selected to be the case study student because she was able to create patterns for all four classes of tree graphs. Many other students were also able to do this for several of the different classes but in the given time period Heidi was the only one to find a pattern for the Caterpillar class. We had to end the session prior to the other students completing this. Heidi was also selected because she sincerely enjoyed the activity and was very willing to share her solutions with the researchers and her classmates during the class discussion period.

Star Graphs

To begin Star Graphs, Heidi quickly worked through labeling the first four distinct graphs in the class (see Figure 2 for the first four distinct graphs). Then she said she did not understand what to do next. We told her to draw and produce the next graph in the Star class by asking her what the next graph would look like. She asked, “Can I draw it?” and we told her yes. Her partner asked if they needed to have six nodes and she responded incorrectly, “You only need five.” She realized her error then changed her answer to needing six nodes. Like the first four graphs, she placed one at the top and put the rest of the numbers in order.

After she drew a graceful labeling for the six noded graph, she asked what to do next. We asked her, “Do you think you could gracefully label any Star graph?” Heidi did not seem to understand the question. We clarified that if there was a Star graph with so many nodes you were not sure how many, could you still label it gracefully. Heidi responded, “Oh.” Again, we clarified, “What if there were thirty nodes?” Heidi stated, “You would put a one at the top and go in a pattern along the bottom.” We asked her to write about it.

To generalize the Star Graphs, Heidi, wrote a description on her page and used her example of a graph with six nodes (see Figure 4). Her description said, “You would put one at the top node and then two, then three and so on depending on how many nodes there are. You label edges by subtracting the number in the nodes also edges labels go in order.”

Heidi was able to find and describe a pattern to label any Star Graph gracefully. In her description of the pattern, she limited herself by drawing a picture of one distinct graph to explain how her pattern worked. She attempted to generalize but this was her first time being pushed to go beyond just developing a solution. However, she did share that she thought all graphs in the Star class could be labeled gracefully following her given pattern.

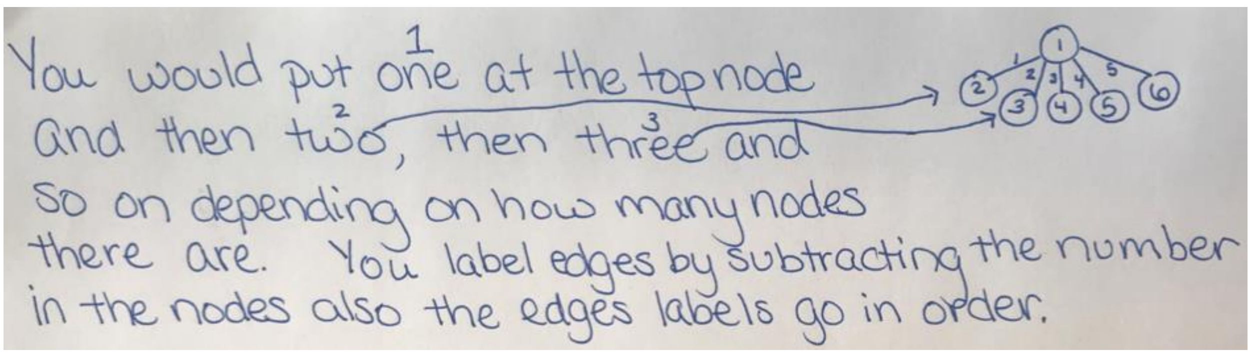

Figure 4: Heidi’s Justification for Star Graphs

Path Graphs

Heidi began working on the Path graphs and labeled the first four distinct graphs in the class. Next, she and her partner talked about what they found. Her partner said that she had the biggest number on top. Heidi said, “No, we put the smallest number on top.” When she was questioned to tell more she said, “The biggest second.” We asked her to write about her pattern. She wrote, “We used one in the first node and then the biggest number in the second node. We also used two in the third node. Then counted down every other node.”

After she finished writing about her pattern, we asked if she would use her pattern to do a Path graph with 50 nodes. She said, “Honestly, yeah.” Heidi went and began drawing and labeling a Path graph where she placed one if the first node, a fifty in the second node, a two in the third node and continued doing she had drawn eleven nodes.

Next, one of the researchers asked her what she would do if she did not know how many nodes. She said, “One would go in the top.” They asked, “What would go next?” She said, “It would be the biggest number.” Next, the researcher asked, “What would go after that?” Heidi looked at her pattern on the other side of her page and said, “Two. And then it kind of depended on…” She stopped talking and looked at her pattern. After several seconds the researcher asked, “Well you had one, the biggest number, two, and what happened with the next number?” Heidi stated, “The second biggest and yeah we did that every time.”

Then Heid went and drew the picture to give a justification, or beginning of a proof, for how she would label any tree graph in the Path class gracefully (see Figure 5). In her justification she completed how she could use her pattern for any Path graph and explained how to label the edges. We saw this as an advancement from her previous explanation of the Star graph where she just described a pattern and referred to one example picture. In this case, Heidi gave a complete justification or what we would consider a proof for a fourth grade student.

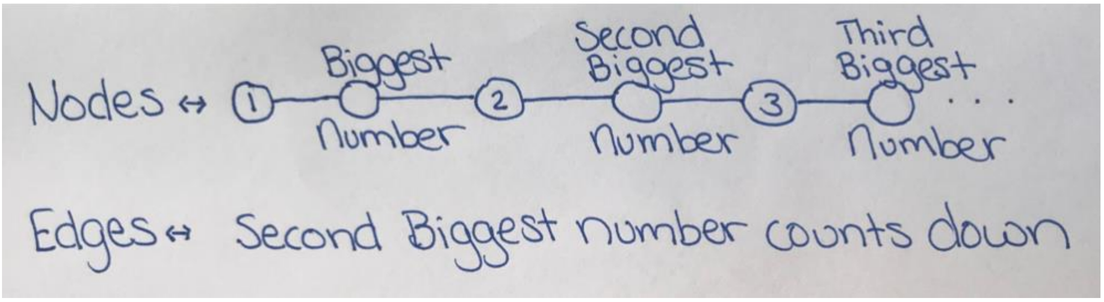

Figure 5: Heidi's Justification for Path Graphs

Double Star Graphs

Heidi began working on Double Star Graphs during the third session. Her partner from the previous two days was absent so she decided to work by herself. After gracefully labeling the first four distinct graphs, she began to write about her pattern. However, on this graph she went back to the idea of using one graph to show her pattern (see Figure 6). She went and wrote how to label the nodes. She wrote "Example Graph." She then wrote "I start the one on the top node. Then take the biggest number and put it below. It goes in a pattern too. From Biggest to smallest."

After she finished, one of the researchers came and she showed them her pattern. The researcher attempted to further the idea with trying to figure out how to label the edges. After a brief discussion they left her with the question of what would the edge that connected the two Stars of the graph might be. Heidi decided that she needed to do a few more graphs to figure it out. She went and drew a Double Star Graph with twelve nodes (see Figure 6). After thinking for a while and looking at her previous graphs in the Double Star class, she furthered this pattern by describing "The top edge is always half of the biggest number. There for whatever the number for the edge is the node is one bigger."

While Heidi was able to find a way to label any graph in the Double Star class she went back to the idea of giving an example graph to explain her pattern. We found this similar to how she labeled her Star Graph and was a slight decline in her advancement of justifications and proofs. However, she did have a more advanced thinking about the graphs than any of her classmates by finding the idea that the edge connecting the two stars of the Double Star Graph was always going to be half of the biggest number and that the node connected to the edge would be one more than half the biggest number.

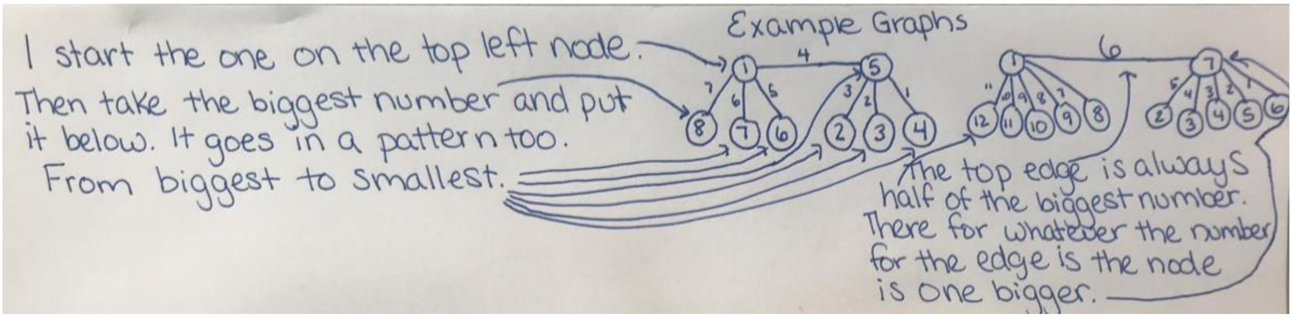

Figure 6: Heidi's Justification for Double Star Graphs

Caterpillar Graphs

Heidi asked if she could start on the next type of graph while her classmates were still finishing the previous graphs during the third session for which she had already completed her pattern. We gave her the Caterpillar Graphs on which to work. She began by completing the graphs with six, nine, and twelve nodes or the first three distinct graphs of Caterpillar class (see Figure 2).

She then drew the next graph in the class, a 15 node graph, and found a graceful labeling. She then went on to write about how she would label any graph in the Caterpillar class. She said, "I always put the one at the top far right node then put the biggest number on the bottom far right node and the second biggest number on the node next to it."

She continued by drawing a picture for how she would label any graph in this type of Caterpillar class (see Figure 7). However, this time she did not do an example of what she wrote as she had previously done for the Star Graph and the Double Star graph; rather, she completed her pattern by drawing a picture where she used the idea of biggest, second biggest, third biggest and so on. To show that her pattern would continue on, she used an ellipsis so show it would continue. She explained this idea as "So on." On this graph, Heidi did not explain how to label the edges but the researcher asked her how she labeled her edges and she verbally said they counted down in order from left to right.

Heidi showed an advancement again in her development on proof and justifications. She was able to complete a pattern and draw how to label any graph in this class similar to how she did for Path graphs.

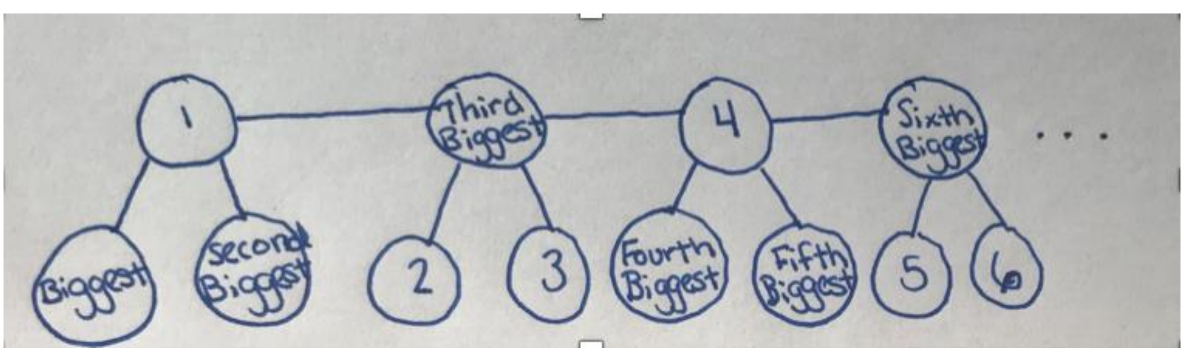

Figure 7: Heidi's Justification for Caterpillar Graphs

Conclusions

Carpenter et al. (2003) stated there are three ways students present a justification or argument, namely: appeal to authority, use an example, or give a generalized argument. We set up the situation so the fourth grade students were not able to appeal to authority because graph theory was not something they had prior knowledge of nor did the classroom teacher. The students tried to present a justification with an example. For Star Graphs, Heidi created an example to explain her pattern. She did this again on Double Star Graphs. However, on Path Graphs and Caterpillar Graphs, Heidi created generalized arguments for each. She even was able to include the ellipsis to show the pattern would continue indefinitely.

As a fourth grade student, Heidi developed through this experience from explaining how a pattern could be labeled through an example and words to creating a generalized argument for graphs in the second type of Caterpillar class. While she did not use the idea of a variable, which would not be common among fourth grade students, she was able to use terminology like the biggest number, second biggest number and so on. Further, on her first attempt at Caterpillar Graphs, she was able to discover that an edge labeling on a graph would be half of the biggest node whereas in previous graphs she only described biggest, second biggest, … numbers. In addition to finding success, Heidi, stated that she enjoyed the challenge of the activity and wondered if we would give her extra graphs to gracefully label.

This study could potentially influence the horizon of teaching because students are not typically given the chance to explore an unsolved graph theory problem in an elementary classroom. Beyond that, they are not typically given a chance to reason through an experience where they have to provide justification or proof. With the case of Heidi, she developed a more in depth understanding of what it meant to prove something with four opportunities. These types of experiences are important because students have the belief they should be able to finish a mathematics problem in five minutes or less (Schoenfeld, 1992) and we need students to understand that any good problem solving will take much longer than five minutes! Further research is needed in this area of discovery; what could happen if we went against the current horizon and shifted our expectations as a community of mathematics learners from a focus on conceptual and procedural understanding to a focus on proving, reasoning, and providing justifications for work?

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IMPLEMENTACIÓN DE UNA ESTRATEGIA DE APRENDIZAJE FUNDAMENTADA EN UN SISTEMA DE GEOMETRÍA DINÁMICA

Daniel Ortiz-May
Cinvestav-IPN
daniel.ortiz@cinvestav.mx

Daniel Aguilar-Magallón
Cinvestav-IPN
aguilarm@cinvestav.mx

Se presentan los resultados de implementar una propuesta de aprendizaje enfocada en la resolución de problemas de variación con el uso de un Sistema de Geometría Dinámica (SGD). Los participantes del estudio fueron estudiantes de último semestre de bachillerato. En particular, interesó documentar cómo algunas 'affordances' del SGD permitieron a los participantes desarrollar e implementar ciertos recursos y heurísticas de resolución de problemas. Los resultados muestran que affordances como el arrastre de objetos, la medición de atributos y la generación y visualización de lugares geométricos fueron importantes, en primera instancia, para resolver problemas de variación con argumentos empíricos y visuales y, en segunda instancia, para formular conjeturas y presentar argumentos algebraicos.

Keywords: Resolución de Problemas, Cálculo, Tecnología, Educación Media Superior

1. Introducción

Un concepto fundamental en el marco de la resolución de problemas es el de heurística. Para Schoenfeld (1985): "las heurísticas son las reglas de oro para resolver problemas efectivamente. Son estrategias básicas para lograr progresos en problemas no familiares o difíciles" (p. 44). Polya (1965) ofrece una amplia lista de heurísticas, algunos ejemplos son: realizar analogías (considerar problemas similares o relacionados), descomponer y recomponer el problema (relajar las condiciones del problema), dibujar una figura o esquema, razonar el problema en forma regresiva, buscar objetivos particulares, obtener el resultado de diferentes maneras, entre otras. Aguilar-Magallón y Poveda (2017) argumentan que el uso de un SGD ofrece acercamientos que motivan diferentes episodios de resolución de problemas que involucran aspectos relacionados con la visualización, exploración y validación de conjeturas además de la generalización y exploración de resultados. ¿Cómo influye e impacta un SGD en el uso de heurísticas para resolver problemas? Esta pregunta es importante porque el SGD ofrece herramientas particulares y funcionalidades (*affordances*) que determinan los procesos de acción y respuesta que puede implementar el estudiante (Mackrell y Bokhove, 2017). En esta misma dirección, Leung (2011) menciona que las affordances que ofrece el SGD juegan un papel importante en el aprendizaje de las matemáticas:

> En el proceso de aprender cómo utilizar herramientas en una tarea matemática, los aprendices gradualmente construyen ideas matemáticas que son moldeadas por el empleo de las herramientas y que dan lugar a conceptos expresables en términos de la herramienta (p. 326).

Para este estudio resultó importante documentar los resultados de implementar una estrategia de aprendizaje diseñada alrededor de ciertos recursos y heurísticas de resolución de problemas mediadas por algunas affordances que ofrece el SGD. Estos resultados se describen en términos de estrategias que mostraron los participantes durante el proceso de resolver problemas de variación. En este sentido la pregunta de investigación que guio el estudio es: ¿Qué estrategias y formas de razonamiento exhiben estudiantes de bachillerato al resolver problemas de variación en un ambiente de aprendizaje basado en la resolución de problemas y uso de un SGD?

2. Marco Conceptual

Al emplear un SGD en como herramienta de aprendizaje en escenarios de resolución de problemas se simplifica la ejecución de heurísticas tales como el análisis de casos particulares y la observación de familias de objetos que cumplen ciertas características (Santos-Trigo y Aguilar-Magallón, 2018); además, un SGD constituye una herramienta útil para desarrollar ideas conceptuales relacionadas con la variación (Leung, 2008). En este sentido, el software y el individuo tienen una relación dinámica que ofrece permisibilidades y restricciones; por un lado, se conciben y abordan fenómenos matemáticos de maneras particulares que moldean el aprendizaje y, por otro, el uso del software delimita y enfoca la atención de los individuos hacia elementos matemáticos específicos (Calder y Murphy, 2018). Estas permisibilidades y restricciones se denominarán en este trabajo como *affordances*. El arrastre, por ejemplo, es una affordance que distingue los SGD y conforma una herramienta esencial que permite a los usuarios percibir la dependencia lógica, en términos del contexto matemático, a través del movimiento de objetos que es producido por el movimiento de otros objetos; esto requiere que la información perceptual sea interpretada en relaciones de condicionalidad. (Baccaglini-Frank y Mariotti, 2010).

Al respecto, Aguilar-Magallón (2018) propone una estrategia de resolución de problemas de variación dentro de un SGD llamada análisis dinámico de relaciones (ADR) basada en el arrastre de objetos y la generación de conjeturas matemáticas de condicionalidad. Esta estrategia es la conjunción de algunas heurísticas de resolución de problemas como relajar las condiciones del problema, analizar casos particulares y extremos, analizar y visualizar patrones e invariantes (Santos-Trigo y Reyes-Martinez, 2018) y affordances del SGD como el arrastre de objetos, medición de atributos de figuras, creación de puntos dinámicos y visualización de lugares geométricos. La estrategia ADR se describe principalmente en tres fases: I) *Representación*, en la cual se construye un modelo dinámico de la situación problemática; II) *Exploración*, en donde se identifican los atributos variables que se desean analizar en términos de puntos dinámicos que los relacionan; III) *Solución empírica*, en donde se visualiza la solución al problema de variación por medio del lugar geométrico del punto dinámico creado en la fase de exploración. La naturaleza ejecutable de las representaciones en un SGD dota de movimiento o variación a los objetos matemáticos que usualmente son estáticos en el papel, de modo que introduce nuevas epistemologías que afectan los enfoques de resolución de problemas de los estudiantes y por ende su proceso de aprendizaje de las matemáticas (Santos-Trigo y Moreno-Armella, 2016).

Así, el objetivo de la propuesta de aprendizaje fue que el participante se apropiara de la estrategia ADR como herramienta para resolver problemas de variación.

3. Metodología

La propuesta se implementó con un grupo de 22 estudiantes de bachillerato entre 17 y 18 años, como parte de un taller de Cálculo durante 5 semanas con dos sesiones por semana. Éstas se desarrollaron en aulas de cómputo, donde cada estudiante contaba con una computadora con el SGD (GeoGebra) instalado. Cada sesión estuvo guiada por la resolución de uno o varios problemas que los investigadores planteaban. Los participantes contaban con un momento inicial para trabajar individualmente o en grupos, posteriormente se realizaba una discusión plenaria acerca de las diversas formas de abordar el problema, en la cual los estudiantes exponían frente al grupo sus acercamientos y explicaciones. Finalmente, los investigadores cerraban la sesión con una síntesis. La propuesta se estructuró en tres etapas principales: una etapa de introducción al SGD, una de instrucción de la estrategia ADR y la tercera etapa, de implementación de la estrategia ADR. Las sesiones agrupadas por etapas se resumen en la Tabla 1.

Tabla 1: Organización general de la propuesta de aprendizaje

Etapa	Sesiones	Descripción
Introducción	1-3	Presentación de los aspectos básicos del SGD mediante actividades de argumentación.
Instrucción	4-6	Empleo de las Affordances del SGD para la generación de la estrategia ADR.
Implementación	7-10	Uso de la estrategia ADR para la resolución de problemas de optimización

Los datos del estudio se obtuvieron del trabajo de los estudiantes a través de sus archivos de GeoGebra, producciones textuales y videograbaciones de las sesiones. Los investigadores analizaron el trabajo de las sesiones y, para los fines de este documento, se seleccionaron episodios representativos de cada etapa, en los que se ejemplifican los procesos e ideas que las caracterizan. En la etapa de introducción, el problema representativo consistió en construir un triángulo rectángulo cuya hipotenusa fuera un segmento dado; en la etapa de instrucción, se presenta el problema de encontrar las medidas de un triángulo rectángulo inscrito en una circunferencia dada cuya área sea máxima y en la etapa de implementación, encontrar las medidas de un rectángulo inscrito en un triángulo rectángulo cuya área sea máxima.

4. Implementación y resultados

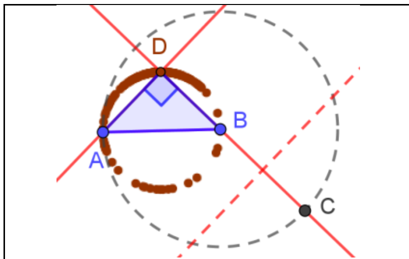
En esta sección se presentan los episodios representativos de cada etapa de la propuesta. Se destacan los procesos cognitivos que las affordances del SGD favorecieron para la consolidación de la estrategia ADR como una herramienta para resolver problemas de optimización.

Etapa de introducción. Se presentaron a los estudiantes aspectos elementales del funcionamiento de GeoGebra (trazo de puntos, segmentos y círculos) que sirvieron para la construcción de objetos geométricos de mayor complejidad: punto medio de un segmento, mediatriz de un segmento, un triángulo isósceles, un triángulo rectángulo y un cuadrado. Las discusiones se orientaron, principalmente, a la validación de las construcciones de los estudiantes, quienes no estaban habituados a este tipo de tareas. En la Tabla 2 se muestra uno de los problemas abordados en esta etapa con una solución presentada por uno de los estudiantes. En un principio las soluciones al problema planteado estuvieron basadas en aproximaciones, pues los participantes construyeron triángulos ABD (con D móvil sobre el plano) que visualmente parecían rectángulos, sin embargo, al mover cualquier vértice este no conservaba las propiedades de un triángulo rectángulo. Esta necesidad por obtener un modelo dinámico robusto (que conserve las propiedades estructurales de un triángulo rectángulo sin depender de los puntos móviles de la configuración) fue resaltada por los investigadores. Gracias a esta discusión, los participantes pudieron encontrar modelos robustos como el mostrado en la Tabla 2.

Tabla 2: Problema representativo de la Etapa de Introducción

Problema planteado	Aproximación dinámica

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

<table>
<tr>
<td>


Dado un segmento cualquiera AB, trazar un triángulo rectángulo que tenga por hipotenusa *AB*
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<td>
Recursos y estrategias: arrastre de puntos, mediatriz, paralelismo.
Solución: Se traza la circunferencia de radio AB con centro en B; se coloca un punto móvil C en la circunferencia. Se traza la mediatriz de BC y la recta paralela a la mediatriz, que pasa por A. Luego, desde B se traza una recta que es perpendicular a la recta que pasa por A. El punto D es entonces la intersección de dos rectas que son perpendiculares, obteniendo así el triángulo rectángulo ABD.
Extensión: ¿Cómo se comporta el punto D cuando el punto móvil C se mueve a lo largo de la circunferencia? Con la traza se nota que D describe una circunferencia de diámetro AB.
</td>
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</table>

A partir de un análisis retrospectivo de la solución, se observa la implementación de una estrategia de generalización importante: con ayuda de la circunferencia con centro en B y el punto móvil C sobre ella se generó una familia infinita de rectas BC. Un fenómeno interesante que se observa en este acercamiento es que el trazo de la mediatriz no era necesario, sin embargo, es posible que el estudiante sintiera la necesidad u obligación de utilizar a la mediatriz, que había sido un recurso abordado en actividades anteriores.

Dado que el movimiento del punto *D* depende del movimiento del punto *C*, se planteó a los estudiantes utilizar la herramienta *trazo* de GeoGebra para estudiar la trayectoria descrita por *D*. Los participantes conjeturaron que la trayectoria seguida por el punto D era una circunferencia que pasa por *A* y cuyo centro es el punto medio *E* de la hipotenusa *AB* del triángulo (ver Figura 1). En este momento, la actividad se centró en demostrar dicha conjetura. Gracias a la herramienta de medición los participantes observaron que la longitud de los segmentos ED y EB eran iguales y constantes (argumento empírico). El investigador presentó un argumento geométrico basado en el rectángulo *BDAF* y la propiedad de que sus diagonales son congruentes y se intersecan en su punto medio.

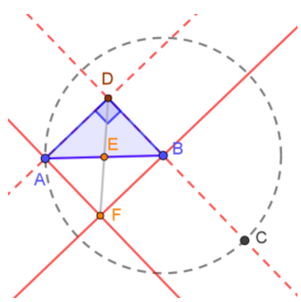


Figura 1: El punto *D* siempre está a la misma distancia del punto *E*.

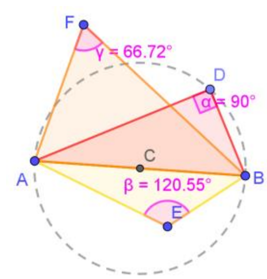


Figura 2: ¿Existe un triángulo rectángulo de hipotenusa *AB* cuyo tercer vértice no esté en la circunferencia de diámetro *AB*?

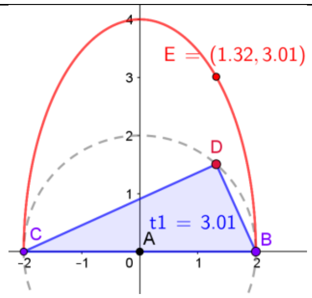
Esto quiere decir que existe una infinidad de triángulos rectángulos cuya hipotenusa es *AB*, sin embargo, el tercer vértice de todos ellos siempre está sobre la circunferencia de diámetro *AB.* De este modo, se planteó a los estudiantes la pregunta inversa: ¿es posible encontrar un triángulo rectángulo de hipotenusa *AB* cuyo tercer vértice **no** esté sobre la circunferencia que tiene por diámetro *AB?* En la Figura 2 se ejemplifica el tipo de exploraciones que los estudiantes realizaron con ayuda de la medición para conjeturar que no es posible: trazaron triángulos con base *AB* y cuyo tercer vértice estuviera dentro o fuera del círculo con centro en *C*. Gracias a esta exploración concluyeron que el ángulo en dicho vértice es obtuso (si está dentro del círculo) o agudo (si está fuera del círculo). De aquí se concluyó que un triángulo *ABC* es rectángulo con hipotenusa *AB* solo cuando *C* se encuentra sobre la circunferencia cuyo diámetro está sobre *AB.*

Cabe destacar que en esta etapa no solo se introdujo a los estudiantes al uso del SGD, sino que también se promovieron razonamientos propios de la resolución de problemas como lo son el análisis y validación de conjeturas.

La mezcla de aspectos tecnológicos y de contenido de esta etapa se observa en el empleo de estrategias de solución mediadas por elementos de la herramienta como la medición y el arrastre de puntos.

Etapa de instrucción. En esta etapa, es necesario que los estudiantes estén familiarizados con el SGD y con las nociones de función y variación. Una pieza clave en la estrategia ADR consiste en construir un punto dinámico relacional cuyas coordenadas corresponden a dos magnitudes variables dentro del modelo dinámico (longitudes de segmentos, medidas de ángulos, áreas, perímetros, etc.). La variación de las magnitudes depende del arrastre de un punto inicial en el modelo dinámico; si el punto inicial se mueve por una trayectoria bien definida (recta, circunferencia, cónica) es posible trazar el lugar geométrico del punto dinámico relacional. En la Tabla 3 se muestra el problema trabajado en la sexta sesión, el cual servirá como ejemplo del tipo de ideas trabajadas en esta etapa.

Tabla 3: Problema representativo de la Etapa de Instrucción

Problema Planteado	Aproximación dinámica
 Analizar los atributos de un triángulo inscrito en un semicírculo.	**Recursos clave:** Triángulo inscrito en un semicírculo, área de un triángulo, lugar geométrico, movimiento controlado, comando de abscisa de Geogebra. **Fase de representación:** Para obtener una configuración dinámica, se traza una circunferencia de radio *AB* y un triángulo con base igual al diámetro *CB* cuyo tercer vértice móvil *D* se mueve sobre la circunferencia. **Fase de exploración**: Definir el punto dinámico *E* cuya abscisa es la misma que la del punto *D* y su ordenada es el área del triángulo *BDC*. El lugar geométrico que describe *E* cuando *D* se mueve es una cónica. En este caso, se ha decidido explorar el comportamiento del área del triángulo. **Fase de solución:** Cuando *D* se coloque sobre el Eje *Y* se alcanza el punto máximo sobre la curva.

Se retomó la construcción del triángulo rectángulo y se empleó como un punto de partida para iniciar un análisis variacional que los investigadores plantearon de manera abierta al grupo: ¿qué atributos de la figura varían y pueden ser estudiados? La primera propuesta fue explorar el área del triángulo en función de la abscisa del punto *D.* En este caso, el punto *D* funge como punto inicial, y su trayectoria de movimiento está bien definida (dentro de una circunferencia). Así, los estudiantes definieron el punto dinámico relacional *E=(x(D), t1)* con ayuda de la barra de entrada de GeoGebra, donde *x(D)* es el comando que calcula el valor de la abscisa del punto *D* y *t1* es el valor numérico del área del triángulo *CBD* (Tabla 3). Empleando la herramienta de lugar geométrico, los participantes trazaron la trayectoria descrita por el punto *E* a medida que *D* se mueve, obteniendo la mitad de una elipse. Los participantes conjeturaron que cuando *D* se coloca *directamente sobre* el punto *A,* se obtiene un triángulo cuya área es máxima y, visualmente, notaron que para cualquier otra posición de *D* el triángulo tenía un área menor. Posteriormente, se les solicitó a los estudiantes trabajar en parejas, o individualmente, en proponer y estudiar otras relaciones y analizar si en ellas existen máximos (o mínimos); algunas de las propuestas fueron:

- La longitud de alguno de los catetos en función de la abscisa del punto *D*.
- El área del triángulo, en función de la altura del triángulo.
- El perímetro del triángulo, en función de la abscisa del punto *D*.

En este reporte, solo se presentará el análisis de un estudiante acerca del primer punto. En la Figura 3 el punto dinámico relacional *E* tiene por coordenadas *(x(D), b)*, donde *b* es la longitud del lado *AD*. El lugar geométrico que describe *E* a medida que *D* se mueve permite observar que *D* puede aproximarse a *B* tanto como se desee, incrementando arbitrariamente la longitud de *AD*, sin embargo, si esos dos puntos coinciden, no se genera un triángulo.

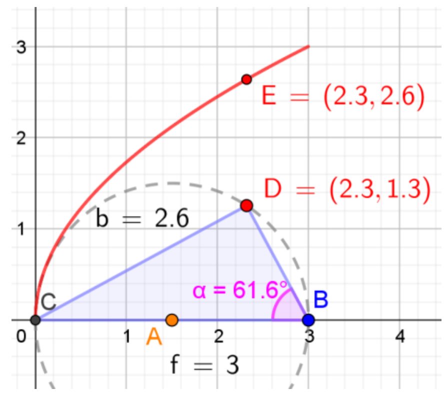


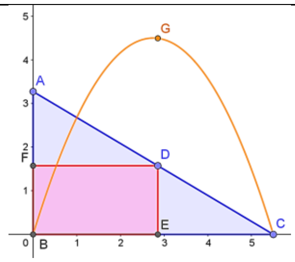
Figura 3: Relación dinámica entre la abscisa del punto *D* y el cateto *b*

Como una exploración adicional, el estudiante identifica la relación *sen*(α)=|*AB*|/|*BC*| Por lo que genera otro punto dinámico relacional *F* que tiene por coordenadas *(x(D), e)*, en donde *e*=*f*·*sen*(α) y *f* es la hipotenusa del triángulo. El estudiante comprueba que su razonamiento es correcto ya que al definir ese punto, *E* y *F* siempre están superpuestos al mover el punto *D*, e incluso cuando el diámetro de la circunferencia inicial incrementa. La producción de este estudiante exhibe la manera en que los objetos observados en el SGD constituyen una plataforma que favorece la abstracción de las variables. Es decir, aunque solo puedan percibirse instancias particulares (por ejemplo, en la Figura 3 |*CB*|=*3*), es posible concebir que las medidas estáticas que observan en la pantalla representan magnitudes que pueden variar libremente. Este estudiante no emplea el valor particular de *AB* para establecer la relación trigonométrica del cateto, aunque *AB* se mantenga constante cuando *D* se mueve, sino que introduce el *nombre* que el SGD asigna a la hipotenusa, pues es consciente de que todos los elementos de la construcción pueden variar.

Etapa de implementación. La estrategia ADR es una potente herramienta para comprender e interpretar problemas de optimización en contextos geométricos. En la sesión 10 se planteó el problema de encontrar las medidas (lados y área) de un rectángulo con área máxima inscrito en un triángulo rectángulo, tal que dos de los lados del rectángulo estén sobre los lados del triángulo (ver Tabla 4).

Tabla 4: Problema representativo de la Etapa de Implementación

Problema planteado	Aproximación dinámica

<table>
<tr>
<td>
Encontrar las medidas de un rectángulo inscrito en un triángulo rectángulo cuyos catetos miden m y n tal que su área es máxima.</td>
<td>Recursos clave: Perpendicularidad, área de un triángulo, área de rectángulos, movimiento controlado, lugar geométrico
Fase de representación: Se traza el triángulo ABC cualquiera, con catetos sobre los ejes. Se construye un punto D móvil sobre la hipotenusa y se trazan las perpendiculares hacia cada eje. El rectángulo BEDF cumple las condiciones iniciales.
Fase de exploración: El punto dinámico G relaciona la posición de D sobre la hipotenusa y el área del rectángulo correspondiente. El lugar geométrico de G cuando D se mueve es una parábola.
Solución empírica: Si D es el punto medio de la hipotenusa, G se ubica en el vértice de la parábola. Los lados del rectángulo BEDF miden la mitad de los catetos del triángulo ABC.</td>
</tr>
</table>

En el proceso de construir una configuración dinámica, debe considerarse que las dimensiones del triángulo y las del rectángulo varían, es decir, se presentan dos formas de variación. Distinguir entre ambas formas de variación es fundamental para la implementación apropiada del ADR. Si bien el problema es general, algunos estudiantes comenzaron analizando el caso particular cuando el triángulo rectángulo dado es isósceles (ver Figura 4). El punto *D* es móvil sobre AB y el punto dinámico G tiene por coordenadas el lado AD (la abscisa) y el área del rectángulo (la ordenada). El lugar geométrico descrito por *G* parece ser una parábola.

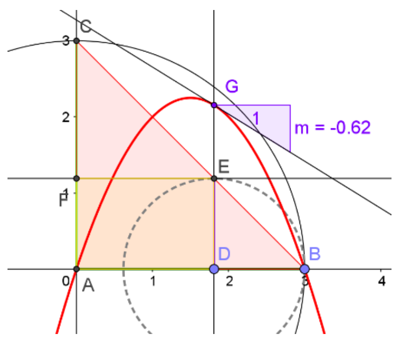


Figura 4: Configuración dinámica del problema del rectángulo inscrito

De manera empírica, la estudiante concluye que cuando el rectángulo es un cuadrado, se alcanza el área máxima y que los lados del cuadrado miden la mitad de los catetos del triángulo. Para apoyar esta idea, la estudiante se da la tarea de explorar el comportamiento de las rectas tangentes a la curva descrita por el movimiento de *G*. Geogebra no admite ciertas acciones sobre los lugares geométricos, de modo que encontrar la ecuación que describe la curva se convierte en una tarea impuesta por la necesidad de interactuar con ella. Su proceso es como sigue:

> Estudiante: Sabemos que para sacar el área de un rectángulo utilizamos base por altura, por lo tanto, en este caso usé la fórmula de $S=x(a–x)$ tomando en cuenta que x es el segmento *AD* la cual tomaríamos como base y $a–x$ es el segmento *DE*. Para justificar esto se hizo una circunferencia de radio *DB* y vemos que la altura es igual al radio del círculo, *DE* y *DB* son iguales.

Cabe destacar que la estudiante es consciente que la gráfica de la ecuación que encuentra y el lugar geométrico descrito por *G* deben mantenerse superpuestas al variar el *AB*, de modo que emplea la variable *a* en la expresión, la cual representa la medida general de los catetos del triángulo. Una vez concluido el caso particular, se procede a verificar si en el caso general la solución se mantiene. Puesto que conseguir una expresión analítica de la función área en el caso general demanda mayor esfuerzo, los estudiantes no consiguieron ofrecer argumentos algebraicos para la solución general, sin embargo, fueron capaces de afirmar que el rectángulo inscrito de mayor área siempre posee un medio del área que el triángulo rectángulo dado y que sus lados miden la mitad de los catetos del triángulo.

Aunque el objetivo de la propuesta nunca fue el de emplear el método analítico tradicional por el cual se resuelven los problemas de optimización (criterio de la primera derivada), el SGD dota de cierta materialidad a los conceptos de variable que podría fungir como una forma de dar sentido a los procesos analíticos por la necesidad de interactuar con ella. En la Figura 4 se muestra una exploración adicional, en la que los estudiantes observan el comportamiento de la pendiente de una recta tangente a la curva en *G*, y a media que se aproxima al vértice, la pendiente tiende a cero. Esto representa un ejemplo del tipo de oportunidades y acercamientos que el empleo del SGD ofrece para ayudar a los estudiantes a interpretar y comprender conceptos esenciales del cálculo.

5. Conclusiones

Parte esencial del trabajo en la propuesta de aprendizaje fue la de introducir elementos pedagógicos en la estrategia ADR. Para conseguir que los estudiantes se apropien del ADR como una heurística en la resolución de problemas, fue necesario crear un campo propicio de trabajo en el que las ideas matemáticas fundamentales fueran desarrolladas junto con la tecnología, de modo que la estructuración de la propuesta en etapas jugó un papel importante. Cada etapa estuvo marcada por la consecución de objetivos necesarios, que contribuyeron a la implementación del ADR. En la primera etapa, se trabajó con tareas de construcción orientadas a prácticas argumentativas, apoyadas en las affordances del SGD: los estudiantes comprobaban sus resultados con ayuda de la medición de segmentos y ángulos, o verificando que sus construcciones se mantuvieran coherentes ante el arrastre de los puntos involucrados; En la etapa de instrucción, fue clave el papel de la parametrización de lugares geométricos en el trabajo de los estudiantes, pues estos procesos fungieron como antesala a la producción de argumentos geométricos y algebraicos. El uso del SGD probó ser de gran ayuda en la abstracción de variables por parte de los estudiantes, ya que estas eran representadas por segmentos cuya variación podía ser percibida de manera concreta. Finalmente, en la tercera etapa, los estudiantes fueron capaces de integrar recursos geométricos y visuales para la construcción de configuraciones dinámicas como representaciones de un problema y, a partir de los elementos ofrecidos por el SGD, obtener conjeturas de las soluciones de problemas de optimización.

6. Agradecimientos

Se agradece el apoyo recibido del proyecto SEP-Cinvestav-12 durante el desarrollo de esta investigación.

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IMPLEMENTATION OF A LEARNING STRATEGY BASED ON A DYNAMIC GEOMETRY SYSTEM

We present the results of the implementation of a set of instructional activities that focused on solving variation problems with the use of a Dynamic Geometry System (DGS). The study was conducted with high school students in their calculus course. Particularly, we were interested in documenting how some affordances of the DGS allowed the participants to develop certain problem-solving resources and heuristics. Results show that affordances such as dragging objects orderly, measuring objects and tracing some objects loci were important, at first instance, in the process of solving variation problems with visual arguments and, secondly, for the formulation of conjectures and presentation of algebraic arguments.

CHARACTERIZING EVOLUTION OF MATHEMATICAL MODELS

Jennifer A. Czocher
Texas State University
Czocher.1@txstate.edu

Hamilton Hardison
Texas State University
HHardison@txstate.edu

Formulating a mathematical model is a dynamic process, and attending to changes in students' models as they engage in modeling tasks is critical for informing pedagogical interventions. In this report, we coordinate constructs from literature on mathematical modeling, quantitative and covariational reasoning, and semiotics to characterize changes in mathematical models. We illustrate the application of these constructs using data from an undergraduate solving a modeling task.

Keywords: Modeling, Using representations

Mathematical modeling remains an important skill for K–20 students (Bliss et al., 2016; CCSSM, 2010), and relatively little is known about how to prepare teachers to interact with a student-modeling task dynamic. This is partly because the field has yet to bridge task-centered analyses common to research with teacher-centered "tips" on what to notice in students' work, because it is not clear what to direct teachers to focus on as students model. A cognitive view of modeling (Kaiser, 2017) can be leveraged to provide just such a link. A detailed exploration of how a mathematical model evolves with an eye toward identifying pivotal moments in students' reasoning that could be used for teacher education and teacher intervention has yet to be undertaken. In particular, it would be useful to teachers and teacher educators to know when, how, and with what level of scaffolding to intervene in a student's modeling process, since the best results from teaching with a modeling approach are obtained when students work out their own solutions (Kaiser, 2017). To make progress in this area, it is first necessary to address the methodological problem of articulating criteria for determining whether, and the extent to which, a student's model has changed. In this methodological paper, we will use micro-analytic techniques to document the changes that a student introduces to his model and network a set of theoretical constructs for explaining those changes.

Theoretical Perspectives

The target construct in this exploratory work is *model evolution*, specifically, revisions that a student might make to his/her model during mathematical modeling. A priori, there are several related theories and attendant constructs that might serve to articulate and trace changes to a model (elaborated below) that offer partial explanations of how individuals' models evolve. The Networking Theories Group (2014) proposed strategies like combining, coordinating, integrating locally, and synthesizing, for conducting parallel analysis of empirical phenomenon. Combining and coordinating involve generating "deeper insights into an empirical phenomenon" (p. 120). Integrating and synthesizing involve development of new theory by building on a small number of already-stable theoretical approaches. Here, we foreground our efforts coordinating theories of symbolic forms, multiple representations, and quantitative reasoning. We use one undergraduate's work on a modeling task to facilitate the theory-building. The end result is a method for using observable indicators to trace the evolution of a model. We then apply the method to identify and characterize significant conceptual hurdles experienced by the student, along semiotic and cognitive (Greca & Moreira, 2001) dimensions.

The mathematical modelling process is often conceptualized as an iterative cycle. One approach to studying individuals' modeling activity is to examine what are termed *modeling competencies* (Kaiser, 2017; Maaß, 2006). This approach is cognitive in nature and addresses how students come to understand a real problem and choose mathematical representations for it as they work on challenging tasks, which encourage accounting for various constraints. These competencies are *not* meant to suggest student abilities, but rather describe *phases* of modeling in a way that opens the process to observation and analysis. Phases include formulating a problem to solve (identifying aspects or characteristics that need to be modeled), systematizing (selecting relevant entities and relationships, identifying variables, making assumptions, or estimating parameters), mathematizing (representing entities and relationships in mathematical notation), analyzing (using mathematical techniques to arrive at mathematical conclusions), validating (evaluating the model and establishing its scope), and communicating (sharing conclusions obtained from its use) (Blomhöj & Jensen, 2003; Blum & Leiß, 2007). Student decisions made during each phase contribute to the dynamic evolution of the model. The systematizing and mathematizing phases can be viewed as model construction whereas validating and verifying can be viewed as a reflective monitoring process (Czocher, 2018). Empirically, the phases do not proceed linearly (Czocher, 2016) and the process draws on a complex interplay of mathematical, nonmathematical, and perhaps even scholastic knowledge (Stillman, 2000). Throughout these phases, the model can be refined, modified, or entirely rejected (and replaced) as it evolves to meet the modeler's problem-solving needs. In the next sections, we lay out additional relevant theories that offer insight into ways models could change.

Greca and Moreira (2001) argued that comprehension of a topic in physics is tantamount to being able to predict phenomena without needing to reference mathematical formalism. They distinguish among physical models, mathematical models, and mental models, while also maintaining that an integration of all three is necessary for building understanding. In their elaboration, "a *physical theory* is a representational system in which two sets of signs coexist: the mathematical signs and the linguistic ones" (p. 107). Physical theories are not direct presentations of observations of phenomena or objects; instead, statements of physical theories are about simplified and idealized physical systems, which they term *physical models*. The role of mathematics, then, is to formalize the theory as statements without semantic content. They characterize the *mathematical model* as a "deductively articulated axiomatic system, which can express the statements of the theory in terms of equations" (p. 108), but also acknowledge that the term may also extend to the mathematical theory the syntactic structure is derived from. *Mental models*, then, are internal and idiosyncratic representations of phenomena, which contrast physical and mathematical models which are socially mediated. Finally, they posit "families" of distinct mental models that serve as explanations for phenomena. For example, explaining a physical phenomenon like motion as interactions of forces rather than as a consequence of a linear, causal agent. Greca and Moreira's (2001) demarcation and explanation of interaction among mental, physical, and mathematical models is compatible with a competency view of mathematical modeling, even extending from physics education to modeling other phenomena of interest. We note some changes in vocabulary. We use *mathematical representation* to refer to an outward expression of an individual's mathematical model (Greca and Moreira's "mathematical model"). *Mathematical model* refers to the attendant conceptual system. Empirically, only the mathematical representations are observable, though changes in these may indicate shifts in students' perceptions of the relevant mathematical or physical concepts.

We take two further positions from mathematics education. First, we hold that mathematical models can be expressed in other conventional forms, besides equations (e.g., tables, graphs, words, etc. (Hitt & S., 2014)). Second, the modeling cycles that are common descriptions of students' mathematical modeling activity tend to conceptualize the modeling process as deriving mathematical expressions and variables from the physical problem. In contrast, Greca and Moreira (2001)asserted that variables in equations have meanings only *after* the they are interpreted through a physical model. From our own experience, we suggest that there may not be a general rule regarding whether the mathematical model or the physical model come first when addressing the kinds of modeling problems that are found in educational research. However, the point is tangent to two other useful theories: Sherin's (2001) elaboration of how individuals use mathematical models as templates to adapt to the problem at hand and Thompson's (2011) theory of quantitative reasoning.

A priori, mathematizing a situation would involve generating mathematical models and assigning semantic meanings drawn from physical models of physical theories. Identifying quantities and describing how they vary is just such a link. Thompson's (2011) theory of quantitative reasoning offers relevant insights. First, Thompson asserts that quantities are mental constructs, not characteristics of objects in the world. It immediately follows that a *quantification process* is carried out by an individual in order to conceive of quantities and that the process is non-trivial. Quantification is taken to mean "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bilinear, or multi-linear) with its unit" (p. 37). One can then also imagine an object, an attribute, and a measure in such a way that the value of the measure takes on different values at different moments. Observing a phenomenon and conceptualizing that there are quantities and that they can vary (or may be constant) is foundational to formulating physical models and articulating physical theories. These mental acts may become quite familiar or nearly automatic if one has much experience in the context. For example, quantities like distance and velocity may be more readily available for high school students than torque, electrical current, or GDP.

Quantitative reasoning entails conceiving of quantities and relationships among quantities. Thus, deriving a mathematical model entails quantitative reasoning. However, conceiving of covarying quantities is a non-trivial mental act. Thompson (2011) explains that covariational reasoning involves conceiving of invariant relationships among quantities whose values may vary independently. The difficulty lies in imagining how a situation can change, the quantities conceived from it can change, but that a relationship among them stays the same. In deterministic language, mathematical modeling entails discovering the invariant relationships that govern the quantities involved. Coordinating quantities and attending to relationships among quantities, variant or invariant, is *covariational reasoning* (Carlson, et al., 2002). It involves identifying ways to combine quantities through operations and trace their changes, rates of changes, and intensities of changes whether they are directly measurable or not (e.g., Johnson, 2015). Relationships can be identified through observation, a priori reasoning, or through knowledge of principles rooted in physical theory. When these relationships are expressed externally in mathematical notation, they become the mathematical representation of a physical model. The mathematical representation brings with it the relevant mathematical model (mathematical concepts, objects, and structures) and the physical meanings of its constituents (attributes, measurements, quantities). Thus, the theory of covariation in conjunction with theory of quantification elaborates an important aspect of how physical models are formalized into mathematical models.

At its core, a mathematical model presents a system of signs used to stand in for a physical system. Naturally, a theory of modeling should attend to semiotic processes that imbue meaning to the signs and to how (perhaps multiple) systems of signs are coordinated. Following Kehle and Lester's (2003) application of Piercian semiotics to mathematical modeling, we view as a process of unification among a sign, a referent (the object the sign stands for), and an interpretant. The interpretant has a dual role; it is the individual's reaction to the sign and object and simultaneously defines the sign/object pairing through the individual's reaction. Generally, interpretants can be actions, emotions, thoughts, or ideas. An interpretant is the fundamental unit of inference between an object and a sign and it is subjective and idiosyncratic.

Sherin's (2001) theory of symbolic forms, which explains how meaning is read from equations, can be construed an extended example of semiosis. A symbolic form consists of a template and a conceptual schema (the idea to be expressed in the equation). For example, _ + _ = _ expresses a "parts-of-a-whole" relationship. The blanks can be filled with a single symbol or a group of symbols representing quantities or combinations of quantities (perhaps related via other symbolic forms). Familiarity with symbolic forms helps individuals "know" to use certain operators (e.g., + or ×) and to know where to place the symbols of quantities in an equation.

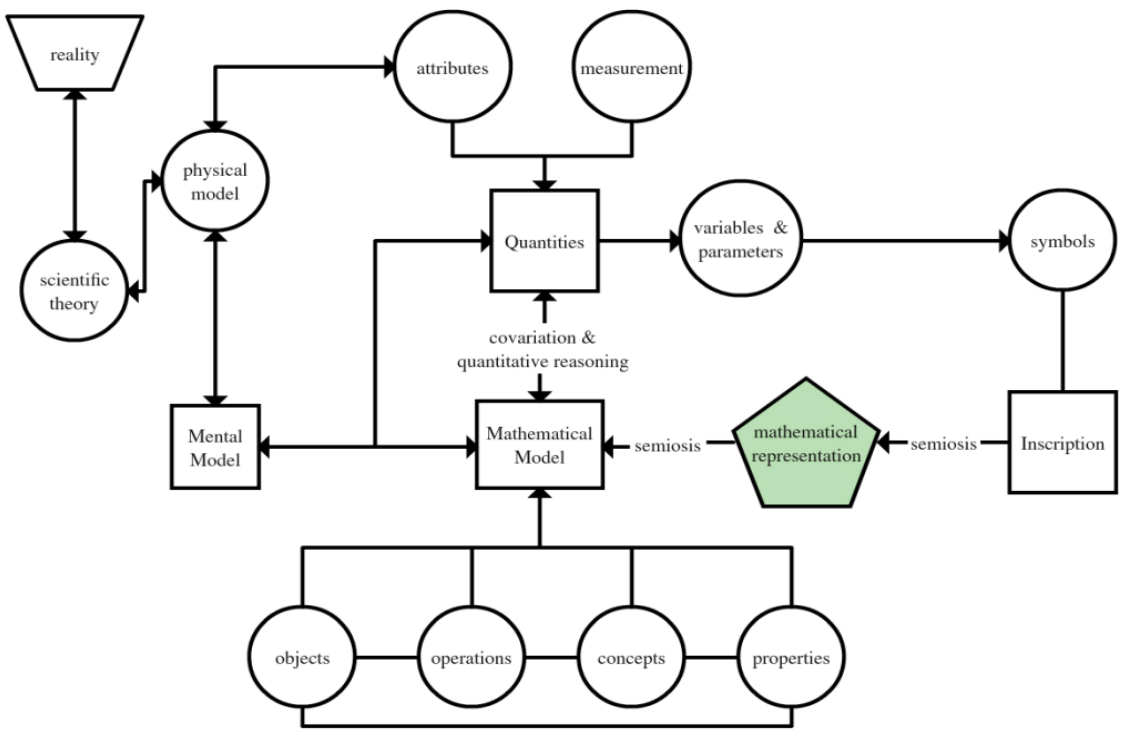

Figure 11: Networking of Major Theoretical Constructs (rectangles), Components (circles), and Mental Processes (arrows) to Hypothesize Sites for Change in How Physical Meaning is Projected onto Mathematical Symbols.

These theories operate at different grain sizes, were developed among different populations, and have different scopes. They are also asymmetrical in terms robustness. However, each layer of theory suggests a dimension of evolution and offers nuance to describe changes in a model at varying grain sizes. The networked theories are shown in Figure 11.We suggest that attending to the aforementioned constructs may produce a more comprehensive and nuanced account of how a student's mathematical model evolves.

Methods

To illustrate the coordination of constructs outlined above, we draw on data generated from a larger project examining instances of students' validating activity as they solved modeling, application, and word problems. Participants ranged from 6th grade to undergraduates. Here, we focus on Merik, an engineering major who had completed Calculus III (vectors) at a large southern university, and his work on The Monkey Problem. Merik's work was purposefully selected to explore the plausibility of the networked theory and articulate criteria for identifying changes in a model because (a) he was exceptionally verbal, (b) he often explained his own reasoning without prompting, and (c) he used multiple approaches to solve the problem.

> ***The Monkey Problem***: *A wildlife veterinarian is trying to hit a monkey on the tree with a tranquillizing dart. The monkey and the veterinarian can change their positions. Create scenarios where the veterinarian aims the tranquilizing dart to shoot the monkey. (And later: create models to represent the situation).*

The networked theory predicts that evolution of Merik's model might proceed along external (symbolic) or internal (meaning-making) dimensions. Thus we conducted three parallel analyses that could highlight opportunities to observe the representation or its meaning changing. The first coded Merik's work according to a modeling competencies framework (Blum & Leiß, 2007; Czocher, 2016) to document the phases of modeling, specifically *validating* activity since it is hypothesized to precipitate changes to the model. The second documented the various representations Merik used to capture and express his reasoning mathematically. The third sought evidence of shifts in meanings of the representations.

To analyze representations, we acknowledged that "what we ultimately observe are the external components (representations), but these cannot be disengaged from the conceptual systems" (Lesh & Doerr, 2003, p. 213). We introduce the term *inscription* to mean writing without implying anything about the inscription serving as a sign. We use the term *representation* to imply that the inscription serves as a sign for Merik. These theoretical commitments necessitated that methodologically we search for overt changes to inscriptions as indicators of changes to the model and separately search for evidence of changes to the cognitively-generated meanings for those inscriptions. To trace changes to representations, we first documented changes to inscriptions by attending to their spatial and temporal organization on Merik's paper. In the first case, we judged his model to have changed if either (1) the system of signs comprising by the representation changed (i.e., the type of representation changed, introducing a symbolic equation after working with a graph) or (2) a new inscription was created in a different location on the page. Each new mathematical representation was called a *parent*. We judged Merik's model to have changed through considering whether Merik's attention switched to another parent and whether there were substantive changes to the parent. To identify substantive changes, we considered (a) whether there was evidence to infer that information or meaning was distributed to the representation or removed from it, (b) whether Merik modified an inscription, or (c) whether Merik modified an inscription in a way suggestive of transporting meaning to or from another parent. We considered these alterations to a parent representation as a *child*. Children amending the same parent were called *siblings*. We recorded the *first time* an inscription was introduced and briefly annotated the nature of the change.

For example, Merik's first inscription appeared at time 0:51 (**Figure 12**). We labeled his *tree representation* as Parent 1 and this first version as Child 1A. At time 2:10, Merik added inscriptions x, θ, y producing Child 1B. Merik's talk indicated that these symbols were signs standing for the quantities *distance between veterinarian to the tree*, *distance from monkey to the ground*, and *angle formed by the veterinarian aiming his dart gun at the monkey*, respectively. His talk about 1A yielded evidence of both an implicit physical theory (kinematics) and an implicit mathematical model (right triangle geometry). The modification of the inscription from 1A to 1B signals introduction of quantities. At 2:38, Merik inscribed a new parent (#2), capturing his ideas that the dart, due to gravity, would not travel along a linear path but rather a parabolic one. The change, combined with his talk, and in addition to introducing gravity explicitly as a quantity, indicated that his mental model and physical model had shifted, necessitating a new mathematical model and mathematical representation. The symbolic form used in Representation 2A was _ - _ . It combined velocity and gravity. Notably absent are an explicit reference to time as a variable (though implicitly it was present in his talk as a quantity) and dimensional analysis. At time 4:25, Merik returned to Representation 1C, substituting in specific numbers for *x*, *y*, and *h* (with *h* standing for "hypotenuse"). We infer these symbols were parameters for Merik that could be fixed momentarily and changed from scenario to scenario.

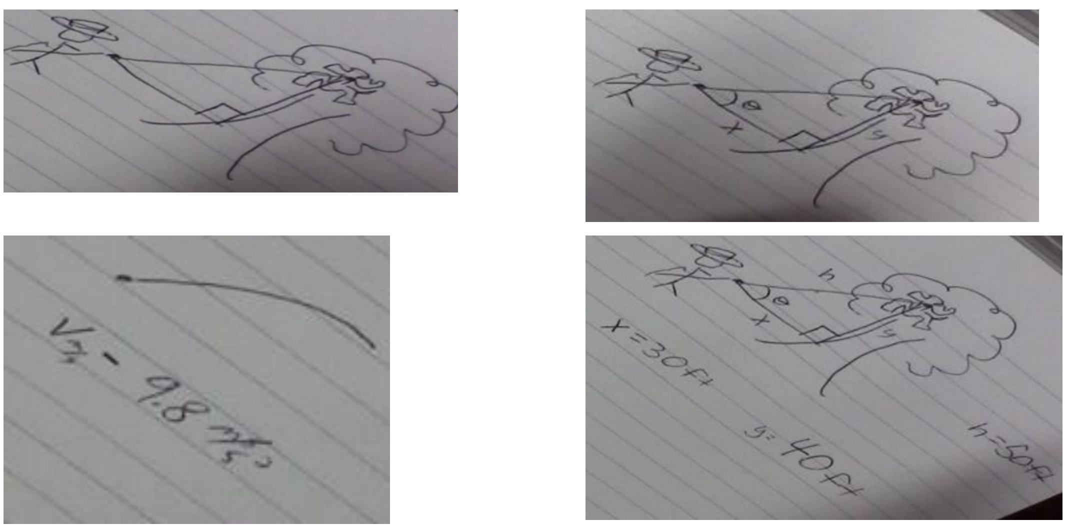

Figure 12: Merik's Representations
Clockwise from top left: Representations 1A, 1B, 1C, 2A

To help us visualize the evolution of Merik's representations, we plotted the parent and child representations over time by recording each time it could be inferred that Merik's attention was on a given sibling (Figure 13). The parent and sibling tracking system allowed us to resolve methodological issues distinguishing representations and models from one another. We next sought shifts in mental, physical, and mathematical models, and quantitative reasoning.

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

Results

Seventy-four total explicit switches in attention and modifications were identified belonging to 11 distinct parent representations (comprising 22 temporally distinct siblings). The introduction of new parent representations signaled either that Merik was refining existing ways of thinking or to introducing new approaches (see Figure 13 for evolution over time). Of the 31 instances of validating identified using techniques from Czocher (2016), only 10 co-occurred with a shift in (any aspect of) the model, suggesting that most adjustments were not consequences of validating. We share three examples of model evolution in detail.

One

At time 9:45, Merik introduced Representation 4A: $f(x) = Ax^2 + Bx + C$, a standard form quadratic equation because he asserted that it would trace the motion of the dart. At his time, he had already introduced x as distance between the veterinarian and the base of the tree holding the monkey. However, he treated the equation as a symbolic form combining quantities for initial position (*C*) and gravity (*A*), evidenced by his substitutions (representation was "$= -10x^2 + Bx + 0$"). He set $f(x)$ = 0 and solved for B obtaining $B = \frac{94}{3}$ and finally representation 4C, $f(x) = \frac{94}{3}x - 10x^2$. His attention shifted back to Representation 1 and he created 1D by inscribing a parabolic arc to connect the veterinarian and the monkey, leaving the hypotenuse of the triangle in place. Throughout this excerpt we infer that Merik was attempting to describe a spatial parabola, yet the substitutions in his symbolic form suggested a temporal component he did not explicitly attend to. By this we mean that his physical model included a trace of the path of the dart which can be described as a parabola. However, his mental model, which helped him to interpret the meaning of the symbolic form included a temporal component to which he did not explicitly attend. Thus, from our perspective, the meanings ascribed to the representations did not align; however, Merik initially indicated no misalignment due to his certainty that the situation could be represented with a quadratic form. It was not until 15:34 that Merik acknowledged that time would play a role in the quadratic equation, noting that "So yeah, um, x would be time, which means I couldn't use, yeah plugging in these [gestures to x=30 and y=40 in representation 1C] did nothing cause this equation is based on time." This analysis foregrounds the importance of explicit quantitative referents for variables when symbolic forms are used to generate mathematical models and that covariational reasoning was largely absent.

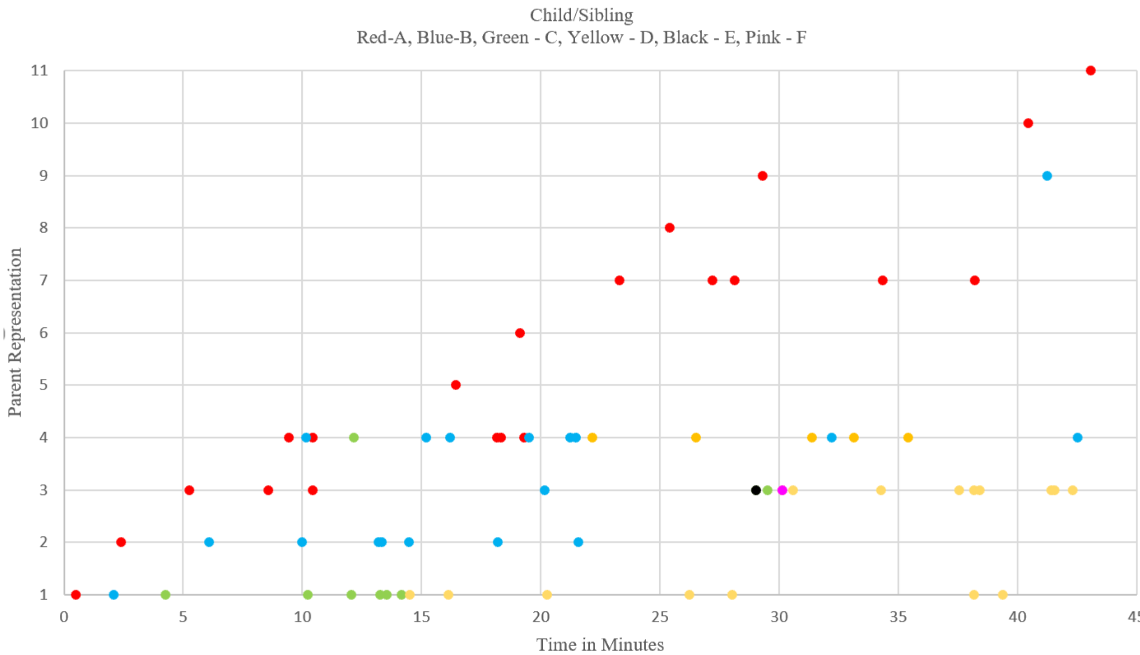

Figure 13: Evolution of Merik's Representations

Two

At 16:45, Merik introduced an analogical problem situation. He described trying to swim across a river with a current. He explained that the situation “applies the same with gravity because of the flow of the river going this way is just your rate of gravity working downwards, but your motion is that way.” Introduction of a sketch of the river (Figure 14) is indicative of abductive reasoning (a type of semiotic inference). The shift leverages a different physical model to support his mental model of how the motion of the dart could be modeled, but still using kinematics as a physical theory.

Three

At 27:18, Merik summarized a difficulty that had emerged in treating the path of the dart as a parabola: he could not use right-triangle trigonometry to determine the launch angle. He acknowledged that it shouldn’t matter that the path was curved. The interviewer intervened with the intention of suggesting an additional mathematical model: angle between tangent vectors. She asked “Can you think of a way that you might be able to find an angle measure between two curves? Have you ever studied anything like that?” She asked him to draw to curves in an xy-plane. He said, “I guess a straight line can be a curve,” and produced representation 3E (**Figure 14**). Merik realized, from a mathematical representation that did not overtly have meaning connected to the monkey problem that he could use tangent lines to curves (via derivatives) to find the angle subtended by the two curves in question. He concluded, “that’s the direction…at that exact instant which means that’s where it’s aimed. So if you do it from the point that leaves the barrel that’s the way…”. Later, at 29:30, he introduced representation 9A, a version of the law of cosines, to express θ. He also confirmed that θ would be a function of x, progressively attaching more information (in the forms of quantities, relations, and dependencies) to his representation. This analysis demonstrates that a shift in mathematical model co-occurred with a static physical theory (kinematics) and static mental mod

el (projectile follows an arc).

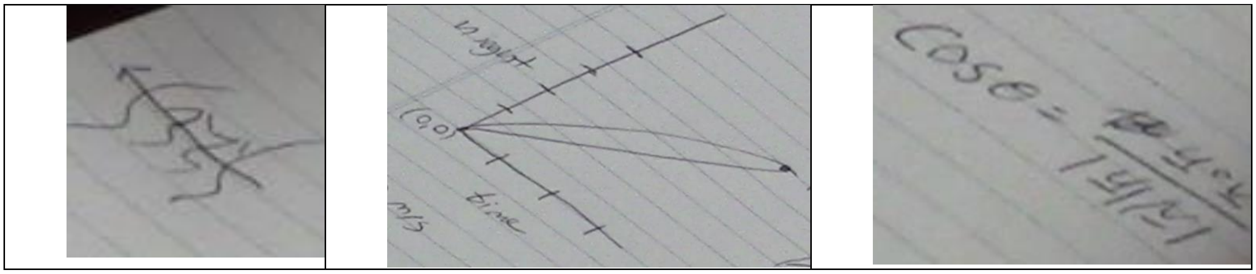

Figure 14: Representations 5A (river), 3E (coordinate axes), and 9A (law of cosines)

Our analysis exemplifies the complexity and richness of mathematical modeling when accounting for the attendant cognitive acts. We have networked theories to hypothesize and then demonstrated empirically several dimensions along which a model can change, including the adding or taking away from the inscriptions themselves, shifting attention among representations, introducing or removing quantities (attributes and how they are measured), semiotic processes (ascribing or shifting meaning of an inscription), mathematical models (including concepts, objects, operations, or procedures), physical models, and mental models. Thus, we demonstrated the plausibility of integrating the networked theory to explain the evolution of a mathematical model by tying together theories of modeling, quantitative reasoning, covariational reasoning, and semiotics.

Shifts in the ever-evolving model are important to identify because they co-occur with the critical competencies of mathematical modeling and are junctures where new ideas can be introduced and developed. We hold that sensemaking and monitoring are ongoing throughout mathematical modeling, noting that tracing representations is only partially predictive of shifts in the model. It will be important to also trace shifts evident in the students' speech that do not co-occur with changes in inscriptions. Though the Monkey Problem did not call for it (Merik worked solely within kinematics), we can easily imagine student-task interactions where a shift in physical theory is called for. We further identified at least two sites for facilitator intervention: one that was used successfully (supporting Merik in finding a mathematical model that would allow him to completely determine an angle) and one that was missed by the interviewer (the mis-alignment between a spatial and temporal parabola). Finally, we posit that the richness of structures is created and maintained through quantitative reasoning. We recommend that future research explore a parallel analysis that would trace quantification as the evolutionary thread rather than representations.

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UNIT DISTINCTION AS A PRE-REQUISITE FOR MULTIPLICATIVE REASONING: A CASE STUDY OF ADAM'S UNIT TRANSFORMATION

Cody Jorgensen
University of Colorado Denver
cody.jorgensen@ucdenver.edu

Amy Smith
University of Colorado Denver
amy.l2.smith@ucdenver.edu

Ron Tzur
University of Colorado Denver
ron.tzur@ucdenver.edu

Heather L. Johnson
University of Colorado Denver
heather.johnson@ucdenver.edu

We address the question: How can a student's conceptual transition, from attending only to singleton units (1s) given in multiplicative situations to distinguishing composite units made of such 1s, be explained? We analyze a case study of one fourth grader (Adam, a pseudonym) during the course of a video recorded cognitive interview. Adam's case provides a glimpse into this process, as he proceeded from solving a prompt-less version of the multiplicative situation to two versions that gave gradually more explicit hints for those two types of units. We postulate an inferred process in which Adam reflected on the relationships between his counting actions (of cubes and towers) and the effect of those actions being different numbers pertaining to different units. We discuss the theoretical importance of explaining this transition and its practical implication of understanding whole number multiplication (e.g., array).

Keywords: Reasoning and Proof, Learning Trajectories, Number Concepts and Operations

In this case study of a fourth-grade student named Adam (pseudonym), we addressed the question: *How can a student's conceptual transition, from attending only to singleton units (1s) given in multiplicative situations to distinguishing composite units made of such 1s, be explained?* Our study is situated within and contributes to the body of research focusing on students' units coordination as a conceptual lens to explain multiplicative reasoning (Steffe, 1992; Norton et al., 2015; Ulrich, 2016). These researchers demonstrated that, to reason multiplicatively, a student needs to coordinate and simultaneously operate on at least two qualitatively distinct types of units: singletons (1s) and composite units comprised of such 1s. Our study further elaborates on this stance in two important ways: (a) pointing to the conceptual prerequisite distinguishing between 1s and composite units plays in then also coordinating them multiplicatively and (b) explaining the conceptual process involved in arriving at this distinction.

Explaining this conceptual transition can contribute to understanding challenges students face when taught multiplication. A student may try to solve an archetypical multiplicative situation consisting of several equal-size groups, such as: How many cubes do you need to build 6 towers, each made of 3 cubes? To solve this problem, the student would need to understand two types of different units are involved: single cubes (1s) and towers comprised of three such cubes each (composite units). This distinction seems necessary also in cases where the units are presented as an array (e.g., 6 rows by 3 columns) with a total of eighteen 1s (e.g., dots). In this case, the student would need to unpack the organization of 18 items into rows and columns as a highly compact form in which she needs to coordinate two different units.

Theoretical and Conceptual Framework

Our framework for this study draws on Piaget's (1985) concept of assimilation, which stresses a learner can only make sense of some information by using available (assimilatory) conceptions. For example, upon reading the problem situation above, a student who is yet to distinguish 1s from composite units may literally see both given numbers (6 and 3) while only "taking in" the number of single cubes (3). Assimilation is thus the starting point for any transition to a new conception, which Piaget (1985) and von Glasersfeld (1995) explained by the mental process of reflective abstraction. Simon et al. (2004) elaborated on this process, in a mechanism they termed *Reflection on Activity-Effect Relationships* (Ref*AER). In a nutshell, Ref*AER consists of four mental steps: 1) setting a goal in the problem situation, 2) calling up and initiating an activity sequence to attain that goal, 3) determining if the effects of the activity match the goal, and if not 4) adjusting the activity sequence as needed. In our example, a student may come to link her or his mental actions (e.g., counting 1s) with the effects of those actions (e.g., noticing that the count of cubes resulted in "3" whereas the count of towers resulted in "6"). Tzur & Simon (2004) further distinguished two stages of mathematical conceptual learning based on the extent to which a learner's new conceptions depends on some prompting or can be independently and spontaneously called up and used by the learner. The scope of this study led us to focus the data analysis on Adam's transition to a prompt-dependent form of distinguishing 1s from composite units

As we noted above, the conceptual transformation on which we focus in this study is rooted in the explanation of multiplicative reasoning in terms of units coordination. Of the six multiplicative schemes that Tzur et al. (2013) identified, our study focuses on the first, called *Multiplicative Double Counting* (mDC). Units coordination that characterizes mDC involves the distribution of items of one composite unit (e.g., three 1s) into the items of another composite unit (e.g., six equal-size units, such as towers made up of 3 cubes each). This coordination of two types of units is essential not only for developing multiplicative reasoning but also fractional reasoning (Norton et al., 2015; Hackenberg, 2013; Steffe & Olive, 2010; Hackenberg & Tillema, 2009). By addressing an initial phase in the conceptual transition from not distinguishing between composite units and 1s that constitute them to ably doing so, this study can help explain how this conceptual basis for coordinating both types of units may come about.

Methods

Setting and Participants

This is a case study focused on examining a conceptual transition – a phenomenon requiring in-depth analysis of the participant's meanings (Stake, 1995) while solving mathematical word problems. A fourth grader (Adam, pseudonym), from a school in a large US city, participated in a cognitive interview to assess his assimilatory scheme for mDC. The school he attended reported a percentage of English Language Learners at 54.2% and students receiving free or reduced lunch at 84.9%.

We chose Adam as a case because of three key reasons. First, he exemplifies the conceptual transition at hand. Second, he is a fluent reader whose mathematical abilities did not seem rooted in difficulties to comprehend the questions. Third, he solved the additive screener problem (see next section) by doubling 7+7=14, then adding 1 more to obtain the correct answer of 15. This solution indicated to us Adam capably distinguished and used composite units and 1s to assimilate and solve additive problem situations. Combined, these three reasons provide a basis for analyzing a transition to a unit distinction required to assimilate (make sense) and solve multiplicative situations involving 1s and composite units.

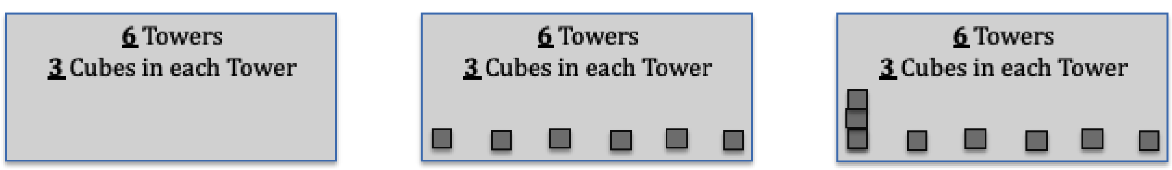
Problem Situations Adam Solved

Our team developed and validated the 5-item assessment, used during the cognitive interview, as a measure of students' mDC scheme (Johnson, et al., 2018). The first item of the measurement, intended as an additive screener question, is followed by four items to measure mDC reasoning. All four mDC items are expressed as word problems and include diagrams to illustrate the given units. Table 1 presents the additive screener and the mDC item (Problem 3 with its four sub-questions) we used in this study of Adam's case. It should be noted that, prior to solving any of those items, Adam literally built a tower of 7 cubes, and then correctly pointed to a diagram of such a tower that was included in the assessment.

Like all four mDC items, Problem 3 begins with a prompt-less version consisting of 4 questions (Figure 1a). Once the child attempts solving that version, the interviewer moves to the next version, which gives a first hint (showing there are 6 composite units) followed by the same four questions (Figure 1b). Finally, the interviewer moves to the third version, which includes a more explicit hint (showing the 6 composite units and the three 1s that comprise the first of them) followed by the same four questions (Figure c). This sequencing of assessment items – progressing from prompt-less to gradually more explicit hints – draws on Tzur's (2007) notion of fine grain assessment.

Table 1: mDC Measurement Word Problems

#	Word Problem
1	For breakfast Ana ate 8 grapes. For lunch, Ana ate 7 grapes. How many grapes did Ana eat in all? Ana ate _______ grapes in all.
3	The picture to the right shows a box. Alex put **6** towers in the box. Alex made each tower with **3** cubes. The numbers on the picture show this. How many **cubes in all** did Alex use to make **6** towers? (fill in the blank): ______



1a: Prompt-less version **1b: First-hint version** **1c: Second-hint version**

Figures 1a, 1b, and 1c: Diagrams Provided Along with Problem 3

Cognitive Interview

The third author of this paper conducted the cognitive interview with Adam. This cognitive interview was one of 40 similar interviews that constituted the last phase of the validation process of the 5-item measure. Accordingly, it required of the interviewer to give no additional support (or probing) beyond what is presented in the written assessment. The protocol for this interview involved first asking Adam if he would like to read the problems on his own or be read to by the researcher. Adam proudly opted for the former. Then, the researcher presented, one at a time, the single page on which each Problem version with its sub-questions is given. Adam read the problem situation and then the sub-questions out loud, while solving each of those questions as he thought proper without any further assistance from the researcher.

Data Collection and Analysis

The cognitive interview was video-recorded and stored along with Adam's written work for later analysis. Three team members (the PI and two graduate research assistants) individually analyzed the video, while taking extensive notes and identifying key moments they would then use to analyze the data. They met to discuss their notes, identified agreed upon key moments, and then then first author (graduate assistant) transcribed those video segments. After scrutinizing the video and transcriptions line-by-line, we chose Problem 3 for the analysis presented in this paper, as it provides the most compelling evidence of Adam's conceptual transition from not distinguishing to distinguishing between 1s and composite units within a multiplicative situation.

Results

We provide evidence of Adam's transformation of his available conceptions that led him to distinguish between single units (1s) and composite units given in a problem situation involving multiplicative relationships. We first analyze data and offer inferences of plausible conceptual sources of his initial inability to solve a task that provided no hints. Then, we articulate how his work on versions of the problem situation that included hints indicates a transformation in his reasoning. We demonstrate his novel distinction between 1s and composite units, which is a conceptual prerequisite for properly operating on both to solve multiplicative problem situations.

We begin with Adam's work on the prompt-less version of Problem #3, which asked to find the total number of cubes (1s) when given 6 towers (number of composite units) each comprised of 3 cubes (unit rate). Excerpt 1 shows Adam's work on the prompt-less version (see Figure 1 above). He read the problem out loud and began answering each sub-question.

Excerpt 1. Prompt-less questions of Problem 3.

A: (Reads the problem aloud, and rather quickly) "The picture to the right shows a box. Alex puts 6 towers in the box. Alex made each tower with 3 cubes. The numbers on the picture show this." (Here, he looks at the box for 2 seconds, then moves on to reading Question 1: "How many towers …?" and incorrectly writes "3". Then, he reads Question 2 silently: "How many cubes per tower …?" and correctly writes "3.")

R: You wanna read that [Question 3] out loud?

A: (Reads Question 3 silently twice: "How many cubes in all?" He looks around and shakes his head) I don't get this.

R: Okay, you can write I don't know, IDK; that's fine.

A: (Looks back and re-reads Question 3 silently, looks up while thinking how to solve it.)

R: Take your time.

A: (Writes, "I don't know," then reads Question 4 silently: "How many cubes are in 2 towers?" He quickly and incorrectly writes "3.")

R: Good? [This question meant, 'Are you good to move on to the next problem?']

A: (Nods, "Yes.")

Data in Excerpt 1 indicate Adam's assimilation of this prompt-less problem situation did not include distinction among different units. To each of the questions, he essentially responded with "3." Whereas this number is a correct answer to the unit rate, it is incorrect when answering the number of composite units (6 towers), the total of 1s (18 cubes), and even the total of 1s in just two towers (6 cubes). We infer from those answers that he could think of the 1s given in the problem (3 cubes per tower), while not yet distinguishing the composite units as six individual items, let alone as six items composed of three 1s each. We thus consider Adam's initial response to Question 3 ("I don't know"), about the total of 1s in all six towers to be genuine. Along with his independent effort to re-read this question twice, the data support our claim that he assimilated the entire problem into a conception that is yet to *distinguish composite units within a multiplicative situation*. We emphasize multiplicative situation, as Adam seemed capable of assimilating composite units in his solution to the additive, screener problem (doubling 7+7 to answer 8+7).

Having completed all four questions in the prompt-less version of Problem 3, Adam continued to its next part, which provides the first hint – a diagram showing the bottom cube of each tower (Figure 2). Aside from this hint, he would then have to answer the same four questions. While Adam's was noticeably slower in his reading of this version than his reading of the prompt-less version, he answered the first two questions the same (incorrect) way. He then also began solving Question 3 similarly. Yet, Excerpt 2 shows that a first perturbation seemed to occur as he further thought of the total of cubes in all 6 towers of 3 cubes each.

Excerpt 2. First-prompt questions.

A: (Reads the problem aloud, noticeably slower pace, touching each word with his pen as he reads it.) "The picture to the right shows a box. Alex put 6 towers in the box. You can see the bottom cube of each ..."

R: (interrupts his reading to orient attention to the hint) Do you see them [the bottom cubes]?

A: (Looks at the diagram, touches one of the cubes with his pen to indicate he does, then continues reading to the end of the problem.) "… 6 towers in the box. You can see the bottom cube of each tower. Alex made each tower of 3 cubes. The numbers on the picture show this."

R: So, can you answer this one?

A: Yah (Reads Question 1 silently. Then, he rereads the given problem situation, and incorrectly writes "3" for number of towers. He then reads Question 2 silently and, similarly to his prompt-less response, correctly writes "3" for the number of cubes in each tower. He moves to reading Question 3 silently (twice), then talk to R as if having an "aha" moment): Wait; so the towers are made of 6? There's 6?

R: (Not confirming Adam's realization directly, but instead restates Question 3) So, "How many cubes **in all** did Alex use to make 6 towers?" You can read it again, or look at the picture, what are the towers and the cubes there. The question is how many cubes **in all** Alex used to make 6 towers.

A: (Writes, incorrectly, "0," then also answers Question 4 incorrectly with "12" cubes as the total in only 2 towers of 3 cubes each.)

Data in Excerpt 2 further support our claim that, while solving the first two versions of Problem 3, Adam's assimilatory conception was yet to include a distinction between units of 1 and other units composed of 1s. In spite of explicitly pointing to the diagram presenting the six bottom cubes of each tower, he still incorrectly responded ("3") to the first three questions. Importantly, an event that could become a turning point in his transition occurred when, upon slowly rereading Question 3, he indicated for the first time a recognition of another numerical quantity that constitutes the problem situation ("Wait; so the towers are made of 6? There's 6?").

This recognition seemed supported by his (a) action following the hint (pointing to the six cubes in the diagram), (b) slower reading of all parts of Problem 3, and (c) rereading of Question 3.

The importance of this perturbing experience, seemingly an "aha" moment for Adam, is that he independently identified the other given number, which he overlooked (did not assimilate) in the previous attempts. His answers to Question 3 and Question 4 were still incorrect. However, his solution to the following, last part of Problem 3, which includes a second, more explicit hint, suggests this realization would contribute to his novel recognition and distinction of the other number as a salient feature of the problem situation.

The final part of Problem 3 included a diagram that shows the bottom cube for all 6 towers as well as all 3 cubes that compose the first tower on the left (Figure 3). The purpose of this hint is to provide a student with a more explicit opportunity to assimilate the two types of units that constitute the problem situation. Excerpt 3 presents Adam's first transition to reasoning with both unit types.

Excerpt 3. Second-prompt questions.

A: (Reads the problem situation aloud in slow pace, touching each word with his pen as he reads it.) "The picture to the right shows a box. Alex put 6 towers in the box. You can see the bottom cube of each tower and all 3 cubes of a tower." (He looks at the diagram, tapping the words about the two different types of units involved (in the box) as he reads them to himself. He then moves on to read Question 1 silently, looks back at the diagram, and *correctly writes "6"* in the blank for the number of towers Alex put in the box. He then reads Question 2 aloud while immediately and correctly answering it) "Alex made each tower with 3." (He writes the answer in the blank and moves to Question 3). "How many cubes did Alex use to make 6 towers?" (He looks at the diagram, makes a circle in the air above the completed tower showing 3 cubes, makes another circle in the air above the next cube that is the basis of the second tower, looks to R, makes circles in the air above the other cubes while likely counting them to six, rereads the question silently, looks up to think more about the question, and finally writes "IDK" for I don't know.)

R: That's okay [confirming IDK as a response].

A: (Reads question 4 silently, looks at that diagram again, and *correctly writes the answer "6"* for the number of cubes in just two towers of 3 cubes each.)

Data in Excerpt 3 indicate the two key aspects of our claim: in Adam's work on the version with a second hint, he was (a) finally able to distinguish between the two units while (b) not yet coordinating them multiplicatively when asked about all cubes (1s) in the six towers (composite units) made of 3 cubes each (unit rate). The first aspect is evidenced in correctly answering Question 1 by identifying the 6 towers (composite units) and in correctly answering Question 2 by identifying the 3 cubes (single units) that composed one tower. We inferred from these two correct answers that Adam has begun a transition to a conception that involves the 1s in each composite unit and the composite units made of those 1s as two distinct types of units.

Having just distinguished the two unit types, operating on both to account for all 1s in six composite units seemed beyond Adam's evolving conceptualization (writing "IDK"). On the other hand, operating on both in Question 4 became possible by this conceptualization. We infer he could then consider the first two towers he might build, each with 3 cubes, similarly to how he used 7+7 when solving the additive screener problem.

Adam's progression through all parts of Problem 3 indicates his transition from inability to an initial ability of distinguishing units of 1 from composite units. We postulate that this transition was initially supported by his reflection on the activity of counting the six cubes in the diagram (Figure 2) provided as the first hint that seemed to enable his "aha" moment. This transition seemed further supported by Adam's reflection on the effects of his explicit actions of counting when studying the problem situation with the second hint. Using two distinct actions of circling above the relevant units, he produced two different effects: counting to three cubes in the first tower and counting to six cubes representing the bottom cube of each tower.

That this transition is not trivial can be emphasized in the sequence of his solutions. Before being prompted, Adam seemed to overlook the multiple units "given" in the problem situation – he seemed capable of identifying only one unit (3), which he used throughout his answers for that problem. Once Adam was given the first hint, he experienced a perturbation that could set in motion his distinction between the units involved in the problem. However, this distinction was yet to be made explicit. Thus, he was still not able to operate correctly with both unit types. With the second hint, Adam's unit distinction had taken on the transformation needed to solve the multiplicative situation. Correctly identifying both units – 6 towers/composite units and 3 cubes in each tower/single units – opened the way to also coordinating them at least when solving Question 4 that dealt with only two towers. His distinction seemed too rudimentary for also coordinating the two types of units when the task involved more than two composite units.

Discussion

We see two key contributions of this study. For research and theory building, it demonstrates how a student engaging in solving mDC problem situations with gradual hints can advance from having no distinction of units to beginning to distinguish between composite units and singletons (1s). That is, the study provides a lens through which to understand the prerequisite role that distinguishing those two types of units serves in advancing to multiplicative operations on such units. Simply put, before operating multiplicatively on two different types of units one must explicitly distinguish them. Moreover, the study goes beyond noting these two, before-and-after conceptual markers (Tzur, 2019), to articulating a mental process involved in the transition from the former (no distinction) to the latter ("seeing" composite units apart from the 1s of which they are comprised). Specifically, we pointed out to Adam's reflection on the effects of his counting activities as a plausible way this transition might have come about (Simon et al., 2004). We do not claim every student would make this critical transition in the same way. However, we believe this study provides a sensible explanation of it. In Adam's case, the role this distinction of units played in his operation on both types of units was manifested in his correct solution to Question 4 (6 cubes in 2 towers of 3 cubes each) of the second-hint version of the problem situation.

For practice, our study demonstrates the importance of developing a way of distinguishing and operating on those two types of units as a basis for learning to meaningfully use arrays as a representation of think whole number multiplication. Adam's case of distinguishing the two units that constitute such arrays can inform teachers about ways their students' may reason is such instances. Students who are yet to make such a distinction would likely reason about such arrays as constituted of all 1s (single items). Such students may hear and assimilate "rows" and "columns" similarly to how Adam read and assimilated the two units given in the prompt-less situation, namely, as 1s. Adam's case also illustrates how students may be able to identify and distinguish between 1s and composite units when they are prompted. Importantly, his case also demonstrates how the students may be able to distinguish between 1s and composite units while not yet being able to operate on those units multiplicatively. We contend that arrays would represent multiplication meaning for students only when they can both make this distinction explicit (as Adam began to do) and operate on the units beyond doubling (as he was yet to do). This information about how the students assimilate and interpret word problems provides a teacher with a conceptual basis to determine where they could go next (i.e., goals for the students' learning) and what activities could be designed to promote the intended advance. Specifically, for students like Adam the goal would be to support their conceptual ability to independently and explicitly distinguish among the units (prompt-less). For students who accomplished such a conceptual ability – the goal would be to promote their multiplicative operations with/on those units.

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A QUANTITATIVE REASONING PERSPECTIVE ON STUDENTS' UNDERSTANDING OF DIFFERENTIALS: AN EXPLORATORY STUDY

Thembinkosi P. Mkhatshwa
Miami University
mkhatstp@miamioh.edu

This paper extends work in the area of quantitative reasoning at the undergraduate level. Task-based interviews were used to examine six high-performing students' reasoning about an application problem that involves differentials, and what their reasoning about the problem revealed about their understanding of differentials. Analysis of verbal responses and work written by the students while solving the problem revealed that a majority of the students were successful in solving the problem. However, three students confused differentials with derivatives. Furthermore, findings of this study suggest that the opportunity to learn about differentials during classroom instruction in the United States is either absent or minimal.

Keywords: Differentials, Quantitative reasoning, Problem solving, Calculus education.

Differentials have numerous important applications in the sciences, especially in physics and engineering (Artigue, Menigaux, & Viennot, 1990; Hu & Rebello, 2013). However, there have been relatively few studies that have examined students' understanding of differentials. Findings of a comprehensive review of literature on students' understanding of various topics in college calculus by Speer and Kung (2016) indicate that there is a dearth of research on students' reasoning about differentials. Of the few studies involving differentials, Hu and Rebello (2013) found, among other things, that when solving physics problems involving integrals, students do not see differentials as representing physical quantities but rather as abstract mathematical symbols. Artigue et al. (1990) reported on high-performing students who confused differentials with limits in a mathematics context. Similar results were reported by Orton (1983). Amos and Heckler (2015) found that students confused differentials with derivatives in a physics context. López-Gay, Sáez, and Torregrosa (2015) found that a majority of the 298 university students in their study had "a quasi-exclusive conception of the differential as an infinitesimal increment" (p. 591). While these studies have provided beneficial information about students' thinking about differentials, there is still much to be explored about how students set up and solve application problems in calculus that involve differentials, which is the motivation for this study. Thus, to build on these studies, we intend to explicitly examine students' understanding of key steps involved in the process of solving application problems that involve differentials.

Theoretical Perspective

This study draws on the theory of quantitative reasoning (Thompson, 1993; Thompson, 1994b; Thompson, 2011). Quantitative reasoning is the analysis of a situation in terms of the quantities and relationships among the quantities involved in the situation (Thompson, 1993). Quantitative reasoning, as used in this study, refers to how students thought about units of quantities, how they used algebraic equations to represent relationships between quantities, how they created formulas for the unknown quantity (i.e., the unknown differential) in the task they were given, how they evaluated these formulas to determine the numeric value for the unknown quantity, and how they created and used diagrams in their solution to the task. Thompson (2011) described three tenets (a quantity, a quantitative operation, and quantification) that are central to the theory of quantitative reasoning. A quantity is a measurable attribute of an object. Examples of quantities (i.e., measurable attributes) in this study include the radius and the volume of the sphere referred to in the task that appears in the methods section. In this task, the sphere is the object.

A quantitative operation is the process of creating a new quantity from other quantities (Thompson, 1994b). The task used in this study provided an opportunity for students to create a new quantity, the unknown differential, through the process of differentiation. Solving this task successfully requires students to find a model (algebraic formula) that relates the quantities in the task. In this task, an appropriate model would be the formula for the volume of a sphere, i.e., $V = \frac{4}{3}\pi r^3$ where V is the volume of the sphere and r is the radius of the sphere. Differentiating this model with respect to r (to get $dV = 4\pi r^2 dr$) would lead to the creation of two new quantities, namely the differentials dV (the unknown differential) and dr. Quantification is the process of assigning numerical values to quantities (Thompson, 1994d). Evaluating dV at the values of r and dr given in the task would lead to the determination of a numerical value for the quantity dV. Our study was guided by the following research question: How can we interpret the theory of quantitative reasoning in the context of students solving application problems that involve differentials? In this paper, the terms "application problem" and "task" are used interchangeably to mean the same thing.

Methods

This qualitative study used task-based interviews (Goldin, 2000) with six students. The interviews lasted for about 30 minutes, on average, and contained three tasks. In this paper, we report on how the students reasoned quantitatively about one of the tasks, adapted from Stewart (2016):

> The radius of a sphere was measured and found to be 10cm with a possible error in measurement of at most 0.05 cm. Use differentials to estimate the maximum error in using this value of the radius to calculate the volume of the sphere?

The students worked through the task while the interviewer asked clarifying questions about their work. After the student concluded their work on the task, the interviewer asked the following questions about the task and the content of their solutions: (a) Have you seen a problem like this before? (b) What did you do to solve the problem? (c) What does your answer tell you? (d) What are the units for the quantity you found? (e) What does each quantity throughout your solution mean? All the students acknowledged having seen or solved a similar task.

Setting, Participants, and Data Collection

The study participants were six high-performing undergraduate students (pseudonyms Brian, Caleb, Jacob, Laura, Lenny, and Mike) who were enrolled in two sections of a traditional calculus I course in the fall semester of 2017. At the time of the study, three students (Caleb, Lenny, and Mike) were engineering majors, Brian was a mathematics education major, Jacob was a business major, and Laura's major was undecided. Only one student (Caleb) had taken a high school calculus course prior to participating in this study. Four students (Brian, Caleb, Jacob, and Laura) were freshman, one student (Lenny) was a sophomore, and another student

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

(Mike) was a junior. The cumulative grade point averages (GPAs) of the six students had a mean of 3.61 on a 4.0 scale.

Data for the study consisted of transcriptions of video-recordings of task-based interviews and work written by the six students during each task-based interview session. In addition, because of the scarcity of research on students' learning of differentials, we created and administered a two-question survey to help gather data that would provide an insight on the teaching of differentials in the United States. Of the 23 surveys we sent to colleagues in different Mathematics departments across the nation, 10 surveys were completed and returned. The surveys covered the following questions:

- Are differentials an optional or mandatory topic for calculus I instructors in your department?
- If you answered mandatory to the preceding question, roughly how may instructional days and/or hours are devoted to teaching differentials?

Data Analysis

Data analysis was done in three stages. In the first stage, we read through each interview transcript and coded instances where students: (1) reasoned about the units of each quantity they used in their solution to the task, (2) mathematized (Freudenthal, 1993) the task, (3) performed quantitative operations (i.e., created formulas for the unknown quantity (differential) through the process of differentiation, (4) engaged in quantification (i.e., determined numeric values for the unknown quantity), and (5) reasoned about the usefulness, or lack thereof, of diagrams they created and used in their solutions to the task. In the second stage of the analysis, we looked for patterns in each of the codes identified in the first stage of the analysis. These patterns included trends in the students' understandings, or difficulties they had in connection with each of the codes identified in the first stage. The common understandings or difficulties in students' reasoning found in the second stage of our analysis provided answers to our research question. In the third stage, we tallied the number of survey respondents who indicated that teaching of differentials in Calculus I is mandatory in their department. We also noted the amount of instructional time (i.e., lecture periods) spent on teaching differentials in these departments. In addition, we tallied the number of survey respondents who indicated that teaching differentials is optional in their department.

Results

Analysis of students' written work and verbal responses to the problem posed in the task revealed that nearly all the students took a similar approach when solving the task. In general, the approach taken by the students consisted of the following sequence of steps:

1. Orienting to the task (e.g., reading problem statement aloud/silently, underlining key words, writing/underlining given quantities, and identifying the unknown quantity).
2. Drawing a diagram (i.e., visual representation of the quantities in the task).
3. Mathematizing: using algebraic symbols to represent quantities, and finding a function that relates the quantities in the task (i.e., the equation for the volume of a sphere, $V =$

$\frac{4}{3}\pi r^3$, where the variable V represents the volume of the sphere and the variable r represents the radius of the sphere).

4. Performing quantitative operations: Differentiating the equation determined in step 3. with respect to r to get $dV = 4\pi r^2 dr$.
5. Engaging in quantification: Evaluating the unknown quantity (i.e., the differential dV) at the given values of r and dr.

Table 1 summarizes students' performance in each of the above-listed steps. The "✓" symbol means that a student correctly carried out the indicated step, the "x" symbol means that a student incorrectly carried out the indicated step, and the "-" symbol means that a student did not do the indicated step.

Table 1: Summary of Results

	1.	2.	3.	4.	5.
Caleb	✓	-	✓	✓	✓
Jacob	✓	-	✓	✓	✓
Mike	✓	✓	✓	✓	✓
Brian	✓	✓	✓	✓	✓
Lenny	✓	-	x	✓	x
Laura	✓	✓	✓	-	-

We remark that in addition to reading the task silently, one student (Laura) underlined keywords (e.g., volume), given quantities such as the possible error in the measurement of the radius, and the statement about the unknown quantity (i.e., the estimated maximum error in calculating the volume of the sphere) while orienting herself to the task. Three students (Laura, Mike, and Brian) each drew a circular diagram they stated was a "picture of the sphere" mentioned in the task. However, none of the students found the diagrams to be useful in solving the task. For example, when asked if creating a diagram was helpful in solving the task, Mike commented, "it wasn't necessary, but I just like to do it." We further remark that three students (Jacob, Lenny, and Mike) assigned the units of cm^3 to the quantity dV/dr suggesting that they confused the quantity of volume (an "amount" quantity) with the rate of change of volume (a "rate" quantity). Three students (Mike, Brain, and Jacob) used "derivative" symbolic notation to represent at least one differential in their solutions, suggesting that they confused differentials with derivatives. Analysis of the survey responses indicated that teaching differentials is optional at four of the 10 responding departments, and only half a lecture period is spent on differentials at three other departments. Three colleagues reported that about two lecture periods are devoted to the teaching of differentials in their departments. The majority of the survey participants were affiliated with at least an R2 institution based on the Carnegie Classification of Institutions of Higher Education at the time of the study.

Discussion and Conclusions

An important finding of this study is that a majority of the six students were successful in solving the problem. Three students confused differentials with derivatives. Similar results were reported by Amos and Heckler (2015) in a physics context. Three students confused an "amount" quantity (volume) with a "rate" quantity (the rate of change of volume). Students' tendency to confuse “amount” and “rate” quantities is well documented in the research literature (cf., Mkhatshwa & Doerr, 2018; Prince et al., 2012; Rasmussen & Marrongelle, 2006). To some extent, the survey results seem to suggest that differentials are either not taught at all or that when they are taught, calculus instructors do not cover this topic in depth. Much work remains to be done with regard to students’ reasoning about differentials, and the opportunities they have to learn about this topic through classroom instruction in the United States.

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MEETING THE COGNITIVE DEMANDS OF PROOF BY INDUCTION: THE CASE OF BEN

Anderson Norton
Virginia Tech
norton3@vt.edu

Rachel Arnold
Virginia Tech
rlongley@vt.edu

Applying mathematical induction within a proof induces high cognitive demand. To investigate this demand, we developed a theoretical framework that integrates action-object theory within a model of working memory. We applied the framework in analyzing interviews with students enrolled in a proofs course. Here, we report results from a student named Ben. We examine ways that holding logical implication as an object offloads working memory so that students, like Ben, can attend to other components of proof by induction.

Keywords: Reasoning and Proof, Undergraduate-level Mathematics

In the Principle of Mathematical Induction, the inductive implication $P(k) \rightarrow P(k+1)$ can be treated as an inductive step, $P(k+1)$, from the inductive assumption, $P(k)$; or, as an invariant relationship between $P(k)$ and $P(k+1)$ for any k (Dubinsky, 1986). This distinction relates closely to Piaget's (1970) definitions of mathematical actions and objects from which Dubinsky (1991) developed APOS theory. Within that framework, Dubinsky (1991) hypothesized that success in proof by induction depends on a student holding logical implication as a mathematical object—"the encapsulation of the process of implication" (p. 111). We adopt a re-framed version of Dubinsky's hypothesis. Namely, we hypothesize that holding logical implication as a mathematical object reduces the cognitive demand of proof by induction, enabling one to attend to other aspects of the proof, such as quantification and domain-specific knowledge.

Working Memory and the Cognitive Demands of Proof by Induction

To deal with the challenging aspects of understanding the inductive implication, Avital and Libeskind (1978) suggested that teachers focus students' attention on how particular cases follow from prior cases. By looking across several cases, students might reflect on the structure of the implication and generalize that the relationship holds between $P(k)$ and $P(k+1)$ for any arbitrary k. Harel (2002) elaborated on this instructional approach and labeled it "quasi-induction."

Reasoning by quasi-induction, a student might show that $P(1)$ is true and that $P(1) \rightarrow P(2)$, $P(2) \rightarrow P(3)$, $P(3) \rightarrow P(4)$, and so on. This leads to the plausible conclusion that $P(n-1) \rightarrow P(n)$. However, the conceptual jump from quasi-induction to formal proof by induction is larger than it may seem. Harel (2002) asserts that this cognitive gap is substantial; he refers to formal proof by induction as an abstraction of quasi-induction. Our framework characterizes this gap as the distinction between the conceptual view of logical implication as an action versus as a mathematical object.

In addition to challenges imposed by the inductive implication, researchers have identified a few other factors that affect student success in proof by induction (e.g., Ernst, 1984). These factors include the importance of the base case, the role of (hidden) quantifiers, and domain-specific knowledge for the conjecture. Shipman (2016) noted that students' mistreatment of quantifiers often leads to the erroneous proof of a "for all" statement by example.

Our framework addresses the limitations of working memory relative to the cognitive demand of tasks (Baddeley & Hitch, 1974; Pascual-Leone, 1970). Existing literature informs us of task factors in proof by induction that place demands on working memory. We contend that the construction of logical implication as an object lowers the cognitive demand of such a task by enabling the student to assimilate the logical implication as a single object, rather than two separate components joined through action. This lowered demand would free up cognitive resources so that the student might attend to other factors, such as quantification.

Working memory relies on a frontoparietal network that includes the prefrontal cortex and the parietal cortex (Harding, Yucel, Marrison, Pantelis, & Breakspear, 2015; Sauseng, Klimesch, Schabus, Doppelmayr, 2005). Specifically, the central executive, which distributes task demands, relies heavily on neural activity in the dorsolateral prefrontal cortex and calls upon cognitive units with neural substrates in the parietal lobe. Figure 1 illustrates this functioning.

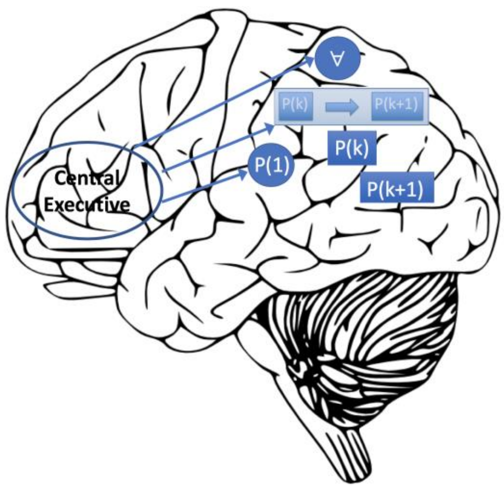

Figure 1: Theoretical Role of Working Memory During Inductive Proving

Research in educational neuroscience indicates a general frontal-to-parietal shift as students develop mathematically. Older and more experienced students rely more on parietal activity when compared to their younger, more novice peers solving the same tasks (Cantlon, Brannon, Carter, & Pelphrey, 2006). Our model accounts for this shift by positing that, over time, sequencing existing cognitive units in working memory can induce the construction of new cognitive units so that fewer units are required to solve the same task. In particular, cognitive units for the inductive assumption, $P(k)$, and the inductive step, $P(k+1)$, might be reorganized into the single inductive implication unit, $P(k) \rightarrow P(k+1)$, as depicted in Figure 1.

Methods

To elicit students' understandings of logical implication and components of mathematical induction, Norton conducted clinical interviews with students from a proofs course taught by the Arnold. A research assistant visited Arnold's class at the beginning of the 2016 fall semester to describe the study and recruit participants. Three students volunteered and were invited to participate. Norton conducted a pair of interviews with each of the participants—one before Arnold's instruction on mathematical induction and one after. All interviews were video-recorded and lasted about 45 minutes.

Interview tasks consisted of three types: logical implication (Type A), components of mathematical induction (Type B), and formal proof by induction (Type C). Table 1 shows tasks, from the pre-interview—the focus of our analysis. More specifically, we focus on pre-interview responses from a high-performing student named Ben.

Table 1: Sample Tasks (adapted from Norton & Arnold, 2017)

Task Type	Tasks
A. Logical Implication	Suppose the statement is true: "If two topological spaces are homeomorphic, their homology groups are isomorphic." Evaluate whether the following statements are true, false, or uncertain. a) [converse] b) [contrapositive] c) [negation]
B. Components of Mathematical Induction	Suppose *P(n)* is a statement about a positive integer *n*, and we want to prove the claim that *P(n)* is true for all positive integers *n*. For each scenario, decide whether the given information is enough. c) *P(1)* is true; there is an integer $k \geq 1$ such that *P(k)* implies *P(k+1)*. d) *P(1)* is true; for all integers $k \geq 1$, *P(k)* implies *P(k+1)*.
C. Formal Proof by Induction	Prove the following claim: For every positive integer *n*, $2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$.

Results

We use Ben's responses to Type A tasks as evidence that logical implication had emerged as an object upon which Ben could operate. For instance, when Ben was given the converse to the statement shown in Table 1, he responded, "I want to say 'uncertain' because I can separate the [original] statement into *P* and *Q*… and the second one tries to reverse that into a *Q* and *P*, which I'm not certain would be true in all cases." Ben seemingly conceived of the original statement as something he could separate into constituent parts, reverse, re-constitute, and then compare back to the original as a totality.

Across tasks, we consistently saw Ben break down implications, manipulate their parts, re-combine them, and compare the result to other implications, as totalities. Such activity indicates that he could treat implications as objects to operate upon (e.g., reverse or negate) and compare. However, the time and effort, along with his appeal to truth tables learned in class, suggest that this treatment of logical implication was a recently emerging development.

Ben's responses to Type B tasks indicate how he coordinates the components of mathematical induction independent of mathematical content. At first, Ben thought scenario c provided enough information to conclude that *P(n)* would hold for all positive integers. He justified his conclusion as follows: "Since it includes 1, you can say *P(1)* implies *P(1+1)*… so *P(2)*, and then you can say the same thing, that *P(2)* implies *P(3)*… and you can just continue along that path for all positive integers." Note that Ben's reasoning fits Harel's (2002) description of quasi-induction. Ben stated that he had never completed a proof like that before, and that "the *k* and *k+1* made me think of some kind of like sequence of numbers just going up." In attending to this case-by-case application of induction, Ben overlooked the existential quantifier. The quantifier's significance remained hidden to Ben until he saw scenario d (which has the structure of mathematical induction) and compared it back to scenario c.

Task C is an inductive proof in a domain-specific context. Ben worked on the task quietly for about four minutes, writing down equations. When asked what he was thinking, he said:

> …my first instinct is to look for some sort of pattern. I did $2 + 2^2 + 2^3$, and that gave me 14, which was 2 minus, then, 2^4, which would be the next term. So, I tried that for 2 plus all the way up to 2^n…and I get some value. And if it works the same as for all of the other values before it in the sequence, then I should be able to add 2 to that to get 2^{n+1}.

Ben used smaller-valued cases to test the statement and to find a pattern. When Norton asked him about his approach, Ben shifted his treatment of these cases to fit the role of a base case. "I guess I might show that, for smaller values, this is true, where you can move the 2 over to where you get, say, $2+2^2+2^3 = 2^4-1$ and then show the pattern holds for other values you add into it." Ben clarified that he wanted to add new terms to the sum to show that the equation would still hold. This approach of "incrementing the exponent by 1," is quasi-inductive.

The limitation of quasi-inductive reasoning became clear when Norton asked, "when does that stop?" and Ben responded, "um, that's a good question." Transitioning from quasi-induction to formal induction would require Ben to generalize the inductive implication to an arbitrary positive integer k, not just specific integers like $k=3$. Instead, Ben focused on the proposition itself, "I think that if you can show that it holds true for small cases and then make it more general by throwing in the n, saying you can go on as far as you want to and this will still be true. But I'm not sure exactly how to express that in proof form." After a few minutes of trying, Ben concluded, "I'm still not sure how you could show that it works for some arbitrary number."

In Harel's (2002) terms, Ben now had an *intellectual need* for formal induction, but his reasoning was presently limited to quasi-induction. So, Norton drew Ben's attention to the Task B scenarios in which Ben seemed to understand inductive arguments. After looking at the scenarios at length, Ben responded, "I'm thinking about the ones where $P(k)$ implies $P(k+1)$ and if you can get some sort of sequence where you can start throwing in values of k, or in this case, n." Here, Ben shifts his focus from $P(n)$ itself to the inductive implication $P(k) \rightarrow P(k+1)$. However, Ben was still unsure how k might first appear when working from specific cases.

Ben tried to translate Task C into the form of Task B scenario d. He was successful in proving the base case; however, he was not able to formalize the inductive implication. Rather than formulating a general inductive implication, he attempted to identify $P(k+1)$ for specific values of k, "So, the next term would be $2+2^2 = 2^3 - 2$." He summarized his goal as follows: "continue the sequence for who knows how many terms and it will still be as true as when you had $P(1)$." Ben's reasoning remained quasi-inductive, at least in this particular domain.

Discussion

For Ben, logical implication was emerging as an object that he could transform and apply in various ways. We saw him apply the inductive implication to arguments about the truth of a proposition, P(n). However, his arguments became problematic as he attended to additional components of proof by induction, including two of those identified in prior research: quantifiers (Shipman, 2016) and domain knowledge (Dubinsky, 1991). Ben could account for both components individually with some cognitive effort, but never simultaneously.

The cognitive effort Ben demonstrated could be instrumental in his development of inductive reasoning. Avital and Libeskind (1978) suggested that quasi-induction could prove fruitful in that development, but Harel (2002) noted a cognitive gap, which Ben exemplified, between quasi-induction and formal induction that would require an abstraction. That abstraction could manifest itself as the objectification of logical implication, which was newly emerging for Ben. Tasks that engage students like Ben in sequencing and coordinating various components of mathematical induction, with the support of figurative material (e.g., written equations), might induce the kind of frontal-to-parietal shift that would further encapsulate logical implication together with its quantification, allowing for flexible application of induction across domains.

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OPPORTUNITIES FOR GENERALIZING WITHIN PRE-SERVICE SECONDARY TEACHERS' SYMBOLIZATION OF COMBINATORIAL TASKS

Lori Burch	Weverton Ataide Pinheiro	Erik Tillema
Indiana University	Indiana University	Indiana University
ljburch@iu.edu	wpinheir@iu.edu	etillema@iu.edu

This paper reports on an interview study conducted with four pre-service secondary teachers (PSSTs) for the purpose of understanding how symbolizing sets of outcomes supports opportunities for generalizing in the context of solving combinatorial problems. This paper examines opportunities for generalizing based on differences in symbolization as well as differences in PSSTs' conceptualization of seemingly identical symbols.

Keywords: Algebra and Algebraic Thinking, Cognition, Design Experiments

Introduction with Supporting Literature

The act of generalizing is central to mathematical reasoning because it is a primary vehicle for the construction of new mathematical knowledge (Amit & Klass-Tsirulnikov, 2005; Lannin, 2005; Sriraman, 2003). As a result, "developing children's generalizations is one of the principal purposes of school instruction (Davydov, 1972/1990, p. 10)." Current mathematics curricula and standards reflect this principal purpose (e.g., Hirsch, et. al., 2007; Lappan, et. al., 2006), and State and National standards focus on generalization (e.g., Council of Chief State School Officers, 2010; Indiana Academic Standards for Mathematics, 2014).

Researchers studying combinatorics have identified symbolizing sets of outcomes as helpful for students to produce all possible outcomes (English, 1991, 1993; Nunes & Bryant, 1996); avoid, correct, and explain common counting errors (Lockwood, 2014); and establish when two ways of reasoning are isomorphic (Maher, Powell, & Uptegrove, 2010). Researchers investigating combinatorial reasoning have not yet explicitly examined how symbolizing sets of outcomes could support students' opportunities for generalizing. Two reasons such an exploration is of interest in combinatorics are: (a) it is common for students to symbolize sets of outcomes for the same problem in different ways; and (b) it is common for students to symbolize sets of outcomes in ways that look identical, but their symbols have different meanings.

Given these observations, the purpose of this paper is to examine differences in the way pre-service secondary teachers (PSSTs)–who were student-participants in a teaching experiment–symbolized sets of outcomes in their solution of combinatorics problems and to identify how these differences afforded and constrained their opportunities for generalizing. The following research questions guide this paper:

1. What different opportunities for generalizing are available to PSSTs when they symbolize sets of outcomes for the same problem in different ways?
2. What different opportunities for generalizing are available to PSSTs when they symbolize sets of outcomes in the same way but conceive of the symbols differently?

Theoretical Perspectives

We use the term *symbolizing sets of outcomes* to include the creation of graphic items (or the use of other figurative material) in the context of a student implementing her schemes (cf. Von Glasersfeld, 1995). We include in our definition items that are conventional ways of symbolizing a set of outcomes like a written or verbal list, a tree diagram, an array, or an empty slot as well as non-conventional ways of symbolizing sets of outcomes like a drawing, tally marks, curved lines, or a demonstration with concrete materials. We follow Ellis (2007) in differentiating between generalizing actions and reflection generalizations. For the purposes of this paper, we consider symbolizing sets of outcomes to be a generalizing action, which is an action that precedes and may support a formal statement of generalization. Therefore, we focus on differences in the way PSSTs symbolized sets of outcomes and how such differences afforded or constrained opportunities for statements of generalization. We do this even in situations where students may not explicitly make formal statements of generalization.

Methods and Methodology

The data for this study was collected using teaching experiment methodology (Confrey & LaChance, 2000; Steffe & Thompson, 2000). The research team consisted of one university faculty member and six graduate students in mathematics education. Participants in the study were four PSSTs who were concurrently enrolled in their second mathematics methods course at a Midwestern university during the fall of 2018.

Participants were paired for interview purposes, with each pair participating in 13 teaching episodes. Teaching episodes were conducted weekly, lasted 60-90 minutes, and were recorded with three cameras. Two cameras recorded the written work of each PSST, and a third captured the interaction between the teacher-researcher and the PSSTs. The goal of the first 9 teaching episodes was to help the PSSTs see combinatorial structure in common algebraic identities, with an aim at helping them gain a progressively more general understanding of the relationship between common algebraic identities and combinatorial structure (Tillema & Gatza, 2016; Tillema & Gatza, 2017). The remaining 4 episodes focused on how the PSSTs could use their work to teach a lesson sequence during their student teaching.

Data analysis included the first and second author independently watching video segments from each of the four participants to analyze how symbolizing sets of outcomes afforded or constrained opportunities for generalizing. The first and second author then discussed their interpretations of these video segments until they reached a common interpretation. These interpretations were then shared with the third author and again discussed until there was a common interpretation. The example in this paper focuses on opportunities for generalizing based on PSSTs' symbolizing sets of outcomes in a combinatorial situation where they were making meaning for the term $10x^3y^2$ in the expansion of $(x + y)^5$. We choose this example because the ten arrangements of three x's and 2 y's was not obvious to the PSSTs who had not yet developed a systematic way to count arrangements for middle terms in the identity (i.e., $10x^3y^2$, $10x^2y^3$).

Results and Discussion

In this paper, we provide one example of how symbolizing a set of outcomes affords different opportunities for generalizing. This case highlights differences in the ways Olive and Aaron symbolized sets of outcomes. We will provide additional examples across the four PSSTs during the presentation.

During the ninth teaching episode, Olive and Aaron generated a list related to the algebraic identity $(A + B)^5 = 1A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + 1B^5$. At this point in the study, Olive and Aaron had established: (a) that A and B were variables representing the number of possible options in a binary situation; (b) that each term on the right-hand side of the equivalence represented 5 selections of either variable A or variable B; and (c) that the coefficients represented the number of ways they could position the As or Bs in 5 slots (e.g., the number of

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different ways to position 3 *A*s and 2 *B*s for the term A^3B^2). They, however, did not have a systematic way to count the arrangements of *A*s and *B*s for the middle terms of the identity (i.e., A^3B^2 and A^2B^3). The teacher-researcher pressed both PSSTs to show the different ways to arrange 3 *A*s and 2 *B*s.

For the A^3B^2-term, Olive focused on the variable appearing fewer times, i.e. *B* (Figure 1).

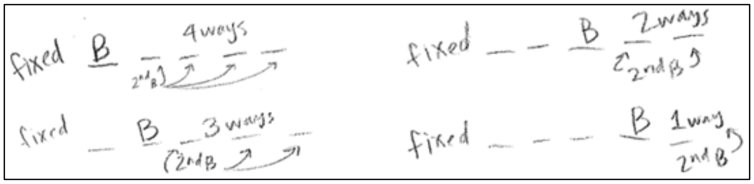

Figure 1: Olive's representation of the 10 variations of A^3B^2.

Olive's way of symbolizing the set of outcomes communicated a systematic organization, and so the teacher-researcher asked Aaron to make a conjecture regarding Olive's thinking.

> She fixed her *B*s and then rotated where the other [second] *B* could be. Because you only have two *B*s. She fixed [a *B*] in the first position and then moved [the fixed *B*] to the second position. But you know that you can't have another B back in the first position because you already counted that for the first one. And then you keep working your way down, rotating over the fixed *B* and counting the other ways, knowing the others behind the fixed *B* were already counted.

Olive agreed that Aaron had adequately explained her method. Without prompting, Olive symbolized her set of outcomes as 4 + 3 + 2 + 1 = 10. After accurately explaining Olive's list, the teacher-researcher asked Aaron to explain his own (Table 1, left).

Table 1: Aaron symbolizing outcomes

Aaron's original list of outcomes:			Aaron's list emphasizing his thinking:		
xxxyy xxyxy xxyyx	xyxxy xyxyx xyyxx	yxxxy yxxyx yxyxx yyxxx	xxxyy xxyxy xxyyx	xyxxy xyxyx xyyxx	yxxxy yxxyx yxyxx yyxxx

Aaron explained that he focused on placing three *x*'s for the outcomes in the first column. He fixed two *x*'s in the first two positions and then rotated the third *x* through the remaining three positions (i.e., either position 3 or 4 or 5) (Table 1, right). In the second column, he repeated this process by fixing *x* in the first and third positions while rotating the third *x* through the remaining two positions. Aaron then fixed *x*'s in the first and fourth positions, placing the third *x* in the last position. Realizing he had exhausted possible arrangements with *x* in the first position, Aaron considered arrangements with *y* in the first position. At this point, Aaron's thinking changed *from* placing *x* in three out of five positions *to* placing *y* in two out of five positions. Aaron stated "the *y*'s are easier to explain. I fixed the first *y* and started the other at the outside and just worked its way in."

Although Aaron interpreted Olive's method before explaining his own, he did not seem to alter his explanation based on new insight provided by Olive's method. We make this inference based on Aaron's reaction to (a) Olive's re-voicing of his thinking and (b) Olive subsequently explaining how she could see her own structure in Aaron's list. During Olive's explanation, Aaron gestured between the two lists (i.e., Olive's and his own) as if attempting to confirm the connection Olive suggested.

Based on Aaron's thinking, the total number of outcomes in his list could be symbolized as (3 + 2 + 1) + 4, where (3 + 2 + 1) represents arrangements when the *x* positions were foregrounded and 4 represents the foregrounding of the *y* positions. However, Aaron did not numerically symbolize his set of outcomes in either way; he simply wrote that there were 10 ways. Writing 10, taken together with Aaron's statement that "the *y*'s are easier to explain", indicates that this mixed method of listing constrained his ability to easily symbolize a structure that he could subsequently generalize with algebraic notation.

Response to Research Question 1

Olive and Aaron worked on the same task; however, they symbolized the set of outcomes differently. Olive's method *explicitly* demonstrated a process, but left the individual outcomes *implied*. The explicit nature of Olive's listing method allowed Aaron to accurately infer the process she used and enabled Olive to independently translate her method into the sum 4 + 3 + 2 + 1 ways to produce the A^3B^2-term. Aaron, on the other hand, listed individual outcomes *explicitly*, switching processes part way through. When Aaron used numerical symbols for his solution he did not use them to show the structure of his reasoning; this is in part because he switched the process for generating outcomes as he created his list. We see these distinctions as affording differences in opportunities to generalize. Olive's explanation suggested she could project how she would reason in new cases whereas Aaron's explanation suggested he would have difficulty projecting his reasoning onto new cases. We took the PSSTs' ability (or inability) to provide explanations that could be applied to new cases as a key indicator of opportunity to generalize.

Response to Research Question 2

Aaron generated his final four outcomes using a new strategy. Aaron could have continued to focus on the position of the *x*'s after exhausting outcomes with *x* in the first position. With this method, Aaron's list (Figure 2) could be symbolized as (3 + 2 + 1) + (2 + 1) + (1)–i.e., as a sum of sums–with the partial sum (2 + 1) symbolizing the number of outcomes with the first *x* fixed in the second position, and the final partial sum (1) as symbolizing the number of outcomes with the first *x* fixed in the third position.

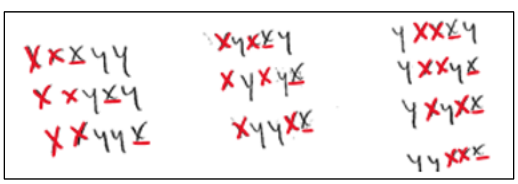

Figure 2: Aaron's list

We see other students, who produce the same list as Aaron but conceive of their list as a sum of sums by persisting with focus on *x*'s, as afforded greater opportunity to generalize.

Our data illustrates both that *different ways* of symbolizing sets of outcomes offer different opportunities to generalize and that different meanings for the *same way* of symbolizing a set of outcomes offers different opportunities to generalize. These findings support the conclusion that careful attention both to the way students symbolize sets of outcomes as well as to the meaning students have for the outcomes they have symbolized is crucial for teachers interested in supporting generalization in the domain of combinatorics.

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EARLY CHUNKY AND SMOOTH IMAGES OF CHANGE IN THE DURATIONAL REASONING OF TWO SECOND GRADERS

Amy Smith
University of Colorado, Denver
amy.L2.smith@ucdenver.edu

Children's daily experiences create the foundation for their development of psychological time—the time of memories (Piaget, 1969). How children reason about this time has not been studied in math education, therefore it is unknown how these conceptions influence children's ability to understand and quantify time. Building from Castillo-Garsow and colleagues' (2012, 2013) chunky and smooth images of change, this study compares how two second graders' reason about time through reflection on their experiences. Semi-structured interviews found that the two demographically similar participants reflected on their lived experiences in very different ways to explain what time is and how to account for it.

Keywords: Elementary School Education, Informal Education, Problem Solving, Reasoning and Proof

Children are exposed to time concepts both formally and informally from a young age. A parent telling their child to "hold on a second" (Evans, 2004) or a family's weekly schedule (Lareau, 2011) are examples of informal time experiences. The intuitive perceptions children gain from experiences such as these establish their early conceptions about time (Piaget, 1969). Through reflection on these experiences, children develop foundational conceptions about what time is and how it can be quantified.

The present research explores the way two second graders explain time as they reflect on their durational experiences (i.e., the length of everyday activities). Utilizing research from Castillo-Garsow and colleagues' (2012, 2013) the cases demonstrate early indications of chunky and smooth reasoning.

Theoretical and Conceptual Framework

Piaget (1971) posited three types of knowledge: physical, social, and logico-mathematical. Physical knowledge is the attributes of objects that can be understood through observation, such as knowing that clocks have numbers. It is purely about the objects, rather than the understandings the individual imparts on the objects. Social knowledge, like physical, exists outside of the individual; however, social knowledge is constructed by people, such as structured start times for daily events like school or a sports practice. Because human constructs are cultural, social knowledge varies from place to place and is a crucial consideration when discussing how children's daily experiences found their understanding of time.

Logico-mathematical knowledge, unlike physical or social, exists only within the mind of the individual. It is the mental relationships constructed through reflection on experiences. When a child is able to recognize that they can cause shorter durations (time intervals) by making their actions quicker, she is reasoning with logico-mathematical knowledge.

Temporal Constructs

Piaget (1969) identified numerous constructs that are coordinated through durational reasoning. Specifically, he indicated that the operationalization of psychological time requires the coordination of two systems: succession of events and colligation of duration.

Succession. Succession is the order in which events occur. Piaget (1969) described duration as a function of succession, in that shorter durations result from quicker successions (and vice versa). So, for example, if two children race to a playground, the child who ran faster will necessarily get to the playground first, precisely because she took less time to get there. The succession of being first before second is the result of the shorter duration.

Colligation. Colligation is the organization of successive durations. It is the means through which "duration acquires a structure" (Piaget, 1969, p. 159). The seriation of durations is inherently successive given the chronology of events. For example, the time taken to fill a jug with water results from the successive partial durations of the flow of water—from empty to full—into the overall duration of "filling up."

Duration. Duration is the intervals of *and* between events. These intervals are naturally successive, operationally colligated, and can be quantified. The notion of the durations of psychological time returns to the earlier considerations of succession—a child's perception of the sequence of events as they actually occurred and how these events are reconstructed in the child's memory. Because perception influences understanding, psychological time is a test of how a child operationalizes the colligation of successive durational memories.

Images of Change

Castillo-Garsow (as cited by Thompson, 2012) "suggested, to operate with change happening in conceptual time, one must extract time from change, so that change happens in relation to time as opposed to happening because of passing time" (p. 11). A child who reasons intuitively about time sees change happening *because* of passing time, for example the bottles filled because time passed but more time passed for one bottle than the other. One who has operationalized time understands change in relation to time, so time passed while the bottles filled, but one filled faster *so* it took less time. Castillo-Garsow and colleagues (2012, 2013) defined two ways to reason about the covariation of time and action: chunky and smooth images of change.

Chunky images of change. When reasoning with chunky images of change, a learner considers only the discrete, countable moments of an event (Castillo-Garsow, 2012). Relating to Piaget's (1969) description of duration, it is acknowledging the intervals between events, but not of the events themselves. For example, when traveling 65 miles per hour, the 65 is the conclusion of the hour because the miles traveled are a result of the hour that passed (Castillo-Garsow, 2012, p. 9). The time passed as a result of the activity.

Smooth images of change. Conversely, when a learner imagines change in progress (Castillo-Garsow, 2012) she is reasoning with smooth images of change. From the previous example, traveling 65 miles per hour is not the result of the time or speed but is instead the continuous relationship between the two. A smooth image of change requires that the learner not only recognizes the start and end of a duration, but also understands that while the duration between is constant, the action(s) that fill the duration can impact the overall duration.

Methodology

The following cases (Yin, 2003) present two children from the same grade of similar demographics and a similar level of mathematical achievement; however, the reasoning expressed during interviews showed two different ways of reasoning about time.

Participants and Procedure

Taylor and Ellis (pseudonyms) are both second-grade, white males of similar backgrounds, from a large suburban city in the western United States. Each come from a two-parent household, with the ability to provide many extracurricular experiences. The two live in the same neighborhood and have been friends since kindergarten. The only difference between them, demographically, is that Taylor is an only child and Ellis has a younger brother. On paper, the two boys appear very similar.

Each participant was video recorded in an individual, semi-structured interview intended to explore how the child understood time (e.g., What is time?), determined various durations (e.g., What is something that takes you a short time to do?), and quantified common experiences (e.g., How long does it take you to brush your teeth?). After every question, the child was asked why or how he knew or came to his answer. It was in this explaining that Ellis and Taylor differed, and a connection was made to Castillo-Garsow and colleagues' (2012, 2013) images of change. These interview were transcribed in their entirety for analysis.

Data Selection and Analysis

With the focus on images of change, each transcript was read thoroughly, and excerpts were pulled that highlighted Taylor and Ellis's reasoning about time through the lens of chunky and smooth images of change. Using a constant comparison analysis (Glaser & Strauss, 1967), each chosen excerpt was narrowed into smaller sections, usually by sentences or sentence fragment and codes were counted for frequency of use by each child, to allow for a cross-case comparison.

Themes were identified correlating with Castillo-Garsow and colleagues' (2012, 2013) chunky and smooth images of change, such as time as distinct points, time as a continuous flow, and time as influenced by action. A Keywords-In-Context analysis (Fielding & Lee, 1998) was performed within each theme, looking specifically at how each child explained their durational experiences. Then, a componential analysis (Spradley, 1979) was performed to more precisely compare the distinctions between the reasoning presented by Ellis and Taylor.

Findings

Ellis: A Case of Chunky Images of Change

From early in the interview, Ellis related time to the units in which it is measured. From his initial definition of time, it became clear that Ellis's reasoning about time as a quantity—that being a thing that can be measured (Thompson, 1994)—was focused not on the duration of the passing time, but rather the beginning and end, the completed chunks (Castillo-Garsow, 2012).

Author (A): What is time?
Ellis (E): time is like, say it's like 5 o'clock..like maybe you have to do something at 5 and then you're done at like say 6:30
A: so if you start something at 5 and end at 6:30, where is the time in there?
E: it's the 5 o'clock and the 6:30
A: so the start and the end?
E: yah, the start and the end

Ellis explained time as only measured units, specifically the start and the end, but gave no indication of the continuous flow between these units as part of the duration. Later in the interview, Ellis demonstrated an understanding of seconds being a consistent passing beat; however, when he had to coordinate his actions with this passing time, his reasoning centered on the individual chunks of time passing rather than the overall flow of the duration.

A: if you walked that same path, and you wanted it to take less time, what could you do?
E: you can do like this (takes huge steps across the rooms as he says with each step) 1, 2, 3

A: so you could take bigger steps?
E: yah, bigger steps but it would take the same amount of time because, like, if you were walking (re-demonstrates taking smaller steps as he counts with every other step) 1, 2, 3 and when you're walking like this (walks again) and you're like (takes huge steps as he says with each step) 1, 2, 3, you're just taking longer leaps instead of shorter leaps
A: so shorter leaps and longer leaps can take the same amount of time, but what if you wanted to make the *time* shorter?
E: shorter(?)
A: how could you make it take less time to walk?
E: take less time to walk(?)

Ellis's focus is heavily on the numerical counting of the steps as a proxy for the duration they fill. These steps indicate a chunky image of change, as there is no consideration for the time during or between the steps.

Taylor: A Case of Smooth Images of Change

From the beginning of the interview, Taylor reasoned about time differently than Ellis.

Author (A): what is time?
Taylor (T):…um, time is, like what you're doing—time can be in days, months, years, seconds, minutes, hours—and time is what you're doing, um, when you pick to do it
A: you said time is what you're doing, later you said it is when, so is it what you do or when?
T: it's both

Taylor mentioned units of time measurement as examples of what time can "be in," however, he attended to both the action and when the action occurs. The qualitative nature of this response demonstrates a more abstract reasoning about time as a quantity. Taylor expanded upon this abstraction of time when asked how he could measure time.

A: how long does it take you to brush your teeth?
T: 2 minutes
A: how do you know?
T: because most people said brush your teeth for 2 minutes or 1 minute because the happy birthday song can usually take you 1 minute—or 30 seconds
A: okay, so you talked about the happy birthday song
T: singing it twice, and so I don't get cavities I say two long names

While not explicitly stating it, the last sentence "so I don't get cavities I say two long names" seemed to indicate that Taylor understands the inverse relationship between his actions and the length of a duration. By saying *two long* names, the duration of his teeth brushing increases, causing him to not get cavities. Taylor seemed to be accounting for time as being changeable through the progress of his action (Castillo-Garsow, 2012); the duration of his teeth brushing was defined not by a set start and stop moment, but through the continuous passing of his actions.

Discussion and Implications

While the reasoning presented by the participants was not intended as covariational, as Castillo-Garsow (2012) had designed these images of change, Piaget (1969) asserted that the understanding of time requires the coordination duration via movement through space, which necessitates a covariational relationship. Ellis and Taylor drew from their lived experiences to explain what time was and how it can be measured; however, their explanations differed greatly in *both* what time was *and* how it can be measured. I suggest that the reason these two seemingly similar children reason so differently about time lies in early construction of chunky and smooth images of change (Castillo-Garsow, 2012; Castillo-Garsow, Johnson, & Moore, 2013).

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GROUP PRESENTATIONS AS A SITE FOR COLLECTIVE MODELING ACTIVITY

Corey Brady
Vanderbilt University
corey.brady@vanderbilt.edu

Hyunyi Jung
Marquette University
hyunyi.jung@marquette.edu

We approach student presentations of solutions to modeling tasks as occasions for whole-class reflection on the rich conceptual work that small-group teams have done in parallel. Analyzing and interpreting these interactions can offer insights into how a classroom group negotiates a shared sense of what they have learned and what they collectively view as "newsworthy" across groups from their recent (and ongoing) model-building. We describe analytical tools to interpret a classroom's work during presentations, and we illustrate their use in a single case. This work offers a foothold for design-based research to harness presentations to improve learning, drive instructional decisions, and illuminate modeling processes at both individual and group levels.

Keywords: Modeling; Problem solving; Communication

Orchestrating presentations and the surrounding discussions can be critical in classroom practice (Ball, 1993; Lampert, 2001), where it can be challenging to avoid a simple "show and tell" pattern (Stein, Engle, Smith, & Hughes, 2008; 2015). In the context of rich modeling problems, we framed group presentations as a scene of second-order, shared mathematical modeling. That is, we viewed interactions around presentations as reflecting the class's shared effort to make sense of what they have learned and to identify continued uncertainty or disagreement. To proceed analytically, we developed a scheme for coding presentations along three *dimensions*, characterizing utterances in terms of (a) how they referred to phases of modeling, (b) how they enacted rhetorical stances, and (c) how they contributed to whole-group-level modeling (e.g., building toward consensus or strengthening a sense of multiplicity in the class's solutions). We present this scheme in Brady & Jung (2019). Here, we focus on a single case, analyzing a class discussion that provoked reflection and shared modeling. We present this case to demonstrate how our coding scheme functions to capture dynamics that reframe presentations as shared modeling opportunities. A single case cannot show changes over time in a class's practices, but it illustrates our method, the dimensions, and connections among them.

Literature Review: In Search of a Group-Level Model of Modeling

Mathematical modeling processes involve the development and refinement of purposeful *models* that describe or provide insight into real-world problem-solving situations (e.g., COMAP & SIAM, 2016; Kaiser & Stender, 2013; Lesh, English, Sevis, & Riggs, 2013; Lesh, Yoon, & Zawojewski, 2007). Lesh, Hoover, Hole, Kelly and Post (2000) defined a *model* as a system used to describe another system. Thus, a *mathematical model* is a system that consists of mathematical elements (e.g., numbers and variables); relationships among the elements (e.g., equivalence relationships); operations and representations that describe how the elements interact (e.g., graphs, symbols, equations); and patterns or rules that show how it can describe another system (Lesh et al., 2000). Developing a mathematical model in the face of a situation, or *mathematizing* reality, involves organizing, quantifying, and/or coordinatizing a real-world situation (Lesh et al., 2007). To characterize students' *presentations* of modeling we consider: (a) features of modeling they refer to; (b) discourse moves they enact; and (c) emergent structure across presentations.

A significant strand in the international research on mathematical modeling has articulated modeling as a cycle with phases (see, Blum, 2015). "Modeling competencies" (Kaiser, 2007; and cf Niss, 2003) are under debate, with a broadly shared caveat that such competencies are not isolated skills, but develop together (e.g., Blum, 2015; Zbiek & Conner, 2006). For example, Galbraith and Stillman (2006) endorse a generally regular image of the modeling cycle, but they show that students can experience barriers in moving from any phase to the next. On the other hand, Borromeo Ferri (2007) found that the work sequence for individual students could be quite idiosyncratic and non-linear, departing from any canonical cycle. In our study of *presentations*, we conjectured that struggles or insights characteristic of model cycle phases (and transitions) might be milestones for groups, but that these might occur in any order.

Our conjectures were also supported by research (a) at the small group level, and (b) on learners' meta-cognitive reflections on their modeling. Czocher (2016) visualized patterns exhibited by *groups* in modeling phases and the transitions between them. She found diversity and non-linearity similar to Borromeo Ferri (2007), but these patterns could be understood in terms of groups' *styles* of problem solving. Research on metacognition by Magiera and Zawojewski (2011) building on work by Wilson and Clarke (e.g., 2004) used students' talk in stimulated-recall interviews about Model-Eliciting Activities or MEAs (Lesh et al, 2000).

As we worked to understand modeling at the whole-class level during presentations we foregrounded *constructivist* and *situated* perspectives on knowing and learning (e.g., Beth & Piaget, 1966; Lesh & Lehrer, 2003) attending particularly to how the specific conditions or framing could affect students' behavior. This led us to begin inductively from cases and focus on characterizing emergent structures of agreed-upon and disputed features of models.

Methods

We present data collected during a two-week summer camp on mathematical modeling for Grade 6-8 students and led by the second author and a mathematics teacher. Twenty-one middle school students (ten females and nine racially diverse students) from five schools participated. On the first day of the camp, the group as a whole constructed norms on interacting with other students (e.g., listen to your classmates, respect others' ideas). Throughout the summer camp, students worked in teams and presented their ideas and solutions to modeling problems to the whole class. The case we present here involved the *Counting Caribou* problem (Lesh & English, n.d.). Students were given aerial photographs of caribou herds and asked to develop a procedure for estimating their numbers. Their "client" was the Alaska Department of Fish and Game. Features of the two sample photographs added to the problem's complexity: (a) there were too many caribou in each photo to count easily; (b) the density of populations differed within and across the photos; (c) some caribous' bodies extended beyond the edge of the photos.

Our analysis focuses on student discourse around six small groups' solutions to this problem, which occurred in a 24-minute exchange at the end of the sixth day of the camp. We selected this case as it was also fundamental in generating our larger codebook for describing references to *modeling phases* and students' *discourse moves* in presentations (see Brady & Jung, 2019).

Findings: Patterns in Discourse across Mathematical Modeling Presentations

We analyzed the class's presentations and discussions of their solutions focusing on understanding moments where questions from the audience provoked shared reflection. The six groups' presentations were very diverse, yet patterns emerged across all of our dimensions. In terms of *modeling phases*, we used the following descriptors:

- **Understand the problem**. Return to the problem statement (including text, tables, or figures) to clarify what constitutes a solution for the Client.
- **Construct / structure**. Frame the problem situation and solution criteria to address them with mathematical tools. Choose a way of looking at the problem.
- **Patch**. Adapt the model in process, responding to "unruly" features of the problem situation that emerge as the initial model is applied.
- **Work mathematically**. Do (or explain) arithmetic or algebraic manipulations.
- **Validate**. Consider the reasonableness of the answer or process. Check extreme or special cases; check assumptions/validity of the mathematical procedures used.
- **Interpret**. Explain the answer as referring back to the context of the problem.

We found that presenters emphasized the Construct/Structure and Work Mathematically phases. In contrast, questioners emphasized the Understand the Problem and Patch phases. This complementarity struck us. We then asked whether we could characterize *forms* of question that had high leverage for raising new ideas. We found questioner contributions of "Seeking Explanations" and "Inquiring about Omitted Features" forms often provoked presenters to introduce novel features of their models, which had been unstated. In particular, unique aspects of the presenters' work did not always show up until students in the audience asked questions of these kinds. We felt that this might be a feature of interactions around presentations that could be supported or learned. In our case, the first instance of this came in the questions for Group 1: Hope, Tim, and Kevin (all proper names are pseudonyms). Their presentation had introduced several key ideas, but many important themes that later arose in the discussion as a whole had not yet emerged. The audience's questions quickly led three of these to surface:

01 Uri: So, I really liked that yours was really exact, but it was also pretty easy and simple. I also have a question. What did you do with the overlapping caribou? Like the ones that are half on the page and...
02 Hope (presenter 1): Oh, those ones we connected.... If it was half a body, we found another half body we would smush it together to make one.
03 Teacher: Any other comments or questions?
04 Irene: Why did you decide to do it in the most crowded...count the ones in the most crowded area?
05 Hope (presenter 1): Well, we kind of like.... All three of us, we counted different areas, so one of them counted up to like 110 or so, some of them counted like to 95 or so. We kind of just rounded them to 100.
06 Tim (presenter 2): Yeah.
07 Uri: So how did you account for.... Like the second one, how did you account for...the instance where there was like none of them...Like the second picture... It's just like...
08 Kevin (presenter 3): We eliminated those squares so I could get an accurate estimate.

The class's questions prompted Group 1 to elaborate on aspects of their modeling that they had left out of their presentation. They also served to identify issues in the public forum that any future presenters should attend to. As such, they represented bids to contribute to an emergent, shared *specification for an adequate solution*. Uri's question surfaced a "patch" that Group 1 had devised as they applied the procedure they had described simply as, "We counted the caribou in one section." Irene called into question the logic of choosing a highly populated grid cell (rather than using a "medium" cell, which emerged with later groups and became a 'standard'). Here, Hope's answer was not fully responsive: instead, she used the question as an occasion to elaborate the counting strategy beyond what Group 1 had presented. (In their solutions, they sampled three *different* crowded grid cells and took a rough average of caribou counts across these.) Finally, Uri's question led Kevin to mention a feature that came to be shared across many groups' presentations: "patching" the grid approach for Herd 2 by removing the number of empty or very sparsely populated grid cells from the count used in the calculation.

These question contributions were coded in our *discourse-moves* dimension as "Inquiring about omitted features of the problem or solution process" (Uri's questions) and "Seeking explanations for aspects of the modeling process" (Irene's question). We find it interesting that the questioning session here operated in collaboration with the presentation to articulate complementary aspects of Group 1's solution and to raise features of the problem and solution that would highlight the diversity of later groups' modeling work and feed consensus and debate.

A more extensive account of our data analysis describes how the sequence of presentations led to the emergence of a shared *solution specification* and a shared *"skeleton" model* at the class level. That is, the class came to agree on what would count as a solution for the Client and also on some key components of good approaches. Without this class-level discussion—constituted across contributions by presenters and questioners—the solution specification and the structural similarities among groups' approaches might have remained implicit. In contrast, this class appeared aware that a shared model had indeed emerged by the end of the presentation sequence. Group 5 began by saying, "Okay. So our method was essentially the same as basically everybody else's method." And Group 6 repeated a similar pronouncement, twice: at the start ("Okay, so we pretty much did the same procedure as everyone else") and after describing a unique aspect of their work ("…and then we did the exact same thing like everybody else").

Discussion and Conclusion

We have explored presentations as occasions where a whole class can identify common ground across their work and identify "newsworthy" elements of each others' approaches. This lens allowed us to ask questions about how the class converted a small-group modeling activity into a second-order modeling experience at the whole-class level. We used a case from our larger study to illustrate the phenomena we are focusing on and to demonstrate our approach.

Reflecting on generalizability, we expect that similar conceptual issues and challenges would face any classroom engaged in processing its small-group work on a rich modeling activity, but that there would be variation in the specific results. This variation might be used to characterize aspects of "expert" group-level modeling practices and/or to guide instructional decision-making. With this in mind, we intend our approach to illuminate the texture of whole-class modeling work, rather than to categorize it neatly or uni-dimensionally. In particular, we expect different phases of modeling to be highlighted in different classrooms or for different activities. For instance, Counting Caribou is a Model Eliciting Activity, or MEA (Lesh et al, 2000), but it places a relatively low emphasis on the details of the client's situation, relative to other MEAs. This might in part account for a relatively low occurrence of Understand the Problem, Interpret, and Validate codes in our *model-phase* dimension.

Having groups present their solutions to the whole class can be critiqued on the basis of the value of the use of time. In fact, Lesh (2010) proposes an alternative approach, modeled on conference Poster Sessions, to allow groups to be exposed to and critically assess alternative solutions. However, our analysis suggests it is possible for a class to engage in collective modeling at the whole-class level, and this may be an instructionally useful experience. Our larger study and future work in applying and interpreting the coding approach presented here promise to offer insights into collective modeling during presentations, and to suggest instructional interventions that could support students in formulating and recognizing high-leverage discourse moves that drive collective modeling processes.

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STUDENTS' VIEW AND UTILIZATION OF PROFESSORS' FEEDBACK WHEN REVISING THEIR PROOF

Nurul Schraeder
West Virginia University
nuwahyuni@mix.wvu.edu

David Miller
West Virginia University
millerd@math.wvu.edu

Feedback on proofs is an important tool that can help students to learn from their proof writing errors. We studied how students view written feedback by having students revise some of their proofs from introduction-to-proof class. We discovered four main findings: 1) students tend to skip reading the feedback when they performed well, 2) students prefer clear, detailed, and descriptive feedback, 3) feedback is not always about fixing students' errors, but it can indirectly influence how they think and reason through future proofs, and 4) doing revisions improves students' understanding. The findings from this study could have an important impact on what type of feedback a professor gives based on their purpose (giving less-detailed feedback so that students can revise their work or giving more-detailed feedback so that students can clearly see what mistakes they made) and how students perceive that feedback.

Keywords: Reasoning and Proof, Feedback on Proofs, Assessment of Proofs, Proof Revision

It is well documented that undergraduate students have difficulty writing proofs (Weber, 2001; Moore, 1994; Sowder & Harel, 1998) and that proof-writing is a primary competence that mathematicians want students to learn in upper-level mathematics classes (Weber, 2004, Zorn, 2015). To help students learn during their proof writing, professors provide feedback on students' proofs during grading with the goal that students will use the feedback to improve their proof writing (Byrne, Hanusch, Moore, & Fukawa-Connelly, 2018; Moore, 2016; Miller, Infante, & Weber, 2018). It is important to investigate professors' feedback on students' proofs because feedback can have a greater impact on students' understanding and writing of mathematical proofs than other instructional practices (Black & William, 1998; Henderson, Yerushalmi, Kuo, Heller & Heller, 2004, Hattie & Timperley, 2007).

Byrne et al. (2018) found to some extent that students may use the feedback on their proofs, but their uses were more rudimentary (quickly reviewing feedback, reviewing feedback before exams, making mental notes when reviewing the feedback). One of the reasons for students' non-use of professors' feedback may be that they are not required to address the feedback or revise their proofs. We see that revisions as a way for students to review and reevaluate their proof productions. While we would argue that feedback can aid students' proof-writing skills and comprehension, we do not know much about the effectiveness of professors' feedback on students' proofs. The goal of this study was to answer the following research questions: 1) how do students utilize feedback during revision and 2) in what ways does the feedback help students revise their proofs and improve their proof comprehension.

Theoretical Framework for This Research

Lyster and Ranta (1997) created a classification of written feedback into six categories, i.e., (1) Explicit correction, (2) Recast, (3) Clarification, (4) Metalinguistic Clues, (5) Elicitation, and (6) Repetition. The study was on French language learning in immersion setting which involved subject-matter lessons and French language arts lessons at the primary level. They found that teachers had a tendency to use the *recast* form of feedback. However, they also found that *recast* is the least effective type of feedback at eliciting correction where the most effective ones were *elicitation* followed by *metalinguistic clues.*

In another study, Byrne et al. (2018) handed three marked proofs with feedback from a professor to each of eight students. They used Lyster and Ranta's (1997) classifications to categorize the feedback. They found that the feedback that was classified as *explicit correction* or *recast* to be the most effective in eliciting the change. The results from Byrne et al. (2018) were in contrast to Lyster and Ranta's (1997) finding where *recast* resulted in uptake only 31% of the time and *explicit correction* about 50% of the time. Byrne et al. (2018) also found that professor comments classified as *clarification* and *elicitation* were the ones that students struggled to understand. In addition, Moore (2016) found that professors have a noticeable variation in how they grade students' proofs and that they provide a lot of feedback that is not always associated with the grade they assigned.

Methodology

Participants

Seven students in an introduction-to-proof class at a large state university in the United States agreed to participate in this study during the Fall 2018 semester. All participants had completed at least the second semester of Calculus. We anonymized the data by referring to the first interviewed student as S1, the second as S2, and so on, and assigned a feminine pronoun for each student.

Material and Procedure

Each participant met with the first author and was videotaped and audiotaped for the first phase of the interview that asked participants general open-ended questions about their views and use of feedback on proofs. For the second phase, we selected four participants (S1 through S4) for further interviews with the first author to discuss their interpretations of the feedback on their graded proofs. These proofs were taken from homework and exams that were selected based the number of feedback and one that the participant had struggled to write a complete proof. The participants were then asked to revise their proof and articulate their thoughts as they revised their proof. We repeated this process for 2 to 3 proofs for each student, totaling 11 participant interviews.

All participants' revision work was recorded on a LiveScribe notebook and backed up using video and audio recordings. After the participants finished working on the revisions, we asked them to describe their approach to revising their proof to get their overall thoughts on their revision work. Once the interview was completed, the first author rewrote the revised proof to anonymize the participants' written proofs and names, and then asked the professor of the course (second author) to provide feedback and regrade the proofs. The interviewer asked the students for another revision only if the revision submitted received a grade of 70% or lower. In the case that the participant needed to revise their proof again, the first author only asked them to revise their proof to see if they would use the feedback. This was followed by asking the participants to describe their approach to revising their proof again.

Analysis

All interviews were transcribed. The authors engaged in thematic analysis (Braun & Clarke, 2006) in the following way. First, both authors individually read over all the data while identifying and highlighting passages that might be of theoretical interest concerning feedback. Both authors met to discuss and compare their findings and developed a list of phenomena for further investigation. Next, each author reread the transcripts again, searching for further participant comments related to each phenomena of interest; we then put this excerpt into a file of comments related to that phenomenon. Through this process, we developed criteria and descriptions of these themes. Finally, the authors individually went back through the data one last time to identify passages that would satisfy the criteria for each of these themes.

This study is different from Byrne et al. (2018) since the participants in our study were reading and writing proofs on mathematical topics that they were working on during the semester, while participants in Byrne et al. (2018) study were not chosen from a specific class and were not reading and writing their own proofs. Also, the comments were written by the professor who was teaching the course, unlike in Byrne et al. (2018), where the feedback wasn't provided by the participants' professor. We believe that these differences are significant to our research due to two reasons. We knew exactly what the feedback meant without having to make assumptions since the professor who gave the feedback was involved in the analysis. Secondly, by having the students read feedback from their own proofs, we can see students' abilities in interpreting comments related to their own proof-writing and proof-comprehension abilities. This will give us a better insight into how students deal with feedback and their perspective toward feedback when revising their proofs.

Results

Reading or Not Reading Feedback Based on Proof Performance

The first important finding was that many participants disregarded the feedback whenever they were satisfied with their performance, regardless of the feedback. For example, S5 remarked "I guess when I got a hundred on it (a proof), I might not read it (the feedback) because I figured I did well" and S1 commented that "if I get like a high enough grade, I'll probably just like brush it (the feedback) off", but did mentioned that she may look at the feedback later. This was reiterated by S4 that "if I've got a lot of papers back and my grade is pretty good and there are other papers that need more attention I'll ignore the feedback on the better ones". Hence the performance can be an indicator on how much students utilize feedback.

Students' View of Professors' Feedback

Most participants claimed that professors give feedback to help them as they learn about mathematical proof. For instance, S3 made the following comment on why professors give feedback, "mostly probably to guide the students into thinking the correct way about how to approach the problem and how to I guess think about the process of solving proof". Therefore, students understood the overarching reasons why professors give feedback.

We also found that the typical feedback that was more favorable to participants were the ones that are clear, descriptive, and detailed. This confirms the result of Amrhein and Nassaji (2010) where they found that students prefer written feedback that requires less of their effort to correct. However, providing extensive detailed feedback can be time consuming and not very thought-provoking. Weber made a point mentioned in Byrne et al. (2018) that "the most important ideas that professors want to convey to students are ways of thinking and proof-writing habits and techniques" (p. 250). However, even though S2 preferred detailed feedback, she somehow understood that "good feedback tells you what you did wrong and usually has a hint of how you could improve it [...] because if it is more detailed, if it's spelling it out for us, we're not as likely to think about it as much."

The interviews with students show that such feedback is most likely considered to be unclear feedback. As a result, we see a gap between what students and teachers prefer on written feedback. On one hand, we see that since students are the recipients of feedback and we need their judgement on the type of feedback that resonates with them. On the other hand, students also may not always recognize the benefit of the feedback when the intention of the professors is not communicated.

Students' Utilization of Professors' Feedback

The interviews confirmed that most students value feedback as something that helps them to improve. But in practice, they would very rarely take the time to revise their proofs based on the feedback they were given, let alone when they received high scores on their proofs. The students see feedback as something that they just need to read, keep in mind, and learn from it.

The revision process gave us a better insight on how students utilize feedback. It is fairly obvious that the revision process forces students to read the feedback (see next finding), but we also found that feedback may have an indirect impact. The feedback can address their general approach, but not give them feedback to address proof specific issues. For instance, S2 made the following remark on one of the comments that she received, *"*It helps me to show a, how the way I was thinking about doing proofs in general was wrong but it doesn't specifically help me to redo this proof", when we asked her if a particular feedback helped her on her proof revision. On the other hand, the feedback can be clear to students but still not sufficient for them to correct their mistakes during revision. For instance, S1 understood the meaning of all the comments and the errors she made but she still could not revise her proof. She claimed "I understand what I did wrong but I still don't understand like how the proof would come together [...] because if he told me like where to start, where should have I started, then I would be able to blow through it [the proof]".

This suggests that, having a clear feedback itself is not enough to enable students to correct their proof errors. It suggested that when they were far from having a good proof, they need some direct feedback to guide them. However, a further investigation is needed to see the effectiveness of direct feedback since we do not have much data supporting this argument.

Revising Improves Students' Use of the Feedback and Their Proof Understanding

The fourth important finding is that students receive much more benefit from the feedback than just reading the feedback during the revision stage. To illustrate this, we note that after S3 was done with her revision work, she said, "I realized what I did properly wasn't 100% correct but I didn't really understand why until I really sat down and looked at what the reasoning for me getting it wrong.*"* Furthermore**,** S2 remarked *"*it [revising her proof] helps reinforce the concepts, it helps reinforce the feedback and it's a more effective form of studying than just looking at old homework actually not only redoing it but having consequences behind redoing it." Thus, a student will not likely utilize the feedback unless they need to revise their proof. From our data, we see that when the students revised their proof, they read the feedback, work to comprehend what the feedback is addressing and use the feedback to guide their revision work.

Conclusion

Reading feedback itself is not enough to be able to correct most mistakes; one needs to comprehend the feedback to learn from it. Having the students revise their proofs results in learning from the feedback. Even though the feedback that a professor gives on a proof is not very clear at first sight, students can improve their proof with work on comprehending the feedback and correcting their errors. We observed that this is a very effective way to utilize feedback and also a way to help students to better understand how to rewrite their proof.

Giving feedback itself is not an easy task. Students with different skill levels will need different types of feedback. For example, a student who has a weaker background will not likely understand very brief feedback and therefore they need more detailed feedback. But this is probably not to be the case for an advanced student. Satisfying the feedback preference for every student is also very unlikely to be achieved. These results suggest that the professors might need to regularly remind students that when they give feedback, they are providing ways of thinking to students and want students to learn from the feedback, even when they do not deduct points for proof writing errors. The professors do this to make the feedback as meaningful and thought-provoking as possible.

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USO DE EXCEL PARA LA MODELACIÓN DE UNA ACTIVIDAD DE CRECIMIENTO POBLACIONAL

Verónica Vargas-Alejo
Universidad de Guadalajara
veronica.vargas@academicos.udg.mx

Luis E. Montero-Moguel
Universidad de Guadalajara
montero_hk@yahoo.com.mx

En este estudio se muestran los modelos que estudiantes de carreras de administración y contabilidad construyeron al realizar una actividad cercana a la vida real, con el uso de Excel. Se analizan las representaciones y las ideas exhibidas por los alumnos cuando usaron el software. La metodología fue cualitativa. El marco teórico fue la Perspectiva de Modelos y Modelación. Los resultados muestran que el uso de la herramienta permitió explorar de manera dinámica conceptos matemáticos como: covariación, tasa de cambio y función exponencial. Los estudiantes formularon conjeturas, argumentos, explicaciones y justificaciones.

Keywords: Modelación matemática, Resolución de problemas, Tecnología.

Las funciones son objetos matemáticos cuya comprensión requiere de conocimiento de conceptos como variación, tasa de cambio, dominio, rango, función inversa (logarítmica), entre otros; su aprendizaje está relacionado con el desarrollo de un razonamiento covariacional (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). De acuerdo con investigaciones basadas en la Perspectiva de Modelos y Modelación [PMM] (Lesh & Doerr, 2003), la resolución de situaciones cercanas a la vida real, puede apoyar la comprensión de conceptos matemáticos como función exponencial (Ärlebäck, Doerr, & O'Neil, 2013).

En esta ponencia se presentan resultados obtenidos al implementar una actividad, diseñada con base en un Crecimiento poblacional (CP) y asociada a la función exponencial. Se analiza el potencial de la actividad en términos de las habilidades y los conceptos matemáticos que los estudiantes revelaron y desarrollaron. Se responden las preguntas de investigación siguientes: 1) ¿qué modelos construyeron los estudiantes de carreras de contabilidad y administración para resolver esta actividad CP? Es decir, 2) ¿qué representaciones, conjeturas, creencias, argumentos y conocimiento matemático utilizaron? 3) ¿cómo apoyó Excel la construcción, modificación y extensión del conocimiento? Se utilizaron los seis principios de diseño de las Actividades Provocadoras de Modelos [MEA por sus siglas en inglés: Model Eliciting Activities] (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003) para analizar el potencial de la actividad CP.

Es importante mencionar que el profesor que diseñó e implementó la actividad CP forma parte de un proyecto mexicano de investigación multi-nivel (Doerr & Lesh, 2003), centrado en la PMM. Este profesor participó en un proceso de formación de doce sesiones, de dos horas cada una, donde el profesor, apoyado por investigadores, a) observó la implementación de la MEA del Problema del avión de papel (Lesh & Doerr, 2003; Universidad de Minnesota, 2008), b) discutió los resultados con investigadores, c) implementó la MEA del Hotel (adaptada de Aliprantis & Carmona, 2003), c) diseñó la actividad CP aquí presentada, y d) preparó e implementó la actividad. En este documento, por falta de espacio, sólo se describen los resultados de la implementación.

Marco Teórico

Aprender matemáticas es un proceso de desarrollo de sistemas conceptuales, que cambian de manera continua, se modifican, extienden y refinan a partir de las interacciones del estudiante con su entorno (los profesores y compañeros) y al resolver problemas (Lesh, 2010). Para desarrollar sistemas conceptuales la PMM (Doerr, 2016) propone estructurar experiencias para que el alumno exprese, analice, pruebe, revise y refine sus formas de pensamiento durante el proceso de diseño de herramientas conceptuales podersoas que incorporen construcciones matemáticas significativas (Sriraman & Lesh, 2006). La PMM sugiere el uso de Actividades Provocadoras de Modelos (MEA) en el aula para promover que el alumno manipule, comparta, modifique y reutilice herramientas conceptuales, para construir, describir, explicar, manipular, predecir o controlar sistemas matemáticamente significativos (Lesh & Doerr, 2003). Interesa que los estudiantes logren desarrollar procesos de matematización; es decir, cuantificar, dimensionar, coordinar, categorizar, simbolizar algebraicamente y sistematizar objetos, relaciones, acciones, patrones y regularidades relevantes (Lesh & Doerr, 2003). Lo importante, por lo tanto, es el proceso de construcción de modelos, más que el modelo mismo ya que interesa que el estudiante desarrolle habilidades y conocimiento matemático.

La PMM propone diseñar MEA’s a través del uso de seis principios (Lesh et al., 2003, p. 43): de significado de personal (Principio de la realidad), de construcción del modelo, de autoevaluación, de externalización del modelo (Principio de la documentación del modelo), del prototipo simple y de generalización del modelo. Estos principios fueron usados para el diseño de la actividad CP y el análisis de los datos recolectados, ya que interesaba desarrollar habilidades y conocimiento matemático en los estudiantes. La tecnología jugó un papel importante por su potencial para la construcción de representaciones.

Metodología

Los participantes en este estudio fueron un grupo de un cinco alumnos (adultos inmersos en el campo laboral con edades entre 24 y 34 años) que estaban cursando la materia de Matemáticas aplicadas a los negocios en el primer cuatrimestre de la Licenciatura en Administración y Licenciatura en Contabilidad. La sesión se llevó a cabo en un aula de cómputo. El Equipo 1 se conformó por tres alumnos (S1, S2, S3) y el Equipo 2 por dos alumnos (S4, S5). S1, S2 y S5 eran estudiantes de administración y S3 y S4 de contabilidad. El estudiante S1 tenía las mejores calificaciones del grupo, la estudiante S4 tenía las más bajas calificaciones y participaba poco en el aula; además, S4 trabajaba como vendedora de seguros en horas extraclase.

Santiago Vázquez, alumno egresado de la carrera de administración de nuestro campus UVM GDL SUR, está haciendo su Maestría en Desarrollo Local y Territorio. Su profesor le solicitó que elabore una carta dirigida a la Secretaría de infraestructura vial sobre el crecimiento poblacional de la zona metropolitana de Guadalajara, para que sea tomada en cuenta en los próximos proyectos de infraestructura vial.

Santiago investigó y encontró que en el año 2018 la población de la zona metropolitana de Guadalajara llegó a 4.299 millones y que el promedio de crecimiento es de un 1.7% anual.

Para que la carta tenga el impacto necesario en la Secretaría, Santiago necesita enviar un procedimiento que permita conocer cuál sería la población en los años 2020, 2022, 2024, 2030, 2040, 2041, 2100, expresar si la variación es constante y saber en qué año habrán 6 millones de habitantes, 7.560 millones, 8.232 millones y para qué año la población se duplicará respecto a la del año 2018.

Ayúdale a Santiago a redactar la carta. Describe el procedimiento de tal manera que pueda ser útil para describir el crecimiento poblacional de cualquier otra ciudad o del mundo.

Figura 1: Situación problema de la Actividad CP

La actividad CP, denominada Crecimiento poblacional en la zona metropolitana, estaba compuesta por tres páginas, las dos primeras contenían la actividad de calentamiento diseñada con base en la problemática del incremento de tránsito vehicular como consecuencia del crecimiento poblacional de la zona metropolitana de Guadalajara. La tercera página (Figura 1) contenía el problema, el cual podía resolverse mediante procedimientos de tipo: tabular (recursivo), tabular (relación funcional), gráfico y algebraico. Los datos fueron extraídos de fuentes gubernamentales (Gutiérrez-Pulido et al, 2011).

La actividad CP se implementó en un periodo de tres horas y media en un aula de cómputo, en dos sesiones. Las fases fueron: 1) individual y luego grupal para la lectura del artículo de periódico, 2) individual, equipo y grupal para la resolución del problema y 3) individual para resolver un ejercicio tipo libro de texto. El papel del docente fue como observador y facilitador. Los criterios de análisis utilizados para describir y analizar los modelos elaborados por los estudiantes en Excel y el potencial de la actividad fueron los seis principios para el diseño de APM.

Resultados y discusión de resultados

Se analiza el potencial de la actividad CP en términos del desarrollo de conocimiento, creencias y habilidades matemáticas.

Principio de significado personal

La lectura individual de la nota periodística permitió que los estudiantes se familiarizan con el contexto y se sintieran motivados para resolver la actividad. Mencionaron su preocupación y reflexión sobre el crecimiento poblacional de su ciudad y la influencia de este fenómeno en los problemas de vialidad de la zona metropolitana de Guadalajara.

Principios de construcción, externalización y documentación de modelos

Modelos iniciales. El equipo 1 elaboró su procedimiento en la hoja electrónica de cálculo. Los integrantes del equipo 2 realizaron primero operaciones en su cuaderno con apoyo de la calculadora; posteriormente, trabajaron en la hoja electrónica de Excel y construyeron dos representaciones: una tabular y una gráfica.

Covariación y tasa de cambio en el modelo del equipo 1. En la tabla elaborada por el equipo 1 los estudiantes detectaron que la población (P) variaba, es decir, era diferente cada año e identificaron cómo variaba. Detectaron un patrón de comportamiento, y escribieron una fórmula recursiva para P. La conjetura de S1 y S3 fue que el crecimiento poblacional era lineal y la tasa (1.7) era constante, No reconocieron la relación exponencial.

Covariación y tasa de cambio en el modelo del equipo 2. El equipo 2, al igual que el equipo 1, tuvo dificultades iniciales para identificar si la tasa de cambio era constante o no. La conjetura de S5 fue que el crecimiento era lineal y la tasa era constante (1.7). Sin embargo, S4, después de realizar un par de operaciones, detectó que la variación no era constante y que debían realizar un procedimiento similar al que hacía de manera cotidiana en su trabajo, como vendedora de seguros. El equipo 2 escribió una relación recursiva. A diferencia del equipo 1, sintetizó todas las operaciones. Es decir, utilizó la población del año dado (actual) para determinar la del próximo año, y obtuvo el crecimiento anual de la población. Este equipo tuvo menos dificultades para identificar el comportamiento exponencial. Aunque S1 (del equipo 1) escuchó a S4 mencionar que la variación no era constante, no la tomó en cuenta; lo anterior, debido a la creencia de que S4 era la compañera de más bajo desempeño en su clase de matemáticas.

Función inversa. Ambos equipos tuvieron dificultades para identificar en qué año habría 6, 7.560 y 8.232 millones de habitantes. Pero, pudieron estimar posibles respuestas a partir las tablas de datos construidas en la hoja electrónica.

El modelo final de los equipos. En la discusión grupal de modelos cada equipo leyó su carta, la cual describía los procedimientos realizados. Cuando el equipo 2 presentó la gráfica, S1 dijo en voz alta: "es cierto, el crecimiento no es constante". Tal como se menciona en la literatura de investigación (Lesh & Yoon, 2004), los estudiantes refinaron su pensamiento sobre el problema. Los estudiantes modificaron el primer modelo lineal y construyeron un modelo de crecimiento exponencial.

El uso de Excel permitió a los integrantes de ambos equipos organizar la información dada en el problema y escribir en lenguaje de la hoja electrónica una relación entre las cantidades, la cual (al ser *arrastrada*) posibilitó analizar cómo variaban las cantidades. En la literatura de investigación (Friedlander, 1999) se considera que la relación recursiva está lejana de apoyar la construcción de una representación algebraica de la covariación. Sin embargo, de acuerdo con el NCTM (2000) es importante que en el salón de clases surja la definición iterativa o recursiva para la función y debe ser comparada con $P(n)$ para ayudar a los alumnos a ver las ventajas y limitaciones de ambas. Esto fue retomado más tarde por el docente.

Principio del prototipo simple y de generalización del modelo

S4 (integrante del equipo 2) identificó que el modelo construido era útil para resolver problemas con otro tipo de contexto. No así los demás estudiantes.

Extensión del conocimiento de los estudiantes con apoyo dirigido por el docente

Después de la discusión grupal, el docente, con base en los modelos construidos, promovió la generalización y escritura de la representación algebraica $P(n) = 4.299(1.017)^n$ para alentar la discusión de los estudiantes en términos de la covariación y uso de la función inversa. En una sesión posterior el profesor propuso la resolución de un problema enunciado en forma verbal, de interés compuesto, del tipo de libro de texto. Los alumnos resolvieron sin dificultades el problema y explicaron al docente el comportamiento de crecimiento exponencial que involucraba, es decir, lograron utilizar su conocimiento para describir una situación problemática en otro contexto.

Conclusiones

Con respecto a las preguntas de investigación planteadas inicialmente, se puede mencionar lo siguiente. El uso de Excel para la elaboración de modelos con representaciones tabulares para resolver el problema permitió a los estudiantes concentrarse en el proceso de revisar cómo variaban las cantidades, al *arrastrar* la fórmula recursiva construida con lenguaje de la hoja electrónica. El uso de la representación gráfica posibilitó que en la discusión grupal los integrantes del equipo 1 modificaran sus conjeturas respecto al comportamiento lineal, e identificaran el comportamiento exponencial. Si bien es cierto, los estudiantes no construyeron una relación algebraica para $P(t)$, pero identificaron la relación recursiva en el lenguaje de la hoja electrónica, la cual les permitió contestar varias de las preguntas planteadas y, además, comprender la situación. El apoyo del profesor fue fundamental para construir y dar sentido a la relación algebraica.

La resolución de la actividad CP, posibilitó que los estudiantes construyeran varios modelos, los manipularan, compartieran y predijeran el comportamiento de la situación. Expresaron sus ideas sobre la variación y tasa de cambio, las analizaron y revisaron; observaron patrones, relaciones y regularidades, es decir, los estudiantes desarrollaron habilidades y conocimiento matemático, lo cual es importante en el aprendizaje de las matemáticas. Un aspecto que no se trabajó en el aula fue la exploración del carácter dinámico de los modelos al cambiar las cantidades iniciales y la tasa de cambio para generar familias de problemas, queda pendiente propiciar y analizar estas acciones en posteriores estudios.

USING EXCEL FOR THE MODELING OF A POPULATION GROWTH ACTIVITY

In this study we show the models that students of administration and accounting careers built when performing an activity close to the real life, with the use of Excel. The representations and ideas exhibited by the students when they used the software are analyzed. The methodology was qualitative. The theoretical framework was the Models and Modeling Perspective. The results show that the use of the tool allowed the dynamic exploration of mathematical concepts such as: covariation, rate of change and exponential function. The students formulated conjectures, arguments, explanations and justifications.

Keywords: Modeling, Problem solving, Technology.

Functions are mathematical objects whose understanding requires knowledge of concepts such as variation, rate of change, domain, range, inverse function (logarithmic), among others; their learning is related to the development of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). According to research based on the Models and Modeling Perspective [MMP] (Lesh & Doerr, 2003), the resolution of situations close to the real life can support the understanding of mathematical concepts as exponential function (Ärlebäck, Doerr, & O 'Neil, 2013).

In this paper we present results obtained when implementing an activity, based on a population growth (CP), and associated with the exponential function. The potential of the activity is analyzed in terms of the skills and mathematical concepts that the students revealed and developed. The following research questions are answered: 1) What models did the students of the Bachelor of Administration and Bachelor of Accounting build to solve this CP activity? That is, 2) what representations, conjectures, beliefs, arguments and mathematical knowledge did they use? 3) How did Excel support the construction, modification and extension of knowledge? The six principles for developing model eliciting activities (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003) were used to analyze the potential of the CP activity.

It is important to mention that the professor who designed and implemented the CP activity is part of a Mexican multi-tiered research project (Doerr & Lesh, 2003) focused on the MMP. This professor participated in a training process of twelve sessions, of two hours each, where the professor, supported by researchers, a) observed the implementation of the MEA of the Paper Airplane Problem (Lesh & Doerr, 2003, University of Minnesota, 2008), b) discussed the results with researchers, c) implemented the Hotel MEA (adapted from Aliprantis & Carmona, 2003), c) designed the CP activity presented here, and d) implemented the activity. In this document, due to lack of space, only the results of the implementation are described.

Theoretical Framework

Learning mathematics is a process of developing conceptual systems, which change continuously, modify, extend and refine from the student's interactions with their environment (teachers and peers) to solve problems (Lesh, 2010). To develop conceptual systems, the MMP (Doerr, 2016) proposes to structure experiences, so that the students express, analyze, test, revise, and refine their ways of thinking during the process of desiging powerful conceptual tools that embody significant mathematical constructions (Sriraman & Lesh, 2006). The MMP suggests the use of Model Eliciting Activities [MEA] in the classroom to encourage students to manipulate, share, modify and reuse conceptual tools to construct, describe, explain, manipulate, predict or control mathematically significant systems (Lesh & Doerr, 2003). The students manage to develop processes of mathematization; that is, to quantify, dimension, coordinate, categorize, symbolize algebraically and systematize relevant objects, relationships, actions, patterns and regularities (Lesh & Doerr, 2003). The important thing, therefore, is the process of building models, rather than the model itself.

The MMP proposed to design MEAs through the use of six principles of instructional design (Lesh & Doerr, 2003, p. 43): personal meaningfulness (reality principle), model construction, self-evaluation, model externalization (model documentation principle), simple prototype, and model generalization. These principles were used for the design of the CP activity and analysis of the data collected, since our goal was to develop skills and mathematical knowledge in the students. Technology played an important role because of its potential for the construction of representations.

Methodology

The participants in this study were a group of five students (adults immersed in the labor field with ages between 24 and 34 years) who were studying Mathematics applied to business in the first semester of the Bachelor of Administration and Bachelor of Accounting. The session was held in a computer classroom. Team 1 was formed by three students (S1, S2, S3) and Team 2 by two students (S4, S5). S1, S2 and S5 were administration students and S3 and S4 accounting. Student S1 had the best grades in the group and student S4 had the lowest grades and participated little in the classroom; In addition, S4 worked as an insurance saleswoman in extra-class hours.

Santiago Vázquez, a graduate student of the management career of our campus UVM GDL SUR, is doing his Masters in Local Development and Territory. His teacher asked him to prepare a letter addressed to the Department of road infrastructure on the population growth of the metropolitan area of Guadalajara, so that it can be taken into account in the next road infrastructure projects.

Santiago investigated and found that in 2018 the population of the metropolitan area of Guadalajara reached 4,299 million and that the average growth is 1.7% per year.

For the letter to have the necessary impact in the Secretariat, Santiago needs to send a procedure that allows knowing what the population would be in the years 2020, 2022, 2024, 2030, 2040, 2041, 2100, expressing if the variation is constant and knowing in what year there will be 6 million inhabitants, 7,560 million, 8,232 million and for what year the population will double compared from 2018.

Help Santiago write the letter. Describe the procedure in such a way that it can be useful to describe the population growth of any other city or the world.

Figure 1: Problem Situation of the CP Activity

The CP activity, called population growth in the metropolitan area, was composed of three pages, the first two contained the warm-up activity, based on the problem of increased vehicular traffic as a result of population growth in the metropolitan area of Guadalajara. The third page (Figure 1) contained the problem, which could be solved by means of tabular (recursive), tabular (functional relation), graphic and algebraic procedures. The data were extracted from government sources (Gutiérrez-Pulido et al, 2011).

The CP activity was implemented in a period of three and a half hours in a computer classroom, in two sessions. The phases were: 1) individual and group to read the newspaper article, 2) individual, team and group to solve the problem and 3) individual to solve a textbook problem. The role of the teacher was as an observer and facilitator. The six principles for the design of model eliciting activities were the criteria for analyzing and assessing the models elaborated by students with the use of Excel.

Results and Discussion

The potential of CP activity is analyzed in terms of the students' development of knowledge, beliefs, and mathematical abilities.

The Personal Meaningfulness Principle

The individual reading of the newspaper article allowed students to familiarize them with the context and to feel motivated to solve the activity. They mentioned their concern about the population growth of their city and the influence of this phenomenon on the road problems of the metropolitan area of Guadalajara.

The Model Construction, Model Externalization and Model Documentation Principle

Initial models. The team 1 elaborated their procedure in the Spreadsheet. The members of team 2 made first operations in their notebook with the support of the calculator; later, they worked on the Excel Spreadsheet and built a tabular and a graph representation.

Team 1, covariation and rate of change. In the data table prepared by team 1 the students detected that the population (P) varied, that is, it was different each year, and they identified how it varied. They detected a pattern of behavior, and wrote a recursive formula. The conjecture of S1 and S3 was that the population growth was linear and the rate (1.7) was constant. They did not recognize the exponential relationship.

Team 2, covariation and rate of change. Team 2, as team1, had initial difficulties in identifying whether the rate of change was constant or not. The conjecture of S5 was that the growth was linear and the rate was constant (1.7). However, S4, after performing a couple of operations, found that the variation was not constant and that they had to perform a similar procedure to the one they did on a daily basis in their work, as an insurance salesperson. Team 2 wrote a recursive relationship. Unlike team 1, it synthesized all the operations. That is, he used the population of the given year (current) to determine next year's, and obtained the annual growth of the population. This team had less difficulty identifying the exponential behavior. Although S1 (from team 1) heard S4 mention that the variation was not constant, he did not take it into account; this was due to the belief that S4 was the lowest performer in her math class.

Inverse function. Both teams had difficulties to identify in what year there would be 6, 7,560 and 8,232 million inhabitants. But, the team 2 was able to estimate possible answers from the constructed data tables in the Spreadsheet.

The final model of the teams. In the group discussion of models, each team read their letter, which described the procedures performed. When team 2 presented the graph, S1 said aloud: "it's true, growth is not constant". As mentioned in the research literature (Lesh & Yoon, 2004), the students refined their thinking about the problem. The first lineal model became a model of exponential growth.

The use of Excel allowed teams to organize the information given in the problem and write in the Spreadsheet language a relationship between the quantities, which (when being dragged) made it possible to analyze how the quantities varied. In the literature research (Friedlander, 1999) it is considered that the recursive relationship is far from supporting the construction of an algebraic representation of covariation. However, according to the NCTM (2000) it is important that in the classroom the iterative or recursive definition for the function arises and it must be compared with $P(n)$ to help students see the advantages and limitations of both. This was taken up later by the teacher.

Simple Prototype and Model Generalization Principle

S4 (team 2) identified that the exponential model was useful to solve a broader range of situations. Not so the other students.

Extension of Student Knowledge with Teacher-led Support

After the group discussion, the teacher, based on the constructed models, promoted the generalization and writing of the algebraic representation $P(n) = 4.299(1.017)^n$ to encourage the discussion of the students in terms of covariation and use of the inverse function. In a later session the professor proposed the resolution of a textbook compound interest word problem. The students were able to solve the problem; they explained to the teacher the behavior of the exponential growth involved, from which, they managed to use their knowledge to describe a problematic situation in another context.

Conclusions

With respect to the research questions initially raised, the following may be mentioned. The use of Excel for the elaboration of models with tabular representations to solve the problem allowed the students to focus on the process of reviewing how the quantities varied, by dragging the recursive formula constructed with the language of the Spreadsheet. The use of graphic representation enabled the team 1 to modify their conjectures, regarding linear behavior in the group discussion and identify the exponential behavior. While it is true, the students did not build an algebraic relationship for $P\left(t\right)$, but they did identify the recursive relationship in the Spreadsheet language, which allowed them to answer several of the questions posed and, in addition, to understand the situation. The teacher's support was fundamental to build and give meaning to the algebraic relationship.

The resolution of the CP activity allowed the students to construct several models, manipulate them, share and predict the behavior of the situation. They expressed their ideas about variation and rate of change, analyzed and revised them; they observed patterns, relationships and regularities, that is, students developed mathematical skills and knowledge, which is important in learning mathematics. One aspect that was not worked in the classroom was the exploration of the dynamic nature of the models by changing the initial quantities and the rate of change to generate family of problems; it is still necessary to promote and analyze these actions in subsequent studies.

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ANALYSIS OF MATHEMATICAL ARGUMENTS IN A 5TH GRADE CLASSROOM

Youngkee Cho
Texas State University
y_c95@txstate.edu

The aim of this study is to provide a new conceptual framework to analyze mathematical arguments using data from a 5th-grade classroom. For the analysis, Toulmin's model of logic and Inglis's distinction between inductive and deductive warrants are employed. A dominant feature of argumentations in this classroom is that taking particular examples is often used to justify universal statements. To evaluate the validity of such arguments, the examples used in the arguments are classified into four categories according their functions: justifier, representative, witness, and conjecturer. The introduction of this new classification allows to more precisely describe the structure of their arguments and re-evaluate their validity.

Keywords: Classroom discourse, Reasoning and proof.

1. Introduction

This study is based on an analysis of one lesson in a 5th grade classroom, exclusively focusing on their mathematical arguments they make during whole-class discussion. The data is a part of the Discussion in Mathematics Classrooms (DIMaC) project (Project PI: J. Bishop). The purpose of this study is two-fold: (1) to find and describe the unique features of arguments of the 5th grade students in one mathematics classroom; and (2) to re-evaluate the force and effectiveness of their arguments by applying Toulmin's layout of logic along with a new specification of 'arguments by example'. There have been various studies of students' mathematical argumentations or arguments, to which Toulmin's layout of logic was applied. (Krumheuer, 1995; Conner et al., 2014; Inglis et al., 2007; Freeman, 2011). This study differs from these previous ones in that it attempts to re-evaluate the validity of relatively younger students' arguments beyond factual descriptions.

The remainder of this paper will proceed as follows. First, in Section 2, the methods I employ in this study are explained. In Section 3, I will share examples of analyses of arguments in the transcripts. Using these examples, we will see that students' invalid arguments from the perspective of the traditional formal logic, can be re-evaluated as valid within Toulmin's framework if we make a proper distinction of their examples in terms of their functions in arguments. In Section 4, the results of the analyses will be described. Finally, in Section 5, I will discuss the implication and potential instructional application of this study.

2. Method

Traditionally, an argument has been analyzed into premises and a conclusion. Toulmin, however, decomposes an argument into six elements: data (D), warrant (W), claim (C), backing (B), rebuttal (R) and qualifier (Q) (Toulmin, 1958). In this new taxonomy of argument, the validity of an argument depends on the backing supporting its warrant. Since the force of backings is determined by the field in which the argument is used, in Toulmin's framework, the validity of an argument becomes relative to the field. One advantage of Toulmin's new model of logic is that it makes a room to re-evaluate the arguments in this transcript as legitimate ones, which the traditional view of logic would ignore or discard simply as invalid.

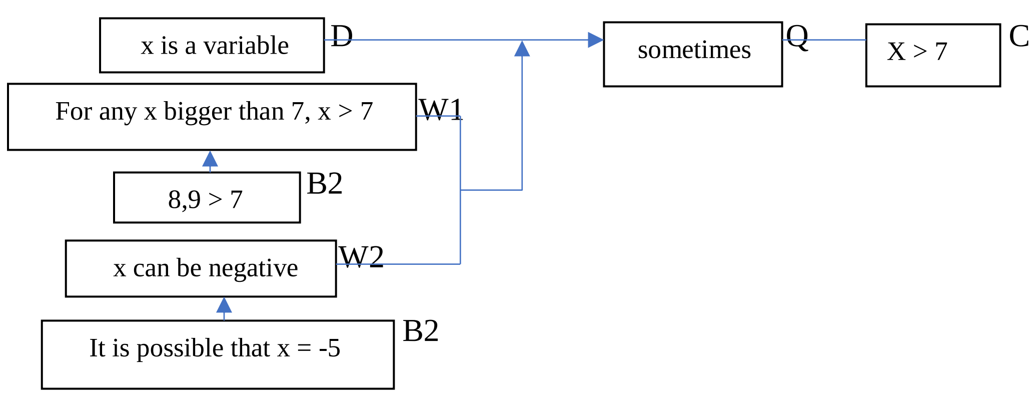
I also adapt Inglis' distinction between inductive and deductive warrants. Inglis defines an inductive warrant as a justification of claim by quantitatively evaluating the claim in one or more specific cases, and a deductive warrant as a form of a formal mathematical warrant, which includes deductions from axioms, algebraic manipulations, or the use of counterexamples. While inductive warrants only reduce uncertainty about the conclusion, Inglis explains, deductive warrants make an argument logically valid (Inglis, 2007). Toulmin and Inglis, however, argue that an inductive warrant paring with its adequate qualifier can make an argument valid as well.

In addition to these tools, I introduce a new classification of examples used in inductive arguments in the transcript. A dominant feature of arguments in the 5th grade classroom in this study is that taking a particular example is often used as means of the justification for universal statements in various ways. For instance, students in the class use a particular number or formula for different purposes. This is illustrated in Table 2. As we will see, this new classification of examples is crucial in evaluating their argumentations in this study

3. Examples of Analysis

Across the entire lesson in the transcript (approximately one hour and twenty minutes), I identified a total of 22 arguments shared during whole-class discussion. Due to its length and the limit of space, I offer only two examples of analyses of arguments. In Argument 1, students are determining whether x > 7 is always, never, or sometimes true. (Note that I am not identifying who [students or teacher] contributes which element of the argument in this analysis; however, in each of the 22 arguments in this lesson students played some role in authoring the argument.)

Argument #1



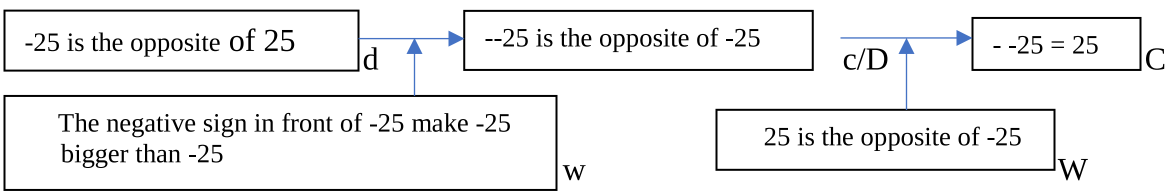
The diagram above illustrates the logical relationship between the statements: The claim C is grounded on the data, D; two warrants, both W1 and W2 support D, each of which is justified by different backings, B1 and B2 respectively; and the qualifier, Q, restricts the applicability of C.

Two things are noteworthy about this argument. First, the whole argument cannot be valid without the qualifier, 'sometimes'; in general, the inductive warrant cannot justify the universal statement without a modal qualifier. Thus, it is an example to show that an inductive warrant paired with an adequate qualifier can justify a universal claim. Second, two inductive backings, B1 and B2 play different roles: While B2 is taken to justify a statement W2, B1 is used only to assure or check the truth of a statement not requiring its justification. According to their different functions, we might call B1 'justifier,' and B2 'witness', respectively.

From a logical point of view, a universal statement cannot be justified by particular instances. However, B2 can justify W2 due to the modality, ‘can’ in W2. W2 can be rewritten as ‘Possibly, x is negative’, and then, B2 can be taken as the evidence for W2. This is another example to show that an inductive warrant or backing can make its argument valid if it is paired with an adequate qualifier. In the case of B1, the context in the transcript clearly indicates that B1 was not intended to justify W1 but just to check the well-known fact which is accepted by all members in the class. It would not be appropriate to require that B1 justify W1 when B1 is not intended to justify W1. Overall, thus, the Argument 1 can be said to be sound even if it contains inductive backings and warrants.

The second example I share comes from a discussion of whether --25 = 25 is always, never, or sometimes true. This argument contains a subargument—the data, warrant, and claim of which is expressed by small letters, d, w, and c, respectively. Note that the claim of the subargument, c, is used as the data of the main argument.

Argument #2



The argument presupposes that the opposite of a number is unique. One important point here is that in d and W, the number, 25 plays different roles. In W, 25 is being used to refer to the number 25. But in d, 25 represents numbers in general. That is, in d, 25 is being treated as an arbitrarily selected number, not as 25 itself. So, we can rewrite d more explicitly as ‘-25 is the opposite of 25 in the virtue of being a number’ or ‘-25 is the opposite of 25 insofar as it is a number.’ Thus, what d really means is that, ‘for any number x, -x is the opposite of x’; even if we substitute 25 with any number, the force of d does not change. In such case, an example is used to represent a group of mathematical objects like numbers. We might call such an example ‘representative.’ Thus, the subargument can be seen as deriving an existential claim, c, from a universal statement, d. Using a particular number as ‘representative’ of a certain group of number is understandable and even justifiable in a circumstance in which students are still learning the concept of variable and only familiar with using concrete numbers.

From these two arguments, we see that examples or instances can play different roles and a proper classification of them may render seemingly invalid arguments valid ones. Another distinguishable function of examples in the students’ arguments is to conjecture the truth of a general statement. Students are sometimes found to use a specific example to determine the truth of a universal statement. For instance, to decide whether a general claim ‘-c > c’ is true, they check the truth of -3 > 3. This ‘conjecturer’ is to be distinguished from ‘justifier’ in that the former is not intended to justify a claim by itself. Thus, as far as it helps or leads students to find a further justification for a universal claim, using it as a warrant for the claim is not problematic.

4. Result

The following are the results and interpretation of the analysis of students and teacher’s arguments in the transcript. Table 1 categorizes the premises used in each argument in this lesson based on (a) whether the argument was inductive or deductive, and (b) whether the premise was used as data, warrant, or backing. Table 1 reveals a general feature of arguments in this 5th grade mathematics classroom. That is, their arguments mostly rely on inductive premises, which function primarily as data and warrants (with few instances of backings). Since the students did not yet fully develop abstract thinking and deductive reasoning, this is understandable. Especially, universal clams and warrants are justified by particular numbers of formula.

Table 1: Inductive vs. Deductive

	Data (36)	Warrant (17)	Backing (1)
Deductive Premises	9	2	0
Inductive premises	27	15	1

In Table 2, I categorize the 42 instances of examples used in arguments based on their function or purpose. The class in this study uses a particular number to (i) represent a certain group of numbers such as negative or positive number, (representative), (ii) confirm the truth of a well-known mathematical fact (witness), (iii) conjecturer whether a statement is true or false for the direction of their further proof (conjecturer), or (iv) directly justify a claim or warrant (justifier). 41 of the 42 examples were used as the premises in the 43 inductive arguments in Table 1. Thus, it can be said that taking an example is the dominant method of argumentations at this stage.

Table 2: Use of Example

Examples	Representative	Witness	Justifier	Conjecturer
42	6	4	31	1

Table 2 shows that the taking an example method is used for different purposes in this classroom. Note that examples are most often used as a representative of some group of numbers. This is because they need to prove a general statement, without the knowledge of deductive reasoning or without knowing how to deal with variables.

Table 3 categorizes the 9 arguments (out of 22 total from this lesson) that use a qualifier by the type of qualifier. Among them, 'sometimes' is most frequently used. We have seen that with the use of 'sometimes', arguments can become valid. Thus, as Inglis points out, with an adequate use of qualifiers, we can preserve the validity of arguments supported by inductive premises.

Table 3: Qualifier

Qualifier	Sometimes	Always	Never
9	5	3	1

5. Conclusion

So far, we have seen the features of arguments in a 5th grade mathematics classroom. One dominant feature of their arguments is that taking an example is a predominant method of justification. We classified such examples into four kinds according to their function in the argument: representation, witness, conjecturer, and justifier. We found that, with that classification, most seemingly invalid substantial arguments in the transcript may, in fact, be valid. We also saw that adopting a qualifier can change the status of an argument.

This study provides a conceptual framework to better understand the nature of mathematical arguments of students who have not yet been exposed to justifying general mathematical statements through deductive reasoning. This framework is expected to allow mathematics educators to develop more suitable instructional strategies to help younger students advance their own valid mathematical argumentations within their current mathematical knowledge.

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VISUALIZING PROPORTIONAL REASONING WITH MEASURED QUANTITIES

Sheri Johnson
University of Georgia
sheri.johnson25@uga.edu

Rui Kang
Georgia College and State University
rui.kang@gcsu.edu

Eun Jung
University of Georgia
eun.jung@uga.edu

Reasoning with measured quantities and developing proportional reasoning are crucial and challenging focuses of middle grades mathematics. Research-based guiding principles (NCTM, 2014) include promoting reasoning with quantities as well as using and connecting mathematical representations. We consider six future middle grades teachers and some of the fine-grained reasoning that occurs while they solve a missing-value proportional reasoning task. Future middle grades teachers were able to use a provided visual display along with a unified view of multiplication to create equivalent proportions.

Keywords: Cognition, Problem Solving, Rational Numbers, Using Representations

Visualization can aid engagement with meanings and concepts that are not readily available through symbolic representation (Arcavi, 2003). In addition, Arcavi (2003) noted that visuals can "group together clusters of information that can be apprehended at once" (p. 218) and noted that "visualization at the service of problem solving, may also play a central role to inspire a whole solution, beyond the merely procedural." (p. 224). Proportional reasoning is both a crucial and challenging concept in the middle grades, and its development, or lack thereof, can greatly influence success for students in later mathematics (Beckmann & Izsák, 2015).

The Common Core State Standards (2010) stress the importance of reasoning with fractions as measured quantities in real-world problem contexts. Quantitative operations are not the same as numerical operations and correspondingly, reasoning with quantities is more complex than reasoning with numbers void of context (Thompson, 2011). Middle grades teachers are required to reason multiplicatively using fractional amounts with designated units, and to guide their students in developing the knowledge to operate with quantities. The present study investigates how future middle grades teachers reason with quantities, as well as how a visual display provides affordances or constraints for reasoning.

Theoretical Perspective

We draw upon theoretical and empirical work in quantitative and proportional reasoning to guide our research design and analysis. More specifically, we consider an account of proportional reasoning (Beckman & Izsak, 2015) that uses a quantitative definition of multiplication (see Figure 1) considering equal size groups where the Product amount (P) is comprised of both units of measure for the Multiplicand (N) and the Multiplier (M).

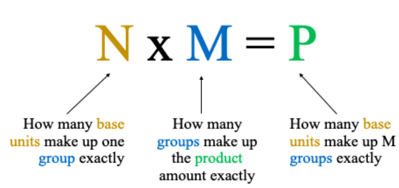

Figure 1: Quantitative Definition of Multiplication: When multiplied together, the multiplier (M) and Multiplicand (N) yield the product P and this can be represented as

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either MxN=P or NxM=P (Beckmann & Izsák, 2015).

The Multiplicand (N) is the number of base units in exactly one group and can be either a whole number or a fraction. The Multiplier (M) is the number of groups, which can also be fractional. The resulting equation is MxN=P or NxM=P, where either order employed in a consistent fashion provides structure for developing reasoning across situations. For this study, we use NxM=P where the number of base units in each group appears first.

Proportional relationships can be seen within this quantitative view of multiplication when there is invariance with one of the three terms, while the other two terms co-vary. For example, if the multiplicand (N) is fixed, there are always 3 cookies on each plate. In this case the multiplier (M) and product (P) can vary (3 plates yields 9 cookies on all the plates, 5 plates yields 15 cookies on all the plates, etc.). Proportional reasoning is often enacted procedurally by both students and teachers and the focus on numerical solutions to standard missing value problems may attribute to the difficulty students and teachers have when explaining it in context. In order to explain why the typical cross multiplication algorithm works in context, a well-developed recognition of situational attributes and a coordinated understanding of relationships is necessary. Using an equal sized groups view of multiplication allows for a more conceptual understanding of proportionality and can help future teachers understand and explain the “why” of cross-multiplication.

This unified view of multiplication works across situations when working with measured quantities. Quantification considers proportionality, as it is “the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attributes measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit” (Thompson, 2014, p. 37). Since a measured quantity must be conceptualized, it seems reasonable that using a consistent name for an object, like an apple, might be helpful to reason with it in context. Additionally, the realization that the same attribute of the same object can be measured in different units entails proportional reasoning.

Research Design

We analyzed transcripts from six future middle grades teachers who were interviewed in the fall of 2016 using a thematic analysis approach (Braun & Clarke, 2006). These future teachers (FTs) were enrolled in the first of a two semester sequence of mathematic content courses, the first with a focus on numbers and operations and the second with a focus on algebra. These semi-structured interviews were video recorded, all written work created during the interview was scanned and the interviews were transcribed verbatim. The data for this study is part of a broader study of future teachers’ reasoning about multiplication and division, fractions, and proportional relationships.

This study considers one particular task, the daffodil problem, given to each of the six participants in part of the third interview. The interviews took place near the end of instruction focusing on multiplication and division, but before instruction on ratio and proportional relationships. Each interview was recorded with two video cameras in order to capture both a close-up view of the participant’s work and a broader view including body language, mannerisms and facial expressions (Hall, 2000). Each of the authors watched the interviews, read the interview transcripts and co-created lesson graphs (Seago, 2004), making analytic notes to document our interpretations of the FT’s reasoning.

The daffodil problem was designed to target a sense of invariance, which in this instance can be seen when the multiplicand (N) remains constant. The problem includes two parts, A and B given to the participants separately. The first part is simply a word problem about a gardening company who plants 224 daffodils per acre and asks how many acres are planted at a park where they plant 392 daffodils. The second part asks the participants to use a visual display (see Figure 2) to solve this same problem.

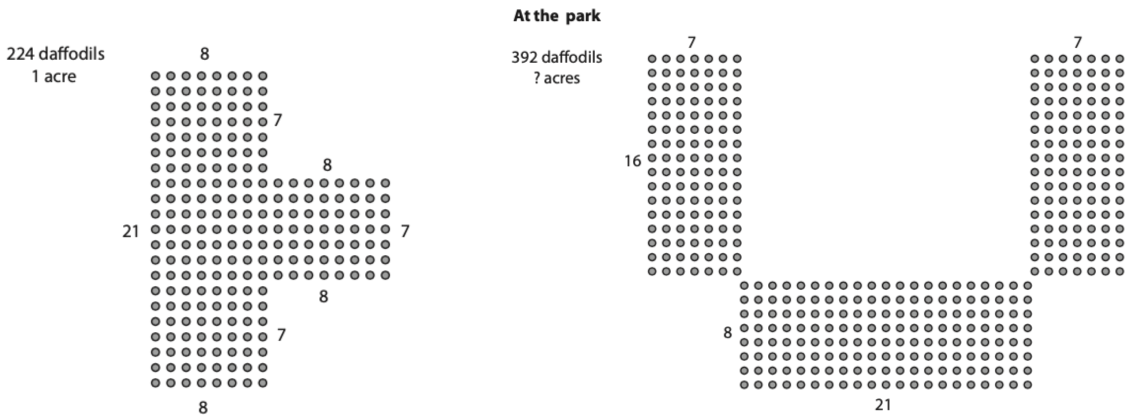

Figure 2: Visual Display for the *Daffodil Problem*, Part B

Results and Significance

We used thematic analysis (Braun & Clarke, 2006) to code the data from the transcripts and identify resulting themes. Our report addresses three themes that our preliminary analysis has identified: quantifying values, selecting base units and identifying equivalence. As we analyzed the transcripts, we use the term “plot” for each of the 7x8 rectangle of 56 daffodils. These are the equal sized groups that allow participants to recognize that the daffodils planted at the park is equivalent to 7/4 acres without carrying out the numerical calculation of 392 ÷ 224.

Overall, the FTs were able to numerically model how to solve part A of the daffodil problem and obtain a correct equation using either division (392 ÷ 224 (3), 392/224 (1)) or cross multiplication (224/1 = 392/x (2)). More of the FTs may have chosen division as opposed to cross-multiplication since they had been exposed to the standard structure of multiplication (NxM=P) in the classroom where division occurs when the multiplier or multiplicand is unknown. Some FTs struggled to explain why their cross multiplication equations worked, one stating “Because I know if you cross-multiply you get an equation… just what I’ve been taught”. All FTs identified this as partitive division where a daffodil is a base unit and an acre is a group. This is consistent with the equation 224*M=392 where the N and P are invariant and the number of groups (M) is unknown.

More interestingly, their reasoning with the visual display showed a range of reasoning and revealed some interesting patterns. Most of the participants began Part B of the daffodil problem by matching like size regions with more of a focus on shape or number. They generally found the whole acre in the park drawing, then determined the value of the “leftover” part, knowing the value is somewhere between one and two. All FTs gave their initial answer in the form of a proper fraction ($1\frac{3}{4}$) which has benefits of being able to see where it falls on the number line. However, it may be this proper form that is stressed in schools that limits their proportional reasoning and the ability to apply the 7:4 ratio to see the 7/4 acres of daffodils that are planted at the park.

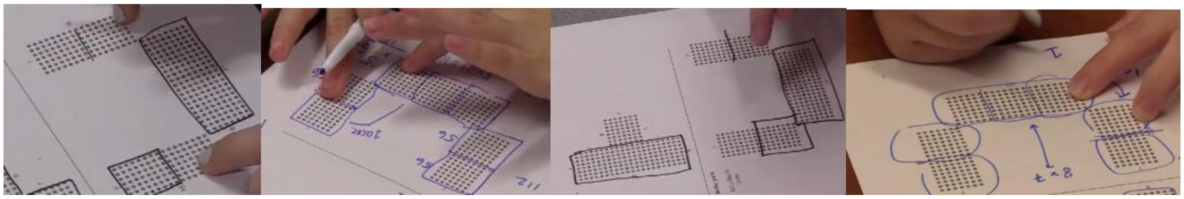

Figure 3: FTs Work Using Plots to Determine Acres

Quantifying an object includes envisioning both an attribute to be measured and a unit of measure. Many of the FTs had difficulty in naming the quantity of a “plot” and this may have impacted their reasoning. Words such as rectangles, blocks, “smallers”, boxes, thingy, and squares were used to identify the ¼ of an acre. Those FTs who identified a name and used it consistently provided more coherent and better coordinated explanations. Additionally, those who identified a number of daffodils (56) for this plot were able to create an equation.

Some of the participants struggled to keep up with the appropriate whole, and two FTs considered the plot to be 1/3 of the whole, as they were using the larger rectangular 8x21 block as the whole instead of this area plus a plot which represents one acre. These FTs self-regulated and found the right relationship, one after receiving a scaffolding question from the interviewer. We consider this to be a constraint of reasoning with the visual display.

Most (5) of the FTs explained the subdivided acre with multiplication by retaining the base units as a daffodil and redefining a group as a plot. However, Molly, who in our opinion had the most coherent explanation and worked flexibly throughout the problem, retained the acre as a group and redefined the base unit as a plot (she consistently called this a square) as opposed to retaining the base unit as a daffodil.

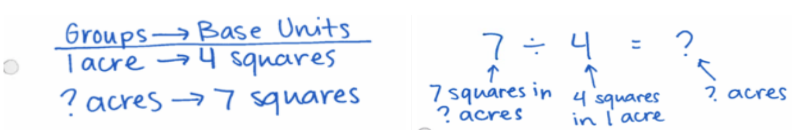

Figure 4: Molly’s s Work Identifying Plots (squares) as Base Units

At times FTs used drawings to come up with relationships, but then did not acknowledge the usefulness of the drawing. For example, Lily was able to use the visual display to see the ratio of plots in an acre to plots in the park as 4:7 but claimed to “use my math rather than these drawings.” Although Lily was unable to produce an equation for part B, which we contribute to her lack of understanding of the measurement attribute, she was able to speak to and represent the equivalence of the relationships (see Figure 5).

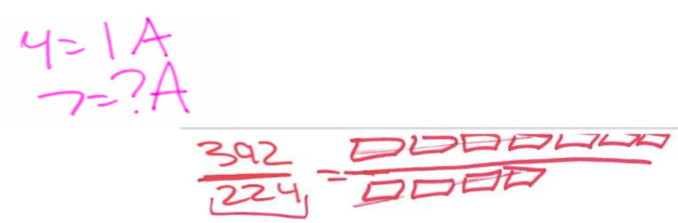

Figure 5: Lily’s s Work Representing Equivalence of Fractions

Conclusion

Research has shown students lack conceptual understanding of proportional reasoning (Keitel, C., & Kilpatrick, J. 2005), and often students and teachers alike will solve missing value proportion problems procedurally applying an algorithm such as cross multiplying. Visual displays can be a useful tool in scaffolding students conceptual understanding, however many practicing teachers are not comfortable using them in instruction (Jacobsen & Izsák, 2015). Our preliminary analysis shows that a provided visual display allows FTs to recognize equal size groups and use this information along with a unified view of multiplication to create an equivalent proportion. The evidence provided here may assist in instilling the need to use representations well beyond the elementary grades.

Acknowledgments

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DIMENSIONS OF STUDENT DOUBTS AND MATHEMATICAL PROBLEM POSING

Priyanka Agarwal
University of California, Irvine
priyanka@uci.edu

The paper reports on student task-based interviews and a teaching experiment on students' mathematical problem posing. The study aims to better understand relationship between students' mathematical doubts and ways in which students craft math problems. Drawing on student small-group interactions and student written-work, the study finds four dimensions of mathematical doubts: pragmatic, empirical, theoretical, and transformational. Definitions, frequencies, and examples of the dimensions of doubts are elaborated. Relationships between the dimensions of doubts and implications for student learning are discussed.

Keywords: Doubts, Problem-posing, Student questions, Mathematical inquiry

Given the importance of mathematical problem-posing for student learning (NCTM, 2000; NRC, 2005), researchers have begun theorizing how to design learning spaces and prepare teachers for problem-posing pedagogy (e.g., Lueng, 2013; Singer, Ellerton, & Cai, 2015). A large part of making progress towards this goal includes seeking clarity of students' epistemic needs (what is it that students are drawn to know and their developing understanding of what can be known and how to know it) and understanding how might students' epistemic needs be related to how students craft math problems. Epistemic needs refer to the feeling of doubt that propels the desire to know and understand something. A doubt is often a feeling of unease, a troubling thought about a situation, a question not yet well-formed, or simply a statement that expresses some kind of uncertainty, wondering, perplexity, or even a conjecture. Thus, I conceptualize problem-posing as student questioning when faced with *doubts* about mathematically obscure situations.

In defining problem posing using the notion of mathematical doubts, I assume that encountering a math doubt is a precondition for student problem posing. This assumption while theoretical in nature is supported by many examples of mathematical work of students and mathematicians in the literature. Dewey (1910) argued that "The origin of thinking is some perplexity, confusion, or doubt" (p. 12) that allows problem finding and reflective inquiry within a socioculturally situated activity. Dillon (1988) referred to student questions that are "expressive of doubt, uncertainty, or perplexity" (p.199) as knowledge seeking tools that allow students to initiate a search for knowing. More recently, Engle & Conant (2002) conceptualized learning as students' efforts to resolve "disciplinary uncertainties". A common theme across these ideas is the recognition that doubts lead to questioning and questioning to new understandings. The notion of mathematical doubts, thus, provides an important lens to investigate the understudied processes of problem-posing. It gives us the leverage to step outside the certainty that so strongly epitomizes the discipline of mathematics and instead investigate the confusions, perplexities, and epistemic needs of students that might engender productive problem-posing.

Using a rich set of data that includes video-recordings, student written work, and interviews; the study asks the following research questions: What types of initial doubts emerge when students explore open unstructured artifacts? What relationship, if any, do the math doubts have with the problems students pose and in what ways do students mathematize their initial doubts to pose meaningful math problems?

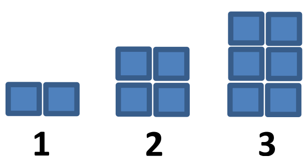
Methods

Research Setting and Data

Data includes task-based paired interviews (Houssart & Evens, 2011) with 64 middle-school students and a teaching experiment in two low-track eighth-grade classes (n=56) in a predominantly working class Latinx neighborhood school. The student interviews were conducted by me. The teaching experiment and the classroom lessons were designed in collaboration with a math teacher. The tasks students worked on during both the interview and the classroom experiment were designed such that students could verbalize their doubts and musings prior to posing a problem. All interviews and classroom interactions were videotaped. Whole-class discussions and 3-4 small-groups in each period were videotaped using flip cameras with an attached table mic. Written work of students was collected to capture their non-verbalized ideas, representations, and conceptual knots.

Tasks

Figure 1 shows the artifact presented to the students during the interview. It consisted of a growth pattern made up of blue blocks placed on a sheet of white paper in the center. Typically, pattern problems (see Parker, 2009; Mathematics Education Collaborative patterning tasks) are used to promote multiplicative thinking, algebraic and functional reasoning and understanding of variables and expressions. Figure 2 shows the border image given to the students during the classroom experiment. It is typically used in a well-known border problem (see Boaler & Humphreys, 2005) as a pre-algebra task to develop knowledge of variables, equivalent expressions, and generalizations. During both the interview and classroom experiment, students were first asked to generate a list of what they notice and wonder about the given artifact or situation. Next they were asked to use their observations and questions from the notice and wonder list to create math problems together with their group-members.



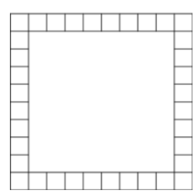


Figure 1

Figure 2

Data Analysis

Data Preparation. 19 video-hours from the interview and 16 video-hours from the classroom along with the written work were analyzed. Prior to the analysis, the data was prepared using Heath, Hindmarsh, & Luff's (2010) three stages: a preliminary review, a substantive review, and an analytic review of the data corpus (see also Erickson, 1985). Content logs cataloguing the participants, time stamps, and a simple description of the students' mathematical activities were created. Specific instances of the phenomenon—student doubts—were found and stored in a separate excel sheet called the *doubt list*. Doubts were any written or verbal statements that reflected a perplexity, query, wondering, or a conjecture. Any statement that reflected student observation of visible or obvious features of the given artifact was not considered a doubt. Doubt list was used to create doubt logs that documented the verbal and nonverbal activities that transpired after the emergence of the initial doubt leading to students posing math problems.

Coding (RQ1). The main purpose of coding the doubt list for both interviews and classroom lessons was to determine if there are any patterns in the data that could more systematically help us understand the nature of students' initial doubts to answer the first research question. The coding generally followed Saldana's (2015) "Four Ductions": deduction, abduction, induction, and retroduction to draw on what we know from the literature about the nature of mathematical thinking and questioning specific to the tasks but also to stay open to the new meanings emergent in the data. I conducted three iterations of coding.

First, I conducted In Vivo coding (Saldana, 2015) that helped gather the mathematical features of the tasks that students focused on and the different ways in which students perceived the given artifact. This, however, did not help shed light on underlying epistemic needs of students, i.e., what students might have desired to know or understand. In the final iteration, I looked at student doubts through the lens of what students might be doing or trying to get at when posing a doubt. Emergent codes included codes such as recalling, generalizing, computing, historical grounding, finding relevance, and asking what-if, why, and how questions. These codes were then thematized into four categories of doubt, which are described in the Findings section. Lastly, I conducted open Process Coding (Saldana, 2015, p.96) to further confirm these themes and to extract the tacit conceptual meanings behind what students were doing or saying.

Processes (RQ2). Using the doubt logs, I selected cases that would distinguish productive problem-posing from the ones that were not productive. Productive defined as problems meaningful enough to allow resolving initial doubts and solutions to which are not immediately apparent. For example, student asking "How many squares?" in the border problem is considered a valid student doubt, but it is not considered a productive problem, since the solution to it can easily be figured by counting. Then, I conducted analyses of small-group interactions for the selected cases to understand the characteristics and the conditions that allowed students to productively mathematize their initial doubts into meaningful math problems.

Findings

Research Question 1

I find four interrelated dimensions of student mathematical doubts: Pragmatic, Empirical, Theoretical, and Transformational.

1. **Pragmatic Doubts.** By raising a pragmatic doubt, students are seeking to understand the purpose, significance, or relevance of the given artifact. A larger question that students appeared to be asking was *What is it for?* Sometimes, rather than questioning its purpose students simply speculated or conjectured it. For example, in the border problem (refer to figure 2), several students in different period and groups said, "It looks like a picture frame", and later used this idea to create a math word problem.
2. **Empirical Doubts.** By raising an empirical doubt, students are seeking to understand specific features and characteristics of the artifact in order to make sense of its non-obvious aspects. A larger question that students appeared to be asking was *What is it?* Once again, many times rather questioning, students conjectured about those empirical features of the artifact. For example, in the pattern problem (refer to figure 1), several

students made conjectured about the relation between case number and the number of cubes in that case: “In each case, the number of the case is multiplied by 2”.

3. **Theoretical Doubts.** By raising a theoretical doubt, students are questioning the facts and characteristics of the artifact as given. Essentially, students were asking: *Why is it the way it is?* Why is it made that way? Resolutions of questions of this nature helped students theorize the reasons behind certain characteristics of the artifact. For example, for the growth pattern several students asked why it only uses blue blocks when the red block as were also available. Students then theorized that blue blocks are to represent even number of cubes while addition of blue blocks can help create another pattern with odd numbers.
4. **Transformational Doubts.** By raising a transformational doubt, students are reaching for new possibilities and transformations of the given artifact. Essentially, students were asking: *What if? How can it be changed?* This dimension of doubt is similar to Brown & Walter’ (2005) what-if-not strategy of formulating new problems. Students were modifying the given properties of the artifact to generate new possibilities.

Research Question 2

Table 1 outlines dimensions, frequencies, and examples of doubts and associated mathematical problem posing. It was found that variation in the frequency of types of doubts students raised depended on the participation structure, the nature of the task and students’ prior familiarity with it, and the ways on which those types of doubts were positioned by the researcher/ teacher vis-à-vis mathematical credence. For instance, the teacher in the classroom in the very beginning explicitly shut down pragmatic and transformational doubts as not worthy of belonging in a math classroom. This influenced students’ diversion towards asking more Analytic and transformative doubts in the classroom as compared to during the interview. Also, it was later found that while students had exposure to growth pattern problems, students had seen the border problem for the first time. This influenced students being able to reach for and imagine new possibilities in the case of the border problem; while for growth pattern problem students asked more familiar and expected questions. Lastly, it was determined that when taken up for problem-posing, each type of doubt had capacity for generating a meaningful math problem as displayed by the given examples in the Table 1.

Table 1: Four Dimensions of Doubts, Frequency, and Associated Math Problem

Types of Doubt	Frequency		Examples—Initial doubts	Examples—Math problems
	Interview	classroom		
Pragmatic (What is it for?)	45%	16%	Can it be used as a picture frame?	How many pictures and of what shape and size can fit in the picture frame?
Empirical (What is it?)	27%	35%	How many border squares are there and how can I tell?	Find the number of border squares in NxN square.
Theoretical (Why is it the way it is?)	22%	13%	Why is the middle empty?	How many ways are there to fit 16 squares or less in the middle?
Transformational	6%	36%	Can they be any another shape?	If they were triangles instead of rectangles: find the relation

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(What if? Can it be changed?)				between case# and the area of the triangle (# of blocks)?

Discussion

A nuanced understanding of the dimensions of students' doubts and it's relation with problem-posing disrupts the deficit identity that is associated with struggling students in math. Findings reveal that even the silliest of a doubt is a result of a learner's epistemic need to know something at a certain time in a certain way and holds a potential for productive problem-posing. Findings highlight the power of cultivating students' doubts for problem-posing, student agency, and for restituting marginalized students' claim to knowledge and knowledge producing practices.

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SUPPORTING MIDDLE GRADE STUDENTS' ARGUMENTATIVE ABILITIES THROUGH COMMUNAL CRITERIA FOR PROOF AND COLLECTIVE ARGUMENTATION

Tye G. Campbell
The University of Alabama
tgcampbell1@crimson.ua.edu

Shande King
The University of Tennessee
nking5@vols.utk.edu

Eugene Glover
The University of Alabama
etglover@crimson.ua.edu

Allison Mudd
The University of Alabama
amudd@crimson.ua.edu

Keywords: Reasoning and Proof

Proof is an integral component of knowing, doing, and understanding mathematics (Schoenfeld, 1994), and current policy suggests that K-12 students should engage in reasoning and proof regularly (CCSSI, 2010; NCTM, 2000). A considerable amount of literature suggests students of all ages struggle to create viable mathematical arguments (Stylianides, Bieda, & Morselli, 2016). Because of this, scholars have suggested multiple supports to aid students in becoming more proficient in proof (Nardi & Knuth, 2017). In this paper, we explore two classroom-based interventions for supporting proof and argumentation in K-12 contexts: communal criteria for proof and collective argumentation.

Communal Criteria and Collective Argumentation

Communal criteria for proof refers to negotiating the standards for what counts as proof in a specific mathematical setting (Stylianides & Stylianides, 2009). When students negotiate the standards for proof, it increases their autonomy in the classroom (Yee, Boyle, Ko, Bleiler-Baxter, 2018), and it explicates the standards for their mathematical community. Collective argumentation is a process wherein students create arguments in collaboration with others as opposed to individually. Specifically, our conceptualization aligns with Brown (2017) who developed the key-word format for facilitating collective argumentation. Brown's (2017) key-word format includes six phases: *represent, compare, explain, justify, agree, validate*.

Methods and Findings

Our study took place during the 2018-2019 school year at a private middle school in the Southeast. The participants were 44 eighth-grade students who were enrolled in one of four different Pre-Algebra classes. For the purposes of our study, Author 1 and Author 2 co-taught each of the four class periods for three days of instruction (55-minute class periods) introducing students to communal criteria and collective argumentation. Over the three days of instruction, students worked on three proving tasks in groups of four, and their arguments were coded according to Stylianides and Stylianides (2009) framework. The preliminary results suggest engagement with communal criteria combined with collective argumentation can help students understand that an argument needs to be general to count as a proof. The groups also generally made progress in their ability to create increasingly sophisticated arguments over the three days of instruction. Future analyses will consider qualitative aspects of students' discourse to understand productive forms of engagement with communal criteria and collective argumentation.

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FACILITATING THE IMMIGRATION DEBATE WITH STATISTICAL MODELING

Ayse Ozturk
The Ohio State University
ozturk.25@osu.edu

Stephen Lewis
Marietta College
lewis.steve.813@gmail.com

Azita Manouchehri
The Ohio State University
manouchehri.1@osu.edu

This study examined a model of teaching with the intent to empower students to reach statistically sound conclusions and develop a personal perspective of the U.S.–Mexico border context and the hardships immigrants face while trying to cross the southern U.S. border. The following question guided our study: How can we advance students' statistical reasoning through the modeling process when adopting the socio-critical perspective of modeling? Our research is grounded by the socio-critical perspective of statistical modeling (Zapate-Cardona, 2018), which suggests that choosing a context that is not only authentic but also socially significant can aid the development of both statistical reasoning skills and deeper awareness of complex real-world social issues. In adopting this perspective, we set out to support our students in investigating the issue of immigration. Students posed their own statistical questions and collected, summarized, represented, and interpreted statistical data to develop an understanding of the issue of immigration between the United States and Mexico.

A teaching experiment (Steffe & Thompson, 2000) methodology was used to study how a teacher guides students through the modeling process, which consists of formulating questions, collecting data, and representing and interpreting results. The research team included the teacher/researcher and an observer who worked with the students twice a week for 60 minutes for 3 weeks with 19 students between the ages of 16 and 18 years old. Each teaching episode was video-recorded in order to examine what transpired during the course of instruction, to prepare subsequent teaching episodes, as well as in conducting a retrospective analysis of the teaching experiment. An instruction unit was designed with six lessons around the issue of immigration between the United States and Mexico. This unit emerged as a result of student interests within the context of immigration. Our research team set out to design lessons as a reaction to these student-driven discussion. The website *http://humaneborder.info* was utilized to guide students in their statistical investigation. The site offers migration data from 1981 to 2018. Students collected and organized data from the website, presented their data to the class, and made informal inferences to interpret the data's results.

The ongoing data analysis has focused on different processes: (1) the representations students used to summarize, organize, and understand the data; and (2) the use of models to make predictions or draw conclusions. Our study confirmed that students' statistical investigation process, from formulating questions to drawing conclusions from datasets, became meaningful to them as they learned more about the context of the issue of migration over the southern border. Overall, three discussion themes emerged from students' investigations on migrant deaths: (1) the primary cause of death, (2) gender differences in deaths, (3) in what regions the most migrants died. Two aspects of implementation were prominent in supporting students' statistical reasoning: using a website to support the statistical investigation process virtually and numerically offered a means to make assumptions, and generate conclusions; and orchestrating reflective classroom discussions to elicit students' reasoning and sense-making supported the development of shared understanding.

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UNDERSTANDING THE GENERALITY REQUIREMENT FOR PROOF ACROSS MULTIPLE DIMENSIONS: THE CASE OF LEXI

Kimberly Conner
University of Northern Iowa
kimberly.conner@uni.edu

Keywords: Reasoning and Proof; Cognition; High School Education

One key component of proof is that they demonstrate the validity of the claim for all cases included in the scope of the statement (generality requirement). Prior studies have described the difficulties secondary students face in recognizing and adhering to the generality requirement, such as becoming convinced of a claim's validity after checking a few examples (e.g., Healy & Hoyles, 2000), not consistently recognizing that a proof demonstrated the statement's validity for all possible cases (e.g., Martin, McCrone, Bower, & Dindyal, 2005), and remaining open to the existence of a counterexample after being presented with a proof (Chazan, 1993). This poster presents an analytical framework for analyzing students' understanding of the generality requirement. I illustrate the framework using a series of arguments produced by Lexi, a ninth grader who participated in a teaching experiment that aimed to introduce proof through developing understanding of the generality and purpose of proof.

I analyzed Lexi's written and verbal work during a series of semi-structured interviews for evidence of her recognition of and adherence to the generality requirement when proving and evaluating geometric and numeric universal claims. Specifically, I a) attended to the explicit and implicit justifications included in her written arguments and used when evaluating provided arguments; b) categorized the purpose of examples used, if any, during the proving process (Ellis et al., 2017); c) identified language used that explicitly referenced the scope of the proof claim or her mathematical statement; and d) noted other aspects of her argument that related to the generality requirement, such as the way she used variables and labeled her self-constructed diagrams during the proving process.

Lexi consistently demonstrated understanding of the generality requirement through her justifications and use of examples, but struggled to reflect this understanding when using variables or labeling diagrams. She routinely included justifications that adhered to the generality requirement within her written arguments, although they tended to be implicit when proving numeric claims (e.g., "Anytime you add two even numbers together, it equals even…"). She also consistently evaluated provided empirical arguments as *not* proofs (e.g., "You can have, like, different examples and it might not prove all of them"). Next, she used examples as a way of gaining initial certainty regarding the validity of the conjecture and to convey a general claim when verbally explaining her arguments (e.g., "So anytime the shape adds up to a multiple of 180, it will tessellate. So like, if it adds up to 720º, that'll tessellate too"). Despite Lexi's prior success in Algebra and understanding of the generality requirement according to the prior criteria, she consistently interpreted variables as placeholders that could be used to generate examples and used variables as placeholders when writing equations in her arguments. She also initially used specific numbers to label her constructed diagram during a proof task, but then revised them to variables after feedback. Viewing Lexi's understanding of the generality requirement across multiple dimensions highlights the complexity of this component of proof and the multiple facets of mathematical arguments it impacts, from the types of justifications used to the way that diagrams are labeled and mathematical relationships are represented.

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AN EXPLORATION OF STUDENTS' ABILITIES FOR DETECTING ERRORS IN NON-AUTHENTIC AND ILL-POSED WORD PROBLEMS

Roberto G. Perez-Delgado
FCFM, BUAP, Mexico
rgperezd@gmail.com

Josip Slisko
FCFM, BUAP, Mexico
jslisko@fcfm.buap.mx

Keywords: Problem Solving, Critical Thinking, Textbook Errors

Much has been said about the need to integrate critical thinking into the curricula and classrooms in all levels of education, however, it is still unclear how to do so. Ennis (1996) outlines specific dispositions and abilities that critical thinkers should possess, such as, but not limited to: analyzing arguments, questions, and their validity, identifying possible answers to a question (if there are any), judging the credibility of sources, and seeking alternative sources of information. We propose that the activity of searching for and detecting errors in word problems can help develop critical thinking skills since this entails all of the abilities and dispositions mentioned above. This work discusses the initial results from a study concerning Mexican middle school students' abilities for detecting errors in non-authentic and ill-posed word problems drawn from current approved and published Mexican seventh grade mathematics textbooks.

We conducted a document review of all 17 current government approved and published Mexican seventh grade mathematics textbooks and found multiple non-authentic and ill-posed word problems. Said problems were categorized as follows: a) Word problems with spelling, grammatical, or typographical errors, b) Non-authentic word problems with artificial or unrealistic contexts, and c) Word problems with multiple answers, or insufficient or irrelevant data. We selected two word problems with spelling, grammatical, or typographical errors and two non-authentic word problems and developed two different questionnaires, each containing one non-authentic problem and one problem with spelling, grammatical, or typographical errors. These questionnaires were applied to 79 Mexican students from grades 7-9 and were aimed at investigating if the students were able to detect the errors in the problems and, furthermore, if they could re-write the problems correctly.

We found that there are students that are able to detect errors in ill-posed word problems when they are invited to do so. Roughly half the students in our study were able to detect spelling, grammatical and typographical errors and some of them were also able to re-write the problems correctly. This suggests that these students already possess critical thinking skills that are not currently being trained because educational programs, curricula, textbooks, and classrooms, in general, do not require students to analyze the word problems contained in their textbooks and determine if they are well-posed. The Age of the Captain problem phenomenon (Baruk, 1985) shows that students tend to try to solve word problems without previously analyzing and determining if said problems are well-posed. Our study suggests that this phenomenon may be caused by a lack of training in critical thinking skills, namely, word problem error detection skills. We propose that word problem error detection may help reverse The Age of the Captain problem phenomenon and, furthermore, help break students' notions that textbooks and the word problems contained in them have no errors.

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OPTIMIZATION IN A NON-SCHOOL SETTING: A DEVELOPMENTAL APPROACH

Cheryl Eames
SIUE
ceames@siue.edu

Kyle Heckert
SIUE
kyhecke@siue.edu

Keywords: Problem Solving, Calculus

Background and Research Questions

One of the biggest focuses of today's mathematics teacher education is problem-solving, and most real-life mathematics problems are based on some sort of optimization. Pennings showed that his dog could solve a classic optimization problem and thus "knew Calculus" (Pennings, 2003). Kaplan and Otten designed an approach for optimization that guided students into a successful use of different representations of the problem (Kaplan and Otten, 2012). There is a gap in the literature when it comes to problem-solving in a non-school setting. Thus, the purpose of this study is to see how well students of different age levels and abilities handle an optimization problem in a non-classroom situation.

Method

Results reported here are part of a larger thesis study about children's performance and strategies for coping with problems involving optimization in a non-school mathematics setting.

Participants

A sample of 30 participants was recruited for the study: 10 students in each of three age groups: Grades 5 to 8, Grades 9 to 12, and ages 18 to 22.

Procedure

Start and finish cones were placed on opposite sides of a swimming pool. The finish was 44 ft. 6 in. down the pool from the start, and the pool was 34 ft. 2 in. wide. Participants were told to get from start to finish as quickly as possible. This was repeated three times.

Data Sources

Data sources include students' run and swim speeds, calculated beforehand using the average of three timed trials, as well as the pool entry point for each student in each trial.

Analysis

Students' swim and run speeds were used to calculate the optimal point of entry to the pool for each student, and deviations from that optimal point of entry were calculated. Multiple regression and two-proportion z-tests were used to determine how well students at different ages cope with a problem involving optimization in a non-school math setting across three trials.

Results and Conclusions

A multiple regression analysis showed that students' deviation from the optimal path was significantly predicted by the model tested ($R^2 = 0.1788$, $F(6, 83) = 3.01$, $p = 0.01$). It was found that deviation from optimal significantly decreased from Age Group 1 to Age Group 3 ($p = 0.042$) but not from Age Group 1 to Age Group 2 ($p = 0.52$). Deviation from optimal significantly decreased from Trial 1 to Trial 2 ($p = 0.007$), but not from Trial 1 to Trial 3 ($p = 0.074$). Deviation from optimal was not significantly predicted by students' self-reported math ability ($p = 0.234$) or swim ability ($p < 0.727$). A two-tailed, two-proportion z-test showed that this difference was not significant with $z = 0.76$, $p = 0.45$, which suggests that students' propensity to jump into the pool too soon did not change in response to repeated engagement with the task. This finding may be rooted in an intuition that the shortest distance between two points is a straight line (Chiu, 1996) rather than reasoning about their swimming and running rates. Future research should explore why children across age groups exhibit a propensity to jump in too soon, with attention to students' thought processes while solving optimization problems in a non-school setting. If we as teachers understand the order of students' thinking and experiences, we can guide instruction appropriately (Kaplan & Otten, 2012).

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THE ROLE OF CREATIVE REASONING IN A TECHNOLOGY-BASED TASK FOR PRE-SERVICE SECONDARY TEACHERS

Kelsey A. Clarkson
Illinois State University
kaclar2@ilstu.edu

Carrie A. Lawton
Illinois State University
clawton@ilstu.edu

Khadijeh Azimi
Illinois State University
sazimia@ilstu.edu

Tami S. Martin
Illinois State University
tsmartin@ilstu.edu

Craig J. Cullen
Illinois State University
cjculle@ilstu.edu

Keywords: Reasoning and Proof, Problem Solving, Technology

According to the National Council of Teachers of Mathematics (2014), teachers should provide students opportunities to use technology to reason about mathematics. Pre-service secondary teachers (PSSTs), then, should engage in tasks to develop their own reasoning skills as they develop fluency with mathematical actions technologies. We considered the nature of PSST reasoning in this context and sought to answer the following question: What role does reasoning play in PSSTs' approaches to a dynamic algebra supported exploration problem?

Hendriana, Prahmana, and Hidayat (2018) found that PSSTs utilized novel approaches and ideas only about 10% of the time when reasoning through mathematical tasks. Reasoning in which the reasoner uses a novel (for them) approach, includes arguments for plausibility of approach or solution, and anchors arguments within mathematical properties would be categorized as creative mathematical reasoning (CMR; Lithner, 2008, 2015). In contrast, imitative reasoning (IR; Lithner, 2008) suffices when a task may be solved by recalling a response or applying an known algorithm to a familiar problem type.

To investigate our research question, PSSTs were asked to describe the path traced by the vertex of parabola $f(x) = ax^2 + bx + c$ as parameters a, b, and c were varied independently (Cullen, Hertel, & Nickels, in press). Through the use of dynamic graphing software, one group of four PSSTs engaged in reasoning about several subtasks of the problem while identifying the path of the vertex as a was varied. We focus on the subtask in which the PSSTs tried to determine the slope of a line for their conjectured vertex path.

By analyzing video and video-based transcription of the group's conversation about this subtask, we identified claims and strategy choices that evidenced CMR and IR. We found that, when engaged in IR, PSSTs' collaboration was productive until an obstacle—such as generalizing an expression for slope—halted their progress. When progress was stalled, PSSTs use of CMR allowed the group to change or modify their problem-solving strategy. For example, one PSST stated, "I think we were supposed to define arbitrary points *A* and *B*, say on that line, they are solutions to, these lines or something." Group members then used IR to calculate the slope of a line between two specific points. A group member identified an obstacle preventing successful completion of the task by stating that the calculated slope was "not general enough." A group member then used CMR when suggesting that they try a pair of arbitrary points and test to see if those points led to the desired outcome. The calculation of slope using arbitrary points was novel (in the situation), plausible, and anchored in mathematical properties.

Hendriana et al. (2018) asserted that CMR was associated with perseverance in nonroutine learning tasks. We found that PSSTs used both CMR and IR to complete this nonroutine subtask; in fact, an interplay between both kinds of reasoning propelled them toward a solution.

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A SELF STUDY OF DESIGNING AGRICULTURAL LIFE SCIENCE MEAS

Quintana M. Clark
Purdue University
qmclark@purdue.edu

Jonathan D. Bostic
Bowling Green State University
bosticj@bgsu.edu

Levon T. Esters
Purdue University
lesters@purdue.edu

Neil A. Knobloch
Purdue University
nknobloc@purdue.edu

Keywords: Curriculum, Equity and diversity, Problem-solving, Modeling

Modeling-eliciting activities (MEAs) are thought-revealing tasks that require students to mathematize real-world situations (Lesh & Lehrer, 2003). This poster shares work from a grant-funded project whose goal is to design, field test, implement, and evaluate MEAs that address agricultural life science issues. These unique MEAs encourage a culturally relevant lens on societal issues in students' local communities (i.e., food security, environment, energy, and health). Writing MEAs is a complex endeavor, especially for a novice MEA writer. Working alongside a team member who has expertise in writing and designing MEAs, the lead author engaged in a collaborative self-study to document the iterative process of MEA design, development, and implementation of four MEAs. Self-study is an "intentional and systematic inquiry into one's practice" (Dinkleman, 2003, p. 8). Self-study necessitates constructive criticism of one's practice, which can then be used to refine and enhance one's practice. The purpose of this poster is to convey results from a self-study by a novice MEA author with the aim of adding to scholarly conversations about MEA development. Data sources for this self-study come from journaling, meeting notes, classroom observations, focus groups, artifacts, reflections, and systematic formal reviews by the research team. The lead author used narrative inquiry (Cresswell, 2012) to capture the MEA design, implementation, and reflection process (Liljedahl, Chernoff, & Zazkis, 2007). One finding from the self-study was that the cultural and community components of the MEA must be developed in tandem with those who understand the objective of the task design. A second finding was that MEA writing became easier over time and benefitted from collaboration with diverse partners. An implication of these results is a possible pathway of MEA development that should be further investigated. A second implication from this work is to document how MEA development teams grow as MEA developers.

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CONNECTING DATA-CLAIM RELATIONSHIPS USING TOULMIN'S MODEL

Amy E. Roehrig	Elisha VanMeenen	Radhika Pathak Kharel
Illinois State University	Illinois State University	Illinois State University
aeroehr@ilstu.edu	emvanme@ilstu.edu	rpathak@ilstu.edu
Naren Kancharlapalli	Craig J. Cullen	Tami S. Martin
Illinois State University	Illinois State University	Illinois State University
vnkanch@ilstu.edu	cjculle@ilstu.edu	tsmartin@ilstu.edu

Keywords: Reasoning and Proof, Technology

Toulmin's model of argumentation is a tool for describing and classifying components of students' mathematical arguments. The model is comprised of six components: *claim*, *data*, *warrant*, *qualifier*, *backing*, and *rebuttal* (Martinez & Pedemonte, 2014; Toulmin, 2003). Each mathematical argument is a unique collection of one or more these six elements connected by relational links (Nussbaum, 2011). Erduran, Simon, and Osborne (2004) used Toulmin's Argument Pattern (TAP; Toulmin, 2003) to identify specific data-claim relationships during classroom activities. In this research, we consider how TAP may be used to trace a sequence of reasoning from conjectures to proof, in a dynamic algebra exploration task.

This instrumental case study involved preservice secondary teachers (PSSTs) enrolled in a content course that focused on problem solving with technology. The task required students to explore the path of the vertex of the quadratic function $y=ax^2+bx+c$ when the parameters were varied (Cullen, Hertel, & Nickels, in press). We focused on the collective reasoning of one group of PSSTs, including Joelle and Jared, and used TAP to model the arguments related to their development of a conjecture about the vertex's path when parameter a was varied.

We model the group's collective reasoning using a Path of Argumentation (PoA), which illustrates a chronological sequence of related arguments from conjecture to proof. Figure 1 illustrates how we linked (with arrows) key components from several TAP claim diagrams, related to the group's reasoning about varying parameter a in the general quadratic, to form a PoA. The collective argumentation evolved from observing that the path of the vertex made a line to then searching for a proof to support this conjecture. Joelle's GeoGebra-supported claim of linearity led to Jared's claim (based on using slider manipulation data) about the slope's dependence on b. Joelle's concern about the limits of relying on GeoGebra data, led to Jared's search for a general proof. Data related to a specific argument, informed the proof development.

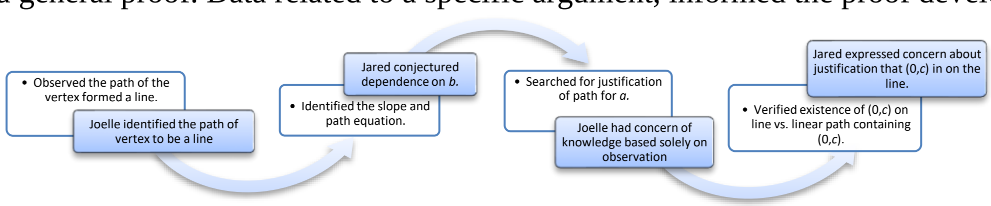

Figure 1: PoA Illustrating Joelle and Jared's Collective Argumentation

We found that the PoA was a valuable way to model PSST's collective argumentation during an extended exploration task, which exposed connections across TAP claim diagrams. Components of reasoning around a single claim appear to guide and influence next steps in an exploration that leads from conjectures to proof, in a dynamic algebra-supported task.

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MODELING FOR SOCIAL JUSTICE: A MODEL-ELICITING ACTIVITY ON GERRYMANDERING

Sandy Spitzer
Towson University
sspitzer@towson.edu

Kimberly Corum
Towson University
kcorum@towson.edu

Kristin Frank
Towson University
kfrank@towson.edu

Melike Kara
Towson University
mkara@towson.edu

Research suggests that for students to develop deep and connected mathematical understandings, they should solve real-world problems in which mathematics is used to model situations and construct solutions (e.g., Pollack, 1969; Lesh & Doerr, 2003). However, for teachers to effectively use mathematical modeling in their classroom, they must develop an understanding of the modeling process and learn to select, modify, and enact modeling tasks. To be prepared to enact modeling in their classrooms, pre-service secondary mathematics teachers (PSTs) need explicit support and education (e.g., Cai et al. 2016). Engaging PSTs in model-eliciting activities (MEAs, or tasks for which the creation and evaluation of a mathematical model is the primary goal) can help them both integrate and apply their mathematics knowledge and prepare to conduct such tasks as teachers (Daher & Shahbari, 2015). Meanwhile, attending to equity and social justice in pre-service mathematics teacher preparation is an urgent concern (e.g., Bartell, 2013; White et al., 2016). Although modeling and attending to equity may seem like disparate skills, the use of MEAs involving social justice contexts can engage pre-service teachers in studying and mathematizing situations to "experience mathematics as an analytical tool to make sense of, critique, and positively transform our world" (Aguirre et al., 2019, p. 8).

The issue of gerrymandering presents a good opportunity for integrating social justice into school mathematics, since it provides an example of authentic mathematics being used to influence public policy. Two actual U.S. Supreme Court cases in 2018 (most notably *Gill v. Whitford*) had significant mathematics content, with much of that math content accessible to K-12 students. Further, since teachers often cite a wish to avoid discussing controversial issues with their students (Simic-Muller, Fernandez, & Felton, 2015), a topic which is facially neutral (like gerrymandering) can be a useful entry point for novice teachers to incorporate social justice issues. All citizens can be concerned about whether or not their vote has an impact; this topic is particularly relevant for high school students who are preparing to vote for the first time.

Although the topic of gerrymandering can address a variety of mathematical topics (including proportions, statistics, etc.), our gerrymandering MEA focuses on the geometry of congressional districts. Many states require legislative districts to be "compact," but there is little agreement about how compactness should be measured. In fact, there are over 30 mathematical models of compactness used by political scientists (Kaufman, King, & Komisarchik, in press). With the goal of engaging PSTs in modeling for social justice, we created an MEA which asked participants to construct their own mathematical measure (i.e., model) of compactness and use their model to identify states which have the most gerrymandered districts. This model-eliciting activity draws on mathematical ideas appropriate for secondary mathematics classrooms, such as area and perimeter, scale, and attributes of shapes. In the activity, PSTs are given data including perimeter and area of congressional districts, as well as printed maps, and challenged to find a

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model which can rank districts by compactness. In a pilot implementation, PSTs struggled to conceptualize the geometric nature of compactness and downplayed its importance. They focused instead on issues of fairness in representation, including ideas of proportionality. This MEA thus provides PSTs with an opportunity to not only engage in rich mathematical practices, but also learn about relevant and pressing social justice issues.

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HOW TO POSE IT: EMPIRICAL VALIDATION OF A PROBLEM-POSING FRAMEWORK

José N. Contreras
Ball State University
jncontrerasf@bsu.edu

Problem posing is an important mathematical activity. Doing mathematics involves creating problems and investigating their solutions. Thus, learning to pose mathematical problems should be an important educational goal. In fact, professional organizations (e.g., National Council of Teachers of Mathematics, 1989, 2000), mathematicians (e.g., Halmos, 1980, Polya, 1945/1973), and mathematics educators (e.g., Brown & Walter, 1983, 1993; Kilpatrick, 1987; Silver, 1994) call for all students to have experiences posing problems. However, the mathematics research community continues to investigate the different aspects and issues of teaching and learning problem posing (Felmer, Pehkonen, & Kilpatrick, 2016; Silver, 2013; Singer, Ellerton, & Cai, 2013, 2015) with the goal of improving students' problem-posing abilities.

During the poster presentation, the author will display a problem-posing framework with examples of problems posed by him and his students using said framework. The framework includes the following systematic strategies to modify a given problem to pose new problems: reversing, proving, specializing, generalizing, extending, and further extending. These strategies have been very helpful in improving not only his problem-posing abilities, but also those of his students. Most of the problems posed involve geometric situations, but the framework is also useful to pose problems in other areas of mathematics.

I will include classical problems to illustrate the problem-posing strategies. For example, the author will use a version of Varignon problem (Let *E*, *F*, *G*, and *H* be the midpoints of the consecutive sides of a parallelogram *ABCD*. What type of quadrilateral is *EFGH*?), among other problems, to illustrate the usefulness of the problem-posing framework. Examples of problems that are related to this version of the Varignon problem are:

- Problem 1: If E, F, G, and H are the midpoints of the consecutive sides of a rhombus ABCD, prove that EFGH is a rectangle. (Special problem)
- Problem 2: *E*, *F*, *G*, and *H* are the midpoints of a quadrilateral ABCD. If EFGH is a rectangle, what type of quadrilateral is ABCD? (Converse of problem 1)
- Problem 3: If *E*, *F*, *G*, and *H* are the midpoints of the consecutive sides of a quadrilateral *ABCD*. What type of quadrilateral is *EFGH*? (General problem)
- Problem 4: Prove that the medial quadrilateral of a kite is a rectangle. (Extended problem)
- Problem 5: ABC is a triangle. Characterize quadrilateral BDEF where D, E, and F are the midpoints of the sides BC, CA, and AB, respectively. (Extended problem to a triangle, which is a degenerate case of a quadrilateral)
- Problem 6: Prove that the points of intersection of the angle bisectors of the consecutive interior angles of a parallelogram ABCD are the vertices of a rectangle. (Further extended problem)

The author will also provide solutions to problems that are supported by proofs in some cases or conjectures supported by empirical evidence in other cases (such as geometric diagrams created with Dynamic Geometry Software or numerical examples).

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Chapter 10:
Precalculus, Calculus, and Higher Mathematics

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

IDENTIFYING STUDENTS' ATTENTIVE FIDELITY FOR CALCULUS INSTRUCTIONAL VIDEOS

Aaron Weinberg
Ithaca College
aweinberg@ithaca.edu

Jessica Tornai
Ithaca College
jtornai@ithaca.edu

Matt Thomas
Ithaca College
mthomas7@ithaca.edu

Jason Martin
University of Central Arkansas
jmartin@uca.edu

Michael Tallman
Oklahoma State University
michael.tallman@okstate.edu

Kimberly Newman
Ithaca College
knewman@ithaca.edu

Growing interest in "flipped" classrooms has made video lessons an increasingly prominent component of post-secondary mathematics curricula. However, relatively little is known about how students watch and learn from instructional videos. We describe and use an eye-tracking methodology to investigate attentive fidelity—the degree to which students attend to the visual imagery that is the subject of the video narration at each moment in time. Our preliminary study suggests that students' attentive fidelity varies widely, but there was no evidence that this fidelity is connected to students' ability to solve calculus problems.

Keywords: Technology, Calculus, Research Methods

Over the past few decades as educators have sought to develop online courses and incorporate active learning strategies into their classrooms, they have turned to videos as a way to deliver lectures outside of a traditional classroom setting (e.g., Lage, Platt, & Treglia, 2003; McGivney-Burelle, & Xue, 2013; White House, 2013). Flipped classrooms have been used to support problem-based learning (e.g., Tawfik & Lilly, 2015) and interactive engagement (e.g., Maciejewski, 2015), and interest in using a flipped classroom instructional format in post-secondary mathematics teaching has grown (e.g., Maxson & Szaniszlo, 2015).

Despite the enthusiasm for using video lectures, relatively little is known about how students engage in the video-watching process, what they learn from watching videos, and how their actions while watching the videos might be connected to learning. Most studies of flipped classrooms have failed to distinguish in-classroom learning from out-of-classroom learning (e.g., Anderson & Brennan, 2015; Schroeder, Micgiveny-Burelle, & Xue, 2015). Weinberg and Thomas (2018) found that students displayed a wide range of ways of interpreting mathematics video lectures and called for researchers to closely examine students' video-watching activity.

The goal of this study is to begin to investigate what students attend to as they watch video lessons. In particular, we will describe a methodology for investigating students' *attentive fidelity*—whether students attend to imagery that corresponds to what instructors would identify as showing the focus of the video at each moment in time. In addition, we will present results from a preliminary study on students' attentive fidelity and learning from calculus videos.

Theoretical Background

One way of investigating students' attentive behavior—that is, the sequence of objects and images in the world that students pay attention to—is to record their eye movements as they engage in activity. Eye-tracking has been widely used in educational research, specifically to examine the ways students interpret curricular resources and, increasingly, to study students' mathematical thinking (e.g., Andrá et al., 2015; Inglis & Alcock, 2012; Lee & Wu, 2018; Mock, Huber, Klein & Moeller, 2016; Ögren, Nyström & Jarodzka, 2017). Eye-tracking offers a substantial advantage over self-reporting methods, which are constrained by students' limited ability to accurately reconstruct and reliably communicate their thinking in the moment.

When people interact with visual stimuli, they make a sequence of *fixations* in which their foveal vision (i.e., the center of the field of vision) is focused on a particular point for roughly 200 to 500ms (Hyönä, 2010). These fixations are connected by *saccades*, which move the foveal vision to new locations without processing information. Fixations may indicate cognitive attention (Andrá et al., 2015), difficulty extracting information (Jacob & Karn, 2003), mental calculation (Hartmann, Mast, & Fischer, 2015), or bored staring.

The *eye-mind hypothesis* (Just & Carpenter, 1980; Rayner, 2009) proposes that there is a positive relationship between what a person looks at and what they attend to. Researchers typically assume that eye movements correspond to cognitive operations (Obersteiner & Tumpek, 2016). However, Holmqvist et al. (2011) noted that it is possible for a person to be thinking about something other than what their eyes are fixated upon. From a slightly different perspective, Abrahamson and Bakker (2016) proposed that eye movements are *part of* the individual's cognitive processes rather than reflecting separate mental processes.

Methods and Methodology

The participants in the study were students enrolled in a first-semester calculus class at either a mid-sized comprehensive college or a large state university. All students in the sections were invited to participate and 13 volunteered to participate in (up to) eight interviews (all reported student names are pseudonyms). Each interview was conducted prior to the calculus topic being discussed in class. For this report, we chose a video about Riemann sums because it involved both procedural and conceptual aspects—that is, it was designed to support students' understanding of how to construct a Riemann sum and how to interpret its computation.

In the interview session, the students were first asked to solve several problems about Riemann sums using a think-out-loud protocol; the interviewer asked follow-up questions to probe their thinking. Then, the students watched a video about Riemann sums and solved several problems that were similar to the pre-video questions.

Participants' eye-movements were recorded with a Tobii X2-60 eye tracker mounted below a 22" LCD monitor. The eye-tracker captured the on-screen coordinate of each participant's fixation at a rate of 60 Hz; we called each time at which coordinates were captured a *moment*. At the beginning of each interview, the eye tracker was calibrated with a 9-point display. The participants viewed the screen without a head restriction from approximately 2 feet away, which corresponded with roughly 0.5-degree accuracy (Tobiipro, 2019).

To identify whether students attended to areas of the screen that correspond to what instructors would identify as the primary foci of each moment of the video, we created one or more rectangular *areas of interest* (AOIs) on frames of the video. We defined two types of AOIs: A *primary AOI* was meant to capture whether or not a student was focusing on the area of the screen that was conveying new knowledge or visual aids for the narration. These included:

- Areas that contained numbers or shapes to which the narrator of the video was referring
- Areas that contained text that the narrator was reading
- Areas that contained deictic animations (e.g., arrows, boxes, or circles being drawn)
- Areas that contained other active animations that illustrated a mathematical concept

A *secondary AOI* was meant to capture areas that were directly related to—or important for understanding—the primary AOI, but were not the immediate object of the narrator's focus. For example, Figure 1 shows a frame from the video with one primary (in red) and two secondary AOIs (in yellow and blue). At this moment, the narrator reads "At 90 km dust accumulates at a rate of 6 mg per km," highlighted by the primary AOI. This speech refers to the numbers 90 and 6 and the units in the table, so these were included in Secondary AOIs because they added relevant contextual information to help understand the subject of the Primary AOI.

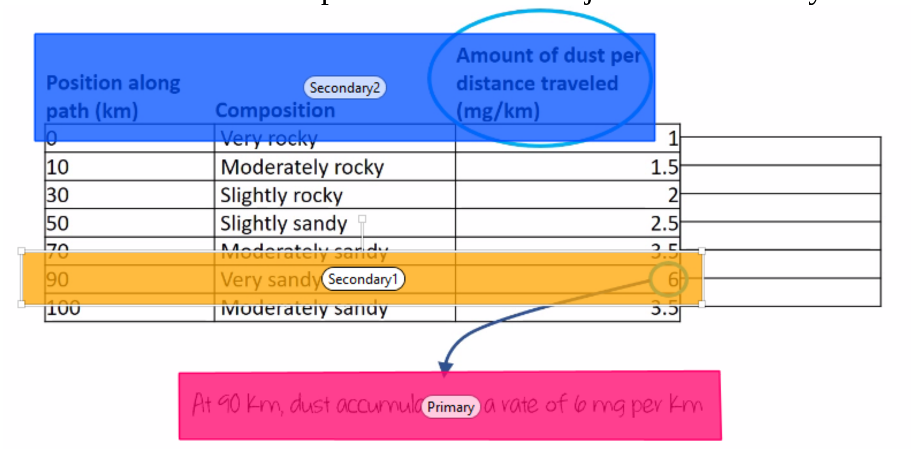

Figure 15: A Frame from the Video Showing One Primary and Two Secondary AOIs

Primary AOIs were defined for each section of the video that described mathematical concepts, but not for sections that introduced the context for the video (dust accumulating on a Mars rover). Secondary AOIs were defined whenever applicable. In order to account for the lack of perfect accuracy in the eye-tracking recording, we created each AOI with roughly a quarter inch border around the intended on-screen imagery. We measured attentive fidelity by determining whether a student's fixation was contained in one of the defined AOIs. The eye-tracker also reported information about whether it was able to locate a fixation anywhere on the screen in each moment, indicating whether or not the participant was looking at the screen.

Results

The video that the participants watched included approximately 12,500 moments in which at least one AOI was defined. Due to hardware or calibration errors, two students had a large percentage of moments (61% and 73%) in which the eye tracker was unable to detect the eye location. Concerns about accuracy led us to remove these two students' data from the study.

Table 1 shows the percent of moments for which the location of each participant's fixation was in an AOI along with whether the participant provided a correct answer to the pre- and post-video questions. The mean for the Primary AOIs was 37% (SD = 10.1%), for Secondary AOIs was 7% (SD=3.4%), and for all AOIs combined was 44% (SD=12.3%). There was substantial variation between the total percentages, with a minimum value of 22% and a maximum of 61%. There was also considerable variation in each participant's split between Primary and Secondary AOIs; for example, 45% of Camila's fixations were in Primary AOIs and only 7% were in Secondary AOIs, while Beth had much closer percentages—34% Primary and 13% Secondary.

There appears to be no relationship between whether a student had a high percentage of fixations on a Primary or Secondary AOI and their performance on the post-video questions. Of the students who got a higher percentage of post-video questions correct than pre-video questions (*n*=6), their total percent of fixations in an AOI was between 22% and 52% with a mean of 40%; the students whose performance decreased (*n*=3) had a mean of 47%.

Table 2: Summary of Results

	Percent of Moments with a Fixation on an AOI			Pre/Post-Video Question Correctness		
Participant	Primary	Secondary	Total	Pre-	Post- Q1	Post- Q2
Alec	20%	2%	22%	Incorrect	Incorrect	Correct
Beth	34%	13%	46%	Incorrect	Correct	N/A
Caleb	50%	11%	61%	Incorrect	Incorrect	Incorrect
Camila	45%	7%	52%	Incorrect	N/A	Correct
Elouise	32%	3%	36%	Incorrect	Correct	Correct
Joaquin	23%	7%	30%	Incorrect	Incorrect	Incorrect
Kayla	43%	10%	52%	Correct	Correct	Correct
Nina	44%	7%	51%	Correct	N/A	Correct
Rory	26%	4%	30%	Incorrect	Correct	Incorrect
Tristan	44%	9%	52%	Incorrect	Correct	Correct
Ursula	41%	9%	50%	Correct	Incorrect	Incorrect

Discussion

In this study we described a new methodology using eye-tracking technology for investigating attentive fidelity—whether students attend to imagery in calculus videos that corresponds to what instructors might identify as the primary foci of the videos. This involved identifying areas of interest throughout the video and measuring students' eye fixations to determine the percent of fixations that were inside these AOIs. If we accept the eye-mind hypothesis, then these percentages indicate the percent of time students were paying attention to these areas of the screen or, similarly, the degree to which this imagery played a role in the students' thinking about the concepts described in the video. Given the nature of the video—in which the Primary AOIs were constantly changing—we believe that it is unlikely that students would be thinking about the Primary AOIs if their fixations were on another part of the screen.

Our preliminary data suggest that students display a wide range of attentive fidelity and that they frequently attend to areas of the screen other than those we believe are most germane to the concepts being described in the video at any given moment. This might suggest that creators of videos need to be careful to incorporate elements that might help direct students' attention to desired areas of the screen. Conversely, it might be the case that there is no way to ensure that students will attend to particular imagery.

Our sample data do not suggest that there is a connection between attentive fidelity and change in performance from pre- to post-video questions. However, with the "coarse" nature of this measure (one pre-video question and two post-video questions) and small sample size, such an inconclusive result is not surprising.

We believe that the construct of attentive fidelity could be a useful tool for analyzing and understanding students' behavior and cognition as they watch instructional videos. Expanding the data corpus to include more videos, more students, and more detailed measures of learning would provide the foundation for generating a new, detailed perspective on student learning.

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FUNCTION COHERENCE IN ADVANCED MATHEMATICS

Kathleen Melhuish
Texas State University
melhuish@txstate.edu

Kristen Lew
Texas State University
kristen.lew@txstate.edu

Sindura Subanemy Kandasamy
Texas State University
s_k252@txstate.edu

Michael D. Hicks
Texas State University
mh1653@txstate.edu

In this paper, we explore students' understandings of functions in abstract algebra. We analyzed students' general conceptions of function and how this understanding was or was not unified with their understanding of particular functions in abstract algebra. We conducted a series of six task-based interviews with undergraduate students from two United States institutions. From these sessions, we analyzed key elements of students' function concept image including metaphors, representations, and evoked example space. We then looked across their activity related to homomorphisms to see the degree of coherence, attention to the homomorphism as a class of function, and the degree to which the students saw functions in abstract algebra as the same or different than functions from prior courses. We found that students demonstrated a variety of levels of function understanding and varying profiles ranging from non-unified to completely unified understanding of function across abstract algebra and earlier contexts.

Keywords: Undergraduate-level Mathematics

Function is one of the core concepts that serves to connect the mathematics curriculum from early algebra through advanced mathematics. The role and importance of function is ubiquitous as documented by mathematics education researchers across varying levels and content in mathematics (e.g., Oehrtman, Carlson, & Thompson, 2008). Furthermore, function has been documented to be quite complex and often poorly realized by students (e.g., Dubinsky & Wilson, 2013). While we know that students struggle with functions at the secondary and early college levels, it is an open question whether limited or incomplete views of functions persist through advanced mathematics contexts including courses like abstract algebra. As such, we report on a study aimed to explore the degree to which undergraduate abstract algebra students demonstrate a coherent understanding of function, and the degree to which their understanding of function plays a role in their activity related to abstract structures found in this subject area. We address the following research questions:

1. What are students' concept images of functions at the end of an abstract algebra course?
2. How is students' general function understanding connected to their understanding and activity related to functions in abstract algebra?

Literature Review

Early research on function understanding established that students and teachers have a wide range of definitions for functions including a normative correspondence definition and more limited definitions such as a function as a rule (e.g., Vinner, 1983; Vinner & Dreyfus, 1989). As Thompson (1994) articulated, even at the undergraduate level, a function is often thought of as

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"two written expressions separated by an equal sign" (p. 24). The focus on a written symbolic rule obscures important aspects of function such as the need for well-definedness (a challenging property for students, Even, 1993). Students often opt for a symbolic rule even when other representations would be advantageous (Knuth, 2000), struggle to move between various representations (Akkoç & Tall, 2002; Schwarz, Dreyfus, & Bruckheimer, 1990), and may not see alternate representations, such as tables, as functions at all (Clement, 2001). A coherent view of function requires both attention and understanding of important attributes, and the successful integration of representations. Additionally, metaphors serve an important role where functions may be understood as machines, collections of objects with directional links (e.g., Lackoff and Núñez, 2000), or as traveling and transforming (Zandieh, Ellis, & Rasmussen, 2017). Furthermore a function may be understood as a process going from domain to codomain or an object that can be acted on itself (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky and Wilson, 2013; Sfard, 1991).

At the abstract algebra level, research has focused on students' understanding of particular function types: isomorphisms and homomorphisms. For example, Leron, et. al (1995) identified the group, the function, and the existential quantifier as three connected areas needed to understand isomorphism. Students in Leron, et. al's (1995) study often compared groups using a concrete set of checks (such as looking at order of elements) rather than focusing on the critical aspect of the function map itself. In terms of function components, Rupnow's (2017) preliminary work demonstrated that students continue to use function metaphors (often inflexibly) when engaging in homomorphism. In general, the research in abstract algebra education raises the testable conjecture that students' function understanding may play an important role in how they engage with more complex functions in the setting of advanced mathematics.

Theoretical Orientation

In this study, we leverage two constructs to address our research questions: *concept image* (Vinner, 1983) and *unified notion of a concept* (Zandieh et al., 2016). A concept image contains all of a students' cognitive structures related to a particular concept including their personal definition, metaphors, representations, and examples. These components may or may not be coherent and aligned. Zandieh at al. (2016) furthered concept image to include ideas of unification across varying contexts, that is, the degree to which students understand "various constructs of a concept as examples of the same phenomenon" (p. 24). We extend this notion to the abstract algebra setting where students may or may not see particular types of maps such as homomorphisms as belonging to the larger category of function. We also expand unified notion of a concept to differentiate between explicit and implicit coordination. That is: (1) a student may be explicitly aware that a particular construct is a type of function, (2) a student may not explicitly address this requirement but implicitly evoke function concept image components in this new setting, or (3) not demonstrate any coordination across the contexts. Additionally, we also incorporate analytic tools to parse aspects of concept image including metaphor, representations, examples, and properties.

Methods

Data Collection

We provided written surveys to four undergraduate-level abstract algebra classes across two large, public universities in the United States. The survey included prompts about definitions, examples, and representations of functions. Then the second portion included prompts related to homomorphisms and kernels. For example:

> **Task 5:** Is ϕ: **Z** → **Z**: $\phi(x)=x+1$ a homomorphism? Why or why not?
> **Task 9:** Let $i=\sqrt{1}$. Consider the homomorphism $\phi(n)=i^n$, that maps **Z** under addition to the set H={i, -i, 1, -1}(a subgroup of **C** under multiplication). What is the kernel of this homomorphism?

After the completion of the survey, six students agreed to participate in follow-up interviews. These interviews lasted between 1 and 1.5 hours and consisted of three types of prompts: (1) elaborate thinking on original survey items, (2) new tasks related to functions and homomorphisms, and (3) reflections on functions in abstract algebra. The interviews were semi-structured. Each interview was audio-recorded and transcribed for analysis.

Data Analysis

When analyzing for students' conceptions of function, student responses were analyzed in terms of the properties that students attended to, metaphors (Zandieh et al. 2017) and representations (Melhuish, 2015) and Mesa, 2004) used, the students' evoked example space for functions and non-functions, and the similarities and differences that the students identified between functions in abstract algebra and in K-12. When analyzing for how students' general understanding of functions may be connected to their understanding of functions in abstract algebra, evidence of explicit awareness of homomorphism as a function, the role that functions played when engaging with tasks related to homomorphism, and the consistency of the students' use of metaphors, representations, and process/object treatment of function were analyzed. All parts of the interviews were analyzed independently by at least two researchers with disagreements resolved through discussion. We created profiles for the six cases based on: their concept image of function and their unification of the general function and homomorphism.

Results

For the scope of these results, we focus on two divergent cases. We note that these cases provided particularly contrasting images. A more thorough treatment of the cases can be found in Melhuish, Lew, Hicks, and Kandasamy (2019).

Case 1: Student D. Fragmented Concept Image and No Unified Function Understanding

Student D understood function as being necessarily tied to an explicit equation. Her function definition was: "equation that will do some kind of operation to an input to get an output." She introduced the idea of a vertical and horizontal line test, but could not determine which was necessary or how these tests may link to what a function is. Her evoked examples were limited to traditional symbolic equations such as $y=x^2$. Her definition for homomorphism then did not include any explicit reference to homomorphism being a function. Rather she wrote down the property: "$\phi(x)\phi(y)= \phi(xy)$". She further responded, "ϕ is what encompasses all of it." We deem Student D's function integration as an *empty symbol* approach. Student D was not able to identify that ϕ was a function without extensive prompting. Further, this lack of function conception persisted into her abstract algebra activity. She explained that $\phi(x)=x+1$ from ***Z*** to ***Z*** is not a homomorphism:

> If you were to apply the homomorphism property, you can't get ... I mean, the left and the right won't equal each other when you apply ϕ to both sides. That's kind of how I interpreted it. Because $\phi(x+1)$ isn't going to equal $\phi(x)$.

Student D manipulated $\phi(x)=x+1$ as an equation (applying something to "both sides") rather than treating this a function where inputs are mapped to that value summed with one. Further, Student D evidenced a limited understanding of function mappings that appeared to interfere with her success on standard abstract algebra prompts. For example, when determining the kernel of a non-injective homomorphism, Student D identified only one element of the pre-image rather than the full set of elements that map to the identity. Such treatment reflects a need for concrete referents and likely an action conception of functions that does not allow for the consideration of multiple elements at a time.

Case 2: Student C. Robust Concept Image and Unified Function Understanding

Student C provides a contrasting case. She was flexible in her understanding of function, and her personal definition for function was coordinated with other aspects of her function understanding. Student C attended to both well-definedness and everywhere-definedness in her definition of function. She also had a rich example space including functions from both secondary and abstract algebra and a variety of representations. Her definition of homomorphism, found in Figure 2, explicates that homomorphism is a function with particular properties. She expanded informally, "[the homomorphism] takes that bunch of things and it maps it to a… bunch of things. But it preserves certain things. So, if you can switch up order, you can still switch up order afterwards. So, commutativity." In contrast to Student C, the mapping element was a primary component of Student D's understanding of homomorphism.

She was then able to leverage this understanding to successfully engage in a number of abstract algebra tasks. In Task 5, Student C leveraged the identity preservation property to address that $x+1$ was not a homomorphism:

> So, if you take ϕ of ... I said here, but it's really $\phi(0)$. Zero is the identity element in the integers. So, and then you say well that's 0+1, that's gonna give you 1, not 0, which is the identity element in the integers. So the identity doesn't map to itself. So, because- that's not gonna be a homomorphism.

Notice the integration of mapping metaphor language as she considers where the identity element maps. Her more robust understanding of function also appeared to play a supportive role in various activities. For example, when identifying the kernel, Student C explained:

> So some element of **Z** that takes i to that power and gives us out 1, which would be the identity element for H. Because i^1 is gonna give us i. i^2 is gonna give us - i, so on. So in order for i to be taken to a certain power and give us 1, it needs to be a multiple of 4.

She was comfortable attending to a multi-element pre-image via holistically considering the homomorphism at as a process.

Discussion

Across our interviews, we found (1) students at the end of an abstract algebra course demonstrated a variety of levels of function understanding, (2) students had varying levels of integrating functions from this setting into their larger function understanding. We shared Student D and Student C to illustrate the extremes of these aspects. Without a strong function understanding or unified view of function, Student D struggled to approach nearly all of the abstract algebra tasks. Student C relied on robust conceptions of function and implicit and explicit leveraging of function understanding when engaged in the abstract algebra tasks.

The results of this study have implications for learning and teaching abstract algebra. First, this study suggest that students may still have incomplete understanding of function even after having completed a course in abstract algebra. Second, students' understanding of function may support or constrain their success in abstract algebra tasks. Instructors may want to explicitly attend to students' understanding of function as they study the specific types of function in abstract algebra. Furthermore, we suggest that researchers, of even advanced mathematics students, be attentive to how the complexities of function may interact and account for student activity and understanding in these settings.

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EXAMINING THE EFFECT OF INSTRUCTION IN THE CONCEPTUAL UNDERSTANDING OF FUNCTIONS IN PRE-CALCULUS STUDENTS

Keshab Raj Adhikari
Syracuse University
kradhika@syr.edu

Duane Graysay
Syracuse University
dtgraysa@syr.edu

Understanding of the function concept is essential for students' success in calculus and in STEM-oriented careers. This brief report shares a small part of findings examining relationships between the teaching of inverse functions in a classroom and students' learning of the concept as displayed in their response to the Function Concept Inventory (O'Shea et al., 2016).

Keywords: Precalculus, Undergraduate-level Mathematics

Proficiency in quantitative reasoning skills is a critical skill necessary to meet the requirements of employment in STEM disciplines (Golfin, 2005). Thus, there has been a growing emphasis on STEM-oriented education in high school and college (Rowan-Kenyon et al., 2012). To enter and develop careers in the STEM disciplines students need to do well in calculus in college. Therefore, college-level calculus has become an essential course for students. But many high school graduates enter college without the understanding or skill needed for college calculus (Carlson et al., 2002; Oehrtman, Carlson, and Thompson, 2008; Monk, 1992; Thompson,1994) and are therefore required to take precalculus courses as a prerequisite. Such courses have the objective of building conceptual and procedural foundations in mathematics which are essential for supporting students' success in calculus (Kilpatrick et al., 2001).

Students are more prepared for college calculus when they have strong conceptual understandings of functions and skills in algebraic competencies associated with functions. This includes cognitive abilities to coordinate two or more related quantities that change together in a functional situation so that students can model realistic problems in their calculus classes (Monk, 1992; Thompson, 1994). Therefore, the extent to which precalculus courses support the study of calculus is related to the extent to which such courses are effective at supporting students' conceptual understanding and procedural fluency (Carlson et al., 2002).

Precalculus courses are often not successful at helping students develop the understandings and skills that support success in calculus (e.g., Carlson et al., 2002). However, it is not clear how features of instruction relate to students' learning in such courses. We investigate this relationship by assessing students' learning of the function concept and by relating that learning to instruction. We investigate the following questions:

1. How do students' understandings of the function concept change over the duration of their enrollment in a precalculus course?
2. How do changes in students' understandings of the function concept relate to the teaching that occurs in their precalculus course?

Related Literature

Mathematics educators and researchers (e.g., Carlson,1998; Thompson,1994) have sought to describe fundamental aspects of understanding the function concept. To understand the concept, students must be able to distinguish between two perspectives on the nature of function: the

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

notion of function as a *correspondence* and the notion of function as a *covariation.* The correspondence perspective defines a function as a correspondence between two non-empty sets that assigns to every element in the first set exactly one element in the second set (Confrey & Smith, 1994; Vinner & Dreyfus, 1989). The correspondence approach to function is essentially a top-down approach in which a teacher or text provides a rule of association and students are supposed to acquire procedural competence for dealing with such formulas (Confrey & Smith, 1994). The pedagogy adopted by this approach emphasizes students' procedural competencies for establishing correspondence between input and output variables (Brumfiel, 1980).

In contrast, the covariational perspective is characterized by students' understanding of links between two simultaneously varying quantities, with a sustained image of both the quantities in one's mind (Saldanha and Thompson, 1998). Carlson et al. (2002) defined covariational reasoning as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). Although it is difficult to teach the concept of covariation to students, Confrey and Smith (1994) express that covariational reasoning is a powerful tool for students to succeed in calculus and higher-level mathematics courses.

Drawing on existing literature on the function concept, O'Shea et al. (2016) outlined a framework of six elements of conceptual understanding of function. These elements are based on students' abilities to identify key properties of the function concept. O'Shea et al. (2016) argued that the framework can be implemented to design instructional strategies for conceptual learning of function as well as to assess students' structural understanding of the function concept in the action- process-object continuum. O'Shea's framework consists of six elements: 1) the ability to distinguish between functions and equations; 2) the ability to recognize and relate different representations of functions and use them interchangeably; 3) the ability to classify relationships as functions or not functions; 4) the ability to have a working familiarity with properties of functions such as one-one/many-one, increasing/decreasing, linearity, composition, inverses; 5) the ability to use functions in context, modelling and interpreting; 6) the ability to engage with covariational reasoning. O'Shea et al. (2016) used this framework of functional understanding to construct and validate a Function Concept Inventory (FCI) consisting of 12 items designed to elicit students' abilities identified by the six elements of the framework.

Research Design and Methodology

Participants were recruited from among all students enrolled in precalculus during a single semester at a midsized private research university in the northeastern United States, along with their respective instructors, including recitation leaders. From among instructors who consented to participate, Guilia (all names in this report are pseudonyms) and the students in her class were selected as participants, in part because Guilia had taught the course for several semesters and was relatively practiced and experienced with the content of the course.

The FCI was administered to students as a pretest and posttest at both ends of a six-week observational period. The corresponding author observed six episodes of Guilia's classroom teaching during the semester, interviewed Guilia about her teaching and her perspective about the precalculus course, conducted three clinical interviews with selected students (Jim, Ryan, and Dikardo, selected because they had completed both pretest and posttest and had consistent attendance habits), and gathered all worksheets that Guilia used during the semester. Data for this study consisted of students' responses in the pretest, posttest, and clinical interviews, transcripts of classroom observations, an interview with Guilia, and Guilia's worksheets.

Pretest-posttest differences in students' responses for each of the multiple-choice items in the FCI were taken as evidence of student learning. Some of the items in the FCI required students to write their responses. For the explanatory items (i.e. item #2, and #7) students' descriptions were compared in an attempt to identify how the students' understanding changed during the semester.

O'Shea et al.'s (2016) six elements of functional understanding were used to code transcripts of classroom observations and clinical interviews for instructional activities with the capacity to support each of the six elements. Findings regarding instruction were then compared to findings regarding student learning to describe how instruction and learning did (or did not) appear to be related.

Findings

For the purposes of brief report, we have selected to describe findings related to a subpart of element 4 of O'Shea's inventory, namely, a working familiarity with the concept of inverse. First, we discuss changes in students' learning about inverse as displayed by their responses to item #7 of the FCI, then we describe how the notion of inverse function was introduced in the classroom instruction.

Finding 1: Students' Learning of the Function Concept Was Limited

Item 7 of the FCI assessed students' understanding of inverse functions (see Figure 2). Regarding students' learning of the concept of inverse functions was concerned, the students' responses to item 7 in the pretest and posttest were similar.

Problem 7. Look at the graphs in (i) and (ii) below.

a) The claim is that Graph (ii) represents a function which is the inverse of the function represented by graph (i). Explain what this means and state whether you think the claim is true. b) Look at the other graph given below and draw, on the same axes, a graph to represent its inverse if you think there is an inverse. Explain clearly why you think there is an inverse or not an inverse.

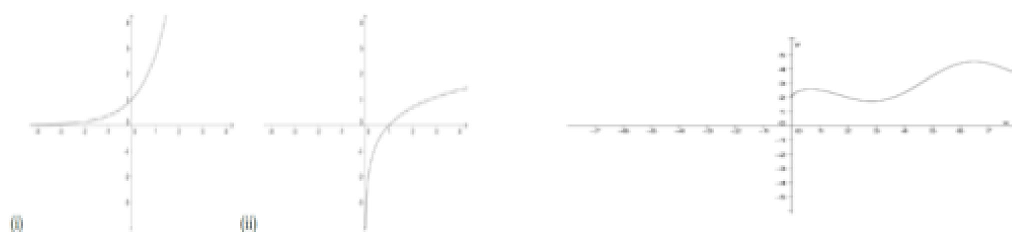

Figure 1: Item 7 from the Function Concept Inventory (O'Shea et al., 2016)

Four students in their pre-test rejected the claim in part (a). Of the ten students who accepted the claim, two described the inverse as a reflection on y = x; the remaining eight students used terms like opposite, positive-negative, reverse concavity, or negative slope to explain the meaning of inverse. 8 students did not respond to part (b). The 6 students who responded either drew an additional graph that was either reflection on the line y = x (the same two students who interpreted it as such in part(a)); on the x-axis; or about the origin.

Similarly, in the post-test two students chose not to respond to item 7. 8 students responded for part (a) but did not respond to part (b) and 4 students responded for both parts. Two of the students rejected the claim for part (a); the other 10 responded positively. The two students who had, in pretest, interpreted the graph of the inverse of a function to be the reflection of the graph in y = x reiterated that interpretation. The remaining 10 students who had responded for item #7 continued to interpret the inverse graph as a reflection on x-axis, y-axis, or both axes and the diagram that they drew for part (b) were not consistent with their interpretations in part (a).

Finding 2: Teaching Engaged Students with Activities Around the Notion of Inverse Function

Guilia's classroom teaching and her classroom materials included attention to the concept of function. In one of the observations, Guilia prompted students to do a graphing activity involving $y = 10^x$ and $y = \log x$:

> Guilia: So, do activity #3, draw the graph of $y = 10^x$ and $y = \log x$ yon the same grid and compare the two graphs. You can use the graphing calculator if you need.

After students completed the task, Guilia led the discussion comparing the two graphs and establishing the inverse relationship of the two functions. Then she asked students to do the following task:

> Guilia: Now, I would like you to investigate the relationship between y = ln x and y = e^x just like we did for log x and 10^x in activity #4. And see whether we can make similar observations.

Guilia used students' work from activity #3 and #4 to lead a discussion about the inverse nature of the pairs of functions in the activities, emphasizing the reverse concavity of inverse functions and how, in each pair of functions, the graph of one is the reflection image of the other across the line y = x. One of the worksheets that Guilia used in class also shows that, besides exploring the graphical nature of the inverse functions, Guilia and her students also examined the algebraic nature of inverses (Figure 1).

Inverse Function Notation

- If we want to emphasize that g is the inverse of f, we call it f^{-1} (read "f-inverse"). To express the fact that the population of birds, P, is a function of time, t, we write $p = f(t)$
- To express the fact that the time t is also determined by P, so that t is a function of P, we write $t = f^{-1}(P)$

Example 1: The cricket function, which gives temperature, T, in terms of chirp rate, R, is T = f(R) = ¼ · R + 40. Find a formula for the inverse function, R = f^{-1}(T).

Recall: The functions f and f^{-1} are called inverses because they "undo" each other when composed: $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$

Exercise 9: Find the inverse(s) of the following functions:

a) $y = x^3 + 3.$
b) $y = 2x + 8.$

Figure 2: Excerpt from In-class Handout on Inverse Function Notation

Finding 3: Instruction Does Not Necessarily Translate into Learning

Students had limited recollection of instruction related to the concept of inverse. Of the three students who participated in the clinical interview, Jim said he did not remember the topic being discussed in the class. Ryan (who was one of the students who correctly responded part (a) of item #7 in both the tests) said his answers were based on learning from a course he did in high school. Dikardo said that he vaguely remembered the topic being discussed in class but was not sure about the answers to item #7.

Conclusions

It is not clear from our data how substantial the attention to the notion of inverse was in Guilia's instruction. However, posttest results show that, in spite of at least two lessons explicitly designed to engage students with the notion of inverse, student learning in this area was limited. This is a concern, particularly given that students reported limited recollection of engaging in the course activities related to inverse. These findings are consistent with the idea that effective instruction requires ongoing, repeated opportunities for students to engage with core aspects of the function concept in order to develop the understandings that are deemed essential to their future success in mathematics.

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FRAMEWORK TO ANALYZE AND PROMOTE THE DEVELOPMENT OF FUNCTIONAL REASONING IN HIGH-SCHOOL STUDENTS

César Briseño Miranda
CINVESTAV-IPN, México
cbriseno@cinvestav.mx

Ernesto Sánchez Sánchez
CINVESTAV-IPN, México
esanchez@cinvestav.mx

The research this report belongs to aims at exploring the functional reasoning of high-school students from a covariation approach and proposing a framework to describe and predict the students' responses to modeling tasks in Dynamic Geometrical Situations. Items were designed for the students to build/construct the corresponding function guided by questions that allowed for the analysis of covariation. The results are shown in a framework consisting of five components, which describe the student's response patterns based on the examples of each component. The framework will provide a basis for the design of tasks involving learning and evaluation of the concept of function for high-school students. It can be an instrument to analyze the students' covariational and functional reasoning.

Keywords: Algebra and Algebraic Thinking, Geometry and Geometrical and Spatial Thinking, High School Education, Modeling.

Introduction

Mathematicians and mathematics educators share the conviction that the concept of function is one of the most powerful and useful notions within Mathematics because of its integrating role and the multiple applications inside and outside this discipline. However, they also know that learning the concept is often slow and fragmented and gives rise to false conceptions. Therefore, understanding how students appropriate the concept of function has been of great interest to several researchers in mathematics education. In addition, the research has become increasingly diverse about functions, exploring learning at all educational levels, covering different theoretical and methodological frameworks (Carraher & Schliemann, 2007; Kieran, 2004; Kieran, 2007; Thompson & Carlson, 2017). Furthermore, in the field of mathematics education there is a lack of consensus regarding which aspects of function to highlight in their introduction to mathematics classes. The function can be characterized in several ways; for example: a rule of correspondence, ordered pairs, a curve in a cartesian plane, or as a covariation of quantities. This research focuses on the first and last characterizations.

Carlson, Jacobs, Coe., Larsen, and Hsu (2002) address the study of functions from a covariational approach and propose a framework "for describing the mental actions involved in applying covariational reasoning when interpreting and representing dynamic function events". Such framework is designed to provide an account of college students' covariational reasoning in order to develop the concept of function to be used in the courses on differential and integral calculus. However, it is convenient to develop and examine how high-school students adopt covariational reasoning without even considering aspects of the Calculus course such as slopes and derivatives. For that reason, the aim of this work is similar to that of Carlson et al. (2002) but in a lower educational level: based on empirical observations and knowledge of literature on mathematics education regarding algebraic reasoning, we seek to propose an initial framework to analyze, predict, and nurture the development of reasoning with and about the concept of function from a covariational approach by high-school students.

Conceptual Framework

This work has been formulated based on an approach of the research on *Grounded Theory* (Glaser & Strauss, 1967/2008; Birks & Mills, 2014), which recommends not to adhere to any pre-established theory but to generate knowledge and build a local and humble theory based on data. This does not mean to address the object without any thought whatsoever, which is impossible, but that we are simply not expecting the reality observed to confirm a theory. Furthermore, we do not assume that a theory chosen in advance will explain the reasonings underlying in the students' responses. This work seeks to understand the students' reasoning from their productions and the patterns of their responses to the questions proposed. Still, some concepts are key to our exposition.

We will restrict the *concept of function*, with a pedagogical objective, to the covariation between two variables that can be represented with one algebraic expression. It is understood that in its more general and precise conception, the concept of function can be constructed based on this expression. Carlson et al. (2002, p. 354) define "covariational reasoning to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other". In our case, we focus our attention on situations in which quantities vary continuously, particularly when a variable is associated to the displacement of a point on a line segment and the other, to the area of a geometric figure. *Dynamic Geometry Situations* (DGS) are geometric constructions that change in a predictable way when a point moves along a curve. An important trait of DGS is that the characteristics of the construction reveal the *mechanism that explains the covariation*, which is understood as a structure that allows for deducing, based on knowledge of the context, the rule of correspondence between the variables. In several dynamic situations, the *mechanism explaining the covariation* is not transparent (for instance, think on the phenomenon of free-falling bodies); still, in the DGSs proposed, only some geometric knowledge is enough to deduce such mechanism.

Method

The participants of the study were sixteen high-school students grouped in eight couples and enrolled in a first Calculus course. Four problem-activities based on different DGSs with progressive difficulty were carried out. Several written questions were formulated in each problem-activity, so that the students could model the situation with the algebraic expression of a function. The research was designed in two phases, each one consisting of nine sessions lasting two hours: (i) The first phase consisted in activities to model each situation using only paper-and-pencil; (ii) the second phase consisted in using GeoGebra to study the models obtained in the first part; students had to introduce the models [rule of correspondence] in the software to obtain and study the graphs. Additionally, parameters of the situations (e.g., length of the initial segment) were changed so that the students extracted conclusions on the family of functions. In this paper, we provide information on the students' reasonings collected during the first activity (first phase of this research); that is, during the paper-and-pencil phase, without using GeoGebra. The authors are working on a more comprehensive report including the second part.

The first DGS administered to the students is:

The following figure (see Figure 1.a) shows the segment $\overline{AB}$ whose length is 10 units. Place a point Q on segment $\overline{AB}$. Draw the square AQCD; the length of each of its sides is segment

$\overline{AQ}$. *Then, draw square QBEF; the length of each of its sides is segment* $\overline{QB}$.

Based on the construction of polygon ABEFCD (see Figure 1.b; not visible for the students), students are asked: (1) find the area of polygon *ABEFCD* for a defined position of point Q; (2) determine on what the value of the area of the polygon depends; (3) identify which variables are involved when point Q is displaced on segment $\overline{AB}$ and determine whether the area of the polygon is constant when point P (Q before) is displaced on segment $\overline{AB}$; (4) find the algebraic expression that allows for the calculation of the area of polygon *ABEFCD* with respect to any point P; and (5) outline the graph of the previous expression.

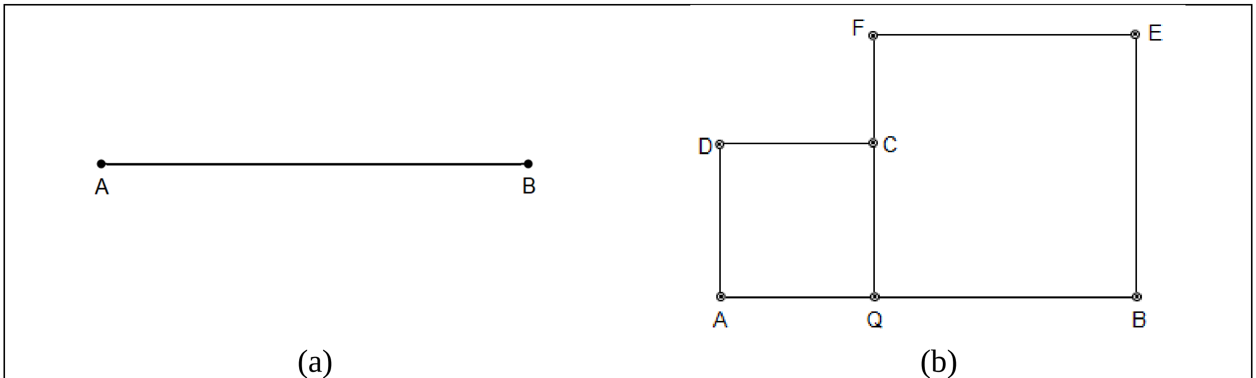

Figure 1: GDS. (a) Initial Segment, (b) Geometric Construction Integrating the GDS

To analyze the data, we considered the principles and procedures of the *Grounded Theory* (Glaser and Strauss, 1967/2008; Birks and Mills, 2011), which is a general methodology of research in social sciences (Holton, 2008) whose objective is to create local theories from data and not logical deductions or other theories previously established. The data analyzed are those responses to the questions formulated and written by the students on their worksheets. These were transcribed into electronic files for handling and were coded, comparing responses between them and grouping those responses when similar characteristics were identified. Through this process, we determined the categories that constitute the proposed framework.

Results

The students' responses to the questions in the item were grouped according to one of the components of the framework described in Table 1.

Table 1: Initial Framework of Covariational Reasoning.

Component	*Description*
I. Ignorance of covariation	The dynamic situation is analyzed statically (fixing a position); then, calculations are made with constant quantities or literals conceived as general numbers.
II. Consideration of covariation	The dynamic aspect of the situation is verbalized, describing how the change of an object (point, quantity, or variable) produces a change in another one (figure, length, area, or volume).
III. Previous analysis of covariation	Dependent and independent variables are identified and the mechanism explaining the covariation is described. Additionally, the range of the

	each variable is recognized.
IV. Representation of covariation	The rule of correspondence is symbolically represented. Representations go from those using geometric notation to those written purely in algebraic notation.
V. Consequences of covariation	From the first representation of the proposed function, other representations, as tabular and graphic, are expressed. Variation ranges are determined, and maximum or minimum values are anticipated.

Figure 2 synthesizes the components that constitute the initial framework of covariational reasoning, which are intertwined, since each level is a consequence of the upper one.

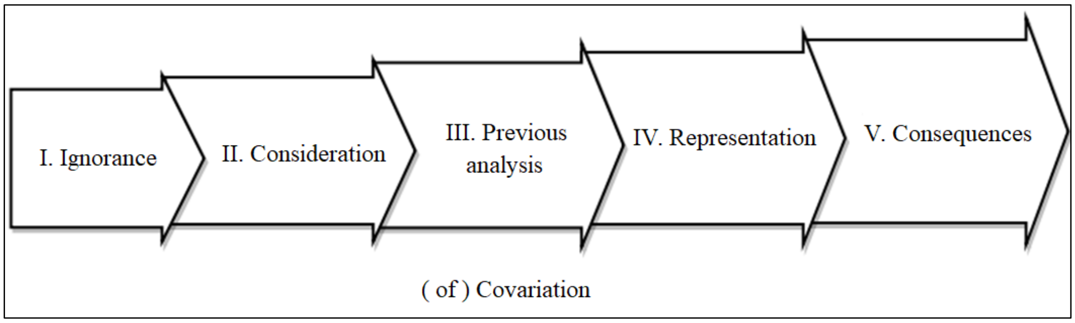

Figure 2: Identified Categories in the Initial Framework of covariational Reasoning

Below are the results obtained from the first task based on the described components:

I. Ignorance of Covariation

A dynamic situation can be and is often modeled fixating it in a certain moment to analyze its structure. Ignoring the covariation means carrying out an analysis of the structure of the situation at a given moment. This is often the first step of a strategy to model such situations.

In the first question after describing the situation, the students are asked to write the area of geometric configuration obtained for a position of a point Q on the segment $\overline{AB}$. All the couples provide a response ignoring covariation, either because they use a constant value for the independent variable or because they do not suggest displacing or changing the initial point Q, although they propose a general expression using symbols of segments or literals.

On the one hand, two couples based their response on the assignation of a numerical value to segment $\overline{AQ}$ and then calculated the requested area based on the outline of the corresponding configuration. For instance, a team assigns $\overline{AQ}$= 3.414634146 and then $\overline{QB}$= 6.585365854, operates with these numbers, and correctly obtains *Area of ABEFCD* = 55.026. Although this procedure includes the data and reflects the understanding of the mechanism explaining covariation, the result leaves no trace of such understanding. On the other hand, two couples use segment notation to express the area of the polygon, but one is not expressed as equation and none of the couples include the data [$\overline{AB}$=10 and $\overline{AB} = \overline{AQ} + \overline{QB}$]. For example, one of the couples writes: $\overline{DA} \cdot \overline{AQ} + \overline{FQ} \cdot \overline{QB}$. Finally, four couples write general expressions; still, none included the data in the equation. As an example, the following response represents these four teams: $A = \overline{AQ}^2 + \overline{QB}^2$. It must be noted that, in this phase, segment expressions are used as general numbers; that is, they express a fixed but undetermined number.

The quality of the responses in this category depends on two aspects: (i) the notation used,

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

better if symbolic and not only arithmetic, and (ii) the addition of problem data to the expression, keeping only the dependent and independent variables, laying the ground to transit towards the algebraic expression.

II. Consideration of Covariation

This category includes the manifestations in which students recognize that the situation is dynamic, alluding to the notion of variable. They can express the idea of covariation in discrete terms; for example, "for values different from $\overline{AQ}$ we obtain different polygons or different area values". They can also implicitly express the continuity of the relation: "When point Q is displaced on the segment $\overline{AB}$, the polygon or the area changes." Question 3, part of the battery of questions asked, promoted the consideration of covariation between point Q and the area of the polygon: *(3) If point Q is displaced on line segment $\overline{AB}$, is the area of the polygon constant?*

For example, a team stated that when "point Q is displaced on the segment $\overline{AB}$, the area grows without being constant and while a square grows, so does its area." Another team states the area is not constant and argue that "when Q is in the center [in the middle of line segment $\overline{AB}$], the area of the polygons is equal. When Q is at the center, that is the smallest area we will get from both polygons. If Q goes from A to the center of $\overline{AB}$, the area decreases and if it goes from the center to B, it increases." Other teams are more explicit. In the following case, the students make a description that does not refer to the area of the polygon, but to the areas of the squares forming the polygon: "If P (previously Q) is in the midpoint of $\overline{AB}$, the areas will be equivalent. If $\overline{AP} > \overline{PB}$, then the area goes from a maximum to a minimum value and after this constant it will grow equally on the opposite side until it reaches its maximum height." The students in this team try to describe covariation observing the changes in the squares forming the polygon. The most developed answers are those that describe the way in which the whole area of the polygon changes. Still, there are others describing covariation by observing the movement of the polygon and not its area.

III. Previous Analysis of Covariation

The fundamental elements of covariation are independent and dependent variables, rule of correspondence, and ranges of both variables. When students identify some or all these elements, we consider they have made a prior analysis of covariation.

The questions *(2) What does the value of the area depend on?* and *(3) Which variables intervene when point P is displaced on line segment $\overline{AB}$?*, foster that students to refer to the variables. It must be noted that, in their responses, no pair of students refers to both variables ($x = \overline{AP}$ and $a = area\ de\ ABEFCD$). Three teams associate the independent variable with the position of point P; for example, one says: "The only variable is P because the two internal segments depend on the position of P." Four teams mention the segment $\overline{AP}$ and a fourth one identifies the magnitude of the segment "*P=distance traveled on* $\overline{AP}$". No team refers to any variable as "*x*". Additionally, students seem to have interpreted the term variable as independent variable and thus the analysis of covariation is partial in all cases.

We must emphasize that the students' responses do not entirely deviate from the context. A majority considers the independent variable as a position or line segment, instead of thinking about it as a quantity associated to the position of the point albeit different from it. However, the most advanced students use letter "*p*" as a quantity that varies and not as a point; no one uses variables "*x*" and "*y*".

IV. Representation of Covariation

An important objective while modeling *Dynamic Geometry Situations* is achieved when the corresponding algebraic expression is obtained. The activities described in the previous components are mere aids of, and subsumed in, the (algebraic) representation activity of covariation. A student who move forward adequately in the activities of the previous components has a good chance of writing the corresponding representation and provide it with a sense related to its dynamic aspect.

The representation activity is promoted with the question *(4) Find the algebraic expression that allows for the calculation of the area with respect to any point P*. All the students propose a representation, and we have identified the group of representations according to the notation used: Geometric, Mixed, and Algebraic, as well as the integration of $\overline{AB} = 10$ and $\overline{AB} = \overline{AP} + \overline{PB}$.

In their representations, most of the teams keep traits of the geometric context in which the problem is formulated. The response $a = \overline{AP}^2 + \overline{AB}^2$ is completely geometric; in such expression, we have conjectured that letter "*a*" is used as a tag and not as a variable (remember that in the section "Analysis of covariation" no student identified the "*area*" as a variable). Additionally, the representation does not include the two data mentioned. Three teams used mixed notation; that is, they combined Geometric and Algebraic notations. For example, a team writes $f(p) = p^2 + (\overline{AB} - p)^2$, but does not include $\overline{AB} = 10$. Finally, two teams use Algebraic notation, although they do not use conventional letters: $p^2 + (10 - p)^2 = a$.

Interestingly, two teams spontaneously provided an algebraic expression that nonetheless alluded the context by using variables "*p*" and "*a*" instead of "*x*" and "*y*". The reluctance to separate the expression from the context is greater in the teams using Geometric or Mixed notation; they tend to overlook the inclusion of the data to the expression. Surely, additional activities are necessary for students to be willing to regard the algebraic expression independently from the context. In the present studies, the activities carried out later using GeoGebra allowed the students to produce expressions using algebraic notation for GDS models. However, paper-and-pencil activities were also convenient for students to reflect on some of the consequences of their representations.

V. Consequences of Covariation

The modeling process consists of representing a situation to study its properties and consequences. Then, the aim is not only to represent covariation but also obtain consequences from it, one of which is (the immediate although not simple) graphic representation that, in turn, allows to obtain more consequences. Other approaches on the concept of function focus on the occurrence of a situation or a data set, then the graph and its interpretation (Leinhart, Zaslavky & Stein, 1990). A consequence in this case would be obtaining the algebraic expression. Even though this approach is fruitful, the one we propose in this work is better adapted to the objectives of an algebra course considering later pre-calculus or calculus courses.

To help students express the consequences of covariation, they were asked: *(5) Outline the graph of the symbolic expression previously obtained.* Figure 3 shows three types of graphs found among the students' responses. A team outlines a graph including two line segments joined at a vertex (see Figure 3.a). The graph did not consider the algebraic expression and was directly derived from a direct (but inadequate) qualitative interpretation of the situation. Another team provided a graph showing a curve segment (see Figure 3.b); still, there is no apparent relation between the written equation and the table built. Five teams provide an acceptable outline of the graph, building the table, plotting the points, and then interpolating the curve (see Figure 3.c).

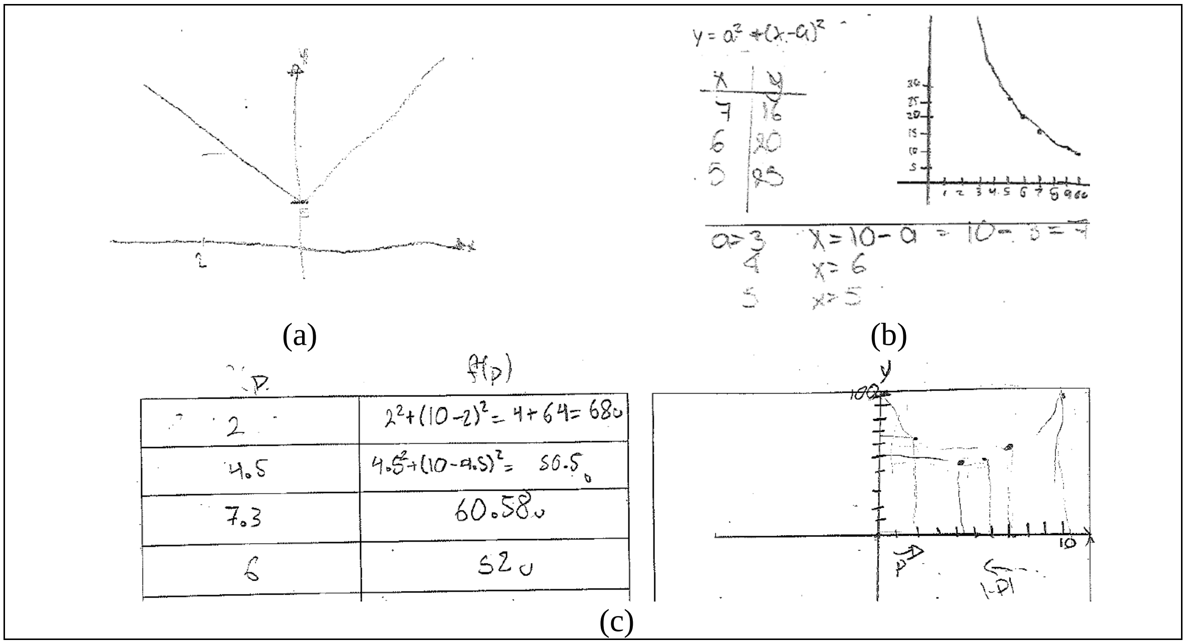

Figure 3: Consequences of Covariation

Conclusions

Once the students have reached a basic level at handling algebra (use of an unknown and formulation of equations to solve problems in static situations), they can model dynamic geometric situations. Therefore, they can progress to building the concept of function, at least in its conception as covariation of quantities. In this research, we characterized different types of actions students take in the process of algebraically modeling a dynamic situation in which they can easily deduce the rule of correspondence from the *mechanism that explains the covariation* [polygon *ABEFCD*]. This methodological decision of reducing the mathematical difficulty with the intention to allow students derive the rule of correspondence highlighted some of the natural and adequate dispositions of the students and reveal their difficulties for using algebra to model dynamic situations. For instance, the notion of displacing the point and coordinating it with what occurs in the polygon naturally leads to the use of symbols as variables. Contrastingly, it seems students have difficulties representing the function independently from the geometric context from which it arises. The approach we propose is part of a teaching experiments in whose development GeoGebra was used. This also explains the aim towards constructing the algebraic expression given that this software can be used to introduce the algebraic expression and obtain the corresponding graph on screen. The analysis and result of this stage is not part of this report; however, we can anticipate it was useful to consolidate the interpretation of the graph of the function and the correct actions and reasoning made in the stage here reported.

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SEEING EXPONENTIAL FUNCTIONS DESPITE REPRESENTATIONAL FLUENCY IN A QUANTITATIVELY RICH QUADRATIC FUNCTION TASK

Nigar Altindis
Syracuse University
naltindi@syr.edu

Nicole L. Fonger
Syracuse University
nfonger@syr.edu

In this study we explored how preservice teachers' (PTs) demonstrated representational fluency and functional thinking in approaching quadratic function tasks within a realizable task context. Participants were secondary school mathematics PTs in their final semester. We employed a multi-case study method with task-based interviews to deeply explore PTs problem-solving activity. The findings showed representational fluency pushed learners to switch back and forward between covariational and correspondence reasoning.

Keywords: Representational Fluency, Covariation and Correspondence Reasoning

Rationale and Research Aim

This research aims to respond to an enduring challenge of supporting secondary mathematics preservice teachers (PTs) in learning about conceptually-oriented approaches to teaching functions. Creating, connecting, translating between, and making meaning of multiple representations, or representational fluency (Fonger, 2019), is an important aspect of PTs' pedagogical content knowledge (Dreher, Kuntze, & Lerman, 2016). Indeed, creating and interpreting representations are "important cognitive processes that lead students to develop robust mathematical understandings" (Huntley, Marcus, Kahan, & Miller, 2007, p. 117). A prevelent notion in math education is that "using multiple representations" supports students' conceptual understanding of mathematics (e.g., NCTM 2014). However, a representational fluency lens on conceptual understanding is by itself insufficient (Fonger, 2019; Lobato, 2013). Indeed, a growing body of work demonstrates that students make sense of functions by reasoning with both covariational and correspondence approaches (Carlson et al., 2002; Confrey & Smith,1994, 1995; Lobato et al., 2012; Thompson & Carlson, 2017). In one study, PTs' covariational reasoning fostered their understanding of the relationship between multiple representations (Moore et al., 2013). Moore et al found "students engaged in covariational reasoning to make sense of and conceive invariant relationships among multiple representations" (p. 472). Moore and colleagues identified a relationship between students' covariational reasoning and their interpretation of multiple representations. The generalizability of this finding to other contexts remains an important area of inquiry.

In this study we investigate PTs conceptual approaches to quadratic function tasks by networking two lenses: representational fluency and functional thinking. We aim to contribute to an understanding of relationships between PTs functional thinking and representational fluency in solving problems in a quantitatively rich context. We ask: RQ1. How do PTs use and connect representations to reason about quadratic functions?; and RQ2. What relationships are observed between PTs representational fluency and functional thinking?

Theoretical Framework and Background Literature

From our stance, a person (e.g., PT) is viewed as active in their meaning-making processes through the activities of creating and interpreting invariant features among representations. We draw on Fonger's (2019) articulation of meaningfulness in representational fluency as an analytic tool to discern the nature of students' problem-solving approaches with multiple representations.

Functional thinking includes students' abilities to generalize and represent functional relationships (Stephens, Fonger, Strachota, Isler, Blanton, Knuth, Gardiner, 2017; Kaput, 2008). Correspondnece, covariation, and recursive are three ways of reasoning about functions (Confrey & Smith, 1991; Thompson & Carlson, 2017); we focus on the first two in this report. From a *correspondence* perspective, students determine output or dependent values (of the range) as related to input values (of the domain) and may identify a symbolic equation between dependent and independent values. From a *covariation* perspective, students reason about relationships of change across two or more quantities (e.g., change in *x* and change in *y*) "describing how one quantity varies in relation to another" (Confrey, & Smith, 1995, p.79). We adopt this framing across symbolic, graphic, numeric, and diagram representations.

Methodology

Data Sources and Mode of Inquiry

Participants of this study were five secondary PTs, in their second semester of a two-semester methods sequence at a private university in the U.S. The authors conducted a 60-minute audio-recorded semi-structured task-based interview (Goldin, 2000) with each PT. Given space constraints and commitment to deeper understanding of participants' thinking, this study reports on only one participant as a case study (Stake, 1995), Eve,who selected task C (Figure 1a). We prompted PTs to think aloud, clarifying meanings of their activity.

Data Analysis Techniques

In this study, we employed an analytic framework for representational fluency (Fonger, 2019), in particular lesser meaningfulness to characterize PTs' discursive activity in creating, interpreting, and connecting multiple representations, and a priori descriptions of students' functional thinking (cf. Stephens et al., 2017) (RQ1). Lesser meaningfulness includes: *pre-structural understanding*; creating and interpreting one representation with unsophisticated thinking; and *multi-structural understanding*; creating or connecting multiple representations with unsophisticated thinking. This framework allowed us to identify relationships between PTs' representational activity and their ways of thinking about function (RQ2) in rounds of coding.

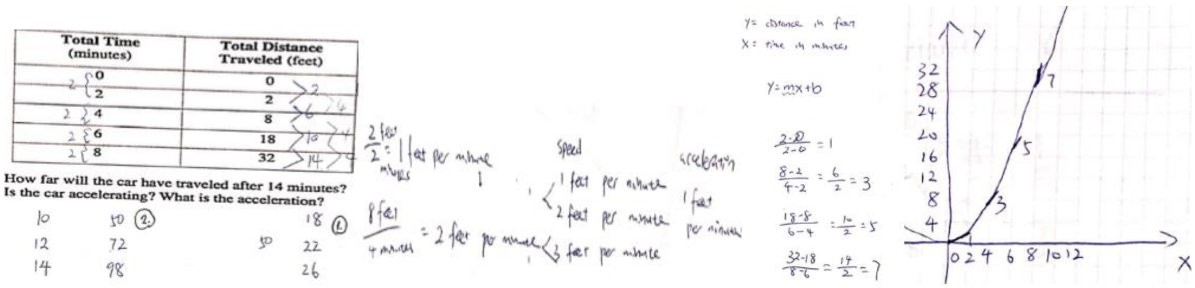
Results

Eve used multiple strategies; covariational reasoning on a table and graph, and correspondence reasoning to create generalized symbolic equation. We discuss each in turn.

Strategy 1: Eve's Covariational and Corresponding Reasoning with a Table

Eve had strong covariational reasoning; to find how far the car traveled after 14 minutes, she extended the table in Task C (Fig. 1a) by finding that the change in the distance was 2, 6, 10, 14, and 18, and the change in the time was 2 for each time interval (see Fig. 1a). While she was extending the table, she recorded both change in the distance and the change in the time as coordinated pairs of quantities. Eve placed her right hand on time and the left hand on the distance column on the table at the same time to represent the covariation across columns. While positioning her hands, she said "So, I see that when the time increased by two every time, the total distance also increased." Eve's covariational reasoning helped support her use of the tabular representation in making sense of this task and finding the distance the car travelled in 14 minutes. To determine if the car was accelerating, Eve employed a corresponence approach by calculating ratios of time and distance, then calculated change in those quantities:

> "I do two feet divided by two minutes [fig. a]. And then I do the same for four minutes. So, I get two feet per minute and then I get three feet per minute. So, I compare the speed between one per minute, two per minute and three per minute. And then I realized that each time the speed is changing by one foot per minute. So, I think the acceleration is just going to be one foot per minute."



(a) (b)

Figure 1: Eve's Quantitative Reasoning with Multiple Representations on the "Car Task"

Also shown in Fig. 1a, Eve divided the total distances and total time for each of: 2 ft/2 min.=1ft/min, 8 ft/4 min.=2ft/min.; 18 ft./6 min.=3ft/min., correspondence reasoning. From these ratios, Eve noted that the speed of the toy car increased by 1 ft/min and concluded that acceleration was 1 ft/min by looking change in total distance over time for each interval. She attended to change in speed, not noticing the time intervals between the computed speeds were 2 minutes each. She seemed to either assume that acceleration is a change in speed or had assumed that the time between the calculated speeds was 1 minute, not recognizing acceleration is a change in speed per unit of time. Thus, while Eve was able to identify patterns by extending and interpreting the table (covariation), and computing ratios of distance and time (correspondence); she had little confidence in distinguishing between average speed, instantaneous speed, and acceleration across the table and verbal representation types, *multi-structural* fluency.

Strategy 2: Eve's Correspondence Reasoning to Create a Symbolic Representation

When asked to solve the task in another way, Eve used a correspondence approach; she investigated a relationship between input (time) and output (distance) variables. Eve labeled y as distance in feet and x as time in minutes, then attempted to find out whether the change in y (distance) divided by change in x (minutes) was constant. She wrote $y = mx + b$ and said:

> "So, if I make y to be the distance and then x to be the time in minutes… So, the speed, the velocity would be my slope because I think that would be m, x plus b. So, if I wanted to find the slope, I can do 2 feet divides by 2 minutes [writes (2-0)/ (2-0) =1], like two, I think that would be exponential. If I do eight minus two [writes (8-2)/ (4-2)], it would be six over two. Which would be [3]. Yeah. Because it's accelerating. So, the slope will not be the same"

Shown in Figure 1b, Eve calculated the "slopes" as 1, 3, 5, and 7, using feet and minutes as units when she was referring to distance and time, realizing the slope was not constant. She reasoned that the slope was increasing, and the relationship between time and distance was exponential. She looked for an exponential equation to model speed even though she noticed initial value is zero, she did not change her claim of exponential growth. She said: "If I try to find the exponential function, it has the form like y equal to initial times the rate of change to a time interval. So, the initial, which would be zero... It does not make sense because if I have zero everything will give me zero." Although Eve saw that the initial value was zero, she did not change her reasoning about exponential growth and wrote $y = \textit{initial value}(\textit{rate of change})^{\textit{time}}$.

Eve expressed an understanding of the meaning of slope and tangent line, quantifying the slope as the speed of the car in a graph. Eve used *correspondence* reasoning to look at the relationship between time and distance, and to find an equation to model the car's speed. She reasoned that the slope would be speed, and from reasoning about ratios of change (*covariation*) she divided the distance by each time interval to calculate the speed. Hence, Eve connected a tabular representation to a symbolic one, without a complete portrayal of the formula. So, we consider Eve' s reasoning as *multi-structural*.

Strategy 3: Eve's Covariational Reasoning with a Graph

In a third strategy, Eve decided to create a graph. Eve labelled y-axis as distance and x-axis as time, and as she plotted the points. After connecting the points on the graph, Eve interpreted the graph as an exponential (Fig 1b). When asked why, she said: "this is going very fast. The speed. So, it's slowly at the beginning, like a small, more flat line, and then the line keeps getting steeper." Eve's visualization of faster speed in a graph was the line getting steeper, so if it increased quickly, then it must be exponential. Eve represented the slopes 1, 3, 5, and 7 for time intervals 0-2, 2-4, 4-6, and 6-8 on the graph, then realized that the function was a concave up curve, *covariational* reasoning. She correctly created a graph that matched the data in the table yet insisted on an interpretation of the function as exponential, *multi-structural* fluency.

In order to observe Eve's ability to connect representations, we asked Eve to represent the change in distance (which she identified as 4 in the table, see Fig. 1b) in the graph. She said, "So the difference is increasing by four each time [referring the change of change in distance in the table, see Fig. 1a], but how does it relate the graph, I do not know.". Although Eve's reasoning showed she was finding average rate of rate of change as constant, she did not recognize the function as quadratic. She also identified this rate as the slope and showed the slopes as tangent lines on the graph. Eve was able to interpret the graph and see "the car is accelerating, 'cause when you see the graph it is going up very fast." However, she could not reason how much the car would accelerate and the distance it would travel after 14 min.

Overall, Eve reasoned that the relationship between time and distance as exponential in graphical and tabular representations. Her interpretation was that the function must be an exponential; she used graphical representation to argue for her interpretation, noting that the graph did not have a constant slope for each time interval (and thus could not be linear), and that it had a non-negative domain for the time variable. Eve's reasoning about rate of quadratic function on a table was clearer than on a graph. As Eve did, greater sophistication in representational fluency supported her to draw connections between covariational and correspondence reasoning.

Discussion and Conclusion

This study contributes to knowledge about how representational fluency and functional thinking may interact in a symbiotic relationship, shedding light on PTs meaningful learning of functions with multiple representations. For instance, Eve's correspondence and covariational approach to tabular representation made the acceleration visible in solving the task. Moreover, interpreting, creating and connecting multiple representations pushed Eve to switch back and forward within covariational and correspondence reasoning. Hence similar to Moore et al.'s (2013) study, PTs' representational fluency is grounded in covariational reasoning. The connection learners make between covariational and correspondence reasoning through representational fluency might develop sophisticated understanding of function concept. Supporting PTs' fluency and functional thinking is a viable approach for teacher education focused on conceptually-oriented approaches to learning. Future research is needed to support PTs understanding of the nature of function families as grounded in rich quantitative tasks.

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SOCIOMATHEMATICAL NORMS REVEALED DURING A PRECALCULUS BREACHING INSTRUCTIONAL ACTIVITY

David Fifty
University of New Hampshire
dri36@wildcats.unh.edu

Dr. Orly Buchbinder
University of New Hampshire
Orly.Buchbinder@unh.edu

Sharon McCrone
University of New Hampshire
Sharon.McCrone@unh.edu

The paper examines sociomathematical norms in a Precalculus course at a four-year public university. The data were collected through a set of instructional activities designed to breach students' expectations about what constitutes an acceptable mathematical solution by exposing them to alternative solution methods. This paper describes one such activity in detail and reports on the nature of a sociomathematical norm that was revealed by the activity: Students characterized an acceptable solution as one that followed procedures modeled by the instructor, while solutions which used other valid mathematical approaches were considered incorrect by the students.

Keywords: Post-Secondary Education, Instructional Activities and Practices

Background / Purpose

Incoming college students who are low-achieving in mathematics are often placed in developmental or remedial mathematics courses such as College Algebra or Precalculus. Research over recent decades describes various shortcomings of such mathematics courses (Stage & Kloosterman, 1991; Hagedorn, 1999; Boyer et al., 2007; Lagerlof & Seltzer, 2009; Bahr, 2013), such as an overemphasis on exposure to remedial content, which often does not require students to change the mathematical practices and habits that contributed towards their need for remediation (Carlson et al., 2010; Goudas & Boylan, 2013). In addition, students, especially those with poor mathematics skills, often prefer a dependent learning style with a procedural focus towards mastering algorithms. Consequently, some researchers have suggested that attempts to improve developmental courses should consider the nature of students' mathematical engagement, focusing on developing students' argumentation skills and reasoning strategies (Chiaravalloti, 2009; Partanen & Kaasila, 2014). This, in turn, requires establishing productive socimathematical norms in the class. These norms help shape how students interact with mathematics, including how they interpret and solve mathematical problems (Voigt, 1995) as well as how students reason and justify their reasoning (Inglis & Mejia-Ramos, 2009). Negotiating sociomathematical norms can help teachers guide students' inquiry and convey how to productively participate in class. This can help focus students' engagement on developing reasoning skills necessary to be successful in future mathematics courses.

As part of a larger study that seeks to characterize students' mathematical engagement in a Precalculus class at a four-year public university, this paper reports on the analysis of sociomathematical norms that developed in such a class. Specifically, we examine the following research question: What characterizes the sociomathematical norm of what constitutes an acceptable mathematical solution in a post-secondary Precalculus class?

Framework

The Emergent Perspective and Norms

This research adopts the view that social and psychological perspectives are essential to characterizing classroom activity, which aligns with the emergent perspective (Cobb & Yackel, 1996). One key social construct in the emergent perspective is *norms*, which characterize mutually established and regulated activity or behavior amongst a collective (Cobb et al., 2001). In a class, norms are not pre-made rules for students to follow but are rather developed through continual student and teacher interaction, either explicitly or implicitly. Even though teachers typically initiate the negotiation of norms, norms are usually based on mutual expectations that are formed as both students and teachers interact with one another (Yackel et al., 2000). *Sociomathematical norms* are specific to mathematical aspects of students' activity (Yackel & Cobb, 1996; Kazemi & Stipek, 2008/2009). One sociomathematical norm of particular importance to this research is: what constitutes an acceptable solution. Research shows that effectively negotiating this norm can help develop a culture to support effective student participation in mathematical discussions (Partanen & Kaasila, 2014).

Methods

Data collection focused on one post-secondary Precalculus class of twelve students, taught by Ethan (all names used are pseudonyms), a first-year graduate student with prior teaching experience. The class largely consisted of students taking the class for the second time, as they did not earn a sufficient grade the previous semester to advance to Calculus 1, and non-traditional students, such as those returning to school or those involved with the armed services.

Typically, norms are widely abided through unconscious acceptance, which can make them difficult to study (Braswell, 2014). To elicit information about social norms, Garfinkel (1967) introduced the methodology of breaching experiments, in which a researcher attempts to violate conjectured social norms. This study appropriates this idea by utilizing a breaching experiment in the form of an instructional activity for students, which was designed to uncover the sustained sociomathematial norms in the class on what constitutes an acceptable mathematical solution.

Students worked on in groups of three or four. Each activity comprised three phases. The first phase asked students to solve a mathematics problem on the content recently covered in class. This phase allowed for the capture of typical mathematical practices pursued by the students. The second phase asked students to cooperatively formulate a grading key by distributing six points in any way they agreed upon. Students were told that they had to explain their point allotment and that they could change it later on if desired.

The last phase of the activity, which is the focus of this paper, asked students to use their grading key to evaluate sample solutions. This phase attempted to disrupt students’ expectations by presenting alternative methods or including mistakes that violate hypothesized normative mathematical conventions established in class. Phases two and three of the activity provided information about the nature of the sociomathematical norm of what constitutes an acceptable mathematical solution.

Data Collection and Analysis

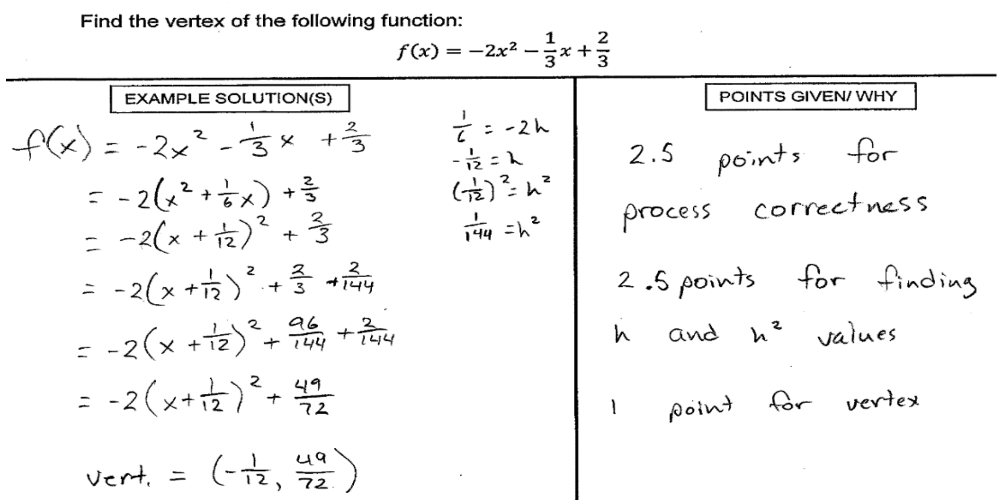
Two groups were video-recorded using 360° cameras. We coded students' activity and conversations during all stages of the activity with consideration to other collected data, including their solutions, grading rubrics, and the grading of sample solution work, to look for evidence of the sociomathematical norm of what constitutes an acceptable mathematical solution. The following results focus on the activity of a group consisting of Darryl, John, Noah, and Sean, which were similar to the other group recorded.

Results

The mathematical topic for this activity was to find the vertex of a parabola. The instructor introduced the activity to the students and described its purpose as an opportunity to explore "different ways to do things … cool ways to solve problems." He stressed that the methods he taught were not "the only way or the best way even" to find the vertex of a parabola.

Phase 1 and 2: Solving the Problem and Creating a Grading Key

Figure 1, below, shows the group's solution and their scoring rubric for assessing the sample work. This solution followed a method referred to as *completing the square,* which was previously modeled by the instructor. The group's rubric depicted their value for completing the square, as most points were rewarded for adhering to this process. This suggests that the group values the process more than the answer. Since the class was not accustomed to exploring different approaches, it is unsurprising that their rubric focuses on the approach that they are acquainted with.



Find the vertex of the following function:

$$f(x) = -2x^2 - \frac{1}{3}x + \frac{2}{3}$$

EXAMPLE SOLUTION(S)	POINTS GIVEN/ WHY
$f(x) = -2x^2 - \frac{1}{3}x + \frac{2}{3}$ $= -2(x^2 + \frac{1}{6}x) + \frac{2}{3}$ $= -2(x + \frac{1}{12})^2 + \frac{2}{3}$ $= -2(x + \frac{1}{12})^2 + \frac{2}{3} + \frac{2}{144}$ $= -2(x + \frac{1}{12})^2 + \frac{96}{144} + \frac{2}{144}$ $= -2(x + \frac{1}{12})^2 + \frac{49}{72}$ vert. $= (-\frac{1}{12}, \frac{49}{72})$ $\frac{1}{6} = -2h$ $-\frac{1}{12} = h$ $(\frac{1}{12})^2 = h^2$ $\frac{1}{144} = h^2$	2.5 points for process correctness 2.5 points for finding h and h^2 values 1 point for vertex

Figure 1: The Group's Solution and Grading Key

Phase 3: Evaluating a Sample Solution

One sample solution, by the fictional student "Frodo," represented a non-standard but valid approach to the problem that only provides the y-coordinate of the vertex (Figure 2). This deliberate flaw was included to determine if students would take exception to how the solution was expressed, as Ethan had emphasized that a vertex was a point with two coordinates.

What was most interesting about students' treatment of this solution, was how the group members fixated on determining the degree to which Frodo followed the procedure that was familiar to them. They were highly critical of Frodo's approach, and were keen to address how the solution deviated from theirs instead of trying to understand it, as the next comment shows:

> John: They're not going to get any other points for it. They don't even have h's in here. So there's 2.5. They don't even have, this isn't the same form to use with h, they don't even have any h's in sight, they completely skipped the h's. So minus 2.5 for that.

The group's reaction provided information about the sociomathematical norm of what constitutes an acceptable solution: an acceptable solution needs to follow the procedures learned in class.

Besides faulting Frodo for not following the "correct procedure," the group further criticized his solution for lacking logic. This was especially surprising considering that Frodo arrived at the same y-coordinate as the group; Darryl attributed this to Frodo being "super lucky."

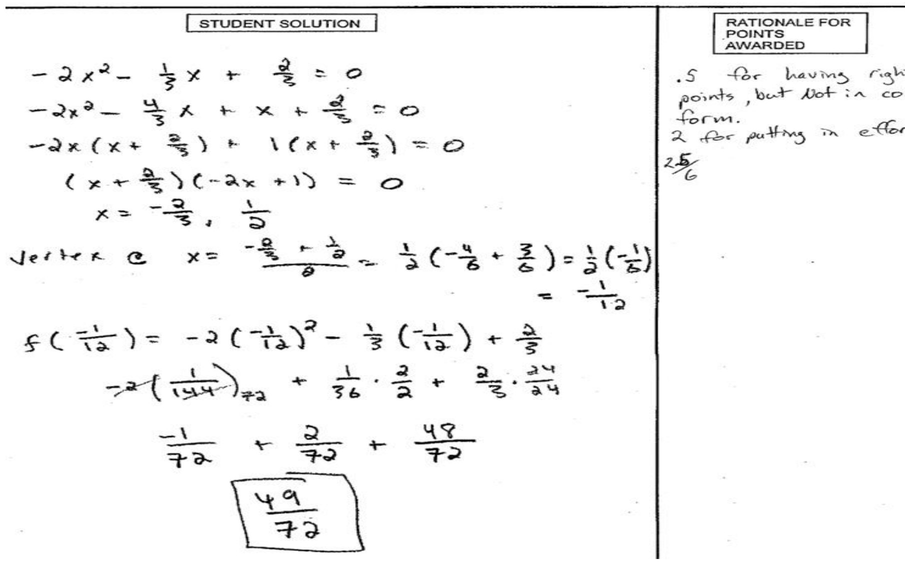


Figure 2: Frodo's Solution and the Group's Grading Rationale

Discussion

One result from this study, that acceptable solutions must represent methods learned in class, displays Ethan's pedagogical approach of having students focus on repeating procedures. Without allowing for investigation into different approaches, Ethan implicitly conveyed to students that the objective of his lessons was for them to learn to reproduce the procedures he modelled. This may have inculcated inflexibility in students to other solution methods, which can be seen by the dismissal of unfamiliar methods as incorrect. This is unfortunate considering one of the goals of developmental mathematics classes is to help students develop flexible reasoning skills needed for subsequent mathematics courses instead of merely having students repeat fixed procedures.

It is important to note that Ethan's instruction did not reflect his own teaching philosophy. He faced several constraints, such as needing to coordinate with other instructors and covering a wide curriculum, which hindered his ability to teach as he wanted. This struggle depicts the usefulness of activities, such as the one described above, that promote student reasoning and flexibility in performing procedures yet still advance the content coverage of the curriculum.

The pedagogical value of this activity was that it allowed Ethan to explicitly negotiate sociomathematical norms to promote more productive mathematical engagement. For example, Ethan began to explicitly dissociate between valid mathematical reasoning and following a procedure, something he noticed the students were conflating during the activity. Because of this, in the following activities, Ethan's students spent increasingly more time investigating the reasoning behind different approaches. These efforts presented opportunities for students to practice drawing on their own intellectual resources.

In addition to identifying sociomathematical norms revealed during breaching instructional activities, this paper is part of a larger project that identifies how sociological constructs develop over the course of a semester. Analyzing multiple breaching instructional activities has allowed us to begin investigating how, for example, the sociomathematical norm of what constitutes an acceptable solution evolved and how this influences students' engagement. Other sociological constructs studied are social norms and communal mathematics practices, which also help characterize students' engagement in the classroom.

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RELATIONSHIPS BETWEEN CALCULUS STUDENTS' STRUCTURES FOR STATIC AND DYNAMIC REASONING

Steven Boyce
Portland State University
sboyce@pdx.edu

Cameron Byerley
Colorado State University
cameron.byerley@colostate.edu

Andrew Darling
Colorado State University
andy.darling@colostate.edu

Jeffrey Grabhorn
Portland State University
jeff4@pdx.edu

Brady A. Tyburski
Colorado State University
brady.tyburski@colostate.edu

We report on results from 20 one-hour task-based interviews with 10 students enrolled in first-term calculus. Each student participated in two interviews. The first interview included tasks designed to assess students' ways of reasoning about fractions in static measurement scenarios (which did not involve varying quantities) and the second interview included tasks designed to assess students' reasoning about scenarios involving quantities that co-vary. We discuss relationships between the ways students coordinated units in static situations and the ways students reasoned about co-varying quantities. The results contribute to the field's understanding of the development of covariational reasoning and the role of constructing three-levels-of-units coordinating structures.

Keywords: Units Coordination, Covariation, Measurement, Calculus Students

Introduction

Steffe's (2017) plenary for PME-NA described children's coordination of units as foundational for their construction of rational number and measurement concepts. How might differences in units coordination relate to differences in older students' reasoning? This paper reports on our exploration of associations between introductory calculus students' ways of coordinating units and their covariational reasoning.

Theoretical Background and Purpose

We believe individuals learn by organizing their experiences to fit within or to extend their existing mental *schemes* (von Glasersfeld, 1995). Schemes are researchers' constructs serving to describe their observations of others' reasoning. A scheme consists of three parts: recognition of a situation, operations (mental actions), and an expected result (von Glasersfeld, 1995). An example is the *iterative fraction scheme* (Steffe & Olive, 2010). A student who has constructed an iterative fraction scheme understands the size of an improper fraction as the result of *partitioning* and *iterating* the size of '1'. For example, they understand 9/4 as 9 copies of 1/4 of 1. If their iterative fraction scheme were *reversible*, the student also understands the amount of 1 to be the result of partitioning and iterating the size of an improper fraction: 1 is 4 copies of 1/9 of 9/4.

In general, reversible schemes are those for which the result of one sequence of mental operations is recognized as a situation for a reversed sequence of operations. Only reversible schemes may become *interiorized,* which allows the recognition of a situation and an expected result of mental activity to be experienced as synchronous. Interiorization allows the results of schemes to become part of more complex logico-mathematical *structures* (Piaget, 1970).

A units coordinating *structure* defines and regulates relationships between transformed units as possible, logically necessary, and reversible (Boyce & Norton, 2017). For example, a student with an interiorized iterative fraction scheme anticipates that there will be iteration of the amount determined by partitioning before actually carrying out the partitioning. This requires assimilation with a units coordinating structure relating the three *levels* of units (e.g., four 1/4 units within one and nine 1/4 units within 9/4, see Figure 1).

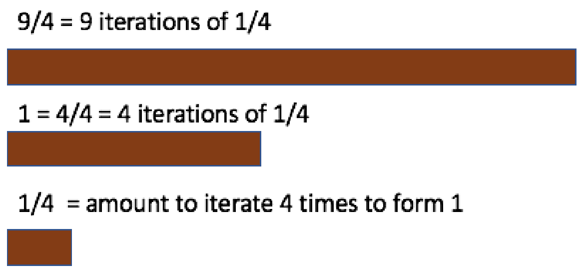


Figure 1: Example of the Form of a Three-level Units Coordinating Structure for 9/4

As part of constructing a three-level units coordinating structure, students learn to coordinate three levels of units *in activity*, which means they "build" an ephemeral third level of units as part of their way of reasoning (Ulrich, 2015). For example, a student who assimilates the task of forming the size of 9/4 given the size of '1' with a two-level structure for coordinating fractional units might form the correct size by *recursively partitioning* the 9/4 into 2 wholes plus 1 part that is one-fourth of a whole, without anticipating that the one-fourth part is one-ninth the size of 9/4 (Steffe & Olive, 2010). Through the activity of partitioning the 9/4 into 1 + 1 + ¼, they may identify the nine one-fourths.

Students who coordinate three levels of fractional units in activity can be successful at school mathematics and persist into college STEM majors (Boyce & Wyld, 2017; Byerley, 2019), although research suggests affordances for students with three levels of fractional units interiorized include abilities to reason flexibly about linear algebraic equations involving unknown values (Hackenberg & Lee, 2015). Such connections between units coordination and algebraic reasoning may explain some differences in students' success in calculus (Grabhorn, Boyce, & Byerley, 2018). In this report we focus on a different connection: relationships between calculus students' units coordination and their *covariational reasoning* (Thompson & Carlson, 2017, p. 440, see Figure 2).

Saldanha and Thompson (1998) described reasoning about covariation as "someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two" (p. 299). A multiplicative object is different from a units coordinating structure in that iteration is not necessarily involved in forming a multiplicative object. The connection to levels of units is the notion of *simultaneity*. Holding in mind an image of two magnitudes as single conceptual entity requires assimilation with a structure of two or more levels. One level involves individual quantities' magnitudes (Thompson, Carlson, Byerley, & Hatfield, 2014), and a second level is the coupling of the two magnitudes. The focus of our investigation is identifying and testing relationships between students' levels of units and their maintaining of multiplicative objects as they are reasoning about quantities that covary.

We theorize that students who assimilate with one level of units can link two quantities' magnitudes, but the act of reasoning about *changes* to either value necessitates a de-coupling—a destruction of the link and lack of a sustained image of simultaneity. We thus expect that students coordinating only two levels of units in activity are limited to pre-coordination of values (or no coordination)(See Table 1).

Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.
Coordination of values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.
Pre-coordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

Figure 2: Levels of Covariational Reasoning (Thompson & Carlson, 2017, p. 440)

Table 1: Theorized Relationships Between Levels of Units and Covariational Reasoning

Levels of Units Interiorized	Levels of Covariational Reasoning Available to Construct
3	Smooth continuous or chunky continuous covariation
2	Coordination of values or gross coordination of values
1	Precoordination of values or no coordination

We theorize that students who assimilate with two levels of units and coordinate three levels of units in activity can reason with either "coordination of values" or "gross coordination of values". Coordination of values involves anticipating the repeating of the mental actions of coupling two quantities' magnitudes to form a sequence of multiplicative objects representing a correspondence without reasoning about the changes in values. Gross coordination of values involves reasoning about the effects of changes to constituent quantities' magnitudes, but these are limited to reasoning about dichotomous changes—one quantity increasing/decreasing and another quantity increasing/decreasing—without coordinating these changes with particular magnitudes of either quantity. Thus in these two levels of covariational reasoning, there is a structural coordination of two values, but not three.

We theorize that the two higher levels of covariational reasoning, in which coordination of values and coordination of changes *are* integrated (smooth continuous covariation and chunky continuous covariation), are each in the purview of students who have interiorized three levels of units. Our purpose is to report on how conjectured relationships held up in clinical interviews of students' levels of units interiorized and levels of covariational reasoning.

Methods

In Fall 2018 the research team conducted one-on-one task-based clinical interviews (Clement, 2000) with 23 students currently enrolled in Calculus I at one of two public universities in the United States, to assess their units coordinating structures and schemes for reasoning with fractions as measures. For example, one task presented a rectangular bar labeled as 7/3, and the student was tasked to create a bar that represented the size of '1'; another task was for students to create a ruler depicting two fictional measures of distance, given that one measurement unit was 5/9 as long as the other. Each interview took approximately one hour, and participants were selected based on convenience. The interviewer's role was to clarify instructions and request elaboration of a student's reasoning, following the approach described in (Byerley, Boyce, Grabhorn, & Tyburski, 2019)

We recruited these 23 students to participate in a follow-up (one hour) interview to assess their covariational reasoning. Ten of the students participated in a follow-up interview, taking place within three weeks of their initial interview. There were three tasks used for assessing covariational reasoning, and we focus on responses to just the first task in the Results. This task (see Figure 3) is a modification of a task in which students reason about the height and volume of liquid in a jug as the liquid evaporates (Frank, 2018; Paoletti & Moore, 2017; Stalvey & Vidakovic, 2015). The other two tasks were modifications of tasks in which students reason about relationships between a bouncing balls displacement from rest, total distance travelled, and elapsed time (Thompson, 2016, p. 451), and a task in which students reason about a ladder sliding down a wall (Carlson et al., 2002, p. 371)

	Imagine this jug has been completely filled with water. It is then left indoors in a sunny window and left untouched until all the water has evaporated. Describe how the height of the water in the jug would change as the volume of the water in the jug decreases. Sketch a graph that gives the height of the liquid in the jug as a function of the volume of the water in the jug. Describe the rate at which the height of the water is changing with respect to volume.

Figure 3: Reverse Bottle Problem Task

Both sets of interviews were video-recorded. Each of the video-recordings was viewed by at least two members of the research team, who independently coded (Corbin & Strauss, 2014) the students' levels of units and levels of covariational reasoning and documented video-evidence to support their assertions. We subsequently met as a team to discuss the outcomes and reconcile our coding, revisiting videos for which there was uncertainty or disagreement. Finally, we revisited our descriptions of students' covariational reasoning for common themes in students' ways of responding to the tasks.

Results

Each of the students we interviewed were assessed as either having three levels of units interiorized or coordinating three levels of units in activity. We had expected that the students who had interiorized three levels of units would be able to reason at the levels of Chunky continuous covariation or Smooth continuous covariation, while the students coordinating three levels of units in activity would be limited to Gross coordination of values or Coordination of values. The overall results aligned with these expectations, except that several students who had interiorized three levels of units did not exhibit continuous covariational reasoning. We elaborate on interviews with five students, indicated by shading in Table 2, to exemplify the similarities and differences of students' reasoning.

Table 2: Levels of Units and Levels of Covariational Reasoning

Pseudonym	Levels of Units Interiorized	Highest Level of Covariational Reasoning
Amber	2	Gross coordination of values
Shawna	2	Gross coordination of values
Jessica	3	Gross coordination of values
Marisa	3	Gross coordination of values
Jacob	3	Coordination of values
Maude	3	Coordination of values
Rebecca	3	Coordination of values
Emily	3	Chunky continuous covariation
George	3	Smooth continuous covariation
Jack	3	Smooth continuous covariation

Amber's Reasoning (Two levels of units, Gross coordination of values)

On the Reverse Bottle Problem Task, Amber's initial graph indicated she was attending to the evaporation of the water, as she plotted her graph from right to left (starting with the point (10, 10), see Figure 4) and described that "the volume of the water would decrease as it evaporates and the height of that would also be decreasing." The interviewer asked how a linear segment of her graph represented the relationship between height and volume, and she first suggested that they were "directly proportional". The interviewer asked whether the volume of water would be the same for two different "cross-sections" of equal height (two regions directly below the top cylindrical portion). Amber decided they were not the same but said, "I don't know how I would draw that… I think I could draw that on a volume to time graph, but just comparing the volume to the height, that one's kind of hard to be able to draw."

Shawna's Reasoning (Two levels of units, Gross coordination of values)

In contrast, Shawna's graph did not indicate reasoning about time, and her graph was plotted from left to right. Like Amber, Shawna's reasoning was focused on three intervals. She explained that "as our height goes up, our volume increases, and once our height hits about the 2/3 mark our volume has a sharp increase because it is getting wider here. Then the volume decreases, because it is getting smaller at the top, so it goes back up like that." Shawna's first graph showed that the total volume in the jug is monotonically increasing. Her second graph (see Figure 4) of the same relationship shows that the volume increases and then decreases and she appeared to mean "cross sectional volume" when she said volume. She struggled to speak precisely about the quantities she was considering. She also tended to speak about one quantity at a time without reference to the other quantity, which we associate with a sequential, rather than simultaneous, coupling of the quantities. For example, she said, "The rate of the change would be increasing and then once we hit the widest part I think it would be decreasing."

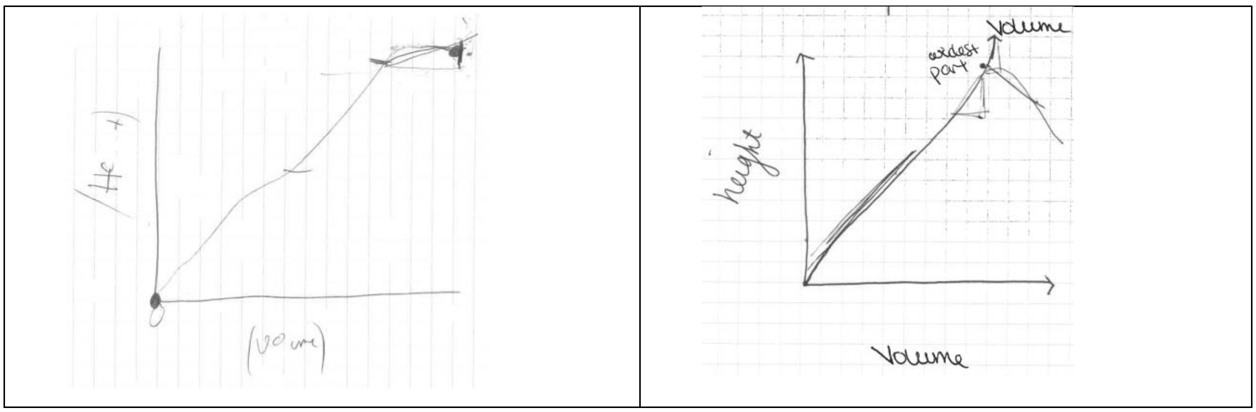


Figure 4: Amber's (left) and Shawna's (right) Graphs on the Reverse Bottle Problem Task

Marisa's Reasoning (Three Levels of Units, Gross Coordination of Values)

On the Reverse Bottle Problem Task, Marisa's responses were similar to Amber's, in that she verbalized both the height and the volume decreasing. As she was preparing to sketch a graph, she labeled the origin, the horizontal axis as volume and the vertical axis as height. She drew arrows parallel to these axes, stating "As x decreases, y decreases" before creating a concave up graph (see Figure 5). These labels and language suggest she was reasoning at the level of gross coordination of values and that values on her x-axis decreased from left to right.

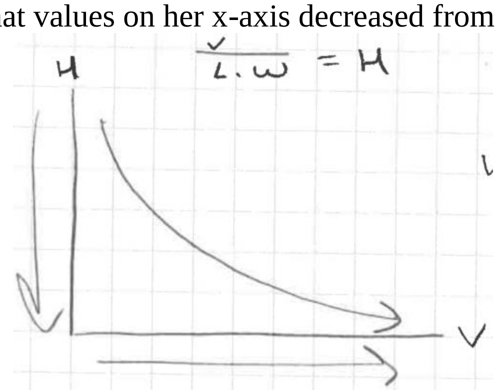


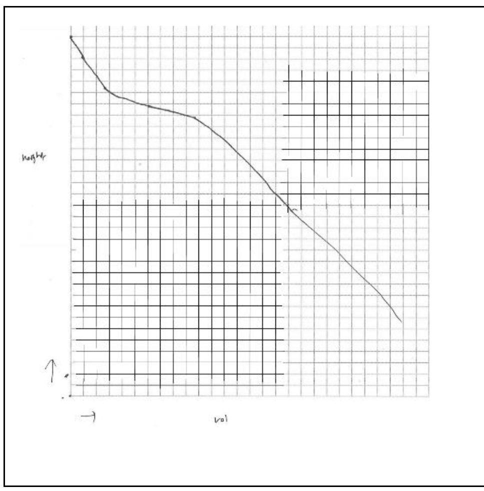
Figure 5: Marisa's Initial Graph on the Reverse Bottle Problem Task

Marisa explained that in her graph, she was attempting to represent her understanding of the meniscus (that water curves up near the walls of a container). After the interviewer clarified that they would measure the height of the water at lowest point of the meniscus, Marisa claimed the relationship between height and volume would be linear. “Every time the volume decreases, the height decreases, by a rate of 3…It’s like a ratio of 3:1, the ratio of volume to height.” Marisa estimated this ratio by noting that the bottle was taller than it was wide so the “height and volume are both decreasing but volume is decreasing at a faster rate”. She did not attend to the varying width of the bottle until pressed. When pressed she did attempt to compare changes in height and volume but found it difficult to create a graph that reflected her ideas.

Jacob’s Reasoning (Three levels of units, Coordination of values)

Jacob’s initial graph attended to three regions from left to right: a linear section of the graph corresponding with the neck of the bottle, the section “becoming wider,” followed by the section with “steady decrease” (see Figure 6). Jacob had partially disentangled time from the situation, in that he represented a relationship between height and volume (not time), but he represented a “decreasing” slope to correspond with the evaporation of water. After the interviewer asked Jacob what the uppermost point on his first graph represented, Jacob decided that since the volume was not zero at the maximum height, he should re-create his graph.

Jacob maintained reasoning about *decreasing* volume and height, suggesting “the start of time is [at the point in the upper right of his graph]” and “the end of your time is [at the bottom left]. His creation of his revised graph is evidence that he had constructed a multiplicative object relating the height of water and the volume of water in the container, coordinating values of height with values of the width of the container (associated with volume). When asked to justify the rates of change depicted in his graph, Jacob correctly explained that the same amount of water would take up less height in the wider portions of the container than at the neck. But when he was asked to justify the concave down portion of his graph, his maintained reasoning about time was problematic. He claimed the rate of change is negative because “both height and volume” are decreasing, and he did not express reasoning about “chunks” of height (or volume) outside of the neck except to compare to the neck.



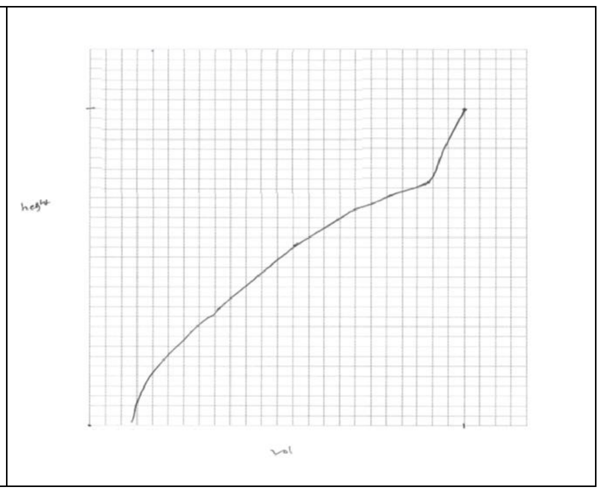


Figure 6: Jacob’s Initial (left) and Revised (right) Graphs of Height-Volume Relationship

George's Reasoning (Three levels of units, Smooth continuous covariation)

George's immediate reaction to the first task prompt was "as volume decreases, the height is also going to decrease. I don't think they are going to be linear." George indicated awareness of the directionality in the change in height and volume with respect to time (their *decrease*). However, in his subsequent activity, he focused completely on the quantities of height and volume and discussed relationships between their increases. He marked units of height on the container (see Figure 7) and represented height on the horizontal axis. To create his graph, he started by explaining that "[the point (1,1)] is specifying a volume unit, it doesn't matter what. At height two, [the region of the container from height 1 to height 2] takes more volume, so it's not going to be [at point (2,2)], it's going to be a little higher". The graph depicted in Figure 7 depicts his intermediate corrections of the right-most portion of his graph, which culminated in him representing and stating that "it should curve, and then go straight at some rate" past the height of 5.

After the interviewer prompted George to revisit the prompt, George explained that he "was just thinking of height versus volume, [he] wasn't thinking of filling or evaporating". He went on to explain that the graph could represent filling or evaporating, stating "I could read this backwards and say the same thing." He explicitly expressed reasoning with smooth continuous covariation, stating "it doesn't have to be between 0 and 1, or 1 and 2, or 2 and 3, the volume keeps increasing over every other unit. Over any equivalent chunk of height, the volume is going to be bigger than any of the previous chunks of height below it. As height goes up, the volume increase is more than the height increase".

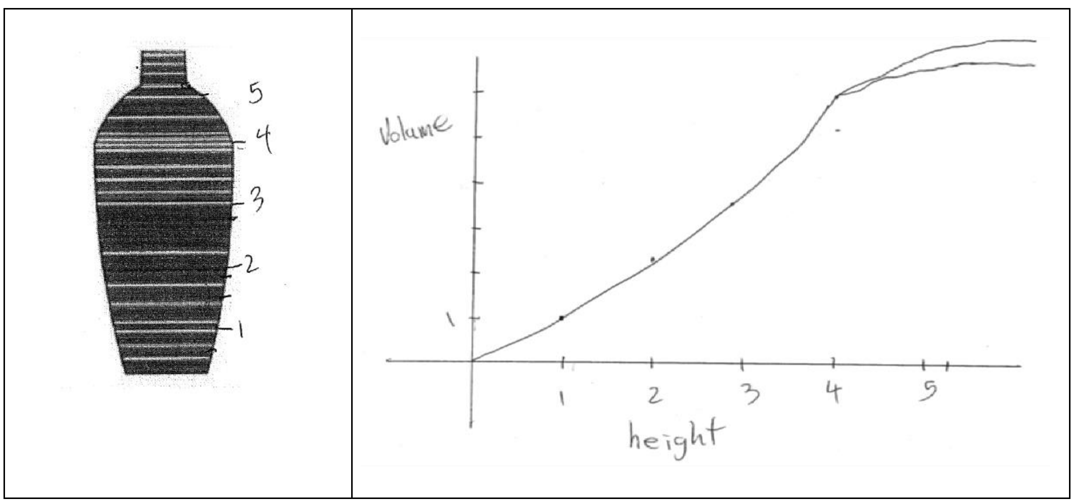


Figure 7: George's Approach to Representing the Covariation of Height and Volume

Conclusion and Discussion

Our anticipation of the relationship between students' structures for coordinating units and their ability to reason with multiplicative objects was affirmed in the sense that neither of the students who had interiorized fewer than three levels of units reasoned at the level of coordination of values or above. However, there were students who had interiorized three levels of units who did not reason above the level of gross coordination of values. Those students were able to form a multiplicative object involving static quantities, as they were able to assimilate with a (unit of (units) of units) structure in which three numerosities are coupled pairwise, but they were unable to maintain links between pairs of quantities' magnitudes associated with dynamic events.

Students with a high level of covariational reasoning were able to do the following without conflating quantities: (1) form a simultaneous coupling of changing quantities *de-coupled from time* and (2) form a simultaneous coupling of *changes* in values that is de-coupled from particular values themselves. Of the students who had interiorized three levels of units; those reasoning with continuous covariational reasoning were able to both (1) and (2); the students who reasoned with coordination of values did one or the other, and the students who reasoned with gross coordination did neither. These results add to our field's understanding of how and why students' construction of three level units coordinating structures is necessary, but insufficient, for continuous covariational reasoning.

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CONTEXTUALIZED INSTRUCTION AS A MOTIVATIONAL INTERVENTION IN COLLEGE CALCULUS

Enes Akbuga
University of Michigan
akbuga@umich.edu

Taylor Acee
Texas State University
aceet@txstate.edu

The purpose of this study is to measure the impact of an intervention on student performance expectations, utility-value, and interest in college calculus courses. Six calculus sections were selected for this study, and three were randomly assigned to take the intervention. Students in the three intervention sections completed calculus tasks that were contextualized to various STEM disciplines. The results indicated that the impact of the intervention on student motivation was not statistically significant. However, student motivation significantly changed over time. Future directions about such interventions is discussed.

Keywords: Undergraduate-level Mathematics; Calculus; Student Motivation; Integrated STEM

Calculus students often ask, "why are we learning this?" Students might not see the value or the connections between course material and their lives (Wulf, 2007; Brophy, 1999). If students are not given the opportunity to see this connection, they might become disengaged, and therefore lack the motivation and develop negative attitudes to study the subject (Clarke & Roche, 2017; Harackiewicz, Tibbetts, Canning, & Hyde, 2014).

Eccles et al. (1983) have developed an Expectancy-value model of achievement-related choices, which includes utility-value as an essential motivational variable. They defined utility-value as "how well a task relates to current and upcoming goals." Aligned with this theory, researchers designed relevance interventions to attempt to modify student motivation (Acee, Weinstein, Hoang, & Flaggs, 2018). For instance, Puruhito et al. (2011) designed an intervention using video segments to increase student motivation in college calculus. Other studies developed writing interventions to help students see the relevancy of psychology course-material (Hulleman, Godes, Hendricks, & Harackiewicz, 2010; Hulleman, & Harackiewicz, 2009; Durik & Harackiewicz, 2007). Other relevance interventions have focused on modifying course content to increase the chances that students find it. For instance, Walkington (2013) developed a context personalization intervention to customize mathematics instruction to secondary school students. Cordova and Lepper (1996) found that contextualization substantially increased student motivation in elementary schools. Overall, relatedly, the purpose of this study is to develop contextualized instruction in college calculus courses and examine its effects on motivation.

Conceptual Framework

Expectancy-value theory (1983) was the primary conceptual framework of this study, which posits that students' subjective task values and expectations of success are major proximal determinants of students' achievement-related choices and motivated behaviors. Accordingly, interest, utility-value, and performance expectations were considered as components of student motivation and chosen as outcomes for this investigation.

The intervention for this study was the implementation of contextualized calculus tasks with science, technology, and engineering applications into an introductory calculus course. The following research questions were investigated: (1) How do the contextualized calculus tasks impact utility-value, interest, and performance expectations in college calculus? (2) How do students' utility-value, interest, and performance expectations change throughout a semester in college calculus?

Modes of Inquiry

Participants

This study took place at a southwestern university in Fall 2017. At this institution, calculus is being taught in the mathematics department and is required for most STEM majors. Participants were enrolled in this introductory calculus course. Eighty-one students responded to both Survey 1 and Survey 3, and 66 students responded to all the three surveys. This data was used for the analysis. The majority of students were male and either White or Hispanic.

Research Design and Procedures

This study followed a quasi-experimental research design. The participants were enrolled in three courses taught by different instructors, each having two lab sections taught by respective teaching assistants (TA). Each lab section was randomly assigned to intervention and comparison conditions (three intervention and comparison lab sections). Therefore, each instructor had both intervention and comparison lab sections, and this helped counterbalance course instructor effects.

The intervention sections were given the contextualized tasks, and the comparison sections were given their regular calculus tasks (typical exercise problems in a textbook). The author coordinated with all the TA's to make sure the comparison groups did not get contextualized tasks. The researcher implemented the tasks in all sections twice during the semester (see Figure 1).

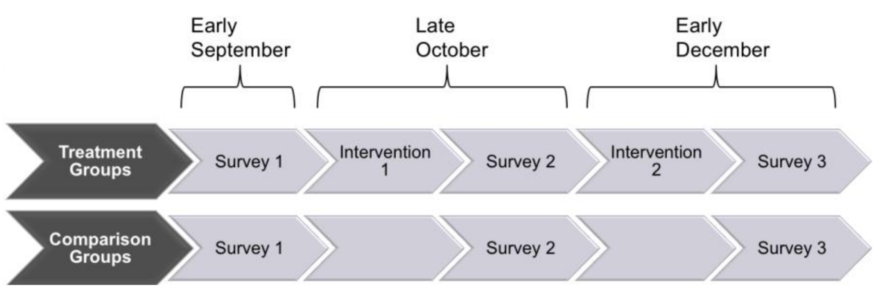

Figure 1: Timeline of the Data Collection

The students were given clear directions about the work on the tasks, and their performance was not graded. Meanwhile, the students in the comparison groups only worked on typical calculus problems with their TA's like any other day.

The data only came from the survey, and the researcher administered the surveys three times (the same survey) throughout the semester. Survey 1 was administered in the second week of the semester to measure scores. Survey 2 was administered around late October since this was the time the first contextualized tasks (Intervention 1) were given. Survey 3 was administered in early December, before the last intervention (Intervention 2) session and final exams.

Intervention

The intervention was comprised of three contextualized calculus tasks with applications to science, technology, and engineering disciplines. The first two tasks were administered to students concerned computer science, and physics (Intervention 1), and the third task concerned engineering (Intervention 2). The tasks required students to practice their calculus knowledge in the contexts of different fields. These tasks were developed as part of a more extensive collaboration of STEM professors at the institution where this study took place, and students were likely to have some of these professors in their future courses.

Instruments

A survey was used in this study to measure motivation—performance expectations, utility-value, and interest, which included twelve *Likert* items. The utility-value (three items) and interest items (six items) were adapted from Hulleman et al. (2010). The performance expectation items (three items) were adapted from Pintrich, Smith, Duncan, and Mckeachie, (1991). *Cronbach's alpha* reliability coefficients for each scale were strong (greater than .90). Exploratory factor analysis was run on the pretest items (Survey 1) to validate the dimensionality and factor loadings. *Principal axis factoring* and *oblique* rotation was used to reduce the data and allow factors to correlate. The items had high loadings on their respective factors and no problematic cross-loadings.

Results

Linear-mixed effects modeling and repeated measure analysis was used to investigate motivation. As an outcome variable for each of the models, the difference in scores of performance expectations, utility-value, and interest was considered.

Effects of Intervention

The first model was run to determine the impact of the intervention on student motivation by considering the instructor as a random effect. Results showed that the effect of the intervention on student motivation, although positive in some cases, was not significant.

Effects of Time and Intervention

Three models—performance expectations, utility-value, and interest were presented for the effects of time and intervention. Time was considered as a variable for analysis as the second research question investigates the change in motivation over time. Results showed that (Table 1) time was a significant factor for change in student performance expectations (p=.03, *eta*=.05). The student expectations keep decreasing for both groups over time (Figure 2).

Table 1: The Main and Interaction Effects of Time and Intervention on Expectations.

	Source	Type III Sum of Squares	df	Mean Square	F	Sig.
time	Sphericity Assumed	4.568	2	2.284	3.609	**.030**
	Greenhouse-Geisser	4.568	1.799	2.540	3.609	**.035**

Note: N=66, participants responded to all surveys.

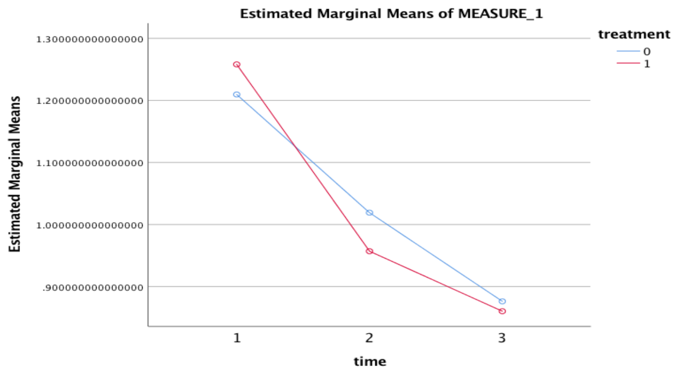

Figure 2: Change in Student Performance Expectations Over Time

When looking at utility-value scores, the impact of time itself, the interaction of time and intervention, and the change in utility-value in between the surveys were not significant. When interest scores examined (Table 2), the impact of time on student interest was significant (*p*=0.04, *eta*=.049.

Table 2: The Main and Interaction Effects of Time and Intervention on Interest

Source		Type III Sum of Squares	df	Mean Square	F	Sig.
time	Sphericity Assumed	3.498	2	1.749	3.275	**.041**
	Greenhouse-Geisser	3.498	1.915	1.826	3.275	**.043**

Note: N=66, participants responded to all surveys.

Discussion

We believe that the research community could benefit from learning about motivation interventions. Our efforts focused on measuring the extent to which contextualized instruction influenced student motivation. The first author implemented these tasks with the expectation that they would make a motivational change towards calculus.

The results showed that the intervention was not significant on student performance expectations, utility-value, and interest overall. Time itself was a significant impact on student expectations, but the interaction of time and intervention was not significant throughout the semester. This result aligns with the national study of calculus courses (Bressoud, Mesa, & Rasmussen, 2015) where they found that composite student attitude (confidence, enjoyment, and desire to persist) in calculus decreased at the end of the semester. This general tendency in calculus courses in the U.S. might explain the students' low-performance expectancy measures in this study.

One unique aspect of this study was the idea of implementing contextualized calculus tasks that were explicitly developed by potential future instructors of students in computer science, physics, and engineering fields. Although the result is consistent with Elliott et al. (2001), where the interdisciplinary approach did not show any significant effect on students, it was different from Marrongelle (2004) where they found that providing interdisciplinary tasks helped students to make connections between calculus and physics.

The study has limitations, but it should be considered as a first step towards designing such contextual instruction as relevancy intervention in college calculus courses. First, the results may not be generalized or extended to populations beyond the group of students that participated. Although the assignment of intervention and comparison was conducted at random, instructors and students were not. Second, the survey that was adapted did not undergo an in-depth validation study. Although the items that were used in the survey were based on previously validated instruments and tested for internal reliability and dimensionality in other settings, it may be limited to calculus usage. Third, the tasks were only validated concerning their content by the authors of the tasks and the researcher. Hence, they may only be appropriate for students at the institution where this study took place.

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SELF-EFFICACY FOR SELF-REGULATORY BEHAVIORS AND STUDENT ACHIEVEMENT IN AN UNDERGRADUATE CALCULUS COURSE

Kari Jurgenson
Iowa State University
karij@iastate.edu

Jeffrey Rokkum
Iowa State University
rokkum@iastate.edu

Amanda Baker
Iowa State University
ambaker@iastate.edu

Heather Bolles
Iowa State University
hbolles@iastat.edu

This study investigated if students' self-efficacy for self-regulation of learning in an undergraduate Calculus I course predicts student achievement and their successful completion of the course. Results indicated that students' self-efficacy for self-regulated learning did significantly predict their achievement in the course. Also, greater self-efficacy for self-regulation was associated with an increased likelihood of passing. The results highlight the importance of educators to teach students self-regulatory skills and build students' confidence in their ability to regulate their own learning.

Keywords: Calculus, Undergraduate-Level Mathematics, Post-Secondary Education

The growing emphasis on preparing future STEM professionals highlights the importance of supporting students' achievement in STEM courses. Thus, university faculty members are interested in ways to support students' academic success, especially first-year students (Kisantas, Winsler, & Huie, 2008) who are learning to adjust to college life. To learn and succeed in college courses, students need to self-regulate their learning (Cohen, 2012) which refers to the "degree to which students are metacognitively, motivationally, and behaviorally active participants in their own learning process" (Zimmerman, 1989, p. 329). Self-regulated learners control their cognition, motivation, emotions, and actions to enhance their own learning process (Pintrich, 2004) by using strategies such as setting goals and progress monitoring (Cohen, 2012).

The National Council for Teachers of Mathematics (2000) advocates that students be active participants in solving mathematical tasks. This active participation involves learning strategies that support self-regulation. However, simply knowing self-regulatory strategies is not enough to ensure their effective use; students also must believe that they can use them effectively, known as self-efficacy for self-regulated learning (SRL) (Usher & Pajares, 2008). Self-efficacy for SRL influences the academic success of children, adolescents, and adults (Bandura, 1986; Klassen, 2010; Zimmerman, 1998, 2002). Self-efficacy for self-regulation raises academic goals, the standards for quality work, and beliefs in one's ability to succeed academically after controlling for instructional level, previous academic performance, and relevant aptitude (Zimmerman & Bandura, 1994; Zimmerman, Bandura, & Martinez-Pons, 1992). Similar to previous studies, we controlled for students' prior knowledge, as measured by the Calculus Concept Inventory at the beginning of the semester in our analyses.

Although many studies investigate students' SRL, relatively few studies investigate students' *self-efficacy for* SRL. Among these studies, we are unaware of any that situate this research in a college mathematics course. The purpose of this study is to examine how students' self-efficacy for SRL in an undergraduate Calculus I course impacts student achievement and their successful completion of the course. The following research questions guided this study.

1. Does self-efficacy for self-regulated learning at the start of the semester predict end of course grades among undergraduate students enrolled in Calculus I, controlling for gender, race, and prior Calculus knowledge?
2. Does self-efficacy for self-regulated learning at the start of the semester predict the likelihood of passing Calculus I, controlling for gender, race, and prior Calculus knowledge?
3. Is there a difference in students' self-efficacy for self-regulation of learning when they begin Calculus I between students who passed the course, students who failed and course, and students who dropped/withdrew from the course?

Theoretical Framework

Students' confidence and beliefs about their ability to self-regulate their learning are key factors in whether they employ self-regulatory strategies (Zimmerman & Cleary, 2006). Thus, one's self-efficacy for SRL is an important predictor of the person's use of self-regulatory skills and strategies (Bandura, Barbaranelli, Caprara, & Pastorelli, 1996, 2001; Bandura, Caprara, Barbaranelli, Gerbino, & Pastorelli, 2003; Bong, 2001; Zimmerman & Bandura, 1994; Zimmerman, Bandura, & Martinez-Pons, 1992) and is related to motivation and achievement in a variety of academic areas and levels of schooling (Bandura, 1997; Pajares, 2007). This emphasizes the difference between having self-regulatory skills and using them regularly. A belief in one's self-efficacy for SRL will support the consistent application of self-regulatory skills even in the face of difficulties, stressors, and distractions (Caprara et al., 2008).

Methods

Participants and Procedure

Within the first two weeks of the semester, students enrolled in a Calculus I course at a large, Midwestern public university (1,758 students) were invited to participate in a survey measuring their attitudes and expectations toward Calculus; 34.3% (n = 606) responded. Of those who responded, 499 students completed all the self-efficacy for self-regulation items on the survey, for a completion rate of 28.4%. Consistent with enrollment patterns for the course, the majority of the students in the sample were in their first year of college (79.4%, n = 396) and were enrolled in a major within the College of Engineering (72.5%, n = 362) at the start of the study. The next most common majors of participants were Computer Science (7.0%, n = 35), Undeclared (4.0%, n = 20), and Biochemistry (2.6%, n = 13). Of the 488 participants who answered the question about their gender, 76.2% were male. Of the 474 participants who reported their race, 81% were white, 6% Asian, 7% Hispanic, and 6% Arabic/Middle-Eastern, Black, Native American, Hawaiian/Pacific Islander, and multiethnic. Of the 492 participants who answered the question about their prior calculus experiences in high school, 39.4% reported that they had not taken calculus in high school. Informed consent was received from all participants.

Materials

Self-efficacy for self-regulation. The scale for measuring self-efficacy of SRL was designed by two of the authors of this paper following the recommendations put forward by Bandura (2006). A total of 7 tasks were specified which represented skills demonstrating their ability to manage their own learning process in Calculus (self-efficacy for SRL in mathematics), and respondents were asked about their confidence for each skill. The skills inquired about represent the abilities of the students to learn mathematics on their own. All items began with the stem "How confident are you in your ability to" and was followed by statements for each item including "learn how to solve a problem by watching my professor solve it," "use the resources available to me (e.g., textbook, online videos, office hours) to learn new material outside of class," and "seek help from my professor and/or teaching assistants when I need it." The responses options to these Likert-scale items ranged from 1 (not at all confident) to 6 (completely confident). A mean score of the respondents' answers to these questions was then created to create one self-efficacy for SRL score which was used in the analyses.

End of course grades. End-of-course letter grades for the Calculus I course were received from the University Registrar for all participants which were then converted to a traditional grade point average scale ranging from 0 - 4.0. To answer the second and third research question, three groups of students were formed: (a) students who earned a passing grade (grade of C- or higher), (b) students who did not earn a passing grade (grade of D+ or lower), and (c) students who formally withdrew or dropped the course.

Covariates. The covariates included in this study are the Calculus Concept Inventory (CCI) developed by Epstein (2006; 2007), gender, and race. Participants completed the CCI, a test of conceptual understanding of basic principles of Calculus, at the beginning of the semester and was used to assess students prior Calculus knowledge and understanding. Information on gender and race was received directly from the University Registrar.

Results

To answer the first research question, we used linear regression to see if we could predict the students' end of course grade from their self-efficacy for self-regulated learning scores (Table 1). Students who dropped or withdrew from the course were excluded from this analysis. The regression model statistically significantly predicted end of course grades, $F(4, 407) = 32.99$, $p < .001$, adj. $R^2 = .245$, and conceptual understanding of calculus, self-efficacy for SRL, gender, and race explained 23.7% of the variance in end of semester course grades. The students' self-efficacy for SRL at the start of the semester significantly predicted their end of semester Calculus I grade, $b = .245$, $t(407) = 5.62$, $p < .001$, as did their conceptual understanding of Calculus, $b = .396$, $t(407) = 8.82$, $p < .001$. Students' gender also significantly predicted their end of semester Calculus I grade, $b = 0.163$, $t\ (407) = 3.656$, $p < 0.001$.

Table 1: Results of Regression Analyses Predicting End of Course Grades From Their Self-Efficacy

	t	*p*	B	F	*df*	*p*	Adj. R^2
Overall Model				32.99	4, 407	<.001	.237
Constant	-0.144	.886					
Gender	3.656	<.001	.163				
Race	-1.881	.061	-.082				
CCI	8.821	<.001	.396				
Efficacy	5.616	<.001	.245				

For the second research question, a logistic regression was used to ascertain the effects of gender, ethnicity, the students' conceptual understanding of calculus, and their self-efficacy for SRL on the likelihood that participants pass or fail their Calculus I class (Table 2). Students who dropped or withdrew from the course were excluded from this analysis. The logistic regression model was statistically significant, $\chi^2(4) = 55.419$, $p < .001$. The model explained 22.7%

(Nagelkerke R^2) of the variance in pass/fail rate and correctly classified 80.5% of cases. Greater self-efficacy for SRL was associated with an increased likelihood of passing just like a higher conceptual understanding of calculus was associated with an increased likelihood of passing.

Table 2: Results of Logistic Regression Analyses of the Likelihood to Pass Calculus I

	B	S.E.	Wald	*p*
Gender	.673	.377	3.190	.074
Race	-.081	.121	.449	.503
CCI	.289	.055	27.445	<.001
Efficacy	.805	.212	14.447	<.001
Constant	-4.048	1.128	12.883	<.001

For the third research question, an ANOVA was performed to test if there were differences between the students who dropped or withdrew from the course, those who failed, and those who passed at the start of the semester for their self-efficacy for SRL (Table 3). Self-efficacy for SRL was statistically different between different the three groups, $F(2, 496) = 17.710$, $p < .001$. Scheffe post hoc analysis revealed a significant difference in the self-efficacy for SRL scores between students who passed the course (N = 380, $M = 4.79$, $SD = 0.65$) and both the students that dropped out of or withdrew from the course (N = 55, $M = 4.34$, $SD = 0.70$, $p < .001$) those who failed the course (N = 64, $M = 4.38$, $SD = 0.87$, $p < .001$). There was no significant difference between the students who dropped/withdrew from the course and those who failed.

Table 3: Analysis of Variance of Self-Efficacy in Self-Regulation and Completion of Calculus I

	SS	*df*	*MS*	*F*	*p*
Between Groups	16.826	2	8.413	17.710	<.001
Within Groups	235.626	496	.475		
Total	252.452	598			

Discussion

This study found that self-efficacy for SRL significantly predicts students' end of semester grade in Calculus I and greater confidence in one's ability to regulate his/her learning is associated with a greater likelihood they pass Calculus I are consistent with previous research suggesting that confidence in SRL contributed to students' academic achievement (Bandura, 1986; Caprara et al., 2008; Zimmerman, 2002). The third result indicates that students who passed Calculus I had more confidence in regulating their own learning at the beginning of the course than those who did not pass or chose to drop/withdraw from the course. We suggest that future research investigates what distinguishes students who pass Calculus I from other students in the course who may be struggling. This would help instructors focus on increasing students' self-efficacy for self-regulatory skills. K-12 educators may want to focus on teaching SRL to equip students for college mathematics. Similarly, college instructors may want to support students' confidence in SRL in Calculus courses. These findings are important for educators teaching freshman-dominated mathematics courses, such as Calculus I, as they must prepare their students to study effectively on their own and encourage their confidence in doing so.

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RE-HUMANIZING ASSESSMENTS IN UNIVERSITY CALCULUS II COURSES

Kelly MacArthur
University of Utah
macarthur@math.utah.edu

Conceptual Perspective

Recent work aiming to heighten equity in math education has focused on "rehumanizing mathematics" (Gutiérrez, 2018). One way to interpret this is as a call to action in implementing humane teaching and testing practices in the math classroom in order to better serve students. One high leverage way to enact this change is to focus on assessment, given the high-stakes nature of grades in college calculus courses.

Research Question & Design

In this study, I explore the question: How does a rehumanizing-math approach to assessment in large calculus 2 classes impact exam scores, student learning and students' attitudes toward calculus?

I changed the structure of all four exams, without altering the content, in two university calculus 2 courses, with about 165 students total. My goal was to decrease anxiety on exams, increase confidence and build community in the class. To work toward achieving that goal, I included a group discussion piece of each exam. To investigate the impact, I examined exam grades, final course grades, attitudinal survey data, and focus group data. As a control group, I compared data from the changed course with roughly the same number of students from a semester prior to the implemented changes.

Data Analysis & Results

The grade averages and standard deviation on the first exam were about the same for the control group and the comparison group. However, from the second exam on, the grade averages for the comparison group were significantly higher than for the control group, while the standard deviations decreased dramatically. By the final exam, the test average for the comparison group was 91% compared to 82.5% for the control group, with standard deviations of 9.6% and 16.5% respectively.

For the survey and focus group data, qualitative analysis identified several emerging themes, including: (a) less anxiety and higher confidence experienced during exams and (b) a community feeling in the classroom, in the rehumanized classroom. Furthermore, data highlighted student views on the importance of rehumanizing math. For instance, when asked what does rehumanizing math mean, one student wrote: "I think teachers/educators can re-humanize math by *making better learning environments* in class, and by *learning how to connect better with people*." (italics added)

Significance

My findings illustrate that this new exam structure helped create precisely the type of classroom where students felt a sense of belonging. I answered the theoretical call to rehumanize the classroom by boots-on-the-ground changes in assessments that produced positive, promising results. This structure engendered a kind, humane classroom, where students flourished doing mathematics.

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COMPOSING THE COMPOSITE FUNCTIONS: EXAMINING STUDENTS' MEANINGS FOR FUNCTION COMPOSITION

Kristin M. Frank
Towson University
kfrank@towson.edu

Rachael M. Talbert
Towson University
rtalbe1@students.towson.edu

Keywords: Algebra and Algebraic Thinking, High School Education, Curriculum Analysis

Piaget (2001) distinguishes between students' actions and the products, or consequences, of those actions in order to characterize the nature of the learner's reasoning. For example, in the context of function composition one can engage in the action of stringing together two functions, f and g, so that the output of the of the function g is taken as the input of the function f. This activity highlights the two-step nature of composing two functions. The product of these actions can be conceptualized as the composite function h, where h(x) = f(g(x)).

While much of the function composition literature focuses on the difficulties students have engaging in function composition tasks (e.g., Bowling, 2009; Engelke, Oehrtman, and Carlson, 2005; Jojo, 2011), in this study we sought to understand the extent to which students' meanings for function composition were limited to thinking about the activity of stringing together two functions as opposed to thinking about the product of those actions, the composite function. In this poster we present preliminary results of a textbook analysis and clinical interviews with undergraduate students.

The first phase of this study consisted of a textbook analysis of 8 textbook (Algebra 2 (n=2), Precalculus (n=3), and Calculus (n=3)). The research team classified the meanings of function composition supported by the examples and homework problems in the book. During the first round of open coding (Strauss and Corbin, 1990) we noticed that while all presentation types (e.g., graphs, tables, and algebraic rules) were used in each book, 25% of the problems focused on students' ability to create the rule for $f \circ g$ given the algebraic rules for f and g and another 28% asked students to determine the value of $(f \circ g)(a)$ given representations of f and g. In other words, the textbook problems focused on the activity of composing two functions.

In the second phase of the study we asked 70 university calculus students to complete a written instrument consisting of six function composition tasks. Preliminary results reveal that calculus students experience little difficulty composing two functions but are less likely to conceptualize the product of their activity as a composite function. For example, after producing the algebraic rule for s(x) = f(g(x)), 65% of students evaluated s(2) by first computing g(2) and then f(g(2)). These students did not use the function rule they constructed for s to evaluate s(2). Interviews with these students reveal that these students did not conceptualize the composite function as a single mapping from x to f(g(x)) and thus did not anticipate that the function s could be defined by a single function rule. Additionally all students experienced difficulty when asked to graph $f \circ g$ given the graphs of f and g with 81% leaving the question blank with some noting they "do not remember how to combine graphs" and others stating the task was impossible because you need to know the algebraic rule for both g and f in order to determine f(g(x)).

Preliminary results reveal that it is nontrivial for students to construct the composite function from their activity composing two functions. We anticipate that students' focus on compositing functions and their limited engagement with the composite function likely impact their understanding of the chain rule and implicit differentiation although further research is needed to verify this conjecture.

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THE REALITY OF COMMON GRADING: ONE ASPECT OF A CALCULUS I COORDINATION EFFORT

Christopher Cornwell	Kristin Frank	Nathan McNew
Towson University	Towson University	Towson University
ccornwell@towson.edu	kfrank@towson.edu	nmcnew@towson.edu

Keywords: Teacher Knowledge, Assessment and Evaluation, Teacher Beliefs, Calculus

As Rasmussen and Ellis (2015) report there is tremendous variation in who is teaching calculus at any given university including tenure track faculty, other full time faculty, and part time adjuncts. As a result, there is variation in the content being taught. And, it seems likely, that an "A" means something different in each section. Thus, in Fall 2018 our mathematics department decided to coordinate Calculus 1 across all twelve sections. We used Rasmussen and Ellis's finding - successful calculus programs at PhD-granting universities had a well-established system for coordinating Calculus I that included a common syllabus, pacing, homework, midterms, and final exams - to inform our university's coordination efforts.

Our coordination included common midterms and common rubrics for grading each exam. The three midterms (limits, derivative rules, applications of derivatives) were co-developed by the twelve full-time faculty members teaching calculus 1 in Fall 2018. The rubrics, on the other hand, were developed by the course-coordinators. After the first exam, the coordinators asked the twelve faculty members to participate in a mock grading where each faculty member would grade the same exam using the assigned rubric. The exam used for the mock grading was developed by a course coordinator based on common student misconceptions documented in the literature as well as her own students' work on the exam.

Ten of the twelve faculty members participated in the mock grading. Faculty grades for the exact same exam ranged from a 53 to an 85. Framed differently, depending on the section that student was enrolled in, the student's work could have earned him/her anywhere from an F to a B on the same exam. In this poster we share the breakdown of the instructors' grades by exam question and focus on aspects of the question and rubric that might have contributed to the range of grades produced by the faculty.

In addition to producing a quantitative metric for comparing instructors' grading practices, this mock grading activity encouraged the mathematics faculty to engage in conversations about students' learning and assessment. While these conversations are often typical amongst mathematics education faculty, the mathematicians involved in the study commented that this was a novel, and enlightening, experience for them. For example, faculty realized that some faculty grade by individual item so that the student's score is the sum of the points they earn on each question. Other faculty believe that the student's score should reflect his/her understanding of the material. Thus, one faculty was perturbed that the student seemed to exhibit "B- level understanding but the grade would be about 65". In addition to reflecting this particular instructor's beliefs about grading, this comment revealed a discrepancy between the midterm/rubric and the instructor's belief about what B level understandings of calculus entail. In the poster we highlight other instructors' beliefs about grading that emerged during this study.

Finally, we conclude by sharing what we learned about creating common rubrics both their importance in the coordinated calculus experience as well as the challenges in designing exams and rubrics that appropriately assess students' mathematical understandings.

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STUDENTS' EVOKED CONCEPT IMAGES OF QUOTIENT RINGS

Valentin A. B. Küchle
Michigan State University
kuechlev@msu.edu

Keywords: Advanced Mathematical Thinking, Undergraduate-Level Mathematics

Over 70% of mathematics majors at U.S. universities are required to take a course in abstract algebra (Blair, Kirkman, & Maxwell, 2018). In fact, between 2014 and 2016, abstract algebra was the most offered upper-division course, running at least biennially, in U.S. mathematics departments (Blair et al., 2018). Although the content covered in abstract algebra courses may vary from institution to institution, the Mathematical Association of America's (MAA) Committee on the Undergraduate Program in Mathematics (CUPM) has published sample syllabi outlining a number of prototypical abstract algebra courses (Isaacs, Bahls, Judson, Pollatsek, & White, 2015). In all of these syllabi, the concepts of quotient groups or rings are present.

Although students' difficulties with quotient groups have been well-documented over the last 30 years (e.g., Asiala, Dubinsky, Mathews, Morics, & Oktaç, 1997; Dubinsky, Dautermann, Leron, & Zazkis, 1994; Larsen & Lockwood, 2013), the study of students' understanding of ring theory—including quotient rings—has been little researched (Cook, 2018). I seek to contribute by reporting on students' evoked concept images (Tall & Vinner, 1981) of quotient rings, as well as the preceding concepts of rings, subrings, ideals, and cosets to answer the question: What are students' evoked concept images of rings, subrings, ideals, cosets, and quotient rings?

In January 2019, task-based interviews were conducted with six students who had completed their first course in abstract algebra, covering ring theory, in Fall 2018 at a large Midwestern public university. Students were granted access to their notes, a compiled set of definitions from their textbook, as well as their textbook to mitigate against memory loss due to the 3-week winter holidays. The interviews were recorded with a Livescribe pen and transcribed.

The task-based interviews are in the process of being coded using a two-cycle coding approach. The first cycle of coding is done using provisional coding (Miles & Huberman, 1994, p. 58) which begins with the researcher "creating a provisional 'start list' of codes prior to fieldwork" (p. 58). My starting list builds on the literature about students' learning of group theory in the expectation that many similarities will exist. To account for differences, I am also building up codes inductively to account for ring-specific findings. All ideas are pulled together in my second cycle of coding, pattern coding, which "identifies an emergent theme, configuration, or explanation" (Miles & Huberman, 1994, p. 69). Pattern Coding allows me to condense my data into manageable units while finding themes and explanations in my data that my previous codes may not have picked up on. Thus, I hope to end up with categories of students' evoked concept images of rings, subrings, cosets, ideals, and quotient rings.

I anticipate that this study will mirror some of the findings from the literature on students' understanding of group theory. For instance, I expect that: (a) students may struggle with the coset notation, (b) confusion around binary operations of rings, subrings, and quotient rings will arise, and (c) that context-dependency of understanding remains an issue. On the other hand, I expect some findings unique to ring theory as students are faced with two binary operations instead of just one. More specifically, I expect that the switch from normal groups to ideals (as the objects by which students "quotient") will pose a challenge to students.

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Chapter 11:
Pre-Service Teacher Education

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

AN INTERNATIONAL STUDY OF PROSPECTIVE SECONDARY TEACHERS' NOTICING OF STUDENT THINKING

Kristin Lesseig
Washington State University

Gregory Hine
University of Notre Dame Australia

Professional noticing of student thinking is considered a useful construct to investigate instructional decisions and activity made by teachers to support learning. This paper highlights the findings from one component of a multi-university (United States & Australia) research project focused on improving pre-service secondary mathematics teachers' (PSMTs) abilities to notice student thinking. The pre-post video component was specifically designed to document PSMTs' abilities to attend, interpret, and respond to student thinking. Results indicate that despite an improvement in ability to attend to and interpret student thinking, there was relatively no improvement in PSMTs' responding scores. This finding, which contradicts Monson et al.'s (2019) study, raises important questions about context and potential intervening factors to which researchers should attend when designing and implementing noticing interventions.

Keywords: Preservice teacher education, Instructional activities and practices, Teacher knowledge.

Introduction

In light of decades of research on how students learn mathematics (e.g., NRC, 2001), a more active vision of mathematics teaching, one that foregrounds student understanding and fosters student participation in the learning process, has emerged (NCTM, 2014). Teaching that uses student thinking as the basis for instructional decision-making, also known as responsive teaching, places significant demands on the teacher. Specifically, teachers need to attend to the substance of student ideas, recognize the disciplinary import of these ideas, and make intentional decisions about how to pursue those ideas (Robertson, Adkins, Levin, & Richards, 2016).

Skills such as interpreting and assessing the mathematical potential in student contributions, necessary prerequisites to responsive teaching, are especially difficult for prospective secondary mathematics teachers (PSMTs). PSMTs are understandably at a very different place than practicing teachers along the continuum of learning to teach (Feiman-Nemser, 2012) and often have had very limited opportunities to analyze student work in their teacher preparation programs (Jenset, Klette, & Hammerness, 2018; Simpson & Haltiwanger, 2017). However, research has demonstrated that PSMTs' abilities to attend to, interpret, and respond to student thinking can be developed through intentional, structured opportunities to engage in this work.

To that end, we sought to replicate a successful intervention, an Interview Module designed for secondary methods courses in the United States (Monson, Krupa, Lesseig & Casey, 2019), in an Australian context. The Australian context was deemed appropriate for two reasons: first, the topic of quadratics (the content central to the Interview Module) is taught to similar-aged students in both Australia and the United States; and second, the teacher education programs at the researchers' universities have similar structures and goals. Before detailing our research study, we briefly review the theoretical underpinnings of professional noticing of student thinking—the competency this intervention is designed to assess and develop—and discuss related empirical research. In this review, we highlight the goals and successes of intervention studies as well as the challenges revealed in research with PSMTs.

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

Theoretical Perspectives and Related Research

Noticing has proven to be a useful construct to investigate instructional moves and decisions teachers make in relation to student thinking (Schack, Fischer, & Wilhelm, 2017; Sherin, Jacobs, & Philipp, 2011). Teacher noticing builds on Goodwin's (1994) definition of professional vision as "socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (p. 606). Subsequently, teachers' professional vision, or noticing, refers to the ways in which teachers are attuned to 'notice' significant events in a teaching situation and engage in principled reasoning when interpreting and reacting to those events (Sherin et al., 2011). With roots in both cognitive and situative perspectives, the construct of teacher noticing has potential to bridge the research divide between teacher dispositions and teaching performance (Scheiner, 2016). In particular, noticing research, focused on situation-specific skills of perception, interpretation, and decision-making, helps reveal complex interactions among teacher knowledge and actions (Sánchez-Matamoros, Fernández, & Llinares, 2018; Thomas, Jong, Fisher, & Schack, 2017).

Our work is situated in a particular aspect of noticing referred to as *professional noticing of student thinking* (Jacobs, Lamb, & Philipp, 2010). Professional noticing of student thinking [hereafter professional noticing] consists of three interrelated practices of (1) *attending* to the mathematics evidenced in student work (writing, verbal communication, or actions); (2) *interpreting* what students understand and/or do not understand based on their mathematical work; and (3) *deciding* how to respond in light of this interpretation.

While the characteristics of expert noticing are still evolving, there is some agreement on the nature of noticing necessary for responsive teaching. In particular, researchers have highlighted the importance of attending to salient mathematical details in a student's approach as well as the ability to connect those details to key mathematical understandings (Jacobs et al., 2010; Sánchez-Matamoros et al., 2018). On the contrary, more novice noticing is characterized by general statements, descriptions, or evaluations that are not tied to specific evidence from students (Jacobs et al., 2010; Mason, 2011). Research has demonstrated that professional noticing can be enhanced through collaborative, structured opportunities to interact with students' mathematical thinking through video (Ding & Domínquez, 2015; Schack et al., 2013), one-on-one student interviews (Monson et al., 2019; Fernandez, 2012) or samples of students' written work (Simpson & Haltiwanger, 2017; Son, 2013). While the majority of intervention studies report positive gains in the noticing abilities of prospective teachers, the results also reveal inconsistent improvements and highlight the complex interaction of noticing with other factors (e.g., teacher knowledge, the mathematical content of the task).

For example, prospective teachers have difficulty distinguishing between procedural and conceptual errors (Ding & Domínquez, 2015) or recognizing evidence of conceptual understanding (Bartell et al., 2013). Studies by Son (2013) and Son and Sinclair (2010) document difficulties prospective teachers had responding to students when conceptual errors were present, with most offering procedural explanations and other teacher-centered responses that did not aim to advance conceptual understanding. Content knowledge was insufficient. This was also evident in Son and Crespo's (2009) study in which PSMTs who were able to unpack the mathematics behind a student's non-traditional approach (i.e. those with strong content knowledge) were actually more likely to offer teacher-centered responses than their peers. In sum, studies consistently demonstrate that responding is the most difficult component of noticing to develop. The one exception is Monson and colleagues' (2019) study in which activities explicitly targeting PSMTs' ability to respond were added to an existing Interview Module.

Purpose of the Research

In this paper we present findings from a multi-university research project focused on improving PSMTs' abilities to notice student thinking within the context of secondary mathematics teaching and learning. The pre-post video component, on which this paper reports, was designed to specifically document PSMTs' abilities to attend, interpret and responding to student thinking. The overarching research question guiding the study was: *What, if any, improvements in attending to, interpreting, and responding to student thinking do PSMTs' demonstrate after completing the Interview Module?*

Research Design

The research is based on an Interview Module, which was developed by one of the authors and colleagues and has been used within the United States to demonstrate gains in PSMTs' noticing abilities. The Interview Module is comprised of a pre-post video assessment, prescribed readings, a one-on-one interview with a secondary student, and a sequence of responding assignments involving analysis of student work samples (Monson et al., 2019).

Participants and Context

Data for this paper come from the pre-post video assessment given to PSMTs enrolled in secondary mathematics teacher preparation programs in two countries. Nine PSMTs completed the assessment, with 6 PSMTs enrolled at one Australian university and 3 PSMTs at one university in the United States. The Australian participants were in their first or second year of their preparation program. Two of the US participants were in their second year of an undergraduate preparation program, though one was returning to university after working in a math-related field. The third had earned a BS in mathematics from a different university and was in her first semester of a 15-month Masters in Teaching program.

Pre-post video assessment. The researchers collected pre-post assessment data, which were based on PSMTs' written responses to short videos shown prior to and upon completion of the full Interview Module. These videos show a Mathematics Teacher Educator (MTE) conducting a task-based interview with a secondary student. In each video, the student is asked to solve two quadratic equations (one resulting in one real solution, the other with two imaginary solutions) which mirrors the student interview PSMTs conducted that focused on solving linear equations. The problems and prompts given to the student are provided in Table 1.

Table 1: Pre-post Video Assessment Problems and Prompts

Question 1:	Solve for x: $x^2-4x+4=0$
Probe:	Could you solve that another way?
Question 2:	Solve for x: $x^2-2x+3=0$
Probe:	Could you solve that another way?

After watching the pre- and post-videos, the PSMTs were asked to independently respond in writing to three prompts, each corresponding to one component in the noticing framework: (1) What do you notice? (Attending) (2) How would you describe what this student understands? (Interpreting) and (3) Describe some ways you might respond to this student and explain why you chose those responses (Responding).

Analysis and Coding

Both researchers used a coding scheme adapted from the module developers (Krupa, Huey, Lesseig, Casey & Monson, 2017), and independently coded PSMTs' ability to attend to, interpret, and respond to student thinking on the pre- and post-assessments. Each question was coded as demonstrating either no evidence (0), limited ability (1), or emerging ability (2). Initially, each researcher coded responses (pre- & post-) from three participants before meeting virtually to discuss any discrepancies in coding. After resolving all coding differences and reaching consensus, the researchers coded the responses from the remaining six participants before meeting virtually again to agree on a consensus score for each response. Reliability in coding was enhanced by the creation and maintenance of an operative codebook with examples and non-examples of responses at each level (Miles, Huberman, & Saldaña, 2013), and any discrepancy was discussed in reference to the codebook until consensus was reached.

Findings

The pre-post video assessment data provided the researchers with an opportunity to look for growth in PSMTs' attending, interpreting and responding skills. While in the larger study these data were compared with other components of the Interview Module, this paper outlines changes in PSMTs' skills as evidenced in the video assessment only. The scores from pre-post video assessment data are presented in Table 2 (all participant names are pseudonyms). The scores represent the following: 0 = No evidence, 1 = Limited Ability, 2 = Emerging Ability. Tables 3, 4 and 5 outline verbatim participant responses according to *attending*, *interpreting* and *responding* skills, respectively, together with the video phase (pre-/post-) and the consensus score (0, 1, 2) given to the response by the researchers. While not necessarily meant as exemplars, these responses have been included to indicate a qualitative range and illustrate our coding levels.

Table 2: Pre-post Video Scores for Attending, Interpreting and Responding

	Attending		Interpreting		Responding	
	Pre	Post	Pre	Post	Pre	Post
Andi (US)	2	2	2	2	2	2
Brook (US)	1	2	1	1	2	2
Rachel (US)	1	2	1	2	2	2
Benedict (AUS)	0	2	0	1	0	1
Georgia (AUS)	1	1	1	1	0	0
Jai (AUS)	1	2	1	2	1	1
Emmeline (AUS)	1	2	2	2	2	2
Steven (AUS)	0	1	0	0	1	0*
Owen (AUS)	1	2	1	2	1	2

Key: * = score decreased from pre- to post; score with gray shading (e.g. 2) = increased; no shading = no change.

Changes in PSMTs' Noticing in Pre-Post Video Assessment

Attending. With the exception of two participants (Andi & Georgia, who remained at the *limited ability* level for pre- and post-), all participants improved their pre-video attending score by at least 1 point in the post-video assessment. Participant Benedict demonstrated the greatest growth in the attending component, scoring at the *no evidence* level in the pre-video assessment and *emerging ability* in the post-video assessment. Benedict attended to the student's affective dispositions in the pre-video assessment (see Table 3), yet documented in detail a number of mathematical terms used by the student in the post-video assessment (e.g., "...knew of two ways to solve a quadratic, factor and formula. Showed/said her understanding of graphing").

Table 3: Examples of Participants' Attending Responses

Participant	Response
Benedict (Pre-video, 0)	Girl always waited slightly to get confirmation about her process when she was uncertain; tried to logically solve it and talked through the process.
Georgia (Post-video, 1)	The student was checking that her factorization was correct. [She] simplified the answer as much as possible without a calculator. The student could think about other ways to answer the question.
Brook (Post-video, 2)	The student correctly factored and solved the first problem. She was also able to sketch a general graph of the equation. Moreover, she realized graphing was an alternative method of solving the equation when asked for another method. The second equation she factored, but in checking herself realized her factoring was not correct. She knew she needed to use the quadratic formula to solve the equation. However, she could not graph the equation. Additionally, the interviewer prompted her to correct a negative sign.

Interpreting. Five of nine participants demonstrated growth in the interpreting component, with the remaining four participants maintaining the same score from pre-video to post-video assessments. Only two participants scored at the *emerging ability* level during the pre-video stage, with this number increasing to five after the post-video assessment. In the pre-video assessment, Rachel only wrote a list of things the student did incorrectly, (e.g. "...the student cancelled a 2 wrong from Problem 3 and she didn't know how to simplify or solve for a negative square root...", scoring a 1 for this response. Rachel's post-video interpreting response was scored as a 2 due to the list of correct and incorrect things the student did, coupled with various statements about what the student understands supported with evidence, (e.g. "...she knows the graphs for parabolas with real solutions, but she needs some help with understanding imaginary solutions and their graphs").

Table 4: Examples of Participants' Interpreting Responses

Participant	Response
Steven (Pre-video, 0)	Jumps straight into the first question. Understood how to solve the 1st question but got the graph wrong when asked what it was (sic) looked like.
Brook (Post-video, 1)	She has a very good grasp of the material. When graphing the first problem, she states the graph would contain two zeros. However, when she sketches the graph, she draws it correctly with only one zero. She solved the 2nd problem correctly using the quadratic equation. She seemed to understand how to graph it but was not immediately able to estimate $2\pm\sqrt{2}$ to pick values for the x-axis.
Emmeline (Pre-video, 2)	Understands factors of factorisation. Perfect square (notion of x^2 and $\sqrt{x}$). Can solve for x, understands inverse relationship of operations. Has an understanding that different method(s) can be used. Knows the quadratic formula. Understands $\sqrt{-x}$, that it isn't possible, therefore that 'no solutions' are possible.

Responding. Eight of nine participants either improved or maintained their pre-video responding score in the post-video assessment (Steven's score decreased by 1 point). Four participants scored at the *emerging ability* level during the pre-video stage, with this number increasing to five on the post-video assessment. To illustrate, Owen scored a 1 in the pre-video assessment after providing a limited rationale of how he intended to respond to the student, (e.g. "Would encourage her to understand a visual representation of the graphs. Might suggest that she considers what's the best approach at the beginning, in case there are short-cuts"). The post-video response from Owen was:

> Can you explain to me your working in solving/using the QF? (help her find the mistake). Is your QF correct? (Direct her to mistake if she doesn't find herself). What does the root of a negative number give you? (Must give understanding as to why it's special).

This response was scored as a 2 by both researchers, as they felt Owen anchored his response and rationale, which he included in parenthesis, to his observations of the student's error in working with the quadratic formula and the result of obtaining a negative discriminant.

Table 4: Examples of Participants' Responding Responses

Participant	Response
Georgia (Pre-video, 0)	Directive questioning -- open questioning to allow her to explain her thought process (e.g., rather than to assume her thought process)?
Jai (Post-video, 1)	I would ask her to check her work to avoid follow-through errors...ask her why she thinks eq. 1 has 2 roots. Might explain why she draws a y-int @ 2.
Andi (Pre-video, 2)	Show her some problems regarding reducing fractions with addition on the top to remind her of why reducing too early gives incorrect answers. To help her graph and aid her in understanding how the first graph looks, I would have her plot points from 0 to 4 to help her visualize the graph. I would also have her plot out points for the graphs x^2=y and $\sqrt{x}$=y to help her understand where she got the idea for her first graph from.

Discussion

As demonstrated in Table 1, the largest improvements in professional noticing were evidenced in PSMTs' ability to attend to student thinking, with seven of nine PSMTs scoring at the *emerging level* on the post-video assessment. Further analysis revealed that while PSMTs mainly attended to mathematical aspects in both the pre- and post-video assessments, they did so with even greater focus after the intervention. Five of nine PSMTs made comments about the interviewer and/or student disposition in the pre-assessment. In particular, PSMTs noted that the 'teacher' did not correct or tell the student if she was correct; but instead prompted her with questions. In the post-video assessment there were no comments about interviewer actions and only one comment that the student in this second video was "quite confident". These promising findings from Australia add to the US literature confirming that noticing skills can be developed through targeted interventions.

Despite improvement in both attending and interpreting, there was relatively no improvement in PSMTs' responding scores. This result is understandable in the US case, given that all three participants received scores of 2 on the pre-video assessment. However, the lack of improvement in scores for the Australian participants is in stark contrast to Monson et al.'s (2019) study in which participants showed the greatest gains in responding. In this regard, the findings from the Australian cohort more closely align with results from an earlier implementation of the Interview Module (Krupa et al., 2017), which prompted the addition of the responding assignments. In that initial study, with 36 PSMTs across three US universities, 38% of participants showed gains in attending, 25% gained in interpreting, but there was no notable change in responding after the intervention. While we are naturally hesitant to draw too many inferences based on the limited sample, we wonder about contextual factors that may have played a role. For example, in the US case, the three participants were the only students enrolled in the methods course, and as a result received more targeted instruction and feedback during the in-class responding assignment. In contrast, 39 PSMTs were enrolled in the course in Australia and thus may have relied more on their peer-group and received less 'expert' feedback when crafting responses to the student work presented. Another possible intervening factor, also potentially related to the disparate class sizes, is the fact that on both the pre- and post-video assessments, the US participants simply wrote more when addressing all three of the prompts. Finally, we wonder whether the in-class responding component of the module was sufficiently outlined in the Interview Module materials to support implementation by an MTE from outside the group. In other words, the different instructors may have enacted parts of the assignment very differently. Given that a key purpose of this study was to investigate the transferability of the Interview Module to a new context, this explanation is a cause for concern that warrants further investigation.

Finally, we note that although there were no significant changes in responding scores (due in part to the ceiling effect with the US participants), we did see qualitative differences in the ways PSMTs chose to respond. Most apparent was the shift toward more student-centered responses. On the pre-video assessment 4 PSMT responses were coded as teacher-centered (2 from the US participants), 3 were coded as student-centered, and 2 as mixed. In contrast there were no responses on the post-video assessment that were coded as strictly teacher-centered, with responses more focused on eliciting additional student thinking or building the student's understanding rather than providing further instruction. For example, there was a shift in language from "I would *show her*…" or "I would *go over* quadratic graphs and what they look like" to "*I would ask* the student…"

This finding is consistent with previous work in which "leaving room for student thinking" was the characteristic PSMTs were best able to meet when crafting responses on the take-home assignment associated with the responding components that were added to the Interview Module (Casey, Lesseig Monson & Krupa, 2018). We speculate that this aspect–allowing students to do their own thinking, rather than jumping in to 'tell'–is not only highly supported throughout the module (e.g., in the course readings as well as the in-class responding assignment), but also is less reliant on subject matter, or pedagogical content knowledge. Son and Crespo's (2009) study with elementary preservice teacher provides some support for this claim.

Limitations

In addition to the limitations alluded to above (e.g., small number of participants, differences in US and Australia class size) the study is subject to methodological limitations that plague noticing research more generally. Specifically, we acknowledge that there may well be differences between what is attended to explicitly versus implicitly (Scheiner, 2016). We accounted for this to some degree by encouraging students to take notes while watching the video, prior to addressing the first prompt. These notes were collected and included in our analysis and scoring. Second, video of a clinical interview (the format employed in the pre-post assessment) provides limited access to information pertinent to the teaching and learning episode (e.g., students' prior experiences or attitude toward mathematics, specific learning goals or associated curricular materials). Without this full context, PSMTs are left to "fill in the gaps", often relying on their own experiences with this topic as high school students. This not only impedes the situational awareness of the PSMTs, but also our abilities as researchers to infer meaning from PSMTs' responses (Nickerson et al., 2017). In future work, we intend to attend more carefully to PSMTs' background experiences and explore potential relationships among PSMTs' knowledge, skills, and dispositions; we encourage other researchers to do the same.

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A TRANSFORMATIVE LEARNING EXPERIENCE FOR THE CONCEPT OF FUNCTION

Michael S Meagher
Brooklyn College - CUNY
mmeagher@brooklyn.cuny.edu

Allison McCulloch
UNC Charlotte
amccul11@uncc.edu

Jennifer Lovett
Middle Tennessee State University
Jennifer.Lovett@mtsu.edu

Milan F Sherman
Drake University
milan.sherman@drake.edu

Functions form a central part of the U.S. mathematics curriculum. A large body of research shows that students at all levels, including preservice secondary mathematics teachers, have difficulties with defining function as a correspondence between two sets with a univalence condition. Those difficulties include privileging algebraic representations and reductive interpretations of the univalence condition in the form of the vertical line test. In our research study, 47 pre-service mathematics teachers provided definitions of function, engaged with an interactive applet that had a non-standard representation of function. The interaction with the applet was effective in initiating a series of dilemmas in their conception of function that resulted in the majority of the participants transforming and/or refining their conception of function.

Keywords: Teacher Education - Preservice, Teacher Knowledge, Technology

Introduction

The concept of function is considered to be one of the most important underlying and unifying concepts of mathematics (e.g., Leinhardt, Zaslavsky, & Stein, 1990; Thompson & Carlson, 2017). Students work with functions from the very earliest grades in pattern exploration, through high school with a formal treatment of functions as arbitrary mappings between sets. There is an extensive body of research on students' understanding of function (e.g., Carlson et al., 2003; Cooney et al., 2010; Dubinsky & Harel, 1992; Even, 1990; 1993, Oehrtman et al., 2008) and much of that research reports that learners (including preservice mathematics teachers) have considerable difficulty identifying functions and in distinguishing them from non-functions. With the emphasis on function in school mathematics, preservice secondary mathematics teachers (PSMTs) must have robust conceptions of function to plan to support their students' understanding. Based on years of research on students and teachers flawed, limited conceptions of function, and on transformation theory (Mezirow, 2000; Taylor, 2007), we designed a task, utilizing advanced digital technology, to meet this need. The purpose of this study is to examine ways in which the task elicited and transformed PSMTs' conceptions of function.

Literature Review and Relationship to Research

Defining Function

In Thompson & Carlson's (2017) discussion of the evolution of the definition of function, they describe how a variation and covariation conception of function came to be replaced, owing to the emerging dominance of a set theoretic conception of variable as used in group theory and other areas, by a correspondence conception of function. This definition "solved problems that arose for mathematicians, [but that] introducing it in school mathematics made it nearly impossible for school students to see any intellectual need for it" (p.422). This abstract correspondence definition is often referred to as the Dirichlet-Bourbaki definition of function and states that a function is a correspondence between arbitrary sets satisfying a univalence condition i.e. each element in the domain corresponds to exactly one element in the codomain.

Thompson and Carlson (2017), citing Cooney and Wilson (1993), as well as drawing on their own review of 17 U.S. Precalculus textbooks, note that a correspondence definition of function is used exclusively in all of these textbooks. Therefore, while we expect that most students (and PSMTs) who have attended U.S. schools to have experience with a definition involving a correspondence between two sets with constraints on the mapping of individual elements (the univalence condition), Even (1993) notes that many students retain a "protypic" (p.96) concept of functions as linear relationships and "many expect graphs of functions to be "reasonable" and functions to be representable by a formula." (p. 96).

Teachers' Understandings of the Function Concept

In addition to content knowledge of functions, mathematics teachers require Mathematical Knowledge for Teaching (Ball, Hill, & Bass, 2005) of functions i.e. teachers should be aware of various representations of functions, many examples of functions and non-functions, and known areas of challenge for students when learning functions. However, teachers' understanding of function, at the content level, is similar to that of school and college students (Bannister, 2014; Even, 1990, 1993; Wilson, 1994). In particular, practicing teachers and PSMTs tend to privilege algebraic representations of functions and emphasize properties of graphs (e.g., vertical line test) in their descriptions of functions and non-functions (Even, 1990, 1993; Wilson, 1994). They also exhibit a limited repertoire of representations on which to draw in helping students understand functions (Bannister, 2014; Hatisaru & Erbas, 2017).

Crucially, teachers' understanding of function has been shown to impact the pedagogical choices they make during instruction. In a study of 152 PSMTs, Even (1993) found they could not justify the need for univalence and did not know why it was important to distinguish between functions and non-functions. Owing to this lack of content knowledge, the PSMTs' MKT was constrained and they limited the exposure of their students to various function representations and emphasized procedures such as the vertical line test in identifying functions.

Learning in Technology-rich Environments

A considerable body of research supports the idea that advanced digital technologies can support learning in general (Tamim et al., 2011) and mathematics concepts in particular (Drijvers et al., 2010; Olive et al., 2010.). Drijvers (2015) argues that "the criterion for appropriate design is that it enhances the co-emergence of technical mastery to use the digital technology for solving mathematical tasks, and the genesis of mental schemes that include the conceptual understanding of the mathematics at stake." (p.15). For example, a good design to allow students to engage with the concept of function will allow the user to experience different kinds of functions and non-functions with enough data to differentiate between the two.

Theoretical Framework

Transformation Theory

Given the preponderance of evidence in the literature that the conception of function of PSMTs is often underdeveloped, our goal was to design a learning experience that problematized those conceptions, required PSMTs to reflect on them, and, ideally, resulted in refinement of their conception. Given that PSMTs come to their methods courses as adults and with a wealth of previous experiences related to the function concept, we turned to an adult learning theory, namely Mezirow's (2000) transformation theory. Transformation theory is an adult learning theory that is consistent with constructivist assumptions and expands on those assumptions by acknowledging the broad predispositions an adult might have toward a concept based on prior experiences, and the role these dispositions play in their meaning making (Mezirow, 2000).

Mezirow (2009) describes four forms of learning at the heart of transformation theory: elaborating existing meaning schemes, creating new meaning schemes, transforming meaning schemes, and transforming meaning perspectives. According to Mezirow (2009), learning by transforming existing meaning schemes and perspectives often begins with a stimulus, a *disorienting dilemma*, which requires one to question their current meaning schemes. However, experiencing a disorienting dilemma alone is not enough to effect a transformation and learning will only occur through critical reflection (Merriam, 2004; Mezirow, 2000; Taylor, 2007).

Sfard (1991) distinguishes between a *concept*, a mathematical idea in its "official form" (p. 3) and a *conception*, "the whole cluster internal representations and associations evoked by a concept" (p. 3). A *concept* is the generally accepted structure of mathematics that has been culturally developed and shared formally among mathematicians for centuries (Pehkonen & Pietila, 2003) whereas a *conception* is a learner's individual, often, incomplete understanding of the concept. In Sfard's definition of conception we see the connection between knowledge and the affective aspects of meaning schemes as defined by Mezirow (2000). Conceptions, then, are the personal side of a concept, one's individual experiences, beliefs, attitudes, and emotions that result in personal definitions, examples, and non-examples of concepts.

Therefore, we aimed to design a learning experience for PSMTs that would trigger a disorienting dilemma related to their conceptions of function and non-function and require critical reflection in the expectation that a transformation of meaning scheme occurs.

A Transformative Learning Experience for the Concept of Function

Given the promise of cognitive roots (Tall et al., 2000) such as a function machine, we set out to design a machine-based experience using representations that were unfamiliar for PSMTs as a stimulus for examining their meaning schemes of function. The applet we designed, built on the metaphor of a vending machine, contained no numerical or algebraic expressions. Our intention was to put PSMTs in a context in which they would not be able to automatically rely on an algebraic, and often procedural, conceptions of functions (e.g., use of the vertical line test).

The Vending Machine applet (https://ggbm.at/X3Cn7npQ) consists of four pages; each with two to six vending machines and asks the user to identify each vending machine as a function or non-function. The machines each consist of four buttons (Red Cola, Diet Blue, Silver Mist, and Green Dew). When a button is pressed it produces none, one, or more than one of the different colored cans which may, or may not, correspond to the color of the button pressed.

Which One is a Function? (pgs. 1 - 3)			
A	Red → Red Blue → Blue Silver → Silver Green → Green	B	Red → Red Blue → Blue Silver → Silver Green → Random
C	Red → Blue Blue → Silver Silver → Green Green → Red	D	Red → Random Pair Blue → Blue Silver → Silver Green → Green
E	Red → Red Blue → Blue & Random Silver → Silver Green → Green	F	Red → Red Blue → Silver Silver → Silver Green → Green

Which Ones Could be Functions? (pg. 4)			
G	All Random	H	Red → Green Blue → Green Silver → Green Green → Green
I	Red → 2 Silvers Blue → Green Silver → Red Green → Blue	J	Red → Red Blue → Blue Silver → No Can Green → Green
K	Red → Red Blue → Blue Silver → Silver Green → Red & Green	L	Red → Red Blue → Red Silver → Silver Green → Silver

Figure 1: Description of Each Machine in the Applet

By removing numeric and algebraic representations, the applet could allow PSMTs to attend to input/outputs and their relationship. We intentionally designed to trigger dilemmas related to known issues from the literature e.g. researchers have shown that students as well as teachers exhibit difficulties identifying constant functions as functions (e.g., Carlson, 1998; Rasmussen, 2000); thus, there is a machine that acts as a constant function, i.e. every button produces the same color can. The purpose of this study is to determine the extent to which we were successful in designing for transformative learning (Mezirow, 2000; Taylor, 2007) related to the function definition. Specifically, we aim to answer the following research question:

> To what extent did PSMTs experience disorienting dilemmas when engaging with the vending machine applet and in what ways did PSMTs' conceptions of function transform as a result of personal critical reflection in the context of the vending machine task?

Methods

Participants and Data Sources

Participants in this study are 47 PSMTs enrolled in a secondary mathematics methods course at four different U.S. universities, ranging from five to 18 PSMTs per university. The PSMTs were all undergraduate mathematics and/or mathematics education majors working toward a secondary mathematics teaching license. The individual degree programs all required at least 36 hours of mathematics, and the PSMTs had all completed at least Calculus II at the time of the study. Every PSMT in the four methods courses took part in the study (N = 55). However, there were some PSMTs that did not have complete data sets (e.g., video had no sound, missing artifacts), these participants were removed, leaving 47 PSMTs in this particular study.

Data for this study consists of all of the PSMTs' work related to the Vending Machine task. During a class the PSMTs completed the task individually and then the class came together for a whole class discussion. Data is focused on PSMTs individual work on the task and includes both written work and video recorded screen captures of their engagement with the applet. Specifically, we collected PSMTs' written pre- and post-definitions of function and their written responses to the Vending Machine task worksheet. In addition, each PSMT captured a screencast of their work on the Vending Machine task as they followed a "think-aloud" protocol while working on the task.

Analysis of Applet Engagement Descriptions

We created a document for each PSMT which consisted of their pre- and post-definitions and a detailed description of the video-recorded screencast. These descriptions included a chronological record of PSMTs' engagement with the applet and verbatim transcriptions of PSMTs' expressed thoughts related to their work. The participant descriptions were uploaded to Atlas.ti and coded for evidence of the occurrence of disorienting dilemmas, triggers for those dilemmas, articulated conceptions of function that were challenged by those triggers, and transformations of meaning schemes related to conceptions.

Dilemmas were identified based on PSMTs' verbal utterances and interactions with the applet. For example, verbal utterances such as, "Ok these are two cans but they seem to be the same. Does it have to be one can of coke, or two cans can still be one output? I don't know. Let's see what others are" were coded as a dilemma. Each dilemma was then assigned a trigger code. Each dilemma was either resolved or not resolved in the PSMTs' utterances. Those that were resolved were coded as having a change in meaning scheme. Trigger codes included both a priori triggers and emergent triggers. Given the personal nature of conceptions of function, we could only code for those that were articulated explicitly in writing or spoken on the screencast. All of the quotations coded for function conception were then open coded to identify themes (Creswell, 2014). Themes for the articulated conceptions of function are shown were: (i) Functions must be/don't have to be continuous; (ii) Functions can/can't be many-to-one; (iii) Functions must map elements in the domain to elements in a defined codomain; and (iv) Functions must be one to one (with or without correct meaning of one to one).

Findings

In our findings we discuss the dilemmas triggered through engagement with the applet, the conceptions of function which were problematized, and the ways in which PSMTs' conceptions of function were transformed as a result of engaging with the Vending Machine Task.

Triggering Disorienting Dilemmas and Transforming Conceptions

From our analysis of the 47 PSMTs' screencast descriptions, we identified a total of 158 dilemmas (i.e. a little over 3 per PSMT on average), with approximately 91% (43 PSMTs) articulating at least one dilemma while engaging with the applet. The machines that triggered these dilemmas were those that produced two cans or no can as an output, those for which different inputs produced the same output, or those for which the color of the can (output) did not match the button pressed (input). Of the 158 articulated dilemmas, 124 resulted in a transformation in a PSMTs' conception of function, where transformation is defined, per our theoretical framework as any one of: elaborating existing meaning schemes, creating new meaning schemes, transforming meaning schemes, and transforming meaning perspectives. The transformations we report on below, i.e. the resolved dilemmas, were mainly "elaborating existing meaning schemes" although there were some instances of "transforming meaning schemes." The three most commonly transformed conception themes were: elements of the codomain, many-to-one, and continuous function. For the sake of brevity, only the first of these will be discussed in this paper.

Conceptions of Elements in the Codomain

The most common conception of function that was triggered was that of PSMTs considering (or not) possible ways of defining the codomain, with 41 of the 47 PSMTs (87%) articulating such a dilemma. We intentionally did not state what elements made up the domain, codomain, and range for the machines in the applet. The machines that produced two cans as an output (D, E, I, K) and no can as an output (J) were designed with the hope that engagement with the machines would trigger a dilemma for the PSMTs and elicit such considerations.

When encountering Machines D, E, I, K, and J, PSMTs typically articulated their dilemma by questioning whether or not no can or two cans could be outputs. For example, PSMT 2's response to Silver Mist not producing an output on Machine J, "Oh, the Silver Mist has no output. [Presses Silver Mist button many times.] That's not broken, right? So, is that ok, that Silver Mist doesn't have an output?" and PSMT 6's response to two cans as an output on Machine K, "Red is red, blue is blue, silver is silver, and green is [red and green cans appear]. That is definitely not a function, because you can't have two. I guess you could have two outputs (sigh) this is very difficult." Additionally, some PSMTs went beyond just verbalizing a dilemma to explicitly discussing the possible output values. One example that typifies how PSMTs explicitly discussed defining the possible codomain elements follows,

> Ok these are two cans, but they seem to be the same. That's an interesting question… That's interesting what we define as output. Does it have to be one can of coke or two cans can still be one output? (PSMT 3, on Machine I).

All of these PSMTs clearly articulated a dilemma related to their conception of types of elements that could possibly be in the codomain.

The first machine where these dilemmas occurred was on Machine D (Red Cola → random pair), and while 20 PSMTs articulated a dilemma on this machine, only 6 resulted in a transformed conception of function. This result can be attributed to the fact that making sense of the two can output was not required to classify the machine as a function or non-function due to the randomness of the output. For example, after deciding that the machine was not a function because of the random nature of the outputs PSMT 23 stated, "And, I'm still hung up on this two can thing, but I really don't know why. And I haven't been able to work through it yet. So, maybe I can explore some more and get back to that."

Most transformations of conceptions of functions related to defining elements of the codomain occurred on Machines I, J, and K (17, 20, and 18 respectively). As PSMTs articulated their transforming conceptions related to the nature of the codomain they either focused on the consistency of the outputs, examples of representations of functions they were familiar with, or their personal definition of function.

Consistency of outputs. Most PSMTs (23 out of 41) who articulated transformed conceptions with respect to codomain attended to consistency as they worked to make sense of what they observed as outputs on Machines I, J, and K. For example, as PSMT 7 engaged with Machine I they explained,

> Ok, now I'm having second thoughts about these two sodas. And like, would Green Dew have two different arrows? So, I guess it depends on how you see your output values. Are the output values just a red soda, green soda, blue soda, silver soda? Or can they be different combinations of those? So, going off of the assumption that it's going off the same output every time, then it's a function. But since it's giving you two different drinks, is it still? Hmm, I'm questioning all my thoughts now. I guess it would depend on how you classify your outputs, so if like getting two different drinks is OK, but as long as it happens every single time that you put this input in then I think it would be OK.

In this example PSMT 7 is considering elements that might be in the codomain, going beyond noting elements of the range that have been observed with the machines so far. PSMT 7 makes the point that whether or not this is a function, depends on how the codomain is defined, ultimately deciding that if it was defined to include pairs of cans then Machine I could be a function because the result of clicking on the Red Cola button is consistently two silver cans. Similar reasoning is evident in the following explanation of Machine J by PSMT 30:

> That's not broken, right? So, is that ok, that silver mist doesn't have an output? … I think its ok, because it's the same output. If it gave us something one time, then I wouldn't be ok with that. So, even though silver doesn't give you anything, by giving you nothing, it is consistently giving you nothing.

In each of these responses, the PSMTs articulated transformed conceptions of the codomain, specifically an elaborated conception that included two cans or no cans as elements. This transformation is a result of considering the importance of consistency in the relationship between input and output elements. Furthermore, all of the PSMTs that attended to consistency when deliberating about how to resolve their dilemma transformed their conception of codomain to include both two and no cans.

Using examples of familiar functions. Other PSMTs worked through dilemmas in which their conceptions of the nature of codomain were challenged by drawing upon examples of familiar representations of function. This is evident in PSMT 34's work on Machine I,

> This one is the most questionable one that I'm the least certain on. We will say that is not a function. I don't know how that would really work on a graph. How that could be expressed as a graph? I think that would basically be saying if I put in a one I would get two 2s out of that. Which is not possible.

PSMT 34 could not imagine how two cans might be represented on a graph, resulting in a transformed conception of codomain in this context that did not include elements other than single cans. PSMT 11 used similar reasoning on Machine J (no can),

> Silver doesn't give me anything. What? … I don't know, it's just the fact that it doesn't give me one out, if that's the reason why I don't think it's a function or…even like a linear basic function. If you put something into it, you have to come out with something. There's got to be some number there. Okay, okay, because you have to have an output.

In this example, the PSMT is trying to imagine the situation as a known function, even a "linear basic function" but is not able to do so. After declaring they cannot think of a function that behaves this way, the PSMT goes further to state that an input must have an output to go with it.

When PSMTs drew on familiar representations to make sense of a dilemma regarding issues with the codomain, they used the familiar representations to determine if a machine was a function or non-function. This resulted in a changed understanding of the codomain in this non-algebraic context by drawing on possible codomains from algebraic contexts.

Drawing on personal definition of function. Drawing upon one's personal definition was also common for the PSMTs when they were thinking about elements of the range and codomain. Consider PSMT 20's explanation of Machine I,

> Immediately that red button is giving you two different cans. Which is not… They are both silvers, but I take that as still two different values even if they're the same value which you can't have. Yeah two different cans we can't have two cans off of one… two ys off one x. Then it's not a function. Although if they are the same can… nah I still think that's not.

Similarly, PSMT 35 working on Machine J said,

> The problem is the silver, because it since it doesn't give you anything. It's like having an x that doesn't go to a y. And, one of the rules of functions is that every x needs a y but not every y has to have an x. So, because silver doesn't give you a can, this makes it not a function.

Both of these examples are evidence of PSMTs transforming their conceptions of codomain. In the latter case, in such a way that the empty set is not included based on their conceptions of the univalence requirement of the definition of function.

Discussion

The purpose of the design of the Vending Machine applet and this study was to elicit PSMTs' many conceptions of functions and attempt to challenge and transform any less robust conceptions to more robust conceptions of functions. The fact that PSMTs exhibited many conceptions and difficulties consistent with research literature on understanding of function suggest that the applet design was effective in this regard. Furthermore, there is evidence in the totality of the data collected that engaging with the Vending Machine applet resulted in most PSMTs reconsidering and refining their conceptions of function in a positive direction.

To enable and facilitate this change we drew on transformation theory (Mezirow, 2000), and Drijvers's (2015) notion of the importance of didactical possibilities in the design of advanced digital technology applications, to guide the creation of an applet to trigger dilemmas that address common conceptions from the literature on distinguishing functions and non-functions. The use of advanced digital technology, allowed us to create a task with which the PSMTs could interact independently and which, with immediate feedback allowed them to formulate conjectures and test those conjectures without having to wait for a class discussion or intervention from an instructor. At the beginning of the task PSMTs articulated conceptions of function that we expected based on the literature. For example, the PSMTs articulated disorienting dilemmas related to their conceptions of univalence (e.g., Even, 1993; Vinner & Dreyfus, 1989), many to one (e.g., Carlson, 1998; Rasmussen, 2000), and continuity (Bezuidenhout, 2001; Tall & Vinner, 1981). With the exception of continuity, we designed the applet to trigger dilemmas related to each of these conceptions. One of the persistent problems noted in the literature is over privileging of algebraic representations (Even, 1990, 1993; Wilson, 1994), putting PSMTs in the context of the vending machines appears to have mitigated this problem.

The ways in which the PSMTs engaged with the applet provided considerable insight into their conceptions of functions and the ways in which they were both challenged and transformed. Given that 43 out of 47 (91%) PSMTs articulated a dilemma related to at least one of the triggers we designed for, we know that we leveraged advanced digital technology to create an opportunity for transformations of conceptions to occur. Furthermore, 41 of the 47 PSMTs (87%) experienced a dilemma related to their conceptions of codomain and the result that 76%

(31 out of those 41) were able to articulate the impact of the definition of domain and codomain on the way the machines would be classified, is significant. Overall, the vending machine task was successful in triggering dilemmas, eliciting critical reflection, and supporting PSMTs in transforming their conceptions of function.

Conclusion

It is crucial that PSMTs have a solid understanding of function, know variations in the definition of function, develop the ability to translate among different representations of functions, and know when to use each definition based on context (Bannister, 2014; Hatisaru & Erbas, 2017). This specialized content knowledge is needed to understand and plan for the diverse student conceptions they will encounter during instruction. While there is a vast literature base on the limited conceptions of functions PSMTs often develop through high school and undergraduate mathematics, little is known about how to transform them after years of building on them in algebraic contexts. The results of this study indicate that by removing PSMTs from familiar function contexts and designing to trigger dilemmas based on conceptions identified in the literature, we can transform PSMTs' conceptions of function in a positive direction.

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TRANSFORMING PRE-SERVICE TEACHERS DEFINITION OF FUNCTION

Milan F Sherman
Drake University
milan.sherman@drake.edu

Michael S Meagher
Brooklyn College - CUNY
mmeagher@brooklyn.cuny.edu

Jennifer Lovett
Middle Tennessee State University
Jennifer.Lovett@mtsu.edu

Allison McCulloch
UNC Charlotte
amccul11@uncc.edu

Functions form a central part of the U.S. mathematics curriculum especially at the high school level. There is a considerable body of research showing that students at all levels, including preservice secondary mathematics teachers, have difficulties with the definition of function as a correspondence between two sets with a univalence condition. Those difficulties include privileging algebraic representations and reductive interpretations of the univalence condition in the form of the vertical line test. In our research study, 47 pre-service mathematics teachers provided definitions of function, engaged with an interactive applet that had a non-standard representation of function, and then provided revised definitions of function. The results of the study show a measurable increase in the participants level of abstraction in their definitions, and an increase in their attention to the univalence condition.

Keywords: Teacher Education - Preservice, Teacher Knowledge, Technology

Introduction

The concept of function is considered to be one of the most important underlying and unifying concepts of mathematics (e.g., Leinhardt, Zaslavsky, & Stein, 1990; Thompson & Carlson, 2017). Students are provided experiences with functions from the very earliest grades, usually pattern exploration, up to and through high school with a formal treatment of functions as arbitrary mappings between sets. Indeed, in the Common Core State Standards for Mathematics the study of function is given its own domain in grades 9-12 (National Governors Association for Best Practices & Council of Chief State School Officers, 2010).

There is an extensive body of research on students' understanding of function (e.g., Carlson et al., 2003; Cooney et al., 2010; Dubinsky & Harel, 1992; Even, 1990; 1993, Oehrtman et al., 2008) and much of that research reports that learners (secondary, post-secondary as well as pre-service and in-service teachers) have considerable difficulty identifying functions and in distinguishing them from non-functions.

Given the emphasis on function in school mathematics, it is important to consider preservice secondary mathematics teachers' (PSMTs) conceptions of function. Since PSMTs must possess robust conceptions of function so they can plan for supporting the development of their future students' function understandings. As such, mathematics teacher educators need to identify methods for eliciting and transforming PSMTs' conceptions of function to meet these needs. The decades of research outlining the details of both students' and teachers' flawed and limited conceptions of function provide grounding for thinking about how to support PSMTs' further conception development. Coupling this vast literature and transformation theory (Mezirow, 2000; Taylor, 2007), a specifically adult constructivist learning theory, we designed a task utilizing advanced digital technology to meet this need. The purpose of this study is to examine the ways in which this task elicited and transformed PSMTs' personal definitions of function. In doing so we aim to add to the knowledge base of designing learning experiences for PSMTs that problematize and support transformation of important mathematical conceptions.

Literature Review and Relationship to Research

Defining Function

In Thompson & Carlson's (2017) discussion of the evolution of the definition of function in the history of mathematics, they describe how a variation and covariation conception of function came to be replaced, owing to the emerging dominance of a set theoretic conception of variable as used in group theory and other areas, by a correspondence conception of function that "solved problems that arose for mathematicians, [but that] introducing it in school mathematics made it nearly impossible for school students to see any intellectual need for it" (p.422). This abstract correspondence definition is often referred to as the Dirichlet-Bourbaki definition of function and states that a function is a correspondence between arbitrary sets satisfying a univalence condition i.e. each element in the domain corresponds to exactly one element in the codomain.

Thompson and Carlson (2017), citing Cooney and Wilson (1993), as well as drawing on their own review of 17 U.S. Precalculus textbooks, note that a correspondence definition of function is used exclusively in all of these textbooks. Therefore, while we expect that most students (and PSMTs) who have attended U.S. schools to have experience with a definition involving a correspondence (or mapping) between two sets with constraints on the mapping of individual elements (the univalence condition), Even (1993) notes that many students retain a "protypic" (p.96) concept of functions as linear relationships and "many expect graphs of functions to be "reasonable" and functions to be representable by a formula." (p. 96).

Teachers' Understandings of the Function Concept

In addition to content knowledge of functions, mathematics teachers require Mathematical Knowledge for Teaching (MKT) (Ball, Hill, & Bass, 2005) of functions in order to be effective, i.e. teachers should be aware of various representations of functions, many examples of functions and non-functions, and known areas of challenge for students when learning functions. However, the situation regarding teachers' understanding of function, at the content level, is quite similar to that of school and college students (Bannister, 2014; Even, 1990, 1993; Wilson, 1994). In particular, similar to students, practicing teachers and PSMTs tend to privilege algebraic representations of functions and emphasise properties of graphs (e.g., vertical line test) in their descriptions of functions and non-functions (Even, 1990, 1993; Wilson, 1994). They also exhibit a limited repertoire of representations on which to draw in helping students understand functions (Bannister, 2014; Hatisaru & Erbas, 2017).

Crucially, teachers' understanding of function has been shown to impact the pedagogical choices they make during instruction. In a study of 152 PSMTs, Even (1993) found they could not justify the need for univalence and did not know why it was important to distinguish between functions and non-functions. Owing to this lack of content knowledge, the PSMTs' MKT was constrained and they limited the exposure of their students to various function representations and emphasised procedures such as the vertical line test in identifying functions.

Learning in Technology-rich Environments

The importance of advanced digital technologies in mathematics education is now well established. Major stakeholders such as The National Council of Teachers of Mathematics (NCTM) have asserted that "It is essential that teachers and students have regular access to technologies that support and advance mathematical sense making, reasoning, problem solving, and communication" (NCTM, 2015, p.1). A considerable body of research supports the idea that advanced digital technologies can support learning in general (Tamim et al., 2011) and mathematics concepts in particular (Drijvers et al., 2010; Olive et al., 2010.)

In explaining the importance of design Drijvers (2015) argues that "the criterion for appropriate design is that it enhances the co-emergence of technical mastery to use the digital technology for solving mathematical tasks, and the genesis of mental schemes that include the conceptual understanding of the mathematics at stake." (p.15). For example, a good design to allow students to engage with the concept of function will allow the user to experience different kinds of functions and non-functions with enough data to differentiate between the two.

Theoretical Framework

Transformation Theory

Given the preponderance of evidence in the literature that the conception of function of PSMTs is often underdeveloped, our goal was to design a learning experience that problematized those conceptions, required PSMTs to reflect on them, and, ideally, resulted in further development and refinement of their conception as articulated in their personal definition (i.e., learning). Given that PSMTs come to their methods courses as adults and with a wealth of previous experiences related to the function concept, we turned to an adult learning theory, namely Mezirow's (2000) transformation theory. Transformation theory is an adult learning theory that is consistent with constructivist assumptions and expands on those assumptions by acknowledging the broad predispositions an adult might have toward a concept based on prior experiences, and the role these dispositions play in their meaning making (Mezirow, 2000).

Mezirow (2009) describes four forms of learning at the heart of transformation theory: elaborating existing meaning schemes, creating new meaning schemes, transforming meaning schemes, and transforming meaning perspectives. According to Mezirow (2009), learning by transforming existing meaning schemes and perspectives often begins with a stimulus, a *disorienting dilemma*, which requires one to question their current meaning schemes. However, experiencing a disorienting dilemma alone is not enough to effect a transformation and learning will only occur through critical reflection (Merriam, 2004; Mezirow, 2000; Taylor, 2007).

A Transformative Learning Experience for the Concept of Function

There is a significant research base that recognizes that PSMTs often have conceptions of function that are inconsistent with the concept itself (e.g., Bakar & Tall, 1991; Breidenbach, et al., 1992; Carlson, 1998; Carlson & Oehrtman, 2005; Rasmussen, 2000; Vinner & Dreyfus, 1989). Given that PSMTs will be responsible for teaching others about function, it is important to try to address this concern through carefully designed learning experiences.

The idea of a *cognitive root* was introduced by Tall et al. (2000) as an "anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built" as they were developing a cognitive approach to calculus (Tall et al., 2000, p.497). As an example of a cognitive root for function concepts, Tall et al. suggest the use of a *function machine* (sometimes referred to as a function box). The machine metaphor Tall and colleagues describe is typically a "guess my rule" activity which is algebraic in nature. Studies using function machines were promising but some students still struggled with connecting different representations and determining what is and is not a function (e.g. McGowen et al., 2000).

Given the promise of cognitive roots we set out to design a machine-based experience using representations that were unfamiliar for PSMTs as a stimulus for examining their meaning schemes of function. The applet we designed, built on the metaphor of a vending machine, contained no numerical or algebraic expressions. Our intention was to put PSMTs in a context in which they would not be able to automatically rely on an algebraic, and often procedural, conceptions of functions (e.g., use of the vertical line test).

The Vending Machine applet (https://ggbm.at/X3Cn7npQ) consists of four pages; each with two to six vending machines and asks the user to identify each vending machine as a function or non-function (Figure 1). The machines each consist of four buttons (Red Cola, Diet Blue, Silver Mist, and Green Dew). When a button is pressed it produces none, one, or more than one of the different colored cans which may, or may not, correspond to the color of the button pressed.

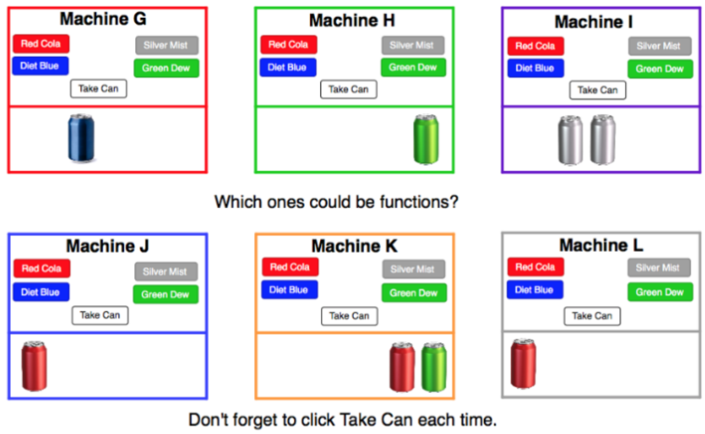

Figure 1: Vending Machine Applet

By removing numeric and algebraic representations, the applet could allow PSMTs to attend to the nature of input and outputs and the relationship between them. We intentionally designed to trigger dilemmas related to known issues from the literature e.g. researchers have shown that students as well as teachers exhibit difficulties identifying constant functions as functions (e.g., Carlson, 1998; Rasmussen, 2000); thus, there is a machine that acts as a constant function, i.e. every button produces the same color can. The purpose of this study is to determine the extent to which we were successful in designing for transformative learning (Mezirow, 2000; Taylor, 2007) related to the function definition. Our research question was:

> In what ways did PSMTs' personal definitions of function transform as a result of engaging with the vending machine task?

Methods

Participants and Data Sources

The participants in this study are 47 PSMTs enrolled in a secondary mathematics methods course at four different U.S. universities, ranging from five to 18 PSMTs at each university. The PSMTs were all undergraduate mathematics and/or mathematics education majors working toward earning their secondary mathematics teaching license. The individual degree programs all required at least 36 hours of mathematics, and these students had all successfully completed at least a second level calculus course at the time of the study. Every PSMT in the four methods courses took part in the study (N = 55). However, there were some PSMTs that did not have complete data sets these participants were removed, leaving 47 PSMTs in this particular study.

Data for this study consists of all of the PSMTs' work related to the Vending Machine task. This current paper represents a subset of a larger project and he subset of data relevant to this paper is written pre- and post-definitions of function on the Vending Machine task worksheet.

Analysis of Pre- and Post-definitions.

Pre- and post-definitions were entered into a spreadsheet for analysis. They were then coded using a codebook which was developed in a previous study and for which reliability was established (Author et al., 2018). All 94 definitions (47 pre and 47 post) were double-coded by two of the four authors using this codebook. To ensure reliability, coding was done in subsets of the data corpus, and coders compared codes, discussed, and resolved discrepancies (DeCuir-Gunby, Marshall, & McCulloch, 2011).

Similarly to Vinner and Dreyfus (1989), each definition was coded in terms of 1) accuracy, 2) focus, and 3) attention to output. In terms of *accuracy*, each definition was assigned a code of correct, incorrect, or close to correct. Key elements of a correct definition were 1) the definition was not limited to a specific type of function (e.g. linear or quadratic), or to a particular representation (e.g., equation), and 2) the definition addressed the idea that functions map each input to one and only one output, i.e., the univalence condition. Definitions coded as close to correct included those that indicated each input has one and only one output, but were not classified as correct because they were not general enough (e.g., the definition limited a function to a particular representation, such as an equation).

In terms of *focus*, each definition was coded regarding whether the definition indicated a function was a relationship (or mapping), an object, or neither. We referred to this set of codes as *focus*, as they indicated how the students "saw" function. We note that our use of the term *object* differs from its meaning in the APOS framework (Asiala et. al., 1996). In general, if a student identified a function with a representation or representations (e.g., "a function is an equation…"), then the definition was assigned a code of object. If the definition referred to a function as a relationship or mapping between variables or sets, it was coded as relationship. Finally, some definitions did not identify a function as an object or a relationship, but simply described some property of a function, e.g., "a function passes the vertical line test," then the definition was coded as neither. Although this code was intended to be mutually exclusive, there were a few definitions that identified a function as both a relationship and an object.

Finally, definitions were coded according to whether or not they *attended to output*. In order for a definition to be coded as attending to output, the definition needed to note something special or unique about the output. For example, "an equation with an input and an output" would not be considered as attending to output, while "an equation where each input has exactly one output" would. In addition, any definition which included mention of the vertical line test was coded as attending to output.

After coding was completed, results for each code were summarized and analyzed for patterns and themes that provided insight to transformations of students' conceptions related to the definition of function.

Results

PSMTs' Personal Definitions of Function

Given the applet design goal of disrupting students' meaning schemes for the concept of function, we noted how many students changed their definition from pre to post (students were given the option of not changing their definition from to pre to post as well). The number and percentage of definitions that were classified as correct, close to correct, or incorrect, pre- and post- engagement with the applet are shown in Table 1.

Table 1: Accuracy of Function Pre- and Post-Definitions

	Correct n (%)	Close to Correct n (%)	Incorrect (n (%)
Pre	4 (8.5%)	9 (19.1%)	34 (72.4%)
Post	7 (14.9%)	18 (38.3%)	22 (46.8%)

While 36 of the 47 PSMTs made a change to their definition, in many cases the post-definition did not change in terms of accuracy. Of those 36 that revised their definitions, 15 PSMTs improved the accuracy of their definition from pre to post, one PSMT's definition degenerated, and the rest of the definitions did not change with respect to accuracy. All 15 PSMTs whose definition improved started with incorrect definitions; three improved to a correct definition, and the other 12 moved from incorrect to close to correct. An example of a change for incorrect to correct is PSMT 17 who changed from "A function describes a relationship between 2 variable where the value of one variable determines the value of the other variable. A function must pass the vertical line test" to "A function describes a relationship between the domain and the range where for each input, or each value in the domain, there is only one corresponding output, or value in the range." With the PSMT noting "I changed this definition so that it focused on the number of outputs." An example of a change from incorrect to close to correct is PSMT 30 who went from "Function: an identity with more than one variable" to "A function is an equation that for every input (usually x) there is one output (usually y). It ceases being a function when multiple outputs exist for one input." The one PSMT whose definition declined with regard to accuracy went from close to correct to incorrect.

In terms of focus, the frequencies and percentage of definitions classified as relationship, object, both, or neither is depicted in Table 2. The notable result and very important with respect to focus is that while *object* was the most common code for the pre-definitions, *relationship* was the most common code for the post-definitions. This change corresponds both with the improvement in accuracy noted above.

Table 2: Focus of Function Pre- and Post-Definitions

Focus	Relationship n (%)	Object n (%)	Neither (n (%)	Both (n (%)
Pre	17 (36%)	19 (40%)	7 (15%)	4 (9%)
Post	20 (43%)	15 (32%)	9 (19%)	3 (6%)

Finally, the classification of attention to output had the most drastic change from pre- to post-definition. 60% (n= 28) attended to the output in their pre-definition and 89% (n= 42) attended to the output in their post definition. All of the 28 PSMTs who attended to output in their pre-definition continued to do so in their post-definition, and 14 of those who did not attend to the output in their pre-definition did so in their post-definition. Examples of PSMTs paying changing to pay attention to output are PSMT 12 who changed from "An expression involving more than one variable," to "An expression involving more than one variable. For each input there is only one output" and PSMT 29 who moved from "A relationship that maps inputs and outputs and has some combination of variables and constants to "A relationship that maps every input to one output consistently."

Overall the number of PSMTs who experienced a shift in accuracy of their personal definition, and who attended to output in their post-definition, and did not in their pre-definition indicates engagement with the applet resulted in transformations of their articulated conceptions of the definition function.

Discussion

The purpose of the design of the Vending Machine applet and this study was to elicit PSMTs' personal definitions of functions and attempt to challenge and transform those definitions. The PSMTs exhibited many difficulties consistent with research literature on understanding of function suggesting that the applet design was effective in this regard. Furthermore, there is evidence that engaging with the vending machine applet resulted in most PSMTs reconsidering and refining their personal definitions of function in a positive direction.

Unsurprisingly, PSMTs' struggled to articulate a complete definition of function, with much focus on objects rather than relationships or mappings as has been shown in previous research (e.g., Breidenbach et al., 1992; Carlson, 1998; Even, 1993). As future teachers of the function concept, perhaps the most concerning issue in PSMTs' articulated definition, prior to engagement with the applet, was that 59% did not attend to the univalence requirement. However, as a result of engaging with the Vending Machine applet 89% attended to the univalence requirement in their post-definitions.

To enable and facilitate this change we drew on transformation theory (Mezirow, 2000), and Drijvers's (2015) notion of the importance of didactical possibilities in the design of advanced digital technology applications, to guide the creation of an applet to trigger dilemmas that address common conceptions from the literature on distinguishing functions and non-functions. The use of advanced digital technology, allowed us to create a task with which the PSMTs could interact independently and which, with the immediate feedback of the machine outputs allowed them to formulate conjectures as they worked and test those conjectures without having to wait for a class discussion or intervention from an instructor. Our use of function machine, in the form of vending machines, as a cognitive root (Tall et al., 2000) proved to be accessible and meaningful for the PSMTs. One of the persistent problems noted in the literature is privileging algebraic representations (Even, 1990, 1993; Wilson, 1994) and putting PSMTs in the context of the vending machines appears to have mitigated this problem.

Conclusion

It is crucial that PSMTs have a solid understanding of function, know variations in the definition of function, develop the ability to translate among different representations of functions, and know when to use each definition based on context. This specialized content knowledge is needed to understand and plan for the diverse student conceptions they will encounter during instruction related to functions. While there is a vast literature base on the limited conceptions of functions PSMTs often develop through high school and undergraduate mathematics, little is known about how to transform them after years of building on them in algebraic contexts. The results of this study indicate that by removing PSMTs from familiar function contexts and designing to trigger dilemmas based on conceptions identified in the literature, we can transform PSMTs' personal definitions of function in a positive direction.

The results of this study suggest that coupling the research base in mathematics education with transformation theory to guide the design of learning experiences for PSMTs is promising, particularly in a technology-rich environment. Transformation theory values and leverages the wide-range of mathematical experiences of PSMTs, combining this with what research has revealed about learners' conceptions of a particular mathematical concept provides mathematics teacher educators a framework upon which to design such experiences. Given these findings, we hope that mathematics teacher educators will consider the use of transformative theory when designing learning experiences for PSMTs and inservice teachers.

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PRESERVICE TEACHERS' USE OF NOTICING PRACTICES TO EVALUATE TECHNOLOGICAL RESOURCES

Sheunghyun Yeo
University of Missouri
syhw6@mail.missouri.edu

Corey Webel
University of Missouri
webelcm@missouri.edu

This study examines how elementary preservice teachers notice children's mathematical thinking and how this noticing influences the evaluation of technological resources. In particular, we explore the aspects of thinking to which preservice teachers attend and how they interpret evidence about children's thinking when using the Spatial-Temporal Math (ST Math) program. Data collection included a group survey administered after an initial exploration of a set of ST Math activities, screencast recordings during which children used and talked about the program, and a reflective writing assignment. The findings of this study show how preservice teachers used their noticing skills (attending and interpreting) in their evaluations of the tool, in some cases prompting them to shift their evaluation on the basis of student thinking.

Keywords: Teacher Education-Preservice, Technology, Instructional Activities and Practices

Introduction

Professional noticing (Jacobs, Lamb, & Philipp, 2010) is an important instructional practice, essential for supporting and extending students' mathematical thinking by focusing on in-the-moment decisions as well as on students' mathematical understanding. Recent studies have connected noticing to the use of technology, including technology evaluation (Smith, Shin, Kim, &, Zawodniak, 2018), technology-mediated teacher noticing (Walkoe, Wilkerson, & Elby, 2017), and developing noticing through the creation of animated teaching episodes (de Araujo et al., 2015). For example, Smith and his colleagues (2018) articulated a technological framework for the evaluation of technological tools by having preservice teachers (PSTs) engage with and reflect on the qualities of the tools. In this study, we build on this work by exploring 1) how PSTs notice the mathematical thinking of students as they engage with a particular tool, and 2) how this noticing influences their evaluation of the tool.

The technological resource we investigated was "Spatial-Temporal Mathematics" or ST Math, a game-based instructional software. We selected ST Math primarily because it was a tool that had been adopted by the district in which most of the PSTs in the program were placed for their junior-year field experience. District policy was that elementary students were to spend 90 minutes each week using ST Math. As a central focus of the mathematics methods course was noticing and responding to student thinking, we wanted to know how PSTs would use these skills in the context of the ST Math program.

ST Math is designed to use multiple dynamic representations of quantities and other mathematical objects to develop students' construction of mental images ahead in space and time (Peterson et al., 2004). In a previous analysis of student's engagement with ST Math, Yeo (2018) found that various ST Math activities ranged widely in their support for the development of mathematical concepts, with some activities providing strong connections to concepts and others requiring only surface level engagement.

This study shifts the focus from student thinking to PST noticing, exploring how novices make sense of student actions and speech when they are engaged with ST Math software. In particular, this study seeks to examine the following questions: (1) To what aspects of children's thinking do PSTs attend when the children are engaged in the ST Math activities, and how do the PSTs interpret evidence about this thinking? (2) How do PSTs draw on their noticing of children's thinking when evaluating the ST Math activities?

Theoretical Background

Teachers are aware of aspects of students' work in the classroom and they use this awareness to make pedagogical decisions (Goodwin, 1994). Awareness and sense-making of student's work have been described as *intentional noticing* (Mason, 2002) and were later expanded to include instructional responses and referred to as *professional noticing* (Jacobs et al., 2010). Professional noticing consists of three components: *attending* to student's mathematical ideas, *interpreting* their understanding, and *deciding how to respond* to their understanding. Studies have shown that PSTs can learn and develop these noticing skills through the support of teacher educators (Sherin & van Es, 2005), and that there are patterns in how PSTs apply the components (e.g., Wieman & Webel, 2019). The noticing framework has been extended to serve as a basis for a set of specific teaching moves (Jacobs & Empson, 2016), with the justification that one cannot act on information that one does not perceive. This idea is echoed in the establishment of "Elicit and use evidence of student thinking" as one of eight central teaching practice endorsed by the National Council of Teachers of Mathematics (2014). Noticing has also recently been extended to the evaluation of technological tools, such as interactive dynamic geometry activities (Smith et al., 2018). Previous studies have shown that teachers tend to evaluate online resources and activities positively with little consideration of mathematical or pedagogical features, but instead attended to surface level characteristic, such as whether students would be familiar with the problem types or if the activities had a game-like interface (e.g., Webel, Krupa, & McManus, 2015). In this study, we focus on how PSTs attend to and interpret children's thinking in the use of technological resources.

Methodology

The participants of this study were 21 elementary PSTs enrolled in a methods course at a Midwest university. Most participants were in their third year of a four-year program. The primary emphasis of the course was to understand how children's mathematical thinking develops in the domain of number and operations (Carpenter et al., 2014) and to develop the core teaching practices of eliciting and responding to children's thinking (Jacobs & Empson, 2016). Additionally, the PSTs engaged in daily one-to-one interactions with an assigned student in 3rd, 4th, or 5th grade (a "Math Buddy") during the whole semester.

In a series of course assignments, PSTs were first asked to explore a specified set of ST Math activities in small groups and respond to some reflection prompts. Then they later asked their Math Buddies, who were familiar with ST Math, to engage the same activities. PSTs were asked to elicit the child's thinking about the mathematics in the same tasks, and then finally to write a reflection paper about this experience. Three ST Math tasks (Figure 1) were chosen: *Pie Monster* (subtraction), *How Many Petals?* (place value), and *Building Expressions* (multiplication and division). These were selected to ensure that all of the children could engage in a developmentally appropriate task, and because, based on our previous engagement with ST Math, we believed they represented a range of opportunities for children to develop conceptual understanding. Specifically, the *Pie Monster* task involves whole number subtraction with various structures, such as start-unknown, change-unknown, and result-unknown (Carpenter et al., 2014) including three types of direct modeled representations. The screen (see Figure 1- left) uses two red-circles to represent the change (subtrahend), seven orange-circles to represent the start (minuend), and the white circles in the Monster's belly to represent the result (difference). When choosing the number of the white circles, JiJi (penguin character) attempts to cross the screen. If a provided answer is correct, the boxes are burnt by the Monster's fire and Jiji can cross the screen. If not, JiJi would go back to the starting place and one trial would be lost. The *How many petals?* task involves two-digit and three-digit place value concepts with the representations of petals (ones), flowers (tens), and a bunch of flowers (hundreds). Each tap on the 'ten' section on the screen (Figure 1-middle) collects ten petals, and so on for each place value. If the the 'ones' section has more individual petals than ten, a flower would be automatically made of the ten petals. Ten tens will automatically transform into a bunch of flowers (hundreds). The *Building Expressions* task involves the relationship between multiplication and division (e.g., 24 ÷ 4 = 6, 4 × 6= 24). A number of green dots must be selected according to the first number of a given number expression and the user decides how to drag the slider to partition the set of dots into the number of pink segments as designated by the second number. The quotient is the number of dots corresponding to each segment.

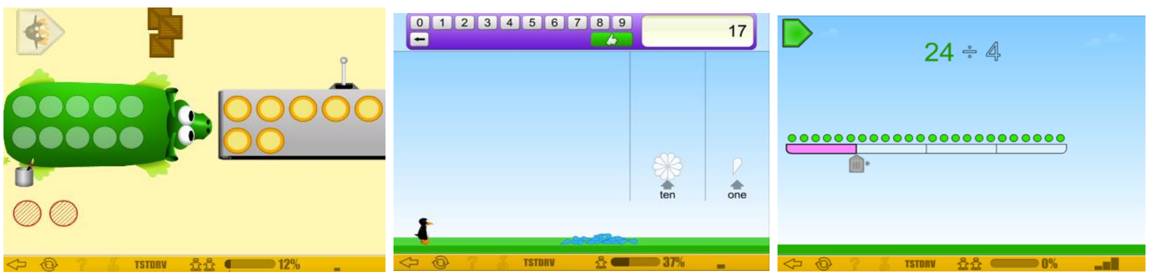

Figure 1: *Pie Monster* (left), *How Many Petals?* (middle), and *Building Expression* (right)

The data collected consisted of three parts: 1) "responses from the exploratory activity," in which groups of PSTs described, for each ST Math task, what mathematical ideas they believed the task was targeting, whether the task provided a "good opportunity" to learn those ideas, and what questions they would ask children to better understand their thinking while engaging in the task; 2) "screencasts" recorded while working with the Math Buddies, which captured manipulations on a tablet device and verbal explanations in real time; and 3) an individual reflection paper in which PSTs described the children's strategies, compared the strategies to how the children solved story problems, and gave an evaluation of each activity.

To address RQ1, PSTs' initial responses regarding the ST Math activities were categorized into *attending* and *interpreting*, and then additional data from reflection papers were coded similarly (see Table 1 for specific codes). In this analysis, we did not include the codes for *deciding how to respond* to focus on PSTs' evaluation of technological resources through the process of attending and interpreting. The screencasting data were reviewed to redefine and modify this coding scheme. Then, we examined responses from the exploratory activity to characterize each PST's initial evaluation of the ST Math activities. To address RQ2, this baseline was compared to their final reflection paper submitted after actually interacting with students. We noted whether the PSTs' evaluation of the ST Math activities changed, and how their noticing of student thinking appeared to influence their evaluation (or not).

Table 1: Noticing Coding Scheme in Exploratory and Reflective Phases

	Attending	Interpreting
Exploratory Phase	-Instructions -Visual representations -Manipulation -Mathematics concepts -Strategies	-Using sense-making to solve tasks -Potential mathematical concepts -Making a connection between representation and concept -Requirement of prior knowledge
Reflective Phase [additional codes]	-Task structures -Learning goals -Verbal explanations -Semiotic actions -Gamified features -Situated context	-Progression of problem-solving strategies -Solving tasks with given representations -Sequence of the tasks

Findings

In this section, we present our analyses of PST's attending to and interpreting children's mathematical thinking in the context of their ST Math explorations, focusing primarily on the *Pie Monster* task. We then discuss PST's noticing positive and negative features of ST Math in relation to potential learning opportunities for children, focusing primarily on the *How many Petals?* and *Building Expressions* tasks.

Attending to Children's Thinking

Attending refers to focusing on "noteworthy aspects of complex situations" (Jacobs et al., 2010, p. 172). PSTs understandably tended to pay more attention to student thinking when reflecting on their interactions with their Math Buddies, but they also showed evidence of noticing when reflecting on their own exploration of the ST Math tasks. That is, they anticipated how students would think about and solve the tasks, commenting on the instructions, the visual representations, ways the environment could be manipulated, the mathematical concepts in the tasks, and strategies students might use. For example, Group A attended to the instructions of the Pie Monster task (Figure 2), which represents whole number subtraction with various number choices.

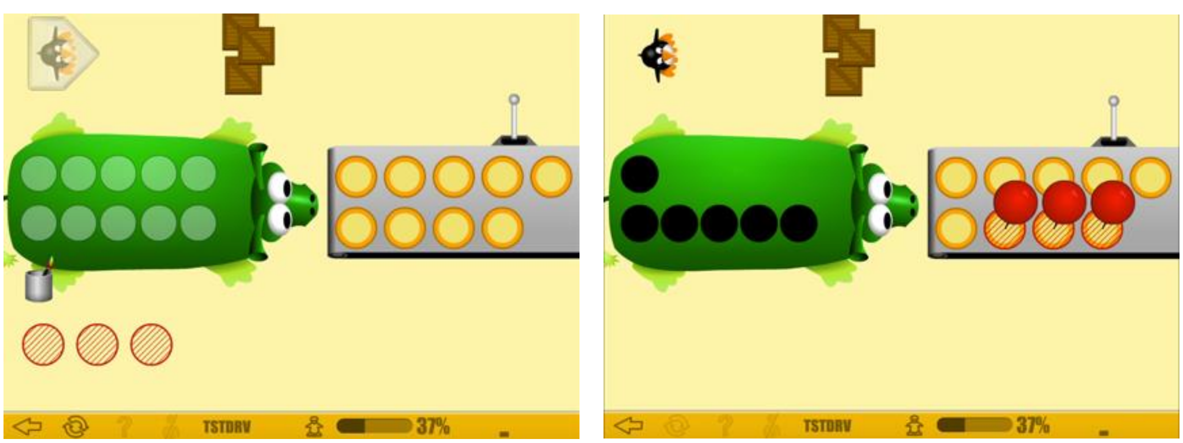

Figure 2: Problem Solving of the Pie Monster Task

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

This group noticed that the activity did not provide any instructions about how to play the game (this is a central design feature of ST Math). This unique feature of ST Math could lead to confusion about how to start and what they are supposed to do: "I feel like the game is simple but the instructions are not there and it can take them a while to figure out what numbers they need to subtract".

The initial attending pattern was expanded through one-to-one interaction with their student. PSTs attended to task structures, learning goals, verbal explanations, semiotic actions, gamified features, and situated context. For example, PST B reflected on her student's understanding of task structures:

> Something I noticed while asking him about the game is that...when the game changed from a result unknown problem to a change unknown problem, he recognized that shift in mathematical concepts.

Since there are various levels in one task, students should be engaged in multiple problem structures with a similar context. PST B attended to this transitioning of tasks and the mathematical structure of the Pie Monster task.

Interpreting Children's Thinking

Interpreting includes reasoning about children's strategies and comprehending their understanding based on details (Jacobs et al., 2010). PSTs' initial interpretations focused on how children might make sense of the tasks, what mathematical concepts they might engage with, how they might make a connection between a concept and the ST Math representation, and what prior knowledge might be required. For example, Group C anticipated a possible way to use sense-making when students solve the Pie Monster task: "Some kids might know how to visually play the game but not understand that they are actually doing subtraction." This group of PSTs anticipated that students might use the visual representations to solve the task without understanding the embedded mathematical concepts (e.g., subtraction).

In the reflection paper, PSTs' interpretations of student thinking included descriptions of how students solved the tasks, how they engaged with different representations, and how the sequence of the tasks impacted students' approaches. For example, PST D noticed the progression of her Math Buddy's strategies. The students used a guess-and-check strategy at first, but this strategy had changed to a counting strategy over time:

> She was beginning to use other strategies that weren't simply guess-and-check, such as counting on. She counted the red circles, then found that amount in the yellow circles. She then counted the yellow circles that were left to find the answer.

PSTs articulated how students' solutions to the ST Math tasks, including their actions and explanations, revealed evidence about their understandings.

PSTs' Evaluations of ST Math

Our data revealed that PSTs' evaluations of the ST Math activities were, in some cases, more negative after engaging in them with students. In other cases, they were more positive, and in the other cases, the evaluations appeared similar.

Increased negative evaluations. Seven PSTs had a positive evaluation based on their initial explorations, but during their interactions with children they began to question whether some features were likely to foster mathematical thinking relevant to targeted concepts. For example, PST E was positive about the potential of the How many Petals? task to develop place value concepts (Figure 3). The major mathematical idea in this task is to recognize 10 petals are the same as 1 flower and 10 flowers are the same as a bunch of flowers.

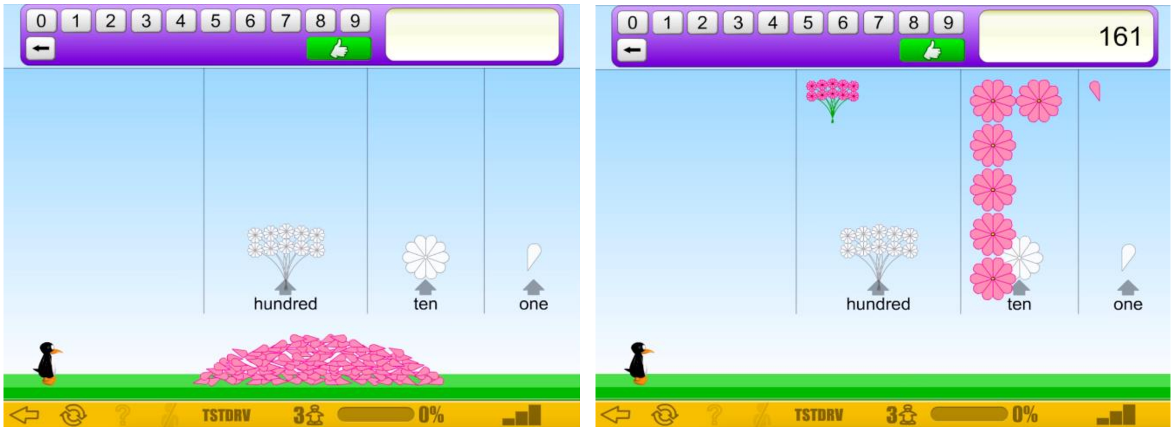

Figure 3: Problem Solving of the How many Petals? Task

PST E believed that the task could provide an opportunity to learn place value concepts (e.g., hundreds, tens, ones): “I think that the students can learn that they need 10 petals to make a flower and that they need to know how many flowers they have.” However, he noticed that his student was able to get the right answers by just tapping the columns repeatedly, and did not demonstrate an understanding of the relationship between different places.

> *PST E*: [Before solving the second problem] So on this one you explain all your thinking out loud and how you do it. So, what’s the first thing you do?
> *Math Buddy [MB]*: So, like if there is a big pile you press tens. These are all tens. And then or if you have not enough tens press ones.
> ...
> *PST E*: [At the third problem] Basically you just keep pressing tens until you run out.
> *MB*: Yeah.
> ...
> *PST E*: [At the final problem] What are you learning on this game you play? So, what do you learn when you do this?
> *MB*: I don’t really know.

Even though the student completed the task successfully, he was not sure what he was learning from the ST Math activity. PST E noticed this lack of understanding of the mathematical concept:

> My understanding is that my buddy just counts the petals and that’s it. You can even hear him clicking on the tablet screen rapidly to get rid of as many petals as you can. To me, there isn’t much learning going on during this game, other than being able to identify where the hundreds, tens, and one’s value is.

Initially, PST E believed that tapping the counting button could help develop an understanding of the relationship between ones, tens, and hundreds. However, when working with his Math Buddy, he noticed that the student was able to mindlessly tap the button until the solution was represented as a number of bunches, individual flowers, and petals. This made him evaluate the ST Math activity negatively ("isn't much learning going on").

Increased positive evaluations. Two PSTs had the opposite shift from a negative evaluation of ST Math to somewhat more positive evaluation, though these were sometimes the result of relatively sophisticated reasoning. For example, PST F initially criticized the *How many Petals?* task, anticipating that students might not use mathematical thinking: "The kids do not really have to do much thinking; they just need to memorize the different flowers." However, after she saw her Math Buddy demonstrate strong understanding in her explanation for why certain ones go in the tens column and others in the ones column, her evaluation was more positive:

> This leads me to believe that she was thinking mathematically rather than just playing the game without thought… I think that she does better at ST math because it is easier for her to visualize, as she uses direct modeling as her primary strategy for solving problems… I do not think ST Math should be discounted, it seems to be a big help for students to refresh on previously learned material.

PST F modified her evaluation of ST Math, focusing on its potential to "refresh on previously learned material." Indeed, the child appeared to be bringing her understanding of place value concepts to the task rather than developing it through the activity. The PST noticed this, and remained negative about its potential to "help students learn new material." Though the PST's evaluation is somewhat softened, we argue that the response shows a fairly sophisticated evaluation that includes some skepticism despite the child's "success."

Consistently negative. Our analysis showed three PSTs remaining consistent in their evaluations of ST Math. For example, PST G initially expressed the concern that her Math Buddy might use only an unsophisticated "counting by ones" strategy, "Because they can always count single units within the 10 petals, so as long as they can count by 1's they can finish the levels." When working with her Math Buddy, however, PST G noticed that the student focused on getting a right answer only without considering other strategies: "She seemed to just pick up patterns of how to pick out the correct answers and numbers to move to the next problem…. I do not believe that there is any strategy of solving the problem besides counting."

PST G was also worried about her student's misconceptions since it was possible to get the correct answers without understanding the base 10 structure of the petals representation: "It also may give students the impression that they understand the content just because they are able to find the pattern of the game and fill in the rest of the answers."

Consistently positive. Only one PST was included in this category. This PST kept evaluating ST Math positively. For example, she expected her Math Buddy to understand and use the relationship between multiplication and division embedded in the given pictorial model (e.g., dots, boxes) in the Building Expressions task (Figure 4). She initially appreciated the potential of the tasks, writing, "Students get to see a visual representation of every step of the multiplication and division process, furthering their understanding."

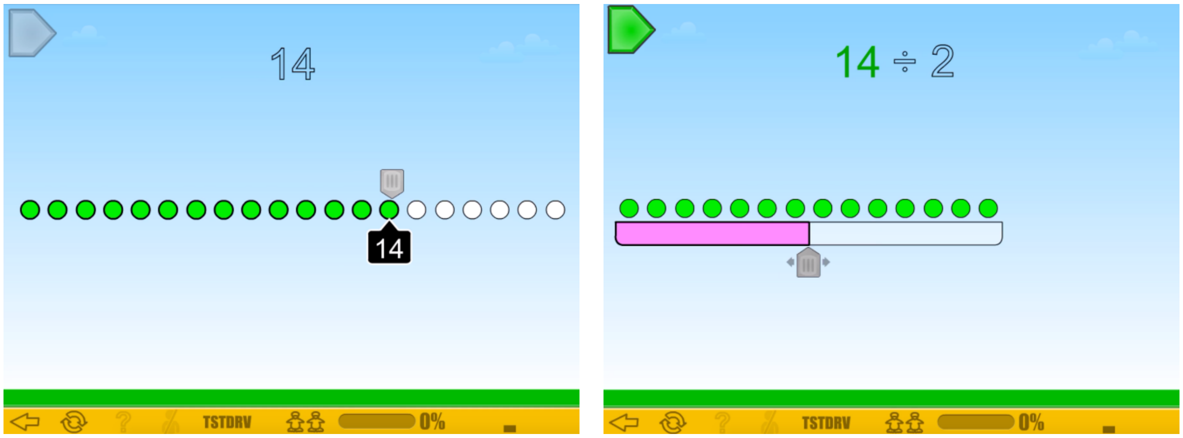

Figure 4: Problem Solving of the Building Expression Task

During the interview, PST H noticed her Math Buddy expressing more of his understanding with ST Math tasks, and eventually concluded that ST Math could provide good opportunities for children to engage in mathematical concepts and thinking: “When working with my Math Buddy I realized that ST Math reveals more about his thinking and understanding than story problems…This allows him to focus on showing his understanding of multiplication and division, or any other concept he is working on, instead of focusing on the words in a word problem.” Perhaps because her Math Buddy was an emerging English Language Learner, the PST interpreted his work in ST Math as evidence that the visual representation could support his mathematical understanding better than story problems.

Discussion

In this study, our data show PSTs’ attending to and interpreting student thinking during the use of ST Math tasks. Furthermore, we saw evidence of PSTs’ drawing on their noticing of student thinking in their evaluations of the ST Math tool, in some cases coming to different conclusions from their original evaluations, prior to working with students (e.g., PSTs E and G). Often, these conclusions were based not just on whether children were able to complete the tasks and answer with correct answers, but rather how they were thinking about the mathematical ideas embedded in the tasks (Dick & Hollebrands, 2011; Pea, 1985). This raises the possibility that developing noticing skill, in general, might help PSTs be better consumers of technology, especially if they are asked to evaluate tools while simultaneously attending to student thinking.

Notably, many PSTs were positive about the *Building Expression* task after the exploratory activity, explaining that the task seemed accessible and helpful in developing the concept with broken-down visual models. However, most changed their evaluation after engaging with students and seeing them struggle to make sense of the connection between the symbolic and quantitative representations. One possible implication is that interactions with real children, with whom PSTs have relationships, can stimulate critical reflection on the value of learning experiences, including those involving new technologies.

We illustrated PSTs’ attending and interpreting using example of ST Math tasks. These data extend earlier studies (e.g., Smith et al., 2018) to use field experiences in elementary schools for the development of technological noticing skills. We were encouraged to see several PSTs take a critical perspective on the use of technological resources based on their noticing of children’s mathematical thinking. We believe that starting with a tool (ST Math) that preservice teachers see in their placement classes increases the relevancy of the activity, and helps us better understand how noticing children's thinking influences their evaluations. We also realize the limitation of this single case would not represent all technological resource and need further studies to investigate the impact of different types of tools.

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"BIG IDEAS" IN SECONDARY MATHEMATICS EDUCATION PROGRAMS

Eryn Stehr
Georgia Southern University
estehr@georgiasouthern.edu

Hyunyi Jung
Marquette University
hyunyi.jung@marquette.edu

Jill Newton
Purdue University
janewton@purdue.edu

Although research and policy documents provide recommendations to inform secondary mathematics teacher preparation, no single study has addressed the "big ideas" of courses in multiple programs and how those big ideas may be interpreted through the lens of recent research and policy documents. To answer this need, we focused on big ideas and course objectives from three courses (i.e., Linear Algebra, Secondary Mathematics Methods, and Teaching in a Diversity Society) taught in four secondary mathematics programs. Major themes emerged related to mathematical content, pedagogy, and issues of equity. We describe findings related to big ideas, course objectives, and their connections to recommendations from current policy documents. Such integration contributes to promoting dialogue related to the preparation of mathematics teachers and informing teacher educators.

Keywords: Preservice Teacher Education, Linear Algebra, Equity

Given the challenges that future mathematics teachers will face in supporting their students for successful learning, preservice teachers (PSTs) must be guided by quality instruction to meet these challenges (Association of Mathematics Teacher Educators [AMTE], 2017). Extant research has addressed aspects of knowledge that teachers need to develop related to both content and pedagogy (e.g., Ball, Thames, & Phelps, 2008; Fuson, Kalchman, & Bransford, 2005). Specifically, teachers need preparation that "covers knowledge of mathematics, of how students learn mathematics, and of mathematical pedagogy that is aligned with recommendations of professional societies" (National Research Council [NRC], 2010, p. 123). Recent policy documents have been written by mathematicians, mathematics educators, and teacher educators to focus on: (a) how secondary mathematics teachers should be prepared (e.g., National Council of Teachers of Mathematics [NCTM], 2014; AMTE, 2017), (b) how mathematics should be taught---to future teachers or any undergraduates (e.g., Mathematical Association of America [MAA], 2018; NCTM, 2014), and (c) how university instructors should address issues of equity and access to mathematics, and communicate these issues to future teachers (e.g., AMTE, 2017; MAA, 2018; NCTM, 2000; 2014). Little is known about how such recommendations are incorporated into programs of study in mathematics for teachers (NRC, 2010). Although these policy recommendations inform secondary mathematics teacher preparation, no single study has addressed the "big ideas" of courses in multiple secondary teacher preparation programs and how these big ideas are interpreted through the lenses of recent research reports and policy documents (e.g., AMTE, 2017; Ball et al., 2008; Fuson et al., 2005; MAA, 2018). This paper utilizes data from a larger study that administered a national survey and conducted case studies to describe opportunities that secondary mathematics preparation programs provided for PSTs to learn about mathematical content, teaching practices, and issues of equity. In this study, we examined three courses required at four universities (i.e., 12 courses) to address the following question: *How do course goals and big ideas of courses in secondary mathematics teacher education programs emphasize areas related to content and teaching practices necessary for future mathematics teachers as recommended by policy documents?*

Relevant Literature

The teaching and learning of mathematics for K-16 students has been studied extensively, leading to detailed conceptualizations and descriptions of essential mathematics content and student experiences. To develop our framework for this study, we examined several national policy documents (e.g., AMTE, 2017; MAA, 2018; NCTM, 2000; 2014).

The authors of MAA's (2018) *Instructional Practices Guide* argued that "professional associations in the mathematical sciences along with state and national funding agencies are supporting efforts to radically transform the undergraduate education experience" (p. vii). The guide provided overviews and vignettes of effective mathematics teaching practices for undergraduate mathematics faculty. The introductory manifesto argued that mathematics instruction should incorporate experiences that allow access to rich and deep mathematics for all students, which requires ongoing change and attention to classroom, assessment, and task design practices.

The standards described in AMTE (2017) inform the preparation of preservice mathematics teachers, including "clearly articulated expectations for what well-prepared beginning mathematics teachers need to know and be able to upon completion of a certification or licensing program and the recommended characteristics for programs to support teachers' development" (p. xii). AMTE built on existing research (e.g., Ball & Forzani, 2011; Hill, Rowan, & Ball, 2005; Shulman, 1986) and policy documents (e.g., NCTM, 2000; NCTM, 2014). For example, AMTE (2017) stated that learning to teach mathematics requires deep understanding of the content they will teach, knowledge of how students reason about mathematics, knowledge of instructional approaches that support students' mathematical learning, and awareness of the societal context in which the content is used in students' everyday life. Similarly, several researchers proposed the importance of knowledge related to mathematical content, student thinking, and instructional approaches (Ball et al., 2008; Hill et al., 2005; Shulman, 1986), as well as the societal context in which mathematics is taught (Turner, Celedón-Pattichis, Marshall, & Tennison, 2009).

Teaching for access and equity is emphasized by researchers and professional organizations, including AMTE (2017), MAA (2018), and NCTM (2000). AMTE, for example, highlighted the importance of equity as the first of five foundational assumptions about mathematics teacher preparation, stating "Although equity, diversity, and social justice issues need to be specifically addressed as standards, they must also be embedded within *all* the standards…we believe that equity must be both addressed in its own right and embedded within every standard" (p. 1). Similarly, NCTM (2000) emphasized equity as the first "principle," highlighting the critical importance of ensuring *all* students have access to mathematics programs that provides quality instruction. MAA (2018) also emphasized inequities existing in our society, encouraging instructors to provide mathematics instruction that increases access to all students. While these standards provide recommendations for mathematics educators to shape their courses, little is known about how these recommendations are enacted in specific mathematics, mathematics education, and general education courses in secondary mathematics teacher education programs.

Method

As part of a larger study, we conducted a series of interviews in secondary mathematics teacher education programs at four universities: Great Lakes University (GLU), Midwestern Research University (MRU), Midwestern Urban University (MUU), and Southeastern Research University (SRU); the institutions were chosen based on the diverse nature of their student populations, the types of communities in which they were situated, and the departmental homes of their secondary mathematics education programs. At each university, the research team selected approximately ten required courses in the secondary mathematics teacher education program based on the likelihood that each course would offer students opportunities to learn algebra content and/or to learn to teach algebra; courses included mathematics, mathematics education, mathematics for teachers, and general education courses. For the purpose of this paper, we examined one Linear Algebra course, one Secondary Mathematics Methods course, and one General Education course related to teaching in a diversity society required at each university.

For each course, we collected a syllabus and interviewed an instructor. We asked each instructor, "What are the goals or big ideas of this course?" For mathematics content courses, we asked a follow-up question, "Do you do anything specific in this course to help prepare future mathematics teachers?" We analyzed course goals and big ideas as reported in the interview and written in corresponding course syllabi under "Course Objectives" or "Course Goals." To clarify statements from the big ideas question and course objectives, at times we examined other elements of a course syllabus or responses to follow-up questions.

To answer our research question, multiple policy documents (e.g., AMTE, 2017; MAA, 2018; NCTM, 2000; 2014) were reviewed by the three authors. We also examined instructor responses from interview transcripts and text from corresponding syllabi, noting emergent themes that were common and different across the courses (Creswell, 2007). We focused on emergent themes that related to recommendations in policy documents. We compared themes to summarize similarities and differences between course objectives in syllabi and instructor responses to interview questions. After writing a summary of responses, we iteratively reviewed their original responses, considering what they reported through the lens of selected policy documents.

Findings

We present findings from each course type in this section: linear algebra, secondary mathematics methods, and teaching in a diverse society. We compare similarities and differences between reported big ideas and course objectives, through the lens of policy documents.

Linear Algebra

The four Linear Algebra instructors understandably described their big ideas, their goals for Linear Algebra, in similar ways. For example, instructors reported focusing on: moving from or between concrete mathematical situations and abstractions (MRU, SRU); studying systems of linear equations and their solutions (GLU, MUU, SRU); eigenvectors, eigenvalues, and eigenspaces (GLU, MUU, SRU); computational applications (MRU, SRU); and learning ideas that are needed in other areas of mathematics and other disciplines (GLU, MUU, SRU). In this section, we describe how the four Linear Algebra instructors reported their intended classroom practices, assessment practices, and course (and task) design practices.

Three of the instructors (GLU, MUU, and SRU) described their modeling of teaching strategies that they believed teachers might notice and use in their own teaching. The GLU instructor explained that in his department, the culture is that mathematicians and mathematics education specialists work well together so he incorporated teaching strategies to be consistent with experiences in pedagogy courses. The MUU and SRU instructors felt their teaching strategies would implicitly support future teachers. Instructors' responses revealed intended instructional practices.

MAA (2018) described several strategies to support access to mathematics for all students through classroom practices. We focused on the use of groupwork, supporting productive struggle, and supporting critical thinking and reasoning. MAA strongly recommended use of small groups as a strategy for collaborative learning. In his syllabus, the GLU instructor urged students to work together on homework. He reported using groupwork extensively in the course, saying that students worked in small groups on activities to consolidate ideas, foreshadow ideas, or discover concepts. MAA described practical tips for supporting productive struggle, practice that is essential in mathematics. In alignment with these goals, the MUU instructor described his structure of class sessions as "lively," with spontaneous discussions of student questions and struggles: “going through the challenges that they also went through and showing how they have overcome that.” MAA described strategies for “responding to student contributions in the classroom,” especially “creating a safe space for incorrect answers” and “focusing on reasoning” (pp. 5-6). The SRU instructor described that he pushed all of his students to explain their reasoning with a focus on explanations, proofs, counter-examples, and holding students accountable for addressing the why rather than just stating facts.

MAA (2018) recommended using multiple forms of assessment, including formative assessment cycles and summative assessment when appropriate. In their syllabi, instructors described their assessments as including: three exams and a final exam (all), weekly homework (GLU, MRU, SRU), weekly quizzes (GLU, MRU, MUU), and computer lab activities (GLU). Although neither assigned points for student journaling, the MUU and GLU instructors reported that they encouraged students to write mathematical journals. The MUU instructor’s syllabus did not indicate his expectation for writing journals; however, he reported that, in the beginning weeks of the course, he frequently told students when certain ideas from their homework or class notes should be written in their journals. After several weeks, he said he would stop pointing out these ideas, expecting students to take ownership of their needs for the journals. The GLU instructor gave a clear description of expectations for mathematical communication and his expectation for students to keep “a well-organized record of all your study notes and completed problems for future reference.”

Providing clear learning goals to students is recommended as a course design practice in MAA (2018). In his syllabus, the MUU instructor listed clear learning goals for students that included attention to content and process; for example, “Be able to apply some technology...to facilitate problem solving.” Regarding task design, MAA drew on Stein et al. (1996) to recommend that students have opportunities to engage in high-level tasks that allow multiple solution strategies: “there is not a predictable, well-researched approach or pathway explicitly suggested by task instructions” (p. 31). MAA also drew on Boaler (2015) to recommend that instructors open tasks to provide open learning spaces using several strategies, including “open the task up to multiple methods, pathways and representations” (p. 40). The GLU instructor reported that he modeled valuing mathematical processes and encouraged students to recognize the potential for alternative, and equally valid, solutions or approaches.

Secondary Mathematics Methods

The four secondary mathematics courses share commonalities and differences in their big ideas. Common features of these courses included the practices of planning and implementing mathematical lessons, analyzing students’ mathematical thinking, and exploring instructional materials. While three courses (GLU, MUU, SRU) focused more on pedagogical content (e.g., learning to assess student learning or identify appropriate questions), the MRU course centered around the reconstruction of school mathematics (e.g., ratios and proportional reasoning, integers) to envision how PSTs would communicate mathematically with their students. In this section, we describe how these secondary mathematics methods instructors provided opportunities for PSTs to develop knowledge and practices.

Most secondary mathematics courses involved field experience components that required the design and implementation of lesson plans. AMTE (2017) recommended that effective secondary preparation programs provide PSTs with multiple opportunities to learn to teach through clinical experiences with coherent, developmentally appropriate contents. The GLU instructor, for example, mentioned a big idea addressing this recommendation: “We spend a great deal of the semester focusing on this cycle [teaching-learning cycle]… you start developing plans…then you implement the plans and then the cycle goes around.” Similarly, the SRU instructor emphasized planning and implementing mathematical lessons as a big idea. PSTs in her course had the opportunity to learn about facilitating classroom discourse and using appropriate instructional strategies in the field, while they concurrently took a methods course in which they discussed and reflected on relevant readings. Learning about teaching through the design of instruction addresses the knowledge of content and teaching (Ball et al., 2008), one of the domains of mathematical knowledge for teaching. In selecting and implementing tasks for teaching, orchestrating effective classroom discussions is essential. PSTs pose purposeful questions to probe students’ mathematical ideas and make mathematical structures visible (NCTM, 2014).

While PSTs designed, implemented, and reflected on their lessons, instructors intended that PSTs develop their understanding of student thinking. Anticipating what students are likely to think about mathematics is relevant to knowledge of content and students (Ball & Forzani, 2011; Ball et al., 2008; Shulman, 1986). AMTE (2017) recommended that well-prepared PSTs are committed to deepening their knowledge of students’ mathematical skills and dispositions. The GLU instructor addressed this need: “As you are talking to the student [during implementing your lesson plan], what information did you gather, what did it tell you about, and then what support did you need to provide as a result of that?” Similarly, the MUU instructor provided PSTs with the opportunity to interview a middle or high school student to ascertain the student’s beliefs about mathematics and knowledge of a particular mathematical topic. PSTs were to develop a series of questions and performance-based tasks that they would pose to the student during the interview. In a writing report, PSTs would describe the student’s mathematical knowledge and beliefs based on their analysis of data gathered during the interview.

All instructors incorporated opportunities for PSTs to explore a variety of instructional materials. AMTE (2017)’s standards address this opportunity that well-prepared preservice teachers analyze and discuss curriculum and standards documents. Knowing about appropriate instructional materials and their characteristics is an essential teacher knowledge to be developed (Ball et al., 2008; Hill et al., 2005). The GLU secondary mathematics methods instructor included a goal that addressed this knowledge: “to acquaint the teacher assistant with available instructional resource materials such as curricula, professional journals, and relevant research.” The MUU instructor also described a specific activity PSTs engaged in during the semester, in which they explored and critiqued textbooks. She said, “I have them look at materials [the traditional course sequence versus integrated math courses] and evaluate them, and that’s the subject of again, usually a class discussion about what they think the opportunities are that are afforded by these textbooks, versus traditional textbooks, the challenges of teaching math in this way.” PSTs in her course then observed the way integrated math curricula were being taught and evaluated the implementations of the curricular.

While all four courses emphasized PSTs' development of pedagogical content knowledge, a course offered by MRU emphasized PSTs' development of specific mathematical contents for teaching. AMTE (2017) stated that learning to teach mathematics requires "a central focus on mathematics" (p. 2) and flexible knowledge of school mathematics. The MRU secondary methods course was the only course among the four in which developing specific school mathematical concepts was a course objective. The syllabus highlighted: "The mathematical topics that we will examine are ratios and proportional reasoning,...and quadratic relationships and factoring. These are BIG ideas in middle school and early high school mathematics, and they are important for reasoning algebraically." PSTs in this course were asked to keep a three-ring binder of problems exemplifying these topics. The instructor stated in her syllabus, "One of your greatest assets in understanding students' mathematical thinking is understanding and deepening your own mathematical thinking." PSTs generated mathematical conversations with each other, reflected on their own mathematical knowledge around these mathematical concepts, and used their mathematical knowledge to design problem sequences for students. They submitted the binder of problems to receive feedback and points for thoroughness, organization, explanations and analysis of targeted problems, quality of problem sequence and discussion, and mathematical correctness.

Teaching in a Diverse Society

Each of the four SMTE programs under review required a course related to diversity; the title of this course varied across program, however to protect anonymity, we gave all courses the generic title: Teaching in a Diverse Society (TDS). What was common across these courses was that they were general education courses, taught by female instructors who were not associated with mathematics or mathematics education; therefore, the curriculum was not subject-specific and students from multiple education disciplines enrolled simultaneously. Several themes emerged from TDS course big ideas and objectives provided by the instructors.

First, the instructors emphasized their attention in the course to highlighting the vast number of ways in which diversity is present in the United States; all instructors highlighted multiple aspects of diversity. For example, when the SRU TDS instructor was asked about the big ideas of the course, she said "we talk about race and ethnicity; we talk about class, gender and sexual identity, exceptionality, like special needs students. We talk about language, geography, religion. And really, my goal at the end is that students would…be ready to teach in a diverse environment." No one aspect of diversity was mentioned in all four TDS courses; however, race, culture, ethnicity, religion, sexual orientation, ability, and language were each mentioned in three of the four courses. The second theme emphasized across the TDS courses was the idea that schools are situated in historical, socio-political, and geographic contexts. For example, the MUU instructor emphasized the local context, including "Understand the impact of family and community in the learning experiences of English language learners in the classroom" as a course outcome. Taking a more national approach to context, the GLU instructor stressed the importance of PSTs' "understanding how their work in the classroom and in the schools is a part of democratic practice in the United States."

The third and most prevalent theme highlighted across the TDS courses was the impact of diversity (theme 1) and historical, socio-political, and geographic contexts (theme 2) on educational opportunities in particular schools and for particular learners. TDS instructors highlighted strategies they used to attempt to mitigate these effects. For example, the MUU instructor discussed opportunities that she offered for PSTs to engage in investigating students' school experiences, including reading and discussing articles such as *Nothing to Do: The Impact* *of Poverty on Pupil's Learning Identities* (Muschamp, Bullock, Ridge & Wikeley, 2009) and *Barbie Against Superman: Gender Stereotypes and Gender Equity in the Classroom* (Aksu, 2005). In addition, the PSTs analyzed U.S. federal laws developed to ensure all students access to education (e.g., Individuals with Disabilities Education Act [IDEA], McKinney-Vento Act [protects the rights of homeless children]).

Several TDS instructors also mentioned critical reflection as an important activity for PSTs in developing dispositions and skills for teaching diverse learners; the MRU instructor made this focus on reflection explicit in her syllabus: "We will explore various realms of diversity…As part of that exploration we will engage in significant reflection, written and oral, personal and collective, challenging our assumptions, and questioning our beliefs." Similarly the GLU course was described as: "grounded in the idea that an essential aspect of good teaching is having the time and space to reflect upon the kinds of issues that impact your pedagogy and instruction."

The attention to the impact of diversity and contextual factors on educational opportunities reported by TDS instructors is well supported by professional mathematics and mathematics education organizations, including AMTE, MAA, and NCTM. In fact, attention to historical inequities in mathematical learning opportunities is highlighted front and center in NCTM (2000) as the first "principle," in AMTE (2017) as "Assumption #1," and in the Manifesto of MAA (2018). NCTM (2000) expressed concerns about pervasive low expectations and tracking practices, and less challenging mathematics curriculum for "students who live in poverty, students who are not native speakers of English, students with disabilities, females, and many nonwhite students" (p. 13). They highlight engaging curriculum, use of technology, enhanced assessment practices, and increased attention to mathematics processes (beyond memorization and symbolic manipulation) as possible mitigators to increase equity in mathematics classrooms.

AMTE (2017) took a similarly strong stance toward the need for teacher education programs' commitment to preparing teachers who have the skills and dispositions to teach all learners: "Assumption #1: Ensuring the success of each and every learner requires a deep, integrated focus on equity in every program that prepare teachers of mathematics" (p. 1). The authors repeatedly emphasized the disparate opportunities resulting from historic discrimination and sociopolitical factors, and stressed the importance of preparing teachers who are advocates for their students with these disparities in mind: "Well-prepared beginning teachers embrace and build on students' current mathematical ideas and on students' ways of knowing and learning…They also attend to developing students' identities and agency so that students can see mathematics as components of their cultures and see themselves in the mathematics" (p. 13). The authors recommend opportunities for PSTs to critically analyze current mathematics education systems, challenge deficit views about student learning, recognize the key roles that identity and power play in mathematics education, and spend time in community settings to learn from and about students, families, and communities.

MAA (2018) continued this call for first, recognition that these systemic inequities exist, and second, action to change the status quo: "Inequity exists in many facets of our society, including within the teaching and learning of mathematics...We owe it to our discipline, to ourselves, and to society to disseminate mathematical knowledge in ways that increase individuals' access to the opportunities that come with mathematical understanding" (p. vii). The authors describe the statistical disparities of underrepresented populations among both mathematicians and university students in mathematics departments, and encourage instructors to beware of implicit and explicit messages being sent to students about who "belongs" in mathematics.

Discussion & Implications

Although recent policy documents provided recommendations about how secondary mathematics should be prepared to learn about mathematics, mathematics teaching, and equity issues in mathematics teaching and learning (AMTE, 2017; MAA 2018; NCTM 2000; 2014), little is known about how such recommendations are integrated into mathematics teacher education programs (NRC, 2010). To promote dialogue related to the preparation of secondary mathematics teachers, this study highlighted ways in which course goals and big ideas in secondary mathematics teacher education programs emphasized areas related to mathematics learning, teaching, and issues of equity and access as recommended by policy documents.

Across all course types, we noticed that many policy and research recommendations were addressed, both explicitly and implicitly. In fact, many of the "big ideas" reported with closely related to these recommendations. For example: The GLU Linear Algebra instructor pointed to the strong mathematics-mathematics education community in his department as leading him to experiment with multiple teaching strategies that he hoped would align with his students' future teaching needs. The MRU secondary mathematics methods course centered around the reconstruction of school mathematics, such as proportional reasoning and integers, to address PSTs' development of big mathematical ideas for teaching. The MUU TDS instructor provided opportunities for PSTs to engage with their students' families and communities.

In this study, we confirmed that, although the four universities required similar versions of these three courses (i.e., Linear Algebra, Secondary Mathematics Methods, and Teaching in a Diverse Society), the courses were also unique. Therefore, the experiences of PSTs across these programs will be different, likely as a result of many factors, including geography, program emphases, and priorities of the course instructor.

This study is intended both to build on existing research (e.g., Ball et al., 2008; Fuson et al., 2005; Hill et al., 2005; Turner et al., 2009) and policy documents about mathematics teacher preparation (e.g., AMTE, 2017; MAA, 2018; NCTM, 2000) and to encourage researchers to explore areas that are less well investigated. We acknowledge that creating programs that are coherent across multiple departments and disciplines is often a challenge. We wonder what creating a culture of communication among the dozens of faculty who support future teachers at each university would look like. We wonder how such a culture could naturally build coherence by discussions asking: What are the big ideas of our program? What are the fundamental ideas we want threaded throughout the program? How do we ensure that students have multiple opportunities to encounter these ideas, building on each other, through the program?

From the findings of this study, next steps would include an investigation of how big ideas play out in written curriculum (e.g., textbooks, course materials, other resources) and enacted curriculum (e.g., classroom instruction) as well as an investigation of how PSTs perceive the opportunities provided throughout their secondary mathematics teacher education programs. What would be the benefits of an in-depth exploration of each of our programs? How would such a conversation get started? How would it be sustained? For example, from a stance of equity, how might TDS instructors interact with mathematics and methods instructors to stimulate conversation about opportunities for discipline-specific, equity-related experiences?

Integration between course goals and policy documents contributes to promoting dialogue related to the preparation of secondary mathematics teachers. Our study presented here, connecting big ideas of multiple courses with recommendations from research and recent policy documents, promises to inform teacher educators, especially those who are new mathematics teacher educators in the field. Future work analyzing our larger data will provide insights into how other areas of recommendations from policy documents play out in practice; this current work is an initial step to illustrate big ideas of required courses commonly offered by the four case study universities.

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CASES OF LEARNING TO RESPOND TO ERRORS THROUGH APPROXIMATIONS OF LEADING WHOLE-CLASS DISCUSSIONS

Erin E. Baldinger
University of Minnesota
eebaldinger@umn.edu

Matthew P. Campbell
West Virginia University
mpcampbell@mail.wvu.edu

The emerging use of approximations of practice in teacher education calls for ways to document teacher candidates' (TCs') skill from their enactments and to determine how TCs' skill develops through such pedagogies. We highlight two cases of secondary mathematics TC development, connecting analyses of two types of approximations—coached rehearsals and scripting tasks—with a focus on the practice of responding to errors in whole-class discussion. Each case illustrates a distinct example of TC development. This work contributes to the research on pedagogies of practice in teacher education by offering approaches for understanding TC practice and development through multiple data sources.

Keywords: Teacher Education-Preservice; Instructional Activities and Practices

The use of approximations of practice (Grossman et al., 2009) in teacher education offers a promising way for teacher candidates (TCs) to develop skill with key aspects of the work of teaching. There exists the need to document TCs' skill from their enactments and to identify the way in which TCs' skill develops through such designs (Janssen, Grossman, & Westbroek, 2015; Shaughnessy, Boerst, & Farmer, 2019). In this paper, we connect analyses of two types of approximations—coached rehearsals and scripting tasks—to investigate the relationships between learning opportunities and evidence of learning captured across time. We use two cases to illustrate what can be learned about TC development through these multiple data sources.

Literature Overview

Leading whole-class discussions is complex work. It includes eliciting, responding to, and building upon student contributions in ways that move the discussion toward mathematical goals (Boerst, Sleep, Ball, & Bass, 2011; Leatham, Peterson, Stockero, & Van Zoest, 2015; Sleep, 2012). While errors—contributions that are incomplete, imprecise, or not yet correct—are key to students' learning process (Brodie, 2014), they are often positioned negatively when teachers quickly correct them or avoid them entirely (Bray, 2011; Santagata, 2005; Tulis, 2013). This can potentially remove opportunities for students to make sense of errors. During whole-class discussion, teachers must also balance the needs of the student who contributed the error with the needs of the rest of the class, who may or may not share that student's conception. For these reasons, we focus on this practice in our work investigating TC learning.

Coached rehearsals, an approximation of practice, can afford opportunities for TCs to develop a vision of ambitious and equitable practice; understandings of students and content; dispositions regarding students, content, and teaching; and a repertoire of practices and tools (Ghousseini & Herbst, 2016). In this approximation, one TC takes on the role of teacher while other TCs take on the role of the students, and the teacher educator provides in-the-moment coaching (Kazemi, Franke, & Lampert, 2009). TCs engage in the interactive work of teaching, such as making sense of and responding to student reasoning during discussion. TCs' enactments during approximations of practice also provide a lens for teacher educators and researchers to assess TCs' developing skill and their coordination of approaches and goals.

We also use "scripting tasks" as an approximation that puts TCs in a position to make sense of and respond to student reasoning. In designing these tasks, we draw on research around scripting classroom interactions (Crespo, Oslund, & Parks, 2011; Zazkis, 2017). TCs are presented with a classroom scenario and then demonstrate, through written dialogue, how they might continue the discussion. These dialogues represent, in part, TCs' imagined response to a particular student contribution. They also represent TCs' sense of how students might contribute further, giving insight into TCs' view of what is reasonable or desirable in a classroom episode.

Perspective on Teacher Learning

Hammerness and colleagues (2005) assert that teachers must be supported in learning communities and enabled to develop tools and practices, vision and dispositions, and understandings. This framework guides our research on TC learning through approximations. Tools and practices encompass a sense of when, where, why, and how to do the work of teaching. Vision represents teachers' sense of where they are going and what is possible in teaching, which are connected to dispositions, which relate to commitments toward professional growth and inquiry into practice. Understandings represent a teacher's deep knowledge of their subject and how to make it accessible to others, including knowledge of how students learn and develop particular ideas. Taking this perspective, we ask the following research question: What forms of TC learning are evident through their work with multiple approximations of practice? In particular, we focus on evidence of TCs' practices and vision as they develop over time.

Methods

Our work is the product of an ongoing, multi-year collaboration situated in secondary mathematics methods courses at two large, public research institutions. At "Institution A", TCs are enrolled in the methods course as part of a yearlong post-baccalaureate licensure program. TCs from "Institution B" are enrolled in a shared methods course across multiple programs. At both sites, TCs completed the scripting task (Baldinger, Campbell, & Graif, 2018a) twice: early in the methods course ("Initial") and then again near the end of the course ("Follow-up"). The scripting scenario we highlight in this paper depicts whole-class discussion around a sorting task (Baldinger, Campbell, & Graif, accepted), where students are asked to sort shapes into examples and non-examples of polygons. The scenario concludes with a student, "Jessie," contributing an error. TCs are prompted to assume the role of the teacher and write a dialogue of how the conversation would continue. TCs also write a rationale for how they constructed their dialogues.

In between the two implementations of the scripting task, TCs participated in coached rehearsals of a sorting IA (Baldinger, Selling, & Virmani, 2016) focused on defining linear functions. For all rehearsing TCs, the mathematical focus and the sets of cards to be used were provided by the teacher educator, with common materials used across sites. In order to ensure that these rehearsals include opportunities for responding to errors, we used "planted errors" (Baldinger, Campbell, & Graif, 2018b). Non-rehearsing TCs contributed these instances of student thinking during the rehearsal, enabling the teacher educator to stay in the role of coach while providing more authentic student voice to the rehearsal. After a set of rehearsals, TCs completed reflections, in part through video annotation, which serve as additional data.

For this paper, we zoom in on the experiences of two TCs, one from each site, purposively sampled to represent the range of TC learning that might be visible through these different approximations of practice. We selected TCs who rehearsed one of the sorting IAs. From Institution A, we highlight the experience of Greg (all names are pseudonyms). Greg was often quiet during class discussions, and showed off his engaging personality during interactions with students. In his sorting IA, students were asked to sort graphical representations into examples and non-examples of linear functions. The planted error involved a student asserting that the graph of a vertical line ($x = 2$) was a linear function because it looked like a straight line.

From Institution B, we highlight the experience of Travis. He regularly exhibited thoughtfulness about the work of teaching in his contributions in the class. In Travis's sorting IA rehearsal, students were asked to sort tabular representations into examples and non-examples of linear functions. The planted error involved a student looking only at the change in the y-values in a table and concluding that the table did not represent a linear function because the change in y was not constant. However, the change in x-values was also not constant, and looking at the changes in x- and y-values together would reveal that the table did have constant slope.

Using the themes developed through our analysis of data from the larger study, we looked holistically across the data in each case for evidence of learning related to each aspect of framework for learning to teach (Hammerness et al., 2005). We considered features of each approximation such as the types of teaching moves used, the representations of student voice, and the way in which each TC attended to the mathematics. We investigated vision and dispositions through exploring TC reflections on their own practice. In presenting these findings, we aim to develop a picture of each case not for the sake of comparing the two cases, but rather to illustrate the range of learning captured across these different approximations of practice.

Findings

In this section we share the cases of Greg and Travis to provide two distinct images of TC learning, and two instances of how that learning can be documented through coordinated analysis across two distinct approximations of practice.

Trying New Practices, Changing Vision: Greg

Scripting task. Greg demonstrated some notable differences in his two responses to the scripting task polygon scenario (see Table 1). In the Initial dialogue, Greg first calls students' attention to the "extra line" in Shape J. This starts a series of funneling questions to get Jessie to quickly correct how shape J is sorted. In contrast, the opening move in the Follow-up dialogue asks for additional arguments in support of classifying shape J as a polygon.

Table 1: Greg's Initial and Follow-up Dialogues

Initial Dialogue		Follow-up Dialogue	
Teacher:	well what about this extra line here? Does it make a difference?	Teacher:	Who can tell me another reason why shape J is a polygon?
Jessie:	Idk, maybe	Student:	Like Rosalia said, all of the sides are straight.
Teacher:	What does it mean to be a square?		
Jessie:	All sides are equal length and opposite sides are parallel	Teacher:	Who can tell me why shape J might not be a polygon?
Teacher:	So does this fit the definition of a square?	Student:	There is that line in the middle so it is not really a square.
Jessie:	No, that line isn't parallel to anything	Teacher:	Can someone expand on what _____ just said?
Teacher:	So what can we say about it being	Student:	Well that segment has one end not

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	a polygon?	meeting any other sides.
Jessie:	that it is not because that line only connects to that one edge and the other lines have to [*sic*] edges	

The Initial dialogue includes only conversation between the teacher and Jessie, making no effort to incorporate other students. In the Follow-up dialogue, though the students are not named, it is possible that up to three different students participate in these few turns of talk. Another interesting feature of the Follow-up dialogue is how it concludes without any move to lead students toward a conclusion. It feels much more like a snippet of a longer conversation, as opposed to the Initial dialogue, which feels in some ways like a completed conversation.

Greg's thinking about why he constructed the dialogues in this way also changed (see Table 2). At first, he intended to draw on the definition of a square in order to correct the sort and move the discussion forward. In contrast, on seeing this scenario for the second time, Greg focused much more on engaging other students. Though he acknowledged Jessie had incorrectly sorted shape J, correcting that error was no longer the focus of his dialogue.

Table 2: Greg's Initial and Follow-up Rationales

Initial Rationale	Follow-up Rationale
The student is having a misunderstanding to what a square is, so I believe it is important for them to re-think what a square is and modify what they are saying about Shape J. From there it might be easier for the students to recognize whether it is a polygon or not.	Well I would want other students to think about why it is possible for J to be a polygon, but then I would also want other students to explain why they think it is not a polygon. Although it is not a polygon I want students to be thinking about both reasonings.

Greg pointed out many of these differences himself. In reflecting on how his response to the scenario changed, he wrote, "My questioning is much different, I tried to expand more one the original ideas rather than going straight to finding the 'right' solution." Greg did not see his Initial responses when he wrote this, but his recollection is strikingly accurate. The shifts in his dialogues and rationale, along with this final reflection, suggest changes in Greg's practices related to responding to errors, as well as his vision for dealing with errors.

Rehearsal. Evidence of Greg's changing practices and vision of responding to errors is clear in his rehearsal. Greg was the second of three TCs to rehearse the sorting IA. The rehearsal included discussion of five cards: two easy-to-sort examples of a linear function, two easy-to-sort non-examples, and one boundary case. The planted error was the first card contributed. Greg responded to the error by experimenting with orienting moves: "It's a straight line. Alright, you are correct, it is a straight line. Does anyone agree with this? Who agrees with this?" After one student agreed, Greg said, "Alright, does anyone disagree with this?" Several students contributed some disagreements, and Greg restated and recorded their reasoning. He then checked back in with the group that originally contributed the error:

Teacher: Okay, interesting. So, going back to your group, what do you think about this? Does your opinion change?

Student: It doesn't because my idea of what a linear function was, was a straight line. And even thought that line is up and down, it is definitely straight, there's no curves.

Teacher: Sounds good. Alright, we'll leave that here for now. We can come back to it.

Greg experimented with a tabling move, allowing the disagreement among the class to go unresolved for the moment as he moved the discussion on to additional cards.

Greg's approach to responding to student thinking was relatively similar for the contributions without errors. He continued to experiment with orienting moves, seeking agreement and disagreement from students. He recorded student thinking and received feedback from the teacher educator about his practice. Later, students discussed the graph of a step function. Some argued that the step function was not a line, and the teacher educator inserted disagreement, saying that the step function had constant slope, and thus represented a linear function. Through this conversation, Greg helped the class discuss the vertical line test as a way to determine whether or not a graph represented a function, and that led to the following conversation:

Teacher: Can someone give me a reason that they disagree that it needs to pass the vertical line test in order to be a linear function?
Student 1: Yeah, because of Graph D [the vertical line graph] that we were talking about earlier. For that one to be a function.
Student 2: But it's not a linear function.
Teacher: But it's a nonlinear function?
Student 2: But it says it's a linear function.
Student 1: Oh yeah, we left it in a gray area. Bah. Alright.
Teacher: Going back to [Graph D], do we all agree that it needs to pass the vertical line test in order to be a linear function? [Some students nodding] Unless someone's going to disagree? Give voice to that? [Pause] So if we come back to this [Graph D], does it pass the vertical line test?

This exchange shows how Greg was able to revisit the planted error while keeping the focus on student thinking. Many of Greg's moves in the rehearsal are consistent with his Follow-up dialogue. Following the rehearsal, Greg reflected that he "learned better ways to facilitate discussions by using better questions." He felt that "The timeouts were quite helpful, as I was able to stop the lesson and go back and fix what I did wrong." This illustrates Greg's intentional work on questioning and his changing vision about how to best respond to errors in the moment.

Complexity in What Gets "Taken Up" from Rehearsals and Coaching: Travis

Scripting task. Travis uses similar moves in both dialogues, but they are used in seemingly more productive ways over time (see Table 3). For example, the Follow-up dialogue starts with an orienting move instead of a question probing Jessie's reasoning. Also, the third line of each dialogue elicits agreement (or disagreement), yet the move in the Follow-up dialogue is not focused only on Jessie's contribution. These changes show shifts in approaching the error—from something to be targeted to something that is part of a broader conversation.

Table 3: Travis's Initial and Follow-up Dialogues

Initial Dialogue		Follow-up Dialogue	
Teacher:	What is your reasoning for determining that Shape J is a square?	Teacher:	Rosalia, can you re-state what Jessie was trying to say about Shape J?
Jessie:	Well it has 4 straight sides of equal	Rosalia:	I think Jessie said that Shape J is a

	length that are connected.		polygon because it is a square.
Teacher:	Does anyone else agree with Jessie, that Shape J is a square, and therefore a polygon?	Teacher:	Alright, does anyone agree or disagree that Shapes Q and J are polygons because they are squares?
Melinda:	I disagree that it is a square and a polygon.	Student:	I don't think Shape J is a square. A square only has 4 sides that are straight lines. That shape has 5 straight lines.
Teacher:	And why do you believe that?		
Melinda:	Well it looks like a square with an extra line inside of it, and that line isn't connected to another line on both sides. So it isn't a square or a polygon.	Teacher:	What do you think about what student said about this shape not being a polygon because it isn't a square?
Teacher:	Could you say in different words why you think it is not a polygon?	Jessie:	If we are looking at the shape as a whole, then it would make sense that it isn't a square. If you ignore that diagonal line, we have a square, but I don't think we can do that after hearing that explanation.
Melinda:	The figure has a side that is not connected to two other sides. (Teacher records this reasoning on the board)		

Another difference is the way the error gets resolved. In the Initial dialogue, since only Melinda's idea is recorded, it suggests that the error has been corrected without any input from Jessie. Alternatively, in the Follow-up dialogue, the teacher checks back in with Jessie after another student contributed a disagreement. While Jessie appears to become convinced with this new information, we see how Travis is considering ways to involve Jessie in that work.

We can connect these observations to Travis's rationales (see Table 4). His Initial rationale was focused on probing Jessie's thinking and eliciting other students' ideas. The rationale also confirms the inference that the last moves were an effort to highlight correct ideas about polygons. The Follow-up rationale details the deliberate decisions being made to use moves that clarify the ideas being discussed, to elicit agreement or disagreement (though without singling out a particular idea), and to go back to Jessie as part of resolving the error in-the-moment.

Table 4: Travis's Initial and Follow-up Rationales

Initial Rationale	Follow-up Rationale
It is always useful to allow students to explain their reasoning out loud. It helps this practice explaining their thoughts in mathematical terms as well as giving other students opportunities to engage with each others' [*sic*] thoughts. Then the teacher asked for another student's opinion to give another student a chance to either restate what has already been discussed or to give a different opinion/thought process on the situation. The teacher asked questions to help the student try to dive deeper into why she did not think it was a polygon and therefore getting more information out	I think it is important to have crucial parts of the discussion be re-stated so that everyone is clear about what we are discussing specifically. I also thought it was better to ask the class to agree or disagree with both shapes Q and J so that it did not seem as if the teacher was singling out one of the cards, giving a cue to the students that the point made about the card was probably wrong. After a correct description was given about the card, it is important to go back to the student who had an incomplete conception about the card to ensure that they understand

there to help students understand the requirements/definition of a polygon.	where their mistake was and why the card can be sorted as a non-example.

Based on his recollection of his Initial response, Travis noted that his focus on “coming back to the original student who made the statement with an incomplete conception” was a change. He “tried to focus the discussion on the goal more than I did the first time,” which might explain the Post-dialogue correcting the error. These shifts suggest changes in Travis’s practices and vision, and also provide insight into what does not seem to change in the dialogue alone.

Rehearsal. Making sense of Travis’s development is supported by considering his rehearsal. Travis was the third of three TCs to rehearse the sorting IA. His rehearsal included discussion of four cards: two easy-to-sort examples of a linear function and two easy-to-sort non-examples. The planted error (Table D) was the first non-example shared and was revisited later by another student. Travis elicited more reasoning about Table D from the student, recorded ideas, and confirmed that he was representing the idea accurately—all moves consistent with his Pre-dialogue. Travis then elicited other non-examples, implicitly tabling the conversation about the planted error. This move was consistent with how he initially responded to all cards.

Travis’s intention to move on was challenged by other students wanting to disagree. After asking for additional non-examples, he allowed a student to comment on Table D. After the student shared a lengthy contribution, Travis turned to the teacher educator and said:

Teacher: Okay, so I guess now this would be - what I’d like to talk about now is sort of the difficulty to get closer down to the definition of it. But I don’t know if I want to talk about that yet.

Coach: Then yeah, I think that’s a sound decision. So, one thing you could have done with [Student] wanting to comment, is you could have tabled that and known to go to [Student] whenever there is an opportunity to raise any questions or disagreements or whatever. But prior to that it seemed that you were willing to move on to the next card. So that’s great.

In his reflection on this moment, Travis expressed wanting to respond to the error in a way that did not make it, “seem as if I single [*sic*] out the one person who makes an incorrect statement with an incomplete conception.” This may explain the tabling move, and also highlights the way Travis negotiated valuing students’ ideas while pursuing the goal. In reflecting on the second contribution about Table D, Travis noted that the student, “made an important distinction here where he began to discuss the how the *x*-values are changing relative to the *y*-values,” but that he got “lost” in the student’s ideas.

This example speaks to the power of considering multiple data sources. Looking only at the rehearsal, we might claim Travis was experimenting with tabling moves as a response to errors. From the scripting tasks, we might claim that Travis did not “learn about” tabling moves and wants to correct errors relatively quickly. Together, we see that Travis’s takeaway seems to be a negotiation of valuing student contributions while also making progress toward goals.

The rehearsal also helps explain Travis’s attention to checking back in with the student who contributed the error. Once students shared reasoning about Table D as an example of a linear function, the teacher educator reminded Travis to check back in with the student who originally initially contributed Table D. Travis later turned to that student:

Teacher: [Student 1], how would you use these change in *y*'s versus these change in *x*'s to show that this is an example?
Student 1: So, I get that the *y*'s and *x*'s are both changing at different rates, but I don't understand like how they're connected. So, like, I get that the *x*'s aren't constantly changing by 1 and the *y*'s aren't, but how does that make it—I still don't understand how that would make it an example.

Travis elicited an explanation from another student about the specific relationship of the changes in *x* and *y* and recorded those ideas. He then checked back in with the original student:

Teacher: [Student 1], does what [Student 2] was discussing there, does that make more sense about how we're relating the change in *y*'s to the changes in *x*?
Student 1: Yeah, I think I get it now, because you have to divide the change in *y* by the change in *x* to find what the slope is. And when you actually do it, it gives you 6 every time. I think I get it now.

Checking in with original student in the Follow-up dialogue had roots in Travis's rehearsal. Reflecting on this moment of his rehearsal, Travis noted that he, "made it a priority to come back to [Student 1] to ensure that he had understood what his errors in thinking were." The check in move helped realize two key aspects of Travis's vision—valuing students' contributions and making progress toward a mathematical goal. Even though the student was not initially convinced, the student was eventually able to articulate the correct idea, which is consistent with Travis's Follow-up dialogue. Looking across these approximations enables us to make more meaningful claims about TC learning and what experiences contributed to that learning.

Discussion & Conclusion

Through this work, we respond to the need to document TCs' skill from their enactments through approximations of practice and how to identify the way in which TCs' skill develops through such designs (Janssen et al., 2015; Shaughnessy et al., 2019). Through our analysis of two TCs' engagement with multiple approximations of practice, we highlight characterizations of skill and a more nuanced understanding of TC development across approximations. The cases of Greg and Travis illustrate different manifestations of skill and development that contribute to sensemaking of TCs' work through approximations—both in-the-moment and over time.

Greg's developing practices, made evident through differences in his dialogues, had direct connections to the moves he experimented with during his rehearsal. While these practices were continually developing throughout the rehearsal, their use in response to errors seemed to shape his vision of how discussions around errors could unfold without the need for immediate resolution. Travis also demonstrated through his dialogues how his practice of using orienting moves continued to develop as he refined the purpose for using these practices. Travis's rehearsal provided opportunities to experiment with moves and experience discussions around errors that play out in novel ways. While this contributed to his developing vision that included a valuing of students' ideas, that was being negotiated with a developing focus on how discussions are moving toward a goal. This resulted in Travis's focus on involving students who contribute ideas in the continued discussion of that idea (particularly involving resolving errors).

A main takeaway from these two cases is how the variety of data sources—across multiple approximations and time—offer a more complete picture of TCs' developing practice and the way a vision of teaching informs that practice. These cases highlight how looking at only one approximation or only at TCs' enactment would result in an incomplete picture. We see this work contributing to the field of research on pedagogies of practice in teacher education by offering approaches for understanding TC practice and development through multiple data sources. While we have focused our work on supporting TCs around the practice of responding to errors in whole-class discussion, we see these findings as having implications for the broader body of work and a focus on other focal practices.

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WHAT ROLE DOES PROFESSIONAL NOTICING PLAY? EXPLORING CONNECTIONS TO AFFECT AND PEDAGOGICAL CONTENT KNOWLEDGE

Cindy Jong
University of Kentucky
cindy.jong@uky.edu

David Dueber
University of Kentucky
david.dueber@uky.edu

Edna O. Schack
Morehead State University
e.schack@moreheadstate.edu

Molly H. Fisher
University of Kentucky
molly.fisher@uky.edu

Jonathan Thomas
University of Kentucky
jonathan.thomas1@uky.edu

This research report describes a study that explored the intersections of professional noticing of children's mathematical thinking (PN), pedagogical content knowledge (PCK), and affective domains with preservice elementary teachers (PSETs). An instructional module on professional noticing, as defined by Jacobs, Lamb, and Philipp (2010), included three components: attending, interpreting, and deciding, was implemented with 170 PSETs. A comparison group of 121 PSETs were enrolled in mathematics methods courses but the PN module was not implemented. PSETs who participated in the module showed significant positive growth in attending, interpreting, and attitudes toward mathematics. While there was no significant change in dispositions toward teaching mathematics, a decrease was observed in PSET deciding and pedagogical content knowledge. However, findings showed promising connections among the constructs.

Keywords: Teacher Knowledge; Affect, Emotion, Beliefs, and Attitudes; Preservice Teachers

Introduction and Objectives

Research on teacher noticing has been prominent in the field of mathematics education over the past decade (Schack, Fisher, & Wilhelm, 2017; Sherin, Jacobs, & Philipp, 2011; Stahnke, Schueler, & Roesken-Winter, 2016). Much of this work has used proximal measures of teaching practices to examine teacher noticing. In addition to teacher noticing, mathematics education research has established strong connections between teacher knowledge and teaching practices (Kunter, Klusmann, Baumert, Richter, Voss, & Hachfeld, 2013). While affect (e.g. attitudes and beliefs) has also been studied extensively in the field, there have been fewer links made between affect and teaching practices in mathematics (Philipp, 2007; Schoenfeld, 2015; Swars, Smith, Smith, Carothers, & Myers, 2018). However, there is a dearth of literature that connects the constructs of teacher noticing, pedagogical content knowledge, and affect in mathematics among preservice teachers. As Philipp (2007) states, "Researchers studying teachers' knowledge, beliefs, and affect related to mathematics teaching and learning are still trying to tease out the relationships among these constructs and to determine how teachers' knowledge, belief, and affect relate to their teaching" (p. 257). In this study, we attempt to examine such intersecting constructs that have the potential to influence teaching practices. Specifically, we explored professional noticing, a specific branch of teacher noticing, with the following research questions:

1. *To what extent can the implementation of a PN module influence PSET professional noticing skills, pedagogical content knowledge, and affect toward teaching mathematics?*

2. *In what ways do PSET professional noticing skills relate to their pedagogical content knowledge and affect toward teaching mathematics?*

Theoretical Framework

Professional Noticing of Children's Mathematical Thinking

Professional noticing of children's mathematical thinking (PN) is a specific branch of teacher noticing defined by Jacobs, Lamb, and Philipp (2010) as the amalgam of three components, attending, interpreting and deciding. Jacobs et al.'s definition of PN is widely referenced in the mathematics education community. It is frequently shortened to PN often leading to confusion with the more general term, teacher noticing (Sherin, Jacobs, & Philipp, 2011), which usually includes only the attending and interpreting components. Schack, Fisher, Thomas, Eisenhardt, Tassell, & Yoder (2013) demonstrated that PN can be developed in preservice teachers with improved responses to prompts related to attending, interpreting, and deciding over the course of a mathematics methods course embedded with specific instruction on PN. Additionally, Fisher, Schack, Thomas, Jong, Eisenhardt, Tassell, & Yoder (2014) previously explored connections between PN and preservice teachers' attitudes toward mathematics. Using the Attitudes Toward Mathematics Inventory (ATMI) developed by Tapia and Marsh (2004) and adapted for preservice teachers by Schackow (2005), Fisher et al. (2014) found significant change from pre to post assessment on three of the four factors of the ATMI. There was, however, no significant correlation found between ATMI factors and PN. This study extends that work by engaging preservice teachers in more complex teaching situations and examining the interactions of pre-service teacher PN with affect, as measured by two domains of the Mathematics Experiences and Conceptions Survey (MECS) (Jong & Hodges, 2015), and pedagogical content knowledge, as measured by selected items of the TEDS-M (Brese & Tatto, 2012).

Pedagogical Content Knowledge

Pedagogical content knowledge refers to that knowledge about content that is specifically needed to teach the content. Shulman (1986) drew upon Ong's 1958 observation that the medieval university made no distinction between content and pedagogy. Shifts in teacher preparation that emphasized pedagogy over content led Shulman to examine the need for content/subject knowledge in one's pedagogical approach. He defines pedagogical content knowledge as "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). Teachers with strong pedagogical content knowledge also are knowledgeable of the potential misconceptions (Shulman, 1986) and speed bumps students will encounter in their learning, preparing instructional environments that allow students to grapple with the content, and ultimately overcome the speed bump. Pedagogical content knowledge has been the subject of much research over the past three decades (Ball, Thames, & Phelps, 2008; Hill et al., 2008; Depaepe, Verschaffel, & Kelchtermans, 2013) following on the premise that Shulman presented his theory as a heuristic method for studying the types and processes of knowledge needed by teachers. Ball, Thames, and Phelps (2008) attempted to refine Shulman's knowledge types, including pedagogical content knowledge, but acknowledged that further research was needed to bring clarification to the definition.

Depaepe et al. (2013) performed a systematic review of research on pedagogical content knowledge in mathematics education seeking to determine how PCK is conceptualized and investigated in empirical studies. The results demonstrated much alignment in researchers' definitions with Shulman's but there were also variations that drew upon other conceptualizations of pedagogical content knowledge (Ball et al., 2008; Hill et al., 2008).

Depaepe et al.'s (2013) conclusions illustrate that some researchers study a cognitive conceptualization of PCK, one in which PCK is within the mind of the teacher and often assessed through a test, and others study a situated conceptualization of PCK, which is related to "knowing-to act", one of Mason and Spence's (1999) four types of knowing.

Affect in Mathematics Education

Affect is used as a broad term that most commonly includes attitudes and dispositions, and are also closely linked to beliefs and self-efficacy, which vary in where they fall along the spectrum from emotion to cognition (Philipp, 2007). Schoenfeld (2015) points out the complexities of understanding affect in different contexts and while asserting that underlying beliefs influence practices, he also states that actions are what ultimately matter. He also urges researchers to examine changes in affective factors over time as he notes that since it takes years to develop beliefs and practices, then one would expect the same for substantive changes accompanied by support over time. Although there are affordances and drawback to various methods that examine affect in mathematics teaching, scholars agree that beliefs and attitudes matter (Aguirre & Speer, 1999; Jacobson & Kilpatrick, 2015; Wilkins, 2008).

Connections: Professional Noticing, Pedagogical Content Knowledge, and Affect

Thomas, Jong, Fisher, & Schack (2017) examined potential theoretical connections between two frameworks: mathematical knowledge for teaching and PN of children's mathematical thinking. They theorized that the skills of PN mediate mathematical knowledge for teaching and teacher responsiveness to children's mathematics. They developed a hypothesized trajectory of effective PN skills as an outcome of a well-developed MKT and responsiveness to student thinking and this research will continue to investigate these relationships. Similarly, Stahnke et al (2016) argue that such "skills display the missing link between mathematics teachers' dispositions (professional knowledge, affective motivational features) and their performance (observable behavior)" (p. 24). They are essentially making a case that PN skills have the potential to bridge affect, pedagogical content knowledge (referred to as "professional knowledge"), and teaching practices. Swars et al. (2018) recently conducted a study on the preparation of elementary mathematics specialists. They examined connections among pedagogical beliefs, teaching self-efficacy beliefs, specialized content knowledge, and observed teaching practices at the end of the program. The only significant relationship they found was between pedagogical beliefs and teaching self-efficacy; however, it might have been partly due to the self-reported nature of the instruments. Newton, Leonard, Evans, and Eastburn (2012) examined preservice teachers' self-efficacy, pedagogical content knowledge, and outcome expectancy before and after a mathematics methods course. Results from their study showed a consistent moderate positive correlation between self-efficacy and pedagogical content knowledge, but no relationship with outcome expectancy.

Methodology

Participants and Context

Participants were preservice elementary teachers enrolled in mathematics methods courses during one of five semesters at five universities (two urban, three rural) in the south-central United States. Treatment group PSETs (N = 170) participated in a module designed by the researchers to develop PN in the context of early algebraic thinking in a whole class setting. The *Examining Essential Expressions in Algebra* (E^3A) module included three 60-minute sessions focused on PN skills through the content of early algebraic reasoning and used complex video vignettes from whole class instruction in elementary mathematics classrooms to prompt discussion about the three components of PN: attending, interpreting, and deciding. Comparison group participants (*N* = 126) completed their mathematics methods course "business as usual".

Data Collection

Participants completed pre and post assessments consisting of three instruments, although not all participants completed all assessments, thus the varying *N*s. PN was assessed using a video-based assessment (*N* = 268). Affect was measured using the MECS (*N* = 149). We focused on the MECS' two-subscales of attitudes and dispositions to capture affect (Philipp, 2007). Additionally, preservice teachers responded to selected TEDS-M items (*N* = 196) chosen for their relation to pedagogical content knowledge in that some items represented student (mis)conception and required participants to identify such and discuss how they might respond instructionally. Selected TEDS-M items also matched the module's algebraic thinking content.

Video-based assessment. We assessed preservice teacher PN through a video clip of an authentic classroom in which children are engaging in the meaning of the equal sign in a whole group setting. The teacher has presented a number sentence, 10+10=__+5 asking the children to determine what number to put in the blank to make the number sentence true. The full class setting requires preservice teachers to attend to multiple understandings as displayed by the various children's responses in close temporal proximity to each other, make sense of the responses, and make an instructional decision based on the responses. The assessment prompts are aligned with the three components of PN in the order of deciding, attending, and interpreting: 1) *Pretend that you are the classroom teacher. What might you do next? Provide a rationale.*, 2) *What mathematical thinking and actions did you observe?*, and 3) *What did you learn about the children's mathematical thinking that influenced your decision in question 1?*. We intentionally prompted participants to decide first in an attempt to capture their in-the-moment thinking, more closely. PSET responses were scored by research team members on a four-point scale using decision trees programmed in JavaScript for automated scoring (see Schack, Dueber, Jong, Thomas, & Fisher, 2019 for details).

Mathematics Experiences and Conceptions Surveys (MECS). MECS is a set of instruments consisting primarily of six-point Likert-scale items (ranging from *strongly agree* to *strongly disagree*) designed to measure various affective factors related to teaching mathematics, such as attitudes, beliefs, dispositions, and self-efficacy over time (Jong & Hodges, 2015). MECS utilized the Philipp (2007) definitions of attitudes as "manners of acting, feeling, or thinking… Attitudes, like emotions, may involve positive or negative feelings. …Attitudes are more cognitive than emotions, but less cognitive than beliefs" (p. 259). It was also informed by his broader description of affect as "a disposition or tendency or an emotion or feeling attached to an idea or object", the main distinction between attitudes and dispositions being a feeling versus a tendency that might be more directly linked to an action. The two subscales used in the study were attitudes (MECSA) and dispositions (MECSD) toward mathematics. MECSA consists of six items, such as: "Mathematics is one of my favorite subjects." and "I look forward to teaching mathematics." MECSD consists of 10 items, such as "I plan to engage students in mathematics discussions." and "I plan to encourage students to share their thinking."

Teacher Education and Development Study in Mathematics (TEDS-M). TEDS-M was a study, funded by the International Association for the Evaluation of Educational Achievement, that examined how preservice teachers were prepared to teach mathematics at both the elementary and secondary levels in teacher education programs across 17 countries (Brese & Tatto, 2012). A major aim of TEDS-M was to capture preservice teachers' knowledge about mathematics. To assess pedagogical content knowledge in our study, we used a subset of eight

TEDS-M released items from the number and algebra content domains. Responses were then scored according to the TEDS-M scoring guide with varying scores of 0-2 based on the problem. Three open-response questions were added to the eight items to assist the researchers in further understanding PSETs' thinking. We did this by simply asking, "Please explain your response." We had specifically selected items within the number and algebra content domains because the TEDS-M post-test was administered immediately after the E^3A module was implemented in an attempt to capture PCK that was closely related to the content goals.

Data Analysis

Psychometric properties (e.g. model fit, factor analyses) of the aforementioned instruments were completed using WINSTEPS (Linacre, 2018) to inform subsequent analyses. Preliminary analyses of MECS' affect subscales indicated that the attitudes and dispositions scales were the most reliable, given the sample size. A Rasch rating scale model was used to examine reliability and model fit of the scales. Both subscales (at both pre and post) had reliability estimates ranging from 0.70 to 0.88. Principal components analysis of the residuals from the Rasch model also indicated strong unidimensionality for both scales, but a few items had poor fit according to INFIT and OUTFIT (Bond & Fox, 2007) and thus were eliminated from the analyses. Preliminary analyses of the TEDS-M items revealed excellent item fit according to INFIT and OUTFIT; however, the TEDS-M showed poor reliability (0.65 for pre and 0.55 for post) due to the limited number of items. This low reliability can reduce power in univariate significance testing such as *t* tests (Kanyongo, Brook, Kyei-Blankson, & Gocmen, 2007), and make the results of multivariable statistical analyses such as multiple regression untrustworthy (Cole & Preacher, 2014). Thus, we present findings including TEDS-M scores with a great deal of caution. While there are major limitations of the claims we can make with the TEDS-M due to the low reliability, we thought it would be beneficial to present results as a way of being transparent.

As PN video-based assessment scores are not interval level data, Wilcoxon signed rank tests were performed to determine whether PN scores significantly changed from the pre to the post assessment. As TEDS-M, MECSA, and MECSD scores are interval level data, a mixed design ANOVA was conducted on pre and post data for these scales to explore changes given the treatment effect and repeated measures aspects of the design. In the event of a significant interaction term, indicating a greater score change for the treatment group than the control group, a paired *t* test was performed to provide a significance test and Cohen's *d* effect size for the score change of the treatment group.

Next, in order to determine whether variables other than the treatment condition contributed to post-test scores, a multiple regression model was employed for each outcome of interest in which all measured variables (e.g. sum scores for TEDS-M and MECS factors) at pre-test were used as predictors. Models were tested with and without interaction terms (e.g., PreAttending × Treatment), but none of the interaction terms had significant regression coefficients and these interaction models did not explain significantly more of the variability in the outcome than the main effects models. Therefore, only main effects models are reported. As not all measures were completed by all participants, missing data were handled using full information maximum likelihood, an approach which utilizes all available data and makes fewer assumptions about the nature of missingness than listwise or pairwise deletion techniques (Enders, 2010).

Because of the way measurement error attenuates correlations, measurement error in the predictors of a multiple regression model can change the strength and pattern of effects. On the other hand, measurement error in the outcomes of a multiple regression model attenuate the strength of effects. Therefore, correcting for measurement error can give a more accurate description of the true underlying relationships (Cole & Preacher, 2014). Due to substantial measurement error in the TEDS-M data, we performed a sensitivity analysis in which results using observed variables were compared to results using a single indicator latent variable (SILV; Hayduk, 1987) technique for correcting for measurement error in the MECSA, MECSD, and TEDS-M variables.

Results

Changes in Constructs

Table 1 provides average pre and post scores of the various constructs examined among PSETs from both the comparison group and those who participated in the E^3A module. For the PN component scores of the treatment group, Wilcoxon signed-rank tests indicated that significant positive changes were observed in attending (Z = -3.219, *p* = .001) and interpreting (Z = -3.961, *p* < .001), but not in deciding (Z = -.384, *p* = .701). Furthermore, since scores for all three components decreased for the comparison group, there is evidence to suggest that the treatment was effective in the areas of attending and interpreting; however, the deciding component warrants further investigation.

Table 1: Descriptive Statistics

	Treatment		Comparison	
	Pre	Post	Pre	Post
	M (*SD*)	*M* (*SD*)	*M* (*SD*)	*M* (*SD*)
Attending	1.03 (0.98)	1.34 (0.92)	0.96 (1.01)	0.83 (0.92)
Interpreting	0.77 (1.11)	1.22 (1.11)	0.88 (1.09)	0.66 (0.94)
Deciding	1.36 (0.81)	1.34 (0.88)	1.55 (0.98)	1.45 (0.99)
MECSA	21.85 (7.55)	24.09 (6.20)	22.73 (7.05)	22.49 (6.31)
MECSD	43.64 (3.95)	44.27 (3.35)	44.73 (2.91)	43.48 (4.47)
TEDSM	9.70 (2.75)	9.45 (2.13)	10.67 (2.33)	9.41 (2.05)

The mixed-effect ANOVAs of the TEDS-M and MECS results are reported in Table 2. On the left half of the table are the results of omnibus tests of interactions; all three interactions are significant, indicating that treatment group scores improved relative to control group scores. The results of post-hoc paired t tests on the treatment group can be found on the right half of Table 2. Affect, consisting of the MECS scales, yielded a significant positive change in attitudes (MECSA), and a non-significant positive change in dispositions (MECSD). Pedagogical Content Knowledge (PCK) included scores from the TEDS-M, in which average scores were lower, but not significantly lower than pre-treatment scores. Note that scores for MECSD and TEDS-M exhibited a treatment effect according to the ANOVA interaction term, but no significant improvement in scores for the treatment group. The significance of the treatment effect was therefore at least partially due to score decreases in the control group.

Table 2: Mixed-Design ANOVA and Post Hoc Tests

	Omnibus test of interaction effect				Treatment group paired *t* test				
	F	df	*p*	$\widehat{\eta_p^2}$	Diff.	t	df	*p*	*d*

MECSA	11.38	1, 146	.001	0.072	2.235	6.172	114	<.001	0.296
MECSD	6.322	1, 146	.013	0.042	0.635	1.762	114	.081	0.161
TEDSM	6.124	1, 162	.014	0.036	-0.254	-1.271	117	.206	0.092

Note. Cohen's d computed using pre-score standard deviation

Connections Among Constructs

Results of observed multiple regression models for all outcome variables are presented in Table 3. When the same models were estimated including SILV corrections for measurement error, all statistical decisions about significance were retained. Furthermore, parameter bias was slight except for the coefficients for the TEDS-M predictor, which were all smaller in the observed model than the SILV model. Accordingly, only the results of the more conservative observed model are reported and interpreted. All models had a significant R^2, however, for the TEDSM and MECSD variables, the only significant predictor was the pre-score of the same variable. The E^3A module, or treatment condition, was seen to be a significant positive predictor of Post-Attending (PostATT), Post-Interpreting (PostINT), and PostMECSA, consistent with earlier reporting of changes in constructs. Furthermore, PreINT significantly predicted PostATT, while PreTEDSM significantly predicted PostDEC. Also, PreMECSD significantly predicted PostMECSA.

Table 3: Standardized Regression Coefficients for Observed Analysis

	Outcome					
	PostATT	PostINT	PostDEC	PostTEDSM	PostMECSA	PostMECSD
	β	β	β	β	β	β
Treatment	.27***	.29***	-.00	.05	.21***	.15
PreATT	.27***	.01	.06	.01	.00	-.09
PreINT	.16**	.35***	-.01	-.06	.05	-.04
PreDEC	-.00	.06	.06	.01	.03	-.07
PreTEDSM	.15	.06	.24*	.58***	.02	.11
PreMECSA	-.05	-.04	.15	.11	.86***	-.01
PreMECSD	-.09	.03	.02	.03	.11*	.46***
R^2	.21***	.21***	.11*	.27***	.77***	.25***

Note. The coefficient for the treatment variable only standardizes the outcome, in order to provide a standardized difference between the two treatment and control groups while controlling for the other predictors. *p < .05. **p < .01. ***p < .001.

Discussion

In this study we endeavored to study the connections of PN, with PCK and affect. Multiple regression analyses confirmed ANOVA results, indicating that participation in the E^3A module was a significant positive predictor of PSETs' attending, interpreting, and affect. In regard to the connections among constructs, we found that pre-interpreting significantly predicted post-attitudes, demonstrating a connection between PN and affect. Pre-dispositions significantly predicted post-attitudes, which makes sense but needs to be further investigated. In addition, pre-PCK significantly predicted post-deciding. While tenuous, this shows a connection between PN and PCK. This was an interesting finding, given that significant differences were not observed from pre to post in either deciding or PCK. Post scores of every component skill, except deciding, were significantly predicted by the pre-score of that component skill. It is also worth noting that results yielded no connections between PCK and affect.

The interaction of PN, pedagogical content knowledge, and affect offers the opportunity to examine the conceptualization of each construct. Our results indicate that connections between PN and affect in mathematics teaching can be detected in a brief period. While it makes sense that dispositions might serve to predict attitudes as related constructs, it is less clear why interpreting might serve as a predictor for attitudes. We suspect that PSETs who have a stronger grasp on interpreting children's mathematical thinking at the onset might be more enthused about teaching mathematics. If the TEDS-M instrument were more reliable, we would be able to say with more confidence that pre-PCK influenced post-deciding. However, both resulted in lower scores and it might be the case that the regression analysis picked up on similar slopes.

We are aware that there are limitations in the context and the instruments of this study. The context is limited by the short duration of the treatment embedded in a semester-long mathematics methods course. While significant changes for the treatment group were realized, this may have been associated with the course in which the treatment was embedded and not the treatment alone. The use of the comparison data is useful in debunking this particular limitation, we recognize that all instructors used in the study teach in diverse ways. The TEDS-M instrument is limited by the reliability due to the low number of items selected. The limitations of the TEDS-M also underscore the need for an instrument that is more reliable in assessing PCK and/or MKT of preservice teachers. Fisher, Thomas, Schack, Jong, & Tassel (2018) previously attempted to assess preservice teachers using the LMT (Hill, Schilling, & Ball, 2004), however, because this assessment was designed for inservice teachers, our results were limited. Given these limitations, we believe reporting these results is important for the advancement of research exploring the interaction of these constructs: PN, affect, and PCK. Our results indicate there are connections amongst the three constructs though not clearly among all three constructs and not among all components of the various constructs. Though limited, this is an encouraging step in the research that should be further explored with an improved assessment instrument for PCK and potentially better measurement strategies for PN.

As noted previously, the literature illustrates that many researchers are exploring such connections. Mired within this research are multiple challenges, not the least of which is the invisible nature of what occurs in a teacher's head. Jacobs, Sherin, and Philipp (2013) referred to teacher noticing as "the hidden skill of teaching" (p. 723). One might argue that affect and PCK are similarly hidden.

An additional challenge to researchers is in sorting through the language used to describe the various constructs. Questions arise. What is simply a language difference and what constitutes a different construct? In what ways is PN alike or different from the situated conceptualization of PCK (Depaepe et al., 2013)? In what ways is PN alike or different from the integrative perspective of teacher knowledge conceptualized by Gess-Newsome (1999) as the integration of subject matter, pedagogy, and context? Thomas et al. (2017) found connections in the theoretical space between PN and MKT. "[W]e contend that one productive lens for considering the practical outcomes of professional noticing is from the perspective of responsiveness. In this instance, responsiveness may be considered a broad manifestation of the coordinated component skills of professional noticing. . .[and] effective professional noticing occurs at the intersection of developed MKT and a high level of responsiveness to the mathematical activities of students" (pp. 13-14).

Ultimately, the goal is to support teachers to enact responsive teaching that benefits the learning of the students. To be responsive, teachers must constantly use converse and diverse thinking. Attending to an individual student's mathematical response requires converse, or very focused seeing (Broudy, 1984; Polanyi, 1966), that considers that specific student's response and the specific mathematics of the moment. However, the teacher must also think diversely of the needs of the whole class of students as well as the mathematics of the moment in context of a larger mathematical concept.

Are these descriptions of teacher practice describing the same or different constructs? Is it that many attempted to understand teacher thinking and practice from different perspectives, using different language? Does it benefit research on teacher preparation to study essentially the same construct under different names? If constructs are different paths to the same end, understanding "the hidden skill [or skills] of teaching" (Jacobs et al., 2013, p. 723), research such as this, that attempts to find the intersections is crucial to deepening our understanding of effective teacher thinking and with that, how to better prepare teachers.

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LEARNING TO TEACH MATHEMATICS FOR SOCIAL JUSTICE: HELPING PRESERVICE TEACHERS CONNECT CRITICAL TOPICS TO STANDARDS

Marrielle Myers
Kennesaw State University
mmyers22@kennesaw.edu

This study examines how preservice teachers select standards, plan tasks, and develop a full lesson plan focused on a social justice context. While traditional studies of this topic involve scholars examining completed lesson plans, this study is unique in that audio data was recorded to capture the types of discussions preservice teachers engaged in while completing social justice lesson plans. Findings indicated that PSTs used keywords to map standards onto elements of the social justice context, that they vacillated between selecting a standard and developing an appropriate task, and that they treated the social justice context as a fixed scenario. I will share examples along with implications for how mathematics teacher educators can better support PSTs as they navigate this complex process.

Keywords: Equity, Diversity, Instructional activities and practices, Teacher knowledge

Introduction

Teaching mathematics for social justice (TMfSJ) is a complex process that involves the coordination of "fair and equitable teaching practices, high expectations for all students, access to rich, rigorous, and relevant mathematics, and strong family/community relationships to promote positive mathematics learning and achievement" (National Council of Supervisors of Mathematics and TODOS, 2016). In this paper, I present findings based on a semester-long project with preservice teachers (PSTs) focused on the Flint water crisis. I begin this paper with an overview of research on teaching mathematics for social justice. Next, I describe the context of the study, the participants, and articulate key elements of the data sources. I then provide excerpts from transcript data that illuminate how PSTs negotiated the complexities of planning mathematics lessons focused on social justice. After identifying themes that emerged in the data, I conclude by discussing the implications of the findings.

Background

While teaching mathematics for social justice is not a new concept in mathematics education, understanding how preservice teachers come to enact this pedagogy is still relatively new to the field. Previous work around this topic has examined PSTs beliefs about teaching injustices or controversial issues (Simic-Miller, Fernandes, & Felton-Koestler, 2015). Gonzalez (2012) articulates four components of TMfSJ. These components include students having access to rigorous mathematics, that mathematics is re-centered on issues related to historically marginalized students, that mathematics is used to critique society, and that we use the knowledge generated from the previous three constructs to advocate for change (Gonzalez, 2012). For PSTs to engage in this type of pedagogy, Felton-Koestler (2017) states that PSTs must experience mathematics for social justice as a learner, see examples of what this could look like in practice, and reflect on their own beliefs about mathematics. Much of the extant literature around TMfSJ calls attention to the fact that PSTs need explicit exposure to examples of these lessons and opportunities to plan lessons for themselves as they will not likely find social justice lessons in their school curriculum (Leonard & Evans, 2012).

Bartell (2013) described secondary teachers' attempts to teach mathematics for social justice. In this study, in-service teachers enrolled in a graduate course designed, implemented, and revised a mathematics lesson that had social justice goals (Bartell, 2013, p. 136). Findings from this study indicated that teachers had varying definitions of what it meant to teach mathematics for social justice ranging from awareness, to cultural exposure, to student empowerment. Another finding from this study was that participants felt tensions between "mathematical goals and social justice goals" while planning (Bartell, 2013, p. 141). I have replicated these findings in my work in that many of the topics elementary PSTs selected when planning a social justice lesson related to their own goals for the lesson (e.g., wanting to expose their students to world issues) as opposed to addressing students' needs (Myers, 2017). PSTs also planned lessons that focused on current events or chose less critical topics (e.g., discussing pet adoption instead of racial issues involved in the adoption of children) because of their belief that young students could not handle this type of mathematics, for example (Myers, 2017).

While previous research has demonstrated that PSTs can design lessons focused on social justice topics, it is unclear how they negotiate the tension between mathematical goals and social justice goals in their lesson planning activities. Further, much of this work has focused on analyzing survey data about PSTs beliefs or examining completed lesson/unit plans (Leonard & Evans, 2012). This study makes a unique contribution to the field in that the focus is on exploring the conversations that PSTs engaged in while planning social justice lessons. After a discussion of the theoretical framework and methods, I will share themes that emerged from the analysis along with sample text from transcripts of the PSTs lesson planning sessions.

Theoretical Framework

This study is grounded in Gutstein's (2003 and 2006) framework for teaching mathematics for social justice. The goal was for PSTs to unpack the Flint water crisis and think about how mathematics could be used as a tool for elementary students to engage with and act upon this injustice. In essences, how could we use mathematics to *read* the Flint Water Crisis? Gutstein (2003) defines *reading the world with mathematics* to mean:

> [Using] mathematics to understand relations of power, resource inequities, and disparate opportunities between different social groups and to understand explicit discrimination based on race, class, gender, language, and other differences. Further, it means to dissect and deconstruct media and other forms of representation and to use mathematics to examine these various phenomena both in one's immediate life and in the broader social world and to identify relationships and make connections between them. (p. 45)

To help students engage in *reading* the Flint water crisis, I provided PSTs with articles, links, and hashtags to read and follow before our discussion. I intentionally selected multiple forms of media (e.g., print media, blogs, social media) as well as publications from different political backgrounds (e.g., CNN, Fox News, and The Root) so that PSTs could see this crisis from multiple lenses and begin to think about how to negotiate various viewpoints.

Another essential framework for this study was the Culturally Responsive Mathematics Teaching (CRMT) Lesson Analysis Tool. This tool was developed as a part of the TEACH-MATH project and was intended to support teachers in critical analysis of how they plan and implement lessons (Drake et al., 2015). Because previous literature highlights challenges PSTs face with maintaining the mathematical goals of the lesson, it was critical to select a framework that explicitly draws the user's attention to mathematical goals of a lesson as well as sociopolitical goals. Additionally, one "critique" of TMfSJ is there "where is the math" question. This framework contains elements of "good mathematics teaching" (e.g., open tasks and discourse) as well as critical components of reading the world with mathematics. This tool is composed of six categories including cognitive demand, depth of knowledge and student understanding, mathematical discourse, power and participation, academic language support for ELLs, and cultural/community-based funds of knowledge (Drake et al., 2015). Each of the six categories is further delineated by a five-point rubric where level one articulates a less sophisticated or appropriate approach to the category and level five represents an ideal demonstration of the construct.

Methods

Context

I conducted this study in a semester-long mathematics methods course. Students were simultaneously enrolled in literacy, science, and social studies methods courses and followed a cohort model. In addition to taking four methods courses, all students completed a 200-hour field requirement during the semester where they were expected to progress from completing structured observations, to one-on-one interviews, to small group activities, and ultimately teach a whole-class lesson.

As the PSTs worked on this lesson planning activity, they had a variety of resources to use while planning. They had electronic access to the state standards, the departmental lesson plan template, documents that listed mathematical supports and scaffolds for ELL students, as well as articles about the Flint water crisis. One graphic that the students were particularly drawn to is shown below:

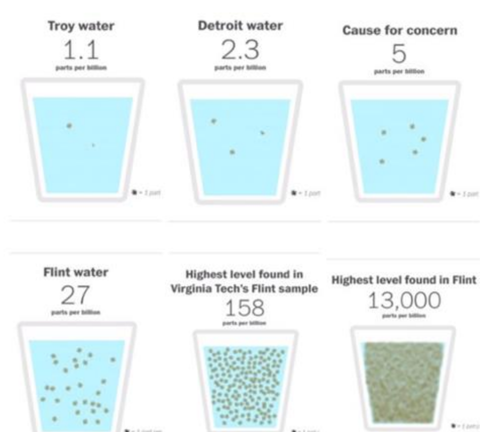

Figure 1: Michigan Water Level Contaminations. Reprinted from "We had to make some cuts": Flint water catastrophe a matter of business, by D. Johnson, Retrieved from http://www.occupy.com/article/we-had-make-some-cuts-flint-water-catastrophe-matter-business#sthash.o2xFPb6m.dpbs. Copyright 2016.

Participants

Seventeen PSTs participated in this study. Of these participants, five identified as African-American, seven identified as White, one as Hawaiian, two identified as Hispanic, one as

Jamaican, and the final student identified as Vietnamese. Three of the PSTs students were bilingual (they learned English as a second language) and all of the students identified as female. The racial makeup of this class was atypical in comparison to other sections of the course offered that semester and the overall demographics of the department.

The students worked in grade-level teams according to the grade level they were placed in for their field experience. There were five groups in the class. Three groups of PSTs worked on first-grade lesson plans, one group worked on a second-grade lesson plan, and the final group worked on a third-grade lesson. In this paper, I will share the results of one subgroup of this study that was made up of three PSTs. Two of the three PSTs in this subgroup identified as White. The third identified as African-American.

Data Sources and Analysis

Data sources for this study included small group lesson plans and audio recordings of lesson planning sessions. The lesson plan template asked candidates to: identify a GA standard; state the problem to be solved (based on the Flint water crisis); identify tools needed; identify polysemous words, cognates, and other vocabulary demands; pose probing questions to ask students; articulate differentiation strategies; as well as create summative and formative assessments. I used this lesson plan template because it is a required template for the department. In the first part of this project, candidates were given three hours in class to work on their group lesson plans.

Both ongoing and retrospective data analysis were used. Data were coded using open coding (Merriam, 2009). After broad themes were identified, I looked for examples of each of the six categories from the CRMT lesson planning tool. Although the tool can be used to "rate" each category of the lesson plan, I will not provide ratings as the focus of this study is to explore the complexities of PSTs conversations as opposed to the final result of what was produced on the lesson plan.

Findings

For this paper, I am going to discuss the data from one of the first-grade groups. A few themes emerged in the analysis. In the subsequent sections, I will discuss each of these themes in greater detail and offer examples from the transcripts and lesson plans to illuminate the case.

Standards vs. Tasks

At the start of the semester, I engaged the class in a discussion around the elementary mathematics standards for the state. We also engaged in a discussion about some of the way's standards are used, and expectations school administrators typically have for how teachers implement standards-based instruction. This is a critical conversation to have with PSTs because in my previous work, I have found that there is significant push back given that mentor teachers believe that TMfSJ falls outside of the standards. Therefore, helping PSTs understand that this potential pushback is coming can help them be proactive about justifying their lessons and the resulting tasks. Additionally, negotiating standards and TMfSJ is an integral part of the political work of mathematics teaching.

As the PSTs continued reviewing the state standards, they would select multiple standards to work with. They would then develop a task aligned with that standard. This process was not linear however, and it appeared that PSTs vacillated between choosing a standard and developing a task. There were a few examples of this in the data:

Student C: So, I guess they could count the number of things in the cup [see Figure 1 above] and then write it out. Could we maybe have them count to understand place value? Like maybe ten particles in a cup equals the Flint water.
Student A: I see where you're trying to go there
Student B: You're trying to use place value
Student A: I'm trying to put that into words
Student C: Have them think of like…
Student A: Have them use the particles like blocks…like base-10 blocks
Student C: Yes…have them understand that…
Student A: I don't know how we would convert all that though
Student B: You would convert it just like you would anything else. You would have one particle [referring to the particles in the cups pictured above]…Instead of trying to like come up with or find something that fits in more complex ones, we can work with what we have. Because we know we can do households affected…like with number of people. Cause you can find that easily…you count…you know what I mean?
Student A: So, should we do word problems?
Student B: Yeah…work with what you can do…no trying to re-invent the wheel.
Student A: Right…so that would be like use addition and subtraction within 20 to solve word problems that involve situations of adding to, taking from, putting together, taking apart, and comparing with unknown

When this clip begins, the PSTs had planned to create a task based on representing and interpreting data. After the PSTs had some difficulty translating this into a task, one of the group members suggested a standard that focused on place value. From there, the PSTs begin to discuss how they could get students to count and graph the particles of water in Figure 1. Another student then suggested that the conversions (particularly the two cups that are represented with decimal numbers) will be too difficult and that the task should include discrete values only. This data highlights that PSTs were trying to coordinate the standard they selected, a possible task, and the cognitive demand of the proposed task. The conversation then shifted to creating a word problem, which is a different standard. This pushed one of the PSTs to now select a standard that addresses solving word problems within 20.

PSTs looked at other research they had done on Flint, and this conversation continued until they settled on a standard and a task:

Student A: I think I would go with the one that has addition and subtraction so that we could do like…you know…x amount of homes had this many or this many homes got infected, how many homes were left uninfected?
Student B: Say that again…
Student A: If you started with the number of homes that were uninfected and then gave them the number of homes that got infected, how many homes were left uninfected
Student B: So, you could say there were this many homes in Flint…50 homes in Flint…20 of them got affected with lead poising due to unclean water. How many more homes have yet to be affected? Thirty. Ok …let's go with that.

Standards as Keywords

Like the previous example highlights, unpacking standards is a complicated task. Initially, some PSTs assumed that they would teach one standard per day until all grade-level standards had been exhausted. As students entered methods courses and started field placements, they were exposed to how standards must be unpacked and that each standard is comprised of smaller learning goals. As PSTs attempted to unpack the standards in this study, they pulled keywords from the standards and tried to map keywords from the standard onto elements of the social justice context. In the following example, you will see that the words *represent* and *interpret* were selected to frame a possible task for students.

Student A: Let's use organize, represent, and interpret data up to three categories
Student B: Represent…you can compare the lead in the water to the amount of people that ended up sick…that can be *represented* on a chart…data…at this rate…you can use one city [referring to Figure 1 above]…you can like *interpret* the data…like if they keep on using this water source and you look at the ratio between the people getting sick and how in a year or two years how much more people would end up getting sick.

Data seemed like a natural first step for PSTs to consider when planning this lesson. One PST continued looking at the standards as the rest of the group was working and noticed that, "It looks like represent and interpret data is under every standard though." The PSTs decided that this standard was too broad and that they should look for something more specific than *representing* and *interpreting*.

Planned Supports vs. Standards vs. The Task

It seemed that the PSTs had settled on a standard and created a word problem that aligned with that standard. The discussion then shifted to what types of tools they would use to support students in developing rich mathematics. Much of their conversation here centered on category two of the CRMT Lesson Planning tool in that the PSTs grappled with what they could provide students with so that students could show their learning. The challenge that emerged here is that as the PSTs tried to engage in this discussion, they found that their planned supports were not necessarily aligned to the standard they selected or the word problem they created.

Student A: Ok…so what tools will we use?
Student B: Modeling? It has one for model with mathematics and it has use appropriate tools.
Student A: Under tools?
Student B: But I'm thinking tools can be rulers, graph paper…
Student A: I want to say like…manipulatives…because yeah, those dots…I think they're just called counters
Student B: So, is graph paper not a tool?
Student A: So, are we going to have them create graphs? Because that is not part of the standard.
Student B: I thought you was comparin' …it could be.
Student A: I feel like you would show them the graph to say 'hey this is what's happening here and this is what's happening here'
Student C: What exactly are we doing?

Student B: The water pollution vs. the people getting sick. Basically, the water pollution vs. the people going to the hospital. I guess I don't know how you'd compare that in one city. That would be hard the hard thing. Ehhhh…I don't know.
Student A: Uhhh…not yet. Well we have the math part…so maybe we can at least… I'm confused
Student B: What are we doing? Let's start with that
Student A: They're gonna do word problems. So, they're not…
Student B: But you can show word problems though…you know what I'm saying?
Student A: So, they can draw it out and use graph paper…so I can definitely see it to be used there. I put down manipulatives and by manipulatives I put i.e. counters. I put down graph paper.
Student B: What did you put next to the manipulatives? You could use white boards? I'm thinking too much into it.

As you see here, the conversation about manipulatives caused the PSTs to question the standard they had selected as well as the goals of the task. To help reconcile this difficulty, the PSTs engaged in a discussion about what mathematical tools were, what tools were necessary for students to solve a story problem, and additional layers about the Flint Water Crisis. As I stated earlier in this analysis, planning lessons related to data seemed like a "safe space" for PSTs to develop tasks. The notion of comparing and representing data re-emerged in this discussion, although it was not related to the word problem the PSTs previously decided on.

Is the Context Fixed?

Recall that a component of teaching mathematics for social justice is that mathematics be used to examine a socio-political context. Moreover, this examination is not solely to better understand the setting but also to create an avenue for advocacy. A key learning that emerged from this study is that although we spent a significant amount of time unpacking the water crisis, once PSTs started planning the lesson, Flint went from being a dynamic sociopolitical context to a fixed context. This was further confounded as PSTs went back and forth about the type of task they wanted to create. At one point, they planned to use existing data to graph. Later in the discussion, they considered the idea that this problem may not be fixed and they wanted to project the type of damage that could occur if the necessary changes to infrastructure did not happen. The PSTs then settled on creating a word problem that used Flint as a context (e.g., infected homes versus homes that are not infected).

As they continued working on the lesson plan, further dialogue emerged around the mathematical goals and the social justice goals for the lesson. The exchange below provides additional insight into this:

Student A: So, one goal should be make sense of problems and persevere in solving them?
Student B: Where are you finding this?
Student A: I am on the GA standards website. I think another goal should be for them to understand the severity of the water crisis since that's what we're…
Student B: But you can't say understand…umm…I remember one teacher saying that when you set goals for students you can't say things like they're going to understand…You can say make them aware
Student C: So, make the students aware of the severity of the water crisis

These examples illuminate different conceptions of a social justice context. For instance, if PSTs treat the context as something static (e.g., a word problem to be solved or fixed data points to graph), the actual goals of TMfSJ may never be realized because no action emerges from these types of tasks. The discussions that we engaged in when unpacking Flint as a class did not carry into the lesson planning discussions. As this last set of data from the transcript reveals, PSTs sought to raise awareness about Flint without an additional step of advocacy or effecting change.

Implications

This study adds to the extant literature in the field by providing details about the types of challenges PSTs negotiate while planning lessons focused on TMfSJ. Although scholars typically analyze completed lesson plans, studying the lesson plan along with the dialogue from the transcripts, provides us with new insights. First, given the complexity of TMfSJ, PSTs can benefit from sustained opportunities to engage in some of these categories independently (e.g., unpacking standards and socio-political contexts) before attempting to create a full lesson plan. For example, if PSTs do not know how to fully deconstruct a socio-political context and see the historical underpinnings of it, Flint and other social injustices will be reduced to a simple context for a problem just like shopping in a store or playing video games. As presented, the PSTs discussions in this portion of the study aligned with level three of category 6b in the CRMT-TM Lesson Analysis Tool. This highlights that while PSTs did plan for opportunities to analyze a socio-political context, they did not intend for students to engage in discussions about how the mathematics could be used to change or transform the problem (levels four and five of category 6b).

Categories one and two of the CRM-TM Lesson Analysis Tool address the cognitive demand in the lesson as well as the opportunities to develop students' depth of knowledge. During this study, it was clear that PSTs own content knowledge directly impacted their ability to plan lessons aligned with level five in each of these categories. For example, the word problem that this group settled on did not provide opportunities for "complex mathematical thinking, [utilizing] multiple representations…or complex understanding" (TEACH-MATH, 2012). Therefore, PSTs need ongoing development of their mathematics understanding as they plan lessons. As I highlighted above, PSTs abandoned one potential task because they did not think they could do the necessary numerical conversions.

Another overall finding from this work is the balance between learning to develop such a complex skill compared to PSTs trying to meet doing a “good job” with this activity (although it was not graded). PSTs were very concerned about demonstrating everything that was required for the assignment and wanted to be sure they were getting it right. PSTs seemed to think that there was one “preferred task” that they should pick to work towards developing a standard. Therefore, there were many times during the conversation that the PSTs wanted to quickly decide on what to fill in for the lesson plan template to satisfy the professor. Mathematics teacher educators must be explicit that the purpose of these assignments is for PSTs to navigate this complex process in a safe space free from assessment.

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GRADE THREE STUDENT'S SOLUTION STRATEGY ON A MULTIPLICATION TASK: PRESERVICE TEACHERS' NOTICING AND REFLECTION ANALYSIS

Amy Roehrig
Illinois State University
aeroehr@ilstu.edu

In this study, I looked at the type of noticing through reflections written by preservice teachers (PSTs) after viewing a video of a grade three student's multiplication solution strategy. The task presented to the grade three student was also completed by the PSTs prior to viewing the video. PSTs were provided with a prompt to complete a reflection concerning the video. Results show that the use of video observations and reflection prompts in conjunction with content lesson produced reflections that were focused on analysis of student strategies and thinking.

Keywords: Problem Solving, Teacher Education-Preservice, Instructional activities and practices

Observations are an integral part of teacher education programs, for they provide a way for preservice teachers (PSTs) to experience a classroom before they have their own (Star & Strickland, 2008). But what PSTs garner from the experience is often different than the intended goal set forth by the teacher educator (Mason, 2008, 2011). The purpose of this paper is to report on the type of noticing made by PSTs using video observations, reflection prompts, and teaching mathematical content in conjunction with the observation.

Introduction/Background

A person's previous experience and beliefs influence what they notice (Beswick & Muir, 2013; Mason, 2011). This knowledge is an issue when PSTs are required to observe classrooms and students as part of their teacher education program. When PSTs observe and reflect on their experiences, they tend to focus on events of the classroom in chronological order, with a large focus on classroom teachers' classroom management of students (Beswick & Muir, 2013; Star & Strickland, 2008; van Es & Sherin, 2002). The noticed event depends mostly on a person's "own experiences of teaching and learning…all influenced by their experiences of the subject and how it is taught and learned" (Beswick & Muir, 2013, p. 29).

A growing field of research in education centers around what teachers notice that non-educators may fail to notice (Roller, 2016). Teachers go through the day being aware and noticing multiple aspects of student learning and understanding. When people "are not attuned or sensitized to" (Mason, 2011, p. 35) something, their attention can be drawn elsewhere. This seems to be the case when teacher educators require PSTs to observe classrooms, teachers, and students. When PSTs are not cognizant of what they should be observing, they often focus on teacher classroom management actions and classroom environment, whereas experienced teachers notice more about student learning (Star & Strickland, 2008).

To better focus PSTs' observations and noticing, several researchers have incorporated videos into their education courses (e.g., Ineson, Voutsina, Fielding, Barber, & Rowland, 2015; Mason, 2008; Roller, 2016). Researchers found that the use of video for observations attuned PSTs to fine details in the classroom setting and navigate the complexity of teaching and learning (Mason, 2011). Roller (2016) allowed PSTs to "re-watch and reflect using various lenses or questions to help tease apart what was effective and not" (p. 480). The purpose of using video for observations was to give PSTs a moment outside of teaching to break down and analyze decisions that occurred during a teaching episode and the effect on student learning (Mason, 2008, 2011; Roller, 2016; van Es & Sherin, 2002).

Another method that researchers have used for focusing PSTs during observations is providing prompts or specific learning goals for the video observation (Beswick & Muir, 2013; McDuffie et al., 2014; Phillip et al., 2007; Star & Strickland, 2008). Phillip et al. (2007) found that giving PSTs the opportunity to analyze student understanding of mathematics before being tasked with teaching students allows PSTs to make connections linking what they “are learning about teaching with what they already know about students’ mathematical understanding” (p. 443). Likewise, Beswick and Muir (2013) found that when PSTs were given prompts prior to viewing a video, they were able to better identify students’ learning.

Lastly, the task of writing reflections about video viewing exercises enables PSTs “to recognize the specific features of problems to which experts know to attend” (van Es & Sherin, 2002, p. 576). Analyzing what is noticed is just as important as the action of noticing. This analysis through written reflection allows PSTs to develop skills in interpretation of student learning and actions that can then be used to inform pedagogical decisions (van Es & Sherin, 2002). McDuffie et al. (2014) found that by providing support such as through prompts and class discussion, PSTs were able to “notice a range of student resources and to analyze how and why these resources enhance learning” (p. 267).

Methods

Participants

Participants were 22 elementary PSTs at a Midwest university, enrolled in a 16-week mathematics content course focused on algebra and geometry. Nine student reflections were selected based on their adherence to the reflection protocol and evidence that they viewed the student video.

Data Collection

A multiplication task was assigned to the PSTs to examine their understanding of non-traditional multiplication solution strategies for finding the total number of mailboxes in a 23 x 29 array. The same task was presented to a grade three student in a one-on-one interview. PSTs were shown the grade three student video interview. A class discussion followed the initial viewing and PSTs were given access to the video and a reflection prompt.

Data Analysis

The constant comparative method of data analysis was used inbuilding a framework for coding data. To enable the development of a coding schema, Corbin and Strauss’ (2015) three phases of coding—open, axial, and selective were used. Data analysis begun using open coding, statements that focused on how PSTs viewed the learning of the student and beliefs about the level of student understanding were selected. Axial coding followed, wherein the extracted statements were grouped according to the focus of the statements (i.e., PST centered, student centered, or a comparison statement). A second survey of the reflections produced more statements of interest using the three focus categories. To finish analysis, selective coding was done, in which the final schema was constructed.

Relations among the statements within and between the three categories were identified and new coding categories were created which led to a two-part coding schema: (1) focus of statement; (2) description statement. Through the process of open coding a total of 88 statements of interest were found and categorized.

Results

The purpose of this research was to determine the types of reflection PSTs provide using video, prompts, and content knowledge. Of the 88 statements, 35% were focused on PSTs' strategies or self, 45% were focused on the student's strategy, and 19% were focused on comparison between the PST and student. The following sections will illustrate results for each of the seven descriptive categories.

Error/Strategy Analysis

Thirty-one percent of the statements were focused on a student made error or an analysis of the student strategy and many of the statements were focused on the student. Only one statement was comparing the PST's strategy to the student's strategy.

Expected Strategy

Statements concerning the student using an expected strategy included only two of the 88 statements. None of the statements focused solely on the student, but instead focused on the PST or was a comparison statement. The comparison statement discussed how the student's method was similar to the PST's strategy.

Unexpected Strategy

Many of the statements categorized as *unexpected strategy*, focused on the PSTs being surprised or interested that a student would use a similar strategy to their own. This is evident in 12 out of 14 statements being focused on the PST. Two statements that were focused on the student were elaborated on the student's explanation of why certain steps were necessary or used.

Insight into Standards

There were few statements focused on insights into mathematical standards because the PSTs' experience with state standards was not a course focal point. Only five of the 88 statements contained language that could be interpreted as centering on standards or a point in learning expectations. For example, one PST wrote, "he started to learn the algorithmic way, but it was interesting to see how he still used the horizontal way" (PST 8).

Insight into Video Viewing

Though viewing the video of a grade three student was part of the reflection process, very few PSTs focused on the video and its specific usefulness in observation or reflection. Of the five statements concerning viewing the video, three were focused on the PST. The theme of the statements fixated on future teaching positions or the degree to which mathematics education has changed since they were elementary school students.

PST Learning/Insight

When PSTs' statements focused on their own learning and insight into multiplication, it was often written as a comparison between theirs and the student's strategy. There was a total of 19 statements, 13 of which focused on the comparison of strategies. PSTs did not provide insight into analyzing the difference between the strategies, instead they focused on what they did compared to the student.

Student Strategy

Statements focusing on the student's strategy but not analysis of the strategy comprised of 17% of the total statements of interest. As expected, many of these statements restated the step-by-step procedure or comment on method done by the student. For example, one PST stated, "he [the student] was able to find his answers fairly quickly in his head" (PST 9).

Connections

When PSTs' reflections were focused on themselves, they were more likely to comment on the student's unexpected strategy or the use of video observations. For example, referencing the use of video observation, a PST commented, "…this video proved how much math has changed since I was in school" (PST 3). When referencing an unexpected strategy from the student, a PST stated, "I did not really know what types of steps or methods a third grader may use to solve these [problems]" (PST 8).

If the reflection statements focused on the student, the PSTs were more likely to be reflecting on the specifics of the student's strategy or attempting to analyze an error or strategy used by the student. One PST wrote, "he was spilling any thoughts he had onto the paper" (PST 1) and "he went for 2 sections of 20x10" (PST 1), when referencing the student's strategy. To analyze an error or strategy, PSTs' commented on the confusion the student had when he was attempting to use the distributive property. For instance, "he would break them [mailboxes] down and then add, but sometimes got confused on how to break them down" (PST 1).

Lastly, PSTs were more likely to discuss their insights into their own learning when they were making comparison statements between theirs and the student's strategy. For example, one PSTs reflected on the beliefs the student has concerning mistakes, "…Grant corrected his mistake and even laughed at his misstep. I oftentimes forget to let myself make those mistakes and laugh when I do" (PST 7).

Discussion

In this study, analysis of PSTs' reflections of a grade three student's solution strategies through video observation in conjunction with mathematical content knowledge was done. My goal was to document the types of reflection statements given in response to a video observation.

It was found that the PSTs' noticing was focused on student learning, possibly due to the alignment of content knowledge and reflection. The experience the PSTs garnered from completing the task themselves enabled them to focus on strategies the student used. Such as, PST 3 stated that they "were surprised that he [Grant] was able to multiply horizontally so easily" when the PST had a difficult time multiplying mimicking his strategy. PSTs were able to analyze and compare the student's work to their own. For instance, PST 9 recognized that Grant "knew the skeleton of the distributive property, but still needs to develop his understanding because he erroneously distributed (9x10)x3 instead of (10x9) + (10x3)." The PST then continued by describing their own use of the distributive property.

These results are like the findings of Roller (2016), who found that PSTs were positively affected by the experience of viewing the video. Both McDuffie et al. (2014) and Mason (2011) found that by providing a focus prior to the observation the participants were more attentive to the observation goals. By providing a prompt, the PSTs were able to focus their noticing when viewing the video. This was evident in the procurement of 88 statements within the nine one-page reflections. Additionally, the wording of the prompt directed PSTs to focus on error analysis, unexpected strategy, and comparison of their strategy to the student's strategy, which was showed by the 31%, 16%, and 22% of the total statements, respectively.

A limiting factor in this study was low number of reflections used to create a coding scheme; therefore, future research is necessary to support the framework. The prompt provided to the PSTs was limiting in focus, further research needs to be conducted on the level of specificity needed in a prompt to produce higher quality PSTs' reflections. This study provided additional insight into the use of video when engaging PSTs in student observations and adds to the growing research on methods to aid in PST noticing practices. Lastly, further research is needed to determine whether an increase in the number of focused reflection statements transfers to better quality observations.

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COUNTING, LAYERS, AND FORMULAS: PSTS' RESPONSES TO STUDENTS' VOLUME MISCONCEPTIONS

Dae S. Hong
University of Iowa
dae-hong@uiowa.edu

Cristina Runnalls
Cal Poly Pomona
ccrunnalls@cpp.edu

The purpose of this study was to explore how elementary pre-service teachers responded to students' hypothetical misconceptions about volume measurement. We carried out both pre-assessments and follow-up interviews with 17 pre-service teachers, with tasks focused on both volume content knowledge and hypothetical student responses to volume tasks. Preliminary findings indicated a preference towards show-and-tell responses, often with a focus correct or incorrect use of the volume formula. While PSTs were frequently able to correctly identify students' learning challenges with regards to volume, they experienced difficulty in addressing those challenges in meaningful ways. Recommendations for supporting pre-service teachers in teacher education programs are discussed.

Keywords: Geometry and Geometrical and Spatial Thinking, Measurement, Teacher Knowledge

In *Principles to Action*, the National Council of Teachers of Mathematics [NCTM] (2014) presented eight highly effective practices for teaching mathematics. One important practice promotes specifically the need to elicit and use evidence of student thinking during teaching. Teachers are required to make in-the-moment decisions when responding to students, using questions to build productive discussions, and interpreting students' thinking in ways that help shape further instruction (NCTM, 2014).

In the present report, we attend to these ideas in the topic of volume measurement. Findings from national and international assessments indicate that American students tend to perform poorly in the measurement domain compared to other mathematical domains (e.g. Mullis, Martin, Foy, & Hooper, 2016). There are many potential reasons for students' learning difficulties in volume measurement, such as development or conceptual difficulties, as well as inadequate curricular coverage. One significant factor in students' learning is the role of teachers' knowledge about volume measurement, especially as it affects lesson design and implementation. The purpose of this study was to explore types of knowledge that PSTs possess with regards to volume measurement, and the ways they use this knowledge in response to hypothetical student work in volume.

Related Literature

Understanding Volume Measurement

To understand volume measurement conceptually requires coordinating several ideas, such as: filling three-dimensional space with same-sized units, counting same-sized units, an iterating layer structure, and linking layer structure to the volume formula (Battista, 2004; Sarama & Clements, 2009; Vasilyeva et al., 2013). While these ideas are fundamentally important in other measurement topics such as length and area, the addition of a third dimension for volume makes them significantly more challenging. When working with same-sized units and volume, students may count only cubes visible on the faces, sometimes double-counting (Battista & Clements, 1996), or may struggle to "see" the structure of layered cubes in three-dimensional space

(Battista, 2004). Unfortunately, without these ideas, students tend to use the volume formula for a rectangular prism, $l \times w \times h$, without knowing why or how it works (Vasilyeva et al., 2013).

Different Types of Teachers' Knowledge

As suggested by the NCTM practices guide and others, teachers need many different types of knowledge to plan and teach effective lessons. For volume measurement, some of the knowledge required may include understanding and linking ideas of space-filling with same-sized units with ideas of iterated layer structure, and knowing and responding to students' well-known challenges in volume measurement (Battista & Clements, 1996; Vasilyeva et al., 2013).

To explore our PSTs' developing knowledge about volume, we employed analysis of their responses to hypothetical student work. Examining PSTs' responses to hypothetical students' errors or invented strategies has been an effective tool in previous studies, where it has been used to explore PSTs' knowledge regarding multiplication (Maher & Muir, 2013), subtraction (Son, 2016a), and ratio and proportion (Son, 2013), among others. Teachers can elicit and use evidence of student thinking if they are able to respond to students' misconceptions, and build on these to have productive mathematical discussions. To explore the mathematical knowledge of our PSTs, as well as to better understand the knowledge necessary for effective teaching of volume measurement, the present study analyzed PSTs' responses to volume tasks and hypothetical student responses to those tasks. Through our analysis, we were interested in exploring types of knowledge that PSTs need to design and develop effective volume measurement lessons. We also hoped that the results of this analysis could help identify areas requiring additional support in teacher education, especially with regards to volume measurement.

Methods

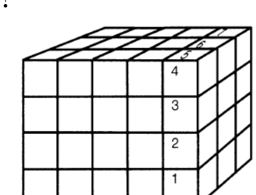
This study took place in a large Midwestern university. The first author taught a 15-week geometry and measurement content course for elementary education majors, designed to address K-8 geometry and measurement topics. A total of 17 participants from the class volunteered for the study. All participants were female, reflecting the overall makeup of the course. There were a wide range of mathematical backgrounds, with highest previous reported mathematics courses ranging from other college mathematics (41%) up to Calculus (41%).

Data Collection

Data collection began immediately before the measurement unit of the course, and continued throughout the length of the unit. A written pre-assessment was first used to examine students' content knowledge and initial responses to hypothetical students' strategies. The pre-assessment consisted of six tasks focused on volume measurement, adapted from previous studies (Battista, 2004; Battista & Clements, 1996; Vasilyeva et al., 2013). Follow-up semi-structured interview explored the same tasks more in-depth. We report on PSTs' written responses and interviews for two of the tasks, both shown in Figure 1

1. The following is a student's response to a volume question.
   a) Do you agree with this response? Why or why not?
   b) Why do you think the student answered that way?
   c) If you don't agree, how could you help the student with the question?

T: How many cubes does it take to build this prism?
S: There are seven layers, so it will be 105 cubes.



2. The following is a student's response to a volume question.
   a) Do you agree with this response? Why or why not?
   b) Why do you think the student answered that way?
   c) If you don't agree, how could you help the student with the question?

T: How many cubes does it take to build this prism?
S: There are ten cubes in the front, and six on the right side, and eight in the back corner. So it should take 24 cubes to build it.
T: So what's the volume of the shape?
S: Length times width times height… so 30.

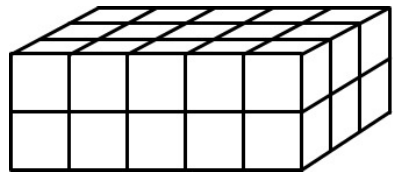


Figure 1: Pre-Assessment Tasks

Data Analysis

We analyzed Tasks 1 and 2 with a focus on participants' assessment of student thinking and use of pedagogical strategies. Our analytical framework for Tasks 1 and 2 was adapted from (Son, 2013), who developed a framework for analyzing PSTs' responses across several studies. The analytic framework served to examine several facets of responding to student errors, including the focus of instruction, form of address, and pedagogical actions. Categories and subcategories used are outlined in Table 1.

Table 1: Analytic Framework for Pedagogical Strategies

Aspect	Categories
Mathematical/instructional focus	Conceptual, procedural
Form of address	Show-and-tell, give-and-ask
Pedagogical action(s)	Re-explain volume concepts, suggest visual cognitive conflict, probe student thinking, recommend partitions, etc.

Mathematical/instructional focus was coded as either procedural, conceptual, or both, with double-coding if a participant elaborated upon strategies that involved both conceptual and procedural suggestions in their answer. Procedural responses focused primarily on use of the formula or correct answers, while conceptual responses attempted to use fundamental ideas of volume to make sense of the task. Form of address was introduced by Son and Sinclair (2010), and was coded as either show-and-tell or give-and-ask. Show-and-tell responses were primarily teacher-centered responses that provided solutions or explanations, typically using words such as *show*, *explain to*, or *demonstrate* (Son, 2016b). Give-and-ask responses were often student-centered, and asked the student to provide explanation for their work. These responses included words such *give*, *ask for*, *suggest*, or *look at* (Son, 2013). Finally, pedagogical actions were specific to the tasks, and including coding for the ways PSTs' responded to the mathematical content of the student response. These actions included revisiting the volume formula, pointing out double-counted layers, and other similar actions.

Results

Written Responses

For Task 1, responses to this student were focused on procedure, and often involved re-explaining the concept of volume through use of the formula. The most common response noted that the student had incorrectly counted cubes, with a focus on double-counting. This response indicated that our PSTs were able to diagnose the student's primary challenge, a good foundation for addressing that challenge. However, only four PSTs went on to mention addressing layer structure for their pedagogical suggestions. This showed that PSTs might not have connected the importance of understanding layer structure to the student's misconception, with over half of participants instead reminding students of the formula for volume to solve the task.

Task 2 was interesting because the hypothetical student did not provide the correct answer initially, but was able to still correctly apply the volume formula. Responses to Task 2 did not show the same reliance on procedural responses as before. The most frequent responses noted that the student was correct, with little elaboration. The most frequent pedagogical suggestion involved introducing the use of manipulatives to *show* or *tell* the student how to count cubes. However, is interesting to note that many PSTs who thought the student was correct did not attempt to provide any pedagogical suggestions, despite evidence of confusion in the transcript. These PSTs may have believed that arriving at the correct answer once was enough to not to explore student's thinking further, and hints at the fact that several of the PSTs in our study viewed using formula correctly as synonymous with understanding volume.

Interview Responses

For Task 1, although approximately half of our PSTs recommended using manipulatives to explore volume measurement, only one PST requested that the student explain his/her thinking. Seven PSTs noted that they might tell or teach the student more about the topic, with many referring to reviewing the volume formula. Additionally, only two PSTs mentioned that they would address or re-explain 3D or layer structure. It appeared that many PSTs were not aware that understanding of layer structure was critical to the task.

For Task 2, unlike their written responses, more PSTs provided pedagogical suggestions, including recommending the use of manipulatives, or an explanation of structure. However, only two PSTs requested the student to explain his/her thinking. These responses show that it is likely PSTs may feel the need to provide further instructions to students, preferring to "tell" or "explain" volume concepts rather than using students' thinking to guide discussion.

Discussion and Conclusions

Our initial analysis shows that encouraging PSTs to not only correctly identify students' learning challenges, but also move towards strategies of addressing them at a conceptual level is a pivotal step towards supporting PSTs in future implementation of effective volume lessons. Our findings also highlight the importance of reminding PSTs that providing mathematically correct answers is not synonymous with conceptual understanding. Continually assessing students' thinking, even after arriving at a correct answer, remains an important and critical instructional practice key to implementing effective lessons. Our preliminary findings also indicate several recommendations for teacher educators. First, although it is natural to think that using the formula might be an easier way to solve volume problems, PSTs need opportunities to understand and explore layer structure and space-filling ideas. Once they are familiar with layer structure and space-filling with same-sized units, they will be more likely to recognize the relationships between these ideas and well-known learning challenges in volume measurement. Since our PSTs were early in their teacher education, it is expected that they lean towards more “show and tell” strategies, especially with a focus on the formula. However, it is important for PSTs to have opportunities to develop familiarity with give-and-ask strategies as well.

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PROSPECTIVE ELEMENTARY SCHOOL MATHEMATICS TEACHERS' PERSPECTIVES OF CHANGES IN THEIR IDENTITY THROUGH A STEM COURSE

Olive Chapman
University of Calgary
chapman@ucalgary.ca

Dianne Dodsworth
University of Calgary
ddodswor@ucalgary.ca

Research has indicated that there is a strong relationship between teachers' identity and their teaching. This suggests the importance to support prospective teachers' development of identities oriented to teaching mathematics from a reform perspective. This study investigates prospective elementary school mathematics teachers' perspectives of changes to their professional identities resulting from their participation in a Science, Technology, Engineering, Mathematics [STEM] education course that incorporated multiple perspectives of STEM education. Data collected through interviews were analyzed qualitatively based on the theoretical perspectives of identity and STEM education. Preliminary findings indicate shifts to their identity based on their view of mathematics, doing mathematics, teaching mathematics, and themselves as becoming teachers of mathematics. This was influenced mainly by one unit of activities of the STEM course.

Keywords: Affect, emotion, beliefs and attitude; Instructional activities and practices

Recent studies on prospective mathematics teachers [PTs] have addressed different issues in relation to their identity including: how they understand teaching and themselves as mathematics teachers (Neumayer DePiper 2012); the development of their identity as they transitioned from prospective to practicing teachers (Jong, 2016); how they positioned themselves in field experience contexts (Mosvold & Bjuland 2016); and how their identity progressed from mathematics methods coursework to field experience (Hodges & Hodge 2017). Some studies focused on change in identity or identity development resulting from PTs' participation in mathematics education courses, mathematics course for PTs, or field experience activities. For example, Heffernan (2016) investigated change in PTs' mathematical identities during a mathematics course for education majors; Kaasila, Hannula and Laine (2012) compared the identity talk of five PTs at the beginning and end of a mathematics education course to identify their interpretative repertoires; and Akkoç and Yeşildere-İmre (2017) investigated the contextual nature of professional identities of three PTs prior to and after participating in field experience activities. The results of such studies provide evidence of the importance to understand and impact PTs' professional identity to enhance their learning in becoming mathematics teachers.

In general, research has indicated that there is an important relationship between teachers' identity, particularly, their beliefs or conceptions of mathematics and mathematics pedagogy, and their teaching of it. However, as Ponte and Chapman (2016) discussed, there are few studies that deal explicitly with PTs' identity and more attention is needed on how to support its development. This paper reports on an ongoing study that contributes to this area of research. The larger study is investigating the impact of a Science, Technology, Engineering, Mathematics [STEM] education course in a teacher education program on elementary school PTs' identity related to mathematics education. The research question addressed here is: what are the PTs' perspectives of how they changed in their thinking (e.g., beliefs/conceptions) associated with their identity about mathematics and its teaching and learning and what aspects of the STEM course supported these shifts?

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

Theoretical Perspectives

The key theoretical constructs framing the study are identity, STEM education, and inquiry-based learning.

Identity

While identity has become an important construct in researching the teacher and in teacher education, it is a complex construct that has been conceptualized in different ways in studies in education (Beauchamp & Thomas 2009; Beijaard, Meijer & Verloop 2004) and more generally (Gee 2001; Wenger 1998). Identity includes our experiences and knowledge, our perceptions of ourselves, others' perceptions of us, and our perceptions of others' perceptions of us (Wenger, 1998). It is also viewed as beliefs, commitments, and intentions adjusted to a particular community (Lave & Wenger 1991), the constellation of attributes, beliefs, values, motives and experiences that people use to define themselves in their professional capacity (Schein, 1978), and the roles, values and norms of the professional group (Berger & Luckmann1966). These perspectives can be used to study different aspects of teachers' identity; however, in this paper, the focus is on their professional identity, which is important because it is a key way that individuals assign meaning to themselves, and it shapes work attitudes, affect and behavior (Siebert & Siebert, 2005). It includes professional attributes, beliefs, values, motives and experiences they use to define themselves as teachers (Schein 1978). In this study, the PTs' professional identity is considered to include their views, beliefs, conceptions, or what they value about mathematics and its learning/teaching and about themselves as the mathematics teachers they want to become. Their perspectives of their identity consist of their views or beliefs based on their explanations of their thinking and experiences in the STEM course.

STEM Education

In the last decade, STEM education has received increased attention as concerns grow about preparing students for a changing global economy and workforce needs. As Bybee (2013) points out, the overall purpose of STEM education is to further develop a STEM literate society. However, STEM education could have different meanings depending on if it is viewed, for example, as an acronym for the four separate disciplines or as interdisciplinary or intradisciplinary that cuts across the boundaries of these disciplines. Bybee also noted that the "purpose of STEM education is to develop the content and practices that characterize the respective STEM disciplines" (p. 4). Thus, for example, a course in mathematics that includes connections to real-world situations to develop deep understanding of mathematics, or a course in which "students are required to demonstrate their understanding of STEM disciplines in a work-based, contextual environment" (Kennedy & Odell, 2013), or a course on robotics in which students are expected to innovate and invent with no explicit focus on developing mathematics or science concepts can all be considered to be STEM courses. The section of the STEM course in this study was a hybrid of these perspectives, structured around three units that focused on mathematics, robotics, and the engineering design process.

Inquiry-based Learning

An inquiry-based learning perspective framed the STEM course. Inquiry-based learning places emphasis on the learners with their understanding as the central focus. "[It is] about asking questions and seeking answers, recognizing problems and seeking solutions: wondering, imagining, exploring, investigating, discussing, reasoning and looking critically at what we find out" Jaworski (2015, p. 30). Students are engaged individually and collaboratively in inquiry tasks to "foster [their] construction of their knowledge through inquiry, exploring, and finding their own path to solution" (Maaß & Artigue 2013, p. 782).

Research Methods

This study, which is in progress, is based on one section of the STEM course offered in Fall 2018 and taught by the second author of the paper. Participants are elementary school PTs in the first semester of their two-year Bachelor of Education program at a Canadian University. They had no post-secondary mathematics and no mathematics education courses prior to the STEM course. They are representative of PTs who disliked mathematics and took the course because it was required for their program. Based on ethics requirement, we are allowed to recruit them after the course ends and final grades are confirmed by the university.

The STEM Course

The course is scheduled for two 90 minutes classes each week for 11 weeks. The learning outcomes for students include*:* (1) Develop a foundational understanding of the nature of discourse in STEM disciplines as related to teaching and learning; (2) Understand and appreciate how the engineering design process can contribute to teaching and learning mathematics and science; (3) Design learning environments in STEM; and (4) Identify concepts and make explicit the connections across disciplines. Key activities of the section of the course being studied were:

Activity (1): Conduct a *concept study* of elementary school mathematics concepts (e.g., whole number multiplication) over 4 weeks. This included exploring alternative meanings, representations, instantiations of concepts; identifying concepts within genuine/rich/authentic contextual tasks; and making connections within and across mathematics concepts. Problem solving/modeling also played an important role. For example, students, in groups, had to identify a real-world scenario (e.g., local forest fire with plans to reforest) and decide on an authentic problem to solve (e.g., how long to reforest a park).

Activity (2): Create a real-world challenge task of their choice and design, build and program a robot to complete the task over four weeks. This included understanding and applying the engineering design process.

Activity (3): Apply the engineering design process to design a STEM inquiry unit for teaching and unpack the mathematics and science concepts contained within it that students will learn.

Data Sources and Analysis

Data for the study consist of interviews, students' course assignments, and instructor's notes during the course. We base this paper on the first four PTs we interviewed before the due date of the proposal. We also focus only on the interviews to offer a sample of the participants' perspectives of changes to their identities in becoming teachers of mathematics. Each participant was interviewed for about an hour. The semi-structured interview included eliciting their thinking about their school experience learning mathematics, pre- and post-STEM course beliefs and feelings about mathematics and doing mathematics, learning in the STEM course, shifts in understanding of mathematics and its teaching and learning, and aspects of the STEM course that contributed to these shifts. All interviews were audio-recorded and transcribed.

Analysis was guided by the research question and theoretical perspectives of identity and STEM education. Using open coding (Corbin & Strauss, 1990), the transcripts were scrutinized to highlight statements that indicated the participants' perspectives of changes in their thinking about mathematics and its teaching and learning and their experience in the STEM course that contributed to these changes. This included coding their pre- and post-course beliefs and feelings about mathematics, doing mathematics and teaching and learning mathematics and aspects of the STEM activities identified as supporting changes in their thinking. The coded information was summarized for each participant and compared for similarities and differences in their thinking/identity. The findings reported here are examples of common shifts in their thinking.

Results

These preliminary findings are a preview of changes to the PTs' identity and of specific aspects of the STEM course they considered to be effective in helping them to achieve these changes to value mathematics differently from the way they experienced it as learners and want to teach it. In particular, there were shifts in their identity based on how they viewed mathematics and doing mathematics, teaching mathematics, and themselves as becoming teachers of mathematics. Examples of these shifts follow.

For their pre-course view of mathematics, all the PTs emphasized that mathematics was about numbers and applying numbers. Consistent with this, for them, doing mathematics meant developing numeracy and working on computations and repetitive worksheets. The main shift in their post-course view of mathematics was extending it to include mathematics as problem solving, connections and reasoning.

Their pre-course view of teaching focused on teacher-directed approaches which were "very un-engaging" for students. Their post-course view shifted to a more student-centered perspective, which included: "engaging students in multiple ways to solve a problem" and "logic of math" (i.e., mathematical reasoning); "offering opportunity for success for all;" seeing "potential to facilitate success for all;" "making it [math] more visible and authentic to students;" and "making it relevant to life of student."

Their pre-course view of *self* as teacher was connected to "terror" of doing and teaching mathematics. Their post-course view shifted to having "increase in confidence" to teach mathematics, being "less intimidating," having more "courage to explore math," and being "grateful to be empowered."

The PTs' perspective of course influence on these shifts included: (1) what they learned; for example, "STEM gives the context for math," "hands-on, real-world math problem and problem solving," "multiple problems from real-world situation," "problem posing," "many ways to understand math," "multiple ways to solve problems" and (2) collaboration; i.e., "group work."

While course Activities 2 and 3 described above had potential to enhance the PTs' math-related identity, the four participants did not refer to them and focused mainly on Activity 1, that explicitly addressed mathematics concepts and problem solving, as the key influence on the changes in their identity. For example, the engineering design process has similarities to problem solving and modelling, but without explicit support to consider these similarities these PTs seemed to not associate this process with mathematics and any impact on their mathematical identity. They also drew on the mathematics they already "knew" in unpacking mathematics concepts in the STEM tasks, suggesting no new learning of mathematics for teaching and shifts in identity. Thus, while the inquiry-oriented approach allowed them to be innovative in designing STEM tasks, their initial identity seemed to play a role in limiting the type of shifts they were able to achieve in their thinking and identity.

Conclusion

Our preliminary results, offered with caution, suggest that elementary school PTs could experience meaningful shifts in their identity for teaching mathematics in an inquiry-based, STEM education course with a component that explicitly deals with mathematics. However, further work is needed to explore the relationship between other components/perspectives of a STEM course (such as activities 2 and 3) and PTs' learning that can result in shifts in their identity development to become teachers who can meaningfully engage students in learning mathematics. We intend to consider this as we complete the larger study and make changes to the course to further investigate and understand this relationship.

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FRACTION-WITHIN-FRACTION: PROSPECTIVE TEACHERS' PERSPECTIVE ON SYMBOLS AND MEANING

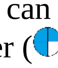
Ofer Marmur	Xiaoheng Yan	Rina Zazkis
Simon Fraser University	Simon Fraser University	Simon Fraser University
omarmur@sfu.ca	xkyan@sfu.ca	zazkis@sfu.ca

We explore prospective elementary-school teachers' attempts to provide signs and symbols with mathematical meaning. The data comprises responses to a task inquiring about the existence of numbers between 1/6 and 1/7, in which the participants were asked to compose a hypothetical dialog between a teacher and students addressing this issue. We focus our analysis on three representative cases exhibiting difficulties in assigning meaning to "6½" when it appears in the numerator or denominator of a fraction representation. These difficulties are examined by utilizing the theoretical lens of "semiotic representation", and explained via the meaning associated with different representations of fractions.

Keywords: Teacher Knowledge; Number Concepts and Operations; Rational Numbers

Among the many difficulties associated with the learning of fractions, we focus on difficulties related to the multiple representations of fractions (e.g., Čadež & Kolar, 2018), and inquire into how learners attempt to provide unexplained signs and symbols with mathematical meaning. We present a case where $\frac{1}{6½}$ and/or $\frac{6.5}{42}$ were proposed as solutions by prospective elementary-school teachers, and examine the teachers' difficulties in accepting the symbolic suggestion as a legitimate solution endowed with meaning.

Theoretical framework

We rely on the notion of *semiotic representation* (Duval, 2006), which suggests that mathematical objects can be accessed only through signs. These signs are not the mathematical objects themselves, but merely representations of mathematical objects, which can be represented in different semiotic systems, also called registers. Regarding fractions, different models are linked to a range of semiotic representations. For example, the same fraction can be represented in a verbal register (three-quarters), symbolic register (¾), and visual register ().

According to Duval (2006), a mathematical activity consists of two types of *transformations* of semiotic representations: treatments and conversions. *Treatments* are transformations of representations within the same semiotic register, while *conversions* are transformations of representations between different registers. For example, a transformation between $\frac{3}{4}$ and $\frac{6}{8}$ is a treatment within the fractional register; and a transformation between $\frac{3}{4}$ and 0.75 is a conversion between the fractional and decimal registers. Duval pointed out that conversions between registers are more complex than treatments, and are accordingly a significant source of learners' difficulties. We use these constructs to address the following research question: *How do semiotic representations of fractions play a role in prospective elementary-school teachers' consideration of the appearance of a non-whole number in the numerator or denominator?*

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Method

The participants were 33 prospective elementary-school teachers enrolled in their last term of an undergraduate program towards a teacher certification. During the course the participants were given several scripting tasks, which included a beginning of a hypothetical classroom dialogue between a teacher and students that the participants needed to extend in a way they found mathematically and pedagogically fit (e.g., Brown, 2018; Marmur & Zazkis, 2018).

Teacher:	Could you give me a number between 6 and 7?
Anna:	We don't have any numbers between 6 and 7!
Maya:	That's not true! $6\frac{1}{2}$ is a number between 6 and 7.
Anna:	Ah, right. I forgot about fractions.
Emily:	Teacher, is there a number between any two numbers?
Teacher:	Good question. What do you think?
Emily:	Hmm, I don't know.
Maya:	I don't think so. There are no numbers between $\frac{1}{6}$ and $\frac{1}{7}$.
Teacher:	…

Figure 1: Prompt for the Scripting Task

The scripting task we discuss here (see Figure 1) presented two pseudo-successive fractions, $\frac{1}{6}$ and $\frac{1}{7}$, and included a student-character claim that there are no numbers in-between. In addition to extending the dialog, the participants were asked to provide complementary commentaries, including a diagnosis of potential sources of error and a description of their own understanding of the mathematics involved. In the design of the prompt, we chose two fractions whose denominators were successive natural numbers, as such fractions are often perceived incorrectly as successive numbers with no numbers in-between (e.g., Vamvakoussi & Vosniadou, 2004). For analyzing the data, we first used inductive content analysis to identify emerging themes in the participants' responses. Subsequently we used directed content analysis (Hsieh & Shannon, 2005) to explain the rejection of solutions in which six and a half appeared in the numerator or denominator, while concentrating on different representations of fractions.

Findings

We focus on three representative cases that illustrate three kinds of obstacles in accepting the six-and-a-half suggestion, based on the provided meaning to its semiotic representation.

Symbolic Representation of Fractions: The Case of Adam

Adam's assignment demonstrated a rather sophisticated mathematical understanding, as it was one of only four assignments that explained how infinitely many numbers between 1/7 and 1/6 could be found. Nonetheless, his script opened with a student-character wondering whether 1/6.5 could be a solution, a suggestion that was immediately rejected by the teacher-character:

Teacher: As we can see, the numbers 1/6 and 1/7 go here and here (Points to exactly where these two numbers fit in on a number line). Now Emily, do you think that there are any numbers that are between 1/6 and 1/7?

Emily: There must be, because there is a gap between these two numbers in our number line, but what could they be? 1/6.5?

Teacher: It cannot be 1/6.5 because a fraction is already a decimal, so we cannot have a number that is 1/6.5. Any other ideas?

In the above excerpt, the teacher not only instantly rejects 1/6.5 as an optional answer, but also provides a perceived reason: “a fraction is already a decimal, so we cannot have a number that is 1/6.5”. Clearly the suggestion is rejected on the basis of representation: 6.5 is not an accepted representation for a denominator. This type of reasoning was also found in additional scripts that proposed an answer with 6.5 in the numerical expression. These responses pointed towards a rejection of 1/6.5 based on the mixture of fractional and decimal representations, revealing a fraction image in which the numerator and denominator must be whole numbers.

Interestingly, later in Adam’s script the student-characters suggested 2/13 as a solution, which is equivalent to the rejected 1/6.5. Nonetheless, neither in the continuation of the script nor in Adam’s mathematical commentary, was 2/13 connected to 1/6.5. Shedding light on this matter, we return to the teacher-character’s statement, “we cannot have a *number* that is 1/6.5”, and suggest that the option of 1/6.5 was rejected not only as a possible answer, but also as a perceivable number. Accordingly, while 1/6.5 can be written as a string of symbols on the page (“1”, “/”, “6”, “.”, “5”), this semiotic representation may not be associated with any numerical meaning for the scriptwriter. In this regard, we suggest that no connection between 2/13 and 1/6.5 was made, as the representation of the latter was not interpreted in the “number register”.

Visual Representation of Fractions: The Case of Sarah

Sarah’s script included many visual representations of fractions aimed at exemplifying the meaning of fractions and equivalent fractions. Attempting to create a visual representation was also the way by which the characters in Sarah’s script tried to make sense of 1/6.5:

Emily: Well, you know, Maya seems to make some sense, when I see the number 1/6 and 1/7, 6 is before 7 and you can’t have make a 1/6.5.

Anna: Yeah, 1/6.5 sounds weird... If I try to draw it, it looks weird:  Wait... I just drew 1 part of 6.5, so it is possible? Oh man, now I’m super confused. My drawing doesn’t make any sense.

We interpret the picture above as an attempt to draw six and a half squares in a row – the six squares on the left, and the last “half square” (rectangle where one side was halved) on the right. Additionally, the most left square is colored in, indicating what we interpret as the “1 part of 6.5”. We suggest that this visual representation provides a sophisticated attempt at making visual sense of 1/6.5, employing a *set model* of fractions, rather than the common *area model* involving pizzas and pies. Interestingly, later in the script, Sarah demonstrated an understanding of the set model, by creating a visual representation of 4/12 as 4 squares out of a total of 12 squares. Nonetheless, this model was rejected when applied to the suggested representation of 1/6.5.

We suggest that the common part-whole fraction-image may be so overpowering to learners, that even when demonstrating part-set understanding, they can only make sense of it as long as the numbers involved are whole. In this regard, we note Sarah’s attempts to make a conversion between different representational registers of fractions, that is between a visual part-whole register and a visual part-set register. Nonetheless, even when producing a suitable visual representation for 1/6.5 in the part-set register, we observe Sarah’s *cognitive impossibility of dissociating* (Duval, 2006) the fraction object and its ingrained visual part-whole representation.

Permissible Operations on Fractions: The Case of Eve

Eve's script included a suggested solution of $\frac{6.5}{42}$, although also here we found a student-character claiming: "I don't think we can have a fraction as a numerator". Similar to Adam's case, the student-characters reached an equivalent solution 13/84, though no connection to 6.5/42 was made. However, as opposed to Adam's case, we suggest that the issue here is not whether "6½ over 42" is a *number* according to the scriptwriter, but whether it is recognized as a *fraction*. When examining Eve's script, we noted that 6/42, 6.5/42, and 7/42 were written together in a row, though only 6/42 and 7/42 were *operated on* as fractions (to reach 12/84 and 14/84), illustrated by small "×2" writings present only next to these two fractions.

Additionally, in the accompanying commentary to the script, Eve suggested 1/6½ as another solution and provided her personal mathematical understanding: "I also knew 6 ½ was between 6 and 7 so inverse 6 ½ and turned it into a fraction: $1 \div 6½ = 1 \div \frac{13}{2} = 1 \times \frac{2}{13} = \frac{2}{13}$." While this calculation is correct, we argue that not only is the expansion of the numerator and denominator by 2 a quicker method to reach the final answer (i.e., $\frac{1}{6.5} = \frac{1\times2}{6.5\times2} = \frac{2}{13}$), but also that the mathematical symbols Eve uses reveal how the numbers are mathematically regarded by her. Rather than representing "inverse 6 ½" in fractional form as $\frac{\text{number}}{\text{number}}$, it is represented with a division symbol "÷" as "$\text{number} \div \text{number}$"; and subsequently operated on in accord with the rules of fraction division. We argue that in order to treat 1/6½ as a fraction, it should be recognized in the "fraction register". If it is not recognized as such – it is treated in the "number" register. It is clear Eve recognizes "1/6½" as a number between 1/7 and 1/6, as evident from her comment above. However, in Eve's own words, the symbolic manipulations are taken so that "inverse 6 ½" could be "*turned* into a fraction". Thus, as mathematical operations are associated with specific registers (Duval, 2006), not being able to regard 1/6½ in both "number" and "fraction" registers, narrows the range of permissible operations on this mathematical object.

Discussion

In the education literature it is well-known that the choice of a symbolic representation can play a crucial role in learners' understanding of a mathematical object. This is also evident in the case presented herein, as the data pointed towards the representation of fractions as an underlying source of explanation for the participants' unwillingness to accept their *own* 6.5-suggestions. Rather, the "odd" fraction representations, whether in symbolic or visual form, seemed to have shaped the assigning of meaning to the suggested mathematical object. In particular, the prospective teachers either: (a) dismissed the proposed answer as an actual number; or (b) struggled to make conversions between different representation registers of fractions; or (c) operated on the object according to division rules rather than fraction rules. Whereas the last option limited the range of permissible operations on the suggested object, the first two options resulted in the full rejection of the six-and-a-half numerical expression.

The overreliance of prospective teachers on the part-whole representation of fractions has been evident in the responses of almost all the participants in the current study. In Duval's (2006) terminology, we suggest it might be a cognitive impossibility of dissociating fractions from this early introduced representation, that is at the core of the difficulty with accepting fractions involving 6.5 as a semiotic representation endowed with meaning. We also note that the language used by many of the scriptwriters to describe their solutions, suggested that the *existence* of fractions between 1/6 and 1/7 depended on the symbolic representation enabling whole-numbers-in-between; as for example illustrated in the following statement: “There is no fraction that can fit between 6/42 and 7/42 but it looks like there could be a 13 that would fit between 12/84 and 14/84.” To conclude with practical implications, we note that in the literature we could not find sufficient attention to the *set model* of fractions (see Sarah’s case). We suggest that further attention to the set model not only could provide learners a way to endow semiotic representations of fractions-within-fractions (such as 1/6.5) with approachable meaning, but could also expand the ways by which they can operate on fractions.

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THE GROWTH OF PRESERVICE SECONDARY MATHEMATICS TEACHERS' TECHNOLOGICAL CONTENT KNOWLEDGE DURING INSTRUMENTAL GENESIS

Xiangquan Yao
Pennsylvania State University
xzy73@psu.edu

Relying on the theory of instrumented activity, this study examined the development of PSMTs' technological content knowledge when solving mathematically rich problems with GSP. Preliminary findings from the study were shared to highlight the dynamic interactions between technological and mathematical knowledge.

Keywords: Technology; Teacher Knowledge

The ubiquity of technology offers mathematics teacher educators opportunities and challenges to enhance the ways in which prospective teacher learn about mathematics and mathematics pedagogy. Serval organizations have highlighted the importance of preparing teachers to teach students mathematics using appropriate technology (e.g., NCTM, 2014; AMTE, 2017). Mishra and Koehler's (2006) framework of Technology, Pedagogy, and Content Knowledge (TPACK) offered a useful way to conceptualize what knowledge preservice teachers need to integrate technology into teaching practices but left the specifics of what lies in each circle to disciplinary researchers. Bowers and Stephens (2011) noted that technological content knowledge (TCK) had received less research attention than other domains of knowledge for teaching mathematics. This study aimed to address this gap and was guided by one research question: How preservice secondary mathematics teachers (PSMTs) develop their technological content knowledge when working with Geometer's Sketchpad (GSP) to solve mathematically rich problems? Following Bowers and Stephens (2011), this study defines technological content knowledge as knowledge of using technology to explore mathematical relations.

Theoretical Framework

Theory of instrumented activity (Verillon & Rabardel, 1995) served as a conceptual framework for this study. Based on a neo-Vygotskian approach, this theory considers situations in which an instrument is used to measure actions between a subject and an object. It differentiates between an *artifact* and an *instrument*. An artifact is merely a bare tool, which is available to the user and has the potential to support a certain kind of activity, but which might be a meaningless object if the user does not know what tasks the tool can support in which ways. The subject does not initially consider an artifact to be an instrument (Artigue, 2002). The user needs to develop an understanding of the affordances and constraints of an artifact and to establish a relationship with the artifact. An artifact becomes an instrument when the user is aware of how the artifact can extend one's capacities for accomplishing a given task and has developed means of using the tool for a specific purpose. Therefore, an instrument consists of both an artifact and the associated mental schemes that the user develops to use the artifact for accomplishing specific tasks. Instrumental genesis describes the process of developing an instrument from the interaction between a subject and an artifact.

Instrumentation and instrumentalization describe a bilateral relationship between the artifact and the user established within instrumental genesis. *Instrumentation* describes processes oriented toward the learner him/herself, in which the learner's mathematical activity and the corresponding emergent conception of mathematics are influenced by the affordances and constraints of a tool. In contrast, i*nstrumentalization* refers to processes directed toward the artifact, in which the learner enriches the artifact's properties acquired momentarily or durably. Through instrumentation, utilization schemes are developed, which are subsequently transformed into techniques for dealing with the task at hand. Utilization schemes are defined as significant and coherent mental schemes aimed to resolve a specific type of task with the use of an instrument (Drijvers, 2013). *Techniques* (a set of procedures used to solve a determined type of problem) are the observable counterpart of the invisible mental schemes. The two dialectical relationships (i.e., artifact and instrument; instrumentation and instrumentalization) are at the heart of the theory of instrumented activity (Figure 1).

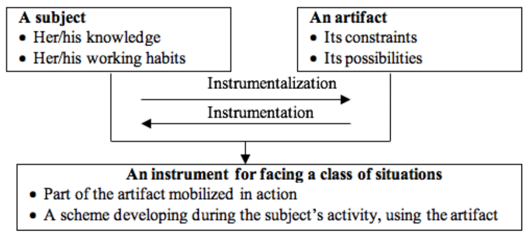


Figure 1: A Representation of an Instrumental Genesis (Trouche, 2018)

Methodology

To investigate the development of technological content knowledge, one-on-one constructivist teaching experiment (Steffe & Thompson, 2000) methodology was utilized. At its core, the teaching experiment is a "conceptual tool that researchers use in the organization of their activities. It is primarily an exploratory tool… it involves experimentation with the ways and means of influencing students' mathematical knowledge (Steffe & Thompson, 2000, p. 273). One of the primary functions of constructivist teaching experiment is the construction and testing of hypothesis regarding human thinking. Such was the case for this study. When planning for the teaching experiment, I considered various types of tasks in order to study instrumental genesis in different scenarios. Some tasks demanded participants to construct GSP sketches, while other tasks invited participants to explore with existing GSP sketches. In some cases solving the problem might only demand the development of a new mathematical idea or new use of GSP, while in other cases new mathematical idea and novel use of GSP were both essential for successful problem solving. Figure 2 presents sample tasks used in the teaching experiment.

The participants of this study were three junior undergraduate students enrolled in a secondary mathematics teacher education program. Chen majored in both mathematics and mathematics education, and Jan and Joe majored in mathematics education. Prior to the study, all the participants have taken courses in calculus, linear algebra, discrete mathematics, abstract algebra, real analysis, and algebra for teachers. None of them had taken an undergraduate course in geometry. While participating in the study, all the participants were enrolled in a course that focused on teaching mathematics in technology-intensive environments, where they learned to use various mathematics-specific technologies (TI-Nspire, GeoGebra, GSP, and Fathom). However, since GSP had not yet been introduced in that course when the study began, and none of the participants had any exposure to it before, the researcher introduced GSP and its basic

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

tools (e.g., "construct", "transform", and "measure" commands) to each participant and provided them time to play with it and get conformable with the dynamic environment.

Each participant participated in 8 teaching sessions, each of which lasted approximately 2 hours. Screen-recording was used to capture the participant's interaction with GSP. Interaction between each participant and the teacher-researcher was video-recorded. GSP files the participants produced in each session were saved. The verbatim transcripts of the recordings were annotated to include snapshots of GSP screen content and non-verbal actions. Instances of instrumental genesis were then identified. An event was identified as an instance of instrumental genesis if a new technique of GSP usage was observed or new mathematical idea was developed through the use of GSP. Each instance was analyzed based on the following categories: techniques used in the activity, prior mathematical knowledge activated in the activity, mathematical ideas newly developed in the activity, cognitive obstacles manifested in the activity, and interaction between the student and teacher-researcher. These categories allowed the researcher to identify techniques and utilization schemes participants developed when working with GSP to solve mathematical problems. The utilization schemes and techniques provide a window to examine the dynamic interactions between instrumentalization and instrumentation and growth of technological content knowledge.

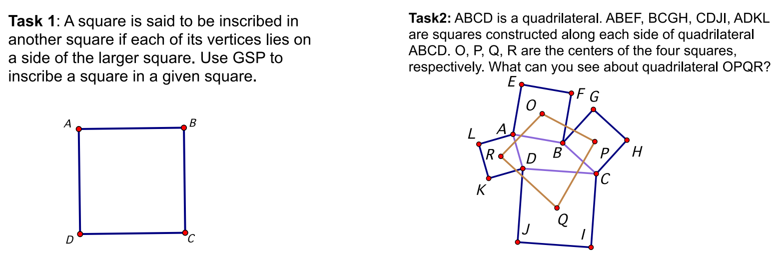


Figure 2: Sample Tasks in the Teaching Experiment

Result and Discussion

Data analysis is still ongoing. This section presents two participants' work on inscribing a square inside a given square to illustrate dynamic interaction between technological and mathematical knowledge and paths PSMTs took in developing technological content knowledge.

The Case of Joe

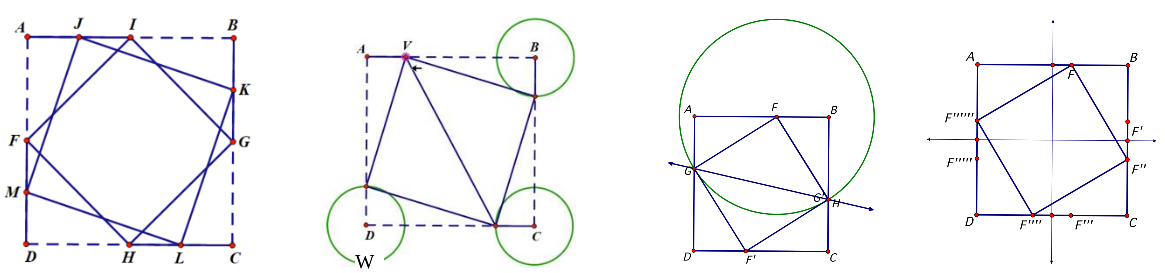
Joe approached the problem by finding midpoints of the four sides of the given square and then connecting the midpoints to form an inscribed square. Joe relied on the Pythagorean theorem to justify why the sides of the inscribed shape were congruent and used the 45°-45°-90° triangle at each corner to explain why angles of the inscribed figure were 90°. In response to the request to find another inscribed square, Joe constructed "midpoints of the midpoints" of sides of the given square and connected the four "midpoints" to form another inscribed square (see Figure 3a). Again, Joe relied on the Pythagorean theorem to justify the congruency of the four sides of the inscribed figure, but was only able to use measurement to verify the angle measures of the inscribed square. Joe then stated that there were infinite many inscribed squares as he could keep finding "midpoints of the midpoints" and thus J could be anywhere on $\overline{AB}$. He also observed m$\overline{AJ} \cong m\overline{BK}$ and the four triangles were congruent by the side-angle-side theorem.

The researcher then pointed out to Joe that he would not be able to drag the vertices of each inscribed squares constructed in his way and challenged him to inscribe a square such that he could change its size by dragging its vertex. The researcher invited Joe to consider the observations he just made about the diagram. Joe knew he would need four congruent triangles at the four corners, but did not know how to obtain them. He put a point V on $\overline{AB}$ and another point W on $\overline{AD}$ and was wondering how to make $\overline{AV}$ and $\overline{DW}$ congruent. The researcher then asked Joe whether he thought about the circle tool and pointed out that there are two ways to construct a circle in GSP ("circle by center + radius" and "circle by center + point." Joe then constructed three circles at each corner of the square by using “circle by center + radius” tool and found the other three vertices of the inscribed square (Figure 3b).

The Case of Chen

Similar to Joe, Chen created an inscribed square by connecting midpoints of the four sides of the given square. When asked whether that was the only inscribed square, Chen picked a point F on side $\overline{AB}$, drew a segment from F to $\overline{AD}$, and rotated the segment 90° around F. By dragging F and G, Chen observed “for every point on AB, it looks there is only one orientation of this angle that would allow having equal length hitting both $\overline{AD}$ and $\overline{BC}$." Based on this observation, Chen constructed a circle with F as the center and FG as the radius. He thought point H was another vertex of the inscribed square. However, when starting to move F, he realized the circle did not always intersect with $\overline{BC}$ and the two segments were not always perpendicular. He dragged F and G to make G' and H coincide. Chen then reflected $\overline{FG}$ and $\overline{FG'}$ across $\overleftrightarrow{GG'}$ to create an inscribed square (see Figure 3c). He seemed to be aware that his square would not pass the dragging test because he picked the point on $\overline{BC}$ visually. The researcher asked Chen what he noticed about the triangle at each corner. Chen stated that they were congruent.

The researcher challenged Chen to make a draggable inscribed square. After a long pause, Chen stated he was thinking of rotating ΔAFG about B for 90°because that would probably give him $\Delta BG'F$. After deleting previous work, Chen did the following: rotated F 90° around B, reflected F' across the line connecting the midpoints of $\overline{AD}$ and $\overline{BC}$ to get F'', rotated F'' 90° about C to get F''', reflected F''' across the line connecting midpoints of $\overline{AB}$ and $\overline{DC}$, rotated F'''' 90° around D to get F''''', and reflected F''''' across the line connecting midpoints of $\overline{AD}$ and $\overline{BC}$ to get F''''''. He claimed the figure $FF''F''''F''''''$ is a square (see Figure 3d). When asked why he decided to carry out a rotation followed by a reflection on each side of the given square, Chen said he was really thinking of rotating the entire ΔAFG, although he only cared about recreating $\overline{AF}$ on each side of the square. The researcher asked Chen if he could simplify his construction steps. Chen was not able to reduce his construction steps.



(a) (b) (c) (d)

Figure 3: Joe’s and Chen’s Approaches to Inscribe a Square

Although both Joe and Chen successfully inscribed a dynamic square, they relied on different mathematical and technological knowledge to obtain the desired configuration and seemed to gain different technological content knowledge. Joe’s initial approached relied on the “midpoint” tool, which allowed him to create multiple inscribed squares and developed a dynamic mental image about the inscribed squares. Relying on this mental image, he used the "circle by center +radius" tool to inscribe a dynamic square inside the given square. In contrast, Chen seemed to rely on geometric transformations frequently. He first used 90° rotation and dragging to create the desired configuration that started a square. Based on the knowledge developed from this non-draggable configuration, he used a series of composition of rotation and reflection to create a dynamic square inside the given square. His problem-solving actions revealed that transformation tools became his technique for creating congruent geometric objects.

The two cases revealed a pattern that describes a relationship between instrumental genesis and the growth of geometric knowledge. This cyclic pattern is as follows: Use particular GSP tools to obtain a static or non-constructible geometric configuration, observe new geometric properties or relationships from this configuration, and use this newly developed knowledge to guide GSP usage and create a more dynamic configuration or an alternative method for constructing a dynamic configuration. It highlights the interactions among learner, mathematical tasks, technological tools, and teacher and creates a dynamic space within which new mathematics knowledge and meaningful ways of using technology emerge.

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MATHEMATICAL MODELING EXPERIENCES: NARRATIVES FROM A PRESERVICE TEACHER AND AN INSTRUCTOR

Sarah Brand
Marquette University
sarah.brand@marquette.edu

Hyunyi Jung
Marquette University
hyunyi.jung@marquette.edu

Regardless of the benefits of engaging in mathematical modeling, few preservice teachers (PTs) have experienced mathematical modeling firsthand. This study offers an example of how to make sense of the interaction between the teaching and learning of mathematical modeling by examining a teacher educator's decision making, her analysis of 36 PTs' learning, and an in-depth narrative from a PT. Findings show the value of engaging with structurally relevant mathematical modeling tasks and considering social issues via mathematical modeling, resulting in task designs which aim to deepen students' understanding of society and mathematics.

Keywords: Mathematical Modeling; Preservice Teacher Education

The benefits of engaging with mathematical modeling (i.e., the process of using mathematics to provide insight into real-world situations) have been presented by numerous research studies (Consortium for Mathematics and Its Applications & Society for Industrial and Applied Mathematics [COMAP & SIAM], 2016). Students develop reasoning abilities, entrepreneurial thinking, conceptual understanding, and procedural fluency when they engage with mathematical modeling (e.g., Blum & Niss, 1991; Lesh, Hoover, Hole, Kelly, & Post, 2000; Zbiek & Conner, 2006). However, despite the advantages of learning mathematical modeling, many teachers have limited experience with learning and teaching the related content and processes (e.g., Burkhardt, 2006; COMAP & SIAM, 2016). The inclusion of mathematical modeling in preservice teachers' (PTs') learning has been recommended so that they may be prepared to engage their future students with authentic problem-solving experiences (Association of Mathematics Teacher Educators, 2017); still, few PTs have experienced mathematical modeling firsthand. Research remains to be done about mathematical modeling learning opportunities provided to PTs, especially on the purpose of such learning opportunities and how learners respond to them. This study offers an example of how to make sense of the interaction between the teaching and learning of mathematical modeling by examining a teacher educator's curricular intentions and her PTs' responses to tasks designed around those intentions. The focus here is on the teacher educator (second author)'s decision-making, her analysis of PTs' learning, and a more in-depth narrative from one of her PTs (first author). Specifically, the research questions are "How does a teacher educator make decisions about implementing mathematical modeling to preservice teachers?" "How do PTs respond to the implementations of mathematical modeling tasks?" and "How does a PT reflect on and narrate specific aspects of the implementation?"

Methods

36 PTs enrolled in three sections of a problem-solving mathematics course taught by the second author (Jung) at a Midwestern university in the U.S. Thirteen teams of PTs solved, designed, and revised mathematical modeling tasks in this course. Specifically, the PTs solved relevant mathematical modeling tasks, including Model-Eliciting Activities (MEAs), which are known to showcase the nature and usefulness of mathematics while developing valuable

everyday skills (Lesh et al., 2000). PTs also reflected on their own learning and relevant literature related to mathematical modeling (e.g., English, Fox, & Watters, 2005). They then collaboratively designed mathematical modeling tasks and revised them based on peer feedback. The course resulted in the following data sources: (a) audio-recordings of individual interviews focusing on PTs' learning of mathematical modeling; (b) 26 team solutions for two mathematical tasks (13 solutions for each task); (c) 25 mathematical modeling tasks designed by PT teams; (d) and PTs' individual reflections on their learning.

Data analysis involved two phases. During the first phase, Jung and her colleagues analyzed data sources collected from 36 PTs and found the overall learning opportunities related to mathematical modeling (Jung & Magiera, 2018). This analysis was used to describe Jung's analyses of the 36 PTs' learning opportunities. The second phase focused on the learning experiences of the first author (Brand). Among the 36 PTs, Brand was uniquely interested in multitiered teaching experiments (Lesh & Kelly, 2000) and the analysis of her learning of mathematical modeling. Brand's products included her team's written solutions to two MEAs, two written mathematical modeling problems designed by her team, and seven individual journal reflections that documented her learning of mathematical modeling. Upon dissecting her solutions to tasks, Brand noted themes embedded within her processes of developing mathematical constructs as she solved or designed mathematical modeling problems. Then, analyzing her reflections, Brand focused on what she learned from each activity and documented evidence of changes revealed from each journal reflection (Strauss & Corbin, 1998). The following section outlines the overall learning experiences of 36 PTs as intended and analyzed by Jung, followed by Brand's narratives of her own learning.

Results

PTs as Problem Solvers

Jung's intention and initial analysis of student work. To engage PTs with mathematical modeling as learners, Jung first selected the Fun on the Field MEA (Chamberlin, 2000), which required PTs to split 15 individuals into three equal teams for a school's field day. To sort the students, four categories of data were provided: each individual's 100-meter dash time, 800-meter run time, high jump height, and whether they passed a fitness test. When she analyzed PTs' initial work on this MEA, Jung found that only five out of 13 teams demonstrated a sophisticated sorting method that considered all four categories of data. Jung asked each team to review other teams' work and provide peer feedback (West, Williams, & Williams, 2013). She then provided the Volleyball MEA as a follow-up activity because it was mathematically connected to the Fun on the Field MEA but required more complex thinking about large datasets. By way of explication, the Volleyball MEA engaged PTs to split 18 individuals into three equal teams for a volleyball summer camp. Compared to the Fun on the Field MEA, the Volleyball MEA required a more involved usage of ranking systems and quantification processes, as it supplied additional data sets, including qualitative data (e.g., coach's comments). Jung's anlysis of PTs' work on the Volleyball MEA revealed that most teams (12 out of the 13 teams) considered all the data sets and used ranking methods to represent their sorting system.

Brand: My narrative. As my partner and I began solving the Fun on the Field MEA, I proposed using a point system which assigned points to individuals' results based on predetermined intervals for each category, excluding the fitness test results. My partner, however, expressed a concern that my procedure failed to differentiate between individuals whose results fell in the same point interval. Suggesting we rank individuals within each category, my partner encouraged me to consider both our avenues for solution. As we continued, my partner and I wanted to ensure that our solution strategy utilized all the data provided. Thus, we needed to determine how to include the participants' fitness test scores. Initially, we chose to assign 1 point to individuals who passed the test and 0 points to those who did not. Upon review, though, we recognized two flaws with this plan. For one, according to our ranking system, a lower score indicated a more athletically inclined individual. Receiving additional points for passing a fitness test would increase that individual's score, making them appear less physically adept than he or she was. To fix this, we quickly decided to add 0 points to an individual's score for passing the test. For our second concern, I was apprehensive about how uninfluential the fitness test results were on the subjects' final scores. I recommended that we add 5 points to the scores of those who failed, while still adding 0 points to the scores of those who passed. My partner agreed that this would generate an appropriate effect on the subjects' final scores.

Overall, my learning from this MEA included growth in my recognition and application of justifiable strategies, open-mindedness to others' skillsets, and receptivity to revisions. I also grasped that mathematical modeling allows for students to continuously revise their models until a desirable outcome is reached. Resultingly, when my partner and I completed the Volleyball MEA, we paid close attention to our peers' feedback and strategies from the Fun on the Field MEA. Having two opportunities to engage in similar MEAs encouraged my peers and me to develop more evolved mathematical reasoning skills and problem-solving strategies.

PTs as Designers: First Task

Jung's intention and initial analysis of student work. Jung later incorporated a problem-posing activity in her course. She also provided PTs opportunities to revise their tasks based on peer feedback, followed by instructor feedback. Her analysis of 36 PTs' invented tasks revealed improvements from the initial tasks to the revised tasks. Most PTs refined their tasks to make them more realistic and mathematically sound. Specifically, they developed diverse mathematical modeling tasks, including contexts such as Christmas tree decorating, a dream vacation plan, and designing a park.

Brand: My narrative. When designing my first modeling task, I frequently evaluated my group's problem to ensure it met the mathematical modeling problem criteria discussed in class and required students to (a) make assumptions about and predict a realistic context; (b) create and verify mathematical models; and (c) provide complex solutions beyond numerical results. Because my group was anxious about meeting these standards in our first original MEA, we chose to create a task which elicited strategies we had applied when solving previous MEAs. The result was an MEA, titled "Ms. Penny's Classroom," similar in mathematical structure to the Fun on the Field and Volleyball MEAs. Specifically, the task required students to split 15 students into three groups of equal academic caliber for a group project.

Despite the similarities between our problem and those we had solved in class, a new strategy could emerge from this problem: using qualitative data as a tool for revision. My team recognized that quantifying comments on students' behavior (e.g., "well-behaved and a great student") was not entirely feasible. We instead chose to use that dataset as a mechanism to adjust our initial groupings. Creating a new MEA thus broadened my supply of problem-solving strategies. Additionally, I developed an understanding of mathematical modeling as an enriching instructional experience because of its relevance to students' realities.

PTs as Designers: Second Task

Jung's intention and initial analysis of student work. Although the contexts of the first tasks that PTs developed were relevant to target students of their choices (e.g., Christmas tree, academic grouping), Jung wanted PTs to consider critical aspects of the real world. She believed it to be crucial that PTs encourage students to critically interpret the world using mathematics (Gutstein & Peterson, 2005). PTs were asked to share unfair or unjust experiences their future students may face in their lives or world. With these experiences in mind, PTs collaboratively developed a mathematical modeling task with a critical context.

Brand: My narrative. My partner and I wished to create a problem which did not ask students to group individuals or items, as we felt we had exhausted this type of context. After much consideration, we recalled the garbage pollution in our city and chose to focus our task on a solution for widespread littering. We also wanted to involve spacial concepts in our problem through the use of a map. Our problem "Preventing Litter in the City" emerged, involving data reasoning strategies we'd encountered in previous MEAs. Further, a new strategy, which we dubbed "grouping," became necessary for my team to both teach and solve our problem. For example, I chose to group two trash cans with every recycling bin in my solution. I also wanted the task to demonstrate my own beliefs that since we should involve "students in social and political conflicts…we should ensure that the implications in our math problems allow this" [Reflection 6]. To do so, I incorporated my students' city of residence in the problem, thereby encouraging them to take action within their own community. Ultimately, by creating this task, I learned the importance of choosing justice-oriented contexts for mathematical modeling tasks. I saw how teaching mathematics can be more engaging, interdisciplinary, and useful for developing students' knowledge of social justice.

Conclusion

Through the reflective processes, the authors illustrated practical and research knowledge that influenced the decisions a teacher educator and a PT made throughout mathematical modeling problem-solving and problem-posing experiences. As Diefes-Dux and Capobianco (2008) proposed, reflection supports a better understanding of the complex, interrelated sets of situations that teaching and learning requires. When this line of thinking is organized and connected, it often leads to new knowledge or action (Diefes-Dux & Capobianco, 2008).

The reflective nature of this self-study contributes to mathematics education in three ways. First, this-self study describes both the instructor's intentions for and analyses of 36 PTs' mathematical modeling experiences, as well as one PT's narrative of her learning. The perspectives from both the instructor and PT provide evidence of learning and changes in action, which would remain uncaptured were only one of these lenses considered. The results show that each participant was an active decision maker (Mundry, Britton, Raizen, & Loucks-Horsley, 2000), acting in different, evolving directions (Lesh & Kelly, 2000). Second, both the instructor's and PT's reflections reveal the value of engaging with structurally relevant mathematical modeling tasks (Ärlebäck, Doerr, & O'Neil, 2013; Doerr, 2016). The PTs used more sophisticated ways of data reasoning, including the analysis of qualitative data, when they engaged with two related mathematical modeling tasks. Brand's narrative detailed decisions she made when solving and posing mathematical modeling problems, ultimately contributing to a better understanding of her process of engaging with complex mathematical modeling activities. Last, the study details the rationales of how a PT created mathematical modeling tasks regarding societal issues. PTs were encouraged to design tasks that aimed to deepen students' understanding of society. Such results around mathematical modeling problem-posing activities extend findings from previous studies about PTs' problem-posing activities focusing on word problems (Tichà & Hošpesovà, 2010) or other modeling problems (Paolucci & Wessels, 2007).

As Brand narrated, choosing socially-aware contexts for a mathematical modeling problem enabled her to see the value of mathematics as a tool for broadening problem solvers' knowledge of the world.

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PREPARING SECONDARY MATHEMATICS TEACHERS TO TEACH WITH TECHNOLOGY: FINDINGS FROM A NATIONWIDE SURVEY

Allison W. McCulloch
UNC Charlotte
allison.mcculloch@uncc.edu

Jennifer N. Lovett
Middle Tennessee State University
jennifer.lovett@mtsu.edu

Keith R. Leatham
Brigham Young University
kleatham@mathed.byu.edu

Nina Gabrielle Bailey
UNC Charlotte
nbaile15@uncc.edu

Samuel D. Reed
Middle Tennessee State University
sdr4m@mtmail.mtsu.edu

The field of mathematics teacher education has been moving collectively towards a common goal of preparing preservice mathematics teachers to teach with technology, which is explicated in the Association of Mathematics Teacher Educators' (2017) Standards for Preparing Teachers of Mathematics*. In this paper we present findings from a national survey of accredited university secondary mathematics teacher education program. The purpose of the study is to describe the current state of the ways in which programs are preparing preservice teachers to teach secondary mathematics with technology.*

Keywords: Technology, Policy, Teacher Education

Since the release of the *National Educational Technology Standards for Teachers* (International Society for Technology in Education, 2000), the field of mathematics teacher education has worked to develop common goals about how secondary mathematics teachers should be prepared to teach mathematics with technology. Recently these goals were articulated in the Association of Mathematics Teacher Educators' [AMTE] (2017) *Standards for Preparing Teachers of Mathematics*, which stated that "well-prepared beginning teachers of mathematics are proficient with tools and technology designed to support mathematical reasoning and sense making, both in doing mathematics themselves and in supporting student learning of mathematics" (p. 11). These standards call for prospective secondary mathematics teachers (PSMTs) to be proficient in using digital tools to solve mathematics problems, to "enhance or illuminate mathematical and statistical concepts," to "explore mathematical and statistical ideas and to build conceptual understanding of these," and know when such tools "enhance teaching and learning, recognizing both the insights to be gained and possible limitations of such tools" (p. 12). Although these goals have now been articulated and disseminated, relatively little is known about the ways program faculty have designed courses to meet these goals.

Given the complex nature of teaching with technology, it is evident that teacher preparation programs need to focus on supporting PSMTs' learning to effectively integrate technology in mathematics classrooms. Although standards exist that purport this importance, there are many ways in which teacher education programs might choose to meet this need. In 2003, Kersaint, Horton, Stohl, and Garofalo reported that 21% of mathematics teacher educators in the U.S. who responded to their survey taught a course focused on technology. A few years later, Leatham (2006) reported that 29% of mathematics teacher educators had courses at their U.S.-based institutions that focused specifically on the teaching of mathematics with technology. However, simply identifying whether or not a course focused on teaching mathematics with technology is offered may be misleading. Secondary mathematics education programs might incorporate teaching mathematics with technology into methods courses, content courses, or even general educational technology courses (Kersaint et al., 2003; Leatham, 2008). Of course, the scope and nature of such integration could vary drastically across programs.

While the extant literature does have suggestions for the design of courses in which learning to teach with technology is a goal (Lee & Hollebrands, 2008; Leatham, 2008), the research on how that is actually being done is limited and largely out of date. In an aim to provide an update of where we are as a field we conducted a nationwide survey of U.S. secondary mathematics teacher education programs. In this paper we answer the following questions: 1) In what types of courses are PSMTs being prepared to teach mathematics with technology? and 2) What reasons do programs give for why they do not have a course specifically focused on learning to teach mathematics with technology?

Methods

We employed survey methodology (Groves et al., 2009) and mixed methods analysis. The survey was designed to elicit descriptive information (using both closed and open-ended questions) about courses designed specifically to address learning to teach mathematics with technology. For example, after identifying whether or not a mathematics specific technology course (typically referred to as a technology, pedagogy, content [TPC] course) is offered within a program, the remaining questions targeted the structure of the course(s) in which teaching with technology is incorporated, course learning objectives, different types of technology utilized and frequency of use, and types of learning activities included. In this brief report we focus on survey items related to the structure of courses.

We identified all accredited university secondary mathematics teacher preparation programs in the U.S. by visiting the department of public instruction website for all 50 states. Once a list was procured, we visited each university website to verify that the university had an undergraduate secondary mathematics teacher preparation program and, if so, to identify a potential contact person. This search resulted in a list of 956 accredited programs. The survey was sent to all 956 universities along with two reminder emails, if necessary. The response rate was 30%, well above that of previous studies focused on this topic (Kersaint et al., 2003; Leatham, 2006). The sample represents a broad cross-section of universities from 49 of the 50 states.

Descriptive statistics were used to describe the various courses in which programs address preparing PSMTs to teach mathematics with technology and to discern overall patterns and differences between programs with dedicated technology courses and those that do not. In addition, we open coded responses to a question posed to non-TPC programs, “A number of institutions offer a course dedicated to teaching mathematics with technology. What are some of the reasons why you do not offer such a course?” to identify themes related to decisions about not offering a technology-specific course.

Findings and Discussion

Of the 290 programs that responded, 71 (25%) reported that they offer a designated TPC course (referred to as TPC programs going forward), 214 (74%) indicated they do not have a designated course, but instead integrate technology across other courses (referred to as non-TPC programs going forward), and two (<1%) reported that technology preparation is not included in their program. It is encouraging that, with the exception of these two programs, the remaining programs (n=288) represented in this study provide a number of opportunities for PSMTs to learn about teaching and learning mathematics with technology. The 25% of programs that currently offer a TPC course is practically the same proportion from a decade ago (29% according to Leatham, 2006). In the remainder of this brief report we focus on the types of courses in which goals related to preparing PSMTs to teaching mathematics with technology are intentionally integrated when a TPC course is not offered and the reasons programs give for why they do not offer a TPC course.

Types of Courses in Which Teaching Mathematics with Technology is Integrated

Of the 214 non-TPC programs, 158 (74%) reported the titles of courses in which they integrate technology. Of those 158 programs, 64 (41% noted they integrate technology in one course, 44 (29%) across two courses, and 50 (32%) across three or more courses. Typically, the courses identified were mathematics methods courses (143 of the 302 courses reported, 47%). However, 83 (53%) were mathematics or statistics content courses, 51 (33%) were general education methods courses, and seven (4%) were field experience courses (e.g., student teaching). For the 41% of programs that indicated they integrate technology in only one course, that one course is typically a mathematics education methods course (70%), with the second most common course type being a general education course (19%). Looking across the number of courses in which technology is integrated, 21 programs (13%) indicated technology was integrated into only mathematics and statistics content courses and 15 programs (10%) indicated technology was integrated into only general education courses.

It is notable that 23% of the non-TPC programs are not integrating technology into a mathematics education course. Given the complexity of integrating technology into mathematics pedagogy, this separation from mathematics specific methods is concerning. In addition, 29% of programs are integrating technology in only a single mathematics methods course. While integrating in a mathematics specific methods course does suggest attention to pedagogical considerations, integration into a single course suggests minimal time is being allocated to this very complex practice in these programs.

Reasons for Intentionally Integrated Courses Instead of TPC Courses

Programs that indicated they did not offer a TPC course (*n*=214) were asked why they did not offer one; 136 (63%) of these programs provided an explanation as to why. The most common reasons for choosing to integrate across courses rather than offering a TPC course were program limitations such as credit limits (35%) and program size (24%). In addition, it is notable that 4% indicated there was no faculty expertise to design or teach a TPC course. One respondent wrote “We currently do not have instructor expertise to design coursework around teaching math with technology.” Thus, it appears that many non-TPC programs do not offer TPC courses because of programmatic constraints rather than purposeful choice.

Table 1: Reasons for Intentionally Integrated Courses Instead of TPC Courses

Reason	Percentage
Program credit limits don’t allow adding another course	35
Low program enrollment	24
Sufficient Attention in other courses	21
Believe integrating across is more impactful	9
Lack of faculty expertise	4
Not required by state/accreditation agencies	4
Faculty does not believe in using technology	2
No faculty consensus on need of such course	2

Although one in five (21%) of the non-TPC programs suggested that program decisions about integrating teaching with technology across methods courses was intentional due to sufficient attention in other courses, their responses make one wonder about the validity of those statements. For example, one respondent explained, “There is the sense that something like technology is best integrated across other courses (I tend to agree with that to some extent), though I cannot say we do a very good job of it.” Another wrote, “Because we incorporate technology across required mathematics courses.” In contrast, some respondents (9%) noted that their intentional integration was done because they believed integration was more impactful than offering a TPC course. For example, one respondent wrote, “To have a course dedicated to technology alone flies in the face of expecting students to use technology at any time in any course in which the technology is deemed appropriate.” The most concerning responses were those that indicated intentional integration was chosen because the faculty does not believe in using technology. This belief was expressed by 4% of the respondents through statements like “I do not believe technology is a vehicle for discovering mathematical principles” and “Because it is not an effective way to teach math.”

Conclusion

Although every program is working within their local constraints to make decisions about how to best address the goals set out by AMTE with respect to preparing PSMTs to teach mathematics with technology, some of the reasons provided for not including a TPC course give rise to issues that should be important to the field. For example, 8% of responding programs indicated they do not have faculty expertise to develop such a course. If we consider that the sample represented in the study likely favors the use of technology in mathematics education, it is highly possible that there are many more programs for which there is not adequate faculty expertise related to learning and teaching mathematics with technology. There are some projects currently underway (e.g., Enhancing Statistics Teacher Education with E-Modules [ESTEEM], Preparing to Teach Mathematics with Technology – Examining Student Practices [PTMT-ESP]) aimed to address faculty expertise with respect to preparing PSMTs to teach with technology, finding ways to ensure these faculty are aware of existing projects and their materials should be a priority.

Preparing PSMTs to teach mathematics with technology remains an area of emphasis in our national standards (AMTE, 2017; NCTM, 2012) and an area where, across our programs, we have yet to converge on a common understanding of the most effective ways to meet our goals. We hope that this paper provides fodder for conversation as we collectively work to prepare all teachers to productively incorporate technology in the teaching of mathematics.

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SUPPORTING PROSPECTIVE TEACHERS' DEVELOPMENT OF RELEVANT AND AUTHENTIC MATHEMATICS STORIES

Eryn M. Stehr
Georgia Southern University
estehr@georgiasouthern.edu

Lindsay M. Keazer
Sacred Heart University
keazerl@sacredheart.edu

Supporting prospective elementary teachers (PTs) to develop mathematics problems with contexts that reflect their students' lives is a challenge that remains to be explored. We began this work by conducting collaborative action research to explore PTs' development of mathematics problems that were relevant and authentic to PTs' own lives. We engaged in iterative cycles of action research by asking PTs to write mathematics stories, followed by our review and discussion of the themes we saw and sharing selected examples with PTs during class discussions. Findings after three cycles of data collection suggest that PTs' mathematics contexts became more story-like and personally authentic, and the authors (mathematics teacher educators) developed a clearer understanding of their goals for supporting PTs' development.

Keywords: Culturally Relevant Pedagogy, Instructional Activities and Practices

Scholars of equity in mathematics education have established that teaching mathematics is not a neutral enterprise, and that teachers' choices of mathematical problems convey messages to students about what issues mathematics is useful for addressing (e.g., Felton, 2010; Gutiérrez, 2007; Gutstein & Peterson, 2013). How teachers learn to identify mathematics problem contexts that mirror students' diverse lives and experiences is not yet fully understood. As mathematics teacher educators, we (the authors) realized that our students (elementary prospective teachers) may not yet have sufficient experiences working with children to connect mathematics to children's cultures beyond superficial ways. Thus, this collaborative action research focused on supporting prospective teachers (PTs) in telling mathematical stories that reflected their own lives and cultures, as a first step towards developing the ability to see relevant mathematics in the world around them and eventually connect mathematics to their future students' cultures.

We began the project from a shared value for supporting the development of PTs' decision-making and creativity about how to contextualize mathematics learning opportunities in meaningful ways. We asked the guiding question: How can we support PTs in telling mathematical stories that are relevant and authentic to their lives, homes, and communities?

We engaged in iterative cycles of action research by asking PTs to write mathematics stories, followed by our review and discussion of the themes we saw. Then we presented examples to PTs during class discussion and prompted them to continue to search for mathematics contexts within their lives. While we each engaged in independent analysis of our own PTs' data, a second and integral part of our professional learning opportunity was collaborative action research (Capobianco & Feldman, 2006): discussing our noticings and wonderings through phone calls and emails as our distinct goals and strategies emerged.

Literature Review

Mathematical problems communicate messages about what is normal in our world and what issues mathematics is useful for addressing (Felton, 2010), and teachers' choices of mathematics problems influence the messages communicated. Some teachers have difficulty developing contexts that connect to students' experiences (Rubel, 2017), or risk making connections to students' cultures in superficial ways (Author blinded).

We see culturally relevant contexts as an important way to connect to students' lives and experiences (Ladson-Billings, 1994). Developing relevant mathematics contexts reveals the power and potential of mathematics as a tool to understand students' culture, community, or the broader world (e.g., Gutiérrez, 2007, Gutstein & Peterson, 2013).

Developing authentic mathematics problem contexts--particularly contexts that are authentic to students--was another idea that motivated our work. We align our definition of authentic contexts with that of the youth-centered perspective described by Buxton (2006), who outlined three perspectives of authentic contexts based on a review of science education research literature. The youth-centered perspective is "grounded in useful truths for solving students' real problems" (Brickhouse, 2001, cited in Buxton, 2006, p. 698). Aligned with this definition, we wanted PTs to develop their ability to identify real problems existing in their own lives that mathematics would be useful for understanding and solving.

As we considered ways to support PTs in developing mathematics problems with culturally relevant and authentic contexts, Radakovic, Jagger, and Jao's (2018) account of PTs writing mathematical poetry inspired us to provide PTs an opportunity to write their lives into mathematics stories. We hoped this open-ended task would encourage their creativity and hone their ability to see mathematics problems that were real within their worlds.

Method

We engaged in a model of collaborative action research (Capobianco & Feldman, 2006), collaborating around a common goal while allowing our voices, philosophies, and teaching methods to remain distinct. Carr and Kemmis (1986) conceptualized teacher action research as a self-critical inquiry into one's practice with the goal of improving and developing a better understanding of practice. Teacher action research enables us, as teacher educators, to study our own practice through cycles of action and reflection (Reason & Bradbury, 2008). Collaboration strengthens opportunities beyond those of individual action research (Capobianco & Feldman, 2006), as teacher researchers support each other in the refinement of their individual inquiries while serving as "critical friends" (Kemmis & McTaggert, 1988).

We conducted action research working with K-8 PTs in two distinct contexts: Eryn's project is situated in a number and operations content course, and Lindsay's project is situated in an elementary mathematics methods course. Eryn's PTs have not yet been accepted into the teacher education program. Her course is the first mathematics-for-teachers course they experience. Over the semester, she hoped to support her PTs to notice mathematics relevant to the course as well as authentic to their lives. Lindsay's PTs are seniors in an elementary mathematics methods course during the semester prior to student teaching. She hoped to help PTs learn to better recognize the mathematics already embedded in their lives and in the local community and write mathematics problems that captured this.

Data Collection and Analysis

Two types of data were collected: data from PTs' mathematics stories and data on instructors' actions and reflections. PT data was collected through PTs' mathematics stories written as classwork, transcribed by the instructors into Excel. Instructor data was collected through email discussions, meeting notes, and researcher journals (writing brief ten-minute reflections after meetings or class sessions). We met for discussion and exchanged emails throughout the semester, before and after each writing prompt was given to PTs.

Data collection spanned 6 weeks as we engaged in three cycles of action research, collecting PTs' mathematics stories in weeks 1, 3, and 5. We conducted interim analysis of the data during the week between each data collection cycle. This consisted of repeated reviews of the data, noticing categories and themes in an informal grounded theory approach (Creswell, 2007). As part of our collaborative action research, we then discussed with each other the themes we were noticing in our data. We each selected individual PT responses that stood out to us and discussed how these responses were distinct from the themes we saw in others. The process of identifying themes and distinct examples within PTs' responses allowed us to better understand differences and clarify our goals for the next steps of our action research. This analysis informed our implementation of a class discussion of PTs' stories the following week: we each shared selected PT responses during class discussion and asked PTs what they noticed about different examples. Then PTs were given the prompt again the following week. A later, second round of analysis occurred after all data was collected, following a more formal approach to grounded theory (Creswell, 2007).

Findings

We present findings regarding PTs' mathematics stories and our learning as mathematics teacher educators, contextualized by describing the actions taken by each instructor in their course. Because the courses and instructors were different, course activities were tailored to the course and the action research strategies that emerged were also distinct.

Author Eryn

During week 1, I asked PTs to write a response to: "Word problems in the classroom are not always realistic. Thinking about the math in this course, write one example of math you have noticed in your home, your life, or your community." To connect this prompt to our mathematics course content, I prompted PTs to incorporate mathematics concepts from the course as they searched for mathematics contexts reflecting their lives. After reviewing responses, I noticed that PTs' mathematics contexts felt impersonal and lacked detail or context. This observation triggered my learning and made me aware that I wanted to push PTs to make their descriptions more story-like, and also to consider how to make them authentic: to solve real mathematics problems that they saw in their lives (Buxton, 2006). After categorizing PTs' responses, I shared a series of selected responses with them in a week 2 discussion. Examples ranged from impersonal stories or stories with minimal description to a detailed story about a PT's interaction with her younger brother around dividing pepperonis evenly between pizzas in a Lunchables.

In week 3, I revised the prompt to emphasize the need for developing story-like contexts, changing the word *realistic* to *authentic*, and prompting for a detailed math story instead of one example. I also shared my own example of a mathematical story from my life. I discussed how situations authentic to one person, are not necessarily authentic to another. After reviewing PTs' week 3 responses, I noticed that many PTs followed a structure and context similar to my story. This noticing helped me refine my goals: to support PTs in developing an ability to recognize new ways that mathematics might be used as a tool to investigate the world around them and problems interesting to them, rather than using my example as a template. As part of my analysis of PTs' responses, I made notes of responses that demonstrated this use of mathematics.

Accordingly, during week 4 I shared selected examples of PTs' mathematics stories that better illustrated my goal of using mathematics to explore their own curiosities in their lives. I also shared the story of students learning fraction operations while studying a problem of concern for them: school overcrowding (Turner & Font Strawhun, 2007). I revised the prompt in week 5 to emphasize using mathematics to "help you make sense of or critically examine situations in your home, life, or community." Many responses seemed to describe ways PTs had used mathematics to make sense of situations in their lives. It seemed that the more personal mathematical stories were written to be more complex and less solveable than a typical mathematics story problem.

Author Lindsay

As part of my mathematics methods course, PTs had the opportunity to read several articles related to relevant mathematics contexts and teaching mathematics for social justice (e.g., Felton, 2010; Turner & Font Strawhun, 2007). At each data collection I gave PTs the prompt: "Word problems in the classroom are not always realistic or connected to peoples' experiences. Let's look for ways to bring in math from our own culture. What math have you noticed in your home, your life, or your community?"

When I reviewed my first set of responses during week 2, I was initially disappointed to see that many PTs' responses felt like generic lists of mathematics contexts that may have been told to them during their socialization into school mathematics (e.g., money, time, cooking), and that sounded akin to typical story problems found in traditional textbooks. In contrast, a few PTs' responses stood out as examples that illustrated mathematics embedded in their personal experiences or uniquely connected to their community. This clarified my understanding of my goals, as these personal mathematics stories (as we began to call them) signified the development of a PT's ability to self-identify connections between their lives and mathematics. Identifying instances of mathematics in their lives seemed quite different from regurgitating mathematics contexts that they had learned from their past school mathematics experiences. In the class discussion the following week, I provided a series of selected responses and asked PTs to comment on what differences they noticed across them. I provided several list-like responses on the first slide, followed by four PT examples that were contextualized in a personal and story-like way. PTs noticed the more story-like nature of the examples that I highlighted and began to emulate that in subsequent data collection opportunities.

After completing the second and third rounds of data collection and continuing to facilitate class discussions around selected examples of PTs' mathematics stories, I saw shifts occurring in PTs' responses. Their responses became noticeably lengthier and developed more personal and story-like qualities. While PTs had developed their story-like descriptions of mathematical contexts, however, I realized that they had not yet given the contexts specific enough quantities to make them into mathematical tasks. This seemed to be a direct reflection on what I had brought their attention to during class discussions, and what I had not yet emphasized. This provided me with a next-steps goal for my action research in the future.

Discussion

The process of conducting collaborative action research around a common goal allowed us to reflect together on our learning as mathematics teacher educators, through the process of developing PTs' lenses to "see" mathematics in the world around them and to tell mathematical stories that reflected relevant and authentic experiences of their own. Our mathematics teacher educator learning as a result of this research centered on three key themes: a) refining our understanding of what relevant and authentic mathematical stories looked like by identifying developing evidence in PTs' work, b) recognizing PTs' desire to simply follow prescriptive instructions rather than hone their creativity, and c) identifying the power of our actions and expressed values on influencing PTs' responses. The repeated action of presenting PTs' with examples of their peers' mathematics stories seemed to accomplish multiple purposes: to value the PT's contribution, as well as provide positive peer pressure by providing new illustrations of the ways that mathematics could be connected to life in authentic and personally relevant ways. These themes will be unpacked in detail during the presentation.

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PROSPECTIVE TEACHERS' COLLECTIVE KNOWLEDGE: SOLVING INTEGER MISSING SUBTRAHEND PROBLEMS

Nicole M. Wessman-Enzinger
George Fox University
nenzinger@georgefox.edu

Laura Bofferding
Purdue University
LBofferd@purdue.edu

We present the collective knowledge used by fifteen prospective teachers' as they solved two integer subtraction open number sentences, with a missing subtrahend, during think-aloud interviews. The results indicate that prospective teachers have a rich, collective bank of strategies for integer subtraction. Their reasoning can be leveraged to explain the procedures they are often characterized as using in literature and that they also used in this study.

Keywords: Teacher Education-Preservice; Number Concepts and Operations; Cognition

"Subtracting a negative is like adding a positive, which I don't really get why." —Karey

Based on responses from PTs, such as in the opening quote by Karey (pseudonym), prospective teachers (PTs) are often characterized as having poor content knowledge. PTs' low or procedural performance leads to calls for more content courses and increased use of tests for certification. However, considering their collective thinking and providing PTs the opportunity to talk about (and share) their thinking presents an encouraging view. This approach challenges the expectation that prospective teachers need to have mastered all content, in this case integer operations, and positions teachers as lifelong learners who have a rich network of collective knowledge to draw on.

Moving Toward a Community of Practice Mentality

Historically, teachers in the United States have worked in relative isolation compared to cooperative countries, such as Japan (Wong, Brittan, & Ganser, 2005). In line with an individualistic point of view, many descriptions of teacher knowledge in the United States paint a grim view of teachers having more procedural knowledge and less conceptual knowledge of mathematics (e.g., Lui & Bonner, 2016; Ma, 1999). However, given the renewed focus on learning communities (Darling-Hammond & McLaughlin, 2011), research—against a new horizon—on teacher knowledge might benefit from a shift away from focusing on teachers as isolated beings to teachers as being part of a community of practice (Lave & Wenger, 1991; Wenger, 1998; Wenger-Trayner & Wenger-Trayner, 2015) where teachers are considered resources for each other. From this perspective, analyzing a community's collective knowledge takes the forefront.

Strategies for Integer Subtraction

Integer subtraction is notoriously famed for the use of rules and procedures in nonsense-making ways (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2018; Piaget, 1948), and PTs often use procedures with integer operations (Bofferding & Richardson, 2013; Kajander & Holm, 2013; Steiner, 2009). However, PTs are capable of reasoning in ways about integer subtraction beyond rules. In addition to procedures and recall of facts for integer addition and subtraction, PTs may draw on strategies that use translation for integer subtraction (Bofferding & Wessman-Enzinger, 2018). Their strategies for translation may include counting, referents to zero, movements with distance, and subitizing distance. PTs may also use other strategies like analogies, where they compare integer addition and subtraction to whole number addition and subtraction. Or, PTs may use counterbalance strategies where they make use of the magnitudes of numbers, trying to "balance" numbers by using additive inverses (Bofferding & Wessman-Enzinger, 2018). Although we have insight into the types of strategies that PTs may use to solve result missing problems (e.g., -3 – 2 = ☐), we know little about the collection of strategies that PTs may produce for part missing problem types (e.g., -3 – ☐ = -5). Furthermore, we need insight into PTs' collective reasoning for different open number sentence types to better understand how to support PTs and how to position them as resources for each other. Therefore, we investigate the following research question: What is the collective knowledge of PTs as they solve integer missing subtrahend problems?

Methods and Analysis

Fifteen PTs (eight secondary, seven elementary) participated in think-aloud interviews (Ericsson, 2006; Ericsson & Simon, 1993). They all had taken some mathematics coursework in the mathematics department prior to participating. Part of the individual think-aloud interviews included solving a series of integer addition and subtraction open number sentences (i.e., ☐ + 1 = -2; ☐ – -7 = 15). We coded each of the open number sentences for the strategies used. Although interviews were conducted individually, we examined the types of strategies generated collectively for understanding collective knowledge. The results presented here focus on the richness of their collective reasoning for the two integer subtraction open number sentences, -2 – ☐ = 4 and -5 – ☐ = 0, where the subtrahend is missing.

Results

Collective Strategies for -2 – ☐ = 4

Overall, the PTs drew on two main strategies for solving this problem, which could be broken down into a few variants, and three (analogy, counterbalance, procedures or recall) supporting strategies (see Table 1).

Algebraic manipulation. The first main strategy was using algebraic rules to solve for the box. Collectively, PTs demonstrated flexibility in manipulating the problem as they initially added a box to both sides, added a positive two to both sides, or subtracted a negative two from both sides. When moving the box to the right side (-2 = 4 + ☐), one PT then subtracted four from each side to get -2 – 4 = ☐. In order to solve the problem, the PT changed it to -2 + -4 = -6. One PT who initially added two to both sides got -☐ = 6. She then multiplied both sides by negative one in order to make the box positive. Finally, a PT who subtracted negative two from each side ended up with -☐= 4 – -2. She continued, "Four minus negative two is the same as four plus two, so negative box equals four plus two, which equals six. So negative box equals six."

Translation. The second main strategy was using translation, either by using zero as a reference point or by counting each individual space between -2 and 4. Interestingly, this strategy provides a conceptual way for solving the right side of the equation when solving -☐ = 4 – -2, a step in solving the problem algebraically when subtracting -2 from both sides. One PT explained why reasoning about the numbers in relation to zero made sense:

> Negative two is on the other side of zero on the number line, and I was trying to get to four, and I was subtracting. So I knew I was going to have to add to move to the right. And so, minus times a minus, I knew I needed a negative number here [in the box] to move in that direction.

Additional reasoning. Beyond using procedural rules, PTs sometimes used additional strategies, often together with algebraic manipulation or translation. For example, one PT referenced a counterbalance conceptual model (hills and holes) and used an analogy to a related positive number problem after adding the box to both sides and subtracting four from both sides (☐ = -2 – 4): "Again, we have a hole of negative two and we are subtracting four more units. So two plus four is six, but because we are going in the hole, that is a negative six."

Table 1: Collective Knowledge and Connections for -2 – ☐ = 4

Algebraic Manipulation	Translation	Additional Reasoning
-2 – ☐ = 4 → -2 = ☐ + 4 → -2 – 4 = ☐ → ☐ = -6 -2 – ☐ = 4 → -2 = ☐ + 4 → ☐ = -2 + -4 → ☐ = -6 -2 – ☐ = 4 → -☐ = 6 → ☐ = -6	Distance from -2 to 4	-2 = hole of 2 units; subtract 4 more units (dig a deeper hole); -6 = hole of 6 units
PTs exhibited flexibility in procedures used. These procedures could be connected to translation strategies. Further, their manipulations of adding 2 or subtracting -2 from both sides could be used to explore different ways of making zero (i.e., additive inverses and subtracting from self).	This strategy provides a conceptual way for solving the right side of the equation when solving -☐ = 4 – -2, a step in solving the problem algebraically when subtracting -2 from both sides.	This strategy provides a conceptual way for solving the left side of the equation when solving -2 – 4 = ☐, the resulting equation when adding the Box to both sides and subtracting four from both sides.

Collective Strategies for -5 – ☐ = 0

Unlike for the previous problem, PTs relied heavily on analogies for this problem with less focus on algebraic manipulations (see Table 2).

Analogy. PTs constructed analogies as the main strategy for this problem. The PTs compared -5 – ☐ to 5 – 5 or other subtraction problems. For example, one PT explained, "Negative eight minus negative eight equals zero, um, I would put negative five in there just again, the quantity minus itself is zero."

Translation. As with the previous problem, PTs counted each space or moved in chunks when using translation for this problem. Two PTs used zero as a referent, stating, "To get negative five to be zero, I would have to add five to get that point." An additional PT explicitly referenced the distance between -5 and 0 as adding 5. Both of these PTs thought additively about -5 – ☐ first and adjusted their reasoning. Another PT used counting:

> I wanted to reach zero, I could do like one, two, three, four, err, it really should be negative

> four, negative three, negative two, negative one, zero. And so these ones are three, negative two, negative one, and that was one, two, three, four, five. And so this one would be five.

This PT wavered between +5 and -5, but ultimately decided on -5.

Algebraic manipulation. One PT used algebraic procedures and solved -5 – ☐ = 0 by replacing the box with an X. Then, the PT added 5 to both sides, changing the open number sentence to -X = 5. The PT multiplied -X = 5 by -1 and determined the solution to be X = -5.

Procedure. The most common procedure that PTs used for solving this problem involved recognizing -5 – -5 is equivalent to -5 + 5. Although many of the PTs used a procedure for solving -5 – ☐ = 0, three of the PTs paired their procedures with other strategies, like translation or analogy.

Table 2: Collective Knowledge and Connections for -5 – ☐ = 0

Analogy	Translation	Algebraic & Procedure
Compare to 5 – 5, -8 – -8, or x – x.	Distance from -5 to 0 is 5.	-5 – -5 = -5 + 5 -5 – x = 0 → -x = 5 → x = -5
Comparing -5 – box to 5 – 5 is a strategy that children often use. PTs made reference to any quantity minus the same quantity (i.e., x – x), which is a form of algebraic reasoning. This type of reasoning could be compared and connected to the other problem -2 – ☐ = 4.	This type of contribution aligns to visual representation of a number line well, which is different from the other strategies. These strategies provide opportunities to about what -5 - -5 looks like on a number line.	The procedures that the PTs used can be compared to the distance strategies they created. The distance models they created matched -5 + 5 = 0 well.

Discussion

The PTs' collective reasoning portrays a rich perspective of the types of knowledge that PTs bring to their classrooms about integer reasoning. The PTs collective reasoning illustrates potential for conceptual understanding (i.e., translation, analogy) to build to procedural fluency (i.e., algebraic manipulation). As mathematics teacher educators and researchers, our work includes facilitating opportunities for making connections among the PTs' reasoning and making the collective knowledge accessible to all students. Beyond that, we need to help PTs use each other as a resource and help them create spaces where they can build on the group's knowledge.

Critical pedagogies offer potential to think about how to incorporate use of collective knowledge. Complex instruction, for example, is based on a teaching-learning perspective where all students are necessary for the thinking and learning of the class (Featherstone, Crespo, Jilk, Oslund, Parks, & Wood, 2011). When using complex instruction "the teacher structures the assignment so that all the children must participate in the intellectual work of problem solving" (Featherstone et al., 2011, p. 4). Building on the research around students' integer thinking and this research on PTs' integer thinking, we could design complex instruction tasks that draw on this collective knowledge. Although developing an instructional task is outside the purpose of the work described in this research brief, we aim to describe the thinking of PTs as intellectual work and hope that describing their thinking will provide a lens into how to support instruction with PTs in the future.

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ENGAGING PROSPECTIVE ELEMENTARY TEACHERS IN STANDARDS FOR MATHEMATICAL PRACTICE WITHIN CONTENT COURSES FOR TEACHERS

Brooke Max
Purdue University
bmax@purdue.edu

Rachael M. Welder
Texas A & M University
rachael@rachaelwelder.com

Since prospective elementary teachers (PTs) in the US will be expected to engage students in mathematics via the Common Core Standards for Mathematical Practice (SMPs), national teacher preparation standards are calling for PTs to have opportunities to engage in these practices during their development of mathematical knowledge for teaching (e.g., AMTE's Standards for Preparing Teachers of Mathematics, 2018). This report investigates the intended curriculum of 82 mathematics content courses for teachers for such opportunities. Findings suggest that most instructors of these courses intend to engage PTs in several of the SMPs, mostly by offering them opportunities to experience learning through lessons planned with the SMPs in mind. However, substantial variance was noticed in the frequency and depth of which the individual SMPs were reported or found evidenced in sample tasks. Implications to follow.

Keywords: Teacher education – preservice, Instructional activities and practices, Standards

Common Core State Standards of Mathematics (CCSS-M, National Governors Association Center for Best Practices and the Council of Chief State School Officers [NGA & CCSSO], 2010) offer standards for K-12 students in terms of mathematical content and practices. The Standards for Mathematical Practice (SMPs) bring attention to the processes and practices that have been recommended by US national organizations for K-12 students (e.g., National Council of Teachers of Mathematics (NCTM) *Process Standards*, 2000; National Research Council *Strands of Mathematical Proficiency*, 2001). The CCSS-M, including the SMPs, are currently adopted by 41 states (NGA & CCSSO, 2019) and included in national recommendations for the preparation of prospective elementary teachers (PTs) (e.g., Association of Mathematics Teacher Educators *Standards for Preparing Teachers of Mathematics*, 2018; Conference Board of Mathematical Sciences (CBMS) *Mathematical Education of Teachers II*, 2012). Because of the increased attention being given to the SMPs in terms of teacher preparation, we investigated the intended curriculum of mathematical content courses for opportunities they afford prospective elementary teachers (PTs) to engage with the SMPs during their development of mathematical knowledge. By gaining insight into the ways in which Mathematics Teacher Educators (MTEs) are affording PTs such opportunities, our community can consider ways of replicating and enhancing this work.

Methods

Data Collection

A subset of data from a larger study on the mathematical content preparation of elementary teachers was analyzed to address the question, "In what ways are PTs being afforded opportunities to engage with CCSS-M SMPs in content courses for PTs?" The data for the larger study (see Max, 2018) was collected through an online survey that was sent to MTEs across the US through the use of various large-scale listservs and social media outlets (e.g., the AMTE email and affiliate list, the STaR listserv, the AERA SIG-RME Facebook page). The survey asked MTEs who are involved in the mathematical preparation of elementary teachers about their educational and professional backgrounds, the teacher preparation programs in which they are involved, and any content courses they teach that are specifically designed for PTs.

For each content course, participants were asked to select all of the SMPs they intentionally address in their courses and to provide general information about the ways in which the SMPs are addressed (by selecting from a list of potential options, including an option to select “other” and explain). They were also asked to identify the SMP that receives the *most* attention in each course and to describe an example of a way in which PTs are engaged in that particular practice. Open-ended responses to this prompt will be referred to as *SMP descriptions* and considered to be *explicit* ways in which MTEs are engaging PTs with SMPs.

Participants were also asked to upload or describe an activity, assessment, or reading related to the content domain (CBMS, 2012) that receives the most attention in each class. While respondents were not asked to consider the SMPs in regard to their content examples, their responses to this prompt were analyzed through the lens of the SMPs to identify ways in which MTEs might be engaging PTs in SMPs throughout tasks focused on content development. This collection of written responses and uploaded materials will be referred to as *content examples* and considered to be *implicit* ways in which MTEs are engaging PTs in SMPs. For more detailed information on survey construction and data collection, please see Max (2018).

Participants

The online survey was completed by 44 MTEs who have experience teaching content courses for PTs and will serve as the sample for this study. This sample represents MTEs from 41 different US universities across 19 states. Thirty-nine (89%) of the MTEs reported having some affiliation within a department of mathematics: 28 (72%) of these MTEs belong solely to a mathematics department, while 11 (28%) have joint affiliations with a department of education. The remaining five MTEs (11%) work solely in departments of education. The participants reported a mean of 10.7 years of experience teaching at the post-secondary level. Thirty-three (75%) MTEs reported prior experience as a K-12 teacher, with a mean of 4.5 years of teaching experience. Ten (30%) MTEs had taught at the elementary level, nineteen (58%) at the middle level, and twenty-five (76%) at the secondary level, with twenty-two (66%) having taught at more than one of these levels.

The respondents reported a mean university program requirement of 2.1 content courses for PTs, worth a mean of 6.7 total credit hours, with 84% of these courses being housed in departments of mathematics. These courses are reportedly being taught by faculty 91% of the time, with staff (17%), graduate students (16%), and adjunct faculty (7%) teaching these courses much less frequently. MTEs also reported over 50% of these courses tending to be taken by first-year college students, with only 22% likely to be enrolled by juniors or seniors. Lastly, all but one of the respondents reported research as being a component of their current position, with 38 (88%) of MTEs conducting research in the area of mathematics education, 3 (7%) in mathematics, and 2 (5%) in a non-math related field.

Analysis

Descriptive statistics and content analysis were used to investigate the collected data looking for evidence of opportunities for PTs to engage in the SMPs in content courses. Using protocol coding (Saldaña, 2016), all SMP descriptions and content examples were coded according to each of the SMPs considered to be addressed. This was done by identifying opportunities that aligned with the actions explicated in the detailed SMP descriptors offered in the CCSS-M standards document (NGA & CCSSO, 2010). For example, for SMP3: Argumentation, the coders looked for opportunities for PTs to "construct viable arguments and critique the reasoning of others…make conjectures… use counterexamples… justify their conclusions, communicate them to others…explain what it [a flaw in an argument] is" (pp. 6-7). Although the participants were asked to discuss activities pertaining to the SMP and content domain that receive the *most* attention in each course, the coding served to identify any and all SMPs that PTs were invited to engage in throughout the activities described. This coding scheme was used because SMPs are interconnected and at times unable to be separated from one another. For example, one respondent described how they address SMP1: Problem-Solving in a task related to finding the volume and surface area of three-dimensional objects. Since this task involved PTs using Geo-Solids, it was also coded as addressing SMP5: Tools. Each of the SMP descriptions and content examples were independently coded by two researchers. The coders cross-checked their work to find an initial 75% agreement; all disagreements were discussed until consensus was achieved.

Results

The 44 participating MTEs provided information for a total of 82 content courses for PTs that they were either teaching or have previously taught. Below, we report findings on our analysis of the ways in which PTs are being provided opportunities to engage in SMPs in these 82 courses. First, we will discuss *explicit* attention given to SMPs found in the intended curriculum of content courses through analyzing SMP selection data and coded SMP descriptions. Afterwards, we will discuss *implicit* attention identified through analyzing coded content examples).

MTEs selected one or more SMPs as being intentionally addressed in 74 (90%) of the reported 82 content courses (n = 74). See Table 1 for the number of courses for which each SMP was selected. For 55 (74%) of the 74 courses that address at least one SMP, MTEs selected a most-addressed SMP (n = 55). Table 1 also shows the number of times each SMP was selected as being the most-addressed SMP for a course. However, we acknowledge that asking participants to choose a single most-addressed SMP may have presented a challenge. For example, one respondent indicated, "Each of them [SMPs] is addressed regularly. It is difficult to say which is addressed the most." In fact, MTEs stated that all SMPs received regular and/or equal attention in 4 of the 55 courses. In these cases, we recorded all 8 SMPs as having been selected (thus the total in the "Most Addressed" column is greater than 55).

Table 1: Attention to SMPs

	Explicit Attention			Implicit Attention
SMP Name	Intentionally Addressed (n = 74)	Most Addressed (n = 55)	Indicated by a SMP Description (n = 39)	Indicated by a Content Example (n = 57)
1. Problem Solving SMP	64	17	21	40
2. Reasoning SMP	63	6	4	10
3. Argumentation SMP	62	30	20	13
4. Modeling SMP	56	9	1	9
5. Tools SMP	62	7	5	26
6. Precision SMP	59	5	4	8

7. Structure SMP	60	8	2	23
8. Regularity MP	54	6	0	6

Information regarding the ways in which SMPs are being explicitly addressed was provided for 69 (93%) of the 74 courses. When asked to select from a list of potential options, respondents reported that 88% (61) of their courses offer PTs the opportunity to experience learning through lessons planned with the SMPs in mind. The remaining provided options, having PTs read the SMPs, facilitating lessons using the SMPs, and creating lesson plans that reference SMPs, were selected for 32 (46%), 16 (23%), and 5 (7%) courses, respectively. Respondents described “other ways” they address the SMPs for 9 (13%) courses, with open-ended responses including “discussion of how these standards play out in class lessons,” “reflecting on their use in lessons in class,” and having a “poster of SMPs on [a] wall in [the] classroom.” Additionally, written SMP descriptions were provided for 39 (53%) of the 74 courses (n = 39). Table 1 presents the results of our coding showing the number of times each SMP was identified as being evidenced within a SMP description.

Content examples were provided by MTEs for 68 (83%) of the 82 content courses on which they reported, including 48 written descriptions of content activities and 20 uploaded activities/tasks. However, only 57 (84%) of the content examples provided enough information for the coders to identify one or more SMPs (n = 57). Table 1 provides the number of times each SMP was coded as being implicitly evidenced within one of these 57 content examples.

Discussion and Conclusion

MTEs reported that all SMPs are being addressed in the vast majority of their content courses for PTs with little variance among the individual SMPs (each being addressed in 73%-86% of the courses). However, when opportunities for explicit and implicit attention were further analyzed, the attention being given to individual SMPs showed greater variation. SMP1: Problem Solving and SMP3: Argumentation were selected much more often as being the most addressed SMP, and consequently showed up the most in the SMP descriptions (for task samples and discussion on the ways in which SMP3 occurred within content examples, see Max & Welder, accepted). These findings are consistent with the national attention these practices have received (e.g., NCTM, 1980, 2000). However, they were only somewhat consistent with the SMPs identified in the content examples, as the use of SMP5: Tools and SMP 7: Structure were noted much more frequently than SMP3: Argumentation. In fact, these two SMPs organically emerged in nearly half of all content examples. On the other hand, our results suggest that it could be beneficial for MTEs to consider additional ways to engage PTs in SMP8: Regularity, SMP4: Modeling, and SMP6: Precision. MTEs identified these three SMPs as receiving attention in the fewest of their courses and they were found to occur the least in the content examples they provided. We encourage MTEs to reflect upon their practice and the ways in which they can afford PTs meaningful opportunities to develop these less-evident SMPs in content courses, as they may not be occurring as organically as others (such as SMP1: Problem Solving). We also encourage MTEs to identify ways in which they are attending to SMPs, perhaps in ways of which they are not aware, so they can connect such occurrences to the SMP language for PTs.

We note that the findings of this work are based on MTEs’ intended curriculum and hypothesize that more (and perhaps more explicit) attention to SMPs could be identified through observing activities being enacted with PTs. Additional research in this direction could be useful in supporting MTEs in finding ways to foster PTs’ working understanding of the SMPs. By helping PTs become more aware of the valuable role each SMP can play in the development of mathematical content knowledge, we may be able to better prepare elementary teachers for engaging their future students in such practices.

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EMERGING SOCIAL AND SOCIO-MATHEMATICAL NORMS DURING TECHNOLOGY-ENHANCED LESSONS

Tuğçe Kozaklı-Ülger
Bursa Uludağ University
tugcekozakli@gmail.com

Hatice Akkoç
Marmara University
hakkoc@marmara.edu.tr

This study aims to explore the way in which social and socio-mathematical norms endorsed by pre-service mathematics teachers change as they move from traditional lessons to technology-enhanced lessons. Two pre-service mathematics teachers participated in this case study. Each participant's four lessons (a total of eight lessons) were video-recorded. Two of each pre-service teacher's lessons were technology-enhanced. Participants were also interviewed after their lessons. The findings indicated that participants constituted new norms and changed some of their endorsed norms during technology-enhanced lessons. Emerging norms were concerned with certain aspects such as justification of mathematical rules and collaboration.

Keywords: Emergent perspective, Social norms, Socio-mathematical norms, Technology-enhanced lessons

Introduction

As teachers move from traditional lessons to technology-enhanced lessons, they also change the nature of classroom interaction such as roles, behaviors and communication (Monaghan, 2001). Rather than reducing the role of the teacher, teaching with technology requires new roles such as choosing available tools, supporting students' participation and constituting new classroom norms (Gardiner, 2002). Therefore, there is a need for investigating teachers' choices, behaviors and norms constituted in technology-enhanced learning environments (Drijvers, 2012). It is also vital to focus on norms endorsed by pre-service teachers since they will be responsible for "creating an intellectual environment where serious mathematical thinking is the norm" (NCTM, 2000, p. 18). Considering this need, the aim of this study is to investigate how social and socio-mathematical norms endorsed by pre-service mathematics teachers change as they move from traditional lessons to technology-enhanced lessons.

Theoretical Framework

This study will explore the social aspects of this new environment, namely social norms and socio-mathematical norms, which are based on Yackel and Cobb's (1996) emergent perspective. Focusing on the individuals and classroom community, emergent perspective explores the development of norms. Norms will be investigated in a comparative manner by exploring them both in traditional and technology-enhanced lessons.

A norm in teaching and learning process is based on expectations and obligations established through interaction between students and teachers (Yackel, Rasmussen & King, 2000; Roy, Safi, Tobias and Dixon, 2014)). It is not only an individual notion, but also a collective construct. Social norms are concerned with aspects of classroom interaction which become normative (Yackel et al., 2000). Whether it is a traditional or reform-oriented classroom culture, social norms are operative for that particular class. For example, an expectation established in a classroom in the way that students should challenge others' thinking and justify their own interpretations is a social norm (Bowers, Cobb & McClain, 1999). The term "socio-mathematical norm" is distinguished from general social norms and refers to normative aspects of mathematical discussions which are specific to mathematical activities (Yackel, 2000). For example, a standard constituted by student-teacher through interaction to evaluate an argument as mathematically sophisticated is a socio-mathematical norm (Bowers et al., 1999). This study will explore how social and socio-mathematical norms change with the existence and use of technological tools in mathematics lessons.

Methodology

A case study was conducted to investigate the way in which social and socio-mathematical norms' changes in technology-enhanced lessons. Participants are two pre-service mathematics teachers (both female) enrolled in a teacher preparation program in a state university in Bursa, Turkey. Participants taught lessons in partnership schools in high school classes with around thirty students during the "Teaching Practice" course. Data collection instruments were observation and semi-structured interview. Since classroom culture might affect norms in a classroom, each pre-service teacher' lessons were observed in the same class, respectively. Participants were also interviewed to better understand their constituted norms. They were asked to explain how they planned to integrate technology into their lessons and compare their technology-enhanced lessons to their traditional lessons. The first author of this paper observed four lessons of each participant: the last two lessons were technology-enhanced. Video recording of all lessons were transcribed and analyzed to investigate differentiation of norms in traditional lessons and technology-enhanced lessons.

Findings

This section presents the way in which pre-service teachers' endorsed norms change as they integrate technological tools into their lessons.

Table 1: Burcu's and Gizem's Endorsed Norms

<table>
<tr><th colspan="2">Burcu's Endorsed Norms</th><th colspan="2">Gizem's Endorsed Norms</th></tr>
<tr><th>Without technology</th><th>With technology</th><th>Without technology</th><th>With technology</th></tr>
<tr><td colspan="2">Students describe ways to solve problems to their classmates and teacher. (SN) (Kazemi and Stipek, 2001)</td><td>Students work individually for finding solutions to problems (SN)</td><td>Students collaborate for finding solutions to problems. (SN) (Kazemi and Stipek, 2001)</td></tr>
<tr><td>Students tell their solutions using mathematical expressions (or mathematically) (SMN) (Williams,2010)</td><td>Students express mathematical properties through whole class discussion. (SMN)
Students examine the validity of rules for different situations. (SMN)</td><td>Students tell their solutions using mathematical expressions (SMN) (Williams, 2010)</td><td>Students explain their solutions using mathematical expressions (SMN) (Williams, 2010)</td></tr>
<tr><td>Students get the right results in their mathematical solutions making use of hints. (SMN)</td><td>Students examine the "reasons" behind mathematical solutions (SMN)
Students make</td><td>Students reach mathematical solutions using the given rules to them. (SMN)</td><td>Working in groups, students examine the "reasons" behind mathematical solutions/ expressions. (SMN)</td></tr>
</table>

	justifications of mathematical rules/statements. (SMN) (Hershkowitz and Schwartz, 1999)	Students make verifications of mathematical rules/statements. (SMN)

The analysis of data indicated various norms some of which were also reported in other studies (see Table 1). The table indicates that participants' endorsed norms have changed in certain aspects such as collaboration, justification or verification of mathematical rules, and relational understanding as they integrated technology into their lessons.We will illustrate this change by focusing on a detailed account of one of the participants, namely Gizem.

Gizem used Geogebra software during her lesson on basic elements of triangles and area of a triangle. She projected Geoegebra drawings on the smart-board and students in groups made drawings using Geogebra which were installed in their tablets. Gizem endorsed the socio-mathematical norm "Students make verifications of mathematical rules/statements" which was also reported by Hershkowitz and Schwartz (1999) and Williams (2010). In her technology-enhanced lesson, rote learning gave way to mathematical verification using the software. This is illustrated in the following excerpt in the context of equilateral triangles:

> Gizem: We have an equilateral triangle but we don't know its height. Let's draw a perpendicular line here onto the triangle and find area of it using the angles. Height from the point "A" to the side "a" divides the side into two equal parts. a/2. The angle A is divided into two, 30 and 30. Side against the angle of 30 is a/2, and then side against the angle of 60 degrees is $\sqrt{3}$, isn't it? Then the area is $a^2\sqrt{3}/4$. Is this valid for the areas of all equilateral triangles?
> Student: At the end, isn't it a formula?
> Student: We find all of them with the formula. Is it gonna be different according to lengths?
> Gizem: We all should be convinced of the validity of this formula. Therefore, let's draw an equilateral triangle using Geogebra and examine the validity of the formula.
> Student: What is the length of the side?
> Gizem: Each group can start with a different length. What do you think?

As can be seen from the excerpt above, Gizem introduced the socio-mathematical norm "Students make verifications of mathematical rules/statements" by emphasizing that they (herself and her students) should be convinced of validity of rules. After giving basic concepts and properties, Gizem moved to Geogebra and use it for verification (See the screen shot on the left in Figure 1). Working in groups, students were involved in this verification process using the software on their tablets.

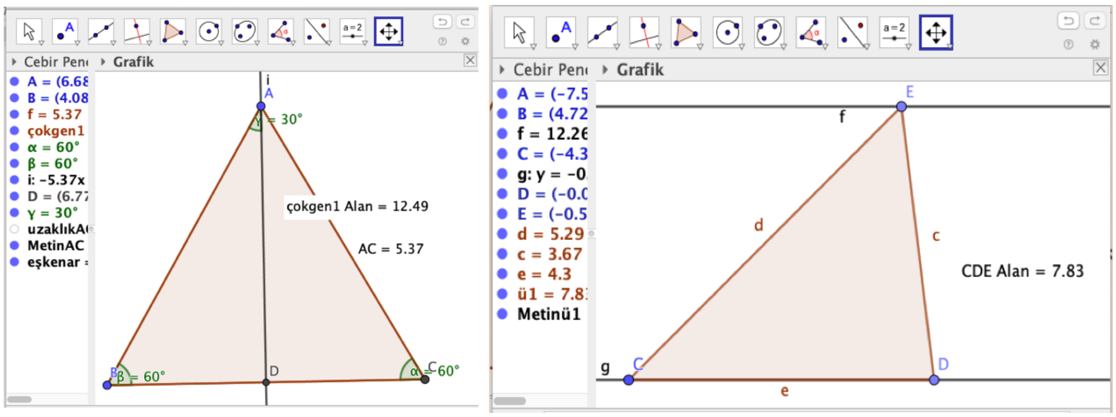

Figure 1: Gizem's Geogebra Constructions During Her Lessons

Another finding was related to the social norm "Students work individually to find solutions to problems" which was replaced by another social norm "Students collaborate to find solutions to problems" in Gizem's technology-enhanced lesson. Although, she did not rely on mathematical software throughout the whole lesson, she encouraged collaboration among students during the whole process. Working in groups in technology-enhanced lessons was negotiated between students and the pre-service teacher and this norm was sustained for other lessons. Students who worked in groups not only focused on solving the problem, but more importantly, examined different solutions and reasons behind them. This situation points out the socio-mathematical norm "Working in groups, students examine the "reasons" behind mathematical solutions/expressions". This norm revealed itself during a discussion on how the area changes when the peak of the triangular area between the two parallel lines varies. Using Geogebra, Gizem constructed a triangle between two parallel lines and moved the upper vertex horizontally and asked her students whether the area remained constant and why. This way, the pre-service teacher helped her students to discover this relationship and reasons behind it working in groups. The socio-mathematical norm "Working in groups, students examine the 'reasons' behind mathematical solutions/expressions" was introduced by the pre-service teacher and negotiated between her and her students without any difficulty. In addition, observational data indicated that Gizem sustained the socio-mathematical norm "Students reach mathematical solutions using the given rules to them" in all her four lessons. She particularly asked for rules and formulas and made this clear for her students. As this norm was sustained, on the one hand, reasons behind these became the focus of discussion in her technology-enhanced lessons.

Discussion

This study explored how social and socio-mathematical norms endorsed by two pre-service mathematics teachers changed as they started to integrate technology into their lessons. By observing participants' traditional and technology-enhanced lessons at the same time, not only pre-existing norms were specified but also emergence of new norms was observed in technology-enhanced lessons. Data also indicated that some of the norms effaced themselves.

One of the remarkable findings is the emergence of new norms with the existence of technological resources in the classroom. Among those norms, one particular norm continued to be introduced and sustained throughout technology-enhanced lessons: Students make verification and justifications of mathematical rules/statements. Considering the fact that pre-service teachers did not give importance to verification and justifications in their traditional lessons, their emphasis on verification and justification in their technology-enhanced lessons became crucial. The emphasis on verification and justification might be due to dragging feature of the dynamic geometry software they used which provides opportunities to verify and justify geometrical theorems and properties. Another remarkable change in norms was concerned with collaboration among students to examine the "reasons". After this norm was introduced, they started to question "why" and "how" mathematical solutions/expressions worked and privileged reasoning.

To the extent that classroom norms constrain and enable learning, it is possible for teachers to estimate which norms they might wish to foster (Yackel, 2000). A new environment can generate new classroom norms (Tabach, Hershkowitz & Dreyfus, 2013). Therefore, teachers and pre-service teachers should be provided guidance. Teacher education programs should monitor pre-service teachers' development with regard to their endorsed norms for successful technology integration and develop an awareness of social and socio-mathematical norms.

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REVEALING TEACHER KNOWLEDGE THROUGH MAKING: A CASE STUDY OF TWO PROSPECTIVE MATHEMATICS TEACHERS

Steven Greenstein, PhD
Montclair State University
greensteins@montclair.edu

Eileen Fernández, PhD
Montclair State University
fernandeze@montclair.edu

Jessica Davidson
Montclair State University
davidsonj5@montclair.edu

We describe an experience within mathematics teacher preparation that engages pre-service teachers of mathematics (PMTs) in Making and design practices that we hypothesized would inform their conceptual, curricular, and pedagogical thinking. With a focus on the design of new tools that can generate new possibilities for mathematics teaching and learning, this Learning by Design experience has PMTs exploring at the intersection of content, pedagogy, and Making. We describe the forms of knowledge brought to bear on their experiences through a case study analysis of one pair of PMTs' Making experience. As the advancement of these forms of knowledge is essential to effective mathematics teaching, these findings suggest the promise of a Making experience within mathematics teacher preparation.

Keywords: Instructional Activities and Practices, Teacher Knowledge, Technology

Objectives

Preservice elementary teachers have been characterized as coming to teacher preparation with limited conceptions of mathematics (AMTE, 2013) and a model of mathematics teaching that largely appeals to rules and procedures (Ball, 1990; Ma, 1999; Thompson, 1984). Unfortunately, this model is not consistent with a pedagogy that supports learning mathematics with understanding. Consequently, prospective elementary teachers' preparation must include opportunities that challenge their current models of mathematics teaching and learning. In this proposal, we present one such opportunity that is centered in the activity of *Making*. We draw on Halverson and Sheridan's (2014) conception of Making as *designing, building and innovating with tools and materials to solve practical problems*. We present a novel *Making-oriented experience* within mathematics teacher preparation that tasks prospective mathematics teachers (PMTs) with designing, fabricating, and evaluating new manipulatives (Post, 1981) to promote learners' mathematical thinking and reasoning. In seeking to determine what this experience might offer prospective elementary teachers as they prepare for the work of mathematics teaching, this project addresses the following question: *What forms of knowledge can be brought to bear on prospective elementary teachers' design work as they Make new manipulatives to support the teaching and learning of mathematics?*

Theoretical Framework

Our theoretical framing is organized around the learning theories of *constructivism* and *constructionism*. These theories recognize that knowledge is actively constructed by a learner, with constructionism adding the dimension that the knowledge be constructed during the process of making a shareable object (Harel & Papert, 1991). We drew from the rich scholarship devoted to teacher knowledge to characterize what forms of knowledge might actually be brought to bear on PMTs' design work (Ball, 1990; Borko & Livingston, 1989, 1990; Cochran, DeRuiter, & King, 1993; Hill, Ball, & Schilling, 2008; Koehler & Mishra, 2009; Mishra & Koehler, 2006; Shulman, 1986). In particular, we took a *Learning by Design* approach (Koehler & Mishra, 2005;

Koehler, Mishra, Hershey, & Peruski, 2004) to leveraging and potentially advancing this knowledge. Learning by design involves the PMTs in the activity of *designing*, or the purposeful imagining, planning, and intending that precedes and interacts with Making. This approach calls upon PMTs to "actively engage in inquiry, research and design" so that they can make "tangible, meaningful artifacts" that represent "the end products of the learning process" (Koehler and Mishra, 2005; p. 135). This approach provides an opportunity to consider the interplay between the evolving artifact and the application of teacher knowledge domains in the artifact's development. The premise for learning by design honors the proposition that it is productive to develop teacher knowledge within a context that recognizes the interactions and connections among these constituent domains of knowledge.

The artifact that a PMT makes, and the design decisions that go into the Making, provide a rich source of data for understanding these knowledge domains if we consider the artifact and its creation from the perspective of designing for mathematical abstraction (Pratt & Noss, 2010). In the case of manipulatives, the designer aims to embed a concept in its design so that it can be made available to the learner for abstraction through their sensorimotor manipulation of the object (Kamii & Housman, 2000; Piaget, 1970; Vygotsky, 1978). This is the task we set for the PMTs in this project: to design a manipulative that is hypothesized to support learners' abstractions of a mathematical concept from concrete tools. Pratt & Noss's case study (2010) offers a proof of concept that learning by design provides a venue for characterizing the interplay between a participant's beliefs and knowledge domains as they are invoked during the design process.

Methods

The study took place in a specialized mathematics content course for prospective elementary teachers. Twenty-six participating students comprised twenty-one groups. The PMTs were given the following task: "The purpose of this project is for you to 3D design and print a new physical tool (or "manipulative") that can be used in teaching a mathematical idea. The design of this tool and a corresponding task will reflect a) your knowledge of what it means to do mathematics and how we learn with physical tools, b) your knowledge of elementary-level mathematics content, and c) your perspective on pedagogy and curriculum in mathematics education." This project had three written components, which comprise the corpus of research data: 1) an "Idea Assignment" that describes a group's initial thoughts about a manipulative they want to work on, 2) a "Project Rationale," which is an account of how their design reflects an understanding of what mathematics is and how learning happens, and 3) a "Final Paper" that describes the purpose of the manipulative, the corresponding tasks that were created, and the group's findings from an intended user's manipulative-mediated engagement with those tasks.

Data analysis proceeded in two phases. In Phase I (more fully reported in Greenstein & Seventko, 2017), three researchers analyzed the PMTs' written work and generated codes (Corbin & Strauss, 2008) that identify forms of knowledge in the PMTs' written work, with initial codes derived from the mathematics knowledge for teaching literature. Intercoder reliability on the codes was calculated at .82. This analysis provided a promising foundation for delving more deeply into the PMTs' written work in order to understand how the knowledge we identified was brought to bear in their Maker projects.

In Phase 2, we took a case study approach (Yin, 2009) with purposeful sampling (Patton, 2002) to identify and select design cases whose reflections bear evidence of *mathematical richness*. In these cases, PMTs articulate and express multiple layers of mathematical detail in describing mathematics content, student thinking, or the use of technology during the project. This sampling technique purposefully mirrors the knowledge frameworks so that we could describe the PMTs' constituent forms of knowledge as reflected in their written work. Crafting narratives of the PMTs' design experiences involved a process of moving between the Phase 1 codes and the PMTs' reflections to intuit the data and weave narratives that illuminate the knowledge brought to bear on the PMTs' design activity. Harnessing the case study's virtue for evoking "images of the possible," (Shulman, 2004; p. 147), we present our narrative for Casey and Mia and their tool called *Minute Minis* (see Fig. 1 for digital design).

Results

We viewed the PMTs' written reflections as containing instances of the knowledge they brought to bear in their design work. For Casey and Mia, our analysis identified various dimensions of mathematical knowing in teaching, such as knowledge of mathematical content, pedagogical content knowledge (PCK), curricular knowledge, and knowledge of how learning works in interaction with manipulatives.

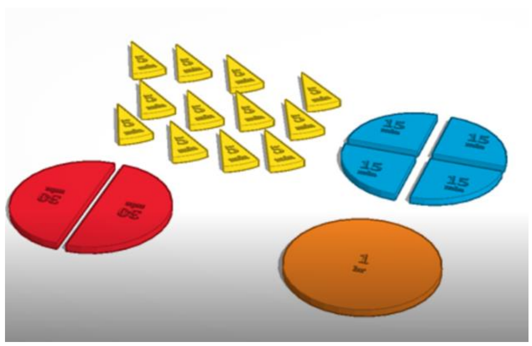

Figure 1: Minute Minis

Casey and Mia were inspired to design a manipulative that could help children reason about the abstract concept of time. In their initial design rationale, they hypothesize about breaking through the ordinarily obscure nature of time to make it more accessible to learners:

> The main goal of this project is to give a concrete representation of the relationship between hours and minutes. Using manipulatives is especially important when exploring new concepts, and sense (sic) time is a very abstract concept, it is especially pertinent that students have something concrete to work with. With these manipulatives, students will be better able to solve addition, subtraction, multiplication and division problems relating to time.

Making such a tool for the shared purpose of learning and teaching was initially and genuinely influenced by questions Casey had about how children were thinking of time in the classroom where she was doing her student teaching:

> Currently, most of the 2nd graders in my class can tell time to the nearest half hour, yet I am unsure of how they know how to do this. Is it just because they know that when the minute hand is pointing at the 6 I say _:30 and when it's pointing at the 12 I say _:00? Or do they have a more (sic) deeper understanding of time and how a clock works?

These considerations reflect how Casey's PCK (wondering about students' current conceptions of the topic of time) inform her design. Over the course of the project, these questions develop into other strands of knowledge that she and Mia use to investigate these issues. That is, the Making of Minute Minis was driven by a desire to transform potentially limited conceptions of time from memorized models into deeper mathematical meanings. Drawing upon other aspects of PCK and of mathematical and curricular knowledge, Casey and Mia take an existing design of fraction circles and use concepts from geometry to amend it for their objectives:

> [We] will be using the same concept of fraction circles, yet instead of labeling them with a fraction, they will be labeled with minutes. For instance, a whole circle will be labeled "1 hour," while two half circles will be labeled "30 minutes." [We] will also have [fraction] circles for 15 and 5 minutes. (see Fig. 1)

One of their key design issues focused on being able to "visually illustrate the concept of minutes as fractions of an hour." Additionally, the circular shape was important to them in ensuring "that students would be able to use the Minute Minis directly on the face of a clock. This would aid [students] in exploring the relationship between where the minute hand is pointing and the number of minutes past the hour." Such a design component was seen as essential in supporting student inquiry of the fractional ideas embedded in the tool so that the child could assemble the fractional pieces to compute time.

We continue to see PCK emerge in Casey and Mia's reflections about tasks teachers can pose to students using their tool. They describe teachers familiarizing students with their tools by asking questions like, "How many 30 minute pieces make an hour? How many 5?" Then they go on to consider and describe how a child might use their tool to investigate how many hours it takes to do four homework assignments each of which takes 45 minutes to finish:

> one 30 minute piece and a 15 minute piece to show 45 minutes, then replicating this 3 other times to show 4 × 45. A child might then notice that they can make 2 wholes – hours in this case – using 4 of the 30 minute pieces and 1 whole/hour using the four 15 minute pieces, leading them to an answer of 3 hours.

Concluding Discussion

We hypothesized that an iterative design experience centered on the task of Making and evaluating a physical manipulative for learning mathematics would provide our PMTs an opportunity to leverage and deepen their understandings (Schön, 1992) of mathematics content, curriculum, and pedagogy. Accordingly, we introduced a pedagogically genuine and authentically open-ended task into a Making context, inviting the interplay between the iterative design of a shareable artifact and the application of teacher knowledge in the artifact's development. Casey and Mia, a pair of PMTs who participated in that experience, shared reflections that illustrate their design in development, and leveraged their knowledge of fractions and area to help mediate a bridge between the abstract and concrete representations of time. By supplementing the traditional focus of instruction about time with a concrete representation that facilitates conceptual connections between a clock face and its underlying area properties, Casey and Mia were able to draw on this knowledge to articulate the mathematical richness underlying this manipulative and its possible uses by a child.

As researchers exploring how design experiences might catalyze new possibilities for pedagogical and curricular change, we positioned PMTs as knowledgeable designers of instruction in a space of technological possibilities. As PMTs assumed the multi-faceted role of teachers designing with technology, they created powerful and innovative tools, and their work demonstrated a rich and mature repertoire of knowledge domains that we are not typically afforded opportunities to see (AMTE, 2013). We propose that the identification and advancement of this knowledge, which is essential to effective mathematics teaching, suggest the promise of a Making experience within mathematics teacher preparation. Future research seeks to illuminate the particular features of the configured world (Holland, Lachicotte Jr., Skinner, & Cain, 1998) of the design environment that contributed to these outcomes, and to assess the impact of the experience on PMTs' identities (Sfard & Prusak, 2005) as designers of mathematical instruction.

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EXPLORING PRESERVICE TEACHERS' NOTICING OF RESOURCES THAT SUPPORT PRODUCTIVE STRUGGLE AND PROMOTE EQUITY

Christine A. Herrera
California State University, Chico
caherrera@csuchico.edu

Christina M. Starkey
Southern New Hampshire University
c.starkey@snhu.edu

Shawnda Smith
Texas Woman's University
ssmith115@twu.edu

Hiroko Warshauer
Texas State University
hiroko@txstate.edu

This study examines the development of preservice teachers' (PSTs) understanding of student resources that support productive struggle and promote equity in a semester-long mathematics content course through video analysis. Our qualitative study examines 39 PSTs in two sections of a mathematical content course for prospective elementary teachers. Findings suggest that PSTs could identify a variety of resources leveraged by teachers and students who engaged in productive struggle, such as translanguaging, peer interactions, and mathematical thinking. Our study also examines the role that instructor feedback plays in the development of resource noticing in PSTs.

Keywords: Instructional activities and practices; Equity and Diversity; Teacher Education-Preservice; Teacher knowledge

Teacher noticing of children's thinking has been recognized as an important skill in classroom teaching (e.g. Jacobs, Lamb, Philipp, 2010; Schack, Fisher, et al, 2013). Research has further extended the focus of noticing to how teachers can support students' understanding, particularly as children are struggling to make sense of mathematics (Hiebert & Grouws, 2007; Warshauer, 2015). The support of productive struggle is among the teaching practices that can promote students' conceptual understanding of mathematics (NCTM, 2014). By productive struggle, we mean when "students expend effort in order to make sense of mathematics, to figure out something that is not immediately apparent" (Hiebert & Grouws, 2007, p. 387). While research suggests that productive struggle is an important component of learning mathematics (Warshauer, 2015; Hiebert & Wearne, 2003), supporting productive struggle in the classroom remains a challenge for teachers to implement (NCTM, 2014). Mathematics content and methods courses can provide an ideal setting for introducing these effective teaching practices to preservice teachers (PSTs) as they engage in their own sense-making of the mathematics (Warshauer, 2015; Roth McDuffie et al., 2014).

Prior studies, including ours, examined how support of productive struggle can be introduced and developed in a mathematics content course for PSTs (Warshauer, et al., 2017). Similar to Turner and colleagues' (2012) findings, we observed PSTs, whose interpretations about the mathematics of the struggle were underdeveloped and not based on the students' mathematical thinking. Their focus on student behavior, participation and/or environmental features were what Mason (2008) called a "fragmented awareness" between the culture/community in the classroom and students' mathematical thinking. There is a need to develop PSTs' noticing of productive struggle concurrently with PSTs' noticing for equity (Erickson, 2011). Elementary teachers need to be prepared to teach in diverse classrooms and simultaneously leverage children's mathematical thinking and their cultural funds of knowledge (Turner & Drake, 2016). Prior research on teacher noticing of equity suggests that teachers who notice equitably perceive mathematical and interpersonal activity inextricably linked and notice individual student participation (van Es et al., 2017; Hand, 2012; Turner et al., 2012, Wager, 2014).

In our current study, we focused on introducing PSTs to how teachers draw upon and leverage resources and knowledge bases to support and empower students' mathematical understanding and productive struggle in inclusive settings (Lynch, Hunt, & Lewis, 2018). Using video clips of classroom interactions between teachers and students, we focused PSTs attention to the resources teachers use to support students in productive struggle. By resources we refer to students' funds of knowledge including mathematical, cultural, community, family, linguistic, interests and classroom peers (Roth McDuffie et al., 2014). Supports to develop the skills of noticing and support of productive struggle were introduced to the PSTs using the Productive Struggle Framework (Warshauer, 2015). We further hypothesized that feedback to the PSTs may provide them additional support to their noticing of students' struggles and the resources teachers capitalize upon in subsequent classroom interactions (Roth McDuffie et al., 2014).

Our research questions for this study are:

1. What resources do PSTs notice are made available in a classroom where students are engaged in productive struggle?
2. How do PSTs connect the resources they notice to supporting students' productive struggle?
3. What is the role of instructor feedback on PSTs development of noticing productive struggle and the resources that support productive struggle?

Methodology

This study was conducted in 2017 over a 14-week period at a public, four-year, Hispanic Serving Institution located in a rural area in the western United States. There were 39 participants enrolled in one of two sections of their final mathematics content course for elementary PSTs, taught by the same instructor. The PSTs completed three productive struggle writing assignments (WA). Each WA allowed the PSTs to reflect on a video episode of student struggles that occurred in a classroom setting and that was supported productively by the teacher. The PSTs were to connect the mathematical content of the video to what they were learning in class, examine teacher responses to the observed student struggles, consider what resources the teacher used to support the students, and decide how productive the struggle was for the student(s). For each WA, the instructor gave feedback to each PST about each component listed above. In this preliminary report, we report results from the analysis of PSTs' noticing of resources from WA2 to WA3.

We coded the two WAs using qualitative content analysis (Hsieh & Shannon, 2005). The WAs were coded inductively to identify themes and compare the PSTs' responses to the expert codes that the four researchers had agreed upon were present in the videos. A sample of the codes were re-analyzed by all researchers to ensure inter-rater reliability. Additionally, we analyzed whether the PSTs went beyond the baseline of listing the resources to describing the connections and/or relationship between or among the resources, the student struggle, and the teacher and student moves. Our coding scheme was an adaptation of van Es' (2011) "Framework for learning to notice", ranging from Level 1 (listing resources and general impressions) to Level 4 (interpretations of relationships between or among the resources, teacher moves, students' thinking, and the mathematical struggle). Furthermore, we analyzed whether the PSTs perceived the use of the resource they noticed as a positive, negative, or neutral. Lastly, we analyzed whether there was a change or no change in PSTs' noticing of resources from the WA2 to WA3 and inductively coded the different groups' feedback from the instructor.

Findings

We report preliminary findings for our research questions. Table 1 includes the type of responses that we noted in the respective WAs and the frequency in which they occurred. We found that 72% PSTs discussed at least one resource they noticed at a Level 3 analysis (awareness with focused noticing on resources with evidence and a discussion on how the resources were used in the interactions) or a Level 4 analysis (analyzing the connections and/or relationship between or among the resources, the mathematical struggle, and the teacher and student moves). Of the 29 PSTs who noticed translanguaging (García, et al., 2017), only 9 of the PSTs had a Level 4 analysis and discussed how translanguaging was a tool that allowed students to convey their mathematical thinking and productively engage in different mathematical struggles. 20 PSTs described the use of translanguaging as a positive resource and only 1 PST had a deficit interpretation of the use of translanguaging in a mathematics classroom. The other 8 PSTs provided neutral descriptions of the translanguaging resource and did not elaborate on the impact translanguaging had on the mathematical struggle.

Table 1: Examples of PST Responses in Writing Assignment 2 and 3

Type of Response	Frequency of Noticing (WA2)	(WA3)	Examples in WA2	Examples in WA3
Peer Interactions	27	31	Group discussion Shouting out ideas Class voting	Classroom community Students compare thinking Group work
Communication	5		Using casual language Thumbs up voting	
Manipulatives	9		Using fingers for subtraction Counting on fingers	
Visual	19	7	Use of white board Visual record of student ideas	Writing on white board
Mathematical Knowledge	20	23	Previous knowledge Knowledge of division Math experiences	Use previous knowledge Knowledge of addition
Experiences	10		Field trip experience	

Translanguaging	29		Spanish to build community Switches between English and Spanish
Use of Examples	8		Additional examples
Non-Resource Teaching Practice	6	Teacher as a mediator Teacher questioning	

The instructor gave individual feedback for WA2 by highlighting the key points the students made and wrote questions and comments, if further interpretations and analysis was needed. The instructor also included a rubric with a score for how well the student addressed resources and the connection to mathematical struggle. 8 PSTs provided Level 1 or 2 description of the resources they noticed in WA2 but improved to a Level 3 or 4 interpretation in WA3. The feedback provided to these PSTs was of the following form: "You noticed a resource was peer interaction. How did the teacher use peer interactions in the struggle to make it productive?" (S1). There were 6 PSTs who provided Level 1 or 2 descriptions in both WAs. PSTs in this category received feedback such as: "You mentioned one resource, the students' knowledge, but did not provide evidence from the transcript or relate it to the students' struggles." (S2). 18 PSTs discussed the resources they noticed at a Level 3 or 4 in both WAs and 6 PSTs' noticing of resources declined from WA2 to WA3. Both groups did receive feedback stating, "You did a good job identifying the different resources and how they were linked and supported the mathematical struggled." (S3). Further analysis is ongoing to determine how and what type of feedback contributes to PSTs noticing of resources to support students' mathematical understanding.

Conclusion

Our findings indicate that PSTs in a content course were able to notice a wide range of resources leveraged by teachers and students who engage in productive struggle. In addition, the PSTs show evidence of developing the skills to analyze the relationships between the resources, teaching moves, students' mathematical thinking, and engagement in productive struggle. The predominant resource that PSTs noticed and discussed at the highest level of interpretations was peer interactions. Consistent with Roth McDuffie and colleagues' (2014) findings, most of the PSTs tended to interpret the resource of students' home languages in a classroom at a lower level. What the PSTs noticed and the level of interpretations provided may be connected to the PSTs' experiences in mathematics classroom. For instance, the mathematics content course leveraged many of the resources, such as peer interactions and mathematical knowledge to support the PSTs themselves as they engaged in productive struggle. Therefore, these were common resources that all the PSTs saw incorporated into a mathematics classroom. However, the integration of students' home languages in a classroom was not a common experience for all the PSTs. The majority of PSTs who did notice translanguaging as resource were bilingual themselves and a subset of those PSTs entering or applying to the multiple subject, bilingual authorization program. This could also explain why the PSTs who did notice this resource, described the use of translanguaging as an empowering resource and believed that allowing students to use their full language repertoire was a way to leverage the students' linguistic wealth and make the mathematical content accessible.

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USING LESSON PLAY TO ASSESS PEDAGOGICAL UNDERSTANDINGS IN AN ONLINE MATHEMATICS METHODS COURSE

Diana L. Moss
Utah State University
diana.moss@usu.edu

Dov Zazkis
Arizona State University
dzazkis@asu.edu

This report describes the preliminary results of a study that aims to acquire new insights about using lesson plays for examining prospective teachers' pedagogical understandings in an online mathematics methods course. In a lesson play, prospective teachers imagine that they are teachers and are interacting with a group of students about a mathematical problem. They write a script between a teacher and the students and provide commentary. Pre-post lesson plays allowed us to chronicle the influence of the activities in the methods course on prospective teachers images of teacher-student interactions. We intend to refine the pre-post lesson play method so that other instructors of prospective teachers can more effectively implement this method in their courses.

Keywords: Instructional Activities and Practices, Teacher Knowledge, Assessment and Evaluation

Mathematics methods courses are intended to deepen prospective teachers' (PTs) knowledge of teaching and learning mathematics. This includes improving PTs' interactions with students through probing students' thinking, valuing students' explanations, and encouraging deeper understanding of mathematics. A core issue that arises in methods courses, especially online mathematics methods courses, is how to assess PTs' understanding. Assessing the development of PTs' pedagogical understanding is difficult because methods courses usually do not include PT-student interactions. These interactions typically occur later in a practicum setting. For this reason, PTs are assessed on their knowledge of research literature and their analyses of episodes (readings or videos) of others' teaching. These episodes serve as proxies to help PTs think about their future teaching and might differ from how they envision interacting in their own classrooms. Additionally, since the practicum is often supervised by someone other than the instructor of the methods course, it is difficult to pinpoint which activities completed in the methods course had their intended effects. Thus, it is challenging to adjust the activities within the methods course using an informed approach. In other words, methods courses aim to improve both theoretical knowledge of pedagogy and practical knowledge of how and when to enact that pedagogy; methods courses tend to focus only on the latter.

Purpose

The primary tool utilized in this study, called a *lesson play* (Zazkis, Sinclair, & Liljedahl, 2013), involves PTs writing scripts of imagined teacher-student interactions, envisioning themselves as the teacher-character in these scripts. The novel variation of this approach is to assign two lesson plays around the same mathematical scenario. One lesson play is assigned at the beginning of the online methods course, before relevant course activities [e.g., reading literature on discourse patterns, questioning strategies, or cognitively guided instruction (e.g., Carpenter et al., 2015, Smith, & Stein, 2011)]. The second lesson play is assigned at the end of the course, after these activities. The comparison of the two scripts allows both researcher

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instructors and PTs to assess how PTs think about approaching teacher-student interactions differently in response to the methods course activities. This approach to assessment might facilitate the researcher-instructor with making modifications to future methods course activities to maximize the positive impact on practice. Our central research questions are:

1. What can be learned about how PTs envision teaching using pre-post lesson plays?
2. What can be learned about using pre-post lesson plays as a means of methods course improvement?

The purpose of this project is acquiring new insights about a relatively novel tool for examining PTs' pedagogical understandings in a mathematics methods course. We aim to refine the pre-post lesson play method so that other researchers and instructors of PTs can more effectively implement it in their online and face-to-face methods courses.

Theoretical Framework Lesson Play in Teacher Education

This study builds on Zazkis et al. (2013) notion of lesson play, an imagined teaching interaction in which a script-writer, typically a PT, creates an imagined idealized dialog of an interaction between a teacher and students, imagining herself as the teacher-character. Lesson play can thus be thought of as what Grossman, Hammerness, and McDonald (2009) called an *approximation of practice* as it allows PTs to approximate components of teaching in a setting with reduced complexity. Past studies that utilize lesson plays provide novel insights into PTs' understanding of mathematics (e.g., Zazkis, 2014, Zazkis & Zazkis, 2016), triangulation of some known results (Cook & Zazkis, 2017) and a lens into PTs' conception of pedagogy (e.g., Zazkis & Koichu, 2015; Zazkis, et al., 2013). We are proposing to expand the use of lesson play to be a pre-post lens into pedagogical understanding, which we expect will make it a valuable tool in the assessment of PTs' knowledge of pedagogy, PTs' knowledge of how and where to enact that pedagogy, instructors' development of activities in methods courses for PTs, and as a research tool for those interested in PT education.

Conceptual Framework

Our conceptual framework draws on four iterative stages of designing good mathematical tasks identified by Liljedahl, Chernoff, and Zazkis (2007). Using these four stages, we will investigate lesson play as a tool for improving methods courses and as a lens into how PTs envision their teaching.

1. PREDICTIVE ANALYSIS occurs before using a new task in a teacher education context. In this phase, predictions are made in regard to pedagogical affordances of a task.
2. TRIAL occurs by implementing the task in a teacher education classroom context and "liberating" the affordances.
3. REFLECTIVE ANALYSIS on the task occurs after the trial to highlight new affordances or constraints of the task.
4. ADJUSTMENT of the task, based on the reflective analysis, occurs to improve the task to meet pedagogical goals.

Methods Participants and Context

We began with a pilot of eight PTs enrolled in an in-person methods course to test pre-post lesson plays. The positive results from this pilot catalyzed expanding data collection to an online, asynchronous, mathematics methods course for elementary school teachers (n = 80). This methods course is delivered 100% online for 15 weeks, the first in a series of two required elementary mathematics methods courses offered through the College of Education, and required for students enrolled in the elementary education program. Prior to taking this course, students take two content courses in the mathematics department designed for pre-service elementary education teachers. In the online methods course, students are required to complete readings, watch lectures and supplemental videos, participate in a group discussion, and take a quiz each week. The course is designed to enable PTs to gain deep content and pedagogical knowledge for teaching rational numbers and proportional reasoning.

Data collection took place within an online methods course for PTs at a land-grant university in the Western United States. The data consists of the pre lesson play assignment from Week 2 and the post lesson play assignment from Week 13 of the online methods course.

The larger research project is in process and will proceed in five phases over two years (Figure 1).

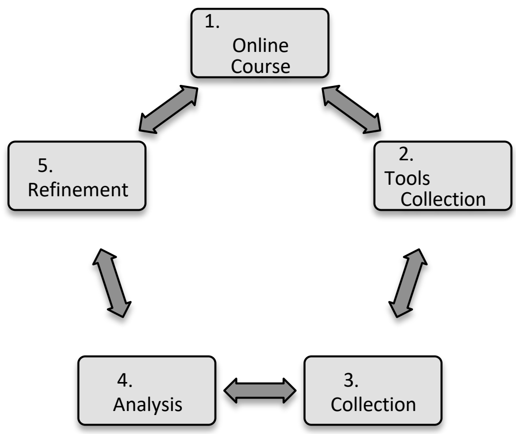

Figure 1: Phases of Data Collection and Analysis

Each phase is described in detail and aligns with the conceptual framework. For the purpose of this brief research report, we share preliminary data from phases 1-4.

Data Collection and Analysis

Phase 1: Analysis of online course (predictive analysis). We analyzed the sequence and content in the modules of the online course and identified places where we can introduce students to the lesson play assignment, the Five Practices for Orchestrating Productive Mathematical Discussions (Smith & Stein, 2011) and the Standards for Mathematical Practice (SMP; CCSSO, 2010).

Phase 2: Building tools for data collection (predictive analysis). We created a pair of matched lesson play assignments that fit within the mathematical backdrop of the methods course and developed questions to guide PTs' commentaries on the lesson play assignments. The purpose of the questions is to determine how the methods course has impacted PTs' thinking about how students learn and how to teach mathematics. For example, a major component of the course is learning the "five practices for orchestrating productive mathematics discussions" (Smith & Stein, 2011) and the SMP (CCSSO, 2010).

Phase 3: Data collection (trial). Using tools developed in Phase 2, the pre-post lesson plays were administered to about 80 pre-service elementary teachers during the researcher-instructor's online methods course. After the PTs completed the post lesson play assignment, they were asked to complete a written assignment to answer reflection questions on similarities and differences that they see between their pre and post lesson plays. These questions included:

- What teaching strategies did you include in your post lesson play that were not present in your pre lesson play?
- How did engaging in the pre and post lesson plays impact how you think about teaching proportional reasoning and ratios?

Phase 4: Data analysis (reflective analysis). We are in the process of identifying repeating themes in participants' scripts, commentaries, and reflections, while acknowledging unique instances. Themes will be organized according to (1) the five practices for orchestrating productive mathematics discussions: anticipating, monitoring, selecting, sequencing, connecting (Smith & Stein, 2011) and (2) the SMP (CCSSO, 2010). We expect additional themes will emerge.

Phase 5. Course refinement (adjustment). We will use what we learned from Phase 4, to refine both the pedagogy related content and the pre-post lesson play assignment with the dual goals of course improvement and more refined data collection. We then repeat Phases 3 and 4 in the next year with the refined methods course activities, lesson play pre-post assignment, and reflection questions to assess the viability of pre-post lesson plays as a course improvement tool.

Results from Pilot

The pre-post lesson plays allowed us to chronicle the influence of the activities in the methods course on how these PTs conceptualized idealized teacher-student interactions. For example, the course involved reading and discussing the five practices for orchestrating productive mathematics discussions (Smith & Stein, 2011), which are actions a teacher can take to manage the direction and progress of an in-class discussion. The interactions between the PTs and methods course instructor revealed that PTs grasped the definitions of pedagogical concepts but not how to apply these concepts meaningfully in a classroom setting. In particular, there were multiple instances where PTs missed opportunities to incorporate particular teacher moves into their scripts, in spite of setting up scenarios where those moves were warranted. This critical information, regarding which discussion practices PTs envision using and in which situation, were gleaned through the lesson play assignment. As such, the pre-post lesson plays provided a new lens for research on PTs' pedagogical understandings. Using lesson plays to assess gaps in PTs' enactment of pedagogical concepts will allow for targeted improvement of the course activities. Thus, pre-post lesson plays might function as a tool for methods course improvement.

Discussion

We expect similar results from the full study of 80 PTs. As this study continues to unfold, we wonder what differences we might see between the pre-post lesson plays in the face-to-face course as opposed to the online course. We expect that our findings will lead to better understanding of developing PTs' knowledge of pedagogy as it relates to teaching mathematics. By studying lesson play as a pre-post lens, we anticipate making significant contributions to mathematics teacher education by developing a tool that instructors can use for refining their online and face-to-face methods courses.

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PRESERVICE TEACHERS' NOTICING IN THE CONTEXT OF 360 VIDEO

Karl W. Kosko
Kent State University
kkosko1@kent.edu

Richard E. Ferdig
Kent State University
rferdig@gmail.com

Maryam Zolfaghari
Kent State University
mzolfagh@kent.edu

Videos of mathematics instruction are commonplace in mathematics teacher education. However, videos' effectiveness in facilitating mathematics preservice teachers' attention is often contingent upon their decomposition and/or approximation of practice. This pilot study examined whether 360 video technology affected the specificity of mathematics attended by PSTs viewing mathematics instruction. Results suggest that viewing 360 videos with a virtual reality headset associates with more explicitly noticed mathematics, while viewing 360 videos without a headset was observed to have similar effectiveness as standard video.

Keywords: Instructional activities and practices; Technology.

Video is commonplace in preservice teacher (PST) education, and has been for the past several decades (Grossman et al., 2009; Sherin & Star, 2011). Use of video in PST education has demonstrated positive effects on PSTs' professional noticing (Jacobs et al., 2010; van Es & Sherin, 2002), and their pedagogical content knowledge (Ball et al., 2008; Brunvard & Fishman, 2007). Video's usefulness in professional training stems from its capacity to present a viable representation of teaching and learning, as well as through application of accompanying approaches to decompose and/or approximate practice (Grossman et al., 2009; Herbst et al., 2016). However, a perennial issue with the use of video, and other similar representations of practice (animations, comics, etc.), is that PSTs often do not attend to key aspects in a teaching scenario with the level of precision that teacher educators desire (Estapa et al., 2018; Gaudin & Chaliès, 2015; Seidel et al., 2010; Sherin & Star, 2011; Stockero et al., 2017; van Es & Sherin, 2002). Stated differently, PSTs are often inconsistent in the quality of their professional noticing because they do not attend to the actions, events, or content that more experienced teachers do.

A recent innovation in video technology has emerged that holds promise for improving aspects of video use in teacher education. The *360 video* format distinguishes itself from traditional video in that it records a spherical view of a scenario, allowing the viewer, rather than the videographer, to select what is viewed in a scenario. In considering the use of this technology in teacher education, we pondered whether a higher sense of immersion translated into improved situational awareness, and thus, improved precision in PSTs' professional noticing. Therefore, the purpose of the present study is to examine whether and how the use of 360 video affects PSTs' professional noticing in the context of elementary mathematics.

Theoretical Framework

Teachers' professional noticing involves identifying aspects in a classroom scenario, relating those aspects to one's professional knowledge and norms, and applying this to reasoning about the scenario at-hand (van Es & Sherin, 2002). Jacobs et al. (2010) defines professional noticing similarly for the context of noticing children's mathematical thinking: attend to children's strategies, interpret children's understandings, and decide how to respond. Much of what is described in the literature on mathematics teachers' professional noticing focuses on *what* PSTs attend to (Schack et al., 2013; Stocker et al., 2017; Teuscher et al., 2017). Since PSTs tend to

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notice different things in a teaching scenario than more experienced teachers (Huang & Li, 2012; Jacobs et al., 2010), and more specific noticings often correspond with improved interpretations and decision making (Jacobs et al., 2010; Teuscher et al., 2017), such a focus in professional noticing research is understandable. This relationship has led to various efforts in improving the specificity of what teachers attend to by examining variations in representations of practice. For example, scholars have compared the specificity of what is visually conveyed in representations by using animations, comic-based representations, video, and written records (Herbst et al., 2013; Friesen & Kuntz, 2018). Although differences in how participants interact with these representations have emerged, such differences do not include the specificity of attending to the content in such scenarios.

To date, the majority of research on representations of practice focuses on variations in how the representation conveys practice. By contrast, we focus on what of practice is conveyable by a representation. We use the constructs of perceptual capacity and embodied interaction to account for this focus. *Perceptual capacity* refers to a medium's ability to allow aspects of the scenario to be perceivable, including but not limited to what is potentially viewable (Ferdig & Kosko, in review). One aspect of perceptual capacity is the idea that a representation can convey a sense of the space in which a scenario occurs (Endsley, 2000; Herbst et al., 2011). Much of the literature on use of 360 videos for professional training focuses on participants' perceived immersion in the context recorded, and has generally found that viewers of 360 videos report statistically significant higher degrees of presence and immersion (Ferdig & Kosko, in review; Harrington et al., 2017; Roche et al., 2017). While perceptual capacity describes a medium's allowance for what is perceivable in a scenario, it does not describe all aspects of interaction with the spatial aspects of a scenario. Scholars studying virtual environments use the term *embodied interaction* to describe approaches that cue the body "to enact certain actions and create physical representations that facilitate conceptual understanding" (Lindgren et al., 2016, p. 174). Cuing the body may be purposeful, or may simply be part of using a specific technological tool (Riva, 2008). In the context of the present study, we consider the use of 360 videos from this latter perspective. Specifically, 360 videos are viewable in two primary ways. Most common is for viewers to watch a 360 video on a screen device (laptop, tablet, phone) and to adjust the perspective either by using a mouse or using their finger, via touchscreen, to adjust left, right, up, or down. Another means of viewing 360 videos is through some form of VR headset (OculusGo, Google Cardboard, HTC Vive, etc.). To adjust the perspective on these videos, viewers move their head left, right, up, or down. Thus, use of the headset better approximates how an individual may move their body if observing a classroom in the field.

To date, we have been unable to locate literature examining the use of 360 videos in the context of teacher noticing. The only studies with empirical data come from Roche et al. (2017) and Ferdig and Kosko (in review). Because 360 video appears to have a higher degree of perceptual capacity, and viewing such videos with headsets may also better support PSTs' embodied interaction, we sought to examine these facets. Thus, the purpose of this pilot study is to examine the specificity of mathematics noticed by PSTs in the context of perceptual capacity and embodied interaction. To fulfill this purpose, the study seeks to answer the following research questions: *Is there a difference in the specificity of mathematics noticed by PSTs when viewing a standard video, 360 video on a laptop, or 360 video with a virtual reality headset?*

Methods

Sample

Participants included 34 PSTs (one male & 33 females) enrolled in an early childhood education program at a Midwestern U.S. institution. Participants were enrolled in their first semester of their senior year; the semester preceding full-time student teaching. Following recruitment, PSTs were randomly assigned to one of three sessions occurring outside of class/program requirements. In each session, PSTs watched a video of a third grade lesson informally introducing the Commutative Property of Multiplication. Participants engaged in the same tasks surrounding the video, with the only difference between conditions being the format of video (standard or 360) and the technological hardware used to view it (laptop or VR headset). In the *standard view laptop* condition, participants viewed a standard version of the video (1280 × 720) on a laptop screen. In the *360 view laptop* and *360 view headset* conditions, participants viewed a 360 video that allowed them to adjust the position of what was viewable in a 1280 × 720 frame. The 360 view laptop and 360 view headset conditions differed in the hardware used to view the 360 video. Those using a laptop were able to adjust the perspective on the 360 video by using a connected mouse to point, click, and move the video view. In the 360 view headset condition, participants each wore an Oculus Go headset and were able to adjust the perspective on the 360 video by moving their head in the direction they wished to look.

Participants in each session completed a 30 minute teacher noticing activity involving three parts. In Part One, PSTs were provided with a synopsis of the task in the video, along with accompanying materials to examine it. The task included a 7 by 8 array (7cm × 8cm) with accompanying Cuisenaire rods, a color-coded length based manipulative (black rods = 7cm, brown rods = 8 cm, etc.). In the video, students were tasked with using only one color of rod to cover the array exactly. After becoming familiar with the task, PSTs wrote predictions of what pivotal moments (moments of interest for the teaching or learning of mathematics) they expected to see in the video. In Part Two, participants viewed the video for the first time and were then asked to type all pivotal moments they had noticed in their initial viewing. In Part Three, participants were asked to view the video a second time and to revise and/or add to their previous list in a new prompt. PSTs were then instructed to select two pivotal moments they believed were ‘most informative for (them) as a future elementary math teacher,’ to describe the most salient part of each pivotal moment, and to share why they selected that pivotal moment. Participants concluded by completing a brief follow-up survey that assessed their perceived immersion and presence. The current paper focuses on initial analysis of written response from Part Three of the noticing activity.

Analysis and Results

PSTs’ written responses, following the second viewing, were analyzed for the specificity participants attended to the mathematics in the scenario. Data included PSTs’ modified list of pivotal moments, as well as their described pivotal moments of focus. This allowed for an initial analysis of attended moments signified by PSTs as significant in some manner. Given that the recorded lesson focused on an informal introduction to the Commutative Property (the property was not formally identified in the video), analysis of PSTs’ mathematical specificity focused on the Commutative Property of Multiplication. The first and third authors analyzed PSTs’ responses developed an ordinal coding scheme (0 = not conveyed; 1 = tacitly conveyed; 2 = implicitly conveyed; 3 = explicitly conveyed). Responses were independently coded before both authors met to reconcile. Prior to reconciling the codes, a Kappa statistic of .651 was calculated, suggesting substantial reliability in the coding process (Landis & Koch, 1977). Descriptive statistics suggest that 24.2% of participants did not describe any aspects related to the Commutative Property. PSTs who *tacitly conveyed* commutativity (39.4%) often described students attempting to fit one set of rods on top of another (i.e., identifying seven rods of 8cm overlapping eight rods of 7cm). PSTs who *implicitly conveyed* commutativity (21.2%) elaborated further by specifying such student actions signaled identifying that 8×7 is the same as 7×8. By contrast, *explicitly conveyed* noticing of commutativity (15.2% of participants) specifically named the property in their written descriptions.

Table 1: PSTs' Level of Noticing the Commutative Property of Multiplication

	None	Tacit	Implicit	Explicit	Total
Standard View Laptop	2 *1.9*	2 *3.2*	4 *1.7*	0 *1.2*	8
360 View Laptop	5 *2.7*	3 *4.3*	3 *2.3*	0 *1.7*	11
360 View Headset	1 *3.4*	8 *5.5*	0 *3.0*	5 *2.1*	14
Total	8	13	7	5	33

Note: Observed values are in regular text and expected values are in italicized text.

We calculated a chi-square statistic to examine differences in participants' specificity of noticing the Commutative Property across viewing conditions (see Table 1). Results suggest that the observed frequencies are statistically significant (χ^2 (df=6) = 18.754, p = .005). This indicates that the relationship between PSTs' mathematical specificity was not independent from the viewing condition. One apparent trend across the three viewing conditions is that in both the standard and 360 view conditions, no participants were observed to explicitly identify the Commutative Property. While all explicit noticing was observed in the 360 view headset condition, none of these participants were observed to convey the Commutative Property implicitly in their noticing. Additionally, participants in the headset condition were observed to have higher than expected tacit noticings than expected by chance, while those in the other two conditions were observed to have lower such frequencies than expected by chance.

Discussion

This study provides a comparison between standard and 360 videos in regards to the specificity of mathematics PSTs attend when viewing representations of practice. Results suggest that PSTs who viewed 360 videos using the Oculus Go headset were more explicit in their noticing of the Commutative Property than participants viewing the same scenario with a standard video, or with a 360 video on a laptop. Thus, this study provides preliminary empirical evidence supporting the use of 360 videos in facilitating teacher noticing of mathematics.

Prior research on teacher noticing has observed improved specificity in professional noticing when engaging PSTs in decomposing or approximating practice (Schack et al., 2013; Stockero et al., 2017; Teuscher et al., 2017), but not when simply responding to a representation of practice (Friesen & Kuntz, 2018; Herbst et al., 2013). As 360 video and VR headset technology evolve and become more commercially available, both the perceptual capacity of such videos and the ability for facilitating PSTs' embodied interaction will also evolve. Future study on the effect of 360 videos should, therefore, consider not only the perceptual capacity of the representation, but the associated embodied interaction that the hardware device may facilitate or limit.

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NEW TEACHERS' UNDERSTANDINGS OF GROUP WORK TASKS VERSUS GROUPWORTHY TASKS IN THE ELEMENTARY MATHEMATICS CLASSROOM

Kathleen Jablon Stoehr
Santa Clara University
kstoehr@scu.edu

Amy M. Olson
Duquesne University
olsona@duq.edu

Recent scholarship in mathematics education has increasingly supported the benefit of providing elementary students with opportunities to engage in groupworthy mathematics tasks in the elementary classroom. This study explores new elementary teachers' understandings of group work tasks versus groupworthy tasks and how they believe the two types of tasks may influence student learning. We highlight how new teachers define group work tasks versus groupworthy tasks, their experiences with groupworthy tasks as students and as preservice teachers, their commitment to offer groupworthy mathematics tasks to their future students, and how they envision supporting students' learning of mathematics through groupworthy tasks.

Keywords: Teacher Education-Inservice/Professional Development, Elementary School Education, Equity and Diversity, Instructional Activities and Practices

Purpose of the Study

All students need "access to high quality mathematics curriculum, effective teaching and learning, high expectations, and the support and resources needed to maximize their learning potential" (NCTM, 2014, p.59). Yet many students are denied equal access to the curriculum as well as have minimal participation in the classroom based on teacher perceptions of their perceived ability. This often results in some students continuously experiencing success while others meet with a series of failures (Cohen & Lotan, 2014; Lotan, 2003). One way to address this critical issue is to create a learning environment that embraces "students' different kinds of problem-solving strategies and intellectual competence" (Lotan, 2003, p.75.) Called "groupworthy" tasks, these activities have the potential to positively impact learning by creating opportunities for all students to experience success (Featherstone et al., 2011; Lotan, 2003). But if teachers are unable to distinguish between traditional group work and groupworthy tasks, the means to provide equitable mathematics learning opportunities for all students is not realized.

Perspective

Many elementary teachers utilize group work as a means to support student learning. Lotan (2003) defines group work as tasks where students "follow clear and detailed procedures to arrive at a correct answer or a predictable solution" (p.72). Often times, students are assigned tasks that can be done individually and do not support group learning. As a result, some students in the group take over and complete the task while other students are denied access to the task.

Featherstone et al. (2011) examined specific ways to offer access to all elementary mathematics students by creating student learning through "*groupworthy tasks.*" Groupworthy tasks are tasks that are complex in nature and offer multiple ways of thinking in order to be solved. Groupworthy mathematical tasks leverage the *mathematics smartness* of all students in mixed-ability groups, thereby providing opportunities for all students to gain deeper mathematical understandings. Groupworthy tasks are designed to mimic real world problems and issues and to be open-ended and challenging. They allow for multiple strategies to be considered when solving them and showcase students' knowledgeable abilities. They require students to work together by assigning different roles or jobs to be completed by the different group members (Featherstone, 2011; Lotan, 2003).

Previous research has not examined new teachers' understandings of group work versus groupworthy work, but such understandings may improve how mathematics methods teachers approach teaching groupworthy tasks. In this study, we explore how new teachers think about these two terms, their experiences with groupworthy tasks, and how they envision implementing groupworthy mathematics experiences for students.

Methodology

The participants in this study were sixteen newly credentialed elementary teachers who had completed a Master of Arts in Teaching degree. The teachers were primarily in their mid-twenties. They had all completed two quarters of elementary mathematics methods as part of their teacher preparation program and had spent an academic year in an elementary classroom. Having been introduced to groupworthy tasks during their mathematics methods courses, the teachers were interested in learning more about how to facilitate this type of teaching in their own classrooms. To meet this goal, they chose to attend a weeklong professional development workshop focused on utilizing equitable mathematics teaching through groupworthy tasks.

Data collected for this study included mathematics tasks, mathematics lesson plans, and individual reflections that centered on creating groupworthy tasks for students. The data was collected over all five days of the workshop. The different types of data were collected to gain an understanding of how the teachers perceived group work tasks versus groupworthy tasks.

We began our analysis with multiple subsequent readings of the data. We then conducted an iterative analysis (Bogdan & Biklen, 2006) to demarcate the phrases that pertained specifically to group work tasks and groupworthy tasks. For each teacher, we identified narratives within their text passages that included key phrases specific to group work tasks and groupworthy tasks. We created tables to illustrate the teachers' understanding of each of the terms. We then looked across the data sources to examine the teachers' experiences as mathematics learners and as prospective teachers with group work tasks and groupworthy tasks. We end our analysis with the teachers describing their rationale for incorporating groupworthy mathematics tasks into their future teaching and the qualities they possessed to support their students' groupworthy learning experiences.

Findings

Table 1 outlines the sixteen teachers' descriptions of group work tasks whereas Table 2 highlights key elements of groupworthy tasks that the teachers identified.

Table 1: Key Phrases Teachers Used to Describe Group Work

Any task where students work or discuss with multiple people	Working with a group on an assignment/task	Working on individual work as a group	Students deciding what they want to do to complete a task	Requires students to complete a certain portion of the work
Students working together as a	Work is usually not evenly split	Some students do the majority of the work &	No shared responsibility	Free for all

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

team to complete a task	nor organized effectively	others do a limited amount	for each other's learning	

Table 2: Key Phrases of Groupworthy Tasks as Identified by the Teachers

Tasks where everyone is learning and increasing the learning for their peers	Tasks where group members are responsible for each other's learning and support each other through the task	Tasks that create learning opportunities for all students	Tasks that involves and engages group members to work collaboratively to achieve a goal	Tasks where everyone gets a specific job, which ensures everyone's participation
Tasks where each member of the group contributes to the solution	Tasks that require collaboration and communication amongst the entire group	Tasks that allow students to help each other to solve a particular task.	Tasks that focus not on simply completing the task but on helping each other understand	Tasks that bring multiple strengths and allow multiple math talents to shine

Experiences with Groupworthy Work as K-12 Students

All sixteen of the teachers reported that prior to entering their teacher preparation program, they had no previous experience or knowledge of groupworthy tasks. However, the teachers all stated that as K-12 students, they had as one teacher described, "done plenty of group work as a student." Another teacher shared that when she engaged in group work as a student "very rarely did my peers try to make sure I understood the task. The focus was always on just completing the task." Upon reflecting on her experiences of group work learning one teacher recalled, "I think I did do tasks in school that could have been groupworthy but they weren't structured and managed in a way that made them so."

Experiences with Groupworthy Tasks During Student Teaching

Six of the teachers reported observing some evidence of groupworthy tasks being utilized in their student teaching placement classrooms. All six shared that the examples of groupworthy tasks they saw tended to be limited to a one-time only experience for students. In addition, they reported of the groupworthy tasks they saw their mentor teachers utilizing in the classroom, the tasks tended to be missing at least one key attribute associated with a groupworthy task. For example, many of the groupworthy tasks did not incorporate roles to ensure that all students in the group had a contribution to make to the final project. All of the teachers stated that they did not see any examples of groupworthy tasks being used during mathematics lessons. The groupworthy tasks they did see were incorporated into social studies lessons.

Characteristics of Groupworthy Tasks

After completing the workshop, the teachers reflected on the features they believed groupworthy tasks should possess. Table 3 outlines their responses.

Table 3: Characteristics Needed to Create Groupworthy Tasks

Tasks that students are excited to solve	Tasks that link to real world connections	Tasks where all students' voices are heard and incorporate different types of students thinking	Tasks that build positive relationships among students	Tasks that require students to persevere and not give up
Tasks that afford the opportunity for students to solve in multiple ways	Tasks that allow students to see that math is everywhere	Tasks that provide ample time for students to learn the concept(s)	Tasks that are engaging and hands-on	Tasks that incorporate all students' ideas

Choosing to Commit to Groupworthy Tasks in Their Future Classroom

All sixteen of the teachers expressed a commitment to provide their students with groupworthy mathematics experiences in their future classroom. They spoke of how groupworthy tasks have the potential to shift the focus from valuing individual learning to focusing and valuing the learning that occurs across group members. By engaging in the different mathematical tasks throughout the week, the teachers also reported how each other's diverse thinking of how to solve the tasks deepened their own mathematics understanding. They also shared that they believed that groupworthy tasks might help to alleviate students experiencing mathematics anxiety, as group members share their mathematical strengths with each other.

Supporting Students' Learning of Mathematics Through Groupworthy Tasks

The teachers shared their teaching strengths that they believed would support the use of of groupworthy tasks in their future mathematics classrooms. Some of these strengths included using groupworthy tasks to listen to all students' mathematical thinking, construct a learning environment where all ideas matter, have the opportunity to facilitate deeper student content knowledge, and thinking "on the spot" if the task needed to be modified. In addition, the teachers believed that they would be able to show students the relevancy of mathematics in everyday life, create hands-on activities to engage their students, build respectful relationships among students, and generate excitement for mathematics learning. The teachers also spoke about modeling the importance of determination and perseverance when learning mathematics.

Conclusion

This study suggests that newly credentialed elementary mathematics teachers have an understanding for how groupworthy tasks have the potential to support student learning and that they envision providing groupworthy mathematics tasks for their students. What remains to be seen however, is to what extent new teachers will embrace groupworthy mathematics tasks with fidelity as they assume the multiple responsibilities of teaching. Data reported in this paper indicate that teacher education programs may want to consider offering more opportunities for preservice teachers to practice groupworthy mathematics tasks. Additionally, it is important that mathematics teacher educators ensure that preservice teachers develop the critical skills to know when they are enacting group work versus groupworthy tasks. An important consideration that this paper highlights is the disconnect between elementary preservice teachers learning about mathematics groupworthy tasks in their mathematics methods course with little evidence of this type of learning occurring in their elementary placement classrooms. This reveals the importance of providing in-service professional development opportunities that teach groupworthy tasks and the necessary critical reflection to enact them well.

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BE BEAUTIFULLY UNCOMFORTABLE: INSPIRING LINGUISTICALLY RESPONSIVE TEACHING OF MATHEMATICS

Mary P. Truxaw
University of Connecticut
mary.truxaw@uconn.edu

Eliana D. Rojas
University of Connecticut
edelasr@gmail.com

This research investigates elementary preservice teacher (PST) education experiences designed to inspire linguistically responsive teaching (LRT) of mathematics. A math methods course involved various opportunities for PSTs to engage in readings, discussion, reflection, and activities related to supporting emerging bilinguals with mathematics. This paper focuses on a lesson designed to promote "beautiful uncomfortableness" by having PSTs experience instruction in Spanish (not the primary language for most of the PSTs) using Mayan numbers (not familiar to PSTs). Data were analyzed qualitatively. A teacher learning cycle to promote LRT of mathematics is proposed and examined.

Keywords: Equity and Diversity; Instructional Activities and Practices; Affect, Emotion, Beliefs, and Attitudes

Objectives/Purposes

Linguistic diversity of students in U.S. schools continues to increase (National Clearinghouse for English Language Acquisition, 2011). Although some might consider mathematics to have limited language demands, research suggests that language is vitally important to teaching and learning mathematics (Moschkovich, 2002, 2013). Further, there is evidence that, overall, U.S. schools are not adequately supporting English learners (ELs) with mathematics (National Center for Educational Statistics, 2015). Clearly, there is an urgent need to prepare mathematics teachers who have beliefs, knowledge, and practices to teach ELs effectively (Lucas & Villegas, 2013). However, because teacher preparation programs have many demands, preparing preservice teachers (PSTs) for linguistically responsive teaching (LRT) may require strategic infusion of experiences (Ewing, 2017; Lucas & Villegas, 2013).

This research investigates experiences infused within an elementary math methods course that were designed to inspire LRT. Throughout the methods course, PSTs engaged in readings and activities related to supporting ELs. This paper focuses on a Mayan numbers lesson designed to promote "beautiful uncomfortableness" by having PSTs experience instruction in Spanish (not the primary language for most of the PSTs) and use of Mayan numbers (not familiar to the PSTs). The lesson was based on an experience that author 1 (A1) had in a Guatemalan school and that was unpacked via cogenerative dialogue with author 2 (A2) (Tobin & Roth, 2005). A teacher learning cycle (TLC) to promote LRT is proposed and examined.

Perspectives

This research is informed by Feiman-Nemser's (2001) professional learning continuum and its application by Lucas & Villegas (2013) to tasks for learning to teach ELs. It is beyond the scope of this paper to explain or apply all components of their work; however, we identify aspects that support our data analysis and a proposed teacher learning cycle (TLC). From among Feiman-Nemser's central tasks of PST preparation, we acknowledge the importance of "analyzing beliefs and forming new visions" (p. 1016). Lucas and Villegas mapped this central task onto "reflecting on and interrogating one's preconceptions about ELL students, language diversity, and the role of other languages in school" (p. 103). We focused on these central tasks because of their potential to help PSTs to shift from deficit thinking about linguistic diversity toward more affirmative, proactive beliefs and LRT practices (Feiman-Nemser, 2018).

Along with drawing from the literature, we build ideas from our own research and experience as mathematics teacher educators, particularly, as we worked to better understand affordances and challenges when the primary language of instruction is not that of some or all of the learners. Our self-study (Truxaw & Rojas, 2013, 2014) helped us to identify a possible ***teacher learning cycle*** (TLC) to enhance LRT (see Figure 1). In the cycle, ***awareness*** (of affordances and challenges) is enhanced through attention to both *academic* components (e.g., literature, research, etc.) and also *experiential* components (e.g., immersion, simulations, etc.). ***Reflection*** on enhanced ***awareness*** can motivate new ***teaching practices*** (that also benefit from *academic* and *experiential* components). Double arrows indicate that movement could *and should* go in more than one direction – ***reflection*** enhances both ***awareness*** and ***teaching practice***. To investigate the TLC in the context of the math methods course, we set up experiences that built from our own and that aligned with ones documented by others (e.g., Ewing, 2017; Gort, Glenn, & Settlage, 2011).

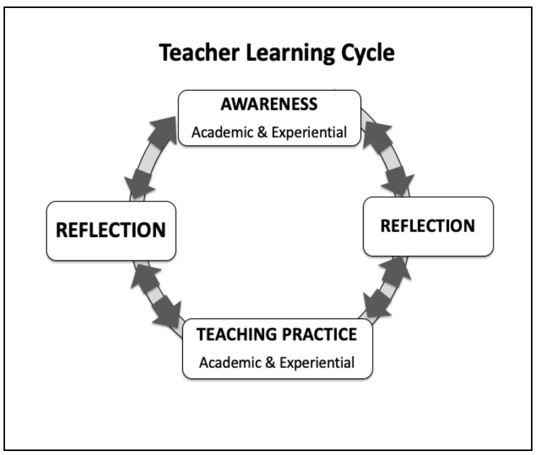

Figure 1: Teacher Learning Cycle

We ask the following research question:
How can LRT be infused within an elementary mathematics methods course?

Modes of Inquiry

Participants were elementary education PSTs (n=122) from three different years (2015, 2016, and 2018) of a math methods course taught by A1 in the semester prior to student teaching. Although the course varied somewhat from year to year, all included readings, videos, reflections, and activities related to linguistic diversity. The focus of this paper is analysis of a Mayan numbers lesson.

Data sources include: selected audio and video recordings of in-class experiences; documentation of readings, activities, and course materials; PSTs' written reflections and work samples; and instructor journals. Qualitative data analysis involved constant comparative methods and thematic coding (Strauss & Corbin, 1990). Selective codes were drawn from our earlier research and from the literature (e.g., Truxaw & Rojas, 2013, 2014; Lucas & Villegas, 2013; Moschkovich, 2013). Open coding allowed additional themes to emerge. Comparisons were made between our self-study learning trajectories and those of the PSTs.

Results and Discussion

We briefly document the focus lesson within the math methods course, noting connections to the TLC. Within the course, the PSTs had experienced readings, activities, and reflection to enhance their *academic* ***awareness*** related to teaching ELs. We recognized that awareness and reflection that were predominantly academic could go only so far in "analyzing beliefs and forming new visions" (Feiman-Nemser, 2001, p. 1016). Infusing powerful experiences, accompanied by reflection, had potential to further enhance ***awareness.*** Therefore, the Mayan numbers lesson was designed to encourage *experiential* ***awareness***. To introduce the lesson, A1 told PSTs that she had had a powerful experience in a second-grade Guatemalan classroom. "I'm going to give you some of the experience I had … If this feels uncomfortable, then I'm doing what I intend to do." (A1, October, 2018). Instruction, slides, and handout were in Spanish. A handout labeled "Numeración Maya," asked students to "Forma el número …" followed by numbers written in Spanish. Many of the PSTs spoke some Spanish (i.e., they were emerging Spanish learners). When the handout, slide, and verbal instructions asked PSTs to "Forma el número trienta y nueve" (form the number thirty-nine), many PSTs wrote "39" on their handouts. A1 then "clarified" that they should "forma el número 39 *usando los numerales Mayas*" (form the numeral 39 using Mayan numbers). After showing the "repuesta" (the answer) using Mayan representations (see Figure 2), A1 provided verbal feedback in Spanish. Next, A1 provided an explanation in Spanish, accompanied by visual representations and Spanish text on a slide (see Figure 3). PSTs were given opportunities to practice with other numbers. A1 circulated throughout the classroom, observing, providing feedback (in Spanish) and asking students to speak only in Spanish. Very few PSTs were able to successfully complete the requested tasks. After the initial lesson, A1 asked students to ***reflect*** about how they felt during the lesson. PSTs shared strong feelings of frustration, feeling unintelligent, and tendencies to shut down and give up. One PST said, "I felt frustrated when you came over and tapped on my paper because I hadn't written anything down. I was like, what do you expect me to do? I don't know how to do it." (See Table 1 for additional PST quotes.) A1 commented, "This is a room of educated, intelligent people … and you felt like shutting down."

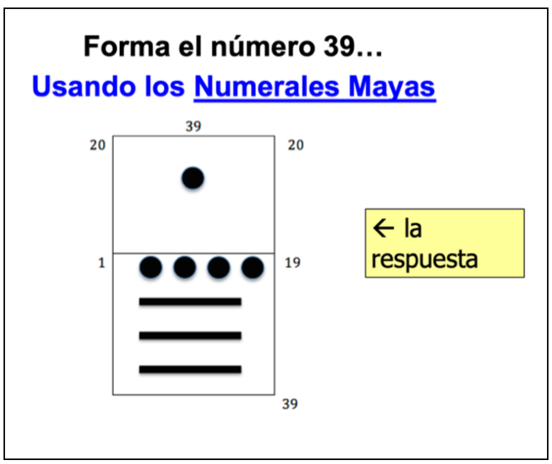

Figure 2: Example "Answer" Slide

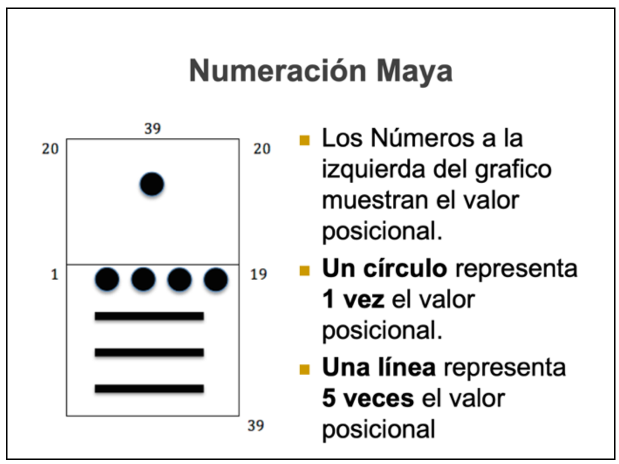

Figure 3: Explanation Slide

Following the group ***reflection*** discussion, A1 provided a worksheet and slides that included Spanish, along with English translations. Additionally, A1 provided more "think time" and encouraged PSTs to speak *in any language* with peers to try to make sense of the ideas. Many more of the PSTs were able to generate Mayan representations. The class then ***reflected*** again about their experience as learners and also about **teaching practice** that might support emerging bilingual students. Further discussion, along with recognizing challenges, acknowledged that students may be held back from learning important concepts and skills because of language. Discussion included focus on keeping cognitive demands high while providing linguistic supports (e.g., Cummins, 2000). Then, small groups worked together to generate and share posters of ideas to support linguistically and culturally relevant ***teaching practice***. One poster included a phrase that inspired the title for this paper, "Be beautifully uncomfortable." Table 1 shows examples of themes and quotes from PSTs that are aligned with our earlier work and the literature. Additional findings will be shared during the presentation.

Table 1: Themes and Evidence from Mayan Numbers Lesson

Challenges	Example PST quotes
Lack of opportunity to reason in one's primary language can hinder sense making.	"I was thinking, 'if she would just put the instructions in English, I'd be able to do this…' A lightbulb went off – okay, if teachers were doing this with ELL learners, it would be so much easier if they could put it in Spanish."
One is likely to appear (and feel) less intelligent than one really is.	"I was disengaged because I was frustrated and confused." "It just makes me want to hug my ELL students."
Academic language is more challenging than conversational language	"Being someone who spoke Spanish, … I was fine. But then … everyone was like, what is it if *yours* is wrong? … I don't know Mayan numbers."
Working to understand even basic instructions can be challenging.	"I thought it was really powerful because it's what a lot of English language learners experience. … it's hard to put into words how someone feels ... I was really confused."
Asking meaningful questions in a second language can be difficult	"I didn't understand how to ask you in Spanish, 'Can you teach me how to do Mayan numbers?'"

Note: PST quotes are from 2015, 2016, and 2018.

Conclusions

This research demonstrates potential for informing teacher preparation programs about strategic infusion of experiences to promote LRT. The TLC provides a scaffold for considering the importance of interactions of ***awareness***, ***reflection***, and ***teaching practice*** – including both academic and experiential opportunities. Indeed, experiences such as the Mayan numbers lesson, scaffolded with the TLC, show potential to promote "analyzing beliefs and forming new visions"(Feiman-Nemser, 2001, p. 1016) and "reflecting on and interrogating one's preconceptions about ELL students, language diversity, and the role of other languages in school" (Lucas & Villegas, 2013p. 103). These are small, but important steps in preparing linguistically responsive teachers of mathematics. Long journeys begin with small steps.

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A CHANGING TIDE? EXPLORING MATH IDENTITY IN PRESERVICE ELMENTARY TEACHERS

Stephanie B. Purington
University of Massachusetts Amherst
spurington@educ.umass.edu

*The math identity that teachers hold can influence how they teach and the math identities that their students develop. This study on preservice elementary teachers (*N *= 13) describes the feelings the participants have toward teaching math, how they developed those feelings, and how they view the identity of "not a math person." Unlike previous studies, the participants in this study expressed very positive feelings about math and about teaching math. Feeling competent in math, having a strong "math for teaching" course, and seeing math as relevant to daily life were the key components to their positive feelings. Although they know that people identify as "not a math person," they believe that no one is born unable to learn math.*

Keywords: Affect, Emotion, Beliefs, and Attitudes; Teacher Education-Preservice

In our culture, it seems that it is perfectly acceptable to claim that one is "not a math person." While the definition of "not a math person" likely varies from person to person, it seems to involve feeling anxious about math, that math is something that one is not destined to master, and not performing well in math. There appears to be a societal acceptance, and even expectation, that some people are born to be math people and others are not. What happens, then, when someone who has self-identified as "not a math person" becomes an elementary school teacher who must teach math?

The purpose of this pilot study was to explore how preservice elementary teachers identify as math doers and teachers through the following research questions.1) How do preservice teachers describe their feelings about and their readiness for teaching mathematics? 2) What experiences do preservice teachers report as instrumental to their feelings about math and readiness for teaching math? 3) How do preservice teachers describe what it means to have an extreme math identity – to be a "math person" or "not a math person?"

Conceptual Perspective and Related Research

This study uses the frame of mathematical identity in looking at how preservice teachers (PSTs) see themselves as learners, doers, and teachers of math. Gee (2000) proposes using identity as an analytic lens for educational research, and he looks at identity as "being recognized as a certain 'kind of person'." How preservice teachers develop their math identity might depend on how quickly math comes to them, how their schools have defined their capacity or abilities, how they have engaged with others about math, or how they see the community of elementary school teachers interact around math. They might have been expected to be passive receivers of mathematical knowledge and so disengaged from mathematics (Boaler & Greeno, 2000). Their identity might also be influenced by the math-related narratives told about them by their teachers, parents, or peers (Sfard & Prusak, 2004).

Gender has been indicated as one factor in the development of a negative math identity, and college students may select majors that reinforce this identity. Most elementary school teachers are female, and elementary education majors have been found to have the highest rate of math anxiety of any college major (Betz, 1978; Baloglu & Kocak, 2006). Palmer (2009) found that only 5% of her female preservice participants came into her math course with a positive math identity. Preservice teachers (PSTs) who have math anxiety often cite their own elementary and secondary teachers as the source of their discomfort with math (Bekdemir, 2010; Jackson & Leffingwell, 1999). Research has also indicated that girls with strong gender role affinity can "catch" math anxiety, which can be part of the "not a math person" identity, from their female teachers (Beilock, Gunderson, Ramirez, & Levine. 2010).

Mathematics anxiety, a fear of all that is related to mathematics (Hembree, 1990),has been shown to reduce a PST's expectations for student mathematics achievement (Mizala, Martínez, & Martínez, 2015), and studies have shown that teachers tend to underestimate girls' math abilities starting in the first grade, and often grade those girls more harshly than the boys in the class (Cimpian, Lubienski, Timmer, Makowski, & Miller, 2016; Lavy & Sand, 2015). These authors further state that this sort of underestimation and poor feedback can lead to waning confidence and engagement, which leads to reduced learning and performance. That in turn leads to forgoing higher level math classes and whole areas of professional exploration.

Those who do not attempt or succeed at higher level math courses are left out of many professions and career paths. The U.S. Department of Labor estimates that in 2018, nine out of the ten fastest growing occupations requiring a bachelor's degree will require the employee to have had significant science or mathematics training (Hill, Corbett & St. Rose, 2010). It is crucial, then, that we better understand the development of math identity in preservice teachers in order to provide a path for them to construct a positive "math person" identity.

Methods

This study was intended to describe the responses of preservice elementary teachers about their readiness and feelings about teaching math. The participants were students (N=13) enrolled in a one-year, dual master's degree and elementary education certification program.

All participants took part in one-on-one, semi-structured interviews prior to starting their student teaching or taking a mathematics methods course. Interview questions were intended to determine how each participant felt about teaching mathematics and how well prepared they felt to teach that subject.

In the interviews, participants were also asked about their level of math anxiety, using the Single-Item Math Anxiety Scale (Núñez-Peña, Guilera, & Suárez-Pellicioni, 2014). Each participant also shared with me a math autobiography, describing their K-16 experiences with math and the areas in which they were most and least confident, written at the beginning of September for their course in Elementary Math Methods.

I analyzed the autobiographies and interviews using thematic analysis (Braun & Clark, 2006), first reading all of the responses to familiarize myself with the data, then creating some initial codes based on my research questions and the topics I was noticing in the responses, and noting where those ideas were mentioned or highlighted. After grouping the similar comments, I looked for patterns within the grouped responses, and coded for themes.

Findings

Analysis of both the interviews and math autobiographies showed that most of the participants had quite positive math identities, contrary to what was found in the previously cited study by Palmer (2009), though many still expressed struggles with higher-level mathematics. In this section, I address their attitudes toward being teachers of math, then focus on the experiences that they report as affecting those attitudes, based on the themes that emerged through analysis across participants.

Teaching Math

It was not surprising that Molly, who majored in math, said that she "felt great about being a math teacher." More surprising was that all but one of the participants gave an answer similar to Molly's. The only participant who did not express excitement about teaching math was Marie, who felt that she did not have the background experiences with children that most of her peers did. Compared to the other subjects, she noted that she felt "a little better" about math.

All of the participants also reported feeling well prepared to teach math, even before taking additional coursework on methods for teaching mathematics or starting their student teaching experiences. Claire, who was homeschooled for most of her early education, noted that "Taking [specific undergraduate math courses for education majors and minors] was a great course in understanding that material and not just teaching kids how to get the answer, but how to understand the math." Molly mentioned that those same courses had led to her feeling very prepared to teach.

Based on the autobiographies and the interview responses, all 13 participants' identities toward teaching math indicated a positive disposition. In order to gain more understanding of their feelings of readiness, I next explored the factors that they said contributed to their feelings about doing and teaching math.

Instrumental Experiences: Competence and Relevance

Competence and relevance were the two ideas mentioned most often when I asked the participants about what led them to feeling so positively about math and teaching mathematics.

Many of the participants mentioned feelings of competence with math, especially at the elementary level, in explaining their positive inclination toward teaching math.

In contrast, several mentioned feeling competent with math up to a point, either high school or college, and then losing that confident identity.

When participants talked about interest in mathematics, it was generally in the context of the math being relevant or having a utilitarian function Ten out of the thirteen participants noted that relevance was an important component in understanding or feeling competent in mathematics. The participants also talked about how keeping math relevant affects elementary school students, and how they want to teach those students.

What it Means to Be, or Not Be, "A Math Person"

Since none of the preservice teachers identified as "not a math person" or even "a math person," I asked them what they thought a student might have experienced or be thinking if they told the teacher "I'm just not a math person." I also asked if they thought that some people were born to be better at math than other people, a common refrain when discussing "math people."

Eight out of the thirteen participants thought that poor teaching quality was a component leading to "not a math person" beliefs or to strong negative math identity. Seven participants mentioned lack of effort on the part of students as contributing to the claim of "not a math person." Most participants felt that people might have some innate quality that makes math learning easier, but that everyone can learn math if they are willing to put in the effort.

I asked the participants what they would say to the student who claimed to be "not a math person," and many of them replied, as Zoe did, that "there is no such thing as a 'math person'." She went on to add that she would tell them "there's lots of different ways to learn math and some people learn it quickly and some people don't, and that's okay."

Because math anxiety has been linked to identifying as “not a math person” (Nosek & Smyth, 2011), I asked participants to rate their math anxiety on a scale of one (none at all) to ten (extremely high). Responses indicated that the PSTs in this study do not suffer from math anxiety.

Discussion

Prior studies (e.g. Cribbs, et al., 2015; Alghazo, McIntyre, & Alghazo, 2013; Chen,McCray, Adams, & Leow, 2014; Uusimaki & Nason, 2004; Bekdemir, 2010) and personal experiences had suggested that many or most of the preservice teachers would claim to be not “math people,” with the related negative feelings and lack of readiness to teach math. Their responses were largely in opposition to that notion. Not one of them expressed hating math or having a lot of math anxiety, and they were almost unanimous in their positive responses about teaching math. The responses given by the participants in the interviews were confirmed in their math autobiographies, which they could not have anticipated I would see. Whether this cohort is an anomaly or a new normal remains to be seen.

As they talked about their readiness to teach math, many of the participants commented that they felt very competent with elementary-level math content, and so felt comfortable with teaching that content. Even those who did not have positive experiences in their own K-12 schooling felt that they possessed the necessary concepts and skills to teach elementary mathematics. The idea that knowing content means readiness for and competence in teaching is common in preservice teachers (Chen et al., 2014), but there is a significant difference between content knowledge and pedagogical skills for teaching content (Ball, Thames, & Phelps, 2008; Shulman, 1986). While the PSTs were excited for their practicum experiences so they could learn about classroom management and teaching techniques, they would likely also find that their pedagogical skills are still developing, and would find that they needed more skills to be the excellent teachers they hoped to be.

Many of the participants identified specific math content courses designed specifically for those planning to teach elementary school as having a strong impact on their readiness to teach math. Several mentioned that they wish they had learned math in this conceptual way in their primary schooling, and that they planned and hoped to teach that way in their own classrooms. The shift away from procedural teaching and learning may have allowed the participants to develop a new sense of agency around mathematics and a more robust math identity (Boaler & Greeno, 2000)

Contrary to my expectations, none of the participants claimed an extreme math identity as “not a math person” or even “a math person.” They were unified in their response that there is no such thing as “a math person,” in the sense that they do not believe that some people are born to be mathematically capable and others are not.

Conclusion

So, is the tide turning for math identity in prospective teachers? While it is exciting to have this cohort of prospective teachers feeling so positively about math, many questions remain about the causes of their feelings and the impact of their positivity. Has the K-12 math education community changed instruction and expectations in a way that encourages a more positive math identity, or is this an anomalous cohort of preservice teachers? A larger study of preservice teachers that included more regions and institutions would be needed to begin to answer that question. Does the positive math identity, positive feeling, and sense of readiness to teach math that these preservice teachers claim to hold mean that they will be stronger math teachers and less likely to trigger math anxiety or negative math identities in their students? How much will the realities of teaching affect the confidence of the novice teacher, and what might it take to repair any losses in their positive math identity? A longitudinal study of their classroom practices and the effects on their students would be necessary to address those questions.

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"YOU CAN'T TEACH AN OLD DOG NEW TRICKS": INVESTIGATING A NEW PRESERVICE TEACHER LEARNING MODEL

Jamie S. Pyper
Queen's University
pyperj@queensu.ca

John Bosica
Queen's University
j.bosica@queensu.ca

Stephen MacGregor
Queen's University
stephen.macgregor@queensu.ca

Many issues co-exist for preservice teachers' learning, such as the incredible amount of mathematics in the curriculum, the challenges of acquiring and teaching 21st century skills and managing inclusive education issues, the diversity of students' abilities, and the increasing knowledge of 'best-practice' pedagogical strategies. A Problem-based Learning (PBL) model for preservice teachers models and aligns with classroom practice and begins with an identified 'problem of practice'. This mixed methods study captures significant changes in secondary school mathematics preservice teacher experiences and explores task implementation over four consecutive PBL cycles by examining preservice teacher beliefs, teacher efficacy, journals from tutors, recordings of PBL task meetings, and preservice teacher thoughts.

Keywords: Teacher Education-Preservice, Teacher Knowledge, Teacher Beliefs

Introduction

Consider three influences on the efforts to improve the teaching and learning of secondary school mathematics: the diversity of preservice teacher backgrounds, student diversity and needs, and the socio-global expectations and experience of being a 21st century learner. These influences often cause tensions within the preservice teacher's own learning. First, there is the intellectual (mathematical) and social diversity of secondary school mathematics preservice teachers that affects their understanding of mathematics and teaching. Second, managing inclusive education issues and the diversity of secondary school students' abilities, personal life histories, and motivation is more than just teaching mathematics, while grappling with the increasing knowledge of 'best-practice' pedagogical strategies. And third, preservice teachers have the challenge of acquiring 21st century skills for themselves while learning how to promote these skills for their students.

Mathematics educators may implement a constructivist learning environment, but often feel they are dragging preservice teachers into some kind of thinking rather than helping them engage in thinking. As one preservice teacher previously stated upon leaving on the last day of class, "I think I learned some things from your course, but you know what they say about teaching an old dog new tricks…", implying she was going to teach in the same transmission manner she experienced as a learner. A Problem-based Learning (PBL) model for preservice teacher learning models and aligns with classroom practice and an inquiry model of inservice professional learning in collaborative learning communities. Beginning with an identified 'problem of practice', PBL inherently answers the questions 'why are we learning this?' 'how is this relevant to my teaching in a classroom?' The research questions for this study are,

1. What is the effect of a PBL professional learning model on secondary mathematics
preservice teacher attitudes and beliefs towards their teaching and students' learning?

2. What is the nature of their teacher efficacy before, during, and after learning in a PBL professional learning model?
3. What is the nature of their mathematics beliefs and problem-solving beliefs before, during, and after learning in a PBL professional learning model?

Literature

This study of PBL as a mathematics teacher education pedagogy draws on literature that examines problem-based learning, problems and problem solving in the context of mathematics and secondary school mathematics teaching and learning, and teacher efficacy and its sources that influence teacher practice.

Problem-based Learning

PBL is a way of learning that incorporates small groups of learners working together, and experts (called tutors) working with the small groups, on relevant and realistic problems the learners might encounter in their professional practice. PBL is not about creating a realistic environment that replicates the learners' professional practice environment, but creating an authentic environment in which learners can develop the necessary knowledge and skills to be successful in their professional practice. The curricula for PBL is designed to "provide students with guided experience in learning through solving complex, real-world problems" (Hmelo-Silver, 2004, p. 239). Five goals underlie PBL's design to help the learner: "Construct an extensive and flexible knowledge base; Develop effective problem-solving skills; Develop self-directed, life-long learning skills; Become effective collaborators; and Become intrinsically motivated to learn" (Hmelo-Silver, 2004, p. 240).

Research on PBL has shown a number of positive and valuable outcomes from this style of learning (Strobel & Van Barneveld, 2009). For example, in comparison with control groups of students not in PBL, learners in PBL can show greater self-regulation and problem-solving skills (Seyan, 2016), as well as improved perceptions of problem-solving ability (Temel, 2014); increased creative thinking ability, self-regulation learning skills, and proficiency in self-regulation (Yoon, Woo, Treagust, & Chandrasegaran, 2014); and an increase in academic achievement (Gunter & Alpat, 2017).

PBL exists in undergraduate and graduate learning. McMaster University in Hamilton, Ontario is widely accepted as the first formal use of PBL in a university setting, starting PBL in its school of medicine in 1965. PBL exists in other notable universities, such as Maastricht University in Maastricht, the Netherlands in undergraduate learning since 1976. PBL has been employed as a learning strategy for graduate or professional school learning, for example, with prospective chemistry teachers, secondary school science teachers, preservice physics teachers, law students, and with a cohort of elementary school preservice teachers in a mathematics methods course at the University of British Columbia. At this point, there does not appear to be a PBL program running for secondary school mathematics preservice teachers.

Problems and Problem Solving

Problem solving is a cognitive process employed when an answer or solution is desired for a problem that does not have an apparent, obvious, or immediate strategy or method. This cognitive process is expressed as goal-directed behaviour as the problem solver selects and applies strategies to achieve the goal of solving a problem (Mayer & Wittrock, 2006). With mathematics, problem solving is valued as a way to increase students' understanding of mathematics (Hiebert & Wearne, 2003), where understanding mathematics means to know about facts and procedures, to know why they exist, and to know when to use them (see ‘knowing-to’ in Mason & Spence, 1999).

There are two key characteristics of problems—routines and structure (Mayer & Wittrock, 2006; Hollingworth and McLoughlin, 2005). Creative thought and critical thinking are required to figure out what can be done to solve non-routine problems, and with ill-defined problems the data is not complete or readily available. Any procedures or steps could be taken and the decisions one makes while problem-solving may not appear to converge towards a solution. Problem design must be completed carefully to consider the many aspects of the intent of the course, the problem, and the capabilities of the students.

Teacher Efficacy

Teacher efficacy, in the context of the professional practice of teaching, emerges from social cognitive theory (Bandura, 1997) and one’s self-efficacy beliefs. Teacher efficacy is a belief a teacher possesses concerning his/her/their capabilities to perform in a way to affect student learning outcomes. Teacher efficacy as used by Tschannen-Moran and Woolfolk Hoy (2001) illustrates the classroom contextual nature of teacher efficacy; their teacher efficacy scale has three subscales: instructional strategies, student engagement, and classroom management.

Teacher efficacy is relatively static, although it can be malleable and change given a gestalt shift or important experience that causes a conscious moment of critical thinking and reflection. Teacher efficacy has been shown to be a powerful construct. In conjunction with social learning theory (Rotter, 1982) and internal and external locus of control (the influence one attributes to achieving a particular outcome), teacher efficacy has been shown to improve student achievement, student motivation, and teacher enthusiasm, commitment, and classroom practice.

Given an appreciation for the impact teacher efficacy has on teacher practice, and on student learning, research is recently focusing on the activities and experiences that might impact teacher efficacy. Exploring these activities and experiences that become the sources to teacher efficacy becomes the foundation to understanding what might be done to improve professional learning.

Theoretical Framework

Teacher cognition (Borg, 2015) is a key pillar to the theoretical framework for unpacking and understanding teacher learning in relation to the mathematics teaching and learning context. Teacher cognition consists of teacher knowledge and beliefs. Teacher knowledge is the present and available knowledge a teacher has at a given time, and Shulman’s (1986) description of teacher knowledge as the body of content knowledge, pedagogical knowledge, and pedagogical-content knowledge, is the seminal articulation of teacher knowledge. Teacher beliefs is being more narrowly defined as teacher efficacy and its sources. Teacher beliefs consist of the evaluation and judgement individuals make that affects the way they interact with the world around them.

Methodology and Methods

A sequential four-phase mixed methods study is used to capture insight into preservice teacher learning relevant to a PBL experience as well as insight into the programmatic implementation of PBL as a learning model within a non-PBL teacher education program. The primary participant group comprises 45 preservice teachers, two-thirds female and one-third male or other gender expressions, in a post-undergraduate secondary mathematics education methods course, at one mid-sized Canadian university. The secondary participant group comprises five PBL tutors (school-board teachers, graduate students, and the course instructor).

Data was captured four times: pre- and post-learning in an introductory learning block consisting of instructor-led workshops, seminars, and collaborative classroom activities (data collection times 1 and 2); and pre- and post-PBL learning block (data collection times 3 and 4). For the preservice teachers, quantitative data came from the 12-item short form of the Teachers' Sense of Efficacy Scale (TSES) (Tschannen-Moran & Woolfolk Hoy, 2001); 18 items from the Beliefs About Mathematics Problem Solving (BMPS) questionnaire (Kloosterman & Stage, 1992) from the Effort, Understanding, and Usefulness subscales; and two ranking questions pertaining to teacher concerns and teacher orientation to teaching. The qualitative data came from short answer questions about (a) concerns preservice teachers expressed at each time, and (b) preservice teachers' perceived program contributions to their learning. Qualitative data collection is ongoing for focus group meetings of PBL tutors.

Data analysis for this study is ongoing. Quantitative analysis will include descriptive statistics, reliability tests, confirmatory factor analysis of both the BMPS and the TSES, and a two-way mixed ANOVA of time in the course versus teaching subject to explore change over time of the average scale and subscale scores from the TSES and the total scores from the three scales in the BMPS. Qualitative analysis will follow a general three-step constant comparison analysis. All data will be examined for relationships across data types that provide details to confirm quantitative results and explain qualitative themes using a convergent parallel design (Li, Marquart, & Zercher, 2000).

Conclusions and Educational Significance

At the time of writing this proposal, data is being collected, and analysis will be complete by the end of summer of 2019. Full results and conclusions will be complete by October 2019 in time for the conference. Preliminary analyses have shown all scales to exhibit an acceptable level of internal consistency (i.e., Cronbach's alpha of .7 or greater). Several statistically significant findings suggest changes in preservice teachers' problem solving beliefs and their teacher efficacy: (a) an increase in the Understanding subscale of the BMPS, $F(3, 129) = 3.866$, $p = .01$, partial $\eta^2 = 0.08$, between times 2 and 4 (which includes practicum and curriculum course work) and between times 3 and 4 (which is over the PBL section of the course); (b) an increase in the Usefulness subscale of the BMPS, $F(3, 129) = 4.664$, $p = .01$, partial $\eta^2 = 0.10$; and (c) an increase in teacher efficacy between times 3 and 4, $F(3, 129) = 3.350$, $p = .02$, partial $\eta^2 = 0.07$. There is clearly something happening with preservice teachers' thinking about problem solving and their teacher efficacy between time points 3 and 4 (i.e., before and after the PBL learning block), as we see positive growth in both teacher efficacy and problem-solving beliefs. While we are not forwarding claims of causality, anecdotal insight from preservice teachers is very positive. Comments made to the PBL tutors and those recorded during the concluding meetings of each PBL task indicate preservice teachers gained valuable learning about mathematics teaching and learning, and comments from the PBL tutors indicate they are observing noticeable professional discourse and learning gains from the preservice teachers.

The 2005 OECD report, "Teachers Matter," states teachers' roles are changing and that teacher learning needs to (a) be evidence based, (b) connect preservice with inservice learning, and (c) align with school needs. Recent research on preservice teacher learning echoes this call for inquiry-based education models and alignment with the professional learning community models of inservice teacher professional development that is classroom practice-based and that begins with an identified 'problem of practice'. PBL offers a different learning model, one authentically articulated by teacher cognition and connected to mathematics problem solving and the problem solving of teachers' daily professional practice. PBL also meets more, if not most, of preservice teachers' learning needs, is authentically inclusive for all students' learning, and promotes the principles of quality learning and teaching of mathematics.

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AN UNDERGRADUATE PRESERVICE TEACHER'S MATHEMATICAL AND PEDAGOGICAL CONTENT KNOWLEDGE OF CONDITIONAL PROBABILITY

Kaitlyn Stephens Serbin
Virginia Tech
ksserbin@vt.edu

Jesse L. M. Wilkins
Virginia Tech
wilkins@vt.edu

It is essential for preservice teachers (PSTs) to develop mathematical knowledge for teaching to prepare for their careers. This study investigated an undergraduate mathematics PST's mathematical and pedagogical content knowledge of probability. We specifically focused on a PST's common content knowledge, knowledge of content and students, and knowledge of content and teaching in the context of probability (Ball, Thames, & Phelps, 2008). We explored how a PST understood concepts related to probability, anticipated and interpreted student reasoning, and used evidence of student thinking to inform her pedagogical response to the student.

Keywords: Teacher Knowledge, Probability

Several tasks performed by mathematics teachers, such as anticipating and responding to students' thinking, designing tasks, and fostering discussion in the classroom, require them to use their mathematical knowledge. It is essential for teachers to hold both mathematical knowledge and pedagogical knowledge and intertwine them in their teaching (Ball, 2000). In this study, we explored an undergraduate mathematics PST's mathematical and pedagogical content knowledge of probability. We investigated this PST's Common Content Knowledge, Knowledge of Content and Students, and Knowledge of Content and Teaching (Ball, Thames, & Phelps, 2008).

PSTs commonly hold the same incorrect conceptions of conditional probability that secondary students commonly hold, including the representativeness heuristic and the Falk phenomenon (Kahneman & Tversky, 1972; Nabbout-Cheiban, 2017). We explored the conceptions that an undergraduate PST held regarding conditional probability, and investigated her occasional use of these incorrect conceptions. We also explored how this PST anticipated students' use of these conceptions and developed pedagogical approaches for helping students better understand conditional probability. These research questions were addressed: How does a PST reason as she performs conditional probability tasks? How does a PST anticipate and interpret students' reasoning on probability tasks? How does a PST respond to student thinking?

Theoretical Framework and Literature Review

We focused on exploring three of Ball et al.'s (2008) Mathematical Knowledge for Teaching constructs: Common Content Knowledge (CCK), Knowledge of Content and Students (KCS), and Knowledge of Content and Teaching (KCT) regarding probability. Several other studies have examined PSTs' understanding of probability (e.g., Nabbout-Cheiban, 2017; Koirala, 2003; Carter & Capraro, 2005; Maher & Muir, 2014). Some PSTs struggle to reconcile their intuitive informal probabilistic reasoning with their formal mathematical knowledge learned in undergraduate courses (Koirala, 2003). Some PSTs exhibit incorrect probabilistic conceptions commonly held by secondary school students. In particular, PSTs often exhibit incorrect probabilistic conceptions regarding the use of the representativeness heuristic (e.g., Nabbout-Cheiban, 2017; Carter & Capraro, 2005; Maher & Muir, 2014). One who uses the *representativeness heuristic* "evaluates the probability of an uncertain event, or a sample, by the degree to which it is: (i) similar in essential properties to its parent population; and (ii) reflects the salient features of the process by which it is generated" (Kahneman & Tversky, 1972). Furthermore, some PSTs exhibit incorrect probabilistic conceptions regarding conditional probability, such as the Falk phenomenon (e.g., Nabbout-Cheiban, 2017). The Falk phenomenon occurs in the context of finding probabilities of a conditioned event that occurs before its conditioning event (Shaughnessy, 1992). Students and PSTs often struggle with the somewhat counterintuitive idea that the probability of the first outcome can depend on the second outcome.

PSTs should be aware of conceptions and intuitions potential students may hold regarding probability. These skills are a part of KCS, which is knowledge of mathematics combined with knowledge of students' conceptions of mathematics (Ball et al., 2008). Maher and Muir (2014) investigated PSTs' ability to identify errors in students' responses to probability tasks and explain what those students could be thinking in the task. PSTs in their study could identify students' errors, but they struggled to appropriately explain why students made those errors.

PSTs should learn how to address students' conceptions (whether correct or incorrect) in the classroom. This skill is an aspect of KCT, which combines knowledge of mathematics content and teaching practices. Watson, Callingham, and Donne (2008) found that some teachers exhibited ease in predicting hypothetical students' ideas on probability tasks, yet the teachers had difficulty in describing suggestions for responding to students' ideas and designing instructional interventions. Chernoff and Zazkis (2011) also found that PSTs struggled with suggesting appropriate pedagogical responses to hypothetical students' answers to probability tasks.

Methods and Procedures

The PST, Lauren, participating in this study was a senior undergraduate majoring in mathematics on the math education track. The participant self-identified as a White female. She was enrolled in a course on mathematics for teachers, in which she learned probability concepts. This course, however, did not address the representativeness heuristic or the Falk phenomenon.

A deck of playing cards has 52 cards and four suits: Spades, Clubs, Diamonds, and Hearts. Spades and Clubs are black cards, and Diamonds and Hearts are red cards. Each suit has 13 cards: Ace, 2, 3,4, 5, 6, 7, 8, 9, 10, Jack, Queen, and King.

1. What is the probability that you will draw a red card, then a black card, then a red card?
2. What is the probability that you will draw a 4, given that the card has an even number on it?

If a fair coin is tossed, the probability that it will land heads up is ½. Let H represent the result of a coin landing heads up and T represent the result of a coin landing tails up.

3. If the coin is tossed 6 times, which of the following ordered sequences of coin toss results is more likely to occur? Choose one.
   a. HTTHTH b. THTHHH c. HTTTTT d. TTTHHH e. These all have an equal chance of occurring.
4. Suppose a fair coin is tossed 6 times, and each toss lands heads up. What would you expect the result of the next toss to be?
5. Suppose a fair coin is tossed 4 times, and each toss lands heads up. Suppose the coin is tossed 6 more times. How many heads would you expect to get in all 10 tosses?
6. Suppose a fair coin is tossed 20 times, and each toss lands heads up. Suppose the coin is tossed 80 more times. How many times do you expect the coin to land heads up out of the 100 total tosses?

A bag has 7 white marbles and 5 red marbles in it.

7. Suppose you drew three marbles (one at a time) from the bag without replacing them. The first marble you drew was red, and the second marble you drew was white. What is the probability that the third ball you drew was red?
8. Suppose you drew one marble and hid it without looking at it. The second ball you drew was white. What is the probability that the first marble you drew was white?

Figure 1: Interview Tasks

A 45-minute semi-structured clinical interview (Clement, 2000) was conducted, in which Lauren performed probability tasks (see Figure 1). We used Ball et al.'s (2008) categories of CCK, KCS, and KCT to design the interview protocol. We designed the tasks in a way that could potentially trigger her to use the representativeness heuristic or bias in conditional thinking (the Falk phenomenon). After completing each task, Lauren was asked how she thought a student might approach the given task. She then read a task solution written by a hypothetical student. Lauren was asked how the student was reasoning about the task, how she would respond to that student, and what influenced her to use that pedagogical approach. Lauren's written work was collected, and the interview was video-recorded and transcribed for retrospective analysis.

We coded Lauren's responses according to whether they demonstrated CCK, KCS, or KCT. The transcript portions that were relevant to CCK were categorized into sections corresponding to her understanding of sample space, her responses related to the representativeness heuristic, and her responses related to the chronology of conditioning and conditioned events. The transcript portions related to her KCS were classified into categories of anticipating student thinking and interpreting student thinking. We categorized the transcript sections relevant to her KCT by the nature of the suggested teaching approach.

Results

Common Content Knowledge

Lauren did not use the representativeness heuristic when predicting future outcomes, yet she did use it when considering the likelihood of the sequences of coin tosses on Task 3 (see Figure 1). She used the representativeness heuristic by claiming that sequences of coin tosses that had a majority of heads or a majority of tails were less likely to occur. She narrowed down her options to the sequences HTTHTH and TTTHHH, since they both have a 50:50 ratio of heads and tails. She decided on HTTHTH as the most likely sequence because having three tails in a row, as in the sequence TTTHHH, is less likely to occur. This incorrect conception could be due to her not recognizing the sequences as comprising a sample space. She did not use the representativeness heuristic on Tasks 5 and 6. Lauren acknowledged that coin-toss results are independent events, so the result of prior tosses does not influence the probability of the result of future tosses.

Lauren did not seem to experience any difficulty on the conditional probability tasks that involved conditioning events occurring before or during the occurrence of the conditioned events. On Task 2, Lauren was not perturbed by the conditioned event and conditioning event occurring simultaneously. Lauren recognized the need to restrict the sample space when working on the probability tasks in which the conditioning event occurred before or during the conditioned event. However, her reasoning in response to Task 8, which involved a conditioning event occurring *after* the conditioned event, was indicative of the Falk phenomenon. She responded to Task 8 with the probability of drawing a white marble on the first draw as $\frac{7}{12}$, without using the information about the color of the second drawn marble. She claimed that the second event does not affect the probability of the first event occurring. She did not use the given information about the conditioning event to reduce the sample size for computing the probability of the conditioned event. The reversed chronology of the conditioning event and conditioned event seemed counterintuitive for her, which led to her bias in conditional thinking.

Knowledge of Content and Students

Lauren anticipated conceptions that students might hold as they approached each probability task. She pointed out a potential source of confusion for students regarding the conditioning event's effect on the sample space in Task 2 (see Figure 1). Lauren thought students might also struggle with forgetting to restrict the sample space after taking out marbles from the bag in Task 7. She acknowledged that students might struggle with knowing when it is appropriate to multiply probabilities of events occurring and when they should use the presence of conditioning events to restrict the sample space. Lauren also anticipated that students could use ideas related to the representativeness heuristic. She described how a student might use the gambler's fallacy on Task 4. She anticipated that students would think "I'm due for a tails up because it's been heads for so long, and I've had a 50-50 shot." Also, for Tasks 5 and 6, Lauren claimed a student might think that the coin would land tails up more often in the remaining coin flips, since they've already had several tosses land heads up. She described that some of these ideas are natural to assume, which indicates her awareness of students' intuitions that could potentially be employed.

After reading a hypothetical student's response to each task, Lauren gave her interpretations of the student's response and provided a potential explanation for the student's reasoning. For example, the hypothetical student's response to Task 2 (see Figure 1) was: *There are four cards with a 4 on them, and 4 is an even number, so the probability is* $4/52$. Lauren recognized that the student did not restrict the sample space when given information regarding the conditioning event. Lauren appropriately described the logic that the student employed in the given response. Lauren demonstrated strong KCS in her ability to anticipate and interpret students' reasoning.

Knowledge of Content and Teaching

In response to the hypothetical student's reasoning, Lauren suggested appropriate teaching approaches for helping the student better understand the concepts in the tasks. One of her suggested teaching approaches involved asking questions to the student about their thought process and using the evidence of student thinking to build on what they already know. Lauren also suggested using the teaching approach of asking questions to lead the student to see their own error. For example, on Task 4 (see Figure 1), the hypothetical student's response was: *The next toss should land tails up because all of the previous tosses have landed heads up. It's time for the coin to land tails up.* In response to this, she claimed, "I'd say okay so you have five tosses landing heads up out of ten tosses total, so that means well we've already had four heads up. So, you're thinking one of the six is gonna be heads up?" Lauren suggested asking questions about the student's response would help the student look for another way to solve the task.

Lauren also described another teaching approach of using simulations and models to illustrate the situations in the tasks. For Task 4 (see Figure 1), she suggested having the students imagine a simulation of flipping a coin six times, putting the coin away, and flipping a new coin, which had an equal chance of landing heads up or tails up. Her purpose in using this was to illustrate that the "coin's last toss doesn't have any influence on the current toss." She claimed that this would help students intuitively understand how the coin tosses are independent events. Lauren also suggested using a model on Task 7 (see Figure 1). She suggested using a drawing of the marbles in the bag and crossing out the drawn marbles to illustrate the changed sample space.

Discussion

Lauren's aspects of Mathematical Knowledge for Teaching (Ball et al., 2008) seemed to be well-connected. Her conceptions of probability seemed to influence her anticipation of students' conceptions and her interpretations of what the student did. She had to draw on her CCK to think of possible ways students might reason about the mathematics tasks. Lauren also used her interpretations of what the student was thinking to design a pedagogical approach in response to the student's reasoning. She seemed to connect her KCS with her KCT. Future research can explore connections between aspects of PSTs' Mathematical Knowledge for Teaching.

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LEARNING TO RESPOND TO ERRORS: EVIDENCE FROM SCRIPTING TASKS

Matthew P. Campbell
West Virginia University
mpcampbell@mail.wvu.edu

Erin E. Baldinger
University of Minnesota
eebaldinger@umn.edu

Sean Freeland
West Virginia University
sean.freeland@hsc.wvu.edu

Foster Graif
University of Minnesota
fgraif@umn.edu

Joshua Karr
West Virginia University
jkarr1@mail.wvu.edu

We document secondary mathematics teacher candidates' (TCs') development of skill responding to errors in whole-class discussions through a scripting task—a constructed transcript of classroom dialogue that follows a provided scenario. We discuss how TCs' responses changed over a methods course and what those changes suggest about TC learning. This work offers implications for research both characterizing and supporting TC learning.

Keywords: Teacher Education-Preservice; Instructional Activities and Practices

A focus on core teaching practices (Ball & Forzani, 2009) has resulted in the growing use of teacher education pedagogies such as "approximations of practice" (Grossman et al., 2009) that have teacher candidates (TCs) enacting teaching in situations of reduced complexity. This raises the need to document TCs' learning through approximations of practice (Shaughnessy, Boerst, & Farmer, 2019). We investigate TCs' development before and after a secondary mathematics methods course through the use of scripting tasks, in which TCs produce dialogues and rationales in response to a written classroom scenario. We describe shifts in how TCs respond to errors in whole-class mathematics discussions. Specifically, we address the following research questions: How do TCs' responses to a scripting task change over a secondary mathematics methods course? What do the changes in TCs' responses to a scripting task suggest about their learning around the practice of responding to errors in whole-class discussion?

Background and Theoretical Framework

In scripting tasks (Crespo, Oslund, & Parks, 2011; Zazkis, 2017), TCs are presented with a scenario and asked to write a dialogue, showing how they might continue the discussion. The scripts then serve as representations of practice that carry meanings beyond themselves (Herbst, 2018). Some scholars have used such dialogues to make claims about skill with practices (e.g., Crespo, 2018; Lim, Roberts-Harris, & Kim, 2018; Rougée & Herbst, 2018). This existing research includes a range of teaching practices and theoretical perspectives.

We have used scripting tasks to document aspects of TCs' skill with the practice of responding to errors in whole-class discussion (Baldinger, Campbell, & Graif, 2018a). Responding to errors—contributions that are incomplete, imprecise, or not yet correct—requires a balance of keeping student reasoning central, making progress toward identified goals, and not conveying problematic ideas. While errors are key to students' learning process (Brodie, 2014), they are often positioned negatively by teachers' responses, such as correcting them or avoiding them entirely (Bray, 2011; Santagata, 2005; Tulis, 2013). The context of whole-class discussion presents additional demands, as teachers must balance the needs of the student who contributed the error with the needs of the rest of the class, who may or may not share that student's conception. This presents different considerations for responding to errors as compared to interview settings (Hallman-Thrasher, 2017) or through written work (Son, 2016).

Teachers must be supported in learning communities to develop vision, understandings, dispositions, practices, and tools (Hammerness et al., 2005). We draw on this framework to attend to and make claims about aspects of TC learning. In the context of responding to errors in whole-class mathematics discussion, tools and practices include awareness and skilled use of moves such as orienting (Chapin, O'Connor, & Anderson, 2013) and tabling (Milewski & Strickland, 2016). Dialogues provide a window into the moves TCs think to use, and the way they use them, in a particular situation. For vision and dispositions, a teacher must consider the nature and role of errors in a way that positions them as formative and sensible and that positions students as sense-makers. Dialogues illustrate TCs' sense of how a classroom situation might play out, either practically or ideally. Finally, the dimension of understandings includes a teacher's sense of how students learn and develop mathematically. A teacher must know that a mathematical idea is incorrect or incomplete, how a student may have come to a particular conception, and how the classroom activity could productively take up that conception without ignoring or quickly correcting it. Dialogues provide access to TCs' understandings based on the way they attend to the mathematics of the scenario and how they represent student voice.

Methods

Our work is the product of an ongoing, multi-year collaboration situated in secondary mathematics methods courses at two large public research institutions focused on how purposeful opportunities to approximate and reflect on practice provide opportunities for TCs to develop skill with responding to errors. This serves as a foundation for our use of pedagogies of teacher education such as coached rehearsals (Lampert et al., 2013) and scripting tasks.

We focus on one scripting task centered on the use of a sorting activity (Baldinger, Campbell, & Graif, in press) intended to refine the definition of a polygon. In the scenario, the teacher asked students for cards they easily knew were polygons. Rosalia says Shape Q (see Figure 1) is a polygon because "it is a square" and that "all the sides are straight lines." Next, Jessie offers Shape J (see Figure 1), comparing it to Shape Q. TCs were asked to construct dialogues (representing the next five to eight turns of talk), starting from Jessie's contribution.

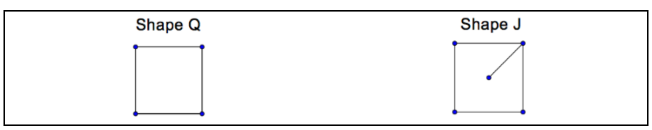

Figure 1: Two Cards Presented in the Scenario as Examples of Polygons

TCs' responses to the polygon scripting task across both institutions were collected twice during the 2017-2018 academic year—once early in Fall 2017 and once at the end of or after the methods course. We analyzed 19 pairs of responses ("Initial" and "Follow-up") from TCs.

Data analysis began with two authors individually coding each dialogue (38 total) using an established coding scheme (Baldinger et al., 2018a). Coding included a set of analytic questions focused on features of the dialogues such as the teacher moves used, the mathematical focus, the introduction of new "students," and whether or not the error gets "resolved" in the short dialogue. Disagreements among the coders were resolved through discussion. We then engaged in a process of analytic memo writing and theme building (Miles, Huberman, & Saldaña, 2013) to identify the most salient features in the dialogues. Next, we reviewed pairs of responses from each TC and constructed analytic memos describing the similarities and differences across the two responses. These individual memos informed a collective memo interpreting how the responses demonstrate TC skill and development over time.

Initial Findings

The Use of Teacher Moves: Developing Tools and Practices

We found TCs used teaching moves differently over time—either using different moves or using moves in different ways—which corresponds to the dimensions of tools and practices.

Teacher revoicing. Instances of using a revoicing move—when a teacher restates the substance of a student's contribution—increased over time. Revoicing moves appeared to have one of two purposes: to reiterate an idea for clarification or to pose more specific prompts that incorporate and connect to student contributions. This latter purpose was more prominent in the Follow-up dialogues. For example, part of a dialogue included the following:

Teacher: Alright, does anyone agree or disagree that Shapes Q and J are polygons because they are squares?

Student: I don't think Shape J is a square. A square only has 4 sides that are straight lines. That shape has 5 straight lines.

Teacher: What do you think about what Student said about this shape not being a polygon because it isn't a square?

In this excerpt, the TC constructs teacher moves that incorporate elements of students' contributions, rather than using a more general revoicing move. This use and refinement of moves is evidence of how TCs are developing tools and practices related to responding to errors.

Tabling ideas. Another move used more often in the Follow-up dialogues was a teacher's decision to table an idea or explicitly move on temporarily from a topic of conversation. TCs can be reluctant to table ideas, particularly errors, so as to not give the appearance of affirming a contribution (Baldinger, Campbell, & Graif, 2018b). Accordingly, most Initial dialogues continually focused on Jessie's contribution, which in some cases led to resolving the error.

Examples from the Follow-up data demonstrated how a tabling move could provide an alternative way to facilitate a whole-class discussion when an error is presented. For instance, one response used a tabling move after some discussion between Rosalia and Jessie, with the teacher suggesting that the card could go in the "maybe pile" and that they could "come back to it." In another response, the TC made the decision to record Jessie's reasoning and elicit other cards as "examples," so as to not "immediately signal that there's something 'wrong' with Jessie's sorting of Shape J." While tabling moves were still only used in some dialogues, the increased use offers insight into how the scripting task can capture such tools and practices.

Orienting student to one another. Instances of TCs using orienting moves—through which a teacher invites other students to reason about, engage with, and/or contribute to a student contribution—were common across both the Initial and Follow-up data. However, the nature of the use of those moves appeared more deliberate and productive in the Follow-up data, serving as evidence of how TCs developed. While orienting moves such as asking students to add on to another student's idea or eliciting agreement or disagreement were common, a notable change in the Follow-up data was the use of "turn-and-talk" moves, particularly in moments of student disagreement. In one dialogue, where students raise the question of whether the "extra segment" in Shape J affects its classification, the teacher responds, “I'd like everyone to discuss in their groups whether the line segment matters in deciding whether Shape J is a square.”

The evolving use of orienting also seemed to correspond with an increase in the average number of students included in each dialogue, and with more dialogues including two or more students, including the introduction of students other than Jessie or Rosalia. This shows that TCs were increasingly aware of the context of the scenario—a whole-class discussion—and were able to use teaching moves that allowed additional students to engage with Jessie’s contribution.

Authenticity of the Representation: Developing Vision, Disposition, and Understandings

We also noted the way in which responses in the Follow-up data were more authentic, either in terms of the situation (e.g., how the error was discussed) or the representation of student voice. We posit this provides insight into the development of vision and dispositions—as TCs come to new ideas of how a discussion can transpire—and understandings—as they represent student voice in ways that are more reflective of what one might hear in the classroom. In an example of this shift, a TC concluded the Initial dialogue (eight total talk turns) with the following:

Teacher: Could anybody give me a reason that Shape J would not be classified as a polygon?
Rosalia: Well if the line from the middle to the corner is a side, wouldn't there be an intersection of more than two sides at a point?
Teacher: That is correct. So what can we conclude about Shape J, and why, Jessie?
Jessie: Shape J is not a polygon, since sides of a polygon intersect at exactly one other side at each endpoint.

This excerpt is reflective of a feature of some dialogues, where a conclusion is reached that Shape J should be sorted as a non-polygon, in part through clearly articulated ideas from Jessie. We see responses like this as representing a vision and disposition that errors can and should be corrected relatively quickly. We also see evidence of an understanding of students that is not authentic—that students who contribute errors can quickly be convinced of alternative ideas.

The dialogue provided by the same TC for the Follow-up highlights the ways in which one can develop their vision, dispositions, and understandings that contribute to more authentic representations of practice. In the dialogue, after Jessie acknowledges the line segment in the middle of the shape, the conversation continues in the following way:

Teacher: Good observation (records reasoning on the board), it appears that there is a line segment from the middle into a corner of Shape J. Does anybody sort this card differently?
Rosalia: We didn't know how to sort it because of the line segment, so we kind of just left it in the middle.
Teacher: That's okay, we can come back to this card a little later.

In addition to the use of revoicing, orienting, and tabling moves, this excerpt illustrates important shifts in other aspects of the TCs’ skill. Notably, the TC presents a student—Rosalia—who is not sure and also imagines a situation where errors and uncertainties can go unresolved. Such shifts in TCs’ vision, dispositions, and understandings are important for their development of skill in responding to errors in whole-class discussions.

Discussion and Conclusion

We highlight the ways TCs' development of skill can be documented through their responses to a scripting task. Importantly, we found that changes in TCs' dialogues provided evidence of learning not just practices and tools, but also vision, dispositions, and understandings. Future analyses will incorporate the corresponding rationales, which will inform inferences made from analysis of the dialogues. Future work will include connecting the analyses from these scripting tasks to other aspects of our broader design, such as rehearsal cycles, to consider threads of ideas to better understand what might be attributed to parts of TCs' development. Future work will also follow up with TCs—through the continued collection of scripting responses, as well as data from student teaching or early career teaching—to understand continued development over time.

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SUPPORTING THE DEVELOPMENT OF FUTURE MATHEMATICS TEACHERS: MATHEMATICS AS HEALING

Jennifer Holm
Wilfrid Laurier University
jholm@wlu.ca

This research examines the preconceived notions that pre-service teachers bring into a mathematics methods course about both how mathematics should be taught as well as their own capabilities in mathematics. Initial responses of pre-service teachers support a procedure-oriented view of mathematics and a range of their own feelings from feeling extremely confident to feeling extremely incompetent. This research examines the initial categories of the pre-service teachers and then the shifts in their beliefs and personal feelings toward the subject through changing their understandings of and about mathematics. Finally, the research looks at the components within the methods course that supported those changing beliefs and feelings. This research starts the conversation about changing the mathematics stories of future teachers through positive mathematics experiences.

Keywords: Affect, emotions, beliefs and attitudes; Teacher education-preservice; Instructional activities and practices

Allen (2016) defines math trauma as something that "stems from an event, a series of events, or a set of circumstances experienced by an individual as harmful or threatening such that there are lasting adverse effects on the individual's functioning and well-being in the perceived presence of mathematics" (PowerPoint slide). This concept of math trauma is very real for many pre-service teachers (PSTs) who are entering a teacher education program. Before they begin teaching others, PSTs need to confront these feelings to avoid perpetuating their negative feelings when interacting with students. To some of the PSTs, these feelings about mathematics serve as a roadblock to prevent their learning mathematics and ultimately causes difficulties in teaching the subject. This research discusses activities that were carried out in the classroom community during the methods courses to help remove the feelings of trauma related to mathematics and support PSTs in changing their feelings about themselves and mathematics while developing their own knowledge of mathematics.

Literature

Philipp (2007) notes that "beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action" (p. 259). As such, beliefs are "situated" and specific to the teacher and student interactions (p. 274). Researchers suggest that past efforts to change teacher practices have failed partially because of not accounting for the beliefs of the teachers being impacted (Grant et al., 1994; Handal & Herrington, 2003). Previous research with PSTs has also shown how the beliefs about mathematics are tied with their own learning about mathematics (Holm & Kajander, 2012). This focus on beliefs is critical in this research as Holm and Kajander (2012) has shown how the beliefs one holds about mathematics does impact how the PSTs interact with the course and learning mathematics.

As mentioned previously, math trauma is the result of past experiences with mathematics that negatively impact the way a person feels about mathematics. There are many different causes and traditions that increase the likelihood of math trauma in learners including the use of labels related to ability and a focus on speed and accuracy (Allen, 2016). Another example is that those who struggle with memorization become anxious even though mathematics is not all about memorization (Willis, 2010). Allen (2016) also cites the long tradition of mathematics being viewed negatively in today's culture adding to the setting for trauma. Boaler (e.g., 2013, 2015) has extensively written about changing the idea that there is a fixed ability in mathematics and works on developing growth mindsets in mathematics classrooms worldwide. The detrimental dichotomy of those who can and those who cannot do mathematics has shown that many PSTs develop this idea that there are people who just cannot do mathematics. Through her research with PSTs, Allen (2016) has created a list of themes related to the causes of math trauma stemming from actual events or people in the past to actual points in the mathematics curriculum that were defined as traumatic. Experiences with math trauma have helped to colour the view of future teachers on the subject of mathematics, so this research looks at this important area to focus on how to support the mental well-being of future teachers and change their beliefs about not only what mathematics is but how to teach it. This research project is grounded in the experiences of PSTs and acknowledges that first dealing with the math trauma became essential before any new learning could effectively take place.

This research project uses a phenomenographical approach to the research by focusing on the stories of the participants. Phenomenography was used because of the attention paid to the individual stories in order to examine themes within the data (Ashworth & Lucas, 2000; Yates, Partridge, & Bruce, 2012). In order to examine the experiences of mathematics within the methods course, as well as the past experiences of the pre-service teachers, general themes were created from the research data, but individual stories of individual participants within the categories are also explored further. This research program was developed based in experiences with PSTs who were so frightened and disengaged with mathematics that they could not absorb the new content and often avoided it completely.

Methods

This research report looks at a subset of data in a larger study focused on the beliefs, values, and opinions of pre-service teachers around mathematics and how to support changing opinions and possibly healing some of the past trauma through a methods course. For the purposes of this research report, two different cohorts (one lower elementary and one upper elementary) in their first mathematics methods course at the beginning of their two-year teacher education program were examined. Early in the course, PSTs set an initial goal for the course and voiced their concerns that they had about completing the methods course in mathematics. Only the 34 PSTs who expressed initial concerns were considered in this data set in order to keep the focus on this more vulnerable population within the total population of 75 PSTs. At the beginning of the semester, PSTs were asked to reflect on their own experiences in mathematics and what they believed mathematics is. Throughout the course, they were asked to revisit what they initially thought about mathematics and their own experiences and how this would impact their futures as teachers. Interviews were then conducted with a subset of the participants who volunteered for a total of 12 interviews. Interviews were used to gain more information about the reflected changes and what participants noted caused them. In order to construct this report, the reflections were analyzed and the rest of the data was coded to add to the initial observations. Tenets of thematic analysis (Braun & Clarke, 2006) were used in order to create codes within the data set that would describe the reasons for some of the changes within the participants, which were later collapsed into overarching themes. In this report, the subset of the 34 PSTs in the two cohorts of a single academic year were examined to provide some initial understandings about pre-service elementary teachers.

Findings

Preliminary findings showed a dramatic shift in most of the PSTs in their feelings about teaching mathematics as well as their perceptions about themselves as mathematics students. In reflecting on the past, all PSTs included in this data set noted some variation that mathematics was about memorizing formulas and being able to find a correct answer. Many participants added that speed was a factor in mathematics, and there was an embarrassment in being slower or wrong that made it intimidating. Many of the participants made a note that they just were not "mathematics people" and would not be able to do it. All PSTs initially commented on how their feelings of success or failure in elementary and high school related to how well they were able to memorize the formulas and perform the operations with accuracy and speed.

Throughout the course, a community of learning was set up to support PSTs who were by and large terrified of mathematics, and a byproduct of this feeling was they hated the idea of coming to a mathematics course or having to teach it in the future. I used the metaphor of "mathematics as healing" throughout the first-year courses and focused on changing the perspectives of the PSTs through increasing their own understandings of mathematics, while broadening the definition of what mathematics is. I also endeavored to avoid the traumatic traditions of the past by allowing flexible due dates, resubmission of assignments, no reliance on speed, and allowing discussion and collaboration in solving problems and engaging with inquiry-based mathematics.

In looking at the participants as a whole, there was a major theme of those who wished they had been taught in this manner throughout school because they finally understood it. They also vowed to teach mathematics in a way that would avoid their future students from feeling like there were mathematics people and those who are not. All of the PSTs cited places in the methods course or their placement experiences as reasons for their changes in opinions, and these experiences related to either gaining understandings of mathematics for themselves or seeing the students gain an understanding of mathematics in a way that was not formula driven. All of the changes were rooted in their newfound confidence in their own understandings of mathematics and, as a result, a greater confidence in being able to teach it.

When speaking with the participants in the interviews and analyzing their responses about reflecting on the course, several themes came up about what impacted their own development: the use of models and manipulatives, seeing that not all of the adults could actually use or remember the formulas, and seeing that mathematics could be fun and enjoyable. In using models and manipulatives, the one activity that came up in most of the stories was exploring the operations with base 10 blocks. The second most common model mentioned was exploring fractions using fraction kits and games. In both of these experiences, participants noted how they never understood the purpose of the steps in the procedures until they saw how it worked with the models. On the first day of the course, an anonymous activity was used to bring out the fact that there was a general dislike or distrust for mathematics (Holm, 2018), and this was mentioned by several participants that it allowed them to see they were not alone. Working with other PSTs who also struggled throughout school became a strong turning point as well in realizing they were not the only ones who were afraid of mathematics. By seeing how the approach of memorizing formulas did not work for others and how there may not be a "math gene" as they were able to gain understandings, PSTs began to consider alternatives for teaching mathematics. Through their experiences, many also mentioned realizing how much they really did not know about mathematics beyond the memorized formulas (if they even remembered them), and they did not even question there may be more to mathematics. The final way the course started making changes was through engaging in games and activities to learn mathematics content. To most of the PSTs, seeing that they were learning so much through the games and activities, they saw the power for their own classrooms and saw firsthand that math can be more than just memorized formulas and drill activities.

Discussion

Initial views of experiences pointed to areas of trauma identified by Allen (2016) in their past that had led to a negative view of mathematics and their own abilities. It should be noted that even in this subset of participants, not all of the PSTs did identify these experiences as traumatic and had a more positive view of mathematics, although there was a fear in their own abilities to teach or learn it. Mainly it was those who were deemed “unsuccessful” in mathematics (either from their own perspectives or someone else’s) who held a negative viewpoint of the subject, although there was a much smaller group who were deemed “unsuccessful” and still held a positive viewpoint of mathematics.

Many of the activities and experiences within the methods course started to change the PSTs’ perspectives about mathematics and themselves as mathematics learners. “Once you reopen doors that were previously closed by negative feelings, math is revealed to students as an accessible, valuable tool to help them understand, describe, and have more control over the world in which they live” (Willis, 2010, p. 15). It was this progress of starting to see mathematics as something useful and that there was another way to teach it that would not perpetuate the traumatic experiences of their past. Many of these PSTs expressed multiple times how they wished they had learned mathematics in this way in their own experiences because it changed the way they felt. As well, many of the PSTs whose initial reflections mentioned terror about the idea of being in a mathematics course admitted in the end that the methods course became their favourite class and that they looked forward to attending every week. In the end, the majority were very excited about getting to teach mathematics and showed a willingness to continue growing and learning in order to support their future students in mathematics experiences that would not be traumatic for them. It was through experiences that changed their views of what it means to be mathematical and increase their own understandings that they felt a shift in perceptions and confidence. Although a few PSTs still expressed concerns over their own understandings, there was a definite positive shift in their feelings that they could teach and learn mathematics with more work and effort.

In considering future directions for methods courses, the observations shared by the PSTs indicated that experiencing mathematics in a way that was fun and honoured their knowledge, while they learned the content deeply, was important for changing their perspectives about mathematics and their willingness to teach mathematics in a different way in the future. As Boaler (2013) notes, “The awareness that ability is malleable and that students need to develop productive growth mindsets has profound implications for teaching” (p. 145) and certainly had profound implications for the PSTs in this study by acknowledging that they too could learn mathematics. Having PSTs acknowledge their traumatic experiences and negative beliefs, while changing their feelings about mathematics in parallel with their understandings, became essential for supporting stronger and more positive future elementary mathematics teachers.

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INTERSECTIONALITY AND MATHEMATICS TEACHER TRAINING: DISTINCT DISTRICT TRAJECTORIES OF DIFFERENT TEACHER SUBGROUPS

Andrew Brantlinger
University of Maryland
amb@umd.edu

Laurel Cooley
Brooklyn College
lcooley@brooklyn.cuny.edu

Matt Griffin
University of Maryland
griff23@umd.edu

This paper reports on the results of a quantitative analysis of retention trends of alternatively certified mathematics teacher subgroups working in low-income urban schools over the course of a decade. We draw on theories of intersectionality and longitudinal data on 620 secondary mathematics teachers from one large certification program to examine the district retention trajectories of different teacher subgroups after three, five, and eight years. We show that, not only do different subgroups have markedly different trajectories, but also that their retention appears to be highly malleable, shaped significantly by both the content of training and the context of school assignments.
Keywords: Policy, Equity and diversity, Mathematics teacher retention

Systems of social stratification based on race, class, and other characteristics are intersectional and enmeshed (Collins & Bilge, 2016). White women are positioned differently by societal institutions than either Black women or White men and, although not determinative, this influences how they understand and act in the world. While qualitative research has taken intersectionality seriously (Frank, 2018), quantitative research in education has not. This study illustrates the promise of intersectionality theory for quantitative research by addressing the following question: *How is the retention of different mathematics teacher subgroups shaped interactively by different approaches to teacher training and initial school assignments?*

Perspective

Consistent with theories of intersectionality and mixed methods research on teacher preparation (Humphrey, Wechsler, & Hough, 2008; Johnson, Berg, & Donaldson, 2005), we view the outcomes of teacher preparation as being the product of interactions between subgroups of new teachers (e.g., young Black females who live in the community they teach in, White mathematics majors from very selective colleges), the schools they work in, and the preparation and support they receive in those schools (see Humphrey et al., 2008; Johnson et al., 2005). Teacher candidates enter educator preparation programs with diverse experiences, strengths, needs, and goals, and, as such, do not uniformly take up and act on the messages they receive during preparation (Humphrey et al., 2008; Johnson & Birkeland, 2008). At the same time, members of particular subgroups tend to react similarly to these messages and differently than members of other subgroups. Finally, we also posit that teacher preparation outcomes depend on how different subgroups, trained in particular programs, integrate with the people and culture in the schools they work in (Meagher & Brantlinger, 2011; Humphrey et al., 2008).

Methods

In this paper, we are concerned with fast-track training of mathematics teachers in a large alternative route program and the programmatic outcome of retention. To model the retention for different mathematics teacher subgroups from a fast-track alternative route program, we drew from data on 620 secondary mathematics Teaching Fellows (SMTFs) who entered the New York

City Teaching Fellows (NYCTF) program in either 2006 or 2007.[1] At that time, NYCTF had four universities and colleges collaborating in secondary mathematics teacher training. As Table 1 indicates, training for secondary mathematics differed across the universities (Author, 2013). UnivB and UnivD emphasized mathematics content and methods whereas UnivA and UnivC emphasized subject-general themes. Of the latter two, UnivA emphasized technical aspects of teaching (e.g., classroom management) whereas UnivC placed a comparative emphasis on theories of adolescent development.

Table 1: Curricular Emphasis of NYCTF's University Partners

NYCTF Partner	Comparative Emphasis of Training	Percentage of Coursework	
		Math Content	Math Methods
UnivA	Practical, Subject-General	23	18
UnivB	Theoretical, Mathematics-Specific	28	37
UnivC	Theoretical, Subject-General	18	19
UnivD	Theoretical, Mathematics-Specific	35	45

We drew on three data sources: (1) *service history data* on SMTFs' school assignments, roles and years of paid service from the New York City Department of Education (NYCDOE), (2) *demographic, academic and professional background information* on SMTFs collected on surveys, and (3) *school demographic and climate (survey) data* collected by the NYCDOE and the New York State Department of Education on the 1600-plus public schools in New York City. We used the service history data to create dummy variables for district retention after three, five, and eight years. We created the teacher background variables using NYCDOE-provided and survey information about SMTFs' age, gender, ethnicity, high school (location), college, and post-secondary degrees. We created the training variables using NYCTF-provided information on initial certification, university assignment information and survey data on the quality of NYCTF's contribution to pre-service training, namely, Clinical Advisory.

We used logistic regression to model SMTFs' district retention at three and eight years (Table 2). We included variables for the characteristics of SMTFs, training, and their first schools. Consistent with our theoretical framework, we included bivariate interactions between independent variables but only kept those that were significant (see Author, under review). To help interpret the interaction terms in the models, we created a taxonomy of SMTF subgroups (i.e., ideal types) (Bailey, 1994). In this paper we focus on two subgroups, namely, on *Young Elite White Outsiders*[2] and *Mature Black Insiders*[3,4].

Results

The regression models indicate that the content of fast-track training is predictive of mathematics teacher retention in urban schools at three and eight years (Table 2). (While we discuss it, we do not present the eight-year model here due to space limitations.) Specifically, these models show that, in shaping retention, fast-track training for secondary mathematics: (1) operates differentially on different teacher subgroups working in particular school contexts, (2) depends on the substance – and not just the fast-track structure – of this training; and (3) has both long- and short-term effects on teachers' district and school retention. When viewed in light of Table 1, fast-track training that emphasizes 'practical' concerns (i.e., their expressed concerns on pre-service surveys) like classroom management is linked to improved SMTF retention generally, although this differs by teacher subgroup.

At the broadest level, the models show that mathematics teacher selection shapes programmatic outcomes like retention. In particular, we find that: (1) Black SMTFs are much more likely than non-Black SMTFs to remain in the district at 8 years; (2) SMTFs from very selective colleges are less likely than others to remain in the district at 3 and 8 years; and (3) SMTFs who started as NYC outsiders[1] have significantly lower estimated rates of three- and eight-year district than the NYC insiders. However, consistent with intersectionality, there also were significant interactions between these and other variables in the final models.

Table 2: Logistic Regression Model of SMTF District Retention at Three Years

	Constant	3.770**
Teacher	From NYC Suburbs[1] (vs. NYC Insider)	0.193**
	NYC Outsider[1] (vs. NYC Insider)	0.215**
	Selective College (vs. Very Selective)	2.035**
	Less Selective College (vs. Very Selective)	1.513
Training	Grades 5-9 Certification (vs. Grades 7-12 Cert)	0.288**
	Top Clinical Advisory (vs. Other Clinical Advisory)	1.758*
Teacher * Training	Grades 5-9 Certification * From NYC Suburbs	5.427**
	Grades 5-9 Certification * NYC Outsider	3.833**
Teacher * School	Asian * Black-Latinx Students Below 61%	0.672
	Asian * Black-Latinx Students Above 98%	0.541
	Black * Black-Latinx Students Below 61%	1.220
	Black * Black-Latinx Students Above 98%	6.927*
	Latinx * Black-Latinx Students Below 61%	0.690
	Latinx * Black-Latinx Students Above 98%	0.716

Figures 1 and 2 illustrate this intersectionality, presenting the "district trajectories" (i.e., estimated district retention over time) of the *Mature Black Insider (MBI)* and *Young Elite White Outsiders (YEWO)* subgroups (see end notes). A comparison of Figures 1 and 2 illustrates the distinctive district trajectories of different subgroups. Examining each figure separately illustrates the malleability of SMTF retention depending on members of particular subgroups interact with the content of training and contexts of their assigned schools.

MBIs have the best estimated retention outcomes in comparison with all other SMTF subgroups. None of their individual characteristics alone account for this. Although Blackness is associated with superior SMTF retention, MBIs have better retention estimates than all other Black subgroups. Compared to MBIs, YEWOs have much lower odds of remaining in the district in the long run (i.e., eight years). Their status as NYC outsiders and as elite college graduates seem to drive YEWO turnover as all of the elite outsider subgroups have high turnover estimates.

There are implications for training above and beyond those for teacher selection. In particular, the regression models point to ways that fast-track training might be modified to improve the retention of particular teacher subgroups. Specifically, *practical training* seems particularly important for retaining elite outsider subgroups. Relative to *theoretical training, practical training* doubles the estimated odds that *YEWOs,* and other elite outsider subgroups (e.g., *Mature Elite Latinx Outsiders*) will remain in the district beyond eight years. While *practical training* is associated with superior outcomes for most SMTFs, including both MBIs and YEWOs, when it comes to shaping retention outcomes, *mathematics-specific training* seems to have worked just as well for a few subgroups (e.g., *Maturing Latinx Insiders*[2]).

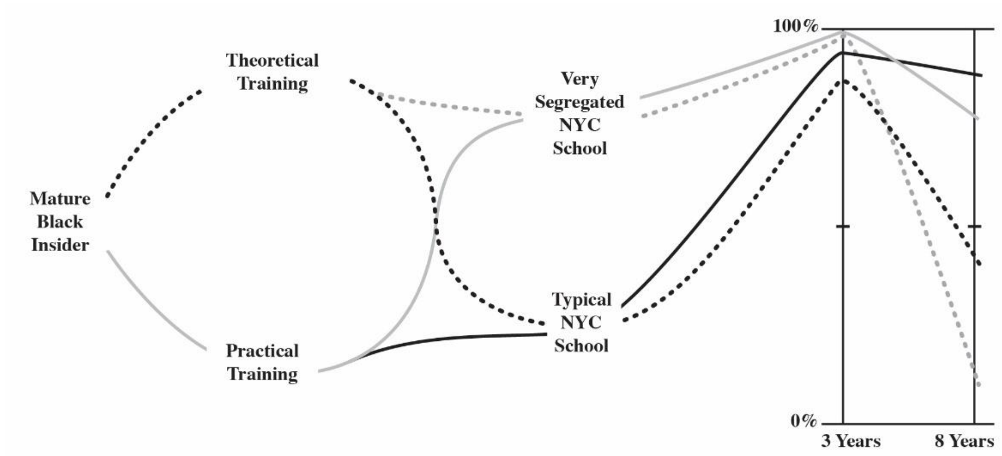

Figure 1: Mature Black Insiders' District Trajectories

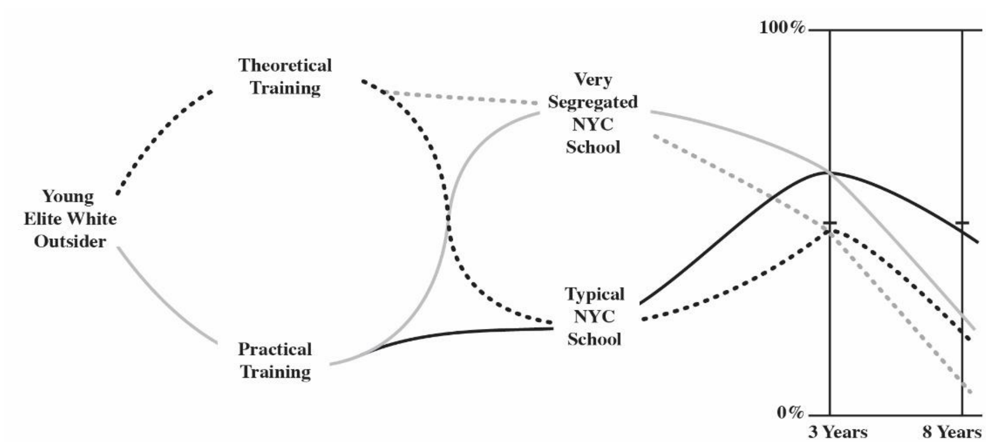

Figure 2: Young Elite White Outsiders' District Trajectories

Significance

Although specific to NYCTF, we believe these results generalize to other secondary teachers selected and trained in fast-track alternative route programs. Some of the results, for example those about practical training, seem particularly relevant to teacher preparation programs that prepare teachers to work in low-income schools serving mostly students of color. That said, the candidate-dependency (i.e., interactivity, intersectional-nature) of the retention results serve to challenge the policy assumption that individual preparation programs can be validated as being either effective or ineffective. Ostensibly proven approaches to teacher training (see TNTP, 2005), mathematics or otherwise, might simply be what works best for middle class, White outsiders, over-represented in teacher preparation programs nationally (Ladson-Billings, 1995).

Endnotes

1. NYCTF's partner universities and colleges are referred to as UnivA, UnivB, UnivC, and UnivD.
2. *Elite* SMTF subgroups include the graduates of very selective colleges (Barrons, 2009).
3. The *NYC Insider* subgroups include those SMTFs who graduated from an NYC high school, the *NYC Outsider* subgroups include those SMTFs who graduated from a high school located 150 miles from the NYC border, and the *Suburban* subgroups include SMTFs who attended high school outside of NYC but located less than 150 miles from the city limits.
4. The *Young* subgroups include those SMTFs aged 21-23 years when they entered NYCTF, the *Maturing* subgroups include those aged 24-27 years old, and the *Mature* subgroups include those who entered when they were 28 years older.

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DESIGNING SIMULATIONS FOR ELICITING STUDENT THINKING: IS IT POSSIBLE TO DESIGN A COMPREHENSIVE STUDENT PROFILE?

Meghan Shaughnessy
University of Michigan
mshaugh@umich.edu

Susanna Farmer
University of Michigan
owenssh@umich.edu

Rosalie DeFino
University of Michigan
rdefino@umich.edu

Timothy Boerst
University of Michigan
tboerst@umich.edu

Prior work has shown that teaching simulations are valid and fair ways to assess novices' skills with eliciting student thinking. In this study, we sought to explore a fundamental question about the use of live, interactive simulations, that is, whether it is possible to design a student profile that sufficiently covers the breadth of questions that preservice teachers might ask during an interaction. Our results reveal that it is possible. Implications are discussed.

Keywords: Assessment and Evaluation; Instructional activities and practices

Successful teacher preparation requires innovations in assessment that can inform preservice teachers (PSTs) and those who prepare them. Of particular importance are assessments that provide information about PSTs' abilities to actually <u>do</u> the core tasks of teaching. This requires combining instructional techniques together with specialized knowledge of the content and insights into students' thinking and development. Although many current assessments of teaching, including observation tools (e.g., Danielson, 2013) and portfolios (e.g., Teacher Performance Assessment developed by Darling-Hammond and Pecheone, 2010), do offer useful information, additional tools are needed to supplement what can be learned through these types of assessments. Further, we must move beyond traditional forms of assessment to create sustainable and fair ways to assess such capability throughout initial teacher preparation.

In recent years, the use of simulations in teacher education has expanded and there is growing interest in the use of simulations for learning and assessment. By *simulation*, we mean a live instance in which a preservice teacher (PST) interacts with an adult whose actions are guided by a profile of a K-12 student's reasoning about content and responding to questions. In such a simulation, PSTs are engaged in the interactive work of teaching and there are opportunities for teacher educators to see and appraise PSTs' developing capabilities and skills (Shaughnessy & Boerst, 2018). This work builds on the use of live, interactive simulations in medicine, dentistry and other professional fields. Doctors and dentists in training engage in simulations of physical examinations, patient counseling, and medical/dental history taking by interacting with "standardized patients." Evaluation of medical students' interactions with standardized patients makes possible common and sustainable appraisal of candidates' knowledge and skills (Boulet, Smee, Dillon, & Gimpel, 2009).

The use of simulations in teacher education has been growing (e.g., Dieker et al., 2014, Dotger, 2015; Dotger & Sapon-Shevin 2009; Mikeska & Howell, 2018; Self, 2018), but there has been limited research on the design of simulations that would be needed for wider-scale use. In this paper, we build on our past efforts to design and study the use of simulations to assess PSTs' skill with eliciting and interpreting student thinking. Specifically, we investigate the robustness of the design of the simulation. In the particular version of simulations that we develop, we design a “student role protocol” which captures the student’s process and understanding and the ways in which they respond to questions about their mathematical work. The “student” uses information from the protocol to guide the ways in which they respond to questions posed by the PSTs. In this study, we examined a student role protocol that we developed for one such simulation and video records of PSTs’ performances. We investigated how and to what extent the student role protocol provided guidance in responding to questions posed by PSTs.

Design Considerations for Simulation Assessments

Since 2011, we have been developing and studying simulations as a means to assess PSTs’ capabilities with eliciting and interpreting student thinking. We have used simulations to learn about the eliciting and interpreting skills that novices bring to teacher education as well as to assess their progress as they move through their professional training. In our simulations, PSTs engage in three stages of work. First, they are provided with student work on a mathematics problem and given 10 minutes to prepare for an interaction. The task for the PST during the interaction is to determine the process the student is using to solve the problem and the student’s understanding of the core mathematical ideas involved in the process. Second, PSTs interact with a “student.” The role of the “student” is carried out by a teacher educator whose words and actions are guided by a student role protocol. As described above, the protocol is a detailed profile of a particular student’s thinking and rules that govern this student’s interactional norms. PSTs have five minutes to interact with the “student,” eliciting and probing the “student’s” thinking to understand the steps she took, why she performed particular steps, and her understanding of the key mathematical ideas involved. In the third stage, PSTs respond verbally to a set of questions that are designed to probe their interpretations of the “student’s” process and understanding and their prediction about the “student’s” performance on a similar problem.

Our assessment development process considers teaching practice itself and how it can be decomposed for the purposes of assessment. We also consider the assessment situation and the opportunities it creates for PSTs to demonstrate their skills in light of a practice-focused developmental frame (see Shaughnessy & Boerst, 2018 for more details). In our simulation assessments, the student role protocol (see Figure 1 for an excerpt) is crucial both for enacting the assessment and for providing consistent opportunities for PSTs to demonstrate their capabilities with eliciting student thinking. We have three main design considerations (Shaughnessy & Boerst, 2018). The first is the *mathematics content* itself that is embedded in the student work sample. The second is the *characterization of the student’s process and understanding*, including the student’s process for solving the problem, the student’s understanding of the process and related mathematical ideas, and the accuracy of the student’s answer. The third is the *student’s way of being,* which refers to the student’s dispositions, interactional style, and use of mathematical language. A student role protocol articulates each of these considerations and this constitutes general guidance for responding to PSTs’ questions. We also script responses to likely questions that are aligned with the three design considerations.

Methods

Thirty-six PSTs enrolled in an undergraduate university-based elementary teacher education program in the United States participated. The assessment was administered at the mid-point of the teacher education program as a regular part of the program. The performances on assessments were video recorded and written artifacts were collected. We used the software package Studiocode© to code the assessment video. We began by parsing the video into talk turns. Then we identified what we refer to as "cases" which contain a question posed by a PST. For each case, we characterized the question posed by the PST using one of three codes:

- *Scripted response available*: A scripted response exists in the student role protocol that could be used to respond to the question posed by the PST
- *No scripted response, but there is general guidance*: No scripted response exists in the student role protocol that could be directly used to respond to the question posed by the PST; however, the general guidance provides information that might be used in response
- *Guidance not available:* There is no scripted response in the student profile that can be used to respond to the PST's question and no guidance for responding to the student is available in the general guidance section

Two coders independently coded each video. Discrepancies in scoring were examined by the full team, referencing the codebook as needed to reach consensus.

$\frac{3}{7} = \frac{6}{14}$ $\qquad$ $\frac{2}{5} = \frac{6}{15}$

$\frac{6}{14} > \frac{6}{15}$

So: $\frac{3}{7} > \frac{2}{5}$

Mathematics topic: Comparison of fractions

Characterization of the student's process and understanding:

The student's process: The student is using the common numerator method to compare fractions. To determine which numerator to use, the student finds the least common multiple of the numerators of the original fractions. The student generates equivalent fractions by multiplying the numerator and denominator of each fraction by the same number. Then the student compares the denominators to determine which fraction is larger.

The student's understanding of the ideas involved in the problem/process: The student knows you have to multiply (or divide) the numerator and denominator by the same number to generate an equivalent fraction but does not understand why that process works. Once the student has common numerators, the student understands that when you have the same number of pieces, you can use the denominator to determine which fraction has larger pieces and therefore which fraction is larger.

Other information about the student's thinking, language, and orientation in this scenario: The student knows of other strategies for comparing fractions, but the student thinks that the common numerator methods works best with the given example.

The student's way of being: The standardized student gives the least amount of information that is still responsive to the PST's question.

Specific responses based on the identified mathematics topic, characterization of the student's process and understanding, and the student's way of being (*a subset of them*):

PST prompt	Response
What did you do first?"	"I wanted to change the fractions so that there would be something in common."
Asks about how 6 was identified as the common numerator	"I wanted the numerator to be 6 because it was the least common multiple."
"Why can you compare fractions by comparing the denominators when the numerators are the same?"	"When you make the numerators the same it means you have the same number of

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* St Louis, MO: University of Missouri.

	parts so all you need to think about is how big the parts are."
"Why is 6/14 greater than 6/15?"	"Fourteenths are larger than fifteenths."

Figure 1: An Excerpt from the Student Role Protocol

Results

Our analysis focused on the extent to which the student role protocol provided guidance in responding to questions posed by PSTs. Across the 36 performances, there were 457 cases in which PSTs posed a question to the "student." In 70% of the cases (319 of the 457), a scripted response was available in the student role protocol. For example, one PST asked, "how did you pick 6 as the common numerator?" There is a scripted response for "asks about how 6 was identified as the common numerator." In another 25% of the cases (112 of the 457), there was no scripted response available, but the "student" could respond by drawing on other guidance provided in the student role protocol, such as information provided about the student's understanding of relevant mathematical concepts and general demeanor. For example, one PST posed an additional problem for the student to solve in which the student generated equivalent fractions which had a common numerator of 12. The PST then asked the student, "How did you use 12 as the numerator?" The response for this exact question was not scripted but the PST could reasonably use the general information available in the profile and the scripted response for a common numerator of 6 to generate a response to the question. Another way of interpreting these findings is that for 95% of the questions posed, there was some guidance available in the student profile for responding. These results suggest that the student profile is sufficient for providing guidance to the "student" for responding in standardized ways.

In the remaining 5% of the cases (26 of the 457), there was no guidance available in the profile for responding to the question. This means that in these cases, the "student" had to improvise to construct a response to the question. We examined how these 26 cases of "no guidance in the profile" were distributed across PSTs. Because the PSTs asked varying numbers of questions, we calculated the percentage of "no guidance" cases for each PST. We found the percentage of instances in which no guidance was available ranged from 0% to 29% with mean of 5% and a standard deviation of 10%. There were 5 PSTs for whom the percentage of cases with no guidance available was higher than 15%. This suggests that cases of "no guidance" were somewhat idiosyncratic and clustered around a small subset of PSTs .

Discussion

Our study establishes that it is possible to design a "student role protocol" with scripted responses that address the vast majority of questions posed by PSTs in a simulation. Further, in the case of this simulation, it was crucial to provide a general frame for how the student was thinking in addition to scripting responses to specific questions. The general guidance was needed to respond to 25% of the questions posed in the study. Further, because interpreting an unscripted question and determining an appropriate response based on guidelines are complex activities, these findings suggest that it is necessary to have a live "student" ready to take up unique lines of questioning. As part of a broader study, we are investigating the fidelity of the implementation (i.e., the degree to which "students'" responses adhere to the profile). Results from our analyses are promising. Individually any of these questions is important, but as a set they constitute what Cohen and Ball (2007) characterize as "scaling in" to establish internal components of the initial intervention that is a crucial precursor to scaling up.

Acknowledgments

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CONVEYING RESPECT FOR STUDENTS THROUGH THE PRACTICES OF ELICITING AND INTERPRETING STUDENT THINKING

Meghan Shaughnessy
University of Michigan
mshaugh@umich.edu

Timothy Boerst
University of Michigan
tboerst@umich.edu

We examine what it might mean for preservice teachers to convey respect for students through their eliciting and interpreting of students' thinking. We report on a conceptualization of what it means to convey such respect and challenges at the beginning of teacher preparation.

Keywords: Instructional activities and practices; Equity and diversity

All students have the right to work with a teacher who is interested in their mathematical thinking and has the requisite mathematical knowledge, pedagogical skills, and commitment to support their day-to-day, lesson-to-lesson, task-to-task learning. Tragically for subsets of the student population – sentences like this have driven national standards (NCTM, 2000), legislation (No Child Left Behind [NCLB], 2002), and research for nearly half a century without redressing disparities in mathematics achievement. Recently scholars have argued that this "achievement gap" reflects serious "opportunity gaps" that must be rectified (Milner, 2010; Horn, 2012; Flores, 2007). Opportunity gaps are created and replicated through mathematics instruction, with instruction defined as the interaction among teachers, students, and mathematics content (Cohen, Radenbush, & Ball, 2003). Studies demonstrate that "no in-school intervention has a greater impact on student learning than an effective teacher" (p.1, NCATE, 2010). Thus, the opportunity gap is a problem of teaching and likely to be, at least in part a function of the preparation of mathematics teachers.

The professional learning of teachers often includes opportunities for preservice teachers (PSTs) to interact with students around mathematics content. Teachers need opportunities to hone their use of mathematical knowledge and practical skills to redress the bias and inequity that produce opportunity gaps. However, because opportunities to interact with children commonly occur in the context of teaching in K-12 school contexts, it is often not possible to predict, let alone ensure, that opportunities to redress the bias and inequity will manifest themselves. In a very real sense there are gaps in PSTs' opportunities to learn to teach in ways that could redress gaps for students in opportunities. Given the challenging nature of reliably providing these experiences, it is particularly important to develop options that can.

Our project focuses on the development of PSTs' capabilities with eliciting and interpreting students' mathematical thinking. The goals of this collective work include enhancing awareness of teaching that could produce opportunity gaps and developing mathematical knowledge, pedagogical practices, and dispositions that enhance equity, access, and inclusion. In particular we focus on attending to and taking up student ideas as such moves are the hallmark of responsive teaching. This is a place where it might be possible to see biased or gap-producing teaching moves. In our past work, we noticed PSTs' attending to and taking up student thinking in ways that appeared to be counter to goals of enhancing equity, access, and inclusion. For example, after learning that a student has used a non-standard (but valid) process to solving a problem, we commonly notice patterns of asking questions focused on why the student has "not done" another process, often a "standard" process with which the teacher either was more familiar or preferred for solving a problem. In some cases, such approaches went further with teachers telling a student that they could not use their valid process, but instead needed to use the "standard" algorithm. When interpreting student thinking, we observed that while PSTs noticed attributes of a student's process or their understanding, they characterized the student's thinking in deficit-focused ways when the student used a non-standard process. Another kind of problematic characterization, of a more general nature, entailed derogatory statements about the student's general mathematical aptitude or skill (e.g., this student is really confused and must have problems in math). We began to wonder about how we could capture the ways in which teachers were respecting students and their thinking as they engaged in the work of eliciting and interpreting student thinking. The choice of these teaching practices is strategic as they undergird much of the work that happens in classrooms. Specifically, our study examined the respect (or disrespect) for students and their mathematical knowledge that was evident when eliciting and interpreting student thinking at the beginning of teacher preparation. We next turn to our conceptualization of the teaching practices of eliciting and interpreting student thinking.

Eliciting and Interpreting Student Thinking

In teaching, "teachers pose questions or tasks that provoke or allow students to share their thinking about specific academic content in order to evaluate student understanding, guide instructional decisions, and surface ideas that will benefit other students" (TeachingWorks, 2016). We conceive of the work of eliciting student thinking as involving: (a) eliciting and probing the student's process and understanding; (b) taking up the student's ideas in questions, including respecting the student and their thinking; and (c) using mathematical language and representations. This work involves teachers listening to and interpreting what students are saying, generating and posing questions to learn more about the student thinking, listening to and interpreting what students are saying. Teachers make sense of what students know and can do based on evidence from interactions and other artifacts of student work. This practice entails: (a) making qualified claims about valued outcomes that can be used as the basis for future action, (b) using evidence to generate and test claims, (c) matching the scope and nature of the claim to the amount and type of information available (d) actively working to prevent bias or distortion, and (e) developing and/or using appropriate criteria to focus or inform judgments.

Using a Teaching Simulation to Formatively Assess Skills with Respecting the Student When Eliciting and Interpreting Student Thinking

Many practice-based professions (e.g. dentistry, law, pharmacy) use simulations to assess novices' knowledge and skill with core elements of interactive work. Simulations are "approximations of practice" that place authentic, practice-based demands on a participant while purposefully suspending or standardizing some elements of the situation. Simulations provide a predictability that cannot be replicated through live work in classrooms, interactivity that cannot replicated through video study, and access to feedback and collective work on practice not replicable through written reflection.

Since 2011, we have been using teaching simulations to study PSTs' skill with eliciting student thinking (Shaughnessy & Boerst, 2018a; Shaughnessy & Boerst, 2018b). In these simulations, a PST interacts with a "standardized student" (a teacher educator taking on the role of a student using a well-defined set of rules for responding) around a specific piece of written work. We design teaching simulations to have a consistent three-part format. First, PSTs are provided with student work on a problem and given 10 minutes to prepare for an interaction. The task for the PST during the interaction is to determine the process the student is using to solve the problem and the student's understanding of the core mathematical ideas involved. Second, PSTs have five minutes to interact with the standardized student, eliciting and probing the "student's" thinking to understand the steps they took, why they performed particular steps, and their understanding of the key mathematical ideas involved. The role of the "student" is carried out by a teacher educator whose words and actions are guided by a detailed profile of a particular student's thinking and rules that govern this student's interactional norms. To ensure standardization of the role, the "student" is trained to follow the highly specified rules for reasoning and responding, including responses to questions that are commonly asked by PSTs. Third, PSTs respond verbally to a set of questions that are designed to probe their interpretations of the "student's" process and understanding and their prediction about the "student's" performance on a similar problem.

We designed this simulation to be one in which a student uses an alternative algorithm for solving subtraction problems (see Figure 1). The process involves writing the value of the minuend and subtrahend in expanded form and then making any necessary trades. When trading, the student works from right to left. The student then subtracts the numbers place-by-place in expanded form, starting with the hundreds place. This student has conceptual understanding of expanded form, the meanings of addition and subtraction, and when, how, and why to make trades. We selected this algorithm because it is one that we anticipated would be unfamiliar to our PSTs. This enabled us to learn about the ways in which they respected the student and their thinking in a context in which the student work was unfamiliar.

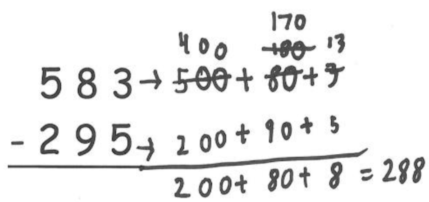

Figure 1: Student Work

Methods

Thirty-two PSTs enrolled in a university-based teacher education program in the United States participated at the beginning of their teacher education program. The simulations were video-recorded. Our analysis focused on respecting students and their thinking. For eliciting student thinking, we used the literature and our research on the work of teaching to identify two characteristics that show respect for students and their thinking: (1) establishing and maintaining a focus on the student's approach while refraining from directing the student to a different process in a way that competes with the student's process; and (2) establishing and maintaining a non-evaluative space in which students can openly share their thinking. For interpreting student thinking, we focused on PSTs' use of a non-deficit focused language to describe the student and their thinking. We focused on three characteristics: (1) characterizing the student's process in asset-focused terms (or ways that are not in deficit terms); (2) talking about the student's process itself without repeated reference to a different process; and (3) characterizing the student's mathematical knowledge and skills in asset-based terms or ways that are not deficit-focused. Two independent coders applied all of the relevant codes to each performance. Disagreements were resolved through discussion and by referencing a code book.

Results

Eliciting Student Thinking

We examined the eliciting of the PSTs to see whether it had characteristics that show respect for students and their thinking. We found that 78% of the PSTs (25 of 32) established and maintained a focus on the student's process. This means that 22% of the PSTs directed the student to a different process in a way that competed with the student's initial reasoning. For example, one PST quickly launched into a series of questions focused on getting the student to solve the subtraction problem without expanding the numbers. Another PST asked a series of questions about the student's process and then told the student that a way to do the problem differently (that would make it easier) would be trade from right to left (rather than left to right). The student indicated that she had seen people do it that way but preferred to trade from left to right. Seventy-five percent of the PSTs (24 of the 32) established and maintained a non-evaluative space in which students could openly share their thinking while 25% of the PSTs employed moves that showed repeated evaluation of the student's thinking such as characterizing as "correct" or "incorrect" each step shared by the student.

Interpreting Student Thinking

We examined the PSTs' interpretations of the student's thinking. In other words, the ways in which they characterized the student and their thinking when we asked them to talk about the student's process, including the student's understanding of the process and the generalizability of the process from a mathematical perspective. We found that 69% of the PSTs (22 of the 32) characterized the student's process in asset-focused ways or ways that were not deficit-focused. In contrast, 31% of the PSTs characterized the student's process using deficit terms. Ninety-one percent of the PSTs (29 of the 32) talked about the student's process itself without repeated reference to a different process meaning that the remaining 9% repeatedly talked about the student's process in terms of a different process. Eighty-four percent of the PSTs (27 of the 32) characterized the student's mathematical knowledge and skills in asset-based terms or ways that were not deficit-focused while 16% characterized the student's mathematical knowledge and skills in deficit terms. When we looked across the characteristics of respecting the student when characterizing their thinking, we found that 38% of the PSTs used one or more moves that we considered to be problematic related respecting the student's thinking.

Discussion

Our study examines an approach for noticing and naming the ways in which PSTs convey respect for students and their thinking when eliciting and interpreting student thinking. As shown in this study, simulations can be designed to raise teaching dilemmas that will surface PSTs' teaching practices, mathematical knowledge, and potential biases, thereby making them available for noticing and addressing. In addition to naming ways to notice the respect of students that can unfold in the work of eliciting and interpreting student thinking, the findings suggest that PSTs at the beginning of teacher preparation are in need of interventions focused on respecting students and their thinking in order to teach in ways that promote access, equity, and inclusion. Further, even those PSTs who did not characterize the student and their thinking in deficit-focused ways were not always carrying out an asset-based approach to characterizing student thinking. In future work, we seek to leverage the simulation to support professional learning at the interactive intersection of teachers, students, and content.

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NOTICING STATUS IN SMALL GROUPS: ATTENDING VERSUS INTERPRETING

Brent Jackson
Michigan State University
bjackson@msu.edu

Byungeun Pak
Michigan State University
pakbyun1@msu.edu

This study explores how eleven prospective secondary mathematics teachers noticed status issues occurring small group while watching a video of mathematics instruction. Analysis of written noticings indicate that teachers' noticings concentrate in attention with teacher, quantity of talk, access to space/tools, and displays of authority. Status deriving from larger social hierarchies (i.e. race and gender) was not included in participants written noticings to reason about the noticed interactions. Moreover, the characteristics of noticings varied whether a student positioned as lower status or higher status was more salient in the video. Implications for teacher educators are discussed.

Keywords: Teacher Education, Equity and Diversity; Teacher Noticing, Group Work

Introduction

Group work has the academic and social benefits on the part of students (Cohen & Lotan, 2014; Davidson, 1990). The careful use of group work in mathematics classrooms has been suggested as one approach to making mathematics instruction equitable across diverse learners (Boaler, 2006; Gresalfi, 2009). Yet, inequitable group work arising from status issues has been documented to limit marginalized students' opportunities for learning (Bishop 2012; Langer-Osuna, 2011). We pay attention to status, "an agreed upon social ranking where everyone feels better to have a high rank within the status order than a low rank" (Cohen & Lotan, 2014, p. 28), because it influences students' expectations about who can participate, and in what ways, which results in inequitable opportunities for learning in small groups.

Successful teachers working with diverse learners notice students' forms of participation and positionings among each other (Hand, 2012; van Es, Hand & Mercado 2017). Given that these teachers attend to status in small groups, teacher preparation ought to develop prospective teachers' (PTs) noticing of status in small groups. To better support PTs, it is necessary to understand what PTs notice in relation to how they focus on details of interactions (e.g., Langer-Osuna, 2011; Wells, 2017) and/or socially constructed categories, such as race or gender (Esmonde, Brodie, Dookie and Takeuchi, 2009). This line of inquiry is not well developed. As such, we argue that PTs are sensitive to noticing status in small groups in various ways; however, additional scaffolding is necessary so that status is interpreted in connection with socially constructed categories that carry influence (Cohen & Lotan, 2014). This understanding may allow teacher educators to promote PTs' noticing abilities.

Theoretical Framework

Previously, we argued that noticing status in small groups can help PTs to reason about how to facilitate equitable learning in small groups (Pak & Jackson, 2018). For example, noticing status, such as a dominant group member that ignores questions from peers, can suggest PTs to intervene in the small group by modeling questioning and extending ideas as a way to respect each other's ideas. Alternatively, a PT might respond by assigning competence to the student's question and highlighting why the particular question is important to consider (Cohen & Lotan, 2014). For PTs, dealing with status in small groups is unlikely to be an easy endeavor (Featherstone, et al., 2011). To help scaffold PTs' noticing in this regard, teacher educators need to offer representations of practices that respond to status as a means to create equitable opportunities for learning.

Following van Es and Sherin (2008), we view teacher noticing as involving two processes: *attending* and *interpreting*. Attending to status means noticing the details in students' interactions. These details include body language (gaze, gestures and body position), authority (decision-maker, task assigner), quantity of talk (how much a student have an opportunity to talk), and access to space/tools (who occupies more space or who has more chance to use tools). Wells (2017) argued gesture and body positioning as "indicators that a teacher should try to notice in a classroom setting which related to the understanding of a group of students" (p. 184). In other research, PTs' noticing via video analysis (McDuffie et al., 2014), one of the foci was how PTs noticed the authority students held working on mathematics as an indicator of status. Cohen and Lotan (2014) and Featherstone and colleagues (2011) suggested quantity of talk, access to space/tools, and authority as details teachers need to attend to.

Interpreting status in small groups extends beyond the details of the interactions to consider how hierarchies developed from socially constructed categories, such as race, gender, and socio-economic status (SES) operate in students' small groups interactions. The social hierarchies in wider society are often reflected in small groups as status (Featherstone et al., 2011; Cohen & Lotan, 2014), which gives rise to the different interactions described in the previous paragraph. Although there is a lack of research on PTs' interpretation of status, Esmonde and colleagues (2009) offer insight as to the importance of students' social identities in small groups. Their analysis showed that "students experience mathematics classrooms as sites for power struggles that are often related to their social identities" (p. 39). As such, marginalized students shared preferences to work with students of the same race, gender, and/or working style because they felt they could learn more in these groups than others. Moreover, teacher noticing is shaped by dominant ideas "that position students from non-dominant communities as mathematically deficient rather than as sense-makers whose ideas should form the basis for further learning" (Louie, 2018, p. 55). Taken together, this research leads us to think of social identities as crucial aspects of what PTs need to notice in order to treat status issues in small groups.

Methodology

Eleven PTs, enrolled in their fourth and final semester in a mathematics methods course, at a large midwestern university participated in this study. In their first methods course they were introduced to the ideas of complex instruction (CI; Cohen & Lotan, 2014) and continually engaged with the ideas from CI by designing group worthy tasks, debriefing videos of group work and identifying aspects of status, among other activities. The data for this study comes from PTs' responses from a class activity after watching the Staircase Problem (Roche, 1996). In this video, high school students work in small groups to determine a closed formula for a growing geometric pattern. First, the PTs engaged in the pattern task. Second, they watched the whole video without stopping (15 minutes). Third, PTs watched the video a second time but in an order so that footage from each of three groups shown in the video was continuous. After watching the video segment of each group, the video was paused and the PTs were asked to draw a status hierarchy in a simple diagram and write a statement answering three questions. These questions were: (1) What evidence of status did you see in the video? (2) Explain why you drew the status hierarchy the way you did, and (3) In what ways did you see status hierarchies reinforced or disrupted?

To analyze the written statements, we drew on the framework of attending and interpreting described above with provisional codes (Miles & Huberman, 1994) of body language, authority, quantity of talk, access to space/tools, race, gender, and SES. After our initial read of the data we added *attention with teacher* as a code because of the frequent references PTs made in their statements. We read each written statement (n=33) and independently assigned the provisional codes to each statement; at times, a single statement referred to the same code multiple times (i.e. mentioned the quantity of talk more than once) and in these instances the code was assigned only once to that statement. Some statements had more than one code, thus, we ended up with 89 noticings, which are phrases that PTs used to describe status from the video.

Next, we analyzed the status hierarchy diagrams in which PTs ranked the students in small groups from the video in terms of status. We noted that in the diagrams to the first video group, one student was unanimously positioned as having lowest status. In the diagrams that correspond to the second and third video groups, two students were each unanimously positioned as having highest status within their respective group. We named these focus students of their respective groups. We then analyzed the written statements and coded only noticings within the statements that directly related to the focus student.

Findings

We present two preliminary findings. First, the PTs' noticings were concentrated in four characteristics related to attending: quantity of talk, attention with teacher, access to space/tools, and authority. Second, the characteristics of noticings varied by whether a focal student was positioned as low or high status. We expand on each of these finding below.

In relation to the first finding, as shown in Table 1, the frequency of the codes used across the 33 noticing statements, the most frequently used codes were attention with teacher (29%), quantity of talk (28%), access to space/tools (18%), and authority (17%). The codes we applied to the teachers' noticings came solely from the codes we characterized as attending in our methodology. It is notable that no noticings could be coded from the interpreting category (i.e. race, gender, SES). Although the PTs specifically used race and gender as markers to identify students in their statements, race and gender were not used to interpret and reason about why the students were positioned in the ways represented in the videos.

Table 1: Frequency of Codes

	Total noticings (n=89)	Focal student 1 (n=16)	Focal student 2 and 3 (n=38)
Attention with teacher	29%	0	39%
Quantity of talk	28%	38%	26%
Access to space/tools	18%	50%	5%
Authority	17%	0	26%
Gaze, Posture, and Gender	8%	12%	4%

For the second finding, the status hierarchy diagrams revealed that in the first video group, focal student 1 was consistently positioned as having low status. As shown in Table 1, the 16 noticings attributed directly to focal student 1 were concentrated in the categories access to space/tools (50%) and quantity of talk (38%). In the second and third video groups, focal students 2 and 3 were each positioned as having high status within their respective group. The 38 aggregated noticings directly attributed to focal students 2 and 3 were concentrated in attention to teacher (39%), quantity of talk (26%), and authority (26%). The noticings attributed directly to the low status student were different from the noticings of the high status students. For instance, when interpreting the interactions of a student positioned as low status, half of the noticings made reference to the ways in which a student was physically denied access to the space in which other group members were working. In contrast, the higher status students were *not* noted as *having* access to space/tool, taking up space, or holding onto tools. The PTs more frequently attended to the students as having higher status by noticing attention with the teacher, often in the form of explaining the current understanding of the group and asking questions. Similarly, attending to the high status students was through disciplinary and directive authority, which allowed the student to decide what the understanding of the group was at a time, ask questions, decide whose ideas to pursue, and direct other students what to work on at a given moment.

Implications and Conclusion

The preliminary findings from this study indicate that, similar to McDuffie and colleagues (2014), PTs are able to see status in small group interactions. What PTs noticed was the ways in which status hierarchies become visible in the small group interactions. The statements that teachers wrote as a part of this study did not reveal the ways in which the PTs interpreted and reasoned about why the status hierarchies exist within the small groups themselves. For example, focal student 1, was a Black male and we found no evidence of PTs hypothesizing race as a possibility of why the student was physically blocked from the ability to more centrally participate. CI is a pedagogical approach that suggests teachers be aware of the ways in which broader phenomena influence classroom interactions and students' opportunities for learning. In terms of noticing, PTs must connect what they see in videos or their own classrooms as instances of broader phenomena (van Es & Sherin, 2008). We wonder how PTs might draw on socially constructed categories to make sense of and reason about students' interactions when considering their own students they are familiar with, versus students they have limited knowledge about via video tape. This suggests that teacher educators should further consider the affordances and constraints of using video from different sources. Moreover, videos that represent students from different backgrounds and identities, do not necessarily ensure that PTs will bring larger ideological structures to hypothesize and reason about the students' interactions. As such, teacher educators need to be prepared to prompt discussion to bring issues of race, gender, SES, and class to hypothesize how these socially constructed categories privilege or constrain access to students' opportunities for learning.

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ONE UNIVERSITY'S STORY ON TEACHER PREPARATION IN ELEMENTARY MATHEMATICS: EXAMINING OPPORTUNITIES TO LEARN

Casedy A. Thomas
University of Virginia
cat6hu@virginia.edu

This multi-case study examines how three elementary mathematics methods instructors in the same teacher education program (TEP) provide their teacher candidates (TCs) with learning opportunities. The findings suggest that the instructors' beliefs and teaching philosophies influence the content that TCs have the opportunity to learn and the nature of the TCs' opportunities to learn (OTL). Through analytic induction, three assertions were developed to understand and explicate: similarities in OTL, differences in OTL, and perceptions about the purpose of the methods courses across the three cases.

Keywords: Instructional Activities and Practices, Teacher Beliefs, Teacher Knowledge

This multi-case study is part of a longitudinal study that focuses on pre-service teacher preparation in elementary mathematics and English Language Arts in a large-scale study involving six preparation programs across three states; a primary focus is on novice teacher perceptions about their experiences and learning opportunities in their TEPs. This study specifically examines how elementary mathematics methods instructors provide TCs with learning opportunities within their courses. Recent literature (e.g., Lampert et al., 2013) unpacks ambitious pedagogies in mathematics teacher education; yet, as documented by Clift and Brady (2005), there still much to learn about the various instructional strategies employed by mathematics methods instructors as well as the learning opportunities afforded to TCs (Cavanna, Drake, & Pak, 2017). Therefore, the study reported here is significant because it helps us to gain a deeper understanding about TCs' OTL in one of the TEPs, and therefore, may point toward what can be done in the future to better prepare teachers in elementary mathematics education.

Conceptual Frameworks

This study operates under Schmidt and colleagues' framework for OTL, defining it as, "the content to which future teachers are exposed as a part of their teacher preparation programs" (Schmidt, Cogan, et al., 2011, p. 140). This framework differentiates the content that TCs have the OTL within their mathematics teacher preparation coursework based upon four categories: mathematics, mathematics pedagogy, general pedagogy, and practical experience (Schmidt, Blömeke, et al., 2011; Youngs & Qian, 2013). In particular, the study is interested in TCs' OTL *high-quality* mathematics instruction as indicated by the National Council of Teachers of Mathematics (NCTM, 2000) process standards and the dimensions of the observation measure, Mathematics-Scan (M-Scan; Berry et al., 2017). Further, I take the position that "Teachers' beliefs influence the decisions that they make about the manner in which they teach mathematics" (NCTM, 2014, p. 10).

Content that Teacher Candidates Have the Opportunity to Learn

In addition to being knowledgeable about indicators of high-quality instruction, TCs need opportunities to acquire mathematics content knowledge (MCK) and pedagogical content knowledge (PCK). MCK has been defined as, "a comprehensive understanding of breadth, depth, connectedness and thoroughness" of mathematics (Ma, 1999; Hine, 2015b, p. 2). PCK is defined as "knowing a variety of ways to present content and assisting students in deepening their [mathematical] understanding" (Hine, 2015a, p. 483). There is also increasing support for mathematics knowledge for teaching (MKT) in teacher preparation, especially in elementary education (Delaney, Ball, Hill, Schilling, & Zopf, 2008). Thus, when instructors in this study are referencing to OTL "methods," they are generally talking about what is widely accepted as PCK or MKT, and when they mention content knowledge, they are talking about MCK. Having OTL MCK, PCK, and MKT helps novices teach for conceptual understanding and tackle students' mathematical misconceptions (e.g., Karp, Bush, & Dougherty, 2014; Cardone & MTBoS, 2015). The National Research Council (NRC, 2001) defines conceptual understandings as "an integrated and functional grasp of mathematical ideas" (p.18).

The Nature of the Teacher Candidates' Opportunities to Learn

TCs' OTL include their exposure to *representations* and *approximations* of practice. *Representations* in the context of teacher education have been defined as teachers having the opportunity to watch and/or read about others engaging in teaching practices (Cavanna, Drake, & Pak, 2017). *Approximations* emerged from the work of Grossman, Hammerness, and McDonald (2009); they essentially entail having safe places to practice what TCs will actually be expected to do as full-time teachers. Approximations can come in various forms such as practicing classroom management, engaging in teaching episodes, creating and/or grading assessments, and making lesson plans.

Research Questions

- How do three elementary mathematics methods instructors implement instructional strategies (teaching practices) in their courses and why?
- How do these elementary mathematics methods instructors provide opportunities to learn for teacher candidates?
- How do these elementary mathematics method instructors perceive the purpose of elementary mathematics method courses?

Methods

Participants and Procedures

The sample for this work includes the three elementary mathematics methods instructors at Robins University (pseudonyms are used for the university and the participants). The names of these instructors are William, Brittany, and Megan. In regard to the participants' level of education, William has completed his M.Ed. and was nationally recognized as an exemplar elementary teacher; Brittany is an associate professor with a Ph.D.; and, Megan is an assistant professor with a Ph.D. All of the instructors are White and have taught at the elementary level. Their ages range from late-30s to late-50s. The instructors taught two different, consecutive courses taken during TCs' fourth and fifth years within a five-year TEP.

Throughout the 2017 Fall semester, I gathered observational data both through classroom observations and "other reportable events." "Other reportable events" in this study includes informal conversations with the participants, e-mail correspondences, and/or brief scheduled meetings. I also collected additional forms of documentation while in the field performing these observations such as classroom artifacts and TC work samples. I conducted eight classroom observations for the three instructors accounting for 20 cumulative hours. The courses were at various times, on different days of the week, yet all classes were two and a half hours long.

Furthermore, I conducted two rounds of interviews with all three instructors. Prior to conducting classroom observations, I individually interviewed the instructors. The interviews were all recorded and transcribed allowing for member checking, and they ranged from 60 to 90 minutes. Following classroom observations, additional interviews were arranged to have the instructors further unpack some of the instructional strategies and learning opportunities that were noted within the observations. These interviews were all approximately 45 minutes.

Data Sources and Analysis

This study drew upon Erickson's (1986) model of analytic induction. As indicated in my methodological log, following each observation, fieldnotes were transferred into write-ups, and all of the audio from the interviews in the first round were transcribed. From write-ups, transcripts, and other reportable data and documentation, analytic memos were written intermittently to document emerging themes and inferences. The audio from the second round of interviews was listened to repetitively to document inferences and supporting evidence which appeared in my final analytic memo. Data sources were triangulated and re-read and re-coded to document emerging patterns and assertions. I compared confirming evidence and disconfirming evidence for each emerging assertion and continued to adjust the assertion until all evidence was accounted for, inclusive of connections between the assertions. Throughout this inductive process, data were reduced to three assertions for the focus of this paper.

Findings

Assertion 1: Similarities in Beliefs. Similarities identified in instructor beliefs associated with teaching philosophies led to parallels in the content that teacher candidates had the opportunity to learn as well as the nature of teacher candidates' opportunities to learn.

***What* TCs have the OTL.** All three of the mathematics elementary instructors engaged in instructional strategies that called for "non-traditional" forms of teaching, believing that mathematics instruction should be student-centered, open-ended, inquiry-based, and highly interactive. Although there were evident differences in instruction based on instructor beliefs (which will be unpacked in Assertions 2 and 3), all three instructors focused on teaching students for conceptual understanding, such that the instructors were pushing back against teaching procedures and rules for the sake of efficiency. All of the instructors discussed how these "rules" or memory tools have to be accompanied with activities that build conceptual understanding; otherwise, they will lead to mathematical misconceptions, especially if the rules expire as the students advance through mathematics (e.g., Karp, Bush, & Dougherty, 2014).

***How* TCs have the OTL.** The two most prevalent types of instructional strategies utilized were *representations* and *approximations* of practice (both have been defined within the theoretical framework). Though representations took various forms across the three classrooms, approximations most often appeared in the form of mini-lessons which the TCs had to plan and present as if they were in the classroom with elementary students.

Assertion 2: Differences in Beliefs. Differences identified in instructor beliefs associated with teaching philosophies led to differences in the content that teacher candidates had the opportunity to learn as well as the nature of teacher candidates' opportunities to learn.

When the methods instructors were asked about their teaching philosophies, they each placed emphasis on different instructional strategies. The observations revealed that even though the instructors collaborate extensively, they each have their own approach and beliefs which impact *what* and *how* TCs have OTL. In Case 1, William focused more (than the other instructors) on OTL that addressed the NCTM process standards (*what*) through reflection and feedback (*how*).

In Case 2, Brittany created OTL that placed more emphasis (than the other instructors) on MCK (*what*) through hands-on activities, technological applications, and interdisciplinary instructional strategies (*how*). In Case 3, Megan created OTL around research in the field (*what*), often having TCs engage in group work linking research and practice while focusing on the presence of multiple mathematical representations (*how*). Mathematical representations (different than those in Assertion 1) include symbols, graphs, pictures, words, charts, diagrams, and physical manipulatives used to demonstrate mathematics concepts (Berry et al., 2017).

Assertion 3: The three methods instructors expressed a common purpose of seeking to influence their teacher candidates' mathematical mindsets. However, the methods instructors each emphasized different (from one another) dimensions of high-quality mathematics instruction in terms of M-Scan when interviewed about the purpose of the elementary mathematics methods courses.

Though there are certainly similarities across the three cases, as witnessed in their desire to influence TCs' mathematical mindsets (e.g., Boaler, 2016), there were noted differences in how the methods instructors talked about and acted upon the purpose of the methods courses. These differences have been unpacked by looking at each case or instructor separately in this assertion. However, it is important to note that even though the instructors emphasized certain dimensions of high-quality instruction more than others when discussing their purpose, that does not mean that they did not address all dimensions of high-quality instruction at various points throughout the semester. William placed the most emphasis on cognitive demand, problem solving, discourse, and explanation and justification; Brittany on structure and accuracy; and Megan on cognitive demand, mathematical representations, mathematical tools, discourse, and explanation and justification. Even though this assertion focuses heavily on interview data, it also compares how participant interview data about course purposes compares to what was observed in the classrooms which was unpacked in Assertions 1 and 2.

Discussion and Educational Importance

In summary, the first two research questions align directly with Assertions 1 and 2. Assertion 1 focuses on teaching for conceptual understanding and combatting mathematical misconceptions (*what* they have the OTL) as TCs most often experience OTL through representations and approximations (*how* they have the OTL). Assertion 2 unpacks the differences identified in instructional strategies and OTL for TCs across the different methods instructors. Assertion 3 helps answer the third research question, addressing how the different methods instructors perceived the purpose of the elementary mathematics methods courses. Within this assertion, I have demonstrated how each method instructor's perception of the purpose of the methods courses aligned with dimensions of high-quality mathematics instruction as indicated by M-Scan, and in doing so, I have indicated differences across the three cases.

This multi-case study may prove influential as we continue to reflect upon the larger, longitudinal study and the development of ambitious instruction with TCs. However, as it directly relates to the purpose of this study, the findings expand upon literature (e.g., Clift & Brady, 2005; Cavanna et al., 2017) on the range of teaching practices in elementary mathematics methods courses and OTL for TCs. Further, this study, like others (e.g., Koedel et al., 2015), continues a larger conversation about an overarching critique in our field regarding how OTL may vary across methods courses, and so, TCs who attend the same TEP may have different experiences and OTL. This is pivotal as we reflect upon the development of high-quality or ambitious instruction and deserves further attention in mathematics education research especially when we contemplate the ways in which generalizations are sometimes made at the program level when unpacking teacher preparation. More qualitative work is needed in the field to examine similarities and differences in OTL across methods instructors for TCs in the same TEPs. Also, more work is needed to examine how these observed OTL correlate with beginning teachers' mathematics teaching practices across various dimensions of high-quality instruction.

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THE ROLE OF FIELD EXPERIENCES IN PROMOTING KNOWLEDGE INTEGRATION

David Barker
Illinois State University
dbarker@ilstu.edu

Matthew Winsor
Illinois State University
mwinsor@ilstu.edu

Tyeler Delrose
Illinois State University
tadelro@ilstu.edu

J. Vince Kirwan
Kennesaw State University
jkirwan@kennesaw.edu

Darl Rassi
Olivet Nazarene University
dsrassi@olivet.edu

In this study, we investigated the types of knowledge secondary preservice teachers (PSTs) gained from their field experiences and then integrated into methods course discussions. In particular, we analyzed the level of detail, stance, and nature of what they noticed in their field experiences and explored the influence this knowledge had on the PSTs opportunity to integrate their knowledge. Our analysis informed us that PSTs typically provide a wide range of general statements about pedagogy and learners when sharing observations from their clinicals, and this sharing resulted in a greater opportunity to integrate their knowledge. Directions for future research will also be discussed.

Keywords: Teacher Knowledge, Teacher Education-Preservice

Background

Secondary mathematics teachers must develop a thorough and connected understanding of mathematics (M), pedagogy (P), and learners (L) in order to plan effective lessons, implement mathematical tasks, and interpret student thinking. Unfortunately, many pre-service mathematics teachers (PSTs) do not have ample opportunities to connect their knowledge during their undergraduate programs. They often see their methods and mathematics courses as discrete, unrelated topics (Ball, 2000). In this paper, we investigate the role that field experiences may play in the important process of knowledge integration.

Field experiences provide PSTs an opportunity to notice student mathematical thinking, which NCTM (2014) has identified as an essential skill for teaching. Van Es, Cashen, Barnhart, and Auger (2017) reported PSTs attending to learning goals, student thinking, teaching strategies, classroom discourse norms, and connections. In addition, the knowledge gained from practice is more naturally integrated. Hiebert, Gallimore, and Stigler (2002) stated that a, "characteristic of knowledge that is linked to practice is that it is integrated and organized around problems of practice" (p. 6). The variety of knowledge that PSTs use in their field experiences provide teacher educators an opportunity to draw upon these field experiences as a means to promote knowledge integration. In this paper, we provide preliminary findings for the following research questions: 1) What knowledge did PSTs gain from field experiences that was shared in their methods courses? and 2) To what extent does the knowledge acquired from field experiences influence PSTs opportunities to integrate knowledge in their methods course?

Theoretical Foundation

For this study, we define *knowledge integration* as the coordinated use of multiple knowledge types in order to reflect on, or make, instructional decisions. Bishop and Whitefield

(1972) proposed that teacher decisions are made using a framework or schema. The main operation of a schema is to store knowledge through a network of connected pieces of knowledge called "elements" (Marshall, 1995, p. 43). The more connections that exist within a schema, the more useful the schema will be. Hence, the process of knowledge integration leads to the development of a more robust schema, resulting in the potential for more informed instructional decisions. The knowledge types used in this study are defined as follows:

> *Knowledge of Mathematics (M)*: knowledge regarding the mathematical content under investigation. Includes the connections and relationships among ideas, the ways and means of justifying and proving these ideas, and conversations focused on mathematics and reasoning about the mathematical topic.
> *Knowledge of Pedagogy (P)*: knowledge regarding the tasks, curriculum, instructional goals, or questions used to further the lesson. Includes comments centered on the lesson implementation or decision-making regarding the flow of the lesson.
> *Knowledge of Learners (L)*: knowledge regarding student thinking. Includes observed as well as anticipated student thinking, conversations about student characteristics, habits, or misunderstandings.

Our definitions of knowledge of mathematics, learners, and pedagogy are related to other models of teacher knowledge, such as mathematical knowledge for teaching (MKT) (Hill, Ball, & Shilling, 2008) or pedagogical content knowledge (Shulman, 1987). However, the definitions in this study are broader in scope, as we are focused more on the *interactions* among these knowledge sets (i.e. knowledge integration).

Methods

Field notes were collected for every session of a senior-level secondary methods course at a teacher preparation program in the Midwest. In conjunction with the methods course, PSTs participated in a ten-week field experience where they attended the same secondary classroom each day. Field notes were initially read to identify individual episodes—sections of the transcript centered on a single idea. Following the identification of episodes, the above definitions (i.e., M, L, P) were applied to code all episodes. If multiple forms of knowledge were used during a single episode, we defined this to be an example of potential knowledge integration. We note that evidence of multiple knowledge types does not ensure knowledge integration, only that the potential of knowledge integration was present.

Our next step in analysis was to further analyze the episodes where the PSTs shared what they noticed during their field experience. Each field-experience-related episode was coded to identify attending statements--statements made by a PST indicating she had focused her attention (Sherin et al., 2011) on something from her field experience. These attending statements were further analyzed and coded based on three aspects: level of detail (general to specific), stance (descriptive, evaluative, and interpretive), and category (objectives, pedagogy, timing/pacing, classroom structure, assessment, student behavior, student thinking, and student participation). The operational definitions for level of detail and stance were adapted from van Es (2011), and the operational definitions for the categories were developed through an open coding process (Miles et al., 2014).

Findings

From the field notes, we identified 57 (out of a total of 268) episodes that contained references to field experiences. During these 57 episodes a total of 146 attending statements were identified. In other words, there were 146 instances where a PST drew upon what they had noticed at their field experience to make a contribution to the methods class conversation. The following episode is an example.

> Matthew then moves to the problem $3(x+2) = 5x - 2$. Meredith says that this problem is troublesome because what if the students are missing appropriate prior knowledge such as the distributive property. Meredith also comments that the symbolism can be confusing to algebra students. Phyllis then shares that at her clinical placement her CT has been working with similar problems. The CT covered simplifying and solving equations in a single day because she assumed that the students already knew the material and it would just be review. Phyllis then shares that the students were confused the next day because they really did not understand.

This episode contains a conversation about the importance of addressing appropriate prerequisite knowledge, and the impact it may have on student understanding. In the middle of the conversation, Phyllis shares an observation she made from her field experience. She describes a situation in which the teacher assumed her students had the relevant prerequisite knowledge, covered the material quickly, and this unfortunately led to student confusion. Although this observation from her field experience did not add any new ideas, it reinforced a connection between the PSTs knowledge of pedagogy and learners.

The above example provides a single episode of a PST bringing an observation from their field experience to their methods class conversation. Table 1 and 2 display a summary of the 146 attending statements that the PSTs brought to their methods course conversations. Table 1 displays the number of attending statements at each level of detail and stance, Table 2 displays the number of attending statement for each category.

Table 1: Number of Attending Statements for Each Level of Detail and Stance

Level of Detail			**Stance**		
L1 - General	L2	L3 - Specific	Descriptive	Evaluative	Interpretive
16	48	9	44	6	23

Table 2: Number of Attending Statements for Each Category

Objectives	**Pedagogy**	**Pacing**	**Structure**	**Assessment**	**Student Behavior**	**Student Thinking**	**Student Participation**
6	25	3	3	7	7	18	3

From the data, we see that the PSTs brought a variety of attending statements to their methods course. As it pertains to knowledge integration, it appears that the field experiences were a significant source of statements regarding pedagogy and learners. However, this data alone does not imply knowledge integration and raises a few questions. The PSTs level of detail was not very specific; does this limit the potential for knowledge integration? Does the stance of the attending statement influence the potential for knowledge integration? Research has shown that when teachers make and justify their decisions they often use their knowledge in an integrated fashion (Barker et al., 2019). Does the act of interpreting lead more naturally to integration than describing? Although we cannot answer these questions with our current analysis, we can share some additional data on the influence these field experiences had on the potential to integrate knowledge.

Table 3 displays the number of episodes that contained the different knowledge combinations (M, P, L, MP, LP, ML MLP) for the episodes that included a field experience connection and those that did not. For instance, the 44 in the next to last column indicates that 44 of the 57 episodes that included a clinical connection resulted in statements that included the knowledge of learners and pedagogy.

Table 3: Knowledge Types Used in Episodes with and without Clinical Connections

	M	L	P	ML	MP	LP	MLP
Episodes with Clinical Connection	0	3	7	0	1	44	2
Episodes without Clinical Connection	4	7	85	2	16	86	11

82.5% of the episodes that included a clinical connection resulted in statements with more than one form of knowledge; 54.5% of the episodes that did not have a clinical connection contained more than one form of knowledge. The data appears to support the idea that including field experiences into methods course discussions can promote knowledge integration. However, for this particular class, the data also reveal that this integration is primarily between the knowledge of learners and pedagogy.

Having looked at the PSTs' integration of their knowledge of mathematics, learners, and pedagogy, we turned our attention to how PSTs might integrate the practical knowledge obtained at a clinical site with the theoretical knowledge gained in a methods course. We observed two distinct processes that appear to support PSTs' integration of theoretical and practical knowledge. The first we have termed *confirmation.* In this process, the PSTs are asked to look for and reflect on a particular aspect of teaching that they had read about in the methods course. Later, as PSTs debrief during their methods class, they are asked to describe events that they observed in their clinical experience that illustrate the particular idea from their readings. This process of "observe and report" allows PSTs to confirm the practical applications of the theoretical idea being examined in class. We view this as an initial step for integrating theoretical and practical knowledge.

The second process that appears to support PSTs' integration of practical and theoretical knowledge we have termed *responding to a dilemma*. In this situation, a PST presents a situation/question/dilemma from their clinical site, the class discusses the dilemma, and then proposes potential solutions. This process often requires the use of theoretical knowledge gained in class to address the challenge found in the practical setting of the clinical experience. It is our current belief that the process of confirmation is most beneficial to PSTs in their initial clinical experiences as they are acquiring the necessary knowledge for teaching. However, we would encourage the addition of the process of responding to a dilemma as they begin using knowledge for the purpose of instruction.

Discussion & Implications

Focused field experiences have the potential to be an important source and mechanism for knowledge integration. Based upon our preliminary analysis, PSTs bring a variety of knowledge from their clinical placements to their methods courses. In addition, episodes that contained references to field experiences were more likely to involve multiple types of knowledge. This coincides with the idea that knowledge that is linked to practice is "integrated and organized around problems of practice" (Hiebert et al., 2002).

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HOW DOES CONTENT KNOWLEDGE AFFECT SECONDARY PRESERVICE TEACHER LESSON PLANNING?

Matthew S Winsor
Illinois State University
mwinsor@ilstu.edu

National organizations such as the Association of Mathematics Teacher Educators recommend that secondary preservice teachers (PSTs) have a strong understanding the content they will teach. Other studies (e.g. Monk, 1994) suggest that the more mathematics courses preservice teachers take after a certain point do not affect their pupils' learning. This study aims to find evidence that participating in mathematics courses tailored for secondary PSTs as recommended by the Conference Board of Mathematical Sciences have the potential to help PSTs gain knowledge that promotes student-centered learning. This paper will share preliminary evidence from a case study that seems to provide evidence of a mathematics experience that promoted a change towards more student-centered teaching.

Keywords: Teacher knowledge, Instructional activities and practices, Teacher education-Preservice

National organizations such as the Association of Mathematics Teacher Educators (AMTE) and the Conference Board of Mathematical Sciences (CBMS) recommend that secondary preservice teachers have a strong understanding of the mathematics they will be teaching (e.g. AMTE, 2017, CBMS, 2001, 2012). Shulman (1986, 1987) as well as Fennema and Franke (1992) proposed that content knowledge, coupled with knowledge of pedagogy and knowledge of learners forms the foundation of teacher knowledge. In what seems contradictory to the claim that content knowledge is essential for teaching, Monk (1994) found that taking more mathematics courses in an undergraduate course of study provided diminishing returns with respect to pupil learning. Understanding how content knowledge affects teaching may facilitate planned experiences for preservice teachers that might help develop their teacher knowledge more effectively.

The objective of this study was to examine how a secondary preservice teachers' content knowledge directly affected their teaching. The literature on the negative effects of inservice teachers' limited content knowledge on teaching is established (e.g. Brown and Borko, (1992); Carlsen, (1993); Mosenthal and Ball, (1992); Stein, Baxter, and Leinhardt, (1990); Van Dooren, Verschaffel, and Onghena, (2002)). Given that preservice teachers are novice teachers, which could obscure how their content knowledge affected their teaching, I decided that lesson planning was a context in which preservice teachers could represent their content knowledge in a lesson without the pressures of actual teaching.

Theoretical Foundations of the Study

Foundational pieces of research on teacher knowledge (e.g. Shulman, 1986, 987) and mathematical knowledge for teaching (e.g. Ball, Thames, & Phlelps, 2008) have moved the field of mathematics education forward by showing that mathematics teaching is more than just being a successful mathematics student in college. Teaching mathematics requires teachers to use their knowledge of mathematics, learners, and pedagogy in tandem (Barker, Lannin, Winsor, and

Kirwan, 2018) to promote mathematics learning. The challenge for researchers is demonstrating how secondary preservice teachers employ their different types of knowledge while teaching.

One potential avenue for uncovering how preservice teachers' content knowledge affects how they teach can be found in Swafford, Jones, and Thorton, (1997). The authors wanted to discover what affect an intervention program designed to increase teachers' knowledge of geometry and student cognition had on teachers' instruction. The intervention consisted of a four-week summer seminar focused on geometry and the van Hiele levels (van Hiele, 1985) as well as six half-day seminars held during the academic year. Participants were asked to plan a geometry lesson before and after participation in the study. Participants were also assessed on their geometry knowledge and their own van Hiele levels before and after participation in the study.

Swafford, Jones, and Thorton (1997) found that teachers' content knowledge grew significantly. Participants' conceptual knowledge of geometry, as measured by the van Hiele levels also increased. Researchers measured the percentage of lesson goals that fell into the four van Hiele levels, level one being the lowest and level four being the highest. For the first set of lesson plans, it was found that 39% of the lesson goals were rated a van Hiele level one and 58% were rated a van Hiele level two. For the second set of lessons there was a change of 73% of lessons goals being rated at a van Hiele level two, which was significant with $p < .001$.

Given the findings from Swafford, Jones, and Thorton (1997), it seems that a lesson plan, connected with a content assessment, is a potential instrument to detect the connections between a preservice teachers' content knowledge and their knowledge of pedagogy. Shulman (1987) suggests that a preservice teacher would connect their knowledge of mathematics to their knowledge of pedagogy during the planning process. Shulman notes that the ideas that teachers understand must be transformed if they are to be taught to students. The transformation of ideas requires a combination of the following steps: 1) preparation of the material in the textbook including the process of critical interpretation, 2) representation of the new ideas in the forms of new examples, analogies, and so forth, 3) instructional selections from a set of teaching methods and models, 4) adaptation of the representations to fit the general characteristics of the students to be taught, and 5) tailoring the adaptations to the specific students in the class.

In each step described by Shulman (1987), there is a connection of different types of knowledge. For example, the representation stage requires teachers to connect their knowledge of mathematics to their knowledge of pedagogical artifacts, such as examples or analogies used in the lesson. Adaptation of representations to fit the general characteristics of the students being taught requires the teacher to employ knowledge of content, pedagogy, and of their students (learners) in tandem. Therefore, a lesson plan seems to be fertile ground for uncovering how the preservice teachers' knowledge of mathematics is connected to and affects their use of other types of teacher knowledge.

Methods of Inquiry

This study took place at a large university in the Midwest. I conducted a case study (Merriam, 1998) of six secondary preservice teachers, three females and three males. Participants were asked to plan an introductory lesson on the topic of functions for a class of high school freshmen before and after participation in my study. I chose the topic of functions because of the substantial work that has been done on preservice teacher understanding of the topic (Cooney, 1999; Even, 1990, 1993; Hitt, 1994; Lacampagne, Blair, and Kaput, 1995; Vinner and Dreyfus, 1989; Wilson, 1994). I also administered a pre and post content assessment focused on the concept of functions.

The main intervention of this study was a series of five study sessions where participants would be able to examine the concept of functions from an advanced standpoint, as recommended by CBMS, (2001, 2012). I used material from Usiskin, Peressini, Marchisotto, & Stanley, (2003) to help participants examine the concept of functions. Each study session was structured to start with a problem to solve. The problem was designed to lead into the topic of the day as well as to generate discussion. The lesson would then proceed into a variety of planned activities. The end of the study session consisted of completing several problems out of the text as a group. Participant solutions were offered and discussed. Finally, the journal prompt was put on the board for participant response and a small amount of homework would be assigned. All study sessions were recorded for later analysis.

Data Analysis

For the purpose of this paper, I will share the data from one participant, Craig. Craig believed that his college mathematics courses were not necessary to help him become a mathematics teacher because they did not have connections to the mathematics that he would teach in high school. Furthermore, he shared that he had learned all the mathematics he knew in high school. I chose to share Craig's data because there seems to be a clear connection between his content knowledge and what he planned in the lessons.

The first type of analysis I completed was examining the participant's definition of functions. I used Vinner and Dreyfus' (1989) categorization of teachers' definitions of functions to categorize Craig's definition of function both in the pre and the posttest. Then, I examined Craig's pre and post lesson plans looking for evidence that Craig's definition of function affected the content of the lessons. In the remainder of the paper I will provide detailed descriptions of shifts in content knowledge as well as how Craig's definition of functions seems to affect his pre and post lesson.

Findings

Pretest and Pre Lesson Plan

Craig's definition of function on the pretest was, "An equation in which each value of an independent variable results in one and only value of a dependent variable." Craig's definition corresponds to Vinner and Dreyfus' (1989) category 5, which says that a function is an algebraic expression, equation or formula. Moreover, in an additional question, Craig noted that, "a function is a specific type of equation." There is no direct mention of univalence (Even, 1993) where each element in the domain is assigned to a single element in the codomain. Craig's definition seemed to narrowly focus on the mechanics of functions whose domain and range are subsets of the real numbers.

Craig's pre-lesson plan seems to be motivated by his definition of functions. Craig's lesson starts by showing students a table of x and y values and then asking students to identify the pattern in the table. He continues with two more tables asking students to supply a y-value for certain given x-values based on the pattern students observe. The next segment of Craig's lesson focuses on examining the graph of $y = 2x$. He asks students to identify the y-values for given x-values from the graph. He then asks students to generate a formula that relates x and y. Craig then notes if a student identified that the graph identified was $y = 2x$ he would tell students that this is a function. The definition that he would give students of function is, "a function shows a relationship between two variables". Craig emphasized in his lesson that he would want to give a definition that was easy for students to remember and conceptualize.

Posttest and Post Lesson Plan

Craig's definition of function on the posttest was, "An operation that maps elements from a set A to at most one element each in set B." He also noted "equations are one way to represent mathematical functions." Craig's definition of functions shifted to what Vinner and Dreyfus (1989) call a correspondence, which notes that there is a relationship between two sets such that every element in the first set gets assigned to only one element in the second set. The concept of univalence (Even, 1993) is much more evident in Craig's definition.

Craig's post lesson plan started with representing sets with two ovals containing elements inside each oval. Arrows connected elements of the first set to elements in the second set in a way that was consistent with Craig's enhanced definition of function. Craig then stated he would give more examples and non-examples of functions (e.g. pets assigned to their owner), asking students to identify the main characteristics of a function. Craig notes that once he was satisfied with students' responses, he would change the elements of the two sets to be numbers and ask if they could identify the rule that assigned elements from the first set to elements of the second set. It seems that Craig connected his old definition of functions with his new definition of functions.

Discussion

Prior to participating with in the study, Craig was fluent with the procedural aspects of functions. He could find patterns, identify graphs that represented functions using the vertical line test, as well as identify 1-1 functions using the horizontal line test. Craig's pre-lesson plan seemed to mirror his definition of functions as well as his procedural fluency with functions. Craig's procedural view of functions was not altered by the required mathematics courses in his degree. Monk's (1994) finding that PSTs taking more math classes do not affect pupils' learning seem to coincide with Craig's beliefs that college math courses were useless and that he had learned all he needed to know in high school.

After participating in five learning sessions modeled after a capstone course (CBMS, 2001, 2012) and taught in a student-centered manner ((NCTM, 1989, 1991, 2000), Craig's definition of functions and the way functions were manifest in his lesson changed. Craig's definition focused more on the idea that a function is a relationship between two sets and less on equations and tables, which was mirrored in his post lesson plan. Perhaps because the study sessions focused on the more conceptual aspects of functions and that they were taught in a manner that was more conceptually focused promoted change in Craig's definition of and lesson on functions. From the evidence, it seems that mathematics courses can be meaningful to future teachers and can affect their teaching if they focus on the conceptual underpinnings of high school mathematics in a student-centered manner.

Moving forward, I plan to expand my analysis to focus on data collected from 45 secondary preservice teachers who participated in the same intervention at two other universities. I will examine questions such as: What evidence does a preservice teachers' lesson plan provide of their use of integrated knowledge? How does participating in a student-centered mathematics course focused on examining secondary school mathematics affect preservice teachers' ability to plan a conceptual, student-centered lesson on functions? How does a preservice teachers' understanding of function develop while participating in a capstone course as recommended by CBMS (CBMS, 2001, 2012)?

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NEGOTIATING DISCIPLINARY IDENTITIES DURING AN INTERDISCIPLINARY COLLABORATION MEETING

Hanan Alyami
Purdue University
alyamih@purdue.edu

Integrated science, technology, engineering, and mathematics (ISTEM) education provides opportunities for interdisciplinary groups to form. To explore the nature of interdisciplinary collaborations, I investigated the disciplinary identity of 11 preservice teachers (PSTs) enrolled in an ISTEM methods course. Each PST joined one of three interdisciplinary groups to collaborate on designing and implementing an ISTEM curricula. For this report, I analyzed the initial collaboration meeting of each group. Through positioning theory, findings indicate that PSTs take a strong disciplinary identity position in their discipline. Their identification in other disciplines varied. These identification levels were negotiated throughout the meeting by shifting either positively or negatively. Future research should investigate potential ways to disrupt negative disciplinary identity shifts, while encouraging positive ones.

Keywords: Disciplinary Identity, Positioning Theory, Preservice Teacher Education

The increased adoption of ISTEM curricula has increased the need to prepare teachers to design and implement ISTEM instruction (National Research Council, 2013). Preparing PSTs for such tasks could be achieved through providing opportunities to experience ISTEM with an emphasis on integration in methods courses (Stohlmann, Moore, & Roehrig, 2012). However, ISTEM methods courses would immerse PSTs who come from an established disciplinary way of knowing in an interdisciplinary community that is not as well-defined as their own disciplinary community (Bybee, 2013). Therefore, methods courses that emphasize ISTEM promote PSTs to collaboratively negotiate the boundaries of their disciplinary ways of knowing (Friman, 2010). To gain insight into the nature of interdisciplinary collaborations, I investigate PSTs' negotiation of their disciplinary identities during an interdisciplinary collaboration meeting. I specifically explore, through positioning theory, the following question: How do PSTs negotiate disciplinary identities as they engage in a collaborative interdisciplinary meeting?

Theoretical Framework

To answer the research question, I take a sociocultural perspective into identity formation. I specifically adopt positioning theory (Davies & Harré, 1990), which describes identity formation through an active process of interactive positioning (of others) and reflexive positioning (of self). Negotiating local expectations, practices, and roles in social activities leads to identity development (Nasir, 2002). Disciplinary identity then is a socially constructed identity that relates to a discipline-specific context. Such an identity can be negotiated through taking up, compromising, or withdrawing discipline-related positions. Harré and van Lagenhove (1999) represent the constant acts of positioning and repositioning with an equilateral triad: position, storyline, and social force. They clarify that social forces are any type of institution (as big as an organization or as small as a classroom group). This framework is adopted because if PSTs use storylines to make sense of their position relative to the surrounding interdisciplinary group context, then such social surrounding could impact their positioning and repositioning acts.

Methods

Context and Participants

Participants in this study comprise 6 undergraduate and 5 graduate students from a variety of STEM disciplines. Participants were enrolled in a 3-credit semester-long ISTEM methods course, designed for PSTs seeking a teaching certificate in one of the STEM disciplines, and for graduate students seeking a graduate certification in STEM education. While the students in the course were pursuing different degree areas at different levels, they are all identified in this study as PSTs because they were all enrolled in a methods course. I was the graduate teaching assistant for the course, which had two more co-instructors from different disciplines (i.e., technology and engineering education and agricultural education). Participants formed three interdisciplinary groups of their choosing (Table 1). Each group collaborated to design an ISTEM unit, which they eventually implemented in a middle-school classroom.

Table 1: Participants Groups, Degree Programs, and Unit Focus

Group Unit Focus	Participants (pseudonyms)	Degree programs
G1 Science	Jenna	Technology and Engineering Education
	Pamela	Science Education
	Faith	Science Education
	Laila	Mathematics Education
G2 Technology	Jesse	Technology and Engineering Education
	Mary	Technology and Engineering Education
	Audrey	Science Education
	Diana	Education Psychology
G3 Technology	Karl	Technology and Engineering Education
	Rosie	Technology and Engineering Education
	Leona	Technology and Engineering Education

Note: G3 was considered interdisciplinary since Leona has an MS in biomedical engineering.

Data and Analysis Process

During the semester, the groups collaborated at least 5 times. An audio recorder was placed on the table in front of each group during their collaboration. The data sources for this report are the three, 30-minute audio recordings of the initial collaboration meetings of each group. The initial meeting prompted PSTs to brainstorm at least three ideas in their ISTEM unit.

Because identity is constructed through the social practice of positioning a person in relation to their self and others (Davies & Harré, 1990), I analyzed PSTs' identity construction through their interactions with each other during collaboration meetings. The analysis explored PSTs' disciplinary identification with their own discipline and in relation to other disciplines. I used a whole-to-part inductive approach for coding (Erickson, 2006), beginning with playing and listening to the entire recording without coding, stopping, or pausing. In this phase, I took preliminary field-notes. I then played the video and used NVivo software as described by Wainwright & Russell (2010) to code the media files. At this stage, the unit of analysis was one or more sentences that formed coherent statements. Using magnitude coding (Saldaña, 2013), I conceptualized disciplinary identity as positions on a continuum of disciplinary-knowledge perceptions expressed by PSTs. The continuum ranged from very weak, on one end, to very strong on the other end. Coding involved how PSTs positioned themselves within the continuum and then the shifts in these positions as the collaboration evolved. I then identified and categorized the storyline and social context of the positioning shifts (Harré & Lagenhove, 1999). Negotiating disciplinary identity involved identifying shifts that occurred in the PSTs self-positioning along the continuum. Positive shifts were associated with changing one's position positively (e.g., starting at a moderate positioning then identifying more strongly as the collaboration evolved). Conversely, negative shifts were associated with changing one's position negatively (e.g., starting at a moderate positioning, then identifying weakly with the discipline).

Findings and Discussion

For their own disciplines, most PSTs had a very strong disciplinary identity. However, relative to other disciplines, there were varying degrees of disciplinary identification positions that were initially taken up. Negotiations of these positions were associated with PSTs shifting these positions (repositioning) during the collaboration, by identifying either more or less with the discipline. I present below the three themes that represent repositioning of disciplinary identification. While these themes were recognized in all three groups, I present them with examples in the form of focused episodes from each group. All three episodes also include evidence of PSTs' very strong disciplinary identity relative to their own disciplines.

Positive Shift: Relating to a Previous Experience of Other Disciplines and/or Finding Interest in Learning Other Disciplinary Perspectives

While many PSTs positioned themselves with a very strong disciplinary identity in their own discipline, some PSTs' initial disciplinary identification positions were moderate to weak in other disciplines. In many cases, negotiating such position involved shifting positively to a moderate or strong position. These shifts occurred by relating to the other discipline through familiarity with a concept or personal experience. For example, Jenna in G1 had a very strong technology education identity, but a weak science education identity. The following excerpt shows the positive shift from her initially weak science disciplinary identification:

> Jenna: I know squat diddly about like earth and space science, especially from the global warming standpoint, I know a lot about space! Because that was the environment I grew up in, I don't have any grand ideas besides just *space is awesome*, but it also depends…
> Pamela: … so I looked at the standards for like earth space and science for eighth-grade, and there's four, there's literally four, there's water cycle greenhouse gases and, maybe how we use the environment?
> Jenna: Okay that makes me sad, earth space and science…
> Pamela: … a lot of them in sixth-grade, then seventh-grade is bio, and eighth-grade is chemistry and physics, um sorry a little bit of bio then chemistry, so she (the cooperating teacher) also says animal classification, I don't know if you are more familiar with that?
> Jenna: Nope! [laughing] So one of the reasons I'm doing this is I want to learn like outside of my field

By relating to her knowledge and love of space, Jenna shifted her science identification positively to moderate. Prior experiences are recognized as a strong influence on the identities of new teachers (Flores & Day, 2006). Another form of negotiating disciplinary identity involves being eager to learn about other disciplines (Shen, Sung, & Zhang, 2015), which Jenna indicated: that she joined the ISTEM methods course because she wanted to learn outside her discipline.

Neutral: No Change in Disciplinary Identification Position

There were some moments where PSTs did not negotiate the disciplinary identification of other disciplines. For example, G2's ISTEM unit was technology based with an integration of science. All the members in G2 had very strong disciplinary identities relative to their own discipline. The following excerpt occurred towards the end of brainstorming some ISTEM unit ideas that were focused on technology and science education, and relates to Diana's identification with those disciplines:

> Diana: Whatever you guys decide, I'm gonna probably go with the person who's doing the introductory lesson, because I really don't know anything about this stuff
> Jesse: That's fine, you'll do great
> Diana: I hate to ask for a special offer, but after you figure out the lesson, I can incorporate maybe a motivational aspect to it, and then I'm gonna do the introductory

Diana had a very strong education psychology identity. She even advocated for that identity through potentially implementing a motivational aspect to her lesson. However, knowing "nothing" about either science or technology disciplines constrained her flexibility. If identity development is viewed as an increased involvement in the social context (Lerman, 2001), Diana's limited willingness to learn about other discipline implies a minute shift in her position.

Negative Shift: Compromising One's Disciplinary Identification Position

To have an interdisciplinary group, everyone could interactively have a very strong identity relative to their own discipline. Yet the following excerpt indicates the varying levels of disciplinary identification during the initial discussion to decide between focusing the ISTEM unit on electricity or manufacturing:

> Rosie: Electricity is not my strong suit, that's for sure, I'm willing to *learn it*, just a fair warning on that
> Karl: I like circuits, circuit chips are fun…
> Rosie: … I'm saying as long as someone is willing to help me with it I'm willing to pick it up
> Leona: Yeah, so, I mean I was an electrical engineering major for my undergrad, and that's ultimately what I would like to go back and teach at the college level, so, I would obviously love to do electricity, but my current research is on perceptions of manufacturing at this grade level so really either way
> Rosie: I am fine with either one, just a fair warning on the electricity stuff, that someone's gotta help me with that
> Karl: I know more about electricity than I do manufacturing, but compared to you, I know nothing, if you majored in it [talking to Leona]

When Leona positioned herself with very strong electrical and manufacturing education identities, she interactively positioned Rosie and Karl with weaker identities. In this case, they can either take up this positioning, or attempt to negotiate it (Harré & Lagenhove, 1999). The excerpt shows how Karl started with a moderate electrical education identity. However, he took up the interactive position that Leona presented by compromising his position, and negatively shifting his electrical education identity from moderate to weak.

Conclusion

In their own disciplines, PSTs in this study had strong or very strong disciplinary identities. However, identification in other disciplines was initially weak to moderate. Positive shifts in PSTs' identification in other disciplines occurred by relating to previous knowledge or personal experiences of the discipline. Positive shifts were also associated with willingness to learn about other disciplines. Negative shifts occurred when a PST compromised their position if another PST expressed a very strong disciplinary identity. The varying degrees of disciplinary identities must be acknowledged when an interdisciplinary group comes together. Further research should look at minute disciplinary identity negotiation and potential ways to disrupt negative shifts in identification while encouraging positive ones.

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TEACHING ACROSS DISCIPLINARY BOUNDARIES: A CASE STUDY OF A PRESERVICE TEACHER TEACHING OUTSIDE HER DISCIPLINE

Hanan Alyami
Purdue University
alyamih@purdue.edu

Teachers of integrated science, technology, engineering and mathematics (ISTEM) curriculum find themselves responsible for teaching disciplines outside their areas of certification. To gain insight into the experience of teaching a different discipline, I explore how a preservice teacher (PST) in an ISTEM methods course facilitated discussion when she micro-taught a lesson from a different discipline. Through a commognitive lens, I examined how Pamela, the PST, facilitated the discussion to elicit learning. Preliminary findings show that when disciplinary boundaries were crossed, discipline-specific words were not recognized or addressed. Teachers implementing ISTEM curriculum could benefit from negotiating authority and participating as learners, through embracing uncertainty in unfamiliar disciplinary discourses.

Keywords: Commognition, Instructional Activities and Practices

Research on ISTEM education is providing evidence of its positive impact on learning (Honey, Pearson, & Schweingruber, 2014). However, teachers implementing ISTEM curricula supplement the teaching of content outside their area of academic certification. In such cases, teachers' disciplinary ways of knowing are challenged (Friman, 2010). In education, integration aims to engage disciplinary communities and merge disciplinary perspectives (Wang, Moore, Roehrig, & Park, 2011). However, while implementing an ISTEM curriculum, some teachers have expressed their preference for teaching their own discipline (Stohlmann, Moore, & Roehrig, 2012). Teachers also expressed seeing themselves as experts in their own discipline but novices in others (Venville, Wallace, Rennie, & Malone, 2002), indicating a sense of competence in teaching their own discipline. Yet policy documents (e.g., National Research Council, 2013) continue to ask teachers to teach content outside their disciplines as part of ISTEM curricula. To gain insight into the development of learning to teach ISTEM curricula, I address the following question through a sociocultural perspective: How does a PST in an ISTEM methods course elicit learning when teaching outside of their discipline?

Theoretical Framework

To explore teaching curricula outside one's discipline, I focus on classroom discourse as viewed by Sfard (2000) as "the totality of communicative activities, as practiced by a given community" (p. 160). Learning in a classroom is conceptualized as a change in discourse (Sfard, 2008). Discourse is described through four characteristics: word use, visual mediators, routines, and narratives. I specifically focus on word use because discipline-specific words have explicit definitions and are used in unique ways in that discipline (Morgan & Sfard, 2016) despite use and overlap of such words in other disciplines and/or colloquial discourses. A teacher crossing disciplinary boundaries might be familiar with discipline-specific words, but not in the specific way the words are used in that discipline. For example, the word *scale* means something different in a mathematics discussion than it does in colloquial speech. Teaching through the commognitive lens involves creating opportunities to elicit and encourage changes in discourse.

Eliciting learning through changing word use involves providing opportunities for learners to expand their discipline-specific words and properly use these words (Tabach & Nachlieli, 2016).

Methods

Context and Participant

A 3-credit ISTEM methods course was the site of this study. The course was designed for PSTs seeking a teaching certificate in a STEM major and for graduate students seeking a graduate certificate in STEM education. There were 6 undergraduate students and 5 graduate students from various disciplines that formed three groups. Only one group included a mathematics education student. The course was taught by two co-instructors from disciplines outside mathematics education (technology and engineering education and agricultural education). I was the graduate teaching assistant for the course and provided resources for mathematics education as required. Students in the course engaged in three micro-teaching sessions, where they taught curriculum for 25 minutes to peers and course instructors. Students in the first micro-teaching session presented a lesson from their own discipline. During the second and third micro-teaching sessions, students presented lessons from an ISTEM unit designed by the groups throughout the semester.

In this case study, I focus on how Pamela elicited learning during her micro-teaching sessions. Pamela is a science PST who focused her first micro-teaching on an introduction to (tectonic) plate boundaries, a topic from the science curriculum. Her third micro-teaching session was focused on scale and scaling, a topic from the mathematics curriculum. Prior to micro-teaching the scale lesson, Pamela asked to meet with me to inquire about her approach to the lesson. During the 30-minute meeting, I encouraged her to consider scale as a proportional relationship between two quantities.

Data and Analysis Process

All micro-teaching sessions were videotaped. Videos of Pamela's first and third micro-teaching sessions are the data source. The analysis in this report explores how Pamela utilized words to create opportunities for learning. I used a whole-to-part inductive approach (Erickson, 2006), beginning with watching the entire video without coding, stopping, or pausing. In this phase, I took preliminary field-notes focused on the teacher's facilitation of the discussion. I then played the video and used NVivo software to code the media files (Wainwright & Russell, 2010). To ensure that the video coding of words was appropriate for the disciplines of mathematics and science, I partnered with a science-teacher educator. Together we identified video excerpts in which Pamela used words to elicit learning.

Findings

In this section, I present some episodes from the micro-lessons. These episodes are representative of Pamela's facilitation of the micro-teaching sessions and her use of discipline-specific terms to elicit learning in her own discipline and in a different discipline. When micro-teaching her own discipline, Pamela elicited information and then provided the discipline-specific terms. Conversely, when micro-teaching a different discipline, Pamela's approach involved attempting to elicit information through discussion, then providing the discipline-specific term; however, that approach evolved to the point that she began to participate in the discussion as a learner.

Learning Science through Word Use

Pamela's science lesson was an introduction to tectonic plate boundaries. The driving question of the lesson was "what happens when plate boundaries meet?" She gave students time to discuss the different possibilities of plate interactions. Students mentioned a few boundary interactions, such as "a plate going underneath the other one", or that plates "rub up against each other, and kinda shear" After the initial discussion, Pamela used student's responses to introduce the appropriate discipline-specific words (i.e., divergent, convergent, and transform). Pamela encouraged students to use these discipline-specific terms during the follow-up activity, where students used a graham–cracker-and-frosting model to mimic and further investigate the plate interactions. Pamela's science teaching shows an intentional choice of lesson structure and ongoing support for the acquisition and use of key terms. She used the earth- and space-science terms that are used to describe tectonic plate interactions: divergent, convergent, and transform. Pamela's familiarity with the science discourse meant that her approach to elicit students' learning of these terms started with eliciting colloquial descriptions of plate interactions. Pamela then linked these descriptions to discipline-specific words and facilitated students' adoption and use of these discipline-specific words to describe plate activity throughout the cracker-and-frosting modeling activity.

Learning Mathematics through Word Use

Pamela's mathematics lesson was a part of an ISTEM unit that she and three of her classmates collaborated on designing. The unit was eventually implemented in an eighth-grade classroom, where students were to design a scale-model prototype of a zoo enclosure. Pamela started the micro-teaching session by attempting to find a scale factor by exploring the relationship between a scale drawing of a building footprint and its footprint in real life. As an inquiry-based approach, Pamela worked to build interest by using the footprint of the building containing our classroom. She handed out an image of the building's footprint (see Figure 1).

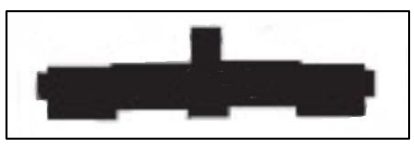

Figure 1: Image of the Building Footprint

To elicit learning of the term scale, Pamela considered approaching the concept by establishing the proportional relationship between the building's footprint on the map, and its actual footprint. The following excerpt includes Pamela's initial question:

> Pamela: Using your ruler, I would like you to measure, um luckily, we are in a building that is kinda shaped like a rectangle, so your best approximation of how long and how wide it is in centimeters
> [gives a minute for students to find the measurements]
> Pamela: So can one group tell me on the map, how many centimeters long the building is?
> Audrey: one!
> Pamela: and how wide is it?
> Audrey: well, the skinniest part is 0.2 but the widest part is 0.3
> Pamela: okay, so let's say it's 0.25, and you guys have a class set of calculators, so can someone tell me what the area of the building is on our map, in centimeter squared?

In this excerpt, Pamela's request for the dimensions of the building reflects her association of the discipline-specific term *area* with a formula for computing the area of regular figures: length × width. Approaching area using the computational formula of length times width has been identified as a constraint that prevents connecting area to the measurement of space (e.g., Baturo & Nason, 1996; Kamii & Kysh, 2006). Pamela's approach to finding scale factor resulted in her use of the discipline-specific term area. Yet calculating area through the length times width algorithm constrained her flexibility when students suggested multiple measures for a dimension of the building. Her move was intended to find dimensions and compute a number to represent area on the map that could be related to the area of the building footprint in real life. In this move she shared her view of area as a computed quantity rather than a word whose use in relation to scale could be discussed.

After finding an approximate area of the building's footprint using the map, Pamela encouraged the students to find the relationship between the building's footprint on the map and its actual footprint. The following excerpt illustrates how that conversation played out:

> Jesse: It's 400 times bigger [referring to the building's footprint being 400 times bigger than its footprint on the map]
> Pamela: How did you get 400? What did you do to the numbers to get 400?
> Jesse: I guess, I don't know, I think I did 0.25, one-fourth of one, and then somehow, I got to 400, from there to 1600
> Pamela: okay, okay, so, 400 times 4 is 1600, and 0.25 is one-fourth, so there's something maybe with 4 there? and that's how you got 4, one-fourth, 400, okay, any other thoughts on how we might be able to relate these two numbers [the building's footprint on the map and in real life]?
> Leona: I divided the 1603.6 [the footprint of the building in square meters] by 0.0025
> Audrey: shouldn't be 0.000025
> Leona: Oh, let me do it one more time then. It's much bigger, it's like 6 million.
> Jesse: Okay that makes more sense, it's much bigger than 1600
> Pamela: and how is this number related to our map? [pointing at the 6 million]
> Rosie: So the actual building is that many times bigger than the measure on the map

Ultimately, the computation of a scale factor became difficult for Pamela, making her micro-teaching of the mathematics lesson less structured compared to the intentional structure of her science lesson. However, this excerpt shows Pamela embracing uncertainty and allowing the discussion to continue. By letting the students converse, Pamela became a learner in the exploration of area. There was room for students to share their ideas and strategies. While the discussion did not lead to an exact number, it did lead to a discussion about the meaning of the relationship between the two numbers, thus leading to the concept of scale.

Discussion

This case study aimed to examine a PST's experience when teaching outside her area. Pamela showed expertise in designing and micro-teaching a lesson from her own discipline. When she crossed the disciplinary boundaries, her performance could be considered equivalent to that of a novice PST (Baturo & Nason, 1996). However, Pamela is not a novice PST. Her familiarity with science led her to identify the discipline-specific key terms prior to micro-teaching the lesson, purposefully integrating these terms in the structure of the micro-lesson for students to adopt and utilize. However, when micro-teaching the mathematics lesson, Pamela's unfamiliarity with that discourse led to a less direct approach to elicit learning of the term *scale*. As the lesson progressed, Pamela embraced uncertainty and positioned herself as a participant in the discussion, and the structure of the lesson became more open and flexible to the students' negotiations about the meaning of scale. If teachers teaching outside their areas do not have to position themselves as knowledge providers or expert participants (Sfard, 1998), there is more potential for learning to occur if they negotiate their authority and position themselves as learners in the discourse. This was evident during Pamela's last episode. Teachers implementing ISTEM curricula need to be aware of how they use and discuss the use of discipline-specific words (Stohlmann et al., 2012). It is especially important for teachers to participate as learners in the discourse, to allow the discourse change and for learning to take place.

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SECONDARY PRE-SERVICE TEACHERS' MEANINGS FOR SUBTRACTING A NEGATIVE NUMBER

Jodie Novak
Univ. of Northern Colorado
jodie.novak@unco.edu

Alees Lee
Univ. of Northern Colorado
alees.lee@unco.edu

Shweta Naik
Univ. of Northern Colorado
shweta.naik@unco.edu

The paper describes 19 pre-service secondary teachers' (PSSTs) responses on a pre-assessment that elicited their meanings for subtracting a negative number. We found several references in their responses that built upon these meanings and here we report on four prominent categories: 1. Subtraction and the negative sign are explained through movement and direction on the number line, 2. Subtraction and negative are the same, and they often cancel each other, 3. Subtracting a negative is the same as adding, and 4. Subtraction is an action. The paper elaborates on these meanings, with examples and a possible reasoning for their emergence.

Keywords: Subtracting integers, Meanings of integer subtraction, Secondary preservice teachers

Operations with integers is a standard part of the secondary math curriculum. Although initial instruction happens in 6th and 7th grade (CCSSM, 2010), many students struggle with operations with integers throughout high school. Consequently, all secondary teachers need a deep understanding of operations with integers to effectively support student success.

Research with pre-service secondary teachers (PSSTs) on operations with negative numbers is less common. The ones that exist deal with different aspects, like representation on the number line, models of teaching, and use of contexts (Cunningham, 2009; Steiner, 2009). These studies make a call for research to bring forward the meanings associated with subtracting integers. In this paper, we focus on what meanings PSSTs associate with subtracting a negative number, which we believe has a bearing on how they will make sense of the representations and contexts and use them in their future teaching.

Subtracting a Negative Number: Meanings and Frameworks

Researchers have recently begun discussing pre-service teachers' efforts to make sense of integer subtraction (e.g. Almeida & Bruno, 2014; Bofferding & Richardson, 2013). The difficulty of subtracting negatives is not only the concern of school mathematics (Schindler et al, 2017) but is seen among adults and university students. They suggest that many concerns with subtracting a negative are associated with the continuum conjectures that students make based on their decade of experience in subtracting positive numbers.

To understand the difficulty in subtracting a negative, researchers have developed frameworks that emphasize the "meaning" of the negative sign. Vlassis (2004, 2008), emphasizes the symbolic aspect and focuses on three uses of meanings of the minus sign: the unary, the binary and the symmetric functions. In the unary function, the minus sign is attached to a number to form a negative number, as in "– 6". A binary use of the minus sign refers to the subtraction operation in arithmetic or algebra, as in "5 – 3". The first two uses correspond to the distinction emphasized in other studies between the use of the minus sign to indicate a signed or directed number and the use of the minus sign to indicate the subtraction operation (Glaeser, 1981). The symmetric use of minus sign is less frequent and refers to taking the additive inverse of a number, as in $-(-6)$. Our study brings out additional details of these meanings that are attached to the symbols, rather the quantities.

Study Design, Participants, Analysis

Data for this study was collected within a math elective course tailored to deepen preservice secondary mathematics teachers' understanding of secondary mathematics topics. The study participants were 19 undergraduate students enrolled in a single section. One participant was an elementary mathematics preservice teacher; eighteen participants were secondary mathematics preservice teachers in the secondary education program. Three students presented as men; the rest presented as women. We refer to all students as preservice secondary mathematics teachers (PSSTs).

During a recent fall semester, the first and second authors co-taught the aforementioned course, beginning with a three-week unit on operations with integers. Before starting the unit, they administered a preassessment to gain information about PSSTs understandings of subtracting a negative integer. The assessment contained a single, open-ended question: "How would you support a student who is struggling to understand why $4-(-3)=7$?" The PSSTs' mathematical responses included diagrams, models, and written explanations. Using a grounded theory approach (Corbin & Strauss, 1990), each author open-coded the responses, marking and commenting on aspects of each response that seemed to portray a meaning the PSST held. One author then collected and analyzed what the others had identified, coming up with 23 references to meanings from the collection. These meanings were then brought to consensus and categorized to portray PSSTs' understandings of subtracting a negative number.

The findings we report here are part of a larger study that examines how PSSTs' understanding of integers and operations with integers has significance to their meaning making process with different contexts and representations and how that changes throughout instruction. This paper specifically addresses the research question: What meanings do preservice secondary mathematics teachers bring for subtracting a negative number?

Results: Meanings and Representations

Given the nature of the question on the assessment, words like "subtraction", "subtract", and "subtracting" pervaded PSST responses as well as phrases like "take away" and "minus(ing)". Here we describe the four meanings we found among the PSST's responses.

Meaning 1: Subtraction and the Negative Sign are Explained Through Movement and Direction on the Number Line

In this collection of responses, we see three uses of direction, usually in reference to a number line. In the first, which we categorize as *subtraction is going backwards*, PSSTs suggest that when one sees a subtraction sign one moves "backwards" on the number line. One description suggested that "all the operations are about which way to go" on the number line, and subtraction is to go backwards. In these elaborations, we assume, consistent with the responses, that moving right is going forward and moving left is going backward.

In the second use, *subtraction and negative both means going backwards*, PSSTs identified subtraction and negative as two separate entities, but they assigned the same direction to both, with language like "If one (-) sign makes you go backwards then 2 (-) makes you go backwards and then turns you around again." This reference to "backwards" is slightly different than the first use; this is a relative "backwards". In the context of $4-(-3)$, a beginning forward direction is assumed, the first subtraction sign suggests move backwards, the second negative sign suggests “backwards again” and that would mean “turning around”.

In the third use, *subtraction says moving left, negative say flip direction*, the PSSTs say things like “The minus sign says to go to the left (←) negative direction, but the negative sign says to flip that direction and go the other way, right (→) positive direction”. All three uses suggest a procedure, or journey on the number line, to arrive at the result of 7, rather than attending to the meaning of the terms and symbols in $4 - (-3) = 7$.

Meaning 2: Subtraction and Negative Are the Same, and They Often Cancel Each Other

We see PSSTs talking about two negatives making a positive in three ways: by canceling, by analogy and by drawing. The first two ways are seen in the literature (Bofferding & Richardson, 2013), but not the last one. PSSTs talk about canceling as a way of manipulating the symbols using language like “when we have two negative signs, they cancel each other out.” Second, the PSSTs appeal to an analogy with the English language where “a double negative turns out to be positive”. In the third example, the PSSTs describe a drawing process they would use to literally turn the two negative symbols into an addition symbol. They describe this with language like “the two negatives make a plus sign” and a picture similar to Figure 1. These responses indicate a reliance on a visual mnemonic rather than a separate meaning for subtraction and the negative sign, which could be problematic when trying to convey meanings to students.

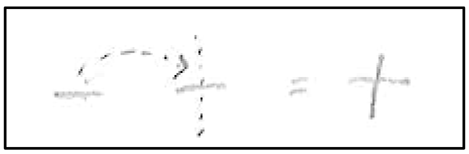

Figure 1: Two Negatives Make a Plus Sign

These teachers did not attend to the meaning of subtraction as an operation nor the meaning of a negative integer as a signed number. Rather, they had non-mathematical strategies for dealing with two negatives when they occur sequentially in an expression.

The language of canceling is common in algebra when students are simplifying algebraic expressions, so the language of canceling is not surprising. What may have started for these PSSTs as a mnemonic to help them remember how to subtract a negative has, at some level, become a meaning for them. The drawing method could be considered a visual mnemonic for remembering the phrase “two negatives make a positive.” What is particularly striking about this meaning is the lack of attention to the difference in the meaning of “-” as a subtraction symbol and as an indication of the sign of a quantity.

Meaning 3: Subtracting a Negative is the Same as Adding

We consistently saw the phrase “subtracting a negative number” in the PSSTs’ responses, although sometimes they use “minusing” or “take away” instead of subtracting. However, the completion of the phrase “subtracting a negative number” is ambiguous in all but one case: “subtracting a negative quantity from another quantity would yield the same result as if you were to add the positive amount of that quantity.” This is the only statement observed that refers to the minuend and language of quantity. The other 4 statements end in a phrase similar to “is the same as adding it” where what “it” is is ambiguous, and which likely refers implicitly to the quantity aspect of the negative number being subtracted.

What’s notable about this meaning is that students are attending to what is being subtracted, whether they use the language of “negative number” or “negative quantity”. Unlike in meaning

2, these PSSTs are distinguishing between when the symbol “-” denotes subtraction and when it is associated with a negative number. In the statement involving quantity, we see an important distinction between the process of the subtracting one number from the other and the result of the process.

Meaning 4: Subtraction as Action

PSSTs portrayed four different meanings for subtraction: take-away, negative direction, going backwards, and adding a negative. The take-away meaning was apparent in one PSST’s use of an integer chip model in which the PSST equated crossing out three negative chips in a picture with “take(ing) away 3 negative chips”. A PSST discussed subtraction, or the minus sign, as going left or in the “negative direction” on a number line. Similar to the negative direction meaning, a PSST stated that “when we subtract, we go backwards on [a] number line”. One PSST chose to redefine subtraction as “adding a negative” to facilitate the rewriting of $4 - (-3)$ as $4 + -(-3)$.

By explicitly assigning an action meaning to subtraction, PSSTs provided a glimpse into how they think about the act or process of subtracting. The first three meanings: take away, negative direction, and going backwards, are closely tied with representations. Researchers have shown that representations assist in reasoning about and making sense of mathematical situations, facilitating student learning (Mitchell et al., 2014), and in communicating thinking (Janvier,1987; Schwartz et al., 1993/94).

Conclusion and Implications

In this study we see PSSTs use a range of language, mnemonics, and representations to describe an understanding of why $4 - (-3) = 7$. As language plays an important role in meaning making and communicating meaning (Vygotsky, 1962), the unstable, inconsistent and ambiguous language across these responses suggests a need for pre-service secondary mathematics teacher programs to include intentional opportunities for PSST to develop rich, consistent meanings of operations with integers. The meanings represented here, if teachers teach these meanings, generally do not distinguish enough between the meaning of subtraction and the meaning of a negative for students to be able to construct their own meaning and make sense of subtracting a negative number.

We hoped that the PSSTs would use the meaning of the symbols in the expression $4 - (-3)$ to explain why the result is 7. Instead, they had a variety of ways to justify switching subtracting a negative three $[4 - (-3)]$ to adding a positive three $[4 + (+3)]$ without attending to the meaning of subtraction or a negative number. That is, they treated $-(-3)$ as a procedure, but they did not attend to $4 - (-3)$ as a process. Unfortunately, some curricula (EngageNY, 2015) do not attend to the meaning of subtracting a negative because they define subtracting a quantity as adding the inverse or the opposite of the quantity. In order for our PSSTs to effectively teach their students the meanings associated with operations with integers, they need to develop those meanings for themselves and understand how their discussion of the pre-assessment task focuses on procedures instead of meanings.

We believe one of the reasons the PSSTs explanations were so varied is due to the nature of the prompt which positioned the PSSTs as teachers, not doers of mathematics. That is, they were not called on to determine what $4 - (-3)$ equals, but to support a struggling student to understand why $4 - (-3)$ is 7. For that specific purpose, they created representations which gave us access to the meanings they have for themselves and which they think might be useful for teaching. By incorporating representations into their responses, the PSSTs recognized the importance of utilizing representations for communicating meanings. This shows promise for the PSSTs establishing meanings closely tied with representations.

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INDIRECT REASONING TASK FOR PROSPECTIVE SECONDARY TEACHERS: OPPORTUNITIES AND CHALLENGES

Orly Buchbinder
University of New Hampshire
orly.buchbinder@unh.edu

Sharon McCrone
University of New Hampshire
sharon.mccrone@unh.edu

We describe an instructional module aimed to enhance prospective secondary teachers' (PSTs') subject matter knowledge of indirect reasoning. We focus on one activity in which PSTs had to compare and contrast proof by contradiction and proof by contrapositive. These types of proofs have been shown to be challenging to students at all levels and teachers alike, yet there has been little research on how to support learners in developing this knowledge. Data analysis of 11 PSTs, points to learning opportunities afforded by the module and the PSTs' challenges with indirect reasoning.

Keywords: Teacher Knowledge, Reasoning and Proof, Instructional Activities, Indirect Proof.

Introduction and Theoretical Perspectives

Reasoning and proof have been at the center of substantial research efforts in the last decades, yet, very few studies have focused on indirect reasoning, proof by contradiction and by contrapositive (Stylianides, Stylianides & Weber, 2017). By indirect reasoning, we mean any instance in which a person makes an argument of the form: "it is not possible because otherwise …". Students use indirect reasoning informally, when checking for errors (Thompson, 1996), and formally, when proving by contradiction or by contrapositive. A proof by contrapositive relies on the logical equivalence between a conditional statement S and its contrapositive $(P \Rightarrow Q) \equiv (\sim Q \Rightarrow \sim P)$. A proof by contradiction assumes the negation of S: $(P \wedge \sim Q)$ and proceeds directly to obtain $r \wedge \sim r$ (where r is any statement), which is a contradiction, implying that S is true. Despite the similarities, (e.g., both have $\sim Q$ in the assumption, and contain direct steps), there are also differences, for example, a contradiction can take many forms, but a proof by contrapositive necessarily ends with $\sim P$ (Yopp, 2017).

The studies on indirect proof suggest that students view these types of proof as less convincing and particularly difficult to construct (Harel & Sowder, 1998; Leron, 1985; Antonini & Mariotti, 2008), specifically struggling to construct negations (Lin, Lee, & Wu Yu, 2003). Recently, more research attention has been directed towards studying students' understanding of proof by contrapositive (Yopp, 2017) and proof by contradiction (Brown, 2018). However, almost nothing is known about what types of interventions or pedagogical supports could help to mitigate students' challenges in this area.

Indirect proof is important in both tertiary and secondary mathematics. For example, proof of congruency of alternate interior angles created between parallel lines and a transversal, or proofs of sums and products of rational and irrational numbers are done by contradiction. Thompson (1996) asserts that teachers can support secondary students' understanding of indirect proof. But to do that, teachers themselves, must have robust knowledge of indirect proof and of pedagogical strategies for teaching it. Our review of educational literature found no studies on whether pre-service secondary teachers (PSTs) have such knowledge, or how to support its development.

To address this gap, we designed and systematically studied an instructional module that aimed to enhance PSTs' knowledge of indirect reasoning, as a part of a larger design-based research project, *Mathematical Reasoning and Proof for Secondary Teachers* (Buchbinder & McCrone, 2018). In this paper we examine the research question: "What challenges and learning opportunities arose from the PSTs' interactions with one of the tasks in this module?"

Methods

Indirect Reasoning Instructional Module (IR Module)

The IR module was the last in the series of four modules in the capstone course, preceded by modules on: direct proof and argument evaluation; conditional statements, and quantification and the role of examples in proving. By the time the PSTs reached the IR module, they had already refreshed and strengthened their knowledge of conditional statements and contrapositive which are critical for indirect reasoning (Buchbinder & McCrone, 2018).

The IR module comprised two in-class activities followed by planning and enacting a lesson in secondary classrooms integrating indirect reasoning. The objectives of the *Indirect Proof Structure* activity were to help PSTs understand the relationships between proof by contradiction and proof by contrapositive. For each of six given proofs, the PSTs were asked to identify the P and Q of the statement, the assumption and the conclusion of the proof, determine if the proof is by contrapositive or contradiction, and if so, identify the contradiction (see Table 1 for examples). To highlight the differences in the logical structure, we included of a proof by contradiction and by contraposition for the same statement (Statements 4 & 5, Table 1).

Table 1: Three Out of Six Items in the *Indirect Proof Structure* Task

Items	Proof Structure
Statement 2: *Let a and b be two real numbers. If ab = 0, then a = 0 or b =0* Proof: Suppose that $ab = 0$, and $a \neq 0$ and $b \neq 0$. Since $a \neq 0$ and $b \neq 0$, we can divide both sides of $ab = 0$ by a and by b. This will result in 1= 0. Therefore, if $ab = 0$, then $a = 0$ or $b=0$.	Proof by contradiction P: $a,b \in \mathbb{R}$; $ab = 0$ Q: $a = 0$ or $b = 0$ Assume: $P \wedge \sim Q$ Contradiction: $1 = 0$ Infer: $P \Rightarrow Q$
Statement 4: *For all integers n, if* n^2 *is even, then n is even.* Proof: Suppose n is an integer, such that n^2 is even, and n is not even. Since n is not even, it is odd, that is, $n = 2k + 1$ for some integer k. By substitution and algebra: $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Now, $(2k^2 + 2k)$ is an integer, since the products and the sums of integers are integers. So, $n^2 = 2(integer) + 1$. By definition of odd, n^2 is odd. Therefore, n^2 is both even and odd. Therefore, if n^2 is even, n is even.	Proof by contradiction P: $n \in \mathbb{Z}$; n^2 is even Q: n is odd Assume: $P \wedge \sim Q$ Contradiction: $P \wedge \sim P$ Infer: $P \Rightarrow Q$
Statement 5: *For all integers n, if* n^2 *is even, then n is even.* Proof: Suppose n is an odd integer. By definition of odd $n = 2k + 1$ for some integer k. By substitution and algebra $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ Now, $(2k^2 + 2k)$ is an integer, since the products and the sums of integers are integers. So, $n^2 = 2(integer) + 1$. By definition of odd, n^2 is odd.	Proof by contrapositive P: $n \in \mathbb{Z}$; n^2 is even Q: n is odd Assume: $\sim Q$ Obtain: $\sim P$ I.e.: $(\sim Q \Rightarrow \sim P) \Rightarrow (P \Rightarrow Q)$

The PSTs worked on the activity in groups of 3 to 4, followed by a whole class discussion, led by the course instructor, the first author of this paper. Next, the PSTs summarized similarities and differences between proof by contrapositive and by contradiction on posters.

Participants, Data Sources, and Analytic Techniques

Participants in the study were 11 PSTs in the capstone course *Mathematical Reasoning and Proof for Secondary Teachers*. All PSTs were in their senior year and had completed proof oriented coursework. Data sources were video recordings of the whole class and of each small groups' work captured with 360° table-top cameras, the PSTs' worksheets and posters. The data were analyzed using open coding (Wiersma & Jurs, 2005) to identify learning opportunities and challenges experienced by the PSTs. Challenges were identified as moments in which the PSTs went back and forth between mathematically correct and incorrect ideas, or verbally expressed their confusion. Learning opportunities were defined as instances where the confusion was resolved in a manner that aligns with conventional mathematical knowledge.

Results

Challenges

Here we identify a few a few challenges that the task elicited.

"A proof by contrapositive is a proof by contradiction." After correctly identifying Statement 4 (Table 1) as proof by contradiction, a group of three PSTs: Jane, Emily and Kelly moved on to Statement 5. They recognized it is the same statement, and correctly identified its P and Q. However, when trying to identify the type of proof a disagreement arose.

Jane: So, it [the proof] follows the contrapositive, but it ends with a contradiction.
Emily: This proof is proving $\sim Q$ implies $\sim P$, which is equivalent to P implies Q.
Jane: How is it not a contradiction at the very last line?
Emily: Contradiction is you start with P and then you come up with $\sim Q$ contradicting what you would expect to happen, right? I would expect n^2 to be odd. We assumed $\sim Q$ so we want $\sim P$. It is equivalent to P implies Q.
Jane: So this [proof] ends with the false statement? n^2 is odd? I think it's a contradiction.
Emily: Contrapositive.

Emily correctly identified the proof as a proof by contrapositive. On the other hand, Jane noticed that the proof ends with $\sim P$ (n^2 is odd) and juxtaposed it with P of the given statement (n^2 is even), perceiving it as a contradiction of the form $(P \wedge \sim P)$. In our analysis we identified several instances of this phenomenon, but a more common confusion was the following.

"A proof by contradiction is a proof by contrapositive." The following example comes from a group of four PSTs: Erick, Abby, Phil and Joel, discussing Statement 2.

Erick: This is probably proof by contrapositive.
Abby: Contrapositive is $\sim Q \Rightarrow \sim P$ … where's the $\sim P$ part, though?
Erick: Right there [points to the paper, see Figure 1], the result would be like 1=0, which I think is kind of the idea that ab can't equal 0.
Joel: For a contrapositive we would need to start by saying if $a \neq 0$ and $b \neq 0$, then $ab \neq 0$, and prove that it is true. In this case, however, they say "suppose $ab = 0$ and $a \neq 0$ and $b \neq 0$, this is "if P and $\sim Q$". So there is no contrapositive whatsoever in this proof.
Erick: I think that they prove the contrapositive true. Because they use the fact that $a \neq 0$ and $b \neq 0$ to show like $ab \neq 0$. […] If you just take out "suppose $ab = 0$", and you get rid of the last line [in the proof], then you have showed that the contrapositive is true. They just put extra things in [the proof] so it would be easier to understand.

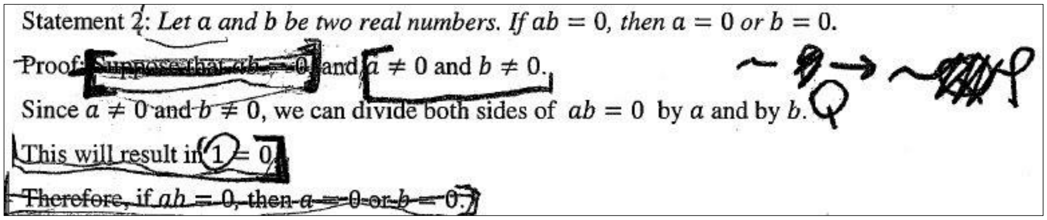

Figure 1: Erick's Worksheet with Crossed Out Portions of the Proof

In this excerpt Erick seems to selectively ignore existing parts of the proof, such as P in the assumption $P \wedge {\sim}Q$, and the conclusion: therefore, $P \Rightarrow Q$, while mentally inserting a non-existing proof line "1 = 0 implies $ab \neq 0$ which he interprets as ${\sim}P$ (Fig. 1). Joel's opposition and correct explanation did little to move Erick away from his conviction. It seems that Erick had a pre-existing assumption that a proof by contradiction is essentially a proof by contrapositive and he sought a way to confirm this. Although Erick was the person who most actively expressed his position, the data shows that other PSTs had similar challenges.

Evidence of Learning

Despite these challenges, the task generated multiple learning opportunities for PSTs to clarify and strengthen their knowledge of indirect reasoning, both in the small group and whole class discussions. The culmination of the activity was PSTs creating posters summarizing main features of proof by contradiction and by contrapositive (Figure 2).

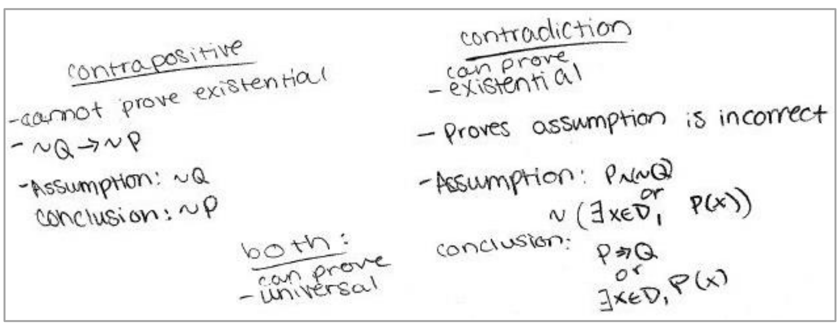

Figure 2: PSTs' Poster Comparing Proof by Contradiction and by Contrapositive

Discussion

The results presented above highlight some of the conceptual challenges and learning opportunities elicited by analyzing the structure of given proofs. These results concur with the literature suggesting that PSTs struggle to distinguish between proofs by contradiction and by contrapositive. We add to the literature by clearly identifying the challenges as confusing these two types of proofs by either: (a) interpreting the conclusion ${\sim}P$ of a proof by contrapositive as contradicting the hypothesis P in the statement, or (b) interpreting a proof by contradiction as a contrapositive proof due to the presence of ${\sim}Q$ in the assumption of both types of proofs. The overarching theme underlying these challenges seems to be "selective noticing" of certain aspects of the proof's logical structure, such as fixating on ${\sim}P$ in ${\sim}(P \wedge {\sim}Q)$, while ignoring other critical aspects such as P in the assumption of the proof by contradiction.

Our data illustrate the pedagogical potential of the *Indirect Proof Structure* task to evoke rich conversations that resulted in multiple instances of learning for the PSTs. We attribute this to the careful selection of the types of proofs, the focus on proof comprehension rather than proof production, comparing across multiple proofs, and summarizing the differences on a poster

(Brown, 2018; Mejia-Ramos, et al. 2012). Going back to Thompson's (1996) aspiration that teachers support students' understanding of indirect proof, we assert that the *Indirect Proof Structure* task in particular and the IR module in general bear the potential to prepare PSTs to carry this practice into secondary classrooms.

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PRE-SERVICE TEACHERS' CONCEPTIONS OF MATHEMATICAL ARGUMENTATION

Hyejin Park
Marquette University
hyejin.park@marquette.edu

Marta T. Magiera
Marquette University
marta.magiera@marquette.edu

Drawing on a situated perspective on learning, we analyzed written, open-ended journals of 52 pre-service teachers (PSTs) concurrently enrolled in mathematics and pedagogy with field experience courses for elementary education majors. Our study provides insights into PSTs' conceptualizations of mathematical argumentation in terms of its meanings. The data reveals how PSTs perceive teacher actions, teaching strategies, classroom expectations, mathematics content, and tasks that facilitate student engagement in mathematical argumentation. It also shows what instructional benefits of enacting mathematical argumentation in the elementary mathematics classroom they perceive.

Keywords: Teacher Education-Preservice, Teacher Beliefs, Reasoning and Proof

Background

For more than two decades, standards documents (e.g., National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) continue placing a great deal of emphasis on the practice of mathematical argumentation in the mathematics classrooms across all grade levels. Previous studies have shown that by engaging in this practice, students develop their mathematical understanding and improve their mathematics achievement (e.g., Cross, 2009; Francisco, 2013). Mathematical argumentation is an important aspect of developing mathematically proficient students, but teachers often view curricular expectations about engaging students in mathematical argumentation as challenging. Graham and Lesseig (2018) noted that "teachers—both novice and experienced—have difficulty incorporating argumentation in the classroom" (p. 173).

Research-based understanding of elementary practicing and pre-service teachers' (PSTs') interpretations of mathematical argumentation is limited. The existing studies on mathematical argumentation have focused predominantly on teachers' perceptions of mathematical argumentation from the perspective of proof (e.g., Martin & Harel, 1989; Stylianides & Stylianides, 2009), teachers' classroom discourse practices in argumentation (e.g., Brown, 2017; Yackel, 2002), or teachers' evaluations of student arguments (e.g., Morris, 2007; Shinno, Yanaginomo, & Uno, 2017). Our work adds to this body of research. We provide a window into PSTs' conceptions of mathematical argumentation by answering the following research question: *How do PSTs conceptualize mathematical argumentation as a pedagogical practice in the context of elementary mathematics classrooms*? Our study builds a foundation for professional development efforts that aim to help PSTs meet the challenges of teaching elementary mathematics with a focus on mathematical argumentation.

Conceptual Framework

This research is grounded in a situated perspective on learning (Lave & Wenger, 1991; Peressini, Borko, Romagnano, Knuth, & Willis, 2004). Using the situated perspective to frame our work allowed us to explore PSTs' pedagogical conceptions of mathematical argumentation in elementary school mathematics as they positioned themselves as future teachers. Consistent with this perspective, we believe that PSTs build their views about mathematical argumentation by negotiating and renegotiating its meaning for themselves as they participate and reflect on their experiences with mathematical argumentation within and across different contexts. In our analysis then, we considered multiple contexts (i.e., mathematics and teacher preparation courses, field experiences) to include PSTs' experiences as both learners and apprentice-teachers.

Methods

Our study draws on data from a larger project conducted in a midwestern university in the United States. The overarching project was designed to explore K-8 PSTs' knowledge development about mathematical argumentation and proof in a teacher preparation program. The data were collected in two different semesters. Participants were two cohorts of PSTs (n = 52) concurrently enrolled in two courses for elementary education majors: a mathematics content course and a first of two mathematics-oriented pedagogy with field experience courses. Curricula of both courses were coordinated and addressed fundamental to elementary school mathematics topics and their teaching. For this paper, we purposefully selected PSTs' written responses to open-ended reflective journals which they completed throughout the semester (see Table 1), and in which they shared their views on mathematical argumentation. Using the qualitative content analysis and constant comparative methods (Glaser & Strauss, 1976; Mayring, 2014) we analyzed 380 responses in total (some participants did not consistently respond to all prompts).

Table 1: Journal Information

Timeline	Journal Prompts & Number of Responses
4th and 9th week of the semester	Thinking about yourself as an elementary school teacher define the term *mathematical argumentation*. How would you explain its meaning to a parent, for example? (J1, P1, 48 responses; J6, P1, 46 responses)
4th, 9th and 14th week of the semester	Describe the practices that characterize an elementary mathematics classroom in which a teacher engages students in mathematical argumentation. What practices could a visitor (e.g., a parent) see observing that teacher? How these practices can support students' argumentation skills. (J1, P2, 45 responses; J6, P2, 45 responses; J10, P2, 48 responses)
5th week of the semester	Are there any areas or topics of study in elementary mathematics that you view as more or less suitable for engaging students in mathematical argumentation? If so which one. Why? (J2, P1, 50 responses)
5th week of the semester	Describe characteristics of mathematical tasks that have high potential to engage students in mathematical argumentation. How are the tasks you described different from tasks that do not encourage mathematical argumentation? (J2, P2, 50 responses)
14th week of the semester	Describe how your experiences this semester influenced your ideas about teaching elementary mathematics with a focus on mathematical argumentation. (J10, P1, 48 responses)

Results and Discussion

Table 2 gives a summary of outcome space that describes PSTs' pedagogical views on mathematical argumentation. Given the space limitation, we only discuss selected results.

Table 2: Results Summary

Major Category	Sub-Category	Number of PSTs
1. Notion of the term mathematical argumentation	Individual perspective	38
	Social perspective	10
2. Benefits of the use of mathematical argumentation in teaching mathematics	For students	26
	For teachers	22
3. Teacher actions that support argumentation	Teacher questioning	33
	Teacher encouragement	13
4. Teaching strategies that promote argumentation	Discussion	38
	Concrete manipulatives or visual representations	19
5. Mathematics content where argumentation can be implemented	Selective topics	30
	All topics	14
6. Tasks that can be used with argumentation	Call for justifications	30
	Open to multiple solution strategies	21
7. Classroom expectations	Student actions	31
	Classroom environment	13

Note. Some of the participants shared more than one view.

While defining mathematical argumentation, the vast majority of our PSTs discussed argumentation from the perspective of an *individual.* They focused on a person's ability to explain and justify the thinking and reasoning used to solve a problem. We illustrate this perspective with an excerpt from PST A38's journal: "Mathematical argumentation is the ability for a student to reach mathematical conclusions through logical reasoning" (J6, P1). A much less prevalent interpretation of mathematical argumentation stemmed from perceiving argumentation as a *social* activity. PSTs with the *social* perspective conveyed the view of mathematical argumentation as a process of communicating mathematical ideas to others to justify, convince, or to provide a challenge. We illustrate this view using PST A19's response:

> Mathematical argumentation is the process of explaining and justifying to others clearly how you got an answer to a particular mathematical problem or question…When questions from others arise, one must be able to answer those questions and must also be able to answer questions of others based on their work if they are unsure about how someone else goes about their answer. (J6, P1)

Across the analyzed journals, 36 PSTs discussed teacher actions which they viewed as essential for engaging students in mathematical argumentation. Most frequently, they attended to teacher *questioning* and teacher efforts of *encouraging* students to participate in argumentation. With a focus on teacher questioning, PSTs often shared that teachers who regularly ask the "*how*" and "*why*" questions engage students in mathematical argumentation by having them to explain and justify their thinking. PST A1's journal entry exemplifies this view:

> A classroom that fosters mathematical argumentation should contain a teacher who is constantly asking his/her students to explain how they reached their answers and why they think it [the answers] make sense… Teachers should be asking their students questions like "how did you come to this answer?," "why does that make sense?," and "how did you know where to start in this problem?" (J1, P2)

PSTs who focused on teacher encouragement discussed that teachers foster mathematical argumentation by prompting students to present their thinking, inviting students to critique each other's reasoning, or correcting misconceptions. PST A29, for instance, wrote:

> A teacher who fosters mathematical argumentation should…encourage students to share their methods of thinking through a problem with the class. [This] promote[s] mathematical argumentation within the classroom and will help students to build their skills in math by getting them to talk to one another and figure out what methods do and do not work for solving math problems. (J1, P2)

Summary Discussion and Conclusions

Our PSTs' largely individual-focused perceptions of mathematical argumentation was clearly visible when PSTs discussed teacher actions in support of argumentation. In their descriptions, only a few PSTs considered how a teacher might support *collective* efforts in which students jointly build on each other's ideas and collectively establish a mathematical claim. The vast majority of our PSTs concentrated on how teachers might encourage individual students to explain and justify their thinking for themselves, to other students, or to teachers. Even while discussing how a teacher might support a group of students, our PSTs painted pictures of individual students developing their own arguments drawing on ideas from others.

We hypothesize that PSTs' experiences with mathematical argumentation in their mathematics content and pedagogy courses could possibly contribute to their largely individual-focused views on mathematical argumentation. Even though in their mathematics content and pedagogy courses instructors frequently engaged PSTs in sharing, analyzing, critiquing, and building arguments collectively, the social aspects of argumentation or any instructional decisions in support of collective argumentation were not explicitly discussed. While mathematical argumentation was also a focal aspect of PSTs' field experiences, culminating activities in which PSTs engaged consisted of one-on-one interactions with students. It might be, then, that in their field experience classrooms most PSTs saw mathematical argumentation from the perspective of individual students focusing on each student's ability to generate arguments.

Drawing on past research which established the relationship between teachers' beliefs and their instructional practices (e.g., Stipek, Givvin, Salmon, & MacGyvers, 2001; Thompson, 1984), it appears reasonable to expect that PSTs with mostly individual-focused views of mathematical argumentation might less likely use argumentation as a pedagogical tool for constructing meaning of mathematics collectively. That is, they might not routinely consider engaging their students in collective examination of assertions and provide them with opportunities to build on and critique each other's ideas. To help our PSTs develop a richer perspective on mathematical argumentation we are now more explicitly draw PSTs' attention to both individual and social aspects of mathematical argumentation in both courses (i.e., mathematics content and pedagogy). Research needs to further examine PSTs' views on mathematical argumentation in relationship to their experiences with mathematical argumentation to provide directions for learning activities that can help PSTs develop richer perspectives on mathematical argumentation.

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CHALLENGING A NEW HORIZON IN MATHEMATICS TEACHER EDUCATION WITH CRITICAL CONVERSATIONS OF EQUITY

Alesia Mickle Moldavan
Fordham University
amoldavan@fordham.edu

Facilitating opportunities for preservice teachers to develop the knowledge, skills, and dispositions necessary to support equitable mathematics instruction, recognize systems of inequity, and advocate for all learners can be challenging. This study reports on the various efforts of mathematics teacher educators to engage preservice teachers in critical conversations of equity in their methods courses. Dominant conversations addressed equity in terms of access, achievement, and identity, while issues pertaining to power, privilege, race, and other systems of oppression called for increased attention. Suggestions for advancing critical conversations in mathematics teacher education are discussed.

Keywords: Teacher Education-Preservice, Equity and Diversity, Classroom Discourse, Instructional Activities and Practices

As mathematics teachers prepare their preservice teachers for today's diverse mathematics classrooms, they must consider ways to embed critical conversations of equity into initial teacher preparation curriculum. Recent initiatives led by professional organizations (e.g., Association of Mathematics Teacher Educators [AMTE], National Council of Teachers of Mathematics [NCTM]) have offered solutions to create, support, and sustain a culture of access and equity in school mathematics (e.g., AMTE, 2017; NCTM, 2000, 2014a, 2014b). With research suggesting that effective mathematics teachers have a comprehensive understanding of equity-based teaching practices (Chao, Murray, & Gutiérrez, 2014), mathematics teacher educators are exploring ways to equip preservice teachers with the knowledge, skills, and dispositions necessary to support equitable mathematics instruction, recognize systems of inequity, and advocate for all learners. This study reports on the various efforts of nine mathematics teacher educators to engage preservice teachers in critical conversations of equity in their methods courses. The following research question is examined: How do mathematics teacher educators address equity in mathematics teacher education?

Theoretical Framework

I used Gutiérrez's (2007) equity framework to guide my inquiry and examine where efforts were made (or not) to engage preservice teachers in critical conversations of equity in mathematics teacher education. The framework describes equity as a complex notion that is comprised of four dimensions (i.e., access, achievement, identity, and power) mapped onto two axes (i.e., the dominant axis and the critical axis). The components of the dominant axis include concepts of access and achievement. Gutiérrez refers to access as the resources provided that enable participation in mathematics (e.g., high quality teachers, rigorous curriculum, inviting classroom environments, appropriate texts and technology). Cautious efforts must be taken to ensure that access does not mean equal resources as such actions perpetuate sameness. Instead, resources must fairly support students' mathematical, cultural, and linguistic needs. The other dimension addresses achievement and how outcomes that historically measure ability (e.g., course grades, course-taking patterns, standardized exams) need to be critiqued as structures that routinely disadvantage marginalized students. Thus, the dominant axis recognizes the dominant, Eurocentric mathematics that enables participation in society and the need to critique society's status quo to understand and interpret other ways of knowing and practicing mathematics.

To challenge the static notions of the dominant mathematics, Gutiérrez (2007) addresses concepts of identity and power, which encompass the critical axis. Identity recognizes the need for students to self-identify with mathematics and draw on personal experiences as well as cultural and linguistic resources (Gutiérrez, 2009). This dimension not only draws attention to students' assets but also recognizes ways students have been (and still are) racialized, gendered, and classed. A close look at power can also assist with examining who is privileged (and silenced) in the classroom and in what ways mathematics can be used to explore injustices and advocate for all learners. Thus, the critical axis acknowledges the need for self-identification in mathematics to self-empower and exercise mathematics as an analytical tool to critique societal norms.

Methods

I used a qualitative case study to critically examine how nine mathematics teacher educators from nine institutions (i.e., public, private, and Historically Black Colleges and Universities) in the southeastern United States addressed equity in their methods courses. The participants identified as mathematics teacher educators responsible for preparing preservice teachers in initial teacher preparation programs. They also indicated that they were knowledgeable (to some extent) of the recent initiatives led by professional organizations to advocate for equity in mathematics teaching and learning (e.g., AMTE, 2017; NCTM, 2000, 2014a, 2014b). Participants were selected using stratified purposeful sampling to offer information-rich cases from varying contexts and settings to facilitate meaningful comparisons (Patton, 2002).

For each participant, I conducted two semi-structured interviews and a follow-up interview for member-checking. I also collected artifacts (e.g., syllabi, course readings, activities, assignments, work samples), which served as corroborating sources of evidence and discussion topics for the second interview. Data sources were coded using in vivo and descriptive code (Saldaña, 2016) and interpreted using Kvale's (1996) meaning-making methods to capture the participants' shared experiences. I also performed analytic memo writing to assist with my reflection and extraction of relevant information to guide my synthesis.

Results

Findings revealed that mathematics teacher educators made intentional efforts to address critical conversations of equity in their methods courses. While concepts of access, achievement, and identity primarily dominated the participants' equity conversations, participants disclosed how they addressed (some more than others) issues of power in terms of privilege, race, and other systems of oppression. Using Gutiérrez's (2007) equity framework to organize the findings, I share examples of the various ways the participants engaged preservice teachers in equity conversations through activities and assignments. The varied efforts reaffirm that equity is a multifaceted concept that is communicated differently based on understanding and experience of how issues of access, achievement, identity, and power can be addressed in initial teacher preparation curriculum.

Access

When asked to describe equity in terms of access, several participants defined access as tangible resources, supports, and self-differentiated tasks that position learners for participation in the mathematics classroom. To illustrate how access is communicated in methods courses, one participant shared how she models a situation in which preservice teachers are asked to complete computations using a non-Western numeral system. Some preservice teachers receive access to supports (e.g., translation charts), while others do not. Following the activity, she engages her preservice teachers in a discussion about the way supports can be used to break down cultural and language barriers to ensure all learners have access to the mathematics. Another participant has her preservice teachers review national and local data reports on homelessness, poverty, hunger, crime, and health status. Preservice teachers create a resource handbook for parents and students in their community to aid with locating agencies and services to assist with homelessness and hunger. The other participants shared how they work with their preservice teachers to create self-differentiated tasks that allow for multiple entry points. The tasks communicate high expectations focused on individualized learning with appropriate supports for all learners.

Achievement

In terms of achievement, several participants commented on the dangers of tracking and how success in mathematics can be used as a gatekeeper to career opportunities. To assist preservice teachers in recognizing the dangers of tracking, one participant has his preservice teachers discuss how school systems are structured in ways that are inequitable for marginalized students. His preservice teachers critique tracking from observations of White students at one end of the hall and Black students at the other. Similarly, a participant facilitates a debate among her preservice teachers that examines how success in mathematics can be used as a sorting mechanism to provide access into colleges and high-paying careers. Another participant assigns an equity report that requires preservice teachers to analyze data from the National Assessment of Educational Progress database for discrepancies in student achievement. Preservice teachers reflect on the ways race, class, and performance attribute to the "achievement gap" and how anti-deficit counternarratives of student success can be used to challenge the discourse.

Identity

The participants also shared specific ways they engaged preservice teachers in equity conversations related to identity. Most of the activities and assignments focused on identifying growth mindsets; incorporating inclusive and culturally responsive mathematics instruction; and recognizing students' mathematical, cultural, and linguistic strengths and resources. For example, one participant has her preservice teachers draw a model mathematician. The preservice teachers talk about ways to picture themselves and their students as mathematicians. Similarly, three participants assign a mathematical autobiography. The activity requires preservice teachers to reflect on their mathematical identity, experiences, and biases. Another participant has her students analyze mathematics textbooks and critique how the word problems relate (or not) to students in the community. Other conversations include reflections on readings about culturally responsive pedagogy and strengths-based thinking.

Power

Not all of the mathematics teacher educators were able to share how they addressed concepts of power in their methods courses. Some participants did not see mathematics as a political activity, while others were hesitant to get into heated debates or were still looking for activities that would help them facilitate a productive discussion. For example, one participant said, "Some of these ideas about equity and social justice in math are things I'm just now learning about more deeply." Another participant shared: "I don't specifically talk about race in my classes, but I do talk about how many people have been neglected and refused opportunities." Another participant communicated his wishes to teach preservice teachers to "recognize that power is a perpetuation of White privilege or White supremacy," but he asserted, "I don't have a nice lesson that's digging into that well yet."

A few participants shared their efforts to engage preservice teachers in conversations pertaining to issues of privilege, race, and other systems of oppression. One participant engages her preservice teachers with an activity called "Race Cards." Preservice teachers are organized in groups where they pull cards of different themes (i.e., attributes, grade-levels, mathematics courses, race, and gender). The attributes describe socioeconomic status, mental and physical health, family dynamics, club involvement, behavior, and sexuality. The groups discuss which students might be more privileged in school based on the descriptors on the cards. The activity assists preservice teachers with identifying stereotypes and challenging the status quo. Four other participants disclosed how they facilitate discussions, often times haphazardly, that encourage preservice teachers to notice privilege, including what mathematics is privileged in society and who is recognized for the mathematics.

Discussion

Mathematics teacher educators are influential in fostering preservice teachers' consciousness of equity-based teaching practices. For preservice teachers to acquire the practices necessary to enable all learners to participate in meaningful mathematics learning that supports the development of positive mathematical identities and challenges oppressive norms, mathematics teacher educators must consider ways to engage preservice teachers in critical conversations of equity. This study provides insight into the various efforts of mathematics teacher educators to address concepts of equity in their methods courses. Mathematics teacher educators used specific activities and assignments to facilitate conversations of equity in terms of access, achievement, and identity. When sharing ways to engage preservice teachers in conversations pertaining to issues of power, privilege, race, and other systems of oppression, most efforts were addressed in vague terms, which is similarly reported by other researchers (e.g., Civil, Bartell, Bullock, & Fernandes, 2018). With awareness that Gutiérrez's (2007) dominant (i.e., access and achievement) and critical (i.e., identity and power) axes are equally important to address equity, mathematics teacher education must look for ways to advance preservice teachers' knowledge about the historical context of mathematics education, including the roles of power, privilege, and oppression that result in inequitable learning experiences for marginalized students. When preservice teachers acquire the critical consciousness to question inequities and advocate for access to high-quality mathematics instruction for all learners, they can serve as role models and leaders challenging a new horizon in mathematics education.

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PROSPECTIVE TEACHERS CONCEPTUAL UNDERSTANDING OF LINEAR FUNCTIONS: PRINCIPLES OR CONNECTIONS?

Erik Jacobson
Indiana University
erdajco@indiana.edu

Fatimah Ahmad
Indiana University
fsahmad@iu.edu

Pavneet Bharaj
Indiana University
pkbharaj@iu.edu

Theodore Savich
Indiana University
tmsavich@indiana.edu

In this empirical study, we report data from 44 elementary PTs' talk about linear functions in a simulated teaching context and our analysis of this talk in terms of the representations involved (i.e., graphs, equations, and tables) and whether it conveyed conceptual understanding or errors. Drawing on two complementary definitions of conceptual understanding from educational psychology and mathematics education, we examine descriptive results about the kinds of conceptions PTs revealed as well as results about the relationship between understanding evident in utterances and types of representations invoked. PT talk about an equation or table alone tended to be more strongly associated with errors than talk about a graph both alone and with other representations. Implications are discussed.

Keywords: Teacher Knowledge, Teacher Education-Preservice, Algebra and Algebraic Thinking

The field of mathematics education has increasingly identified the potential and promise of addressing algebraic ideas in the elementary grades, and this trend has important implications for elementary teacher education (e.g., Soares, Blanton, & Kaput, 2006). The wide adoption of the Common Core State Standards for Mathematics (CCSSM) and inclusion of algebraic thinking as one of the strands in Grade K-5 is a recent milestone. Moreover, the Grade 6 CCSSM standards include algebraic equivalence and graphing linear function (e.g., 6.EE.B & 6.EE.C) and many states license elementary teachers through Grade 6.

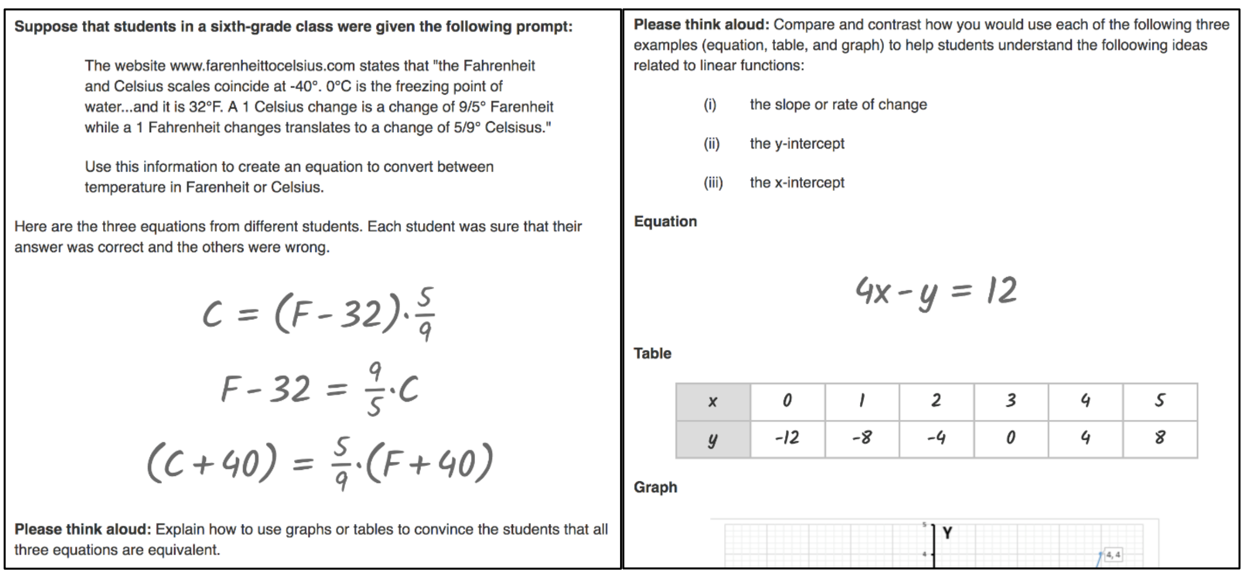
There is little published work about elementary prospective teachers' (PTs) understanding of linear functions and their representation in equations, graphs, and tables, perhaps because algebra has been restricted traditionally to the middle and secondary grades. Even for those elementary teachers who do not teach the Grade 6, knowledge of algebraic ideas provides insights for teaching more central content in the elementary curriculum. First, teachers can anticipate and support mathematical connections students will make in the future (cf. horizon content knowledge; Ball, Thames, & Phelps, 2008). Second, knowing algebra provides elementary teachers with "mathematical peripheral vision" (p. 4) that can support interpretations of students' understanding of arithmetic and instructional responses (Cho & Tee, 2018).

To address this gap in the literature, we asked PTs to discuss connections within and between multiple representations of linear function in a simulated teaching task. As teacher educators, we were also interested in the relationship between different representations of linear function and mathematical errors—a key indication of low mathematical quality of instruction (Hill et al., 2008). Drawing on a framework that defines understanding as connected knowledge and using statements rather than individuals as the unit of analysis, we sought to answer the following research questions. (1) *How is elementary PTs conceptual understanding of linear function* *related to the representations they invoke?* (2) *Are statements involving more representations less error-prone than those involving fewer representations?*

Theoretical Framework

Algebraic expertise involves the flexible use and coordination of powerful conceptual tools including functional thinking, graphing, and symbolic notation. Our work is informed by the definition of conceptual understanding provided by Hiebert and Lefevre (1986): "a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 3). Lesh (1979) and Lesh and Doer (2003) refine this definition by distinguishing five types of representations and identifying conceptual understanding as the ability to move flexibly between representations. We refine further by distinguishing between transformations between representations of the same type (e.g., equivalent equations) and translations between different representations (e.g., equations and graphs, c.f., Duval, 2006).

A complementary definition of conceptual understanding is the explicit knowledge of the general principles in a domain that underlie routine procedures (Crooks & Alibali, 2014). For example, one general algebraic principle is that linear functions describe two quantities changing at a constant relative rate. Translation rules (e.g., “Equivalent representations describe the same rate of change.”) and transformation rules (e.g., “Adding the same value to both sides of an equation maintains equivalence.”) are principles that underlie equivalent representation procedures. We designed our analysis to attend both to flexible use of multiple representations as well as to explicit knowledge of general principles.



(a)

(b)

Figure 1: Transformation Task, Q1 (a) and Translation Task, Q2 (b).

Data and Methods

This study was part of a larger project conducted at a large Midwestern public university. Participants (n = 44) were recruited from an elementary math methods course and participated in a 2-hour research procedure that included pre-assessment, intervention, and post-assessment. In this paper, instead of making claims about individual PTs or the intervention, we combined data from both pre and post occasions and use an *utterance* (defined below) as the unit of analysis to provide descriptive insights pertaining to PTs conceptual understanding of linear equations.

Data

The data from the pre- and post-test conceptual understanding assessments were used in this report. Each assessment involved two tasks and had two versions with slightly different numbers. The two versions were counterbalanced so everyone answered both versions, but half did so in the opposite order. In the Transformation Task (Q1; Figure 1a) the PTs were asked to use graphs or tables to convince three students that three equations in different forms were equivalent. In the Translation Task (Q2; Figure 1b) the PTs were asked to discuss the rate of change and intercepts across three different representations of linear function.

Coding

To analyze the assessment data, the videos of the PTs' responses were transcribed and divided semantically into *utterances*: a one-to-several sentence unit of talk expressing a single main idea. Transcription data from a total of 43 PTs were obtained with data for 42 PTs having both pre and post occasions. (Some data were missing because of equipment failure.) The researchers used a constant comparative method of qualitative coding to define codes for each utterance. Two team members read through the data for each PT in succession, identifying new ideas with new codes and using the growing code book to mark previously identified ideas. New codes were discussed with the full team each week until agreement was reached. The final codes consisted of 36 codes for the Transformation Task (Q1) and 40 codes for the Translation Task (Q2). Each code was further categorized by conceptual understanding and by representation type.

Analysis and Results

As stated earlier, the unit of analysis for this study is a PT utterance. Out of 1470 utterances, we coded 607 utterances related to equations, graphs, and table representations. The rest of the utterances were either too imprecise to code (n = 85, 42 for Q1 and 43 for Q2) or non-mathematical (e.g., reading the task aloud, n = 778). To address the research questions, we first illustrate codes for each task then present statistical tests of distributional independence.

Transformation Task

We used four codes to capture conceptual understanding related to graphs on Q1. These codes reflected different levels of understanding among the PTs. First, "Graphs are the same" (0221LH.q1.1, n = 25) was used when PTs clearly stated that the graphs for the equations in Q1 would be the same without articulating how or why the graphs would be the same. The second code "Equal slope means equivalent" (0222MS.q1.3, n = 9), was used when PT's argued that graphs of the equations would be the same because they have the same slope; the PT might say "all the graphs have the same slope and that's why they look the same." The third code "Graphs will be same as slopes and intercepts will be same" (0228MD.q1.2, n = 4) reflected a deeper understanding of graph equivalency, because it included an explicit check of an additional feature of the graphs: the intercepts. The fourth code, "The graphs are the same because they have the same points" (0228MS.q1.1, n = 7) was used for an even more precise claim. One PTs justified by saying, "we will get the same point value on the graph each time".

Last, we discuss an error code that occurred when PTs were only discussing the equation representation. For Q1, two of the equations were in the slope-intercept form, while the third eq. was in the point-slope form. The code "Unclear ideas about point-slope equation" (0227MY.q1.1, n = 4) captured PT confusion when comparing the point-slope and slope-intercept forms. One PT stated for the third equation that "I don't know how this one goes with the first two." In another example, a PT said that the "first two are the same but the third equation is not similar". While the PT was able to see that the first and second equations are the same, the PT did not see how the third equation could also be equivalent.

Translation Task

We used several codes for Q2 that were related to graphs and understanding linear function in the context of graphs. Many PTs described graphs as a good visual representation of the relationship between two variables (0224MS.q2.2, n = 12). Another set of codes captured the observations that graphs demonstrate the rate of change between two variables as well as the point where the line intersects either of the axes (0221MH.q2.1, n = 29; 0228AE.q2.1, n = 18; 0228MS.q2.2, n = 44; 0222MS.q2.2, n = 40; 0224MS.q2.3, n = 8; 0410EL.q2.3, n = 1). By contrast, PTs exhibited numerous misconceptions when talking about the meaning of coefficients. For example, some PTs stated that coefficient of x (or y) in an equation represents the x (or y) intercept (0331VL.q2.2, n = 6, 0331VL.q2.7, n = 4), or considered the coefficient of x as the value of slope even when an equation is given in a standard form (0303LB.q2.1, n = 3). Although talking about equations alone often revealed errors, utterances that coordinated equations and graphs provided strong evidence of understanding. For example, PTs connected the coefficients of variables in an equation with a feature of the graph related to slope or intercept (0222MS.q2.6, n = 6, 0310TM.q2.1, n = 2, 0228MD.q2.1, n = 4). These utterances were relatively rare, yet they substantiated clear cases of the effective use of multiple representations while discussing the concept of linear equations.

Table 1: Code Frequency, Error Rates by Task and Representation

	Transformation (Q1)		Translation (Q2)			Error Rates		
Representation	CU	Error	CU	Error	Totals	Q1	Q2	Total
Equations Only	56	4	44	23	127 (21%)	7%	34%	21%
Graphs Only	47	10	148	0	205 (34%)	18%	0%	5%
Tables Only	24	3	34	0	61 (10%)	11%	0%	5%
Equations & Graphs	8	1	11	0	20 (3%)	11%	0%	5%
Equations & Tables	24	4	26	12	66 (11%)	14%	32%	24%
Graphs & Tables	3	0	31	0	34 (6%)	0%	0%	0%
All Representations	6	0	88	0	94 (15%)	0%	0%	0%
Total	168 (28%)	22 (4%)	382 (63%)	35 (6%)	607	12%	8%	9%

Tests

Table 1 shows code distribution by task, representation, understanding and error rates. We used Pearson's Chi-squared test with Yates correction to draw conclusions from these data. First, there were utterances for all possibilities of representation use, but the distribution between these categories was not uniform ($\chi^2 = 276.63$, $df = 6$, $p < 0.00001$). Second, there was no evidence in our data that error rates significantly differed for the Transformation versus the Translation Task ($\chi^2 = 1.205$, $df = 1$, $p = 0.273$). Third, utterances involving a single representation did not differ significantly in error rates from utterances involving multiple representation ($\chi^2 = 0.571$, $df = 1$, $p = 0.450$). Finally, utterances involving graphs had significantly fewer errors than those not involving graphs ($\chi^2 = 37.288$, $df = 1$, $p < 0.00001$).

Implications & Conclusions

The results of this study contribute in at least two ways. First, we provide a descriptive account of major themes of elementary PTs' knowledge for teaching linear function in the context of equations, graphs, and tables. This is the only such study of which we are aware, and the more than 30 distinct qualitative codes for each task therefore contribute by providing detailed evidence of the breadth and range of how elementary PTs in one program talk about linear function. Second, we provide results about the relationship between erroneous teacher talk and its representational content. No errors were evident in utterances linking all three representations, yet these data suggest that it is not simply more representations that support accuracy (Finding 3) but rather that one specific representation—graphs—may play a particularly powerful role in supporting accurate talk about linear function (Finding 4). More research is needed to understand how universal both the descriptive coding and the relative importance of graphs are among other populations of preservice elementary teachers.

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TOWARD A PRACTICE-BASED FRAMEWORK FOR DEVELOPING PRESERVICE ELEMENTARY TEACHERS' MATHEMATICAL AUTHORITY

Priya V. Prasad — University of Texas at San Antonio — priya.prasad@utsa.edu
Victoria J. Barron — Boerne-Champion HS — victoria.barron@boerneisd.net
Raquel Vallines Mira — University of Texas at San Antonio — raquel.vallinesmira@utsa.edu

To be mathematically empowered, preservice teachers (PSTs) must rely on their own sense-making in mathematics in order to develop robust mathematical understandings. They must have an internal sense of mathematical authority with which to reason and justify. Mathematics teacher educators (MTEs) must support PSTs in establishing their own sense of mathematical authority. In this brief report, we present our preliminary work on building a practice-based framework to help MTEs assess how mathematical tasks can contribute to PSTs' mathematical empowerment.

Keywords: Teacher Education-Preservice, Algebra and Algebraic Thinking, Teacher Knowledge

Introduction

For students (and preservice elementary teachers) to be mathematically *empowered*, they must rely on their own sense-making in mathematics in order to develop robust mathematical understandings; that is, they must have an internal sense of mathematical authority to reason and justify. Thus, it is of particular importance for mathematics teacher educators (MTEs) to support pre-service teachers (PSTs) in establishing their own sense of mathematical authority. This goal is twofold: 1) to support PSTs to trust in their own mathematical self-efficacy they will need as teachers, and 2) to give PSTs opportunities to experience a mathematics class in which mathematical authority is shared, and not concentrated in the instructor or a textbook encouraging them to develop such dynamics in their own future classrooms. However, many college students are likely to depend on an external mathematical authority to validate their own thinking (Povey & Burton, 2003; Schoenfeld & Sloane, 2016).

In addition, while work has been done to define PSTs' content knowledge (e.g. Thanheiser et al., 2013) or develop their mathematical habits of mind (e.g. Magiera, van den Kieboom, & Moyer, 2017), there is little evidence in the literature that can help MTEs explicitly support PSTs' to rely on internal mathematical authority. We present our preliminary work on building a practice-based framework to assess the extent to which mathematical tasks for PSTs can contribute to their mathematical empowerment.

Theoretical Framework

When students are doing mathematics, they rely on a mathematical authority that drives their decisions about their course of action when solving a problem. Depending on where that authority is located in a classroom or how students confer (or defer to) it, the exercise of mathematical authority, that is, "'who's in charge' in terms of making mathematical contributions" (Gresalfi & Cobb, 2006, p. 51), can vary greatly. Schoenfeld and Sloane (2016) describe that, when mathematical authority is situated within the learner (i.e. internally), it results in a "personal ownership of the mathematics they can certify" (p. 62). In other words, learners and instructors co-create the knowledge they are exploring (Povey and Burton, 2003). Reinholz (2012) proposes three mutually supportive skills that make up the exercise of an internally situated mathematical authority: students who rely on an internal mathematical authority (1) *explain* their reasoning, (2) *justify* their conjectures, and (3) *assess* their work once they find a solution, just as “mathematicians use these skills to derive authority from the logic and structure of mathematics, rather than relying on some other authoritative source” (p. 242). However, students can potentially rely on an external authority by seeing mathematics as outside their control. In this case, the learner views mathematics as a subject that is fixed and unchangeable. Such students leave mathematics to the “experts”. In addition, the “authority for the learner rests in the content,” rather than internal judgement (Povey & Burton, 2003, p. 244).

When students view instructors as the sole arbiter of mathematical correctness, they are surrendering their authority to the instructor, which detaches them from the material they are learning. In order to decrease students’ reliance on external mathematical authorities and foster their dependence on internal ones, instructors must shift their roles in the classroom (Gresalfi & Cobb, 2006; Stein et al, 2008; Webel, 2010). Rather than having instructors present themselves as a “‘dispenser of knowledge’ and arbiter of mathematical ‘correctness,’” they should take on the role of the facilitator or “engineer of learning” while allowing students to construct their own understanding of the mathematics they are learning (Stein et al., 2008, p. 4).

It is imperative for PSTs to understand how to support their future students in exercising mathematical authority, but they must first develop their own robust internal senses of mathematical authority. PSTs need to be mathematically empowered in multiple ways, with mathematical authority and self-efficacy, as well as with an understanding of how and when to cede mathematical authority to their students in their own future classrooms. Thus, it is a task of mathematics teacher education to provide models of teaching for mathematical empowerment. In this paper, we present preliminary work on a framework for analyzing and revising mathematics tasks for elementary PSTs so that they better support PSTs’ mathematical empowerment.

Methodology

This preliminary framework is being developed on data collected about a visual patterns task implemented in a mathematics content course for elementary PSTs at a large, Hispanic-serving southwestern university. This task was within the context of a larger project based on the Continuous Improvement (CI) model for course improvement developed by Berk and Hiebert (2009). This model maps out the following steps for task design and revision: (1) design a task that targets a specific mathematical concept and identify the learning goal, (2) develop hypotheses about anticipated student responses to the task, (3) implement the lesson and observe the implementation, (4) collect and analyze student data on pre- and post-assessments, and (5) revise the task for future implementation. While designing the visual patterns task, the research group identified a specific learning goal: “*PSTs will be able to use a variable to represent an index/input in a sequence of visual patterns and build an expression in terms of the variable to represent an arbitrary step in the sequence”.* After administering a pre-assessment based on the work of Warren and Cooper (2008; see Figure 1), the research group designed a groupworthy mathematical task (Lotan, 2003) to address the learning goal (see Figure 2).

Consider the following growing pattern:
1. Using a variable, write a general expression that tells how many tiles are in a step of the pattern.
2. Justify your pattern by describing the growing pattern in words and explaining what the variable represents.

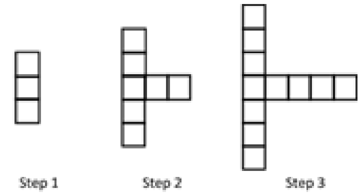

Figure 1: Pre- and Post- Assessment to Visual Patterns Task

The lesson was implemented in two classes by two different instructors. As they worked through the task, PSTs' small-group interactions were audio-recorded and synchronized with the group's written work. The task was followed by a post-assessment that was identical to the pre-assessment.

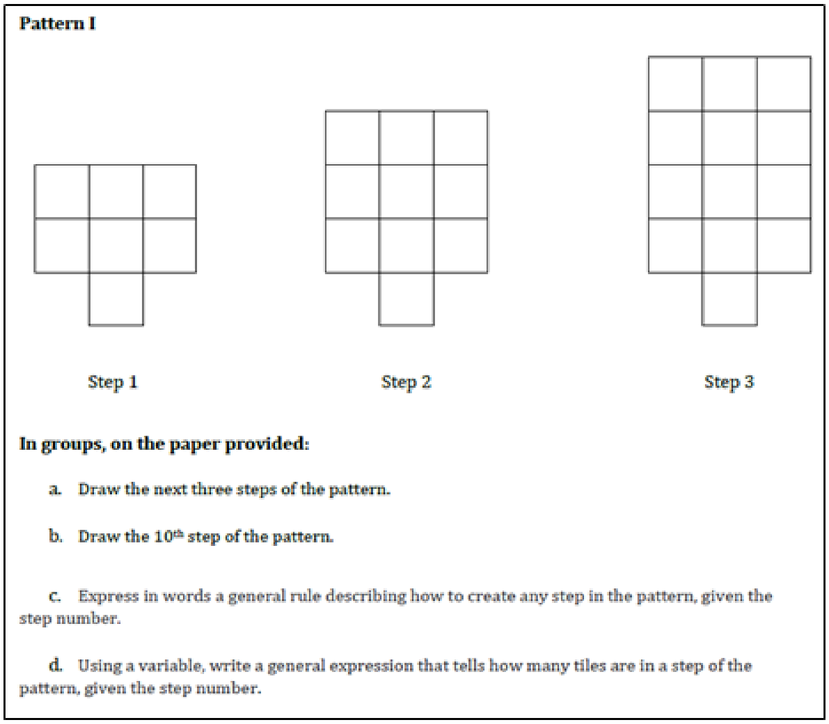

Figure 2: Visual Patterns Task Given to PSTs, from Boaler (2015)

Findings from Classroom Implementation

As anticipated by the pre-assessment results, most groups of PSTs used one of two strategies during their exploration of the task: (1) building a linear relationship by first generating table of values or (2) plugging in the values of the initial step and the difference into the formula for arithmetic sequences. Although both strategies led most groups to develop correct expressions (see Table 1), the PSTs largely relied on the authority of the arithmetic sequence formula or the familiarity of the procedure for developing a linear equation from a table of values to justify their reasoning. Thus, in the first iteration of the task, most groups of PSTs were not fully exercising their mathematical authority by explaining, justifying, and assessing for themselves, even though the structure of the task created opportunities for all three (Prasad & Barron, accepted).

Table 1: Pre- and Post-Assessment Outcomes

	Pre-Assessment ($N = 49$)	Post-Assessment ($N = 49$)
$x + 4$	17	3
$3x + 4$	6	1
Different incorrect expression	6	6
Only preliminary numerical/verbal investigation shown	8	3
Correct expression (e.g. $4x - 1$, $3 + 4(x - 1)$, etc)	12	36

Discussion: Developing the Framework

Despite most PSTs developing correct expressions on the post-assessment, the task did not meet the larger course goal of mathematically empowering PSTs by supporting them to rely on an internal mathematical authority. After observing the initial lesson, the research group revised the task in the following ways: (1) introduce the concept by giving PSTs an expression and the description of a pattern, and multiple choices of patterns that numerically fit the expression but not the description to encourage them to attend to the visual attributes of the patterns, (2) prompt PSTs to investigate only one linearly growing pattern whose visual attributes were clearly growing with respect to the index, taken from Beckmann (2008), (3) extend the ideas by including a quadratically growing pattern, challenging PSTs to move beyond previous understandings and the reliance on their knowledge of linear relationships, and (4) prompt PSTs to develop a growing visual pattern that fits a given linear expression.

The researchers noticed that, while the pre- and post-assessments effectively assessed how well the task met the content standard outlined in the expressed learning goal, they did not provide any information for assessing how and when PSTs relied on an internal mathematical authority in order to reason about the task. In order to better evaluate the revised lesson, we propose a preliminary framework based on the work of Reinholz (2012). It is our intention that this framework be *practice-based*, that is, based on data collected during the practice of teaching and of potential use to MTEs while planning and facilitating lessons, since not every MTE can collect and analyze data for every lesson they teach. This preliminary framework is decomposed into three phases:

1. *Planning:* Does the task provide opportunities for exercising all three of the mutually supportive skills that make up mathematical authority? In other words, does the task motivate students to explain their thinking, does it have opportunities built into it for students to assess conjectures for themselves, and does it require students to justify their thinking?
2. *Implementation:* Does everyone engage in exercising the three mutually supportive skills that make up mathematical authority? This can be addressed at the group level and at the individual level. Group-level assessments can be made easily as MTEs circulate around the classroom engaging with different small-groups. In cases where groups are not explaining, assessing, and justifying for themselves, it becomes important to identify to what forms of external authority the students are turning.
3. *Reflection*: Once the potential sources of external authority have been identified, the task can be revised to mitigate their influence.

An important underlying assumption to this framework is that PSTs can only become mathematically empowered when they have the opportunity to work in small groups on groupworthy tasks (Lotan, 2003). Moreover, some portions of the framework can be addressed by establishing classroom norms that expect students to explain their thinking and utilize questioning practices that expect and motivate students to justify their conjectures.

The next steps in developing this framework is to use it to assess the data collected from the revised visual patterns task that the research group wrote and implemented. We hope to refine the broad recommendations so that they can be of practical use to MTEs.

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PRESERVICE TEACHERS' ANTICIPATED STUDENT ERRORS AND RELATED PLANNED QUESTIONS

Susie Morrissey
Mercer University
morrisseysusie@gmail.com

Gorjana Popovic
Illinois Institute of Technology
gorjana.popovic@gmail.com

Ozgul Kartal
Univ. of Wisconsin-Whitewater
ozgulkartal@gmail.com

Elementary and special education preservice teachers (PSTs) were given an assignment to create lesson plans that included anticipated student solutions that were incorrect or partly correct, as well as planned questions that would help students overcome those errors. Authors categorized preservice teachers' anticipated errors and types of planned questions. Results showed most PSTs focused on procedural errors without considering underlying conceptual understanding. Authors concluded that methods courses need to be explicit in discussing types of questions and focus on questions that require students to clarify and justify their reasoning.

Keywords: Instructional activities and practices, Teacher knowledge

Introduction

Association of Mathematics Teacher Educators (AMTE, 2017) and National Council of Teachers of Mathematics (NCTM, 2014) recommend writing lesson plans that include anticipated student responses to assigned tasks along with the teacher's own responses, because anticipating student responses allows teachers to tentatively plan follow-up questions and instructional moves. Anticipating student responses was included in the first-ever comprehensive Standards for Preparing Teachers of Mathematics (SPTM) put forth by AMTE in 2017. SPTM built on existing standards and research about mathematics teacher preparation and are intended to motivate researchers to investigate areas that are not well understood. According to the standard, C.3.1. Anticipate and Attend to Students' Thinking About Mathematics Content, instead of demonstrating their approaches to a problem or correcting errors, teachers should first try to see problems through their students' eyes, anticipate, understand, and analyze students' varied ways of thinking, and respond appropriately (AMTE, 2017). Anticipating both correct and incorrect solutions to mathematical tasks is emphasized as a key strategy to successfully implement effective mathematics teaching practices that were developed by NCTM (2014). In particular, anticipating both correct and incorrect responses is elaborated as a focal planning practice for three of the eight effective mathematics teaching practices in Smith, Steele, and Raith (2017), and Huinker and Bill (2017): (1) implement tasks that promote reasoning and problem solving, (2) facilitate meaningful mathematical discourse, and (3) elicit and use evidence of student thinking. Elicit and use student thinking is closely linked to another effective teaching practice, pose purposeful questions, in NCTM (2014). Posing purposeful questions enables teachers to determine what students know and think, to adapt their instruction to meet students' different levels of understanding, and to help students make connections among ideas.

The value of anticipating student answers is recognized in mathematics education literature; however, little research has been done. The researchers that have explored the importance of anticipating student answers in teaching and learning of mathematics (e.g., Ball, & Bass, 2000; Hill, Ball, & Schilling, 2008; Thompson, Christensen, & Wittmann, 2011) found that preservice and beginning teachers have difficulty anticipating a variety of correct student solutions. Anticipating incorrect approaches, as well as correct approaches, and planning to include incorrect approaches in class discussions is discussed as an avenue to promote a classroom culture in which mistakes and errors are seen as important reasoning opportunities, to encourage a wider range of students to engage in mathematical discussions with their peers and the teacher (Smith, et al., 2017). Boaler and Brodie (2004) found that questions that encourage students to clarify and justify their reasoning were critical in allowing students to build understanding of the ideas. Likewise, Emerson (2010) found that higher level questioning activities were related to an increase in students' conceptual understanding of mathematical concepts and connections.

In this study, which is part of a larger study to analyze how PSTs plan to enact effective teaching practices, we explore the ways they anticipate incorrect student solutions and pose questions when planning to implement a mathematical problem-solving task.

Theoretical Foundation

In 2008, Stein, Engle, Smith, and Hughes proposed a pedagogical model for conducting whole-class discussions after students had completed work on cognitively challenging tasks. The model specifies five key practices: anticipating possible solution strategies students may employ to solve a high-level cognitively challenging task, monitoring student responses while they work on the task, selecting student work to be presented, sequencing selected student work in such a way to maximize the chance to achieve mathematical goals for the discussion, and making connections between student responses. Since then, anticipating student answers has been discussed as part of orchestrating productive classroom discussions as a crucial initial step for selecting and sequencing practices. Yet, researchers mostly focused on selecting and sequencing practices. For example, Meikle (2014) categorized preservice middle school teachers' rationales for selecting and sequencing choices, in an attempt to understand and improve their selecting and sequencing practices. Silver, Ghousseini, Gosen, Charalambous, and Strawhun (2005) examined teachers' transformation through a year-long professional development experience about the use of multiple solutions in the classroom discussions. Teachers noted that they had in the past always tried to suppress any discussion of wrong answers or flawed approaches, but now they came to an understanding that it is "really important to take a look even at wrong answers." Hallman-Thrasher (2017) investigated preservice teachers' responses to unanticipated incorrect solutions to problem-solving tasks in the enactment phase, and found that they all experienced difficulty in helping students question or leverage an incorrect idea toward a more productive strategy. For example, their response was often encouraging students with correct solutions to explain their ideas to students who had incorrect solutions or inefficient strategies. The authors indicated that PSTs did not treat student ideas as objects of inquiry.

All levels of questioning are important, as questions that gather information and probe thinking are necessary for teachers to determine what students know, while questions that encourage reflection and justification are important to reveal student reasoning. Effective questioning attends to students' thinking by asking students to explain and reflect on their thoughts (NCTM, 2014).

Our work with PSTs explored and characterized their anticipated wrong or flawed solutions, and analyzed the questions they planned to use to help students overcome those errors, in an attempt to inform mathematics teacher educators.

Data Analysis

The sample was comprised of 88 PSTs in a math methods course for a mix of elementary and special education majors. The analysis involved categorizing the erroneous solutions PSTs anticipated, as well as categorizing the proposed questions to help students evaluate and judge their own work, and eventually understand why it was invalid. One researcher reviewed the preservice teachers' work and proposed initial categories for the anticipated erroneous solutions and planned questions. Another researcher was asked to use those identified categories to categorize the preservice teachers' work. Any discrepancies between the researchers were discussed and categories and codes were revised as necessary. The process was repeated until 100% agreement was achieved.

Results

The problem-solving task assignment specifically required PSTs to identify difficulties their students may have that would cause them to not get started on the problem or lead them to incorrect or partially-correct solutions, and to plan questions they would ask to help students overcome those difficulties. Seven main categories of difficulties were identified: (1) student error not specified; (2) student makes a calculation error; (3) student chooses an inappropriate operation, formula, procedure or unit; (4) student incorrectly applies a formula or procedure; (5) student forgets to do something; (6) student cannot get started or is stuck; and (7) student's answer and explanation do not match. Questions were identified as focusing on (1) information and/or procedures, which asks students to identify numbers, shapes, key words or procedures to use in solving a problem; (2) context, which asks a student to identify information given in a problem; (3) meanings, which asks a student to explain the meaning of information or the reasoning for procedures; (4) representations, which asks students to use a different representation to explore a problem; or (5) general questions about concepts or procedures with no connection to the context of the problem. Note that many PSTs proposed more than one category of difficulty and question types.

Twenty-eight of 88 PSTs' work fell into the first category of anticipated difficulty, student error not specified. These PSTs planned questions to help a student overcome the difficulty without specifying the error or difficulty the student may have. Consequently, most questions they planned probably would not support student learning. PSTs primarily planned general questions (50%), such as "How did you get your final answer? Can you show me each step? Could you show me another way using manipulatives?", or questions that focused on information and/or procedures (57%), such as "What operation would you use to try to solve this?" PSTs also planned questions focused on meanings (18%), representations (21%), and context (25%).

Twenty-two PSTs anticipated that students would make a calculation error. For example, "Come up with incorrect factors [of number 56]" or "Started on the wrong number and/or counted incorrectly in the table." Accordingly, the majority (59%) of these PSTs had questions focused on correcting calculation errors that were coded as focusing on information and/or procedures, such as "Can you show me how you counted up from this number?" or "How did you figure out how much money the two items cost together?" Other categories included general questions (27%), and questions focused on meanings (18%), representations (18%), or context (9%).

The third category of anticipated difficulties, student chooses inappropriate operation, formula, procedure or unit, consisted of 56 PSTs' work. These PSTs predicted that students would, for example, "use addition instead of multiplication" (inappropriate operation) or "draws 7 groups of 5 instead of 5 groups of 7" (inappropriate representation) or "confused area and perimeter of a rectangle" (inappropriate formula). The questions these PSTs planned included, "what in the word problem made them think to add the two numbers together" (focus on information and/or procedures); and "Can you read your answer aloud?" (general question). Most (57%) PSTs proposed questions that were coded as focused on information and/or procedures, while about 43% of PSTs proposed general questions with no focus. Questions also focused on meanings (29%), such as "does dividing the angles we know make sense if we are trying to find one missing part?"; representations (29%); or context (25%).

Twenty PSTs predicted that students may make a mistake when applying a procedure or a formula. For example, "didn't carry out, didn't place decimal point correctly," "a mistake that might be made while using a factor tree would be circling a number/numbers that are not prime, like circling a 4 when really, 4 can be broken down further," or "performed equation wrong." The majority (95%) of these PSTs had questions that focused on information and/or procedures, such as "can you explain to me how you did your adding?" and "does the new fraction look like double the old fraction?" Seventy-five percent of these PSTs planned general questions, 55% planned questions focused on meanings, 50% planned questions focused on representations, and 10% planned questions focused on context.

Twenty-one PSTs anticipated that students would forget to do something. Examples include, "compared the numbers in the tens place first and stopped comparing the rest of the number," or "created table correctly, forgot to take into account subtracting the initial expense." These PSTs planned questions to remind students that there was something more that they needed to do. About 71% of these PSTs proposed questions that were focused on information and/or procedures, and 43% proposed general questions. Other questions focused on meanings (33%), such as "what made you conclude that the tens place comparison will correctly compare both multi-digit numbers?"; context (24%), such as "I would ask the student to tell me who ate all the cupcakes"; or representations (10%).

Only four PSTs anticipated solutions where the answer did not match the explanation. Two of those PSTs proposed questions coded as focused on information and/or procedures, including "which amount is bigger, $1.90 or $1.92?" and "what criteria are you using to round it to 55,000?" Finally, the majority of PSTs (82) planned questions for students who were stuck or could not get started. Those questions varied in type, such as those that asked students to focus on the context of the problem (proposed by 23%), such as "how much money does Oliver have in his pocket?", and questions that asked students to focus on information and/or procedures (57%), such as "do you group these numbers?" These PSTs also planned questions focused on representations (37%), such as "can you use one of your math manipulatives to help you solve the problem?"; general questions (57%); and questions focused on meanings (6%).

Overall, the majority of PSTs proposed questions that focused on information and/or procedures (86%). General questions were planned by 77% of PSTs, 42% proposed questions focused on meanings, 52% planned questions focused on representations, and 40% planned questions focused on context.

Conclusions

Authors found that PSTs realized many common procedural errors or mistakes students would make. However, the anticipated errors and the level of questioning indicate that most PSTs did not consider that students may have underlying gaps in their conceptual understanding. A focus on procedures without addressing conceptual understanding is in contrast to the recommended effective teaching practice of building procedural fluency from conceptual understanding, which notes that procedural fluency builds on a foundation of conceptual understanding (NCTM, 2014). Therefore, it is recommended that methods courses should be explicit in discussing types of questioning and focus on questions that support student learning by requiring them to clarify and justify their reasoning.

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MATHEMATICS TEACHER EDUCATION IN THE AGE OF TWITTER: A CRITICAL TOOL IN ELEMENTARY MATH METHODS

Joanne Baltazar Vakil
The Ohio State University
vakil.30@osu.edu

Theodore Chao
The Ohio State University
chao.160@osu.edu

Higher education classes have gradually seen the rise in the use of Twitter for assignments and informal discussions. This paper describes recent research on the use of this platform as a pedagogical tool, demonstrating strategies instructors can implement to promote rich and engaging dialogue of mathematical topics. The potential for utilizing Twitter in Elementary Math Methods courses is illuminated with its use as a discussion board, a means to cultivate a math teacher identity, a forum for sharing classroom ideas, and a connection between pre-service teachers and the in-service teaching community. Data from student surveys and interviews of undergraduate and graduate education students who use Twitter in their classroom are shown.

Keywords: Math identity, Instructional activities and practices, Technology, Community

The landscape of higher education programs is rapidly changing as social media platforms permeate into the curriculum and are required for activities and assignments (Knowlton & Nygard, 2016) and used as a source of "push" technology for instructors to send out course information (Tang & Hew, 2017). One popular type of social media, launched in 2006, is the microblogging platform Twitter, which allows for the communication through posts that can be explored, followed, replied, and forwarded (Gao, Luo, & Zhang, 2012). Twitter's design of instantaneous interaction mimics real-time conversation (Fentry, Boykin, & Vickery, 2017), enabling instructors, through "pedagogical tweeting," (Menkhoff, Chay, Bengtsson, Woodard, & Gan, 2015, p. 1295) to assess students informally. Twitter has increasingly become a focal point of interest to researchers exploring the value and potential of social media in higher education classrooms (Tur, Marin & Carpenter, 2017).

A meta-analysis of fifty-one empirical studies assessed the potential interactions and challenges Twitter offers in an education context (Tang & Hew, 2017). They found that Twitter was beneficial for participants who realized the platform's potential, understood learning outcomes and expectations for its use, and proactively used it for class (Tang & Hew, 2017). A three-phase content analysis of twenty-one studies on microblogging identified classroom challenges such as unfamiliarity and reluctance in working with Twitter, an abundance of distracting information, and lack of participation of "lurkers" (Gao, Luo, & Zhang, 2012, p. 792)

The affordances and limitations of Twitter inform Pre-Service Teacher (PST) educators of how to help their students navigate the increasing societal norm of instantaneous communication. Promoting Twitter can cultivate a math teacher identity within elementary PSTs, provide a forum for sharing classroom ideas, and a connection between PSTs and the in-service teaching community. Disregarding this critical tool will leave students unprepared as they find themselves with 20th century skills in a 21st century world.

Theoretical Framework

With the ubiquity of Twitter and its options of sharing messages, images, and videos, it is necessary to incorporate such prevalent practices into academic coursework. Students, for the most part, are digital natives (Prensky, 2001), though not necessarily "inherently better versed in technology" (Lieberman, 2017). Studies have reported how widely adopted social media is with this generation (Arceneaux & Dinu, 2018), spending one to two hours per day on it (Ozen et al., 2018). Hence, students must be engaged with the technology they are familiar and comfortable with (Denker, Manning, Heuett, & Summers, 2018). Denker, Hermann, & Willits (2015) describe the phenomenon of the polymediated classroom, an extension of the physical college classroom into virtual space "where people are simultaneously communicating online and off" (p. 1). Drawing from Sociocultural Learning Theory (Vygotsky, 1978) and Situated Cognition Theory (Brown, Collins, & Duguid, 1989), we heed the caution against an "ersatz activity," an abstract task that does not align with 'ordinary practices of the culture'" (Brown, Collins, & Duguid, 1989, p 34). Retaining decontextualized classroom encounters that do not include social media may be considered irrelevant to students, and "as a result, students can easily be introduced to a formalistic, intimidating view of math that encourages a culture of math phobia rather than one of authentic math activity" (Brown, Collins, & Duguid, 1989, p. 34).

Our theoretical framework uses the additional lens of Communities of Practice (Wenger, 1998) wherein a community of PSTs and in-service teachers is fostered by using the Twitter domain. The shared interest of teaching and learning by education practitioners results in a dynamic, interactive exchange of ideas among participants, which can lead to connections and relationships in the virtual and real-world settings.

Dearth of Research of Twitter in Mathematics Teacher Education

Research on how higher education instructors utilize Twitter abounds, with studies on its use as a tool for scholarly practices (Veletsianos, 2012; Mollett, Moran, & Dunleavy, 2011; Alerpin, Gomez, & Haustein, 2018) and conference backchanneling (Risser & Waddell, 2018). Instructors who implement Twitter in the classroom tend to share common characteristics of being "reform-minded, open to change, and interested in restructuring their local communities environment to include web-based and social media tools" (Forte, Humphreys, & Park, 2012).

Though research has demonstrated how engaging (Hirsh, 2012; Alim, 2017) and motivating (Rico & Feliz, 2016) Twitter is as a tool for creating and sharing knowledge (Kassens-Noor, 2012; Junco et al., 2011) and collaboration (Tur, Marin, & Carpenter, 2017), limited studies have focused on its use in teacher education. Emerging research details the historical development of #MTBoS (Larsen & Liljedahl, 2017), a grassroots network of mathematics and science teachers which provides a space for nontraditional professional development and a community of practice (Risser & Bottoms, 2018; Willet & Reimer, 2018). Baucom, Ashe, & Webb (2018) report how they have "leveraged Twitter to support the statewide implementation of new mathematics standards" (p. 126). Goals of this investigation include analyzing the discourse engendered within Twitter as the forum for a weekly Math Methods discussion board, observing how this platform cultivates a math teacher identity. Additionally, this study addresses the perceptions PSTs hold concerning the mandatory use of Twitter for class as they employ it to access other forums that actively shares classroom ideas, connecting the PSTs to the in-service teaching community (Carpenter, J. P., & Morrison, S. A. (2018)

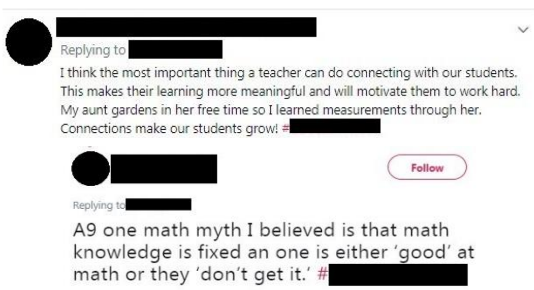
Methods and Data Sources

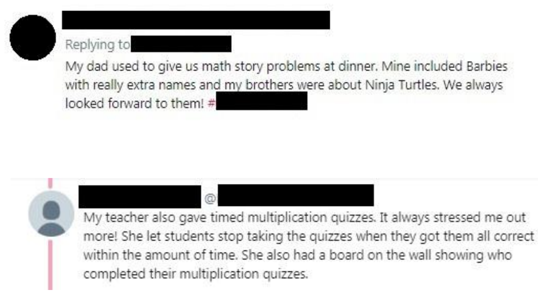
Participants (n= 4) for this qualitative study include undergraduate and graduate PSTs at a large, mid-Western university. Students were enrolled in an Elementary Math Methods course which required the use of Twitter. The students either tweeted directly to the professor's weekly questions about assigned article readings or to a group of 2 or 3 student moderators who tweeted the article questions directly to their classmates. Participants consented to allow the researchers to review the PSTs' Twitter feeds. They also completed a 9-question online Qualtrics survey and took part in a semi-structured, audio-recorded interview for 15- 30 minutes.

Sample Results and Discussion

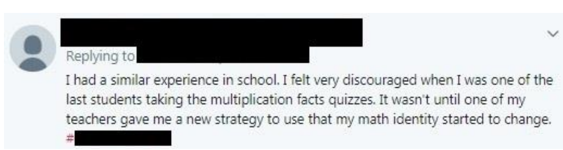
Class Assignment Twitter Feeds

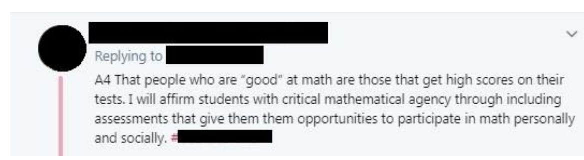
After reading chapters from Aguirre, Mayfield-Ingram, & Martin's (2013) text on identity, PSTs tweeted math stereotypes and assumptions they previously encountered. The response tweets acted as a springboard to deeper discussions asynchronously online and face-to-face.





PSTs addressed of their own informal math experiences, leading to a discussion on personal beliefs, as well as family support and involvement with their development of mathematical concepts. A conversation about math myths emerged and how some PSTs managed to overcome these initial conceptions.





By providing an opportunity for PSTs to share memories of their own experiences, a deep discussion about math anxiety tied into a class reading by Johnston (2010). The tweets exhibited an emerging understanding of mathematical agency (Aguirre et al., 2013) and provided a starting point for PSTs to begin unpacking issues of equity and social justice in the classroom.

Initial Survey Responses

The 9- question Qualtrics survey responses revealed that half of the participants did not previously have a Twitter account and that using this form of social media required at least 2-3 weeks for three of the participants to "feel comfortable in navigating Twitter." The instructor designated time during class to establish a course hashtag (Lowe & Laffey, 2011), model the tweeting process, guide students towards creating a professional account, and highlight hashtags of interest to PSTs such as #ElemMathChat, #iteachmath, and #MTBoS. Preparing students in this manner helped them "fully exploit the resources channeled through Twitter" (Tang & Hew, 2017, p. 111). Research by Rohr & Costello (2015) recommend addressing expectations for quality tweets by rewarding students higher marks for comments that build on other responses.

When asked if the participant felt if "tweeting for weekly discussions helped create ideas among you and your classmates?" and if they felt that "Twitter was an appropriate forum for discussing math problems or problem solving?" all but one participant agreed. Finally, all but one participant agreed that "Tweeting for weekly discussions works better than a traditional text-based discussion board" The participant who disagreed selected the answer "No, a combination would work better." The feedback from the participants enjoins future research to investigate the limitations of Twitter as an effective discussion board platform.

Individual Semi-structured Interviews

When asked "What aspects of using Twitter for weekly class discussions did you like during the course?" the following responses by participants (pseudonyms are used) were made.

> Megan: *I liked the fact that it's not completely cut off from the real world around us; people from education world can see our posts; the goal is not to prove that we read the article, it was to respond what we had read.*
> Andrew: *I like that it is easier to respond and post discussion post questions. It is kind of hard navigating and posting… There are more functions to Twitter, you can post a picture.*
> Jackie: *Examples everyone brought in; their experiences all their co-teachers would do… our mathematical experiences … I felt I came to realize there were mutual feelings.*

One participant noted her difficulties in responding to the Twitter assignments. Her concerns echo Tang & Hew's (2017) suggestions for further work into measuring student behavioral and performance patterns in order to "derive more meaningful lessons to design appropriate Twitter activity to engage different students" (p. 113).

> Hillary: *It was [pause]… fast; I feel like I could respond and forget about it…. Not normally my goal…I struggled to get …something out of the twitter discussion.*

Though she failed to see the benefits of using Twitter as a student, she understood the advantages of using it as a future teacher. When asked, "What are some reasons why you would or would not continue to use Twitter for math discussions after the course?" Hillary answered on a more positive note.

> Hillary: *I don't know if I would stop it… I like how you can make connections with other teachers around world… it's like the new research… getting advice from them.. what you can do differently if or giving advice back.*

Feedback for the course use of Twitter was addressed in the final interview question, "Can you elaborate on your survey question answer to #9, what recommendations do you have concerning the use of Twitter in an online or face-to-face STEM classroom?

> Megan: *We should include [and] take note of posts by influential members the education field; experience more authentic, use Twitter as a means to advocate for things that are important; retweet posts by [Superintendent of local public school]…*

> Andrew: *I really like how we have been using it for math… it would be interesting to use it for ELA. It would be fun …book discussions… [about] science and math…*

Two students provided constructive criticism by pointing out the need for more organization and reducing class time for reviewing themes and ideas expressed during the online interaction.

> Jackie: *[Using Twitter for class was] Good and bad… each week a different group was in charge… confused…kind of told us too late; just trying it out; due dates making sure are set.*
> Hillary: *I would drastically reduce the amount of time used to discuss Twitter in class…. Bulk of time on own time and also in class [would be] cut back on that and only pose one question at a time on Twitter and give students another way to discuss it if not their preferred learning style.*

Significance

The samples of student tweets in this study exemplified how Math teacher educators can weave topics of math identity and equity into Twitter conversations, as will be further elaborated in the presentation. The interviews offered insightful perspectives of PSTs which can inform Math teacher educators designing future courses incorporating Twitter as a weekly discussion board. The reflections demonstrated that overall, PSTs held positive perceptions of Twitter as a critical tool connecting them to a community of practice including stakeholders in the education field and that its feature of making "connections with other teachers around [the] world" could be extended as a forum to "advocate for things that are important."

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TEACHING PROSPECTIVE TEACHERS TO NOTICE HIGH LEVERAGE PRACTICES

Melissa A. Gallagher
University of Louisiana at Lafayette
mgallagher@louisiana.edu

Annick Rougée
Washington State University, Tri-Cities
annick.rougee@wsu.edu

Jessica Jensen
Cal Poly, San Luis Obispo
jjense11@calpoly.edu

Jennifer Suh
George Mason University
jsuh4@gmu.edu

This paper reports on the results of an intervention targeting prospective teachers' (PTs') noticing of high leverage mathematical practices (using precise mathematical language, giving mathematical explanations, using students' mathematical contributions, and remediating student errors). PTs completed a pre-post video assessment in which they watched a short clinical interview and answered prompts about what they attended to, how they interpreted the student thinking and instructional moves, and how they would respond if they were the teacher. Results indicated that the changes PTs' overall noticing skills from pre to post was approaching statistical significance, but that they still did not reach high levels of sophistication.

Keywords: Teacher Education-Preservice; Instructional activities and practices

Objectives/Purposes of the Study

The complexity of the classroom requires teachers to process a cacophony of stimuli. Sifting through these stimuli, determining which need immediate response, which can be addressed later, understanding what these stimuli mean in terms of student understanding, and determining the best response to move students towards instructional goals are the sophisticated noticing skills that expert teachers employ constantly throughout the day. Researchers have theorized a learning to notice framework (van Es & Sherin, 2002) which outlines levels of sophistication of what and how teachers notice. Part of the skill of noticing involves choosing which instructional practices to employ when responding to stimuli. Some researchers have advocated the teacher education programs should focus on teaching prospective teachers (PTs) certain "high leverage practices," or practices that have the most impact on student learning so that they are prepared to implement these practices effectively (Grossman Hammerness, & McDonald, 2009; Lampert et al., 2013).

In order to prepare PTs to sift through the complexity of classrooms, develop more sophisticated noticing skills, and learn about high leverage practices, we developed five lessons to specifically instruct them in these skills. The purpose of this study is to evaluate if the PTs' noticing skills improved after participating in these lessons. The following research questions guided this study:

1. If PTs receive explicit instruction in the skills of noticing (attending, interpreting, and responding), do their noticing skills improve?

2. After participating in the intervention, which instructional practices do PTs notice most?

Theoretical Perspective

Our study is framed by two main areas of prior research: teacher noticing and high leverage practices. Research on PTs' noticing often examines what they attend to in the classroom and how they interpret those events (van Es & Sherin, 2002; Mitchell & Marin, 2015). Sometimes research on PTs' noticing has also examined how they would respond to specific classroom events (Jacobs, Lamb, & Philipp, 2010). Research on using high leverage practices with PTs has mostly focused on helping PTs decide which responses to make in the moment, focusing on developing PTs' knowledge of specific instructional practices (Ball, Sleep, Boerst, & Bass, 2009). We see a need to first teach PTs how to notice and then give them the tools of the high leverage practices to support their responding.

Noticing

Experts within a profession notice different aspects of their environment (Mason, 2002). Noticing has been conceptualized slightly differently by different researchers (Jacobs & Spangler, 2017). For the purposes of this study, we follow Jacobs and colleagues' (2010) conceptualization of noticing as a nested model wherein teachers must first attend, then interpret, and then respond to classroom events. We combined this conceptualization of noticing with the Learning to Notice framework (van Es & Sherin, 2002) as we created the lessons for this study. The Learning to Notice framework posits that there is a predictable trajectory through which teachers progress as they learn to notice (van Es & Sherin, 2002): (1) teachers describe literal events in the classroom and make judgmental statements; (2) teachers make some comments at level 1, but begin to highlight noteworthy events in the classroom, without providing specific evidence for their interpretations of these events; (3) teachers identify important events and can interpret those events; and (4) teachers make connections among the important events they noticed and important principles of teaching and learning and begin to "offer pedagogical solutions based on their interpretations" (p. 581). Lastly, we integrated the ideas of Barnhard and van Es (2015) that PTs can engage in attending, interpreting, and responding at different levels of sophistication when viewing video of their own instruction (Figure 1).

Skill	Low sophistication	Medium sophistication	High sophistication
Attending	Highlights classroom events, teacher pedagogy, student behavior, and/or classroom climate. No attention to student thinking.	Highlights student thinking with respect to the collection of data from a scientific inquiry (science procedural focus).	Highlights student thinking with respect to the collection, analysis, and interpretation of data from a scientific inquiry (science conceptual focus)
Analyzing	Little or no sense-making of highlighted events; mostly descriptions. No elaboration or analysis of interactions and classroom events; little or no use of evidence to support claims.	Begins to make sense of highlighted events. Some use of evidence to support claims.	Consistently makes sense of highlighted events. Consistent use of evidence to support claims.
Responding	Does not identify or describe acting on specific student ideas as topics of discussion; offers disconnected or vague ideas of what to do differently next time.	Identifies and describes acting on a specific student idea during the lesson; offers ideas about what to do differently next time.	Identifies and describes acting on a specific student idea during the lesson *and* offers specific ideas of what to do differently next time in response to evidence; makes logical connections between teaching and learning.

Figure 1: Levels of Sophistication for Noticing Skills (Barnhard & van Es, 2015).

Expert teachers notice more than novice teachers (Huang & Li, 2012; Jacobs et al., 2010). Furthermore, PTs can learn to notice (Hiebert, Miller, & Berk, 2017; Mitchell & Marin, 2015; van Es & Conroy, 2009). For example, Mitchell and Marin (2015) created a video club in which four PTs viewed video of their own instruction over the course of 10 weeks. In advance of each video club meeting, the PTs viewed and scored a video of either their own or of their peers' instruction using the Mathematical Quality of Instruction (MQI) rubric (Hill et al., 2008). During the meetings, an MQI-trained rater facilitated a discussion of their scores. After participating in these sessions, Mitchell and Marin found that the PTs were more likely to notice aspects of mathematics as well as specific aspects of the MQI. The PTs were also less evaluative and judgmental regarding what they noticed. Our study builds on this work by using the MQI to frame the high leverage practices, but differs in how we organized the lessons, with attention to four specific practices and with all lessons structured around supporting PTs' skills in attending, interpreting, and responding.

High Leverage MQI Practices

The High Leverage Practices were born out of the desire for a shared professional curriculum that focuses instruction for prospective mathematics teachers on the essential knowledge and skills needed for practice (Ball et al., 2009). The goal of this instruction is to engage PTs in doing these practices, rather than simply hearing and talking about them. The in-depth teaching and learning of these practices in methods courses allows for PTs to practice and develop these skills in a low-risk environment, prior to enacting them in a classroom with students (Ball & Forzani, 2010). However, even with a shared curriculum, it can be difficult to distinguish when a PT's practice has reached a level that is appropriate to enact in a classroom. The Mathematical Quality of Instruction (MQI) framework was created to be able to identify and analyze mathematical features of the work of mathematics teaching (Learning Mathematics for Teaching Project, 2011). The MQI framework consists of seven constructs, which include: richness and development of the mathematics, responding to students, connecting classroom practice to mathematics, language, equity, and presence of mathematical errors, all of which are pivotal skills in enacting practices that engage students in meaningful learning of mathematics. Each construct has between four and seven individual instructional practices which represent important mathematical qualities of teaching. Based on our previous research with using the MQI with PTs (Gallagher & Suh, under review), we decided to focus on four instructional practices: mathematical language, mathematical explanations, using student mathematical contributions, and remediating student errors and difficulties.

Methods

This mixed methods study integrated qualitative and quantitative analysis of PTs' noticing. For the first research question, qualitative analysis techniques were employed first to stay close to the data in determining the levels of sophistication of the three noticing skills, and quantitative techniques were applied subsequently to explore whether there were changes in the sophistication of PTs' noticing after participating in the intervention. To answer the second research question, we employed qualitative techniques to investigate which aspects of the MQI were noticed most by the PTs after participating in the intervention.

Context and Participants

Participants from this study were sampled across three different university sites (the Pacific Northwest, the South and the Mid-Atlantic). All were enrolled in an elementary mathematics methods course (which range from a one to two semester sequence) at their respective universities. At the conclusion of the semester, five PTs (who agreed to participate in this research) were randomly selected from each of the three university sites for a total of 15 PTs in this sample.

Noticing Lessons

The authors designed five lessons all of which included an activity to activate prior knowledge, a short discussion of the high leverage practice of focus (e.g., mathematical language) and video or transcript analysis using the noticing framework (with the high leverage practice of focus as a lens for noticing). The first lesson introduced PTs to the noticing framework and the three noticing skills: attending, interpreting and responding. The subsequent four lessons focused on the following MQI practices: using precise mathematical language, giving mathematical explanations, using students' mathematical contributions, and remediating students' mathematical errors.

Data Collection

A week or two before the first noticing lesson, PTs completed an online module in which they watched a video of a clinical interview of a student converting a mixed number ($3\frac{3}{8}$)into a fraction greater than one (San Diego State University Foundation, Philipp, Cabral, & Schappelle, 2011). After the video, PTs answered the following prompts in writing: (a) "Please describe in detail what you noticed", (b) "Please explain what you learned about Rachel's understanding.", (c) "Please explain what you learned about the teaching", and (d) "If you were the teacher, what would you do next? Be specific and give your rationale." PTs completed this same task within a week of the last noticing lesson.

Data Analysis

To evaluate the first research question, we began by defining what it would mean for noticing skills to improve. Using the learning to notice framework (van Es & Sherin, 2002) and the levels of sophistication for noticing (Barnhard & van Es, 2015) as a starting point, we began to consider what it would mean for PTs to attend, interpret, and respond at different levels of sophistication. After reading through those frameworks, we then compiled the PTs' responses to the pre and post assessment in a spreadsheet, separating responses to each of the four prompts. We then read through the responses for six PTs and began to level them as low (1), mid (2), and high (3), adding comments justifying each level. We met regularly through this process, comparing our levels and justifications in order to build a rubric (Table 1). We realized that we were asking the PTs to attend to and interpret both teacher and student actions and utterances, so we created categories for each of those in addition to a category for responding.

Additionally, through these discussions we realized that there were particular moments to which we were expecting the PTs to have attended, interpreted, and responded. While focusing on the student, these moments seemed to be related to important mathematical ideas the student shared or a key mathematical error that she made. Thus, we termed the moments we expected the PTs to have attended to with respect to the student as *critical mathematical moments*, or moments in which important mathematical ideas (both understandings and misunderstandings) arose in the student's work or talk. When we focused on the teacher, we realized that the corresponding moments were focused on the pedagogical moves the teacher used in response to the student's critical mathematical moments. We called these *critical pedagogical moments*, which we define as teacher moves in response to the student's critical mathematical moments.

Once we created the rubric and agreed upon the scoring of the six PTs' responses, three of the authors scored the remaining nine PTs' responses. To establish reliability, two participants scores were double-coded by pairs of researchers; 90% of researchers' codes were in agreement. The codes that differed were decided by the third researcher. We then calculated the average of PTs' scores across the five categories of noticing (i.e., attending to the teacher, interpreting the teacher's actions and utterances, attending to the student, interpreting the students' actions and utterances, and responding) for their pre and post lesson assessment. Lastly, we used a paired samples t-test to examine whether there were differences in the PTs' noticing from pre to post.

With regard to the second research question, in order to determine which of the four high leverage MQI practices were noticed most, two of the authors coded the post-assessments of all participants with four a priori codes: mathematical language, mathematical explanations, uses student contributions, and remediating student errors. The two authors met to discuss how they planned to code the data and decided that the practice must be explicitly mentioned by the PTs in order to be coded. After these decisions were made, they divided the assessments and coded them independently. Once all data were coded, they examined each other's coding to be sure they had applied codes consistently.

Table 1: Rubric for Scoring Levels of Noticing

	Low	**Mid**	**High**
Attending - Teacher	No mention of teacher moves OR Describing teacher actions in general terms (i.e., "asked questions")	Describing teacher actions/utterances specifically, but they may not be related to the critical pedagogical moment	Describing teacher actions/utterances specifically, including those related to the critical pedagogical moment
Interpreting - Teacher	Interpretation without evidence, either because they didn't attend or because they were interpreting about something outside of the video	Interprets teacher actions/utterances but they may not be related to the critical pedagogical moment	Interprets teacher actions/utterances and how they contributed to student thinking around the critical pedagogical moment
Attending - Student	Did not attend to mathematical aspect of student thinking (i.e., critical moment around a misconception/partial understanding/thinking) OR attended to critical mathematical moment but very generally (i.e., not in detail)	Attended specifically to students utterances/actions AND attended to critical mathematical moment but did so vaguely (i.e., noticing that there was an error and describing it in general terms)	Attended to critical mathematical moment with details and examples capturing exact student actions and utterances
Interpreting - Student	Interprets st thinking but not around the critical mathematical moment with a general explanation (i.e., "student is confused about the process")	Interprets st thinking around the critical mathematical moment with a general explanation (i.e., "student misapplied the steps in the procedure, but she does have conceptual understanding of fractions")	Accurately and specifically interprets student thinking with regard to the critical mathematical moment (i.e., "student misapplied the steps in the procedure, but she does have conceptual understanding of fractions, including 8/8 in one whole and 3 wholes in 3 3/8")
Responding - Teacher response to student	General, future action without rationale or that does not address the critical mathematical moment	One general response that addresses the critical mathematical moment but doesn't necessarily connect conceptual and procedural understanding	Multiple ways to respond with rationales that make sense to address critical mathematical moment OR one in-depth and specific response with a rationale that addresses the critical mathematical moment and makes connections between procedural and conceptual understanding

Findings

For the first research question we investigated whether or not there were statistically significant differences from pre to post on PTs' overall noticing. Means and standard deviations for each category of noticing are included in Table 2. A visual examination suggests that PTs improved in their attending skills to both the teacher and the student, but that their interpreting and responding skills may not have improved. Furthermore, only three PTs scored a 2 (mid-level sophistication) on interpreting the teacher's actions at post, and the rest scored a 1 (low-level sophistication). This indicates that they were either not interpreting the critical pedagogical moments at all or making superficial, vague interpretations not connected to the pedagogical moves or mathematics, even though many of them were able to attend to those moments (four students scored a 3 or high-level sophistication, on attending to the teacher at post, indicating that they had correctly attended to the critical pedagogical moments). We also found that when discussing what their next instructional move or response to the student would be if they were the teacher , their plans were likely to be general and not provide concrete next steps (e.g., "I would give Rachel another number to change into a number greater than one, before moving on."). The paired samples t-test examined whether there were differences from pre to post assessment on the PTs' overall noticing scores, as overall noticing was the only continuous variable. This test indicated that differences on overall noticing were approaching statistical significance, $t(14) = 2.06$, $p = .059$, from pre ($M = 1.29$, $sd = 0.31$) to post ($M = 1.52$, $sd = 0.28$).

Table 2: Means and Standard Deviations of Levels of Noticing at Pre and Post (N= 15)

	Pre-Assessment Scores	Post-Assessment Scores
Attending-Student	1.27 (0.46)	1.80 (0.86)
Interpreting-Student	1.27 (0.46)	1.20 (0.41)
Attending-Teacher	1.20 (0.56)	1.73 (0.59)
Interpreting-Teacher	1.40 (0.51)	1.47 (0.52)
Responding	1.33 (0.49)	1.40 (0.51)
Overall	1.29 (0.31)	1.52 (0.28)

With regard to the second research question, we found that remediation of student errors and misconceptions was the high leverage MQI practice noticed most often by the PTs in the post-assessment (Table 3). This makes sense as the video was chosen because it illustrates a student who had a procedural misconception but was able to leverage her strong conceptual understanding to complete the conversion of $3\frac{3}{8}$ to a fraction greater than 1. When we examined the PTs' comments regarding remediation of student errors and misconceptions, we found that they were divided on whether to provide conceptual or procedural remediation in response to the student's misconceptions. This seems to have stemmed from difficulty interpreting the critical mathematical moments; no PTs scored at a 3 (high-level of sophistication) for interpreting the student's actions or utterances. An unexpected finding from the analysis of the post-assessment was that not all of the PTs had the mathematical content knowledge, even by the end of the course, for converting a mixed number to a fraction greater than 1. In particular, two of the PTs thought that the problem in the video was $3 \times \frac{3}{8}$.

Table 3: Frequency of MQI Practices Noticed in the Post Assessment

High Leverage Practices	Frequency
Mathematical Language	10
Mathematical Explanations	8
Uses Student Mathematical Contributions	11
Remediation of Student Errors and Difficulties	17

Discussion/Conclusion

An important contribution of this study is the development of the rubric which examines levels of sophistication of the skills of noticing in mathematics, building off the work of Barnhard and van Es (2015), and extending this work by adding criteria for attending to and interpreting the actions and utterances of both the student and teacher. Also, in comparison to other learning to notice frameworks (Barnhard & van Es, 2015; van Es & Sherin, 2002), this rubric focuses on how PTs attend, interpret, and respond to the critical mathematical and pedagogical moments. Within the complexity of the classroom, teachers need to sift through all the stimuli and determine what is most important to attend to. They must then interpret and respond to these critical moments. After participating in this intervention, more of the PTs noticed these critical moments. Supporting PTs' ability to attend to these critical moments is an important first step in developing effective and responsive practitioners. We found that it was essential for us, as researchers, to identify the critical mathematical and pedagogical moments before we began coding. In spite of the fact that the critical mathematical and pedagogical moments will vary for every teaching episode that is viewed, we believe this rubric can nevertheless be applied across different teaching episodes, so long as those moments are identified at the start.

Although the PTs did seem to improve in their overall noticing skills, particularly in attending, and they did notice the high leverage MQI practices which were explicitly taught, we also found that their interpretations of what the student needed next or of the implications of the teacher's instructional decisions did not reach high, or even mid-level, sophistication. We also found that their responses were likely to be general and not offer concrete next steps. Thus, we are working to revise our lessons to include more emphasis on interpreting student thinking and interpreting teachers' instructional decisions. We are also making revisions that will push PTs to think more specifically about their instructional responses given these interpretations.

The findings of this study have important implications for teacher educators as well as researchers. Teacher educators should consider how they can explicitly teach PTs to attend, interpret, and respond to high leverage practices and students' mathematical thinking. This may involve helping the PTs to identify critical mathematical and pedagogical moments in teaching episodes in addition to unpacking each of these practices. Furthermore, although our findings were approaching statistical significance from pre to post for PTs after participating in the intervention, it is possible that these differences were due to the normal maturation of the PTs. In order to determine if differences could be attributed to the intervention, future research should compare the change in levels of noticing of PTs who participated in the intervention to those whose classes did not receive the intervention.

In order to support PTs in becoming effective teachers, they need to be able to determine what they should attend to amid the cacophony of classroom stimuli. They need to be able to interpret those critical moments in light of principles of teaching and learning, and they need effective instructional practices that they can implement in response to those interpretations. By explicitly teaching PTs to attend to and interpret critical mathematical and pedagogical moments, and by giving them specific high leverage practices that they can use to respond to those moments, teacher educators can support the development of effective teachers.

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EXPLORING ASPECTS OF PRESERVICE TEACHERS' MATHEMATICAL LITERACY THROUGH MATHEMATICS AUTOBIOGRAPHIES

Jo Towers
University of Calgary
towers@ucalgary.ca

Ayman Aljarrah
University of Calgary
aaljarra@ucalgary.ca

Lyndon Martin
York University
lmartin@edu.yorku.ca

Herein we present some of the findings of a broader research study exploring students' lived experiences of learning mathematics in Canadian schools. In this paper we explore aspects of preservice teachers' mathematical literacy as revealed by their mathematical autobiographies. More specifically, our exploration is driven by the question: what forms of mathematical literacy, if any, are revealed by preservice teachers' mathematical autobiographies?

Keywords: Affect, emotion, beliefs and attitudes; Teacher knowledge; Instructional activities and practices.

Introduction

According to the National Council of Teachers of Mathematics (NCTM, 1989), the notion of mathematical power (or mathematical literacy) "is based on the recognition of mathematics as more than a collection of concepts and skills to be mastered; it includes methods of investigating and reasoning, means of communication, and notions of context" (p. 5). The NCTM (1989) determined 5 goals for students that reflect the importance of mathematical literacy: (1) learning to value mathematics, (2) becoming confident in their ability to do mathematics, (3) becoming mathematical problem solvers, (4) learning to communicate mathematically, and (5) learning to reason mathematically. These goals for students are all dependent and contingent on promoting students' access to the power of mathematics, thus, a crucial role for mathematics teachers is to engage students in doing and understanding mathematics to help them recognize its power and become mathematically literate. Fulfilling such a role "requires effective mathematics teaching in all classrooms" (NCTM, 2000a, p. 17) necessitating deep understanding of the field of mathematics, commitment to students as learners and human beings, and the development and use of a variety of pedagogical strategies. In addition, teaching that has the potential to promote students' mathematical literacy, "requires a challenging and supportive classroom learning environment" (p. 18), and a continuous seeking for improvement. Having this in mind, this paper explores preservice teachers' mathematical literacy as revealed by their mathematical autobiographies.

Literature Review

Mathematical literacy has been a domain of great interest for educational systems all around the world. In its 2003 Assessment Framework, the Program for International Student Assessment (PISA), which was launched by the Organisation for Economic Co-operation and Development (OECD) in 1997, defined mathematical literacy as

> an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD, 2003, p. 15)

According to the PISA 2003 Assessment Framework, mathematical literacy refers to students' abilities and skills in analysing, reasoning, and communicating ideas effectively "as they pose, formulate, solve, and interpret mathematical problems in a variety of situations" (p. 24). We believe that the NCTM's (1989) goals associated with the transformative process of *becoming mathematically literate* are, ultimately, dependent on engaging students in processes of doing and understanding mathematics and that how students learn mathematics shapes their lived experiences of what it means to become mathematically literate. Our decades of experiences in the field of mathematics education have convinced us that two of the NCTM's (2000a) principles for school mathematics are critical in shaping students' development of mathematical literacy: *equity* and *learning*. The NCTM's (2000a) principles and standards document suggests that *equity*—high expectations and strong support for all students—and *learning*—learning mathematics with understanding, actively building new knowledge from experience and previous knowledge—are fundamental for high-quality mathematics education. Thus, in our analysis of the participants' mathematics autobiographies we focus on what these participants seem to understand about the conditions of the teaching environments that shaped their experiences and how these conditions map onto these two principles: equity and learning.

Methodology and Methods

This report is part of a broader research study, the main purpose of which is to gain a deeper understanding of students' lived experiences of learning mathematics in Canadian schools, and to help teachers and schools to understand their roles in shaping students' mathematical identities. To fulfil the objectives of the study, we adopted a narrative inquiry methodological framework. According to Clandinin and Connelly (2000), "narrative is the best way of representing and understanding experience" (p. 18).

We gathered 'mathematics autobiographies' from kindergarten to post-secondary students, as well as from members of the general public, in two Canadian provinces. In these autobiographical narratives participants described their feelings about learning mathematics, the images of mathematics as a discipline that they hold, the kinds of mathematics teaching they have experienced, and their dispositions and identities in relation to this important discipline. Some responses were gathered during face-to-face interviews and some were submitted anonymously via a website. Many of the latter type of submissions were in multimedia form (e.g., Powtoon, VideoScribe, etc.).

In this report we draw on data from 48 preservice teachers who were participating in one of two Bachelor of Education (B.Ed.) programs in Alberta and Ontario. In our analysis of the study's dataset we have used several approaches, including thematic analyses (e.g., Towers, Takeuchi, Hall, & Martin, 2017), application and/or development of new analysis techniques (e.g., Hall, Towers, & Martin, 2018; Plosz, Towers, & Takeuchi, 2015), and focused interrogations in which we use selected portions of our extensive dataset to shed light on particular issues and questions of relevance to the field (e.g., understanding the unique experiences of very young children in Canadian classrooms (Takeuchi, Towers, & Plosz, 2016; Towers, Takeuchi, & Martin, 2018). The analysis presented here results from addressing our dataset in the latter of these ways and explores the question: what forms of mathematical literacy, if any, are revealed by preservice teachers' mathematical autobiographies?

Findings

In this section we use examples from several preservice teachers' mathematics autobiographies to highlight what these participants seem to understand about the conditions of the teaching environments that shaped their experiences and how these conditions map onto two of the NCTM's principles for school mathematics: equity and learning.

Equity

As noted by the NCTM (2000a), excellence in mathematics education requires equity—high expectations and strong support for all students. The significance of this principle was evident in the narratives of many preservice teachers as they reflected on the importance of being a teacher who strongly supports all students and believes every student can learn mathematics and as they gave examples from their own histories as learners that showed why this principle had meaning for them. For example, one student, Amy, described a pivotal moment in her Grade 5 year when she summoned the courage to approach the teacher to discuss a poor grade she had received on a test. She recalled "his kind and encouraging words, along the lines of 'It's just one test, don't worry about it. I know that you can do much better.'" This participant noted the validation she felt by recognizing that her teacher knew (and communicated to her) that she had potential, even though she experienced some struggles.

Another participant, Carly, described being a successful student until Grade 11. She painted portraits of two of her early teachers and the environments they created that she felt contributed to her feelings of success. These environments included elements such as student choice, opportunities for success, making real life connections, use of manipulatives, encouraging multiple strategies, having high standards for all, and supporting student risk-taking. This is how she described her Grades 3 and 8 teachers:

> By asking interesting questions [my Grade 3 teacher] constantly had us engaged and thinking about math. With a large focus on asking questions ourselves, we felt comfortable and ready to take risks. The authentic relationships each of us had with [him] made us feel important and successful.
>
> [My Grade 8 teacher] set high expectations for his students while offering support. He saw the strengths in his students and believed they could all excel….He encouraged his students to become comfortable with a multitude of tools and strategies. [He] valued the learning differences of all his students. By expecting the best of his students and scaffolding where needed, fewer students gave up on themselves.

By contrast, when students felt unsupported or undervalued in the classroom, the effects were just as profound. Many described classroom environments where there was little opportunity for discussion, answers were dismissed as either right or wrong, there was no support, questions were discouraged, and only students perceived as 'strong' were valued. Here is how Carly described her encounter with such a classroom environment in Grade 11:

> [My Grade 11 teacher] didn't know my name. I was on my way to being a mathematician until I entered this classroom. Unlike my previous classes I dreaded going here. He seemed to favor the students that excelled in his class and cared little for the ones that did not….I was afraid to ask questions. When I fell behind, I gave up.

Learning

As noted by the NCTM (2000a), students must learn mathematics with understanding, actively building new knowledge from experience and previous knowledge. A student we heard from earlier, Amy, described realizing during her preservice teacher education program that the kind of mathematics teaching to which she had been exposed throughout her education followed a script common to North American classrooms (Stigler & Hiebert, 1998) wherein a preferred procedure was first demonstrated by the teacher and subsequently practiced by students. Reflecting on her learning in these classrooms she was now able to recognize that she "might have been able to understand the problem better if [she had been] given an opportunity to figure it out on [her] own" and that it might have been easier for her to "solve the problems in her own way as opposed to following the 'correct' procedure."

Another student, Sammy, reflected that after reading Stigler and Hiebert's (1998) piece on teaching as a cultural activity she "realized that [her] Grade 11 teacher did not take the time…to observe what [was] being taught from the perspective of the student [and instead] followed the 'show, tell, do' model where [students] were shown [a] few examples to grasp the basic concept and [then told to] practice with other given questions" resulting in the participant feeling she wanted to be "done with math" as soon as she reached the end of Grade 12.

Looking back over her history of learning mathematics, another preservice teacher, Jenny, reflected that she did not feel engaged by the majority of her teachers and they did not allow her to make sense on her own and understand what she was learning. For her, mathematical reasoning was not "encouraged or respected" for most of her educational journey. Only in Grade 10 did she encounter a teacher who engaged her and "allowed [her] to use [her] own creativity to solve math problems." This lament—the restrictive, procedural, and formulaic approach to mathematics adopted, and in some cases rigidly enforced, in school mathematics classrooms—was a common feature of many of the mathematics autobiographies we collected. Many participants reported on the recent recognition (during mathematics education coursework) that the mathematics they had learned in school was entirely memorized and rote rather than deeply understood and, in many cases, such as for Abigail whose comment is below, they reported feeling that something had therefore been taken from them in their formative experiences of learning mathematics.

> It became evident to me, that I don't ever remember being in a mathematical classroom environment such as this. I feel as though, had I been, my perspective on math may have been much different. I believe I would have benefitted from being able to discuss math with other students and learn about the different ways my classmates went about solving their math problems; ultimately building on our math skills together. As opposed to sitting at my desk on my own feeling confused and helpless.

Discussion and Conclusion

These participants' stories show that, often through coursework in their teacher education programs, these preservice teachers are developing sophisticated understandings of the kinds of classroom environments that reflect the NCTM's (2000a) equity and learning principles. Such classrooms are built on mutual respect, support for risk-taking, high expectations, and collaboration. These principled classrooms also honour students' own skills of analysing, reasoning, and communicating ideas effectively (OECD, 2003) and hence privilege the growth of mathematical understanding and foster mathematical literacy. "Research has solidly established the important role of conceptual understanding in the learning of mathematics" (NCTM, 2000b, p. 2) and developing such conceptual understanding enables students to "become competent and confident in their ability to tackle difficult problems and willing to persevere when tasks are challenging" (p. 2), two important goals for a mathematically literate citizenry. Our work shows that having preservice teachers explore, through mathematics autobiographies, their own histories of learning mathematics can help them to recognize and unpack important principles of learning and teaching mathematics and come to understand and be able to articulate aspects of their own mathematical literacy.

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SECONDARY PRE-SERVICE TEACHERS' IMAGES AND INTERPRETATIONS OF STUDENT MATHEMATICAL THINKING

Dawn Teuscher
Brigham Young University
dawn.teuscher@byu.edu

J. Matt Switzer
Texas Christian University
j.switzer@tcu.edu

Teachers utilizing student mathematical thinking is important when teaching, yet many inservice teachers find it difficult to implement. The Standards for Preparing Teachers of Mathematics (AMTE, 2017) outline the knowledge, skills, and dispositions that beginning teachers should have after graduating including the importance of attending to and interpreting student mathematical thinking. In this paper, we present results from two focused video analysis assignments that our pre-service teachers engaged in to identify their images of student mathematical thinking and their ability to attend to and interpret student mathematical thinking.

Keywords: Teacher Education – Preservice, Instructional Activities and Practices, Noticing, Student Mathematical Thinking

Utilizing student mathematical thinking (SMT) is important when teaching mathematics, yet, it is challenging for many teachers to utilize SMT in ways that help students make sense of the mathematics. Mathematics teacher educators (MTEs) have used video clubs to improve in-service teachers' focus on SMT and researchers have found that video clubs help teachers refocus from teacher actions to student thinking (e.g., van Es & Sherin, 2010; Walkoe, 2015). Others have assisted inservice teachers to improve their ability to attend to and use SMT while teaching yet attending to and using SMT in-the-moment of teaching is still challenging.

The Association of Mathematics Teacher Educators (AMTE) recently released *Standards for Preparing Teachers of Mathematics* (2017) providing guidance of the knowledge, skills, and dispositions that beginning teachers should have upon graduating. One attribute emphasizes preservice mathematics teachers' (PSTs) ability to "anticipate and attend to students' mathematical thinking and mathematical learning progressions" (AMTE, 2017, p.18). We affirm this standard but wonder if changes in the types of activities that our PSTs complete during their programs might assist them in attending to and interpreting SMT.

We have goals in our undergraduate mathematics education program for PSTs to learn to anticipate, attend to, interpret, and use SMT. In each of our PSTs six mathematics education courses they spend time anticipating, attending to, and interpreting SMT with a focus on the importance of using SMT while teaching. Many activities (e.g., analyze authentic student work, watch video clips of real classrooms) that we use are common in other teacher education programs. However, we have found that our PSTs do not attend to or interpret SMT in productive ways (Teuscher, Switzer, & Morwood, 2016). Therefore, we deem it important to reconsider the types of activities our PSTs complete to assist them in learning to attend to and interpret SMT more productively.

Purpose of the Study

In this paper, we argue that the types of activities that PSTs engage in during their program may influence their images of SMT, which can assist them in attending to and interpreting SMT. We present results to answer the research question: *How do our PSTs attend to and interpret* *student mathematical thinking prior to student teaching?* We strive to make our PSTs' images of SMT explicit to illustrate how specific activities may afford PSTs opportunities to focus on SMT during their program, which we hope will assist them as they enter the teaching profession.

Theoretical Framework

Noticing is defined as being aware of something (Notice, 2019). Researchers have found that teachers, including PSTs, tend to notice classroom features that may not be important, yet were important to the teacher who noticed them (e.g., things on the wall, how the students were seated). Therefore, if we want PSTs to notice specific features, it is essential to guide PSTs' noticing so they become aware of important distinctions in the classroom. We frame our view of noticing with the professional noticing framework (Jacobs, Lamb, & Philipp, 2010) and focused video analysis (FVA) (Teuscher, Leatham, & Peterson, 2017); and our view of SMT with the *MOST* analytical framework (Leatham, Peterson, Stockero, & Van Zoest, 2015).

The professional noticing framework (Jacobs et al., 2010) is "a set of three interrelated skills: attending to children's strategies, interpreting children's understanding, and deciding how to respond on the bases of children's understanding" (p. 172). In our study, we focus on PSTs' attending to and interpreting of SMT. Jacobs et al. (2010) described *attending* to student's strategies as teachers' attention to the mathematical details of students' strategies and *interpreting* student's understanding as teachers' connection of student strategies with research on student understanding. For our study, we describe *attending* as PSTs' attention to the SMT in the video instances and *interpreting* as PSTs' inferred understanding of the attended SMT.

FVA is intentional and directed noticing (Teuscher et al., 2017) consisting of PSTs watching, identifying, and *analyzing* video instances using a specific coding framework that unpacks a targeted teaching practices (e.g. *Practice of Probing Student Thinking, MOST*). FVA differs from noticing in that PSTs do not respond only to what they notice in video clips, use guided questions to focus their attention, nor do they focus on broad teaching practices. Rather PSTs are given short video instances to analyze specific teaching practices, in this study instances of SMT.

We view SMT through the lens of the *MOST* Analytic Framework (Leatham et al., 2015). A **MOST** – "**M**athematically Significant Pedagogical **O**pportunity to Build on **S**tudent **T**hinking" (Leatham et al., 2015, p. 91) – is SMT that, if discussed with students in the class, could move SMT forward. For our study, we view the SMT identified by the PSTs as their written transcript of the student's words or actions in the context of the classroom discourse.

Methods

Participants consisted of seventeen PSTs, in a unique U.S. undergraduate program, who were enrolled in their capstone methods course prior to entering student teaching. The PSTs' take six mathematics education courses (17 credits) focused on connecting pedagogy and content (e.g., student thinking, assessment, task design, algebra, geometry). Upon entering their capstone course PSTs had completed four courses with various assignments analyzing written student work and discussing SMT in video clips as a class. At the beginning of the capstone course, the PSTs completed two FVA assignments about SMT within a week of each other. The FVA assignments were designed to gauge PSTs' ability to attend to and interpret SMT and make PSTs' images of SMT explicit for the instructor (first author).

FVA Assignments

FVA 1. PSTs analyzed seven pre-selected video instances (i.e., 10 - 20 secs) of SMT from various mathematics lessons that the authors selected. Prior to watching the instances, PSTs were provided the context and location in the lesson; the lesson goals; and the mathematics tasks that students in the video engaged for each instance. After watching the video instance PSTs transcribed the student mathematics and decided if they could infer the SMT. If PSTs inferred the SMT, they provided a written interpretation of the SMT. If PSTs could not infer the SMT, they wrote a question to ask the student to clarify the SMT.

FVA 2. PSTs analyzed five extended video instances from FVA 1 that included the teacher's responses. PSTs were familiar with the lesson context from completing FVA 1. After watching the extended instances, PSTs answered three questions: (1) What evidence do you see that the teacher attended or did not attend to the SMT, (2) How did the teacher press or not press SMT, (3) How did your initial inferences of SMT change based on the conversation in the classroom?

Analysis of Data

All 17 PSTs' written responses to FVA 1 and 2 were collected and analyzed to gauge the PSTs' ability to attend to and interpret SMT and make PSTs' images of SMT explicit. FVA 1 responses were analyzed to determine if the PSTs attended to SMT (i.e., did they accurately transcribe what the student said). The PSTs' interpretation of the SMT in FVA 1 were categorized as (a) valid and complete, (b) valid and incomplete, or (c) incorrect. The PSTs initial interpretation of the SMT were analyzed to determine if, and why, their interpretation changed after watching the extended video instances in FVA 2.

Results

We report results from one of the seven video instances to provide a sense of PSTs' ability to attend to and interpret SMT and provide an example of PSTs' images of SMT at the beginning of their capstone methods course. First, we introduce the context of the video instance and then we share the results for PSTs attending to and interpreting the SMT.

Context of Video Instance

This instance occurred near the end of a middle-grades mathematics lesson about quantities (i.e., measurable attributes of an object (Thompson, 2011)). The teacher had two boards in the front of the class that served as racetracks. One board was longer than the other and the boards were placed at different inclines. The teacher placed a car on each board, let them go, and asked the students, "*Which car went faster and how do you know? Can you prove to me that one car went faster than the other car?*" The lesson goals were for students to (1) identify different quantities and (2) coordinate two quantities to solve a problem.

Attending and Interpreting Student Mathematical Thinking after FVA 1

All 17 PSTs attended to the SMT by accurately transcribing the student statement "*If you times for each board, you could see how long it takes to get down at the end over how long the board is*." Three PSTs provided valid and complete interpretations that the student recognized the need for two quantities (time and distance) to find the speed and dividing time by distance would help her find the car's speed, which differed from how they calculated speed. Six PSTs provided a valid and incomplete interpretation by only interpreting that the student realized she needed to coordinate two quantities (time and distance) to find the speed of the car. Seven PSTs provided an incorrect interpretation, stating that the student understood speed as distance over time. One PSTs wrote that he could not infer the SMT and included a question to ask the student. Overall, only three PSTs (18%) provided valid and complete interpretations of the SMT.

Changes to PSTs' Initial Interpretation of SMT in FVA 2

Of the eight PSTs who interpreted the SMT in FVA 1 either incorrectly or did not infer the SMT, six changed their interpretation in FVA 2. Table 1 provides two PSTs' initial interpretation and their justification for changing their interpretation, which are similar to the other PSTs' responses. Five of the PSTs' justifications stated that the teacher's actions of recording the SMT on the board helped them recognize their interpretation of SMT was incorrect. Four of the PSTs' justifications implied that they made assumptions that resulted in incorrect interpretations. Therefore, we see in PSTs' initial justifications that they were not decentering (Piaget, 1955; Teuscher, Moore, & Carlson, 2015) to interpret the SMT.

Table 1: PSTs' Explanations for Change in Interpretation

PST	Initial Interpretation	Justification for changed interpretation
SD	The student is trying to get to *an inches per second kind of solution*, where the higher the number will tell which car was faster.	I originally though that the student was doing distance over time and trying to find the speed that way. *Here with the teacher writing exactly what she is saying on the board, it helps to see that she was thinking of setting something up like that but did not have a full understanding of how speed is calculated.*
KB	*This student understands the relationship s=d/t.* But you can't tell if she understands that that gives us an average speed of each car, not necessarily the fastest speed.	*I initially did not recognize that this student actually divided the time by the distance.* But having the student explain what she meant and put it up on the board made me realize more what the student was actually thinking. *I assumed the student understood that speed can be found by dividing distance by time, but she was not necessarily making a connection to speed.*

Of the six PSTs with valid and incomplete interpretations of the SMT only two changed their interpretation after FVA 2. These two PSTs extended their interpretation by building on their initial FVA 1 interpretation. One PST explained that she had assumed the student stated distance over time, yet after FVA 2 recognized that the student stated time over distance. The other PST explained that the student knew she needed two quantities to find speed but was unsure how to relate distance and time. Interestingly, most PSTs with a valid and incomplete interpretation did not change their interpretation after completing FVA 2.

Implications

These results suggest that the types of activities that PSTs engage in during their programs can influence their images of SMT. Although the PSTs had completed multiple courses focused on SMT prior to entering their capstone course, most did not provide a valid and complete interpretation of the SMT. Yet, after FVA 2, 65% of PSTs provided a valid and complete interpretation of the SMT. We hypothesize that separating videos into shorter segments for PSTs to attend to the SMT may allow PSTs to (1) focus on the SMT and (2) make sense of the SMT prior to watching the teacher response to that SMT. We hypothesized that having PSTs first transcribed SMT would help them interpret the SMT, yet we found that they interpreted the SMT by filtering it through their own way of thinking, rather than decentering.

Having PSTs transcribe and interpret SMT in the FVA assignments required them to analyze the video, which revealed their own thinking. Then having the PSTs view the video instances with the teacher response caused a natural dissidence for many PSTs because they had already decided the SMT from FVA 1. The FVA 2 and the PSTs' FVA 1 analysis became the teaching tool to assist them in decentering as they interpreted SMT. While the results of our study are optimistic, our PSTs did not meet the AMTE standard to attend to SMT even though they had multiple courses explicitly focused on SMT. Therefore, further research on the types of activities that will enhance PSTs' opportunities to attend to and interpret SMT during teaching are needed.

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NOVICE ELEMENTARY TEACHERS' VISIONS OF THE TEACHER'S ROLE: A LONGITUDINAL COMPARISON OF TWO CASES

Temple A. Walkowiak
NC State University
tawalkow@ncsu.edu

Emily C. Elrod
NC State University
ekelrod@ncsu.edu

Frederique C. Yova
NC State University
fcyova@ncsu.edu

The purpose of this longitudinal study was two-fold: to compare the development of two novice elementary teachers' visions of the role of the teacher during mathematics instruction and to examine factors that might explain the variation in their vision trajectories. Data was collected for three years over the course of teacher preparation and the first year of teaching. The two participants were purposefully selected because their vision trajectories were similar early in their preparation program but then diverged during the latter part of the program and first year of teaching. Findings indicate differences in initial self-confidence in mathematics, mentor teacher relationships, and support in the form of curriculum and colleagues. Implications for teacher education and induction programs are discussed.

Keywords: Instructional vision; Affect, emotions, beliefs, and attitudes; Elementary education

Teachers' visions, as outlined by Hammerness (2001) are "a set of images of ideal classroom practice" (p. 143). When preservice teachers begin their teacher preparation programs, they certainly have a vision of classroom instruction, typically based upon their own experiences in school, also called the "apprenticeship of observation" (Lortie, 1975). Often, their experiences in school mathematics have not been aligned with the vision of the National Council of Teachers of Mathematics (NCTM, 2014) and the mathematics teacher educators who are preparing them. Therefore, we would expect their visions of their work as teachers of mathematics to evolve as they move through their teacher preparation program and into their teaching career. However, little is known about elementary teachers' development of instructional visions in mathematics during the preservice and induction years. Much of the past work has focused on experienced teachers in middle or high school settings (Hammerness, 2008; Munter, 2014) or in other disciplines such as science (e.g., Carrier, Whitehead, Minogue, & Corsi-Kimble, 2018).

Herein, we refer to preservice and first-year teachers as "novice teachers." The purpose of this study was to compare the development of two novice teachers' visions of the role of the teacher. Specifically, we examined two novice teachers over the course of three years, from the beginning of their teacher preparation program through their first year of teaching. The research questions guiding this study were: (1) For two novice elementary teachers, how did their visions of the role of the teacher in mathematics instruction develop as they progressed through the teacher preparation program and first year of teaching?; and (2) What factors might explain the differences between the two teachers in their visions trajectories?

Theoretical Perspectives

The current study draws on the framework proposed by Munter (2014) on a vision of high-quality mathematics instruction. Munter (2014) developed multiple rubrics for assessing vision, one of which focused on vision of the role of the teacher. Drawing on standards-based practices in the mathematics education literature, Munter's (2014) framework includes five progressively sophisticated levels: motivator, deliverer of knowledge, monitor, facilitator, and more knowledgeable other. As one moves up the framework, the descriptors are more aligned with a vision of standards-based teaching practices as advocated by NCTM (2014).

In their release of *Principles to Actions* (*P to A*), NCTM (2014) presents a “unified vision of what is needed to realize the potential of educating all students” (p. vii) in mathematics. In fact, much of this publication is devoted to the role of the teacher and the students in relation to a set of eight “Mathematics Teaching Practices.” The writers of *P to A* also suggest “unproductive beliefs” about the teaching and learning mathematics that may act as obstacles to achieving the vision outlined in the publication. By assessing novice teachers’ evolving visions as we undertake in the current study, we aim to surface, understand, and confront such obstacles that are part of teacher development early in the career.

Methods

Research Setting Context

This work is situated in the context of a larger research project designed to evaluate a STEM-focused elementary teacher preparation program. The larger investigation included tracking 245 novice teachers over time with more intensive data collection for 19 case study participants. During their junior and senior years, teachers enrolled in the program complete three full-time semesters of elementary education coursework accompanied with field experiences and one final semester of full-time student teaching in a classroom where they were placed the entire academic year (total hours for field experiences = 926 hours). The novice teachers take two mathematics methods courses (K-2 and 3-5) during the junior year. McIntyre and colleagues (2013) outline details about the teacher preparation program. In their methods coursework, there is an emphasis on the use of cognitively demanding tasks, mathematical representations, and discourse.

Participants

The two participants, Parker and Blake, were purposefully selected from the larger group of 19 case study participants (Walkowiak et al., 2015). Specifically, they were selected because their vision of the teacher’s role in mathematics instruction evolved differently over time. At the beginning and end of their first year of professional coursework (junior year), Parker and Blake had similar visions of the teacher’s role. However, at the end of their senior year and first year of teaching, their visions were different. Therefore, we purposefully selected Parker and Blake to investigate potential factors that might explain the variation in their visions trajectories. Table 1 displays the participants and the grade level of their classroom experiences. We utilize gender-neutral pseudonyms and female pronouns to protect the confidentiality of our participants.

Data Collection

Data was collected from Parker and Blake at four time points (five interviews), displayed in Table 1, about their vision of the role of the teacher in mathematics instruction (e.g., *What the teacher does and what the students are doing during mathematics instruction are really important. Describe what you think the teacher should be doing, and describe what you think the students should be doing most of the time.*)

We also collected interview data at each of the time points about the participants’ beliefs, contexts, and perceptions. We coded the questions that related to their confidence for teaching mathematics, contextual factors in their field placements and first-year-of-teaching schools, and their perceptions of the teacher preparation program (e.g., *Tell me your perception of how the math instruction at your school fits with the impressions you developed in your preparation program.*) Finally, at the conclusion of every semester in the program (n=4), Parker and Blake completed surveys about their field placement experiences.

Analysis

To code responses to the visions questions, we used one rubric (Role of the Teacher) from the Visions of High-Quality Mathematics Instruction (VHQMI) instrument (Munter, 2014). All responses were coded by two researchers who then discussed their codes. In cases of disagreement, discussion focused on reaching consensus. The rubric includes five levels (Levels 0-4) that correspond to how the participant described the role of the teacher. At the level of *motivator* (0), the primary role of the teacher is to entertain, to make learning fun, and/or to make emotional connections with students. The next level, teacher as *deliverer of knowledge* (1), indicates the role of the teacher is delivering the mathematical knowledge accurately and clearly, in a direct way. The teacher as *monitor* (2) is to provide demonstration on how to do a certain type of task but then give students time to work and talk together. The teacher monitors and redirects students if they are moving down an incorrect or nonproductive solution path. The teacher as *facilitator* (3) typically involves a teacher launching a task, students working in groups to figure it out, and the entire class discussing their strategies. The teacher asks probing questions to help them in their thinking, but the teacher does not directly tell the students what to do. Finally, when a teacher is described as a *more knowledgeable other* (4), the teacher and students share the mathematical authority, and the entire community of learners constructs the knowledge together.

For participants' responses to interview questions on beliefs, contexts, and perceptions and to field placement surveys, we utilized an open coding approach looking for themes across data sources for a participant.

Table 1: Participants, Data Collection Time Points, and Visions of Teacher's Role

Participant	TP1 Vision	Junior Year	TP2 Vision	Senior Year	TP3 Vision	1st Year of Teaching	TP4 Vision
Parker	1	Fall:1st Grade* Spring: 5th Grade*	2	3rd Grade*	3	4th Grade	3
Blake	1	Fall: Kindergarten* Spring: 5th Grade*	2	5th Grade*	1	2nd Grade	1

TP1: Introductory Interview
TP2: End-of-Junior-Year Interview
TP3: End-of-Senior-Year Interview
TP4: Math Lesson Interview & End-of-1st-Year Interview
*Field Placement Surveys every semester (n=4) during junior and senior years.

Findings

Vision Trajectories

Table 1 includes Parker's and Blake's visions of the teacher's role in mathematics instruction at each of the four time points using Munter's (2014) framework on a scale of 0-4. Both of the participants started their professional coursework at the beginning of junior year describing the teacher as the *deliverer of knowledge* (1), or as Blake stated, "the teacher should have a good example for [the students]." After the mathematics methods courses at the end of the junior year, they both described the teacher as a *monitor* (2). For example, Parker stated, "the teacher should be showing them why they're doing it….and making sure they really understand the concepts."

At the end of the senior year, Parker continued to progress in her vision of the teacher's role as she described the teacher's role as *facilitator* (3), and her vision remained at this level at the end of her first year of teaching. At that time point, Parker described the teacher's role to "guide [the students] by asking questions and pushing them further" while they "create their own ways to solve the problem." In contrast, Blake did not progress in her vision of the teacher's role at the end of the senior year and first year of teaching. Instead, her vision moved back one level to describing the teacher as the *deliverer of knowledge* (1) because "the teacher should always model at least once or twice what the concept is" (End of 1st year of teaching).

Potential Influencing Factors

Three factors emerged that could potentially influence variation in vision trajectories.

Mentor teacher relationships. Knowledge of the mentor teacher relationships during their student teaching is very limited; however, it is worth pointing to the complexity of these human relationships, particularly since both novice teachers emphasized the importance of the field experiences in their development. On her field placement survey, Parker complimented her mentor teacher for using "manipulatives and other hands-on activities" and for "keeping the students engaged" during math, but she also critiqued her mentor teacher's "negative attitude toward teaching." Blake provided only positive feedback about her mentor teacher at the end of her senior year, but during her first year of teaching, she stated that they "weren't the best fit." It is not clear how these complexities play out in their evolving visions, but their perceptions of the importance of field experiences suggest these relationships are likely pivotal.

Initial self-confidence in mathematics. One noticeable difference between Parker and Blake is their initial self-confidence in mathematics. In the initial interview, Parker says she is "high" in confidence in mathematics (and voices this confidence in subsequent interviews) while Blake indicates she knows she "needs to work on math" because she "was never the best math student." When considering their vision trajectories, perhaps Blake's background in mathematics that she perceived negatively impacts her progression in her vision (Valencia, Place, Martin, & Grossman, 2006).

Support in the form of curriculum and colleagues. Parker's and Blake's support during the first year of teaching seems to be an explanatory factor for Parker landing at *facilitator* and Blake at *deliverer of knowledge* at the end of the year. Parker mentions the "views and adopted curriculum" in her school are related to her confidence and interest in teaching math. The school uses *Investigations*, and the math coach's vision of instruction is aligned with standards-based instruction. She also talks repeatedly about the principal and math coach shared that she is obviously well prepared to teach math in comparison to other new teachers. In contrast, Blake points to her colleagues who have "done a lot of things one way for a long time" and many who follow the teacher-created district curriculum "closely." She verbalizes wanting "more professional development in science and math." It seems that Parker's level of support in relation to standards-based mathematics instruction is much stronger than Blake's; this difference could explain the variation in their visions at the end of their first year of teaching.

Discussion

Hammerness (2008) pointed to the importance of a teacher's instructional vision in the instructional practices actually utilized in the classroom. Therefore, allocating time and space in teacher preparation and induction programs for novice teachers to explicitly articulate and discuss their instructional visions seems important, particularly if we want to overcome or avoid some of the obstacles that may hinder progression of their visions. Furthermore, these perceived "obstacles" could actually become affordances (e.g., mentor teacher relationships) if we broaden our definition of teacher educator and include university-based *and* school-based teacher educators (i.e., mentor teachers, math coaches) in discussions about a novice teacher's evolving instructional vision toward one that is aligned with standards-based instruction (NCTM, 2014).

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PROSPECTIVE MIDDLE SCHOOL TEACHERS USE OF DOUBLE NUMBER LINES IN A CONTENT COURSE

Eric Siy
University of Georgia
ericsiy@uga.edu

In this report, I describe how using double number lines shifted over the course of a year in a content course for prospective middle school teachers. I analyzed classroom data to characterize the relevant features of the double number lines the teachers used to solve division and proportion problems. The prospective teachers used three different sized units in their double number lines. I identify how the teachers created these units.

Keywords: Teacher Knowledge, Rational Numbers, Using Representations

Introduction

Researchers have echoed the importance of representations in mathematical thinking (Cuoco, 2001; Janvier, 1987). With representations, students can make sense of problems and the information in the problem, document solution strategies and concepts, monitor oneself (Lobato, Hohensee, & Diamond, 2014), communicate one's thinking (Roth & McGinn, 1998), and provide access to mathematics particularly if a student's language is not privileged in class (de Araujo, Roberts, Willey, & Zahner, 2018). In this study, I investigate the use of double number lines (DNLs), a type of representation that explicitly appears in the sixth grade Common Core standards. DNLs are introduced to help students solve rate and ratio problems (i.e., 6.RP.3). The DNL is an inscription where two number lines are drawn where each number line is partitioned into units of a quantity and students can coordinate two amounts of quantities.

Research on DNLs is sparse (Küchemann, Hodgen, & Brown, 2011; Orrill & Brown, 2012). Based on the few studies on DNLs, researchers have found DNLs are useful in supporting students' reasoning about situations involving scaling (Küchemann et al., 2011) and teachers' reasoning about proportional situations (Orrill & Brown, 2012). Partitioning and creating composed units, critical actions in understanding the multiplicative concepts can be supported by using DNLs (Orrill & Brown, 2012). Research on teaching with DNLs is more rare than controlled laboratory settings. In this study, I contribute to this body of work by describing how DNLs were used in a class for prospective middle school teachers to solve different multiplication, division, and proportion problems. Specifically, I identify how prospective teacher-generated DNLs changed over time and the conditions which the changes occurred.

Theoretical Framework

Definition of Representations

In this study, I frame representations as culturally and historically rooted artifacts whose meanings grow out of how communities have interacted with an inscription over time. Saxe's account (2012) of cultural forms and functions describe how historically rooted artifacts change over historical time. Forms are socially rooted artifacts perceivable by the community such as the base-27 system of the Oksapmin peoples (Saxe, 2012) or a number line (Saxe, de Kirby, Le, Sitabkhan, & Kang, 2015). Functions are how forms are used by the community. Over time, the community may preserve historical forms but may also change them in order to serve new functions. For example, as students use a number line form in earlier grades, the form shifts to accommodate rational numbers while preserving whole numbers. I focus on DNLs as representations although others were found in the data (e.g., equations).

Definition of Multiplication and Proportional Relationships

The class analyzed in this study used a multiplier-multiplicand definition for multiplication notated by equation $N \cdot M = P$ (Beckmann & Izsák, 2015). In this equation, the value N denotes the number of bases units in one group (the multiplicand), the value M denotes the number of groups (the multiplier), and the value of P denotes the number of units in M groups.

Data and Analysis

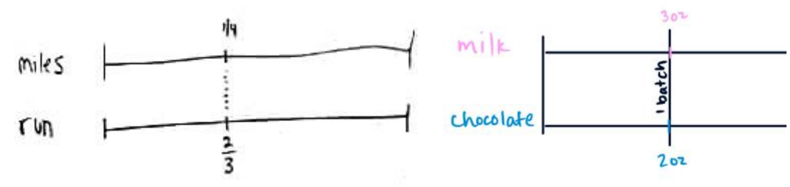
I analyzed fourteen lessons from a mathematics content course for prospective middle school teachers (PSMTs) focused on numbers and operations, including fraction operations. The 13 PSMTs in the course were predominantly white women. I also acted as a participant-observer and sat at one of the tables asking questions to support, extend, and connect PSMTs ideas. One stationary camera captured the whole class and another camera followed the teacher during small group work. Each table had an iPad where PSMTs can write notes and present their drawings. The primary analytical techniques were modified from Saxe et al., 2015 and focused on identifying forms and functions of the representations. To identify a form, I characterized the set of geometric inscriptions the PSMT used to explain their reasoning such as a tick mark or a line. I identified functions by using the PSMTs verbal explanations and annotations to explain what each form represented.

Results

In this section, I present the four forms PSMTs used to draw their DNLs. Each form corresponded to a different sized increment which relied on number choice in the word problems. The first form was used in order to establish relationships in the problem. Once the relationship was drawn, the PSMTs decided how to partition this correspondence in order to create a new correspondence to increment.

Correspondence Line

Each form began with a correspondence between two amounts of two different quantities. The PSMTs would draw a tick mark on both number lines and write down the corresponding amounts in the problem which PSMTs the product of a multiplication problem, the multiplicand, or a ratio. For example, consider Sophie’s initial DNL (Figure 1a) while explaining her strategy for Sue’s Run Problem, which reads “So far, Sue has run 1/4 of a mile but that is only 2/3 of her total running distance. Plot Sue’s total running distance and determine how many miles it is.” She drew a tick mark for one-fourth of a mile and two-thirds of a run and connected them with a line indicating correspondence between the two amounts showing the product or the amount of base units in M groups. Similarly, Molly began her DNL (Figure 1b) with a correspondence to begin her drawing for the Hot Cocoa Problem where PSMTs were told “A hot cocoa mixture has three ounces of milk and two ounces of chocolate. Draw a whole bunch of mixtures.” She connected two tick marks, one indicating three ounces of milk and the other indicating two ounces of chocolate, with each tick mark on a separate number line.



(a) **(b)**

Figure 16: Sophie's and Molly's Correspondence Lines

After the PSMTs drew the correspondence, they would either iterate this correspondence or partition the correspondence to create a new unit to iterate.

Composite Unit Increments

PSMTs incremented the correspondence they initially established during lessons on ratio and proportion similar to Lamon's (2007) building-up strategy. Consider the first lesson where the instructor posed the Hot Cocoa problem. After Molly drew her correspondence (Figure 1b), she then iterated this correspondence to show the mixture of required but also additional mixtures of the same flavor. She then showed amounts of one ounce on the corresponding number lines.

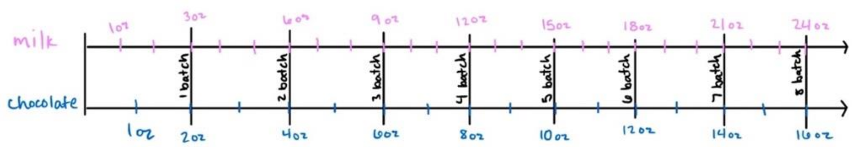


Figure 17: Molly's Drawing for the Hot Cocoa Problem

She also used this form while solving the Scooter Problem, a missing-value proportion problem, which reads, "A scooter is going 3/4 of mile every 4 minutes at a constant speed. How far does the scooter go in 12 minutes?" as seen in Figure 4. She initially drew the correspondence of three-fourths of a mile and four minutes. She then iterated the correspondence twice in order to get to 12 minutes. She explained she counted three-fourths three times in order to obtain two and one-fourth miles as the distance travelled in 12 minutes.

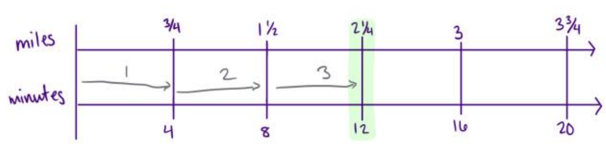


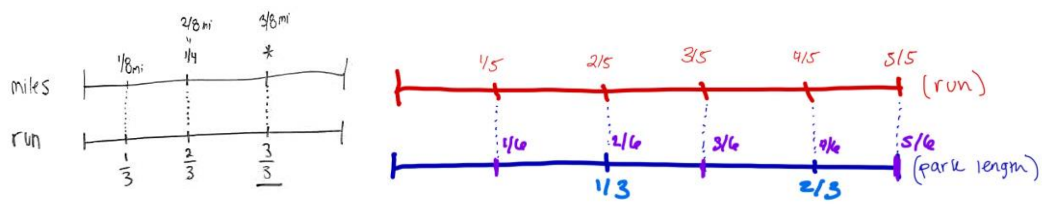
Figure 4: Molly's Drawing for the Scooter Problem

Lamon (2007) described this strategy as a building-up strategy where a PSMT creates a unit (i.e., unitizing) and built up to identify other correspondences. She noted this strategy as an entry point to proportion problems even for young PSMTs. In this study, I also note this strategy as an entry point to missing-value proportion problems where the amount given is divisible by the corresponding amount in the given ratio such as how twelve minutes is divisible by four.

Unit Fraction Increments

PSMTs created DNLs the initial correspondence is first partitioned to create a unit fraction amount which is subsequently iterated. Each unit fraction is paired with a corresponding amount in another quantity by partitioning the amount in the same number of partitions. This form emerged mostly during fraction division lessons and some ratio and proportion lessons.

To illustrate, Sophie used this form to complete her drawing for Sue's Run Problem early in the Fall. After drawing her unit correspondence Figure 16, she marked three-thirds on the run line and drew a line to an amount on the miles line, labelling it with a tick mark. She said she needed to find the amount at this tick mark and marked it with an asterisk. She partitioned the interval from zero to three-thirds in three parts. Knowing that one-third is half of two-thirds, she obtained the amount one-eighth as the corresponding amount on the miles line by getting half of two-eighths on the top number line. She finally iterated the resulting interval three times to obtain three-eighths as the amount represented by the asterisk (Figure 2a). She used a similar DNL in the Spring for a word problem, "If 2/5 of my total run is the distance of 1/3 of the length of the park, how far is my running distance in terms of the park length?" She began her DNL by drawing a Correspondence Line between two-fifths on the top and one-third on the bottom number line. She first identified that one-third was equivalent to two-sixths. She partitioned both the top number line and bottom number line further to show the correspondence between one-fifth of the run and one-sixth the park length. Finally, she iterated this correspondence five times in order to get five-fifths of a run and five-sixths the park length.



(a) (b)

Figure 18: Sophie's Drawing for Sue's Run Problem and 1/3 ÷ 2/5.

Unit Increments

PSMTs drew DNLs by partitioning the initial amount correspondence to create a new correspondence where one of quantities is one which is subsequently iterated. This form emerged during proportion lessons where PSMTs were solving missing-value proportion problems. In particular, the form emerged when the given value is not divisible by the corresponding amount in the ratio. Consider Andrew's work for the Hot Cocoa problem. Unlike Molly (Figure 3), Andrew wanted to draw more amounts and a general strategy (Figure 3). In his DNL, he chose to partition two ounces of chocolate in two and the corresponding amount of three ounces of milk to obtain three-halves ounces of milk. During small group, he explained partitioning something into two parts was "easier" than partitioning something into three. Once he obtained three-halves ounces of milk, he explained, "we can kind of build that up by one, every time an ounce of chocolate builds up, we get three-halves ounces of milk to go along with it each time." Notably, he incremented one ounce of chocolate to a generalized amount *n*. Correspondingly, he iterated the three-half ounces of milk to a generalized amount $\frac{n \cdot 3}{2}$.

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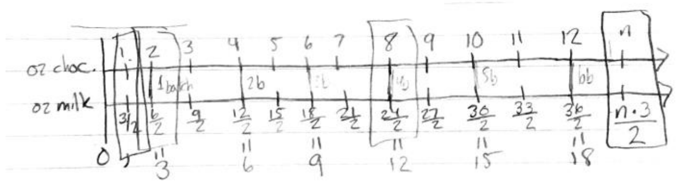

Figure 3: Andrew's Drawing for the Hot Cocoa Problem.

Discussion and Conclusions

In my analysis of the drawings, the forms for the double number lines provided me with a characterization of the different sized units to iterate. The PSMTs' forms followed a particular strategy. First, they established a relationship between two amounts of two quantities to begin the double number line. They used the product in partitive division problems and the given proportion in proportion problems to determine these relationships. They then identified a relationship they needed answer the problem. With a relationship, they selected the quantity and unit they wanted to use to increment. Orrill and Brown (2012) also found coordinating units and partitioning to be critical pieces of knowledge for teachers when engaging with double number lines. This report adds to Orrill and Brown's findings by identifying the conditions where the different size units emerge. Although there were three different-sized increments used by the PSMTs, a consistent strategy was to count up by "ones." In earlier grades, the number line is used to support the order of a counting sequence i.e., one, two, three. I argue that double number lines allow a coherent use of counting similar to earlier experiences especially when coordinating different amounts of quantities. The "one" in this case however where different size i.e., one batch, one of a quantity, one-*n*th of a quantity. This demonstrates the importance of working with number lines, particularly before middle school content and double number lines in order to imbue productive meanings for intervals on the number line.

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PLANNING FOR EQUITY: ANALYZING THE CULTURAL RELEVANCE OF PRE-SERVICE TEACHERS' LESSON PLANS

Nickolaus A. Ortiz
Georgia State University
nortiz1@gsu.edu

Trina Davis
Texas A&M
trinadavis@tamu.edu

This scholarship is based on research conducted by members of the KATE Project, an NSF funded grant that employed strategies in the context of a mathematics problem-solving course. We utilized a rubric designed by KATE researchers to analyze 29 PSTs' lesson plans for their alignment with the tenets of culturally relevant pedagogy (CRP). Our findings indicated that most PSTs' lesson plans were identified in a superficial category, and that the PSTs did particularly well with indicators within the tenet of academic excellence and critical consciousness. Additionally, four themes were developed from the rationale provided by the PSTs detailing why they felt their lesson plans were aligned with CRP tenets.

Keywords: Teacher Education-Preservice, Equity and Justice

The current scholarship extends research conducted by members of the KATE (Knowledge for Algebra Teaching for Equity) Project, a 5-year NSF funded grant that employed strategies during a mathematics problem solving course to enhance preservice teachers' (PSTs') knowledge for algebra teaching for equity (Kulm et al., 2016). Participants in this study were given the directive to design a mathematics lesson, including a problem-solving scenario, that utilized a culturally relevant teaching scheme and would later be taught within a 3D virtual environment known as SecondLife. Also an element of this directive, we asked PSTs to articulate why the lesson plan fit the teaching scheme. Our goal here was to take a different approach from previous scholarship which has described examples of culturally relevant pedagogy (CRP) (e.g., Ladson-Billings, 2006; Tate, 1995), or those that solely evaluate lesson plans for evidence of CRP (e.g., Lemons-Smith, 2013; Leonard, Napp, & Adelke, 2009), and highlight the rationale provided by PSTs regarding why they felt their lesson plans aligned with this teaching scheme, in ways similar to how scholars have discussed PSTs' justification for using social justice oriented mathematics lessons (Felton-Koestler, 2017; Myers, 2015). We sought to understand how PSTs honored students' cultures, even in cases where the culture that teachers addressed was not exclusively predicated upon students' racial or ethnic identities. This contention does not insinuate that race must be addressed to make lessons culturally relevant, but it does infer that CRP would be diminished if discussions of race are consistently overlooked for more superficial and terse interpretations of culture, or in favor of more neutral terms such as diversity. The following research question guided the study: How successful were PSTs in implementing culturally relevant tenants into mathematics lessons designed for a virtual environment?

Literature Review and Theoretical Framework

Though a thorough discussion of the literature is beyond the scope of the present treatment, we rely on the scholarship of Ladson-Billings to explore academic excellence, cultural competence, and sociopolitical consciousness, the three tenets of CRP (1994, 2006), and assert that there are both similarities and differences within scholarship related to CRP and critical race

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theory (Larnell, Bullock, & Jett, 2016; Tate, 1995). Scholarship related to CRP can be grouped into two major bodies of research, described here as manifestations. The first of these manifestations is related to a body of work focused on the teacher's delivery of the lesson. This scholarship describes how teachers are culturally relevant/responsive in the ways in which they engage their students and how they deliver instruction. For example, this was the case in Bonner and Adams (2012) and Johnson, Myamekye, Chazan and Rosenthal's (2013) descriptions of a culturally responsive teacher, one who expressed concern for their students and connected with them through forms of communication that were familiar and empowering, especially to Black children. The second manner in which CRP is manifested, and the focus of this current scholarship, suggests that culturally relevant tenets should be implicit in the lesson plans and mathematics activities. We offer, teachers must be intentional about using liberating and empowering curricula and tasks. This is an idea that was consistent in the work of Ares (2008) who insisted that the context of the problem and nature of the interaction that were facilitated in her classroom were both deliberate and culturally relevant for a classes of predominantly Black and Puerto Rican students familiar with traffic patterns in New York.

Nevertheless, the lesson plan informs delivery and delivery informs the lesson, thus, we argue, CRP tenets should be captured within the lesson plan, the very artifact that allows for a more nuanced and deliberate portrayal of ideas, activities, and goals. The contention of this study is that PSTs are more likely to honor the pedagogy when they put preconceived thoughts into these artifacts and how the concepts connect to students' lived experiences.

Methodology

Archival data used in this study were collected by the KATE Project members in the school year 2013-2014. We developed the Mathematics Rubric for Implementing Cultural Relevance (M-RICR) to analyze 29 lesson plans designed by the PSTs in this problem-solving course, and operationalized CRP in terms of 11 indicators across the three tenets, as seen in Figure 1.

Academic Excellence	**AE1: Teacher draws on issues or contexts that are meaningful to the student.** *Possible source of evidence: Rationale for context selected*
	AE2: Teacher indicates the purpose for students learning present content. *Possible source of evidence: Teacher introduction of the problem and the rationale for context*
	AE3: Teacher utilizes students' skills and/or acknowledges and builds on their initial knowledge. *Possible source of evidence: Devision of plan and looking back.*
	AE4: Teacher uses appropriate mathematical discourse. *Possible source of evidence: Understanding the probem and extension*
Cultural Competence	**CC1: Teacher draws on or uses cultural artifacts as learning tools.** *Possible source of evidence: Questions to probe*
	CC2: Teacher includes the role of family as a knowledgeable and capable source for support and learning. *Possible source of evidence: Questions to probe and extension*
	CC3: Teacher integrates or allows recognition of students' culture and embraces it. Possible source of evidence: Understanding the problem
Socio-Political Consciousness	**SC1: Teacher encourages the students to engage in the world critically to better understand their social position, as well as others.** *Possible source of evidence: Extension, rationale for context, looking back*
	SC2: Teacher highlights multiple mathematics perspectives or approaches. *Possible source of evidence: Looking back*
	SC3: Teacher encourages critical reflection and inquisitive or open-ended thinking. *Possible sources of evidence: Extension*
	SC4: Teacher engages students in mathematics problem solving, where they identify and investigate social problems, and plausible solutions. *Possible sources of evidence:*

Content validity for this instrument was established through review by three content experts in math and CRP, and revision among research members after a pilot study. For each lesson plan,

Figure 1: M-RICR Indicators

we recorded the presence of the 11 indicators by scoring each with a 1, 2, or 3, establishing what Thompson (2006) described as ordinal data; the scores denoted no, partial, or full implementation of the indicator respectively, with the biggest distinction between a score of 2 or 3 being the explicit attention PSTs dedicated to an indicator. The sum of scores across indicators produced the final score of CRP implementation, hereafter referred to as CR Score, and thus can be identified as intervally scaled data (Thompson, 2006). The CR score depicts a continuum that the PSTs are on and how they represent the tenets of CRP as a collective whole within their lesson plans. Borrowing from the work of Lemons-Smith (2013), we utilized categories which subsumed each of the 29 lesson plans based on this CR Score. Lesson plans with a score of 15 or less were categorized as absent level of cultural relevance. Similarly, superficial, moderate, and substantive levels of implementation were based on ranges 16-21, 22-27, and greater than or equal to 28 respectively. CR categories were based on lower (CR Score=11) and upper bounds (CR Score= 33) of the M-RICR. The cutoff score for the substantive level, for example, was 28 because PSTs could achieve this score by receiving a 3 on all but approximately half (i.e., 5) of the indicators, with the remaining indicators having a score of 2. The cutoff score for the other CR categories were similarly calculated. Descriptive statistics and ratios were utilized to see trends in CR category designation. Additionally, a content analysis was used to determine the justification that PSTs provided regarding why they felt their lessons were culturally relevant, and these responses were open coded and grouped into themes (Glaser & Strauss, 1978).

Findings

Most of the lesson plans (n=18) were placed in the superficial category and only one was identified as substantive. The range for the CR scores was 16 points with the lowest CR score being a 15 and the highest being a 29. Of the 29 total lesson plans analyzed, the average CR score was a 20.4, a score that is consistent with the superficial category. Cultural competence was found to be the least implemented among the three tenets of CRP, as evidenced by lower frequencies of full implementation on the M-RICR indicators illustrating this tenet.

Data from the content analysis led to the development of four themes (i.e., familiarity, age group, for all, and real-life) depicting the PSTs' justification of their lessons as culturally relevant. Perhaps the most prevalent theme (occurring in 23 of the PSTs' lesson plans) was based on familiarity. It was common for the PSTs to suggest that their students would be familiar with the context or the items that were referenced in the problem, and thus they would have no barriers to learning the concepts. Familiarity could be conveyed among school activities, food, national issues, nature, or even resources and activities within school. The second theme dealt with age group. Many of the PSTs said that their lessons were culturally relevant for a group of middle school students. Another theme that arose was that their lessons would be applicable to most, if not all, of their students. It was common for the PSTs to state that everyone in their class would have some knowledge or familiarity with the context that was used in the problem, and thus the lessons they created were culturally relevant for ALL of their students. Lastly, it was very common for the PSTs to say that the lesson was culturally relevant because it was a problem that was related to a real-life scenario, one that could or has occurred in students' lives.

Discussion

Our findings mirror those of Lemons-Smith (2013) in that most of the PSTs' lesson plans fell into the mid-range categories. Though the categories created by Lemons-Smith have qualitative descriptors by which group membership was determined, and our CR categories are numeric based cut-off scores, there are some overlaps in what we have described as substantive lesson plans. For Lemons-Smith, a substantive lesson plan used a variety of methods, such as using students' backgrounds, their families, communities, and both lived and out-of-school experiences, to draw upon their informal knowledge and to make connections. Gladys, the only PST in this study whose lesson plan was identified as substantive, demonstrated many of these characteristics in her lesson plan. The anchor, as described in her rationale, reflected the informal knowledge that students brought into the classroom about how to use the technology she referenced in the problem, as well as the reality that some of them may not be afforded the same access to these items or even the privilege of having allowance.

We leave the reader a few thoughts about these themes which will be further articulated in a forthcoming paper. First, familiarity as described by these PSTs, may be too large of an assumption, considering familiarity may differ greatly when we consider access. Regarding theme two, the very cursory way of thinking about students in middle school does not seem to have the ability to draw in or directly highlight the *needs* of middle grades students. Thus, it is easy to disregard more thorough understandings of culture, as well as the sociopolitical consciousness that might occur among middle grades students who frequent places such as a school cafeteria. We believe the notion of "for all" to be the exact opposite of what Ladson-Billings (2006) outlined in CRP. It appears that the PSTs were on a mission to make their lessons universal and to standardize the lesson, yet the impetus behind using CRP is not to dilute it beyond any personal identification that students have with the content, the idea is that students can see themselves, their questions, their interests, and their realities within the curriculum (Ortiz, Capraro, & Capraro, 2018). Lastly, by much of the rationales that were provided in this study, the greater portion of mathematics curricula would be perceived as culturally relevant, because one need not search far to find a mathematics context which occurs in real life.

Conclusion

We began this study by insinuating that ideas of diversity and equity tend to be much more neutral conceptualizations, and that those who are seeking to become more culturally relevant in their pedagogy are less often approaching it through a racialized lens (Milner, 2018). We also argued that though not necessary to implement CRP, if the PSTs in the current study were making no ties to culture from a racialized lens, we were interested in understanding which lens or conception of culture they were targeting in their lesson plan with the mathematics problems they developed. What is revealed in this work is that researchers and teacher educators may learn a great deal from acknowledging the ways in which PSTs make sense of CRP when given the opportunity to reflect on the way they have implemented it within at least one of two manifestations. Additionally, mathematics lesson plans can and should be developed with the intention to demonstrate attention towards each of the tenets associated with the scholarship, and this may allow for more substantive lesson plans to be developed. And lastly, contexts and lessons within mathematics that devote some attention to ideas of race and racism should not be avoided for the sake of circumventing difficult dialogue (Stinson, 2017); a racialized lens may help to better serve a more nuanced understanding of this pedagogy for PSTs, particularly related to cultural competence and sociopolitical consciousness, and may serve education in ways that discussions of diversity or equity are not adequately equipped to resolve (Martin, 2019).

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EXAMINING SECONDARY MATHEMATICS AND SPECIAL EDUCATION PRESERVICE TEACHERS' ENGAGEMENT IN MATHEMATICS CONSULTATIONS

Samuel L. Eskelson
University of Northern Iowa
samuel.eskelson@uni.edu

Elizabeth K. Hughes
University of Northern Iowa
elizabeth.hughes@uni.edu

We engaged secondary mathematics and special education preservice teachers (PSTs) in mathematics-specific consultations as a way to prepare to teach students with special education needs. After engaging in the consultation process, the PSTs created lesson plans and reflected on the process. We analyzed the PSTs' lesson plans and reflections for evidence they attended to issues connected to mathematical knowledge for teaching (MKT), special education knowledge for teaching (SEKT), and a synthesized blend of MKT and SEKT known as mathematics special education pedagogical content knowledge (M-SEPACK). We found that despite being asked to focus on their students' mathematical and special education needs, PSTs tended to focus on general pedagogical issues. We discuss lessons we learned as mathematics teacher educators and implications for those who wish to engage their PSTs in similar work.

Keywords: Instructional Activities and Practices, Teacher Knowledge, Equity and Diversity

Study Purpose and Research Questions

Lawmakers and writers of educational policy have pointed to the need for equitable learning opportunities for students with special education needs (SEN) for nearly three decades (e.g., Australian Disability Discrimination Act, 1992; Individuals with Disability Education Act (IDEA), 2004; UNESCO, 2009). As a result of this emphasis in educational policy, teacher preparation programs are tasked with preparing their pre-service teachers (PSTs) with knowledge and skills to meet the distinctive needs of *all* of their students. Though the call for such teacher preparation is not new, teaching special needs students has been identified by teachers in several countries as their greatest need for professional development (Schleicher, 2012).

Recently, researchers and mathematics teacher educators in four countries have begun a collaboration to jointly explore possible methods for preparing PSTs to better meet the needs of their students with SEN (van Ingen et al., 2019). Their work has focused on engaging general education elementary PSTs with special education PSTs in mathematics-specific consultations as a way to meet the needs of students with SEN. We build on and extend the work of these collaborators by engaging secondary mathematics and special education PSTs in the consultation process and reflect on the process of facilitating this collaborative experience as mathematics teacher educators (MTEs).

This paper will address the follow research question: To what extent do mathematics and special education PSTs attend to issues connected to mathematical knowledge for teaching (MKT) (Ball, Thames, & Phelps, 2008), special education knowledge for teaching (SEKT) (van Ingen et al., 2019), and a synthesized blend of MKT and SEKT known as mathematics special education pedagogical content knowledge (M-SEPACK) (van Ingen et al., 2019) during consultations? We also provide lessons learned as MTEs and implications for those who wish to engage their PSTs in similar work.

Theoretical Framework

The Case for Teacher Consultations

There are several forms of teacher collaborations, one of the most common of which is co-teaching (Mastropieri et al., 2005; Weiss, Pellegrino, Regan, & Mann, 2015). Co-teaching can take on different forms (Sileo, 2011), but a common form is when the general education teacher and the special education teacher simultaneously assume instructional roles in the classroom with the intention of supporting one another in providing inclusive education to students with SEN. While this form of teacher collaboration can be effective, many schools do not have the resources to allow for co-teaching for all students with SEN. Consultations between general education teachers and special education specialists can serve as an alternative to co-teaching to support teachers in providing learning opportunities for students with SEN, particularly when resources do not allow for instances of co-teaching.

Teacher consultations can take varying forms and can follow different models (e.g., Richards, Hunley, Weaver, & Landers, 2003; Truscott et al., 2012; Wesley & Buysse, 2004). While these models differ, they do share some common elements. We use a synthesized model of the common elements of teacher consultations (van Ingen, Eskelson, & Allsopp, 2016). This model consists of: (a) initiate rapport building, (b) negotiate consultation relationship, (c) identify the problem, (d) develop recommendations, (e) finalize recommendations and solidify plan, (f) implement the plan, (g) evaluate the plan, (h) learn from results, (i) check back, and (j) re-engage.

M-SEPACK Framework

To evaluate the focus of the PSTs' consultations and reflections on their resultant teaching, we needed to gauge the extent to which they focused on thinking specific to mathematical knowledge for teaching as well as to teaching students with SEN. To do this, we utilized the M-SEPACK Framework (van Ingen et al., 2019). This framework differentiates between MKT, SEKT, and M-SEPACK. as well as identifies multiple sub-categories within each of these types of knowledge.

Methods

Participants

The participants of this study were 34 secondary mathematics and special education PSTs enrolled in a mathematics methods course and a special education methods course, respectively, at a mid-sized regional comprehensive university in the Midwest United States. This was the second of two mathematics methods courses for the mathematics PSTs and the second of two special education methods courses for their special education counterparts.

Consultation Process

As part of their methods courses, the PSTs were also participating in field experiences in local schools. For this study, the PSTs were asked to select one or more students in their field placement classrooms who had been identified as students with SEN. They engaged in two rounds of consultations. During the first round, the mathematics PSTs selected mathematical content to be addressed in an upcoming lesson to be taught to their focus student. After identifying the target student and choosing mathematical content, the mathematics and special education PSTs met in pairs. The mathematics PST shared the information about the target student and the mathematical content to be taught. We also asked them to ask the special education PSTs questions regarding how they could best meet the mathematical learning needs of the focus student, taking into account their specific special education needs, when teaching the selected mathematical content. The special education PSTs then responded to these questions and shared recommendations and information of what the mathematics PSTs could do to best support their students in engaging with the mathematical content. Based on these recommendations, the mathematics PSTs then modified their lessons to incorporate their colleagues' suggestions. The second round of consultations worked similarly, but this time the special education PST selected a focus student with whom they would work and mathematical content they would be teaching to this student. They then considered mathematical questions to ask the mathematics PSTs regarding the lesson in order to utilize their specialized mathematical knowledge for teaching. The mathematics PSTs then provide suggestions for teaching the content of the upcoming lesson.

Data Collection and Analysis

We collected the mathematics PSTs' final lesson plans as well as their written reflections regarding the consultation process. We specifically asked the PSTs to address the following as part of their written reflections:

- What ideas, strategies, and/or content knowledge did I incorporate into my lesson as a result of the consultative session?
- What was the outcome of the lesson, specifically related to ideas, strategies, or content that you incorporated as a result of the consultative session?
- What would you have done differently?
- What are the next steps for instruction?
- What are the Big Ideas you are taking away from this collaborative experience?

We coded the data from the lesson plans and written reflections using the M-SEPACK framework noting evidence of MKT, SEKT, and/or M-SEPACK. This analysis provided us with insights regarding the focus of the PSTs' consultation meetings and the resulting impact these conversations had on the mathematics PSTs' lesson plans.

Results

Our preliminary analyses found that while the PSTs were asked to focus their consultations on the mathematical aspects of their work with the target student and their specific special education needs their consultations tended to focus on students and teaching moves more generally. The majority of PSTs' lesson plans and reflections were focused on teaching in a more general sense. For example, one mathematics PST clearly identified a specific mathematical learning goal. However, none of the provided recommendations or dialogue during the consultation were connected to that goal. Additionally, the teaching strategies that the PSTs discussed were often not specific to working with students with SEN but would be seen as effective techniques for working with all students whether or not they had been identified as having special education needs. Below are samples of evidence we coded for the various categories of knowledge.

- General Pedagogy: "making sure handouts are not overwhelming, guiding students with options, intentional grouping, and being flexible." (mathematics PST 4)
- MKT: "I made a chart on the board with the numbers 1 through 12, those numbers squared, and then those numbers cubed. I thought having a table with all of the numbers

was a good way for the whole class to see how squared numbers and cubed numbers are different." (mathematics PST 12)

- SEKT: "getting students up and moving around, especially if you have students who have ADHD. It allows them to expend some energy while still working on math activities." (mathematics PST 13)
- M-SEPACK: "my consultant advised me that it is important to clearly articulate what is being asked from what is on the board and that it is important to also provide a visual clue to go along with it as well. This looked for me like reading math word problems out loud, highlighting both vocally and physically key terms and words in the problem for students to draw equations from." (mathematics PST 10)

Discussion and Significance

While the focus of the consultations, lesson plans, and reflections was not always as targeted as we would have hoped, the PSTs, did, however, feel that the collaborative experience was beneficial for their learning as part of a teacher education program. One mathematics PST commented, "My special education group mates helped me see that by analyzing my students' needs first and then creating a lesson around them, I could reach my students much more efficiently." (mathematics PST 6). As mathematics teacher educators, we concur that the consultation process was beneficial. From this experience and study, we have found at least two key lessons for mathematics teacher educators to take into account when engaging PSTs in consultations.

It is important to have a framework in which the PSTs can ground their discussions during the consultations. The mathematics PSTs were quite familiar with the Mathematics Teaching Practices from *Principles to Actions* (NCTM, 2014). When consulting with and reviewing the mathematics lesson plan of their special education PST partner, they were asked to focus on two of the eight Mathematics Teaching Practices which they felt were most relevant to the lesson and provide the special education PST with suggestions for modifications to the lesson with respect to those two practices. For example, the math PST might focus on "implementing tasks that promote reasoning and problem solving" during the consultation by suggesting ways to modify the task so that it promotes reasoning and allows multiple entry points and varied solution strategies. The special education PSTs also had a framework with which they came to the consultation, the Curricular Adaptations Decision-Making Process (Udvari-Solner, 1996). While both sides came prepared to the consultation and a guide with focus questions was provided, we find additional specific guidance and support for structuring the consultation conversation would be beneficial.

It is also critical to ensure that all PSTs are able to engage in each component of the consultation process. Due to scheduling issues, some mathematics PSTs used previously developed and taught lesson plans to discuss with the special education PSTs. Their reflections were noticeably weaker than their peers and did not contain much evidence of SEKT or M-SEPACK.

Mathematics education and special education experts have called for more collaborative experiences during teacher preparation for both general and special education PSTs (Boyd & Bargerhuff's, 2009; Karp, 2013). This study is a response to these calls and provides valuable insight for MTEs and educational researchers as they consider how to design and facilitate these valuable experiences.

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TEACHER CANDIDATES' UNDERSTANDING OF EQUITABLE MATHEMATICS TEACHING

Kelly K. Ivy
University of Maryland
kwatson2@umd.edu

Angela Stoltz
University of Maryland
astoltz@umd.edu

Tarik Buli
University of Maryland
tbuli@umd.edu

Imani Goffney
University of Maryland
igoffney@umd.edu

Ebony Terrell Shockley
University of Maryland
eterrell@umd.edu

This exploratory study aims to unpack the equitable teaching practices that teacher candidates (n=55) attribute to their undergraduate elementary mathematics methods course at a four-year research-intensive institution, in the Mid-Atlantic region of the United States. We used a multiple-case approach which included the teacher candidates' end of course surveys and final project video data of mathematics teaching. We compared teacher candidates' understandings of and intentions for using equity focused practices to their actual teaching practices at the end of the course. Our findings demonstrate the need for ongoing support for enactment of equitable teaching practices which has implications for teacher preparation and teacher induction.

Keywords: Instructional activities and practices, Equity and diversity, Teacher knowledge

When preparing elementary mathematics teacher candidates to become high-quality mathematics educators, we challenge them to take a social justice stance that requires "fair and equitable teaching practices, high expectations for all students, access to rich, rigorous, and relevant mathematics, and strong family/community relationships to promote positive mathematics learning and achievement" (NCSM/TODOS Joint Statement, 2016). We do this by embedding equitable teaching practices that foster teacher candidates' development of research-based pedagogies to support culturally, socially, and linguistically diverse students in elementary mathematics classrooms, into our math methods course. Therefore, the following research questions guided this study:

1. What features of equitable mathematics teaching do teacher candidates attribute to having learned in their equity focused math methods course?
2. Do teacher candidates who earned a level "4" code for survey question 9 provide observable evidence of the practices described in their survey responses in their final project videos?

Related Literature and Framework

This study embraces sociocultural theories of learning, which consider the construction of knowledge occurring in and out of human interactions with one another and the world around them (Crotty, 1998). Therefore, we maintain that "what is learned cannot be separated from how it is learned" (Gresalfi & Cobb, 2006, p. 50). This work also incorporates a critical perspective aimed at disrupting the conditions many students from culturally and linguistically diverse backgrounds face in their mathematics courses. As such, we attribute achievement gaps to the "adverse conditions under which some children are often forced to learn, the privileged conditions afforded to others, and how forces like racism are used to position students in a racial hierarchy" (Martin, 2009, p. 300).

Methods

Our study investigates teacher candidates' interpretation of equitable mathematics teaching practices and how they planned to implement said practices in their mathematics teaching. The study took place at a large, predominantly White, research-intensive, 4-year university where pre-service teachers apprenticed in culturally and linguistically diverse teaching placements.

Equity Focused Elementary Mathematics Methods Course

This methods course, designed[1] by the third author, helps teacher candidates develop a set of practices for teaching mathematics that promote equitable access to the content, social justice, and positive mathematics identities. First, we incorporated Math Strong activities (Aguirre, 2018, with permission) which provided teacher candidates opportunities to confront negative stereotypes that dominate discourses around teaching multicultural students (Goffney, 2018). We used literature to introduce equity focused pedagogies that disrupt these negative stereotypes and corresponding status issues. Examples of status issues include: redefining "math smartness;" assigning competence; strategically positioning/repositioning students as math smart; creating purposeful groups that are heterogeneous, establishing and assigning student roles; incorporating group-worthy tasks; providing scaffolds for multiple entry points, and eliciting and interpreting student thinking (Teaching Works, 2018; Thames and Goffney, 2019) through the use of accountable talk and talk moves (Chapin, O'Connor & Anderson, 2003; Featherstone et al., 2011; Kazemi & Hintz, 2014). This literature helped them build a repertoire of equitable teaching practices while providing clear evidence of their necessity. Through pre-service teachers' enactment of these practices in their teaching placements, we provided a space for them to develop norms and structures aimed at disrupting status issues, such as heterogeneous grouping with assigned student roles, incorporating students' first language, and using complex instruction with multiple entry points (Featherstone et al., 2011; Kazemi, & Hintz, 2014; Moschkovich, 2007, and others).

Study Participants

Fifty-five teacher candidates enrolled in our elementary mathematics methods courses during the fall semester of their senior year of their initial certification program participated in this study. The majority of the participants were white females (*n=38*), and over half of them transferred from two-year or four-year institutions (*n=35*). More than 10% of them were age 24 or older (*n=8*), and they represented a variety of racial/ethnic backgrounds (two or more races, *n=1*; Black, *n=3;* Asian, *n=8;* Latinx, *n=14*).

Data Sources

To answer the research questions, we analyzed data collected from teacher candidates at the end of the course from two different data sources. The first source was a ten question, open-ended survey that was administered online via Qualtrics software (2016) from which we analyzed the responses to one specific question: *How has this course developed my practice to teach with an emphasis on equitable teaching practices that enable culturally and linguistically diverse students to develop knowledge and skills that promote mathematical proficiency?* The second data source was video data from recordings of teacher candidates' final project, where they led a whole class mathematics discussion.

Analytical Techniques

We employed qualitative research methods, starting with open coding of the survey responses to identify features of a high-quality response. These features included the use of specific language that referenced learned equitable teaching practices and referencing how these practices support mathematics proficiency among culturally and linguistically diverse students. We developed initial ratings of responses using values between 0 and 3 and conferred about data observations and individual ratings for each response, ensuring at least 80% inter-rater reliability. Data was ranked as "0," if there was no direct response to the question, as "1," if the response incorporated equitable teaching language but provided no specific examples of equitable teaching practices, as "2," if the response incorporated equitable teaching language with examples for equitable teaching practices but made no reference to the impact they have on student learning, and as "3," if the response incorporated equitable teaching language with examples for equitable teaching practices and discussed the impact they have on student learning. In the last phase of survey data coding, each member of the team re-coded all 3s as either a "3" or a "4" based on whether the responses were well developed, including detailed, specific, and observable practices. Again, we conferred to ensure at least 80% inter-rater reliability.

After identifying the most developed, detailed, and observable survey responses, we looked for alignment between survey responses ranked as "4" and the participants' actual teaching. We created unique video coding rubrics for each participant based upon the specific observable teaching practices they attributed to learning in the course. We calculated the percentage of alignment between survey responses and observed practices for each participant.

Findings

After analyzing the survey data, we found that of the 55 total participants, only 9 earned a level 4 rating. Four of the 9 had 100% alignment between their survey responses and actual teaching, and 8 of the 9 had over 50% alignment. However, one of the 9 had 0% alignment. It is important to note that the directions for the final project required participants to consider features of equitable mathematics instruction when designing and implementing their lessons but did not include assigning participants the task of aligning their teaching with their written survey responses. In Table 1 below, we present the findings from our analyses of participants. The table includes the exact wording used by each participant in their survey response and the percentage of alignment between their survey response and their teaching in the video.

Table 1: Percentage of Alignment between Written Responses and Video

Name	Individual Codes	Alignment
Kathy	Use multiple means of expression and delivery; manipulatives & hands-on experience	50%
Rose	Make classroom environment culturally diverse and accepting; use multiple means of expression to solve problems & manipulatives	0%

Karina	Teach math vocabulary through pictures and consistency; make math problems culturally relevant; student conversation and expressive learning through language is a math tool	100%
Mary	Create an environment of respect; equitable groupings; use talk moves and accountable talk; students answer questions and participate in discussions	100%
Angela	Use consistent math language; anchor documents and sentence starters; celebrate math smarts and attend to strengthening math identities; use peer/group work with assigned roles; include context that highlights a variety of cultures; build a classroom community	60%
Pavika	Differentiate instruction; use manipulatives; support positive identities	100%
Ellen	Use different instructional formats (whole group, pairs, small groups); multiple entry points in a task; assign roles in small groups	67%
Hope	Talk about different cultures in math class; connect students' culture and lived experiences to mathematical topics/ lessons; use multiple strategies for showing math smarts	67%
Adelina	Use strategies geared toward emergent bilingual students; shared mathematical experiences; informal dialogue (with movement into formal dialogue); mathematical representations and symbols	100%

Conclusion & Discussion

During the course, many teacher candidates produced high quality work, generating responses, reflections, and other written and oral productions that indicated substantial knowledge about issues of equity, features of equitable teaching practices, and the importance of equitable mathematics instruction in supporting culturally and linguistically diverse students' math proficiency. Nonetheless, only 9 of the 55 participants earned a level 4 rating. We are interested in examining these responses in greater detail to identify trends and patterns that might be useful for instructors of future sections of this course and school districts working with new and early career teachers.

The video data was most compelling. Of the 9 lessons we watched, 8 had over 50% alignment between the survey responses and the observed practices. Although 8 of the 9 participants were mostly aligned, the number of minutes and number of actual instances of equitable teaching varied considerably among these 8 lessons. Additional analyses will more closely approximate the amount of time spent enacting equitable mathematics teaching in each of these lessons to understand degrees of sophistication among these lessons better and further refine our ideas. Surprisingly, one of the teacher candidates who earned a level 4 rating on her survey response offered no evidence of any of those practices in their lesson. We hope to expand our knowledge on what factors might influence alignment by investigating in further detail the strongest case of alignment (Mary) and the case of misalignment (Rose). We are also interested in learning more about the degree to which our observations of their teaching align with their own observations of their teaching. In conclusion, our exploratory study on the alignment between our elementary mathematics methods teacher candidates' understanding of and intentions to incorporate equitable teaching practices and their actual practices demonstrate the need for ongoing support for enactment. These results have implications for teacher induction, given that our teacher candidates were in the senior year of their four-year certification program.

Endnotes

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SUPPORTING PROSPECTIVE ELEMENTARY TEACHERS' NON-CIRCULAR QUANTIFICATIONS OF ANGULARITY

Hamilton L. Hardison
Texas State University
HHardison@txstate.edu

Hwa Young Lee
Texas State University
hylee00@txstate.edu

In this report, we (a) describe a lesson comprising a series of research-based tasks we designed to foster critical ways of reasoning about angle measure and (b) report upon the impact of these tasks on prospective elementary teachers enrolled in a geometry content course.

Keywords: Instructional Activities and Practices, Teacher Knowledge, Geometry and Geometrical and Spatial Thinking, Angle Measure, Quantifying Angularity

Angle and angle measure are pervasive topics in mathematics curricula from elementary school through higher education. Elementary students' (e.g., Lehrer, Jenkins, & Osana, 1998) and prospective teachers' (Akkoc, 2008) challenges with measuring angularity have been documented. In contrast, few researchers have examined how to foster productive conceptions of angle measure. We view angularity as a quantity—an individual's mental conception of a measurable attribute of an object or situation (Thompson, 2011). Quantities are mental constructions that exist in the minds of individuals, and the process of conceiving a quantity is referred to as quantification. The few researchers who have examined fostering productive quantifications of angularity have advocated measuring angles by measuring circular arcs (e.g., Moore, 2013), which we refer to as circular quantifications of angularity. For example, to say an angle, A, has a measure of $n°$ means A intercepts an arc that is $n/360$ the circumference of any circle centered at A's vertex. Although productive, we do not view circular quantifications of angularity as an appropriate starting point for elementary students' angle measure instruction. In the CCSSM (2010), angle measure is explicitly addressed in Grade 4, and many elementary students cannot yet coordinate these kinds of multiplicative comparisons (Steffe, 2017), much less generalize these comparisons as holding across all possible circles centered at an angle's vertex. In this report, we (a) present a series of research-based tasks we designed to foster critical ways of reasoning about angle measure that do not necessarily rely on multiplicative comparisons of arc lengths and (b) report upon the impact of these tasks on prospective elementary teachers enrolled in a geometry content course.

Previous Research and Task Design

In our research at the secondary level, we found students can develop powerful quantifications of angularity without necessarily attending to multiplicative comparisons of circular arc lengths (Hardison, 2018); these non-circular quantifications of angularity leverage partitioning or iterating (Steffe & Olive, 2010) the interiors of angles. For example, when we asked Bertin, a ninth-grader, how to make a one-degree angle he explained, "if you get a ninety-degree angle [*gestures a right angle*], you can divide that into nine so it would be like ten degrees each and then you can divide each one of those into ten." Bertin's description for making one-degree angles did not reference circles or arcs; instead, Bertin's reasoning involved positing a right angle as a 90-unit angular composite and recursively partitioning (Steffe & Olive, 2010) the interior of this angular template to produce 90 one-degree angles within the right angle.

From working with Bertin and other ninth-graders in a teaching experiment (Steffe & Thompson, 2000) informed by the principles of quantitative reasoning (Thompson, 2011) and focused on understanding, as well as fostering, students' quantifications of angularity (Hardison, 2018), we abstracted several critical ways of reasoning about angle measure, many of which had not previously been emphasized in extant empirical literature. These critical ways of reasoning include (a) attending to the object and attribute being quantified (i.e., the openness of an angle), (b) imagining motion throughout the interior of an angle model (e.g., opening a pair of hinged chopsticks), (c) using enacted or imagined superimposition to compare the openness of angle models, (d) establishing re-presentable templates for familiar angles (e.g., bringing forth a right angle in visualized imagination), (e) positing these familiar templates as composite angular units (e.g., a right angle is a 90-degree angle), and (f) applying extensive quantitative operations to the interior of angles (e.g., partitioning or iterating) to make and measure other angles in various units, including standard units like degrees.

Based on the aforementioned findings, we designed an introductory lesson comprising tasks to engender these critical ways of reasoning about angularity for use in our geometry content courses for prospective elementary teachers (PSTs). We thought working to engender these quantifications of angularity in PSTs would be productive in two regards. First, we observed ninth-graders could productively quantify angularity in these ways; therefore, it seemed promising that the PSTs had already developed or could develop similar quantifications. Second, we viewed these quantifications of angularity as instructional goals PSTs could productively work toward fostering in their future students.

Three of the tasks from our introductory lesson on quantifying angularity are shown in Figure 1. The Chopsticks Task was designed to allow for PSTs to enact motion through the interior of angle models through opening or closing hinged chopsticks and to engage in superimposition for comparing angularity; in this task, PSTs' worked with partners, and partners had hinged chopsticks of different lengths. The Right-Angle Task (in GSP) was designed to help PSTs conceive of familiar angular templates as composite units and insert angular units into these templates. The Making and Measuring Task involved using transparency paper to engage PSTs in partitioning and iterating the interior of angles to make and measure angles.

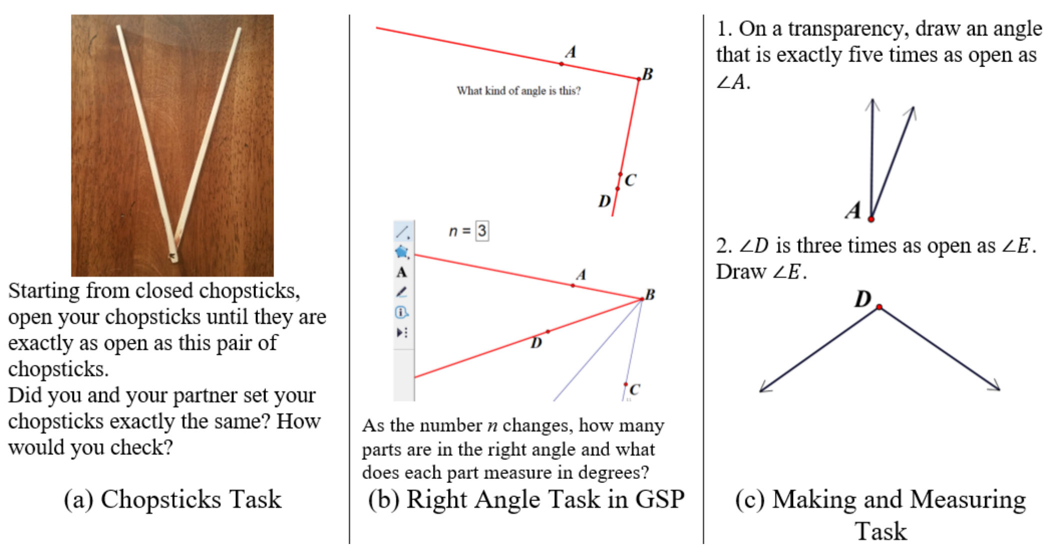


Figure 1: Three Tasks from the Lesson

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

Pilot Study and Methods

In Spring 2018, we piloted the lesson with 31 PSTs in one section of a geometry content course at a large public university in the southern United States. One month after the lesson when PSTs took their first written exam, which contained Prompt 2 shown in Table 1, we noticed many PSTs explicitly referred to at least one of the tasks as an activity they would use to help a student better understand angle measure. For example, some PSTs described opening hinged chopsticks or using superimposition with transparency paper to verify angular congruence. Having found the lesson impactful, we implemented the lesson in Fall 2018 in three sections of the geometry content course with a total of 51 PSTs. Information regarding PSTs' initial ways of reasoning about angle measure at the onset of the course was collected via a first day Introductory Questionnaire. We taught the lesson, which contained the tasks shown in Figure 1, during the 2nd and 3rd days of class. Then, we gave a first test around the 5th week of the semester and a comprehensive final exam at the end of semester. We collected data from these and other written assessments throughout the semester. In this report, we foreground PSTs responses to two prompts on the assessments summarized in Table 1. Our goal was to determine the extent to which the lesson impacted PSTs' conceptions of angle measure.

Table 1: Prompts and Written Assessment Timeline

Prompt	Introductory Questionnaire (1st day)	First Test (~5th week)	Final Exam (end of semester)
1. Without a protractor, describe how you would make a 1° angle.	X	X	X
2. [Given a picture of two congruent angles A and B with different side lengths] Erica says angle A is bigger than angle B. Make sense of her reasoning—why might Erica think this? How might you discuss angles with her? (Modified from Beckmann, 2017)		X	

Findings and Discussion

In this report, we first focus on PSTs' responses to Prompt 1 (describe how you would make a one-degree angle), which was given on each written assessment shown in Table 1. We classified PSTs responses to Prompt 1 into five categories, which are exemplified in Table 2. A response was coded as *Blank/IDK* if the PST either left the question blank or wrote something to the effect of "I don't know;" as *Unclear/Other Attribute* if the process described was unclear to us or if the PST attended to an attribute other than angularity (e.g., length); as *Comparative* if the PST provided a comparative description without inserting angular units; as *Inserting Units Undefinitively* if the PST inserted angular units into a familiar template (e.g., right angle) through partitioning but the partitioning did not produce a one-degree angle from our perspective; and *Inserting Units Definitively* if the PST inserted angular units into a familiar template and produced a one-degree angle from our perspective.

Results are summarized in Table 3. From the majority of Introductory Questionnaire responses (55%), we inferred productive elements of quantifying angularity (i.e., Comparative or Inserting Units categories); however, only 18% of the PSTs considered inserting units into a familiar angular template on the Introductory Questionnaire, and only 10% definitively inserted units to produce a one-degree angle. In contrast, after the lesson, the majority of PSTs (79% on the First Test and 93% on the Final Exam) described inserting units into familiar angular templates when describing how to make a one-degree angle. Moreover, 59% and 74% of PSTs

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

definitively inserted units to produce a one-degree angle on the First Test and Final Exam, respectively.

Table 2: Categories and Examples of PSTs' Responses for Making a 1° Angle

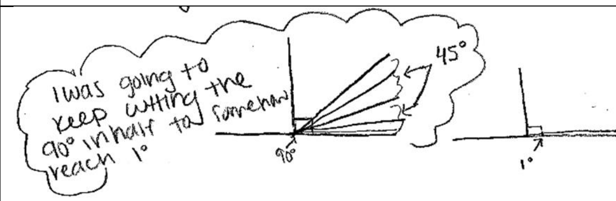
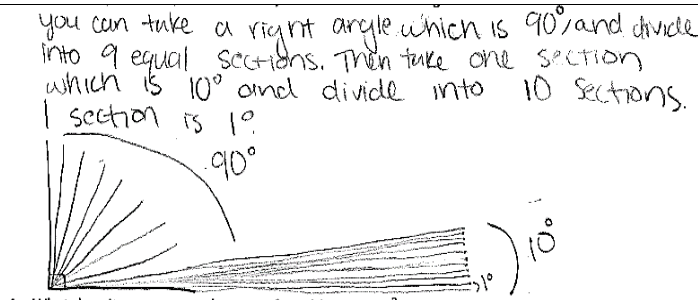
Response Category	Example
Blank/IDK	"I would not know how to do this accurately."
Unclear/Other Attribute	"You would have to try to make it how you would think it looks and then label it."
Comparative	"I would make one horizontal line and start another line at one point going BARELY above horizontal."
Inserting Units Undefinitively	
Inserting Units Definitively	

Table 3: PSTs' Responses for Making a 1° Angle Across Assessments

Response Category	Introductory Questionnaire	First Test	Final Exam
Blank/IDK	22%	4%	0%
Unclear/Other Attribute	22%	4%	6%
Comparative	37%	12%	0%
Inserting Units Undefinitively	8%	20%	19%
Inserting Units Definitively	10%	59%	74%

In their responses to Prompt 2 on the First Test (Table 1), over a third of the PSTs suggested enacting superimposition to verify angular congruence or suggested using motion throughout the interior of hinged chopsticks to help Erica understand angle measure. For example, one PST responded, "A way to discuss this problem with Erica would be to create a transparency of both angles and overlap the two to show they are equal and that the length of the rays have little to do with their degree measurement." Another PST responded, "…I would also demonstrate using different lengths of chopsticks, overlapping them to show their angle of rotation is the same." We took such responses as indications that the tasks may have impacted PSTs' future pedagogical actions in addition to influencing their quantifications. Looking across the assessments, we found the lesson had a productive impact on our PSTs' quantifications of angularity during the semester. In particular, these tasks supported PSTs' development of productive ways of quantifying angularity without foregrounding multiplicative comparisons of circular arc lengths. Further work is needed to investigate the permanence of the modifications

PSTs made in their ways of reasoning and what role, if any, these modifications play when PSTs enact future lessons on angle measure. In closing, we recommend educators (a) encourage non-circular quantifications of angularity by implementing tasks like those presented here and (b) celebrate individuals who develop non-circular, as well as circular, quantifications of angularity.

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QUANTITATIVE REASONING WITH TRANSFORMING FRACTIONS

Eun Jung
University of Georgia
eunjung@uga.edu

The question of preservice teachers' knowledge of fractions warrants further investigation, since having a firm grasp of the different interpretations of fractions is a challenge for both students and preservice teachers. This study seeks to investigate the understanding of ten preservice teachers' of quantitative reasoning with fractions in the context of measurement division by making use of task-based clinical interviews. The results showed that preservice teachers' construction of different levels-of-units structures plays a crucial role in their reasoning regarding measurement division. The study shows that only the participant who constructed a unit-of-units-of-units structure understood measurement division tasks involving fractional quantities conceptually as the transformation of one quantity to another quantity.

Keywords: Cognition, Teacher knowledge, Rational numbers, Teacher education-preservice

In the past few decades, a growing number of mathematics education researchers—particularly those who hold to a constructivist theoretical framework—have approached students' construction of fractional knowledge in terms of schemes and operations. Students require not only the construction of fractional schemes but also the flexibility of interpreting fractions differently depending on their mathematical contexts. Since then, substantial research has further emphasized that there are multiple interpretations of fractions (e.g., Kieren, 1976; Behr et al., 1993, 1997), and numerous books have been written reflecting this point of view on teaching rational numbers (e.g., Charles & Zbiek, 2010; Lamon, 2012). Major work has been done studying the subconstructs of rational numbers (see Behr et al., 1993; Behr et al., 1997; Kieren, 1976), and this research has led to the division of rational numbers into the subconstructs of part-whole, ratio, quotient, measurement, and operator.

Different researchers have spoken of the operator role of fractions from various angles and with differing nuances. A few studies have already addressed some of the many factors that may be considered in regard to understanding fractions as an operator. Among these, some have dealt with students' algebraic reasoning (Hackenberg, 2010, 2013; Lee & Hackenberg, 2014), finding that only those students who had constructed the third multiplicative concept could use fractions as multipliers. At least two empirical studies have dealt with teachers' limitations in using fractions as an operator (Behr et al., 1997; Izsák, 2008), addressing the subject in the context of fraction multiplication. Few researchers have investigated teachers' and preservice teachers' limitations in conceptually understanding fractions as an operator (e.g., Behr, 1997; Ball, 1990; Ma, 1999), and still fewer researchers have investigated the underlying cognitive factors behind these limitations (e.g. Izsak, 2008). The current study concerns itself primarily with the former question, although with an eye toward the underlying cognitive factors as well (to the extent that the data and its analysis may offer insight in this area). In my usage, constructing fractions as operators requires that a student can abstract the operation involved in using a fraction as a "multiplier," and can on at least a conceptual level anticipate the outcome, if not the precise output of the operation, without first having to perform the mental operation itself. A measurement division task in this paper involving the transformation of one fraction into another fraction was designed to investigate preservice teachers' use and understanding of fractions as operators.

Quantitative Reasoning with Fractions

The Recursive Partitioning Operation

A recursive partitioning situation involves a partition of each part from another partition. When a child is asked to produce, say, one-third of one-seventh from a unit bar, and to determine the size of the result in terms of the unit bar, he or she can engage in the task in such a way that partitions the unit bar into seven parts, pulls one out of the parts, partitions this part into three parts, pulls one out of these parts, and determines that the part he pulled out is one twenty-first of the unit bar because he understands that partitioning one of the seven parts into three parts implies partitioning each of the seven parts into three parts without implementing the partitioning.

The Parts-within-wholes Fraction Scheme

The operation that involved this fraction scheme is only a partitioning operation, and a disembedding operation is not available for a child who has constructed this fraction scheme, which means that he or she cannot mentally take out parts from the whole without destroying the whole.

The Iterative Fraction Scheme

Children who have constructed the iterative fraction scheme use an operation that enables them to produce three levels of units to understand improper fractions, such as eleven-ninths. As a result, children comprehend that eleven-ninths is a composite unit containing a fractional whole (nine-ninths) and two-ninths and that each ninth of the eleven-ninths is both a part of the fractional whole and independent of the whole as well.

Research Design

The present research is designed to address the following questions: How does a middle grades preservice teacher reason with measurement division quantitatively?

1. What can her reasoning with measurement division reveal about her construction of fractional knowledge?
2. What is the nature of her struggles with measurement division tasks?

The study uses a case study design, examining how situations involving fraction division are represented (e.g., with drawn models, double number lines, etc.) by a preservice middle grades mathematics teacher in a mathematics education department at a major public university located in the southeastern United States. The primary data source is a series of clinical interviews conducted over the course of two semesters using 10 volunteer participants from a mathematics methods course. More specifically, data was collected through hour-long, one-on-one interviews as well as classroom observation notes and homework submissions.

Data Excerpts

I will present three of the 10 preservice teachers who participated in the study: Cassie, Jordan, and Morgan. The tasks used for this study were making a 1/4-bar using a 1/3-bar, a 1/4-bar using a 3/5-bar, and 1/5-bar using a 7/4-bar.

Task. How could you use a one-third bar to make a one-fourth bar?
Make a bar, and leave it on the left side of the screen. Copy the bar, and leave it on the right side of the screen. Make one-third of the right bar. Make one-fourth of the right bar. What could you do to the left bar to make it resemble the right bar?

The given task was how to use 1/3 of a bar to make 1/4 of a bar**.** Students use the computer program Java-bars, which allows students to make a bar, partition it, pull out a segment, and iterate it. Though not every three student coordinated three levels of units, they were able to state the size of a bar in terms of another given bar – if A is a 3-part piece and B is a 5-part piece, they would know A is three-fifths of B, and B is five-thirds of A – and also know that the numerical result of seven divided by five is seven-fifths.

Morgan used the 1/12-part as a co-measurement unit of both bars, and made a 1/4-bar by pulling out one part of the 4/12-bar and repeating it twice.

M: Okay. If I wanted to use one-third of the bar, of this bar, to make one-fourth, I would have to do, they have to be, okay. So, I have one-third and one-fourth, I think of making them into twelfths and making both unit bars into twelfths. So what I would do here [3/3-bar] is I would partition it from left to right four times [partitions each part of the 3/3-bar and the 1/3-bar into four parts horizontally] and what I would do here [4/4-bar] is partition it left to right three times [partitions each part of the 4/4-bar and the 1/4-bar into three parts horizontally].

Figure 1: Partitioning Each 1/3 into Four Parts and Each 1/4 into Three Parts

M: So that I know that at the end even though they're shaped differently, because I use the copy part of it, I know that really I just have to have one-fourth of this bar [the left 12/12-bar] just because they're shaped differently. I need for here, I had, this is one-twelfth, two-twelfths, and three-twelfths [indicates each part of the first column of the right 12/12-bar]. So that is three-twelfths, and that is equivalent to one-fourth. So here I have one-twelfth, two-twelfths, three-twelfths, four-twelfths [indicates each part of the 4/12-bar] and that's equivalent to one-third. So then I would pull this out [pulls out the first part of the 4/12-bar], and I would repeat it [copies it twice, resulting in making a 3/12-bar], so now right here [changes the color of the 3/12-bar from red to blue] is three over twelve, which is equivalent to one-fourth. So, yeah.

Figure 2: Pulling Out Three Parts from the 4/12-bar to Make the 1/4-bar

In sum, Morgan created a unit bar, copied it, and then pulled out one-third and one-fourth from each bar by partitioning two unit bars vertically into three parts and four parts, respectively, and pulling out one part from the 3/3-bar (the unit bar) and from the 4/4-bar (the copy of the unit bar). When she was asked to make the 1/4-bar using the 1/3-bar, her immediate response was to construct "a similar unit," one-twelfth. She partitioned each part of the 3/3-bar and the 1/3-bar into four parts horizontally, resulting in making a 12/12-bar and a 4/12-bar, and then partitioned each part of the 4/4-bar and the 1/4-bar into three parts horizontally, resulting in making a 12/12-bar and a 3/12-bar. Morgan counted the number of 1/12 parts in the 4/12-bar (originally the 1/3-bar) and in the 3/12-bar (originally the 1/4-bar). She then pulled out one part from the 4/12-bar and repeated this twice to make the 1/4-bar.

Discussion

Cassie only constructed a part-in-fraction scheme, which means that she could make sense of fractional parts in the fractional whole, but could not pull out the parts from the fractional whole without mentally destroying the whole. Thus, she could not discern the relationships among the fractional wholes and the parts she pulled out, the 1/3-bar and the 1/4-bar, from the identical fractional wholes. She tried to directly compare the parts, the 1/3-bar and the 1/4-bar, without referencing the wholes by using trial-and-error. Even when she pulled out three-fourths of the 1/3-bar, Cassie did not anticipate that three-fourths of one-third of the unit bar is the same as one-fourths of the unit bar. Only after determining the size of the two fraction bars to be the same through visual comparison could she explain why they were the same by using her procedural calculation of simplifying three-fourths of one-third, rendering it as one-fourth.

Jordan understood that the co-partition of two fractions can transform one fraction into another fraction. Jordan had constructed the operation to produce a two-levels-of-units structure, meaning she still needed to engage in an activity to conduct a recursive partitioning operation to figure out the result of her partitioning, rather than anticipating the result prior to the task. After solving the first task in thirty minutes, Jordan was able to solve the next task, making a 1/4-bar using a 3/5-bar, by using the result from the previous task and conceptualizing a 1/20-bar as the co-partition of one-fourth and one-fifth and partitioning the 1/5-bar into four parts rather than twenty. However, Jordan had not constructed an iterative fraction scheme and could not implement the reasoning she used to solve the previous task to solve the problem involving an improper fraction, making a 1/5-bar using a 7/4-bar, even though the structure of this task is the same as that of the previous two tasks she had solved correctly.

Morgan partitioned the 1/3-bar and 1/4-bar horizontally into fourths and thirds, respectively, resulting in the construction of differently shaped 1/12-bars from those two bars. However, her construction of a recursive partitioning operation allowed her to be able to determine the size of both parts, one part being one-third of one-fourth and the other being one-fourth of one-third, with both being one-twelfth. When moving on to the third task, making a one-fifth bar using a seven-fourths bar, her construction of an iterative fractions scheme and an equipartitioning scheme of connected numbers allowed her not only to anticipate the number of parts needed to partition the one-fourth bar and the one-fifth bar into fifths and fourths to find the co-partition of two bars, but also to determine the fraction of the seven-fourth bar that the one-fifth bar is.

Conclusion

To carry out these measurement division tasks successfully, prior to a given activity, preservice teachers need to construct various fraction schemes involving the splitting operations for connected numbers and the recursive partitioning operation. To help preservice teachers like Cassie and Jordan, we need to provide appropriate tasks so that they can engage in fractional operations—a disembedding operation for Cassie, and a recursive partitioning operation and for Jordan.

These findings may guide further research that could shed additional light on the construction of fractional reasoning, both in preservice teachers and in primary school students. Considering the foundational importance of being able to understand fractions as operators and transforming one fraction into another, at minimum we need to ensure that preservice teachers have a strong grasp of this fundamental fractional concept in order to teach middle school students effectively.

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STEPPING UP INSTEAD OF STEPPING OUT TO TEACH ELEMENTARY MATHEMATICS DESPITE EXPERIENCES OF MATHEMATICS ANXIETY

Kathleen Jablon Stoehr
Santa Clara University
kstoehr@scu.edu

Keywords: Teacher Education-Preservice, Elementary School Education, Affect, Emotions

Mathematics educators agree that elementary teachers should possess confidence and competence in teaching mathematics (Beilock et al., 2010; Gresham, 2018). Yet many preservice teachers (PSTs) pursue careers in elementary teaching despite personal repeated experiences of anxiety when asked to learn or do mathematics or when asked to teach mathematics (Olson & Stoehr, 2019). This study analyzes how elementary PSTs who experienced mathematics anxiety (MA) during their K-12 years began to think about coping strategies to combat MA as they prepared for teaching mathematics in their future classrooms.

The participants were seventeen PSTs who were enrolled in a twenty-week elementary mathematics methods class. The PSTs selected for this study wrote a mathematics autobiography that spoke clearly about feelings of MA they experienced while learning mathematics. Other data sources included reflections written after course assignments, and a prompt that specifically asked PSTs to share how they would teach a mathematics lesson that created anxiety for them.

An iterative analysis (Bogdan & Biklen, 2006) of the data was performed followed by an emergent coding scheme (Marshall & Rossman, 2006) to demarcate the words and phrases that pertained specifically to MA in learning mathematics and teaching mathematics. Narratives within transcripts and text passages that included key words specific to MA, mathematics teaching anxiety, and possible coping strategies for dealing with mathematics teaching anxiety were identified. Analytic memos (Maxwell, 1996) were written to summarize key patterns.

Findings of the study revealed the types of coping strategies PSTs who have encountered MA while learning mathematics considered utilizing when confronted with teaching a content area that created anxiety for them. Examples included acting confident, reviewing the lesson standards, backwards mapping the lesson, asking other teachers, the mathematics coach or people outside of education for help, asking students to explain the content, being honest with students when feeling unsure, and knowing mistakes will happen and embrace them.

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MEASURING CHANGE IN PRESERVICE TEACHERS' MATHEMATICS MINDSETS

Alexa W.C. Lee-Hassan
University of Illinois at Chicago
aleeha2@uic.edu

Keywords: Preservice Teacher Education, Measurement, Affect, Emotion, Beliefs, and Attitudes

Introduction

Research indicates that a teacher's fixed, deficit mindsets about their own mathematical abilities (Beilock, Gunderson, Ramirez, & Levine, 2010) and their students' abilities (Horn, 2007; Wilhelm, Munter, & Jackson, 2017) can undermine their attempts to support conceptual understanding and student equity. A key challenge for fostering change in teacher mindsets is that current tools treat the concept categorically—individuals are sorted as having fixed, growth, or neutral/unclear mindsets, ignoring the potential for gradations within categories. Thus, shifts in mindsets are invisible unless they cross category boundaries. This study proposes that measuring mindsets on a continuum is a more productive perspective when researching mindset change. Pre-service teachers (PSTs) completed surveys at the beginning and end of one of their first education courses, and a unidimensional Rasch model was used to rank questions and participant responses along a continuum: strength of fixed mindset.

Methods

The data used in these analyses were collected from 57 undergraduates enrolled in two sections of a mathematics content course for PSTs. The PSTs answered eleven multiple-choice questions adapted from the established mindset literature (Rattan, Good, & Dweck, 2012; Yeager et al., 2016), and three related, open-ended questions. The multiple-choice questions were fitted to a longitudinal Rasch model. Open-ended survey responses were analyzed using frame analysis to supplement the quantitative analysis.

Preliminary Findings and Implications

The Rasch analysis showed that both the survey questions and the PST mindsets could be mapped onto continuums, but there was a mismatch between the ranges of the two distributions. Almost all of the PSTs disagreed or strongly disagreed with almost all of the questions, especially the questions drawn from Rattan, Good, and Dweck (2012) which asked about a hypothetical struggling student. The misalignment could be ascribed to a difference in participant population—the 2012 study focused on graduate student teaching assistants, who may have had different attitudes towards teaching than the current study's PSTs. This explanation is supported by Gutshall (2014) in which almost all of the PST participants had growth or neutral mindsets, especially when considering a hypothetical student. This finding could indicate that the PSTs in the current study all have growth mindsets. Some of the PSTs' qualitative responses, however, echo fixed beliefs expressed by teachers in Horn (2007) and Wilhelm, Munter, and Jackson (2017), which indicates that they may hold fixed beliefs not addressed by the multiple-choice questions. Future work will examine whether adding questions based on those beliefs will result in a survey that better measures the range of PST mindsets.

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PROSPECTIVE SECONDARY MATHEMATICS TEACHERS PROPOSED IDENTITIES AS ENACTED IN IMAGINED DISCUSSIONS

Eric Siy
University of Georgia
ericsiy@uga.edu

Jonathan Foster
University of Georgia
fosterjk@uga.edu

We describe the results of an analysis of how prospective secondary mathematics teachers imagine discussions play out and argue that imagined discussions are illustrate their identities. We found four roles they imaged a teacher may enact in discussion around tasks. This provides ways prospective teachers think about their identities as teachers prior to field components.

Keywords: Instructional Activities and Practices, Teacher Knowledge

Teacher education programs (TEPs) provide experiences where prospective teachers (PSTs) negotiate their identities or roles and enact them in interventions such as coursework and student teaching. In this report, we focus on an intervention where PSTs imagine a classroom discussion. Crespo, Oslund, and Parks (2011) argued using such interventions provides insight into PSTs' "developing mathematics teaching practice that would not have been possible to gain with other and more typical forms or representations" (p. 130). We present results from an analysis aimed at characterizing PSTs' identities as inferred through their practice in hypothetical situations. Although teacher educators may want to provide more "authentic" experiences (e.g., student teaching, field experiences), solely relying on or overplaying these experiences may "[interfere] with the opportunity to learn. Being situated in a classroom restricts attention to the sort of teaching underway in that particular class" (Ball & Cohen, 1999, p. 14). The data were collected from ten PSTs in a methods course in a TEP geared towards certifying mathematics teachers to teach grades 6-12. We collected two sets of assignments where the PSTs anticipate discussions based on prompts (Crespo et al., 2011). Based on their responses, we identified four roles as described below where there are two sub-roles within the second role.

1. **Teacher as the Source of Mathematics.** We found some PSTs placed the teacher's role as the thinker and doer of mathematics and the student's role is to question the teacher.
2. **Teacher as a Connector.** We found PSTs positioned the teacher's role as a presenter of data in order to make connections. We identified two sub-roles. First, the PSTs explained the teachers' actions of introducing new data was to "establish some common ground" or relate to or connect student's previous understandings. The teachers presented ideas they identified something they assume has been taught in the past. Second, some wrote about teachers presenting data in the form of an algebraic pattern meant to be generalized.
3. **Teacher as a Facilitator.** We found PSTs positioned the teacher's role as a facilitator of conversation between students where the main role was to elicit student thinking and select a focus to press student thinking.

Engaging PSTs in the activity of imagining how a conversation provides teacher educators to glean how PSTs view a teacher's role (c.f., Cooney, Shealy, & Arvold, 1998) We provided some ways PSTs positioned the teacher's role, and thus their identity, in discussions with students. Largely, the PSTs demonstrated a capacity to respond to student thinking and facilitate discussion. We note the PSTs' rationale for guiding the students are well-intentioned We note that although these discussions are not "authentic," we show that PSTs identities as mathematics teachers, for the most part, align with their images of "good" teaching and that the onus of expanding and reinforcing these images lie in the judicious selection of experiences mathematics teacher educators choose to provide.

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ON THE JOURNEY TOWARDS EQUITY-BASED TEACHING: PRESERVICE TEACHERS' PERSPECTIVES ON GOOD MATHEMATICS TEACHING

Naomi Jessup
Georgia State University
njessup@gsu.edu

Latisha Jones
Georgia State University
ljones130@student.gsu.edu

Keywords: Teacher Education-Preservice, Equity and Diversity, Teacher Beliefs

Equity-based mathematics instruction is supported by policy documents and national reform efforts that includes responsiveness to students' backgrounds, experiences, and cultural knowledge (NCTM, 2008; 2014). Also, the Standards for Preparing Teachers of Mathematics (Association of Mathematics Teacher Educators, 2017) calls for preservice teachers (PSTs) to question practices that produce inequities for students' learning experiences and outcomes. These efforts recognize the importance of attending to students' identities and power dynamics within mathematics classrooms (Nasir, 2016) that are at the core of "good" mathematics teaching.

Good mathematics teaching that is equity-based ensures diverse students have access to the tools and resources needed to engage in high quality mathematics and rigorous curriculum which in turn provides measurable improvements in student achievement, classroom participation, and participation in the math pipeline (Gutiérrez, 2009). Additionally, good mathematics teaching attends to the cultural and linguistic identities of all students and power dynamics that shape children's mathematical identities and learning opportunities (Gutiérrez, 2009).

Promoting equity-based teaching notions of good teaching requires intentionality in methods course experiences. PSTs are encouraged to develop classroom environments where children can participate equitably (Yeh, Ellis, & Hurtado, 2016), challenge the traditional narrative of what it means to be good at math (Crespo & Featherstone, 2012), and critically examine inequities that shape the expectations, interactions, and the kinds of mathematics students experience (Battey & Leyva, 2016; Martin, 2006; Nasir, 2016). However, in planning meaningful experiences that propel PSTs on a journey towards equity-based teaching, little is known about PSTs perspectives towards good mathematics teaching and what it means to be a good math student. This study sought to understand PSTs perspectives on good mathematics teaching that attended to notions of equity-based teaching.

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EXAMINING OUR QUESTIONS: DIFFERENCES IN RELATIONAL QUALITIES BETWEEN MATHEMATICAL AND PEDAGOGICAL QUESTIONING

Alyson E. Lischka
Middle Tennessee St. Univ
Alyson.Lischka@mtsu.edu

Signe E. Kastberg
Purdue University
skastber@purdue.edu

Susan L. Hillman
Saginaw Valley St. Univ
shillman@SVSU.edu

Keywords: Instructional Activities and Practices, Preservice Teacher Education

Questioning is a core practice in teaching that has been closely examined in the teaching of mathematics (e.g., Davis, 1997; Nicol, 1998; Parks, 2010), but is less closely examined in the teaching of teaching (Loughran, 2007). Because questioning in a mathematics methods class serves as an opportunity for prospective teachers (PTs) to learn from the instantiations of questioning practice (LaBoskey, 2007), examination of the questioning of mathematics teacher educators (MTEs) can lead to improving the practice of instructors and the experiences of PTs.

We work within self-study methodology, which is improvement aimed (LaBoskey, 2007) and characterized by openness, collaboration, and reframing of perspectives and ideas (Samaras & Freese, 2009). We explore our methods questioning practice through the lens of relational practice (Kitchen, 2005a, b) and examine ways our questioning contributes to relationships between MTEs and PTs. We take as a question any statement made by the instructor which generates a response from PTs (Dillon, 1981). We previously characterized our questioning according to the assumptions (Dillon, 1990) existing in our questions (Kastberg, Lischka, & Hillman, 2019). Building on that work, we explored differences between MTE's questions asked in teaching about learner's mathematical ideas and those asked in teaching about pedagogical ideas to explore: *What differences exist in our questioning as relational practice when the focus of the questioning situation is learners' mathematics versus teaching about teaching?*

Data gathered from this study include transcripts of audio recordings gathered during one semester of methods classes of each of the three authors along with recordings of our conversations in which we discussed our questioning in those classes. Analysis of our questioning across three different contexts, in which we coded our questions according to the Dillon (1990) framework and according to a relational teacher education framework (Kitchen, 2005a, b), resulted in findings of differences in MTEs' attention to and use of experiences of coming to know about learner's mathematics and mathematics pedagogy.

Two excerpts in Signe's data, drawn from the same lesson in October 2018, exemplify the differences we have found across our data. When questioning PTs about the mathematics in a video excerpt, Signe accepted and used PTs' ideas about learners' conceptions of mathematics (e.g. interpretations of the equal sign) to facilitate discussion of concepts. Signe did not reveal the complexity of her own process of coming to know about learners' mathematics. In contrast, within the same lesson, Signe shifted to a pedagogical discussion of giving written feedback in which she revealed her insecurities about and ongoing process of learning to give feedback. Signe further asked PTs for their advice about her feedback practice. Signe's mathematics focused questioning episodes were informed by her experience of becoming an MTE, but not revealing of that experience. In contrast, her experience of becoming an MTE was explicitly shared in pedagogical focused questioning episodes. Pedagogical episodes were longer and contained evidence of building relationships in the way Signe shared her ideas and the ways PTs responded to the discussion. We posit that MTEs' attending to and using their experiences of coming to know in mathematics and in pedagogy can mediate relationships with PTs.

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TEACHERS MAKING MANIPULATIVES TO PROMOTE PEDAGOGICAL CHANGE

Jessica Davidson
Montclair State University
davidsonj5@montclair.edu

Eileen Fernández, PhD
Montclair State University
fernandeze@montclair.edu

Steven Greenstein, PhD
Montclair State University
greensteins@montclair.edu

Keywords: Instructional Activities and Practices, Teacher Knowledge, Technology

Preservice elementary teachers have been characterized as coming to teacher preparation with a model of mathematics teaching that is not consistent with a pedagogy that supports learning mathematics with understanding. Consequently, they must be provided with opportunities that challenge this model. In this proposal, we present an investigation into a *Making-oriented experience* that tasks prospective mathematics teachers (PMTs) with designing, fabricating, and evaluating new manipulatives for use in teaching. We addressed the following question: *What forms of knowledge can be brought to bear on prospective elementary teachers' design activity as they Make new manipulatives to support the teaching and learning of mathematics?*

Constructivism and *constructionism* (Harel & Papert, 1991) frame this study and conjure an image of how learning works as one experiences it in the context of constructing a shareable artifact. In addition, we draw from the teacher knowledge literature (Ball, 1990; Borko & Livingston, 1989, 1990; Hill, Ball, & Schilling, 2008; Koehler & Mishra, 2009; Mishra & Koehler, 2006; Shulman, 1986) to characterize knowledge that might be brought to bear on PMTs' design work. Then, we take a *Learning by Design* approach (Koehler & Mishra, 2005) to advancing this knowledge by engaging PMTs with the task of designing manipulatives that are hypothesized to support learners' abstractions (Piaget, 1970; Pratt & Noss, 2010) of mathematical concepts from concrete tools.

The study took place in a specialized mathematics content course for prospective elementary teachers. Twenty-six students participated. The data consists of three written components, an "Idea Assignment," a "Project Rationale," and a "Final Research Paper." We took a case study approach (Yin, 2009) to analyzing this data, with purposeful sampling (Patton, 2002) to identify design cases whose reflections provide evidence of multiple layers of mathematical detail, knowledge of student thinking, or the use of technology during the project.

Harnessing the case study's virtue for evoking "images of the possible," (Shulman, 2004, p. 147), a poster session would provide an ideal venue for offering conference participants the opportunity to interact with the PMTs' 3D-printed manipulatives, and for sharing the narrative we crafted of Casey and Mia, who developed a tool called *Minute Minis*. Their narrative demonstrates how they drew on a breadth of knowledge domains to articulate the mathematical richness underlying their manipulative and the support it could provide a child to makes conceptual connections between a clock face and its underlying area properties as representations of fractions.

As researchers exploring how design experiences might catalyze new possibilities for pedagogical and curricular change, we positioned PMTs as knowledgeable designers of instruction in a space of technological possibilities. As PMTs assumed the multi-faceted role of teachers designing with technology, they created powerful and innovative tools, and their work demonstrated a rich and mature repertoire of knowledge domains that we are not typically afforded opportunities to see (AMTE, 2013). We propose that the identification and advancement of this knowledge, which is essential to effective mathematics teaching, suggest the promise of a Making experience within mathematics teacher preparation.

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CHANGES IN ELEMENTARY PRESERVICE TEACHERS' UNDERSTANDINGS OF MATHEMATICAL PRACTICES AFTER AN APPROXIMATION OF PRACTICE

Christina Koehne
Texas State University
crz7@txstate.edu

Keywords: Teacher Knowledge, Instructional Activities and Practices, Informal Education

Quality classroom experiences centered around professional practices are essential for teacher education (Ball & Forzani, 2009; Grossman et al., 2009). However, the complexity of the classroom must be considered when introducing preservice teachers to such an experience (Doyle, 1977)), because preservice teachers (PSTs) do not focus on salient events, classroom practices, or student understanding unless taught about the events and practices on which to focus (Star & Strickland, 2007; Sherin & Han, 2004; Sherin & van Es, 2005; Jacobs, Lamb, & Philipp, 2010). Therefore, in working in a two-week summer math camp (MathKidz) for elementary and middle school students and professional development (PD) that focuses on and supports certain teaching practices, PSTs receive a unique opportunity that prompts them to view and explore these teaching practices.

National policy documents such as *Standards for Preparing Teachers of Mathematics* (AMTE, 2017), *Principles to Action* (NCTM, 2014), *Principles and Standards for School Mathematics* (NCTM, 2000), and *Common Core State Standards for Mathematics* (CCSSM, 2010) have listed various key teaching practices that every teacher, including novices, should be familiar with before entering the classroom. Several of these practices align with the *Guiding Principles* that MathKidz curriculum values. Thus, this poster will focus on elementary PSTs' noticing of the practices of supporting students' justification, the use of multiple representations, student autonomy, allowing students to make and self-correct errors, use precise language, and develop perseverance in problem-solving after their involvement in MathKidz.

Three elementary preservice teachers (ePSTs), each with 1, 2, or 3 years of experience with MathKidz, participated in a research study regarding their views, implementations, and justifications for supporting these practices. After the first day of camp, before the professional development seminar, the ePSTs responded to a written survey about their views and values of each teaching practice. At the end of the study, the ePSTs participated in a clinical interview involving four video tasks centered around these practices. During the tasks, the ePSTs were asked to comment on what they noticed in the videos and how the teacher supported the students. My analytic framework grounded on a corpus of research regarding these practices was then applied to the ePSTs responses. Examples of this will be presented on the poster. In applying the framework, I found that some categories of the framework remained unmentioned by some of the ePSTs, which could potentially mean that the ePST was not considering certain connections or moves when operationalizing and defining the practice. In addition, the ePSTs reviewed their pre-surveys and commented on how their knowledge of the practices had been changed.

This poster will report on the initial understanding of ePSTs views regarding the practices mentioned here, as well as their views and understandings of the practices after implementing and reflecting on them during a two-week summer math camp. This poster will highlight and match the relationship of the ePSTs initial and changed understandings to my framework definitions and categorizations operationalized in this study.

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REHEARSALS AS A PLANNING TOOL IN PRESERVICE TEACHER PREPARATION

Robert J. Matyska, Jr.
Indiana University
matyskar@iu.edu

Keywords: Instructional Activities and Practices; Teacher Education-Preservice; Teacher Knowledge

It is a widely-accepted truth that quality teaching unlocks greater opportunity for all students to learn mathematics (Arbaugh, 2010). Professional knowledge and beliefs about teaching and learning are key components that determine a teacher's practice; fostering the development of this practice is a central component of teacher preparation (Ball & Forzani, 2009; McGraw, 2010; Stigler & Hiebert, 1999).

The heart of this work involves identifiable elements known as core practices that are connected to the aims of learning and consist of "strategies, routines, and moves that can be unpacked and learned by teachers" (Grossman, Kavanagh, & Pupik Dean, 2018, p. 4). This study targets one approximation of practice in preservice teacher education known as the rehearsal, a robust teacher education pedagogy for supporting novice teacher learning and practice (Grossman et al., 2009; Kelley-Petersen et al., 2018).

This study is guided by two research questions:

1. How do preservice teachers learn to launch lessons?
2. How are representations and approximations of practice useful in developing practice?

This poster will present findings from tasks stemming from a semester-long methods block taken by undergraduate students in an initial-licensure program. Comprised of three components—a course on middle-school mathematics methods, a supporting seminar course, and a field-based observation and teaching experience—novice teachers utilize the Lesson Planning Protocol (Smith, Steele, & Raith, 2017) and develop skills in establishing lesson goals, defining essential questions, and designing instructional activities targeting ambitious teaching (Lampert, et al., 2013). A typical lesson is designed in three phases: a launch, an exploration, and a summarizing discussion. Launches are targeted for rehearsal because they are seen as moves that can be unpacked and mastered by novice teachers, occur with high frequency in teaching, and have the potential to support both student and teacher learning (Grossman, Hammerness, & McDonald, 2009). Following the rehearsal, the preservice teachers enact their lesson as a component of their field experience in a local middle school classroom; after a debriefing session with the teacher educator/researcher, a written reflection by the preservice teacher completes the process.

Data sources include preservice teacher lesson plans, video recordings of the rehearsal session and ensuing discussion with peers, a second recording of the enactment in the field, the written reflection, and a one-on-one follow-up interview at the end of the semester to gain insight into the perceived usefulness of the rehearsal process in developing competency as a classroom teacher.

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REFLECTION AS A TOOL FOR ENHANCING SELF-EFFICACY IN ELEMENTARY MATHEMATICS PRESERVICE TEACHERS

Brooke Becker
University of Northern Iowa
bkbecker@uni.edu

Keywords: Teacher education-preservice; Instructional activities and practices; Affect, emotions, beliefs, and attitudes

Although student-centered mathematics instruction, enhances students' deep understanding of mathematics (NCTM, 2014), many teachers fail to implement this approach due to the unexpected challenges they face and inability to overcome them (Marbach-Ad & McGinnis, 2009). One reason it may be difficult for teachers to shift towards student-centered instruction may be related to their teaching self-efficacy. Higher self-efficacy may increase the likelihood that teachers will implement student-centered instruction and try new strategies (Temiz & Topcu, 2013; Zee & Koomen, 2016). Wyatt (2016) emphasizes the importance of reflection in the development of teacher self-efficacy in that encouraging reflection in teachers may indirectly increase their use of student-centered mathematics instruction by increasing their self-efficacy. The purpose of this study is to investigate the effects of an intervention targeting preservice teachers' (PSTs') reflection with the intention of enhancing their mathematics teaching self-efficacy.

Research Questions and Methods

This qualitative action research study is guided by the following research questions: What is the nature of PSTs' reflections on mathematical tasks as they as they engage in the use of prompted reflection focused on students' understanding of mathematical content? How does PSTs' mathematics teacher self-efficacy change after the implementation of prompted reflections focused on students' understanding of mathematical content?

The intervention took place during an eight-week mathematics methods course in a small, Midwestern university with seven elementary PSTs. PSTs' engagement in reflection was measured through written responses to video prompts focused on student understanding and the role the teacher plays in this development. Reflection responses to videos, open-ended responses to questions adapted from the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI), and interviews will be compiled separately and analyzed using open coding and analytic induction (Bogdan & Biklen, 1992; Merriam & Tisdell, 2015).

Results and Discussion

Similar to findings in other studies, preliminary results suggest there was a shift in PSTs' reflection to focus on student understanding (Santagata & Yeh, 2014; Sherin & Han, 2004; van Es et al., 2017; Wilkerson et al., 2018). Although a weaker association, a shift in mathematics teaching self-efficacy is observed, similar to findings in previous literature (Noormohammadi, 2014; Ross & Bruce, 2007; Yekyung Lee & Ertmer, 2006). This study promises to add literature to the mathematics education community as it can inform PST educators about the use of videos in mathematics methods courses. Specifically, the use of videos to develop PSTs' thinking through reflection prompts focused on student understanding and teacher interaction.

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PRE-SERVICE SECONDARY MATHEMATICS TEACHER CANDIDATES' REFLECTION ON THEIR OWN TEACHING

Emily Elrod
North Carolina State University
ekelrod@ncsu.edu

Heather West
North Carolina State University
hmwest@ncsu.edu

Keywords: Teacher education – preservice; Instructional activities and practices; Teacher knowledge; Affect, emotions, beliefs, and attitudes

Research has shown that video analysis has helped teachers develop stronger mathematical content knowledge, instructional practices, and a deeper understanding of students' thinking (Sztajn, Borko, & Smith, 2017; van Es, 2012). Providing teachers with the opportunities to analyze videos of their teaching has been identified as a powerful tool to activate growth and facilitate change in their own classroom (Seidel, Stürmer, Blomberg, Kobarg, & Schwindt, 2011). In addition, integrating a reflection component not only helps teachers critically examine and think about their own mathematical practice, but also encourages them to take action in order to strengthen their instruction (Schön, 1983). In recognizing the benefits of video analysis, building these skills during teacher preparation programs may provide the springboard needed for life-long, reflective teaching practice. Analysis of video-recorded classroom lessons can support teacher noticing by providing explicit opportunities to focus on student thinking and discourse more deeply than is likely to occur in the moment of teaching (Santagata, 2009).

The purpose of this pilot study was to investigate the following research question: How does reflecting on video samples of their own teaching factor into secondary mathematics preservice teacher candidates' anticipation of student teaching experiences? As such, this study was situated around the conceptual framework of noticing, which is characterized by the following three key components: the ability to determine what is noteworthy during a teaching episode, connecting those events to the underlying conceptions and principles of teaching and learning, and utilizing context and experience to interpret the situations (Sherin & van Es, 2005). Research shows that noticing is imperative for teacher growth and demonstration of expertise, thus it is a valuable component of teacher preparation programs (Sherin & van Es, 2005).

An intrinsic case study design was utilized, focusing on two secondary mathematics pre-service teacher candidates (PST) to investigate their teaching reflections and development of noticing through interviews. As part of a senior methods course, the PSTs completed two micro-teaching assignments that required them to plan, teach, and video record lessons in their field placement classroom, a public high school mathematics classroom, then complete a written reflection after reviewing their own video recorded lesson. Document analysis was completed on the reflections through the lens of the conceptual framework, utilizing *a priori* coding, along with open coding. Furthermore, following each micro-teaching, one-on-one semi-structured interviews, focused on the affordances of the video-recorded lessons for reflection and the impact on future pedagogical decisions, were completed with each PST. The interview transcripts were analyzed using the noticing framework. Three themes emerged from data analysis: focus on teacher actions, student-teacher relationships, and events that inform future teaching practice. The data support how video provided explicit evidence for the PSTs, offering opportunities for noticing of and reflection regarding student engagement, teacher facilitation, and pedagogical decisions that can support future teaching. Further, PSTs desire to continue to obtain video evidence from their classrooms to help support their reflective teaching practice.

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POST-LESSON DEBRIEF SESSIONS: WHAT PRESERVICE SECONDARY MATHEMATICS TEACHERS NOTICED

Raymond LaRochelle
University of Delaware
rlaroche@udel.edu

Michelle Cirillo
University of Delaware
mcirillo@udel.edu

Keyword: Teacher Education - Preservice

We share findings related to our efforts to implement a University Teaching Experience (UTE; Bieda, Visnawathan, McCrory, & Sikorskii, Accepted) for preservice secondary teachers (PSTs). In our implementation of the UTE, PSTs plan a lesson in pairs for a university pre-calculus course with guidance from the secondary methods instructor and the precalculus instructor. The pair then teaches the lesson while their peers observe and take notes about the student thinking that occurs as well as evidence related to two research questions identified by the PST pair. Afterwards, the teaching and observing PSTs debrief the lesson. Two cycles of the UTE are conducted so that each pair teaches twice. Herein, we discuss what the PSTs noticed during the UTE with respect to the issues of practice (i.e., research questions) that they selected.

Data for this study come from the first year of implementing the UTE model at a large mid-Atlantic public university. Fifteen PSTs participated in the study, which took place during their enrollment in the first of two methods courses. The UTE, which served as an on-campus field placement for the methods course, occurred in a university precalculus class.

To investigate what PSTs noticed with respect to the issues of practice, we analyzed video-recordings of the lesson debriefs using a professional vision framework (Sherin & van Es, 2009). Goodwin (1994) defined professional vision as "socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (p. 606). Following Sherin and van Es's (2009) method of analysis, we first divided the debrief transcripts into idea units (i.e., sections of the discussion in which one particular idea was discussed). We then coded each idea unit for *what* the PSTs noticed. This involved coding for *who* they talked about (teacher, students, or other), and the *topic* of the conversation (classroom management, climate, pedagogy, or mathematical thinking). Finally, we coded for *how* the PSTs noticed, in particular, whether PSTs were descriptive, evaluative, or interpretive.

Results indicated that the opportunity to reflect on research questions during the debrief gave PSTs many opportunities to attend to and interpret the precalculus students' mathematical thinking that occurred during the lesson. For example, here, a PST reflected on the research question, "What were students' first thoughts when attempting to solve [$0.01=1{,}000^{2x+1}$]?":

> One table I saw tried to rewrite 1,000 as 0.01 because… I think some people want to take the smallest number and make that the base, like, for instance, with 5 and 125 [points at previous problem, $5^{x-2}=125$] you take the 5, so like, so there was one table that I went to that was like 'I don't know how to write 1000 as a base of 0.01.'

In particular, we noticed how this PST made interpretive comments about the math students' mathematical thinking, and even discussed underlying reasons for the students' question.

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GRACE IN LEARNING: A NECESSARY CONDITION OF TEACHING MATHEMATICS METHODS AS AGAPE

Joel Amidon
University of Mississippi
jcamidon@olemiss.edu

Anne Marie Marshall
Lehman College
anne.marshall@lehman.cuny.edu

Rebecca Smith
University of North Alabama
rsmith27@una.edu

Keywords: Equity and Diversity

Teaching mathematics methods as agape, or teaching mathematics methods an as act of unconditional love (see Amidon, 2013), means acknowledging learning as happening in relationship with others (Lave & Wenger, 1991). This pedagogical ideal emerged from a study guided by the research question "What was learned in a elementary mathematics methods learning module designed to promote awareness and agency around issues of equity and diversity in the elementary mathematics classroom?"

In an attempt to identify what was learned in a module organized around data of disproportionate outcomes across racial/ethnic groups in the final course taken in high school (Battey, 2013) we identified and organized twenty-four learnings into four learning families. The learning families include technical learning, relational learning, critical learning, and learning bandwidth. Technical Learning describes instances where pre-service teachers (PST)s shared they had learned how to execute and/or create processes and products for student learning. Relational Learning described instances where PSTs shared what they learned about students and content to develop processes and products for student learning. Critical Learning referenced learning about equity in the teaching and learning of mathematics and the relationship of mathematics to the world. Finally, Learning Bandwidth, is defined as capacity related to the desire and sources of learning for the PST.

The culmination of this study, resulted in describing a kind of teaching that would foster this kind of learning, what we define as *teaching mathematics methods as agape*, with a necessary condition, and a product of such teaching, being *grace in learning*. Borrowed from Su's (2014) construct of *grace in teaching,* we define grace in learning as being undeserved favor in promoting an ideal relationship between PSTs and the equitable teaching of mathematics. Understanding learning to be the result of interactions between different degrees of knowers within a community of practice, *grace in learning* is both given and produced, by both PSTs and mathematics teacher educators in developing awareness and agency around issues of equity and diversity in the elementary mathematics classroom, or advancing toward being a more central participant within the community of practice (Lave & Wenger, 1991). In our poster we will illustrate both the necessity and production of *grace in learning* within the data from our study.

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PRESERVICE TEACHERS APPROXIMATIONS OF PRACTICE: PLANNING FOR AND PRACTICING WHOLE CLASS DISCUSSIONS

Lucy A. Watson
Belmont University
lucy.watson@belmont.edu

Jennifer N. Lovett
MTSU
jennifer.lovett@mtsu.edu

Allison W. McCulloch
UNC-Charlotte
allison.mcculloch@uncc.edu

Charity Cayton
East Carolina University
caytonc@ecu.edu

Lara K. Dick
Bucknell University
lara.dick@bucknell.edu

Keywords: Teaching and classroom practice; Preservice teacher education

The complexity of effectively orchestrating a whole class mathematical discussion is well documented as is the importance of developing this practice (Boerst, Sleep, Ball, & Bass, 2011). For a whole-class discussion to be mathematically meaningful, teachers need to purposefully select and sequence students' solutions to directly address the intended mathematical learning goals. (e.g., Smith & Stein, 2018). Learning to orchestrate such discussions is not trivial (Boerst et al., 2011). In describing their support of elementary PSTs learning to orchestrate discussions, Boerst et al. (2011) note the importance of learning through "doing practice" rather than just analyzing practice (p. 2849), but "doing practice" is difficult when methods courses do not include field experiences. To address this need, we designed a learning experience for an introductory methods course to support PSTs in developing skills needed to orchestrate effective whole class discussions.

To practice orchestrating a whole class discussion by breaking it into a string of "smaller and increasingly specified" pieces (Boerst et al., 2011), PSTs engaged in four sequential activities - 1) analyzing multiple samples of authentic student work, 2) selecting a specific sample of student work as their "favorite wrong answer" to be used to launch a whole class discussion, 3) scripting out a whole class discussion utilizing the student work they analyzed, and 4) creating an animation that captures the way they envision the whole class discussion would occur. Throughout this sequence of activities, PSTs' were expected to justify their choices based on a stated mathematical goal. The purpose of activities one and two was to set the stage by engaging PSTs in practices that would occur prior to launching a discussion, and given that by nature discussions are interactive, activities three and four provided progressively more detailed opportunities to *approximate* (Grossman, 2009) those interactions.

This study focused on PSTs' approximation of preparing for and orchestrating a whole class discussion for which the mathematical goal is to define function. The written responses, scripts, and animations of 21 PSTs were analyzed. Preliminary results showed all 21 PSTs focused on the learning goal in the planning of the discussion and made explicit connections to the learning goal throughout their approximations. Further analysis is being conducted using Boerst et al. (2011) Framework for Teacher Questions to examine the ways in which this series of approximations supports early PSTs in practicing orchestrating discussions. In addition, we plan to compare the findings between the script and the animation to better understand the ways the medium of the approximation of practice supports PSTs articulation of a whole class discussion. Findings from this study will be important for mathematics teacher educators designing effective tasks for developing this complex practice.

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DEVELOPING TASKS FOR INVESTIGATING PRE-SERVICE TEACHERS' UNDERSTANDING OF RANDOMNESS

Nenad Radakovic
Joshua Hertel
College of Charleston University of Wisconsin La Crosse
radakovicn@cofc.edu jhertel@uwlax.edu

Keywords: Instructional activities and practices, Teacher knowledge

In this presentation, we investigate the informal notions of randomness held by elementary preservice teachers (PSTs). Understanding randomness is essential for developing a robust understanding of probability and statistics (Moore, 1990). However, studies suggest that most individuals do not possess a deep understanding of randomness (e.g., Ayton, Hunt, Wright, 1989). Historically, there has been a wide debate among philosophers and mathematicians on the meaning of randomness. In the literature, there is general consensus that random phenomena are the ones that cannot be explored and explained through deterministic or causal means (Beltrami, 1999). In the research reported here, we sought to gather information on the informal understanding of randomness held by PSTs with a focus on two ideas. First, although individual outcomes cannot be predicted, the set of all possible outcomes, as well as the long-term distribution of outcomes, can be determined. Second, a sequence of outcomes generated by a random process often appears "clumpy" in that similar outcomes are grouped together rather than evenly spaced. Our investigation was guided by two questions: 1) What intuitive ideas do preservice teachers hold about randomness? 2) How can an instructional task be developed to engage students in thinking critically about their own understanding of randomness?

Methods

We presented 18 PSTs enrolled in an elementary content course a task with the following structure: 1) Imagine a scenario using a generating device (e.g., a fair coin, six-sided die) and predict what a sequence of outcomes will be. There were two contexts provided: a well known coin-flipping task and lesser-used task in which students were asked to color a hexagonal grid; 2) run through the scenario using the generating device and record results; 3) compare the imagined sequence versus the experimental sequence; 4) judge whether or not a sequence was imagined or generated using a device. We analyzed student responses to identify patterns and themes using open coding (Corbin & Strauss, 2008). Common words, ideas, or phrases were assigned codes and an iterative process was used to refine these codes. A secondary analysis was then conducted to identify themes across the data.

Findings and Discussion

Two broad themes emerged: (1) student intuitive ways of reasoning about randomness and (2) student perceptions of structure in randomness. Within the first theme, we found three subthemes: judging randomness based upon frequency conflating and confounding theoretical and experimental probability, focus on determinism. Within the second theme, we found that the presence of structure was used as both a reason for judging something to be random as well as against judging something to be random. Judging the presence of structure as that is not random is consistent with Batanero, Arteaga, Ruiz, and Roa's (2010) findings that PSTs view randomness as unpredictability and lack of patterns. Using the presence of structure and patterns as indication of randomness has not been fully explored in the literature. With this study, we hope to describe the challenges of overcoming non-normative notions of randomness, and the feasibility of using such tasks to support students in developing conceptions of randomness.

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THE UTE MODEL: DEVELOPING PRE-SERVICE TEACHERS' VISIONS OF HIGH-QUALITY MATHEMATICS INSTRUCTION

Kristen N. Bieda
Michigan State University
kbieda@msu.edu

Fran Arbaugh
The Pennsylvania State University
efa2@psu.edu

Michelle Cirillo
University of Delaware
mcirillo@udel.edu

Keywords: Secondary Teacher Preparation, Field Experience, Instructional Vision

The *University Teaching Experience* (UTE) model is a field experience where secondary mathematics PSTs teach in a first-year undergraduate mathematics course coinciding with their first methods course (Bieda et al., Accepted). Teacher educators mentor PSTs in planning and teaching in the UTE, supporting PSTs to enact ambitious teaching practices early in teacher preparation. We report results of our assessment of PSTs' vision for high-quality mathematics instruction (VHQMI) at the outset of the UTE, which inform mentoring of PSTs during the UTE.

Methods

We used the VHQMI protocol (Munter, 2014) in semi-structured interviews with participants during the first month of the semester in which they were participating in the UTE. The VHQMI probes PSTs' beliefs about the teacher's role, students' engagement, the nature of mathematical tasks, and the nature of classroom discourse in classrooms with high-quality mathematics instruction. We analyzed transcripts using Munter's (2014) rubric.

Results

In responding to the question "Can you describe what classroom discussion would be like if instruction was high quality?" all PSTs' responses scored in the top two levels of the rubric. Also, PSTs generally indicated high-quality mathematics instruction involves using tasks with opportunities for mathematical sense-making. When asked "What things should the teacher do for the instruction to be high quality?" PSTs indicated that teachers should be *knowledgeable others*, *facilitators*, or *monitors* (Munter, 2014) rather than imparting knowledge to students. However, when asked about what students' engagement would be like for high-quality mathematics instruction, their responses reflected norms of direct instruction. These findings suggest attention to students' engagement with tasks and evidence of students' learning may be particularly critical for PSTs as they reflect upon their practice in the UTE.

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EXAMINING CREATIVITY IN A STEAMING HOT CAMP: IMPACT ON STUDENTS AND PRE-SERVICE TEACHERS

Dittika Gupta
Midwestern State University
dittika.gupta@msutexas.edu

Margaret Anderson
Midwestern State University
maggyderbyshire@gmail.com

Aniya Harrelson
Midwestern State University
akh.akh327@gmail.com

Timothy Hinchman
Midwestern State University
Timothy.hinchman@msutexas.edu

Keywords: Creativity, STEAM, Problem-based

Research states that STEAM problem-based learning increases student creativity (Pang 2015; Russo, 2013; Siew, Chong & Lee, 2015) and critical thinking skills (Russo, 2013). Creativity is described as an individual's thoughts, behaviors, and products that can be considered novel, effective and whole (Amabile, 1999; Henriksen, Mishra & Mehta, 2015). STEAM has been defined as, "the collective expertise from many disciplines to pose and solve problems in a manner which foregrounds the problem, not the discipline" (Quigley & Herro, 2016, p. 412). STEAM learning thus provides opportunities for creative thoughts and processes through incorporation of multi-sensory tools and a hands-on problem solving approach (Taljaard, 2016).

Additionally, the dynamic relationship between materials and the teacher can have a significant impact on student knowledge and creative performance. Myers & Torrance (1961) further state that a teacher's responsiveness directly influences the number of creative products and novel ideas generated in the classroom. Pre-service teachers (PSTs), training to be teachers, need to be aware and knowledgeable in the skills of developing lessons that foster problem solving, reasoning, and creativity. Motivated by these arguments the researchers sought to examine the creativity ideation of elementary school students after attending a two week summer STEAM camp and that of the facilitating PSTs working as camp counsellors. The researcher questions that provided focus to the study are: to what extend does elementary school students' creativity change as a result of attending a two week-long STEAM camp incorporating problem-based learning and what is the impact of facilitating a two week-long STEAM camp incorporating problem-based learning on pre-service teachers' self-efficacy?

Research Design

The study uses two research designs. The quantitative research design is used to assess school students' creativity using the Runco Ideational Behavior Scale (RIBS) self-assessment. This 23 item survey is administered at the start and the end of the 2-week STEAM camp and statistical analysis using one-tailed t-test is conducted. A parallel convergent research design is used to gauge the impact on PSTs creativity and self-efficacy through quantitatively analyzing the results from the RIBS self-assessment, Mathematics Beliefs Scale (Usher, 2008) and a Science Teaching Efficacy Belief Instrument or S-TEBI (Enochs & Riggs, 1990), and open and axial coding of the daily reflections.

Results

The results of the study show positive impact of STEAM camp on school students through self-assessing of their creativity. The results also show positive impact on PST's efficacy and creativity through cross examining the results from different surveys and reflections.

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EXAMINING STUDENTS' SHAPE THINKING IN DIFFERENTIATING EXPONENTIAL AND QUADRATIC RELATIONSHIPS

Madhavi Vishnubhotla
Montclair State University
vishnubhotm1@montclair.edu

Despite the emphasis laid by national mathematical organizations (NCTM, 2000; CCSSM, 2010) on understanding exponential and quadratic relationships, there is limited research on students' meanings of these relationships. While a growing body of research has documented that middle school students can reason covariationally (Thompson, 2017) to construct meanings of quadratic (e.g., Ellis, 2011) and exponential relationships (e.g., Confrey & Smith, 1994, 1995; Ellis, Özgür, Kulow, Williams, & Amidon, 2015), there have been no reported findings on pre-service teachers'(PSTs) meanings of these growth patterns. In this study, I sought to gain insights into PSTs' meanings of these relationships and explore their ways of reasoning to differentiate between the two growth patterns when presented with graphs representing quadratic and exponential relationships.

I conducted one-on-one task based semi-structural clinical interviews (Goldin, 2000) with four undergraduate PSTs at a large university in the Northeastern United States who were chosen on a voluntary basis. Several tasks entailed half parabolas, exponential growth and decay graphs. I asked the students to describe the relationship represented by the graphs and justify their explanation. Each interview lasted for approximately 60 minutes. At the time of the interviews, the PSTs had completed at least two semesters of calculus courses. I analyzed the interviews with a goal of characterizing to what extent students conceptualized a graph as an emergent representation of two quantities' values varying simultaneously (Moore & Thompson, 2015) and reasoned about the amounts of change (Carlson et al., 2002) of one quantity for amounts of change in the other quantity.

I found that when attempting to describe the relationship or function the graphs represented, all PSTs engaged in static shape thinking. They tried to recall a shape from memory and conceived the graph as a picture rather than attending to the emergent trace and the covariational relationship. This finding is consistent with Moore and Thompson's (2015) description of static and emergent shape thinking. While two students employed other ways of reasoning, making a table of values from the graph, they inspected the physical properties of the shape of the curve to conclude that the graph could represent either a quadratic or exponential relationship. Students did not attend to the amounts of change in the two quantities to identify the graph as representing a quadratic or exponential relationship. Conclusively, all students' ways of reasoning relied completely on the shape of the curve and their reliance on shape thinking could not provide ways to differentiate between these two function classes.

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EXAMINING PRESERVICE TEACHERS' USE OF MULTICULTURAL LITERACY IN ELEMENTARY MATHEMATICS INSTRUCTION

Alesia Mickle Moldavan
Fordham University
amoldavan@fordham.edu

Keywords: Teacher Education-Preservice, Instructional Activities and Practices, Culturally Relevant Pedagogy

When mathematical concepts are related to cultural experiences, students can engage in meaningful learning that promotes interest and appreciation for diverse cultures. Multicultural literature can be used to teach students about mathematics in culturally relevant contexts (Chappell & Thompson, 2000; Leonard, Moore, & Brooks, 2014). Teachers engaged in this work must recognize mathematics as a cultural construct and have opportunities to design and implement lessons with such texts (Iliev & D'Angelo, 2014; Sleeter, 1997). This study reports on preservice teachers' experiences participating in a microteaching activity that used multicultural literature to communicate mathematical concepts addressed in an elementary classroom. The following research question is examined: How do preservice teachers use multicultural literature in elementary mathematics instruction to foster culturally relevant mathematical connections?

Participants in the study included 30 preservice teachers enrolled in an elementary mathematics methods course at an urban university in the northeastern United States. In the beginning of the course, the preservice teachers were introduced to culturally responsive pedagogy and the notion of contextualizing mathematics teaching and learning to students' lives (Gay, 2002, 2009; Ladson-Billings, 1995a, 1995b). Next, the preservice teachers were grouped in pairs and tasked with designing a mathematics activity with reference to a multicultural text of their choosing. To guide the text selection and the design of the activity, the preservice teachers were asked to reflect on the ways in which the text and the activity: (a) portrayed cultural authenticity, (b) depicted cultural diversity as an asset, and (c) promoted culturally relevant mathematical connections (Harding, Hbaci, Loyd, & Hamilton, 2017). Following the submission of their written reflections, the preservice teachers taught their lessons to their peers. The peers gave constructive feedback to one another and assessed how the overall experience influenced their future use of multicultural literature in elementary mathematics instruction. The collected reflections and peer feedback were analyzed using in vivo and descriptive code (Saldaña, 2016).

Findings revealed that the preservice teachers used multicultural literature to design activities that empowered students to see themselves and their peers as mathematicians. The practice of using such texts to facilitate cultural connections also influenced preservice teachers' awareness of and confidence in using cultural practices to relate mathematical concepts. Several preservice teachers shared how prior to the microteaching activity they had not thought to use such texts to teach mathematical concepts, especially if the concepts depicted in the texts (e.g., numerical representations, computations) differed from the dominant mathematics studied. One preservice teacher described how her experience gave her the confidence to challenge other texts used in her field placement. Another preservice teacher disclosed how he planned to use multicultural literature in his mathematics classroom to help students identify cultural assets and remove potential barriers. Recommendations are shared for equipping preservice teachers with the knowledge and skills needed to select and use multicultural literature to recognize mathematics as a cultural construct practiced beyond what we see in the near horizon.

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TEACHER PREPARATION PROGRAM OUTCOMES: A COMPARISON OF MATHEMATICS AND SCIENCE INSTRUCTIONAL PRACTICES

Temple A. Walkowiak	Heather M. West	James Minogue
NC State University	NC State University	NC State University
tawalkow@ncsu.edu	hmwest@ncsu.edu	jminogu@ncsu.edu

Keywords: Elementary school education, Instructional activities and practices

The National Council of Teachers of Mathematics (2014) has emphasized the importance of teaching for conceptual understanding through cognitively demanding tasks and rich discourse (2014). The Next Generation Science Standards (NGSS Lead States, 2013) call for teachers to utilize similar practices in science. As such, elementary teacher preparation programs aim to develop these practices among candidates, but little is known about whether instruction looks similar across disciplines. Typically, research on teacher candidates has focused on practices within a single discipline *during* the preparation program (e.g.,Spitzer et al., 2011), despite calls to conduct investigations after graduates enter their careers (Cochran-Smith & Zeichner, 2005). The current study fills these voids by examining instructional practices of second-year elementary teachers to understand whether practices in mathematics translate to science.

The participants in this study were purposefully selected from a larger group of 49 second-year elementary teachers who all graduated from the same teacher preparation program and provided videos of their mathematics and science instruction (three videos per discipline). The two participants, Dalton and Emerson, were selected because they taught the same grade level (fourth grade) and demonstrated above average facilitation of mathematical discourse (in relation to peers) as measured by an observational measure (Walkowiak et al., 2014). Videos were watched independently by three researchers who took detailed notes, wrote lesson memos, and met to discuss the discourse, tasks, and source of control of learning (teacher versus student).

Across disciplines, Dalton shared control with her students of their learning, whereas Emerson maintained a high level of control. Both teachers tended to use tasks of low cognitive demand in mathematics (Stein et al., 2009); however, Dalton's strong use of questioning heightened her instruction around these tasks. She probed students' thinking and encouraged them to make generalizations, thereby giving them some control over their own learning. Dalton's science tasks were inquiry-oriented, while Emerson's tasks focused on scientific facts and procedures. Across both disciplines, Emerson's instruction was often guided by videos (e.g., BrainPop). Emerson seemed to value student-to-student talk by giving students opportunities to talk to each other, but interestingly, assumed an extremely passive role during these interactions.

Findings from this study suggest some translation of practices across disciplines. Dalton and Emerson each demonstrated unique discourse practices that were encouraged in the program (e.g., probing, student-to-student talk). With the exception of Dalton's inquiry-oriented tasks in science, tasks tended to be lower level. A distinguishing characteristic in this study was the source of control of learning. Determining the factors that may influence control (e.g., teacher knowledge) is important for future research.

Acknowledgments

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Chapter 12:
Student Learning and Related Factors

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

"GETTING BETTER AT STICKING WITH IT": EXAMINING PERSEVERANCE IMPROVEMENT IN SECONDARY MATHEMATICS STUDENTS

Joseph DiNapoli
Montclair State University
dinapolij@montclair.edu

Supporting students' in-the-moment perseverance is a vital objective in mathematics education because it promotes learning with understanding. Yet, little is written about if and how student perseverance can improve over time. I examine the specific ways in which secondary students improved their perseverance as they engaged with challenging mathematical tasks over six weeks. The results show that encouraging students to initially attend to their conceptual thinking can prolong productive effort upon impasse and explicitly improve selection of problem-solving strategies and affect regulation. These findings suggest learning environment designs that provide consistent opportunities for students to practice (and improve) their perseverance.

Keywords: Perseverance, Problem Solving, High School Education, Metacognition

In the context of problem-solving, perseverance is initiating and sustaining in-the-moment productive struggle in the face of mathematical obstacles, setbacks, or discouragements. The notions of tolerating uncertainty and overcoming obstacles have long been recognized as key processes supportive of learning with understanding (Dewey, 1910; Festinger, 1957; Polya, 1971). These ideas have been echoed for mathematics learning because students make meaning through productive struggle, or as they grapple with mathematical ideas that are within reach, but not yet well formed (Kapur, 2010, 2011; Hiebert, 2013; Hiebert & Grouws, 2007; Warshauer, 2014). Additionally, reconciling times of significant uncertainty (i.e., a perceived impasse) is critical for mathematics learning. The processes of struggle to approach, reach, and make continued progress despite a perceived impasse puts forth cognitive demands upon the learner that are conducive for development of conceptual ideas (Collins, Brown, & Newman, 1988; VanLehn et al., 2003; Zaslavsky, 2005). As such, supporting in-the-moment student perseverance has been made explicit as a way of improving teaching and learning in mathematics education, with the expectation that such support will nurture students' perseverance to improve over time (CCSS, 2010; Kilpatrick, Swafford, & Findell, 2001; NCTM, 2014).

Supporting and Improving Student Perseverance with Mathematics Tasks

Several research efforts have sought to make explicit classroom practices that support student perseverance with challenging mathematics, yet little is known about how such practices can help improve student perseverance in specific ways over time. Studies aiming to unpack the nature of productive struggle (DiNapoli, 2019; DiNapoli & Marzocchi, 2017; Kapur, 2009, 2011; Sorto, McCabe, Warshauer, & Warshauer, 2009; Warshauer, 2014) generally have found that providing consistent opportunities for students to engage with unfamiliar mathematical tasks encouraged more variability in problem-solving strategies and greater learning gains, compared to providing consistent opportunities to engage with more procedural mathematics. Other researchers (Bass & Ball, 2015; DiNapoli, 2016, 2019; Kapur & Bielaczyc, 2012; Stein & Lane, 1996) have explored the nature of perseverance by investigating the effects of implementing classroom tasks with familiar entry points yet a complex structure. In general, there is empirical support for students leveraging opportunities within such low-floor/high-ceiling tasks to persevere in their efforts despite challenge and seemingly make mathematical progress. Additional scholarship has focused on the role of teacher feedback to encourage in-the-moment perseverance (Freeburn & Arbaugh, 2017; Housen, 2017; Kress, 2017; Sengupta-Irving & Agarwal, 2017). A synthesis of the findings from these works suggests non-leading teacher questioning encouraging student metacognition can facilitate more independent thinking and creative problem-solving during times of confusion.

Building from the previously mentioned literature, a recent study (DiNapoli, 2018) developed an operationalization of perseverance called the Three-Phase Perseverance Framework (3PP) (see Table 1), an analytical perspective by which perseverance can be qualitatively described and measured. The 3PP reflects perspectives of concept (Dolle, Gomez, Russell, & Bryk, 2013; Middleton, Tallman, Hatfield, & Davis, 2015), problem-solving actions (Pólya, 1971; Schoenfeld & Sloane, 2016; Silver, 2013;), self-regulation (Baumeister & Vohs, 2004; Carver & Scheier, 2001; Zimmerman & Schunk, 2011), and making and recognizing mathematical progress (Gresalfi & Barnes, 2015).

Using the 3PP, this study (DiNapoli, 2018) investigated how prompting algebra students to create an artifact of their personal conceptualization of a mathematical task (Anghileri, 2006) could support perseverance at times of impasse. The results show how scaffolding tasks in this way encouraged making an additional attempt at solving via re-initiating and re-sustaining mathematically productive effort at impasse significantly more so than on tasks without such scaffolding. Participants articulated that the conceptual thinking recorded after engaging with the scaffold prompt acted as an organizational toolbox from which to draw a fresh mathematical idea, or a new connection between ideas, to use to re-engage with the task upon impasse and to continue to productively struggle to make sense of the mathematical situation. Participants were persevering in problem-solving cyclically, with each additional attempt as a new opportunity to productively struggle with a given task scaffolded by their own conceptual ideas (see Figure 1). Without recording their conceptual thinking on non-scaffolded tasks, participants felt frustrated after a setback and often gave up without making an additional attempt at solving.

Table 1: Three-Phase Perseverance Framework

Entrance Phase	
Clarity	Objectives were understood
Initial Obstacle	Solution pathway not immediately apparent
Initial Attempt Phase	
Initial Effort	Engaged with task
Sustained Effort	Used problem-solving heuristics to explore task
Outcome of Effort	Made mathematical progress
Additional Attempt Phase (after perceived impasse)	
Initial Effort	Engaged with task
Sustained Effort	Used problem-solving heuristics to explore task
Outcome of Effort	Made mathematical progress

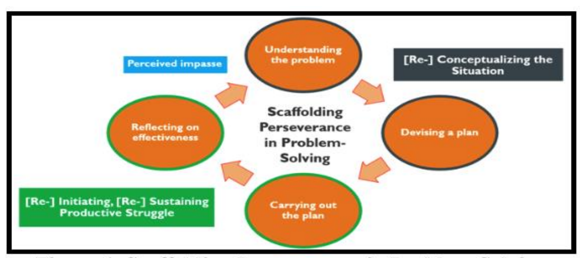

Figure 1. Scaffolding Perseverance in Problem-Solving

Despite the recent research on ways to support student perseverance in the moment, questions remain about whether these practices help nurture student perseverance to improve over time. There exists some work that shows evidence of student improvement in measures of grit (Polirstok, 2017) and time-on-task (Niemivirta & Tapola, 2007), however such work relies heavily on summative outcome variables that reveal little about the ways in which learners were challenged, overcame setbacks, and developed mathematical understanding, if they did at all (DiNapoli, 2018). Research on perseverance can produce insights into effective practices by which to learn mathematics with understanding, yet much of the empirical evidence of student

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perseverance have been situated in single points of time with little or no exploration of how those perseverance experiences may be related or demonstrate signs of specific improvement. Thus, the present study aimed to address these lingering concerns about whether and how student perseverance, when supported properly, can improve over time.

Methods

The participants for this qualitative study were 10 ninth-grade students from one suburban-area algebra class in a Mid-Atlantic state. They were purposely chosen to have demonstrated, via pretest, the prerequisite knowledge necessary to initially engage with each mathematical task included in the study. Each participant was observed engaging with five tasks across six weeks, approximately one per 7-10 days. These tasks were rated as analogous by the Mathematics Assessment Project because of their low-floor/high-ceiling structure, two objectives, and generalization requirements. Three tasks were randomly chosen to be scaffolded with conceptualization prompts (Anghileri, 2006), and two tasks were randomly chosen to be non-scaffolded. The conceptualization prompt embedded into the scaffolded tasks was "*Before you start, what mathematical ideas or steps do you think might be important for solving this problem? Write down your ideas in detail*." Each participant worked on these set of five tasks in a random task order. The results of this paper unpack participant perseverance across the three scaffolded tasks: Cross Totals, Sidewalk Stones, and Skeleton Tower. For context, Cross Totals asked students to generalize rules about how to arrange the integers 1-9 in a symmetric cross such that equal horizontal and vertical sums would be possible or not possible; Sidewalk Stones asked students to generalize rules about an evolving two-dimensional pattern of different types of stones; Skeleton Tower asked students to generalize rules about an evolving three-dimensional tower of cubes. All tasks are available for view at www.map.mathshell.org/.

For each task and participant, I conducted think-aloud interviews while they worked on a task and video-reflection interviews immediately after they finished working. Additionally, once a participant had engaged with all five tasks (and thus all five think-aloud interviews and video-reflection interviews), I conducted an exit interview to elicit reflections on the overall experience working on the five tasks. In all, I conducted 11 interviews with each participant, or 110 interviews in total for this study. I adopted an inductive coding process (Strauss & Corbin, 1990) using the 3PP to capture the ways in which students were persevering, or not, across the five tasks (DiNapoli, 2018, Table 1). The 3PP considered if the task at hand warranted perseverance for a participant (the Entrance Phase), the ways in which a participant initiated and sustained productive struggle (the Initial Attempt Phase), and the ways in which a participants re-initiated and re-sustained productive struggle, if they reached an impasse as a result of their initial attempt (the Additional Attempt Phase). A participant was determined to have reached a perceived impasse if they affirmed they were unsure how to continue (VenLehn et al., 2003). Mathematical productivity was determined based on if the participant perceived themselves as better understanding the mathematical situation as a result of their efforts (Gresalfi & Barnes, 2015).

I used a point-based analysis with the 3PP to help inform deeper investigation of the ways in which participants persevered. Each participants' experiences with each task were analyzed using the framework, and each component in the Initial Attempt and Additional Attempt Phases were coded as 1 or 0, as affirming evidence or otherwise, respectively. Since each task had two objectives and six components per objective, there were 12 framework components to consider, per participant, per task. Thus, 3PP scores ranged from 12 to 0, depicting optimal to minimal demonstrated perseverance in this context, respectively. I conducted regression analyses to compare the ways in which 3PP scores were changing over time for participant work on scaffolded tasks and on non-scaffolded tasks. I also inductively coded interviews to uncover from the participant perspective how and why their perseverance may have been changing over time. I enlisted help from two independent coders to analyze participant perseverance and their reasons for doing so. Our inter-rater reliability was 93%.

Results

Overall, participants' perseverance across mathematical tasks improved over time, more so on scaffolded tasks than on non-scaffolded tasks. Participants' demonstrated perseverance on the three scaffolded tasks improved in quality over time as evidenced by increasing mean 3PP scores (see Figure 2). Participants' demonstrated perseverance on the two non-scaffolded tasks also improved in quality over time. However, the average rate at which scores on non-scaffolded tasks improved was three times less than the rate of improvement of scores on scaffolded tasks.

A simple linear regression was calculated to predict participants' 3PP scores on their second scaffolded task based on their first scaffolded task (see Table 2). A significant regression equation was found ($F(1, 8) = 58.593$, $p < .001$), with an R^2 of .880, indicating participants' 3PP scores on their first scaffolded task explained 88% of the variance in their 3PP scores on their second scaffolded task. Also, participants' 3PP scores on their first scaffolded task was a significant predictor of their 3PP scores on their second scaffolded task, with a one-point increase on their first scaffolded task predicting a .727-point increase on their second scaffolded task. A multiple linear regression was calculated to predict participants' 3PP scores on their third scaffolded task based on their first and second scaffolded tasks (see Table 2). A significant regression equation was found ($F(2, 7) = 10.741$, $p = .007$), with an adjusted R^2 of .684, indicating a participants' 3PP scores on their first and second scaffolded tasks, together, conservatively explained 68.4% of the variance in their 3PP scores on their third scaffolded task.

Table 2: Summary of Regression Analyses of Perseverance Scores

Task Type	Comparison	R^2	Adjusted R^2	B	SE(B)	Sig. (p)
Scaffolded	1st → 2nd	.880	.865	.727	.095	< .001
	1st/2nd → 3rd	.754	.684	.385/.234	.348/.449	.007 (.305/.618)
Non-Scaffolded	1st → 2nd	.553	.497	.680	.216	.014

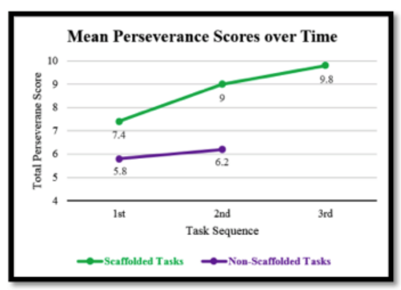

Figure 2. Mean Perseverance Scores Over Time

A simple linear regression was also calculated to predict participants' 3PP scores on their second non-scaffolded task based on their first non-scaffolded task (see Table 2). A significant regression equation was found ($F(1, 8) = 9.879$, $p = .014$), with an R^2 of .553, indicating participants' 3PP scores on their first non-scaffolded task explained 55.3% of the variance in their 3PP scores on their second non-scaffolded task. Also, participants' 3PP scores on their first non-scaffolded task was a significant predictor of their 3PP scores on their second non-scaffolded task, with a one-point increase on their first non-scaffolded task predicting a .680-point increase on their second non-scaffolded task. Thus, participants' 3PP scores were significantly improving over time, more so on scaffolded tasks than on non-scaffolded tasks.

During their exit interviews, participants revealed they noticed improvements, over time, in their engagement with the challenging mathematical tasks. Most participants (8 out of 10) mentioned they thought they were getting better, somehow, as they had more practice with these types of problems. Several participants (6 out of 10) mentioned their improved work on tasks specifically prompting them to conceptualize the situation prior to starting, i.e., the scaffolded tasks. Arguably the most noticeable improvement, from the participants' point of view, was affective in nature. Many participants (7 out of 10) explained in their exit interview that they felt like they were getting better at handling the stress of the situation as they reached impasses within their work on mathematical tasks they did not know how to solve. Some participants (4 out of 10) reported cognitive gains, believing the way they were thinking about the mathematics was changing for the better, over time, and that their problem-solving skills were improving.

Illustrative Case: Sandra's Perseverance Improvement across Scaffolded Tasks

Unpacking representative participant Sandra's experiences across her scaffolded tasks generally illustrates how perseverance improved over time, mostly by supporting participants to make a more quality additional attempt at solving. Sandra encountered Cross Totals first (first overall), then Sidewalk Stones (second overall), and then Skeleton Tower (fourth overall). Importantly, Sandra passed through the 3PP Entrance Phase on all three tasks by affirming she understood all of the objectives, but was not immediately sure how to achieve them. She earned a 3PP score of 6 for her perseverance on Cross Totals, a score of 9 on Sidewalk Stones, and a score of 12 on Skeleton Tower (see Table 3). Like her peers, Sandra made no additional attempt at solving while working on her first task, but progressively improved her perseverance after a setback in the Additional Attempt Phase as she had more experiences with tasks necessitating productive struggle. For comparison, Sandra earned 3PP scores of 6 for her work on both non-scaffolded tasks. She encountered non-scaffolded tasks third and fifth overall.

Table 3: Sandra's 3PP scores for Cross Totals, Sidewalk Stones, and Skeleton Tower

INITIAL ATTEMPT PHASE	Cross Totals (S, 1st)		Sidewalk Stones (S, 2nd)		Skeleton Tower (S, 4th)	
Evidence of Perseverance	Obj. 1	Obj. 2	Obj. 1	Obj. 2	Obj. 1	Obj. 2
Initial Effort	✓	✓	✓	✓	✓	✓
Sustained Effort	✓	✓	✓	✓	✓	✓
Outcome of Effort	✓	✓	✓	✗	✓	✓
ADDITIONAL ATTEMPT PHASE	Cross Totals (S, 1st)		Sidewalk Stones (S, 2nd)		Skeleton Tower (S, 4th)	
Evidence of Perseverance	Obj. 1	Obj. 2	Obj. 1	Obj. 2	Obj. 1	Obj. 2
Initial Effort	✗	✗	✓	✓	✓	✓
Sustained Effort	✗	✗	✓	✓	✓	✓
Outcome of Effort	✗	✗	✗	✗	✓	✓
Total Perseverance Score	**6**		**9**		**12**	

Notes: S = Scaffolded Task; Obj. = Objective; 1st = First task overall; 2nd = Second task overall; 4th = Fourth task overall

Cross Totals. Sandra began Cross Totals (her first scaffolded/overall task) by responding to the scaffold prompt and initiated her effort toward both objectives by stating her plan to reason about the magnitude of integers. She sustained her effort toward both objectives by exploring the logical ramifications of distributing different integers within the cross. Sandra perceived she was making mathematical progress in better understanding the situation by which cross totals may be possible or impossible. She found one example of a possible cross total (see Figure 3a), but soon after reported she had reached an impasse. During her think-aloud she said, "I know this one works, but it's just one example. These things aren't for all cross totals. It wants me to get it in general. I don't know." Although she recognized she had more work to do to better generalize the situation, Sandra decided to record two admittedly incorrect rules based on what she had found and not to make an additional attempt at exploring the problem. Sandra clarified this decision during her video-reflection interview: "In this moment I was just overwhelmed. I felt any other rules, general rules, would be more complicated to get and I just ran out of ideas."

Sidewalk Stones. Sandra's work on Sidewalk Stones (her second scaffolded/overall task) showed specific evidence of perseverance improvement compared to her work on Cross Totals. She similarly began by recording her ideas under the scaffold prompt about the mathematical relationships at play and ways to explore them. Importantly, Sandra revealed during her think-aloud that she was planning ahead when she said, "I was still stuck, or couldn't find a general rule, I could make up examples and check them." Sandra initiated her effort toward both objectives of the task by stating her plan to make a table of values representative of the different kinds of stones. She sustained her effort toward both objectives by searching for patterns within the table that could help her discern how the gray and white stones were changing. Despite some perceived progress, Sandra reached a perceived impasse about the generalization objectives. During her think-aloud she said, "How do I know about Pattern #*n*? This general stuff is hard." She clarified this moment during her video-reflection interview: "I was starting to see the gray pattern, but I had no clue about Pattern #*n*. I was thinking about quitting. I got farther than I thought, but general stuff, it's hard, like finding #*n*. I was stumped."

Despite the urge to quit, Sandra decided to keep working by amending her current plan and changing strategies. She said during her think-aloud, "Well, I guess I can make another example like I said I would." Sandra was referring to her scaffold work, in which she stated that if she "was still stuck" with the objective of generalization, she could "make up examples and check them." Thus, drawing from her earlier conceptualization work prompted by the scaffold, Sandra decided to make an additional attempt at solving Sidewalk Stones. She re-initiated her effort by choosing a different problem-solving heuristic, drawing a partial diagram of a new pattern of stones, and re-sustained her effort by amending her table of values and searching for a pattern amongst all available data (see Figure 3b). Sandra ultimately did not successfully find two general rules, and, by her own admission, she did not believe she better understood the mathematical situation as a result of her additional efforts. Yet, compared to her engagement with Cross Totals, Sandra's perseverance with Sidewalk Stones was much improved primarily because she included in her initial conceptualization work a plan for if she got "stuck." Sandra's backup plan helped her make an additional attempt at solving the problem upon a perceived impasse, an additional attempt she did not make one week earlier with Cross Totals.

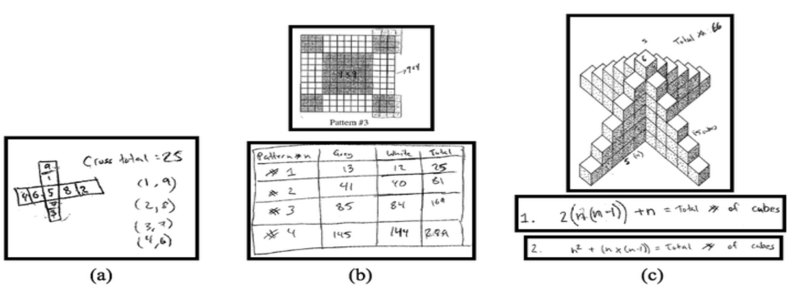

Figure 3. Samples of Work from Sandra's Engagement on Scaffolded Tasks

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Skeleton Tower. Sandra's work on Skeleton Tower (her third scaffolded/fourth overall task) showed even further evidence of perseverance improvement compared to prior tasks. Under the scaffold prompt, she first recorded her mathematical ideas and problem-solving plans, which included "adding up all the cubes" and "knowing the area of a square, s^2, to maybe get an equation." During her video-reflection interview, Sandra clarified that she was preparing for the two generalization objectives: "I thought maybe the s^2 would help me get a general answer with an equation. Something I could just plug height into. The general stuff has been hard in all these problems so I thought using variables might help." Sandra initiated her effort toward finding one general rule by stating she would look for patterns in the diagram, and sustained this effort by counting the cubes in various parts of the tower and reasoning about how to algebraically represent these parts if the height of the tower was n. Sandra visualized piecing two legs of the tower together, stating in her think-aloud, "it's like two squares put together if you flip two of the legs." Eventually, she posited, "I think $(n\text{-}1)^2$ x 2 could be my rule for the total blocks because that's like two squares made up of the legs." Then, Sandra tested her equation and realized it was incorrect. Despite her ample perceived mathematical progress on this task, Sandra shared that she was "annoyed that it wasn't working" and was "stuck."

Despite the perceived impasse as a result of her first attempt to find one general rule, Sandra eventually decided to make an additional attempt by revisiting her past idea about the area of a square. During her video-reflection interview, she clarified, "Well I had that plan to use s^2 and I saw it here and really thought it was a good idea, so I decided to try it again, to maybe think about it another way." Sandra's earlier conceptualization work of Skeleton Tower helped her rethink about deconstructing the tower into squares and algebraically modeling the situation by considering the expression s^2. During this time when she was most "annoyed", it was revisiting her scaffold work that encouraged her to make an additional attempt at solving. From here, Sandra re-initiated her effort by deciding to draw parts of the tower separately, and re-sustained her effort by thinking-aloud about how to piece together the deconstructed parts of the tower. After diligent exploration she exclaimed, "Oh! It's not a square, it's a rectangle!" During her video-reflection interview, Sandra clarified these moments: "I wasn't sure that I was doing it right, but I kept going here. I knew I had a good plan and then, boom, it happened. I figured it out." Sandra went on to use algebraic expressions to model her discovery and ultimately wrote a correct rule that generalized the situation (see Figure 3c). While recapping her success during her video-reflection interview, Sandra cited the importance of incorporating a general equation in her initial conceptualization. She said, "Having an equation (Area = s^2) in mind from the start was a big help. It helped me keep going and helped me with the general part."

Unlike her work on Cross Totals and Sidewalk Stones, on which she worked toward both task objectives simultaneously, Sandra worked toward one objective at a time on Skeleton Tower. This meant that after her breakthrough above, she essentially started over to try to generalize the situation in a different way. Sandra used new strategies and a different point of view to persevere a second time on Skeleton Tower. As she did before, Sandra admitted she had reached another impasse during her first attempt toward the second objective, but ultimately changed her point of view to overcome the obstacle and make an additional, successful attempt (see Figure 3c). Compared to her work with Sidewalk Stones and Cross Totals, Sandra's perseverance with Skeleton Tower was much improved mainly because she planned to work with variables from the outset, during her scaffold work, to generalize the situation. Sandra's more-refined initial planning helped her make an additional attempt at times when she was most frustrated and go on to make progress and solve the problem.

Overall experience. Sandra's experiences with her scaffolded tasks helped illuminate the ways in which participants were improving in their perseverance. Sandra's perseverance gains were most noticeable in the improved quality of her additional attempt at solving a task after reaching an impasse. Sandra's scaffold work – recording her conceptualization of the mathematical situation at hand – played a role in why her perseverance changed because the types of ideas she recorded changed over time as well. During her exit interview, Sandra explained how she thought she was improving in the way she engaged with scaffolded tasks:

> I got better at writing out my ideas in those problems. When you keep doing it you get better at the planning stuff. All the problems ask for a general rule, and I started to learn about how to do that. I was ***getting better at sticking with it*** (emphasis added), too. Like after getting stuck or making a mistake. You just have to get into the problem and maybe even make some mistakes to figure it out. That got easier for me, not getting too annoyed after mistakes.

This perspective is indicative of how cognitive and affective changes impacted participants' perseverance improving over time for scaffolded tasks. Sandra was "getting better at sticking with it" because she better prepared strategies for generalization objectives and better regulated her frustration at key moments during problem-solving. All participants encountered impasses with the scaffolded tasks, yet, as they had more experience in situations requiring perseverance and requiring planning, they persevered more and noticed cognitive and affective improvement.

Sandra earned 3PP scores of 6 on both of her non-scaffolded tasks. She did not demonstrate any evidence of specifically preparing for mathematical generalization, nor did she report in any of her interviews specific ways she was changing how she prepared. Also, Sandra did not mention if she felt better about handling the stress during work on the non-scaffolded tasks, even though they were the third and fifth task with which she worked. Sandra's experiences with non-scaffolded tasks was illustrative of most participants in this study. This suggests that exposure to such tasks warranting struggle did not alone improve participants' perseverance in specific ways, but responding to the initial scaffold prompt, in which participants attended explicitly to conceptualizing the situation, played an influential role in perseverance improvement.

Discussion and Conclusion

Several effective practices for supporting student perseverance have been made apparent in recent research, yet there is little evidence to show if and how student perseverance can improve over time – a vital objective of most reform efforts. This study extends previous research by explaining how and why student perseverance improved over time, and by unpacking this process from the student point of view. These results suggest that developing students' perseverance for solving challenging mathematics tasks may be possible through the process of deliberate practice, a systematic effort to improve performance in a specific domain (Ericsson, 2016). Through this study's design, participants essentially deliberately practiced to improve their perseverance. They worked toward specific objectives of challenge, demonstrated appropriate prior knowledge, invested their full effort and attention to make progress on these tasks (relying heavily on self-control to not give up at moments of impasse and continue to persevere), were a self-source of feedback when recognizing a setback and modified their efforts accordingly, and had opportunities to repeat and practice working through these processes every week. With the scaffolded tasks, this repeated opportunity helped students refine their strategies over that time to learn to persevere in more effective ways. Although students also had a chance to repeat the processes of deliberate practice with non-scaffolded tasks, the data showed that they did not refine their strategies in the same high-quality ways as they did with scaffolded tasks.

Interpreting the apparent perseverance improvement in this study through a lens of deliberate practice makes clearer the process by which perseverance can be nurtured and developed, over time, in students. In mathematics education, deliberate practice has been studied primarily in the context of helping students learn specific skills and developing competencies in particular mathematical domains. Yet, findings in this study suggest mathematical practices like perseverance, in addition to domain-competencies, are malleable and able to improve through processes of deliberate practice, especially with support systems in place that encourage initial conceptual thinking. More work is needed to replicate these findings in different contexts. Still, teachers should take away from this study the key tenets of a learning environment conducive of developing more perseverant learners. No students are always perseverant, but regularly providing learning opportunities that encourage conceptual thinking and prize their productive struggle can support them "getting better at sticking with it" to make meaning of mathematics.

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TIERING INSTRUCTION ON SPEED FOR MIDDLE SCHOOL STUDENTS

Amy J. Hackenberg
Indiana University
ahackenb@iu.edu

Fetiye Aydeniz
Indiana University
fayedniz@iu.edu

Robert Matyska
Indiana University
matyskar@iu.edu

A design experiment with 18 students in a regular seventh grade math class was conducted to investigate how to differentiate instruction for students' diverse ways of thinking during a 26-day unit on proportional reasoning. The class included students operating with three different multiplicative concepts that have been found to influence rational number knowledge and algebraic reasoning. The researchers and classroom teacher tiered instruction during a 5-day segment of the unit in which students worked on problems involving speed. Students were grouped relatively homogenously by multiplicative concept and experienced different number choices. Students operating at each multiplicative concept demonstrated evidence of learning, but all did not learn the same thing. We view this study as a step in supporting equitable approaches to students' diverse ways of thinking, an aspect of classroom diversity.

Keywords: Rational Numbers, Instructional Activities and Practices, Middle School Education

Today's middle school mathematics classrooms are marked by increasingly diverse ways of thinking (National Center of Educational Statistics, 2016). Traditional responses to such diversity are tracked classes that contribute to opportunity gaps (Flores, 2007) and can result in achievement gaps. Differentiating instruction (DI) is a pedagogical approach to manage classroom diversity in which teachers proactively plan to adapt curricula, teaching methods, and products of learning to address individual students' needs in an effort to maximize learning for all (Tomlinson, 2005). DI is rooted in formative assessment, positions teachers and students as learners together, emphasizes engaging all students in creative thinking, and requires teachers to clarify big ideas for instruction in order to make effective adaptations.

One big idea in middle school mathematics is proportional reasoning (Lesh, Post, & Behr, 1988; Lobato & Ellis, 2010), and a great deal of research has been conducted in this domain (Lamon, 2007). One key finding is that computing to solve problems involving ratios (e.g., by cross-multiplication) does not indicate proportional reasoning and may mask it (Kaput & West, 1994; Lesh et al., 1988). A second finding is that students often double, triple, halve, etc., two quantities together as they learn to solve problems involving ratios (Kaput & West, 1994; Lobato & Ellis, 2010); in other words, students treat the two quantities as a *composed unit* (Lobato & Ellis, 2010) and operate on the quantities multiplicatively. For example, students who are creating mixtures of lemonade with the same flavor as 2 T powder with 3 cups water will double each quantity. A third finding is that operating multiplicatively on a composed unit is still different from thinking of the ratio as a *multiplicative comparison* (Lobato & Ellis, 2010), where students know, for example, that in the recipe the amount of concentrate is always 2/3 the amount of water. In this way of thinking, a person is measuring one quantity with the other, and doing so can lead to ideas about rates (Steffe, Liss, & Lee, 2014; Thompson, 1994).

One context for working on proportional reasoning with students is speed (Lobato & Siebert, 2002). In a teaching experiment with nine 8^{th}-10^{th} grade students, Lobato and Siebert gave a distance value and time value for one character to walk in a computer simulation and asked students to determine a different distance value and time value for another character to walk at the same speed. The researchers found that this problem engaged some students in constructing ratios. For example, ninth grade student Terry explained why traveling 2.5 cm in 1 s was the same as 10 cm in 4 s because he could see the 10 cm-4 s journey as made of up four 2.5 cm-1 s segments, and that each small segment was ¼ of the total journey. Similarly, in a teaching experiment with seven 7th grade students, Ellis (2007) found that such tasks supported students' construction of rate and slope in cycles of generalizing and justifying activity. Across both experiments students showed evidence of one hallmark of the construction of speed as a proportional relationship between distance and time, notably that "partitioning a traveled total distance implies a proportional partition of total time required to travel that distance" and vice-versa (Thompson & Thompson, 1994, p. 283).

The purpose of this paper is to report on this research question: What influences did tiering instruction with speed tasks have on a class of 18 regular seventh grade mathematics students during a unit on proportional reasoning? The report comes from a 5-year project to study DI and relationships between middle school students' rational number knowledge and algebraic reasoning. In the last phase of the project, the research team partnered with middle school mathematics teachers who had participated in a year-long study group to explore differentiation. In fall 2017 a seventh grade mathematics teacher and the project team designed to differentiate during a 26-day unit, and the teacher and first author co-taught the unit. The data for this paper comes from five episodes in which the students experienced tiered instruction focused on exploring speed to support reasoning with ratios. The instruction for these five days was adapted from Lobato's Math Talk project (mathtalk.sdsu.edu) and utilized the geogebra app Races developed by Bowers (https://www.geogebra.org/m/J434Kb54).

Theoretical Frame

In this section we present our definition of DI, our view of mathematical thinking, and a tool we use to understand students' diverse ways of thinking, students' multiplicative concepts.

Definition of DI

Our definition of DI is *proactively tailoring instruction to students' mathematical thinking while developing a cohesive classroom community* (cf. Tomlinson, 2005). For us, "tailoring instruction" requires posing problems that are in harmony with students' thinking and posing challenges at the edge of students' thinking (Hackenberg, 2010). We view a "cohesive classroom community" as students and a teacher who are working together to foster everyone's mathematical learning (Lampert, 2001; Tomlinson, 2005), who regularly talk about their ideas (Sherin, 2002), and who hold diverse points of view that are valued (Bielaczyc, Kapur, & Collins, 2013).

Mathematical Thinking and Interaction

As teacher-researchers we don't have direct access to students' mathematical thinking or points of view. So, we organize our experiences with students' thinking by describing and accounting for it using our constructs: operations, schemes, and concepts. *Operations* are the components of *schemes*, goal-directed ways of operating that involve a situation as conceived of by the learner, activity, and a result that the learner assesses in relation to her goals (von Glasersfeld, 1995). For us, mathematical learning involves a learner making reorganizations, or *accommodations,* in her schemes in on-going interaction in her experiential world.

Indeed, interaction is a core principle of our view of mathematical thinking and learning (Piaget, 1964; Steffe & Thompson, 2000). We find it helpful to think about two non-intersecting domains of interaction (Steffe, 1996): intra-individual interactions of constructs within a person, such as accommodations in schemes, and individual-environment interactions of which social interactions are a major part. Social interaction, such as student-student and student-teacher interactions, can open possibilities for accommodations and make operations and schemes apparent via verbalizations, non-verbal expression, drawn representations, or mathematical notation. Similarly, the construction of particular operations, schemes, and concepts can dramatically influence how a student interacts with others in a classroom (e.g., Hackenberg, Jones, Eker, & Creager, 2017). However, interaction of a particular kind in one domain does not directly cause interaction of a particular kind in the other (Steffe, 1996).

For us, *concepts* arise from re-processing the result of a scheme so that students can use it to structure a situation prior to acting (von Glasersfeld, 1982). Broadly speaking, students enter middle school operating with three different multiplicative concepts that significantly influence rational number knowledge (Norton & Wilkins, 2012; Steffe & Olive, 2010) and algebraic reasoning (Olive & Caglayan, 2008; Tillema, 2014). Transitioning between these three concepts requires substantial accommodations that can take two years (Steffe & Cobb, 1988; Steffe & Olive, 2010). Steffe (2017) estimates that at the start of sixth grade, 30% of students are operating with the first multiplicative concept, 30% with the second, and 40% with the third.

Students' Multiplicative Concepts

We view students' multiplicative concepts in terms of units coordination (Steffe, 1992). *Units* are discrete ones (Ulrich, 2015), lengths, or measurement units. As children progress in their construction of number and quantity, they organize units into larger units, such as composite units (units of units). A *units coordination* entails distributing the units of one composite unit across the units of another composite unit. For example, consider this problem: The length of the balance beam measures 8 skewer lengths. There are 7 toothpick lengths in a skewer length; how many toothpick lengths will measure the beam's length?

Students operating with the first multiplicative concept (MC1 students) solve this problem by counting on by 1s past known skip-counting patterns, tracking the total number of toothpick lengths and skewer lengths. For example, they might know that two 7s is 14. Then they count on by 1s to 21 for the amount of toothpick lengths in 3 skewer lengths. And they keep going. These students think of the result, 56 toothpick lengths, as a unit consisting of 56 units. But they don't see a multiplicative relationship between 1 toothpick length and the 56.

Students operating with the second multiplicative concept (MC2 students) do see a multiplicative relationship: The 56-unit length is 56 times 1 unit. These students also see the 56 toothpick lengths as eight 7s, or 8 units of 7 units of 1, which is three levels of units. However, as they work further, they think of 56 as a unit of 56 units of 1; the three-levels-of-units structure does not remain for them.

Students operating with the third multiplicative concept (MC3 students) can see what MC2 students see, but as they work further, they continue to view the 56 units as 8 units of 7 units of 1. This view is helpful if the number of skewer lengths is not a whole number. For example, if the distance were 8 ¼ skewer lengths, MC3 students are able to reason that to measure the distance in toothpick lengths they need eight 7s and ¼ of 7.

Method, Data Collection, and Data Analysis

To launch the experiment, we observed in two seventh grade pre-algebra classrooms: a participating class taught by Ms. W and a comparison class taught by a different teacher. Following observations, 38 students consented to participate: 18 out of 20 in the participating class, and 20 out of 21 in the comparison class. Before the unit began, we sought to develop initial understanding of students' multiplicative concepts and fractions knowledge and to select focus students: six from the participating class (1-3 operating with each multiplicative concept) and six from the comparison classroom. To gather initial information, we administered three written assessments of students' fraction schemes and operations (Wilkins, Norton, & Boyce, 2013) and multiplicative concepts (Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins, 2015).

We used results of the written assessments, as well as our classroom observations, to select 30 students for 40-minute individual interviews prior to the start of the unit. In the interviews we developed deeper understanding of students' thinking, and sometimes we revised our assessment of a student's multiplicative concept. Following the interviews, we had 11 MC1 students (5 in the participating class), 17 MC2 students (9 in the participating class), and 10 MC3 students (4 in the participating class). We selected as participating focus students two MC1 students, three MC2 students, and one MC3 student, as well as six "matched" comparison focus students.

For this paper we have done two analyses. First, we developed second-order models of the six participating focus students; a second-order model is a researcher's constellation of constructs to describe and account for another person's ways and means of operating (Steffe & Olive, 2010). For each student, we repeatedly reviewed video of the student's three interviews; video of the student's activity during the targeted five days of class; and the student's written work, including a quiz on Day 17 and a unit test. To create the models we wrote summaries, interpretations, and conjectures, which we discussed at bi-weekly research meetings with a 6-member research team. We debated and questioned interpretations, coming to consensus through discussion. Each model is a description of the student's operations, schemes, and concepts, with accounts of accommodations that occurred and the individual-environment interactions (e.g., particular small group discussion) that were involved in the accommodations.

Second, we have engaged in close description and analysis of the classroom activity during the targeted five days. We have organized documents of the work of all 18 students across the five days in order to articulate trends and patterns in student ideas and responses.

Findings

Summary of Days 9-13

The unit consisted of three investigations: quantifying orangeyness (Days 1-8), quantifying speed (Days 9-18), and understanding percentages (Days 19-26). By Day 9 we had conducted formative assessment of students' reasoning with ratios during the quantifying orangeyness investigation. We found that MC1 students were not fluidly iterating two quantities as a composed unit (Lobato & Ellis, 2010), while MC2 and MC3 students were. So, we thought that tiering instruction at the start of the quantifying speed investigation would help us target students' current thinking with ratios and could support them to make advances. Students worked in small groups that were relatively homogenous by multiplicative concept from Day 9 to 13.

On Days 9 and 10 students articulated how to measure fastness and how they knew one car was going faster than the other in the Races app (Figure 1). Subsequently they worked on tasks where they were to make the red car go slower than the blue car if both traveled the same distance, and then if both traveled the same amount of time. On Days 11 through 13 students were given a distance value and time value for the blue car, and they explored how to make the red car go the same speed using a different distance value and time value. They were to justify their claims with pictures and explanations. Here we tiered instruction by selecting distance and time values strategically for different thinkers as shown in Table 1.

Now we show how students at each stage worked on the same speed task with a focus on members of the group that made the most progress: a group of three MC1 students in which we focus on Emily, a group of three MC2 students in which we focus on Lisa and Sara, and a group of two MC2 and two MC3 students in which we focus on MC3 student Joanna.

Figure 1: Races App with Blue Car (top) and Red Car (bottom)

Table 1: Students' Multiplicative Concepts and Numbers for Same Speed Task

MC	Same Speed Task (Blue car goes)
1	18 mi in 3 min; yields a whole number unit ratio (6 mi per 1 min)
2	15 mi in 6 min; yields a unit ratio that is a mixed number with ½ (2.5 mi per 1 min)
3	15 mi in 9 min; yields a unit ratio hard to work with as a decimal (5/3 mi per 1 min)

MC1 Students: Emily and Groupmates

When Emily and her two groupmates tried to find a distance and time for the red car to go the same speed as the blue car traveling 18 mi in 3 min, Emily suggested 9 mi in 6 min, 18 mi in 6 min, and 18 mi in 2 min. She seemed to be, primarily, halving or doubling either quantity but not operating on both together.

Then a groupmate suggested 36 mi in 6 min. Emily ran that race and was visibly excited when the cars kept pace with each other. She seemed suddenly subdued when the red car continued traveling after the blue car stopped, but the group concluded that the cars had gone the same speed and that doubling each number "worked."

The first author, Ms. H, asked the group to draw a picture to justify why traveling 36 mi in 6 min was the same speed as traveling 18 mi in 3 min. No one initially had ideas. Emily said, "I know how to tell, but I don't know how to show it." She explained that doubling each number (18 and 3) meant you could then divide each number (36 and 6) by 2, and "it's almost like they're the same number in a way." Ms. H acknowledged this idea but asked them to think about the quantities because that would help them develop stronger ideas about speed.

Ms. H asked if they could draw something to represent each journey. Emily's pictures evolved the most, so we focus on her. First Emily drew a segment to show each journey, identified by labels (Figure 2, left). When asked whether she could show the idea of doubling with the lengths, Emily drew a second picture (Figure 2, right).

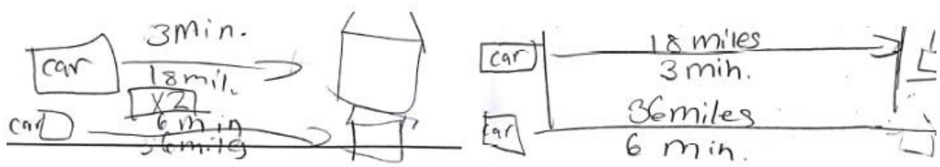

Figure 2: Emily's First Picture (left) and Second Picture (right)

Despite the discussion about doubling and Emily's emphasis on the importance of doubling, the second picture showed lengths about the same size. Ms. H asked whether the journeys were the same size, and Emily said no. She then extended the 36 mi-6 min segment but did not make it exactly twice as long as the 18 mi-3 min segment, in part because she reached the edge of the paper (Figure 3). She identified that there was supposed to be another 18 mi-3 min segment next to the first one, making up the 36 mi-6 min segment. She was about to draw a more exact picture when the period ended. So, Emily went from not knowing how to draw a picture to beginning to show how the 36 mi-6 min journey consisted of traveling the 18 mi-3 min journey twice. She presented this idea to the whole class the next day. In her follow-up interview on a similar question she began by showing two different journeys with equal lengths and then self-corrected to produce a picture showing relative size, which is evidence of learning for her.

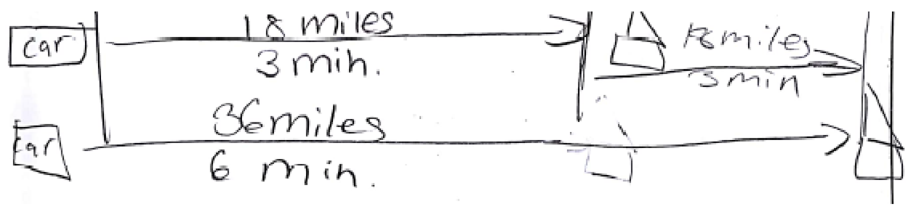

Figure 3: Emily's Third Picture

MC2 Students: Lisa and Sara

When Lisa and Sara tried to find a distance and time for the red car to go the same speed as the blue car traveling 15 mi in 6 min, Lisa suggested 14 mi in 5 min and then 15.1 mi in 6.1 min. When neither worked, both students said it was "impossible!" Ms. H asked them if it was really not possible for two cars to travel the same speed but different distances and times. Sara said: "They probably could, but I can't figure it out." Then she added, "unless you double it." They ran a race where the red car traveled 30 mi in 12 min, and they both seemed excited to find that doubling the quantities produced the same speed. "I figured the system out!" proclaimed Sara. Lisa added that it might be possible to triple both quantities or use other multiples.

Like Emily and her groupmates, Lisa and Sara found it challenging to explain why doubling would work. However, in contrast with Emily, Lisa's first picture showed that the 30-mi distance was twice the length of the 15-mi distance (Figure 4, left). In discussing the picture, Ms. H pointed out that in Lisa's picture it looked like the car traveling 30 mi went a trip of 15 mi in 6 min and then another trip of 15 mi in 6 min (tracing the trips with her pen).

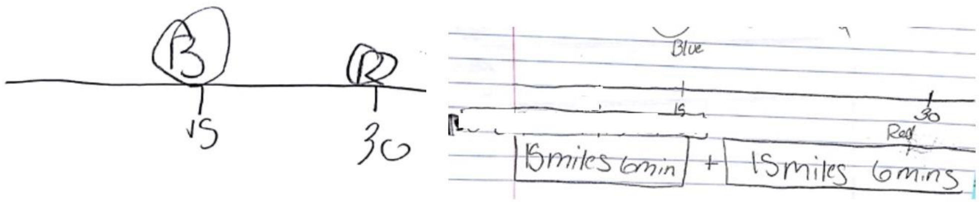

Figure 4: Lisa's First Picture (left) and Second Picture (right)

Lisa agreed and drew another picture (Figure 4, right) that showed the 30 mi-12 min trip as consisting of two "15 miles 6 min" segments added together. In a whole class discussion the next day, Lisa stated the idea that to go the 30 mi-12 min journey "first you'll need to do one 15 miles in 6 minutes and then you'll need to add another 15 miles in 6 minutes." Sara also used this multiple-trip explanation to explain solutions to this and other problems.

When Ms. H asked if they could find smaller distance-time pairs that would produce the same speed, they halved both quantities and indicated that they could continue to halve to find more same speed pairs. We note that Lisa and Sara did not consider dividing the distance and time values by numbers other than 2 without further teacher questioning. Nevertheless, during this instructional segment they went from not knowing how to generate same speeds to using at least whole number multiples and halving to do so, and from not knowing how to justify same speeds to using multiple-trip explanations. They sustained these ways of generating and justifying same speed pairs in their follow-up interviews.

MC3 Student: Joanna

When Joanna's group of four began discussing distances and times for the red car to go the same speed as the blue car travelling 15 mi in 9 min, Joanna quickly suggested 5 mi in 3 min. Her groupmate Mark suggested 16 mi and 10 min, adding one unit to each quantity. Joanna argued that 15 and 9 "reduced" to 5 and 3 but 16 and 10 "reduced" to 8 and 5, so 15 and 9 and 16 and 10 "wouldn't be the same ratio to each other." Using the app to test Mark's suggestion, the group determined that the red car travelling 16 mi in 10 min would actually go slightly slower than the blue car. Then Joanna stated that any numbers "where the miles would reduce to 5 and the minutes would reduce to 3" should work "because they're the same ratio to each other." She suggested 10 and 6 as another pair that would give the same speed.

To justify her claim, Joanna drew a distance line and time line (Figure 5). She partitioned the lines into three equal parts of 5 miles and 3 minutes. Then she used her picture to justify that when the red car travels 5 mi in 3 min, it goes the same speed as the blue car; it just stops earlier. Upon questioning, Joanna elaborated that 5 mi-3 min segment was 1/3 of the blue car's journey. To Joanna, the 15 mi-9 min trip was a unit that could be partitioned into 5 mi-3 min segments, and she saw that any trip made from a multiple of these segments would have to be the same speed as the blue car, a general way of thinking. She created this general way of thinking by determining the smallest whole number pair of numbers that could make the 15 mi-9 min trip.

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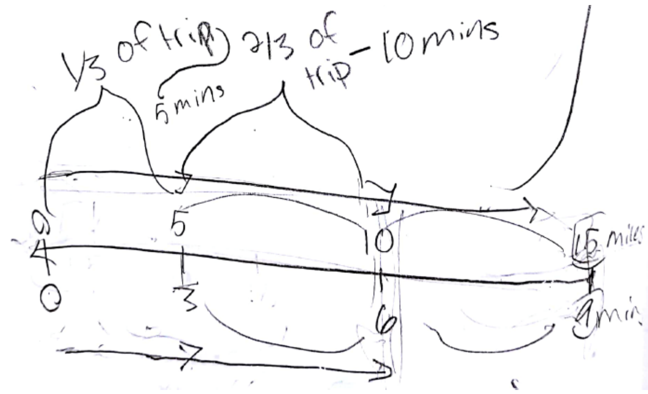

Figure 5: Joanna's Picture

Discussion and Conclusions

Now we point out some similarities in the students' ways of thinking, as well as differences that relate to students' multiplicative concepts. Notably, both Emily and Lisa were not sure how to create same speeds, and both took up groupmates' suggestions to double. However, Lisa's picture indicates that she conceived of doubling the quantities in a way that showed two smaller trips fitting into the larger trip. These relationships were not evident for Emily without interaction and support from a teacher to try to show relative size in her picture.

Students' multiplicative concepts are explanatory here. That is, both Emily and Lisa could repeat a distance and time to create a trip with double the distance and time. But Lisa appeared to have imagery of that larger trip as consisting of two smaller, equal trips—the smaller trip was both a part of the larger trip and also separate from it. This imagery is consistent with having constructed a disembedding operation where a unit (in this case, a segment representing a distance-time pair) can be both a part of a composite unit and also separate from it—a hallmark of the second multiplicative concept (Steffe, 2010). In contrast, Emily did not show obvious understanding of these embedded relationships. This phenomenon is consistent with MC1 students who conceive of a length as consisting of parts (smaller lengths), but once the original length has been separated into parts, they do not reunite the parts to create the original length or see the parts as being embedded in the original length (Hackenberg, 2013; Steffe & Olive, 2010).

In contrast to Emily, Lisa, and Sara, Joanna partitioned her distance and time quantities and seemed to view it as a logical necessity to partition each quantity proportionally (Thompson & Thompson, 1994). Her insight was that any numbers that were in a ratio of 5 to 3 would produce the same speed, so she saw more generally that 15 mi in 9 min was just one journey that was made from a multiple of 5 mi in 3 min. Her multiplicative concept can help account for her insight. That is, Joanna could view numbers and quantities as three-levels-of-units structures prior to working with them in a problem solving situation. So, she saw both 15 mi and 9 min as units of 3 composite units: 15 mi was a unit of 3 units of 5 mi, and 9 min was a unit of 3 units of 3 min. Being able to see both quantities in this way facilitated her thinking about how, since each 5 mi-3 min segment would have to be the same speed, then three of them strung together would be the same speed. Ultimately, she saw that any trip made from a multiple of this smallest whole number pair would produce the same speed.

Now we comment on the different numbers the students worked with. All distance-time pairs required taking thirds to get to the smallest whole number pair that could create the same speed (Table 1). Yet MC1 and MC2 students did not take thirds of their quantities: They doubled, tripled, and halved. We anticipated that many students initially would double and halve, and that is precisely why we gave pairs that could be “thirded” to produce the smallest whole number pair that could create the same speed. Thus, we learned that these number pairs were good choices in the sense of not being completely transparent to students. In addition, they supported MC3 students like Joanna to reveal the structural way she viewed and operated on both quantities.

Other researchers have used students’ ways of thinking about speed as an avenue for supporting the construction of ratios and rates (Ellis, 2007; Lobato & Siebert, 2002; Thompson, 1994), which are extremely important mathematical ways of thinking in secondary school. However, to our knowledge researchers have not investigated how students with different multiplicative concepts construct these ways of thinking, as well as how to differentiate instruction for these thinkers in the same classroom. In this seventh grade classroom tiering instruction was successful in supporting the learning of each of these three thinkers, although what each thinker learned was different. Thus, we view this study as a step in supporting equitable approaches to an important aspect of classroom diversity, students’ diverse ways of thinking. This kind of DI is an important component of inclusive, antiracist classrooms in which “equity is a priority” (Michael, 2015, p. 82) because all students are seen as mathematical thinkers and get what they need to learn.

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BOB'S ADDITIVE REASONING: IMPLICATIONS FOR KNOWING FRACTIONS AS QUANTITIES

Kristi Martin
North Carolina State University
kmartin8@ncsu.edu

Jessica H. Hunt
North Carolina State University
jhhunt5@ncsu.edu

Fractions are one of the most difficult areas of mathematics for all students and especially for students with learning disabilities (LD). An incomplete understanding of fractions during the elementary and middle school may be why some students view fractions as "really small" or "less than 1" compared to others who view them as multiplicative magnitudes. We analyzed how one student with LD worked on tasks involving both whole numbers and fractions using the framework of units coordination. Through our analyses of 14 teaching experiment sessions, we determined that he was operating on two levels of units with whole numbers, but only one level of units with fractions. We argue that this limited his ability to operate multiplicatively with fractions. However, his identified LD did not appear to impact his work with the fractions tasks he was given. Implications for future research and instruction are shared.

Keywords: (Dis)ability Studies, Rational Numbers

Background and Study Objectives

Fractions are one of the most relentless areas of difficulty in mathematics for all students (Siegler et al., 2010) and are especially for students with learning disabilities (LD) (Cawley & Miller, 1989). When asked to place fractions in order from least to greatest on two separate assessments, middle school students with LD answered only 47% and 1% of questions correctly, compared to 85% and 60% by students without LD (Mazzocco & Devlin, 2008). Similarly, researchers report fourth and fifth grade elementary students with LD begin their study of fractions with a diminished conceptual understanding compared to their peers (Geary et al., 2008) and show significantly less improvement in their ability to solve problems, estimate, and apply computational procedures with fractions over time (Hecht & Vagi, 2010). An incomplete understanding of improper fractions during the late elementary and middle school years may be the driving force behind why some students view fractions as "really small" or "less than 1" versus other who view them as multiplicative magnitudes (Resnick et al., 2016).

In this study, we present the whole number and fractional reasoning of one student with LD across 14 teaching experiment sessions. We utilize a three phase qualitative analysis to illustrate how this student evidenced his understanding of fractions as quantities through his interactions with varied tasks. Through our analyses of these data, we raise questions about the child's thinking and what his apparent knowing and learning were relying upon. The research questions addressed are (1) How does one child with LD conceive of whole number and fractions as quantities across varying task types? and (2) What persistent ways of reasoning were apparent across the child's activity and how did these contribute towards this child's knowledge of fractions?

Conceptual Framework

We think of students' understanding of fractions as quantities as a result of their propensity to multiplicatively coordinate units(s). Units coordination refers to how students create units and maintain relationships with other units (Norton, Boyce, Ulrich, & Phillips, 2015). It explains transitions children make from additive operations to multiplicative operations. Norton et al. (2015) explain that a student uses one level of unit when she conceives of situations such as five iterations of four by counting on from the first or second set by ones and double-counting the number of fours to reach a stop value (e.g., 4, 8, 12, 13-14-15-16, then 17-18-19-20). A student who uses two levels of units might conceive of five units of four as two units of four plus three units of four (e.g., three 4s is 12; 13-14-15-16; 17-18-19-20). This student breaks apart the composite unit of five into three and two and uses each of those parts to arrive at the solution. This student also sees "20" as both 20 ones and 5 fours all at once, so finding six units of four would not require reconstructing the first five units of four- the student would count-on from 20. Using three levels of units involves three related or coordinated units: (a) one unit of 20 that contains (b) five units of four, each of which contains, for instance, (c) four units of one. Students who anticipate three levels of units demonstrate flexible, strategic reasoning with each of the units.

Fractional units can also be coordinated. Steffe (2002) hypothesized that students would reorganize whole number unit coordinating to develop similar understandings with fractional units. Put differently, students use their whole number units coordination to conceptualize fractional units as a result of equi-partitioning whole units. The size of each fractional unit is one that when iterated n times would result in the size of one (Olive, 1999). To understand a non-unit fraction (m/n) as a number, students must conceptualize m/n as equivalent to m x $1/n$, n of which are the same value as 1. When the number of iterations of $1/n$ exceeds n/n, students must accommodate their part-whole reasoning from thinking of $1/n$ as one out of n total pieces to thinking of $1/n$ as an amount iterated more than n times without changing its multiplicative relationship with the size of 1. Thus, a multiplicative conception of fractions as quantities involves coordinating three levels of nested units: 10/4 as 10 times (1/4) *and* 1 is the same value as 4 times (1/4). Ten-fourths involves both a unit of 1 and a unit of 1/4 coordinated with 1. If students name the size of proper fractions by counting the number of parts of size $1/n$ within the bounds of a whole of n/n, they do not yet iterate $1/n$ (Tzur, 1999). This limits students' propensity to understand unit fractions beyond the size of the whole in a multiplicative manner.

Methods

"Bob"

The data utilized in this study illustrates the case of one fifth grade student who we refer to as "Bob." Eleven years old at the time of the study, Bob was an outgoing child who often imagined creative ways to solve problems and was eager to interact with problematic situations. Inclusion criteria for the study were as follows: (a) individualized education program goals in mathematics, (b) a cognitively-defined label of learning disability (LD) with working memory as the dominant cognitive factor, (c) identification through clinical interview data that the child had constructed at least a parts within the whole concept of unit and non-unit fractions, and (d) identification by the classroom teacher as 'non-responsive' to supplemental, small group intervention from a textbook or supplemental curriculum in advanced fraction concepts.

For Bob, school-based instruction included opportunities to shade pre-partitioned wholes to reflect a designated number of parts. Bob extended this activity to represent various fractional quantities using area, set, and linear models. These experiences were designed to help Bob see the relationship between one of the equal parts and the whole. Area, set, and linear models were also used during school-based instruction to represent equivalent fractions as well as to identify given fraction representations as equivalent. Procedures for fraction operations followed.

Furthermore, although Bob was able to construct unit fractions and identify shaded parts of wholes (e.g., identify one part shared within a partitioned item as one-seventh; determine the share of 4 objects among 3 people as one whole and one third); he could not yet draw parts outside of and relative to the whole. He did not yet conceive of non unit fractions as a summation of unit fractions (¾ = ¼ + ¼ + ¼) or a multiplicative relation with respect to a unit fraction (e.g., ¾ as three iterations of ¼). Instead, ¾ was three of four parts.

Study Design and Data Sources

We report on five sessions conducted as part of a larger individualized teaching experiment of 14 sessions (Steffe & Thompson, 2000) because these sessions provided evidence of how Bob coordinated units with whole numbers and fractions. We also conducted two formative clinical interviews that serve as pre-and post-measures (Ginsburg, 1997). Each interview and instructional session lasted about 30 minutes. The second author was the researcher-teacher. The first and second authors both engaged in the retrospective analysis after data collection concluded. The instruction was conducted in a one-on-one setting in the school library. The classroom was equipped with one circular table, unifix cubes, pens, paper, and chart paper. Sessions took place after school and were in addition to Bob's regular math class time. We collected three sources of data: video recordings, written work, and field notes.

Data Analysis

Data analysis occurred in three phases. The first involved ongoing analysis of video data and written work immediately following each teaching session. The ongoing analysis led to planning the following teaching episode and consistent modification of hypothesis regarding the child's mathematics and goals in activity. The focus was on generating (and documenting) initial hypotheses as to what conceptions could underlie the child's apparent problem-solving strategies during these critical events. These hypotheses led to (a) designing subsequent teaching episode(s) and (b) guiding retrospective analysis.

The second phase of data analysis included a retrospective identification of broad indicators, or stages, of conceptual growth after the sessions concluded (Leech & Onwuegbuzie, 2007). We conducted a line by line examination of the transcribed data, student work, and notes with a side by side of video data to consider Bob's thinking within each session. Analysis of each teaching episode focused on expanding our model of Bob's thinking. We coded segments within the session for indications of Bob's units coordination and operations at the time, using prior research as a framework. Bob's gestures and explanations (e.g., from post-episode field notes) were incorporated into the analysis to provide a thick depiction (Geertz, 1988) of his current operations— served as further triangulation. The third phase involved fine-grained analysis to closely consider how Bob advanced his thinking from one stage to the next and the difficulties he experienced (Tzur, 1999).

Results

"I Just Added Two"

During the first instructional session, Bob was presented with the task of making "double" of several whole numbers of squares. He doubled one and two squares to make two and four squares, yet when asked to double four squares, he made six squares. His explanation for doubling both two and four was, "I just added two."

<u>Session 1: Make double of four squares</u>
Instructor: Could you make double that [points to line of four squares]?

Student: Yeah.
Instructor: Maybe – [student traces one square on each end of line of four squares] What'd you do there?
Student: I just added two.
Instructor: You added two. Hmm, gotcha. So, what's this whole length called? [runs finger along line of six squares]
Student: [tapping pencil on desk] One or one, two, three, four, five, six, six. [pointing at squares, but not counting one to one]

We questioned whether Bob was coordinating whole numbers multiplicatively. Moreover, it was unclear to us whether Bob connected "doubling" with two levels of units (e.g., viewing four as two units of two). This wondering would permeate throughout the teaching sessions.

"One, Two, Three, Four"

The next several sessions continued to provide contradictions in how Bob was coordinating units. For instance, in the transcript below, Bob shares three French fries among four people. At first glance, it seems as if Bob is utilizing two levels of units multiplicatively to consider the resulting quantity. Yet, as the transcript shows, Bob coordinated units with whole numbers and fractional units utilizing additive reasoning.

<u>Session 5: Share three fries among four people</u>
I: Could they each get a whole one? Could they get less than a whole? Remember these are really big fries.
Bob: So one of them you could cut in half [makes a chopping gesture]... two of them could be cut in half.
I: Two of them need to be cut in half, okay.
Bob: And then we need to have one more French fry.
I: What did you do to that one?
Bob: And then now we'll have to split that one up… into… fourths so if I did one [uses finger to mark first line and second line]– that's…
I: Are you showing me the lines?
Bob: So that's 1, 2, 3, 4 [uses fingers to show lines] and there and there [marks lines].
I: Like that? Nice. There you are [draws a smiley face on paper]. How much do you get?
Bob: [draws a smiley face on one of the halves and one of the fourths]
I: So you get this part and this part? [points to pieces labeled with smiley faces] How much is that? Bob: ½ [points to the one-half].
I: What's this part called?
Bob: 1/4. [points to the one-fourth]
I: So would it be fair to say that you get one fourth and one half of a fry?
Bob: [measures ¼ piece and adds it to the end of "his" half] This is how much I will get.

Bob coordinated four sharers across two wholes by partitioning each whole into two parts. At the time, we viewed this as evidence that Bob utilized two levels of units to create four from two twos multiplicatively. However, to coordinate fourths with respect to one whole, Bob seemed to count by ones as opposed to partitioning into halves and then fourths (Confrey, Maloney, Nguyen, & Rupp, 2014). Furthermore, although Bob was able to draw the two fractional pieces together on one strip, he did not yet see the parts as a continuous quantity (i.e., as three-fourths).

Thus, Bob coordinated whole numbers on two levels additively and fractions on one level additively.

“Two Out of Five”

In Session Six, Bob displayed part-whole knowledge of fractions to consider two-fifths as two parts out of five parts. He was able to use his part whole knowledge to create one-fifth using the computer program JavaBars and iterate the one-fifth to make seven-fifths. Yet, his language shows that he views the created quantity as seven as opposed to seven fifths.

<u>Session 6: This bar is two-fifths. Can you make seven-fifths?</u>
Bob: Two-fifths [splits bar into two], so it’s two… it’s two out of five, so maybe if I… That’s two – every one out of five, two out of five…
I: Mm-hmm.
Bob: There’s two now and, well, uh, and… and then that one. [pulls out one fifth]
I: What’s that one? [points to pulled out one-fifth]
Bob: One-fifth.
I: It’s one-fifth.
Bob: And it –
I: We want seven-fifths.
Bob: Seven fifths… one out of five, seven-fifths, so we want… and then that is… [repeats 1/5 four times, pauses to count]. Okay, you just do that. [repeats two more fifths to make seven fifths] One, two, three, four, five, six, mm, six, seventh. But this is how big it is.
I: That’s how big it is?
Bob: Yeah.
I: That’s how long? [student accidently repeats an additional one-fifth] Whoops.
Bob: Uh… one, two, three – eight-fifths. Eight-fifths, whoa.
I: Yeah, a question for you. See this eight-fifths? How many times bigger is that than the two-fifths?
Bob: Mm… three.
I: Three times bigger?
Bob: Because ... these two come together [points to two one-fifth sized parts]. So, these two come together, and these two come together, and these two come together [points to third and fourth, then fifth and sixth, then seventh and eighth one-fifth parts]. And that; two, four, six – and then those two.

In this excerpt, Bob interprets two-fifths as two pieces out of five pieces, with each piece representing one-fifth. This thinking promotes him to pull out one fifth from the two-fifths displayed; he then repeats the one-fifth he removed from two-fifths four times to make five fifths. He pauses, then repeats the part two more times to create seven fifths. We interpreted Bob’s reasoning as additive; he adds one more to achieve seven one fifths. Bob’s justification of the relationship of eight fifths to two fifths was *three more times,* or three extra pairs of two, as opposed to four times as large. Furthermore, we argue that, for Bob, the parts are not fifths, but *ones* that are part of a whole comprised of five units. Evidence for this claim rests in his consistent numbering and counting of the parts as whole numbers and only naming the parts with fractional language when prompted.

“Six Times Six is 12”

Because of his persistent additive reasoning to this point, in Session 11 Bob plays “Please Go and Bring for Me” (Tzur et al., 2013), a game designed to promote multiplicative units coordinating with whole numbers. Below, Bob considers how to coordinate five iterations of a unit of six.

<u>Session 11: Make five towers of six</u>
I: Okay. Five towers of six. Okay. So, um, how many towers did you bring?
Bob: Um, five towers of six. [brings towers back to table (2 purple, 1 green, 2 pink)]
I: Okay, so five towers, and how many in each tower?
Bob: Six.
I: So how many cubes did you bring in all?
Bob: Six [slides over 1 tower], 12 [slides over 1 tower], 24 [slides over 2 towers], 24…, 25, 26, 25, 26, 27, 28, 29, 30 [counts on fingers], 30.
I: Thirty cubes. Can you tell me how you figured that out?
Bob: I just showed – so, six times six is 12, and then 12 plus 12 is 24, and here and I already have a six, which is 30 [moves towers as he talks about them].

Bob conceived of five towers of six as two, two, and one tower with visible cubes. Interestingly, he builds the towers using three different colors of cubes, suggesting that the groups of two created in his towers belong together. This seems to support his counting of the cubes as a “double” (i.e., “Six, 12, 24”). Bob seemed to conceptualize doubling as *adding the same group*, supporting his breaking apart of the composite five into two, two, and one. Thus, he could access a two level structure additively, utilizing his way of doubling for a few iterations. His way of doubling breaks down with larger numbers. Other sessions continued to show sustained use of additive reasoning to solve problems.

Later in this session, the researcher-teacher confirmed understanding of how Bob utilized doubling in both whole number and fractional situations. For this task, Bob again used the computer program JavaBars. Below, Bob considers how to coordinate fractional units beyond a whole by employing his conception of doubling.

<u>Session 11: This ribbon is one-eighth of a yard long. What does twice as much ribbon look like?</u>
I: I want you to pretend this is a piece of ribbon, and you own a store where you sell ribbon in pieces. And the pieces that you sell are one-eighth of a yard long.
Bob: Okay.
I: I'm thinking about two times as much as that.
Bob: [repeats the bar]
I: Okay. What's the name that you give that?
Bob: One-eighth. I mean two-eighths.
I: Two-eighths. How do you know it's two-eighths?
Bob: 'Cause it's two out of eight, and I had just added one, so it's two out of eight.
I: Now, I'm thinking about an, an amount that's two times as much as this [points to Bob’s two-eighths].
Bob: Okay.
I: What would that look like?
Bob: That big. [repeats one-eighth piece twice; adds on to two-eighths]

I: Okay. That big. Um, how much of a whole yard is that?
Bob: Four fourths – I mean, four-eighths.
I: Four fourths, four-eighths, which one?
Bob: I think four-eighths.
I: Four-eighths? How do you know it's four-eighths of a yard?
Bob: Um, 'cause it's four pieces out of eight.
I: Four pieces out of eight?
Bob: Uh-huh.
I: Okay, um, tell me more about that. How do you know it's four pieces out of eight?
Bob: 'Cause there's four, and there's supposed to be eight.
I: How do you know there's supposed to be – ?
Bob: 'Cause that's how big a yard is, and I know, and I'm the manager.

Previous tasks had shown Bob using his reasoning about doubling, but in this task, he explicates his notion of "double" as "add one more group." Furthermore, he displayed his persistent part whole concept of units (e.g., four pieces out of eight; two parts out of eight parts). Bob's activity confirmed our hypothesis that his additive view of whole numbers and fractions was a salient and persistent way of reasoning.

"Then it Would Be a Ninth"

In the post assessment, Bob shows the limitations of his whole number, additive reasoning. He was able to split three-eighths to create a one-eighth but continued to name fractional units as whole number units. The limitations of utilizing the whole number reasoning and equal groups reasoning becomes evident in Bob's conceptualization of one whole. When asked to create one and one-eighth (as opposed to an improper fraction name such as nine-eighths), Bob loses the unit structure.

<u>Post Assessment: This is three-eighths. Make one and one-eighth.</u>
I: This is three-eighths. [gives new piece]
Bob: Three. This is three.
I: That's three-eights.
Bob: Three-eighths. It's three-eighths.
I: What are you doing there?
Bob: Making three parts [folds a strip of paper].
I: You mean three in there? Okay. And then you're gonna use one of them?
Bob: So then, then it's one-eighth now. [has piece folded into thirds] One.
Bob: Two. Three. Four. Five. Six. Seven. Eight [traces eight of the one-eighths in a line].
I: I want you to make one and one-eighth.
Bob: One and one-eighth [traces one more of the one-eighths separately from the line of eight].
I: Oh. So this is one? [runs finger along line of eight] And that's one-eighth? [points to one piece]
Bob: Mm-hmm.
I: Okay. Would it be the same thing if you had made this one-eighth down here? [points to end of line of eight pieces] Would it be the same or different?
Bob: No. Different.
I: That'd be different.

Bob: Then it would be a ninth.
I: Would it be a ninth? How can it be a ninth?
Bob: Because it would at one. [pause] No and the nine it wouldn't. Or it wouldn't work. 'Cause, 'cause this is one entire length [points at line].
I: Can you circle the one whole?
Bob: There [circles one-eighth].
I: Can you circle the one whole?
Bob: This one? [circles the line of eight one-eighths]
I: Where's the one whole that you made? So if I tacked on another eighth ... I couldn't do that? [motions to one-eighth and puts it at end of line again]
Bob: Nope. You'd have to make these smaller.

Discussion

Throughout the sessions, researchers worked to reorganize Bob's whole number unit coordination to support understanding of fractional units. We hypothesized that Bob would use his whole number units coordination to consider fractional units as multiplicative quantities. To that end, we designed various situations to elicit multiplicative reasoning, both in whole number and in fractions. Two key findings emerged from the analysis related to Bob's reasoning.

First, Bob did utilize his whole number reasoning across situations, whether the situations elicited fractions or not. Yet, because his reasoning was largely additive in nature, this placed limitations on how Bob conceived of fractions as numbers. Namely, Bob's pervasive use of a part whole notions of fractions limited his understanding of fractions as quantities. Second, we saw little to no evidence of Bob's cognitive difference (i.e., working memory) apparent across the sessions. For instance, Bob did not seem to have a need to re-run his activity or re-verbalize his thought processes in the midst of learning. For Bob, the single disabling factor in his reasoning seemed to rest in his pervasive part whole notion of fractions and overreliance on additive reasoning.

Implications from the work include the need to provide students with LDs rich experiences to develop multiplicative reasoning in both whole number and fractions, including tasks that utilize a three level structure in early learning experiences. This student with LDs seemed to "need" more experiences developing multiplicative conceptions as opposed directive forms of instruction that emphasized facts and procedures.

Acknowledgments

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Equity Statement

Our research is based in the notion that students with disabilities can and do advance their notions of multiplicative reasoning in whole number and fractions. Addressing conceptual advancement for this underserved population addressed equity because historically these students have been denied access to opportunities to grow their conceptual knowledge of mathematics, limiting school and life outcomes.

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BACKWARD TRANSFER EFFECTS ON ACTION AND PROCESS VIEWS OF FUNCTIONS

Charles Hohensee
University of Delaware
hohensee@udel.edu

Sara Gartland
University of Delaware
sgart@udel.edu

Laura Willoughby
University of Delaware
lwilloug@udel.edu

This study was conducted to gain understanding about potential influences that learning about quadratic functions has on high school algebra students' action versus process views of linear functions. Pre/post linear functions tests were given to two classrooms of Algebra II students (N=57) immediately before and immediately after they participated in a multi-day unit on quadratic functions. The purpose was to identify ways that their views of linear functions had changed. Results showed that on some measures, students across both classes shifted their views of linear functions similarly. However, on other measures, the results were different across the classes. These findings suggest that learning about quadratic functions can influence students' action or process views of linear. Furthermore, the instructional differences between classes provide insights into how to promote those influences that are productive for students' views.

Keywords: Algebra and Algebraic Thinking, High School Education, Learning Theory

A well-established and widely-held idea in the mathematics education research community is the importance of the relationship between prior ways of reasoning and new learning (e.g., Bransford & Schwartz, 1999; Roschelle, 1995; Vosniadou & Brewer, 1987). However, most of this prior research focuses on the foundational role that prior knowledge plays in new learning. In other words, this research has primarily examined the influence that prior ways of reasoning can have on new learning. This is typically referred to as the *transfer of knowledge* (Lobato, 2008). What has yet to be well examined, especially in the context of mathematics education, is the influence that new learning can have on prior ways of reasoning.

We use the forward and backward direction to distinguish between the two kinds of influences mentioned above. Specifically, we use *forward* to describe influences that go from prior ways of reasoning to new learning and *backward* to describe influences that go from new learning to prior ways of reasoning. While, forward influences (also known as *forward transfer*) have been a well-researched construct in mathematics education, backward influences is a new idea for mathematics education research. Our research addresses this gap by examining backward influences in real algebra classrooms.

We know of only a handful of studies about backward influences in the context of mathematics education, including Hohensee's (2014, 2016) studies on middle school students reasoning about functions, Macgregor and Stacey's (1997) study on secondary students' reasoning about algebra symbols, Young's (2015) study on AP calculus students' reasoning about differentiation and integration, and Moore's (2012) study on undergraduates' reasoning about calculus concepts.

Despite the limited research on this topic, these studies have revealed that backward influences can be unproductive or productive. *Unproductive* backward influences are when students' prior ways of reasoning become muddled or shift to a lower level, because they learn something new (e.g., Macgregor & Stacey, 1997). More theory and research on backward influences are needed to find ways to prevent or at least mitigate unproductive backward influences. *Productive* backward influences are when students' prior ways of reasoning become refined or enhanced because they learn something new (e.g., Hohensee, 2014). More theory and research are needed to uncover ways to promote productive backward influences. The goal for this study was to contribute, as an early step, to developing more understanding about backward influences that occur in high school algebra classrooms, so as to inform ways to inhibit unproductive backward influences and promote productive backward influences.

Theoretical Framework

Our theoretical framework has two parts. The first part involves how we conceptualized backward influences by new learning on prior ways of reasoning. The second part involves the theoretical perspective that guided our study about ways to reason about functions.

Conceptualizing Backward Influences by New Learning on Prior Ways of Reasoning

Backward influences by new learning on prior ways of reasoning in the context of mathematics education have not been well-studied or theorized. However, in other domains, backward influences have been regularly referred to as a form of transfer of learning called *backward transfer*. For example, the effect that learning a second language has on individuals' ability to produce and comprehend their native language has been conceptualized as backward transfer (e.g., Cook, 2003). Therefore, we conceived of backward influences in the context of mathematics education as backward transfer.

Broadening the conceptualization of transfer to include backward influences is a new idea for mathematics education. A theoretical implication from this broad conceptualization is that perhaps one of the well-articulated mathematics education theories of transfer may be a suitable candidate to extend to include backward transfer. Among the theories we considered, Lobato's (2008) *actor-oriented transfer* (AOT) *perspective* is a suitable candidate because of the emphasis in the definition on transfer as an influence. In particular, according to AOT perspective, transfer is "the *influence* of a learner's prior activities on his or her activity in novel situations" (p. 169, emphasis added). An assumption underlying the AOT perspective is that transfer has occurred whenever a learner's prior activities influence their activities in a novel situation (i.e., forward influence), regardless of whether, from an outside-observer's perspective, the new activity involves normative or non-normative performance.

Based on the AOT perspective, we defined backward transfer as the "influence that learning something new has on a learners' prior ways of reasoning about a different or related concept" (Hohensee, 2014, p. 136). Consistent with the AOT perspective, we consider any backward influences by new learning on prior ways of reasoning, regardless of whether they lead to more- or less-normative performance, as cases of backward transfer. A primary reason to study backward transfer in mathematics education contexts is because of the potential that backward transfer unintentionally undermines or weakens learners' prior ways of reasoning (i.e., leads to less-normative performance). Understanding more about backward transfer could enable mathematics educators to develop instructional approaches that minimize unproductive effects.

Note that Young (2015) and Moore (2012), cited previously, adopted the same extension of the AOT perspective to conceptualize backward influences, while Macgregor and Stacey (1997) conceived of backward influences differently (i.e., as interference of learning).

Theoretical Perspective on Ways to Reason about Functions

Within the field of mathematics education research, a number of perspectives on ways of reasoning about functions have been put forth. The perspective we used for this study was the APOS perspective on ways of reasoning about functions (Breidenbach, Dubinsky, Hawks, &

Nichols, 1992). APOS stands for *action, process, object* and *schema*, and represents four ways to view functions (as well as other mathematical concepts). For our study, action- and process-views of functions were most relevant. An action view is described as “any repeatable physical or mental manipulation that transforms objects (e.g., numbers, geometric figures, sets) to obtain objects” (p. 249), and as “a static conception in that the subject will tend to think about it one step at a time (e.g., one evaluation of an expression)” (p. 251). In contrast, a process view is described as “[the] total action can take place entirely in the mind of the subject, or just be imagined as taking place, without necessarily running through all of the specific steps” (p. 249) and as “a dynamic transformation of objects according to some repeatable means…a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done” (p. 251). According to APOS Theory, an action view is a necessary precursor to a process view in the development of conceptions of functions.

One reason the distinction between action- and process-views of functions was important for our study was because, as Breidenbach et al. (1992) point out, “many individuals will be in transition from action to process…the progress is never in a single direction” (p. 251). In other words, an individual can change between action- and process-views of functions. For our study, we wondered if students’ views of linear functions would change because of influences by new learning about quadratic functions. Our research question was the following: *In what ways do algebra students’ prior ways of reasoning about linear functions change, if at all, along the dimension of an action versus process view, after they participate in an instructional unit on quadratic functions?* Next, we describe the methods we used to address this question.

Methods

Participants and Setting

The participants were Algebra II students from two classrooms at two high schools in the Mid-Atlantic region of the US. Both schools were ethnically diverse and drew from an urban population. All students in both classes volunteered to participate in the study (24 in Class 1; 33 in Class 2; N=57). The students reflected the ethnic diversity of their respective schools. The study took place from March to May of the school year. Each class had an experienced teacher: Ms. Henry (Class 1) had 8-years of teaching experience; Mr. Anderson (Class 2) had 17-years of teaching experience.

Procedure

The study procedures followed a pre/post format, in which pre- and post-tests on linear functions bookended an instructional unit on quadratic functions. Before the instructional unit, all students took a 45-minute paper-and-pencil linear functions pre-test. Also, four students from each class participated in one-on-one pre-interviews about their responses on the pre-test.

Students then participated in a multi-day instructional unit on quadratic functions taught by their regular algebra teacher. All lessons were observed and video recorded.

After the instructional unit, all students took a 45-minute paper-and-pencil linear functions post-test. Finally, the same four students from each class who participated in the pre-interviews, participated in one-on-one post-interviews about their responses on the post-test.

Linear Functions Pre- and Post-Tests

We used two versions of the linear functions test. Students were randomly sorted into two equal groups and assigned to either take Version A as the pre-test and Version B as the post-test, or vice versa. Each problem on Version A had a structurally similar problem on Version B. The purpose of having two versions and varying the order was to control for the possibility that students would do better on the post-test if they took the same test pre and post, and for the possibility that items on the versions were not comparable. Each version had one problem based on a graph, one problem based on a table and one problem based on a picture. Results from the picture problem are presented in this report (see Figure 1 for the picture).

Instructional Unit

Each teacher taught their instructional unit on quadratic functions using the Integrated Mathematic II curriculum and were given no direction by the researchers about how to teach the unit. Both teachers typically lectured for part of the class, gave students a seatwork assignment and held a whole-class discussion about the solutions. Ms. Henry's instructional unit was comprised of 16 lessons, and each lesson was 70 min. Mr. Anderson's instructional unit was comprised of 11 lessons, and each lesson was 45-80 min., due to a rotating block schedule. Both teachers seated students in groups. Both teachers were well liked by their students and their lesson were conducted in an orderly manner.

A difference in teaching styles was that Ms. Henry typically used an online platform that students accessed with laptops (i.e., Class Lab), that allowed her to monitor student responses in real time. She would then go over students' responses with the class and provide feedback. In contrast, Mr. Anderson typically had students go the board and write down their solutions to problems. Then, he would go over the responses with the class and provide feedback.

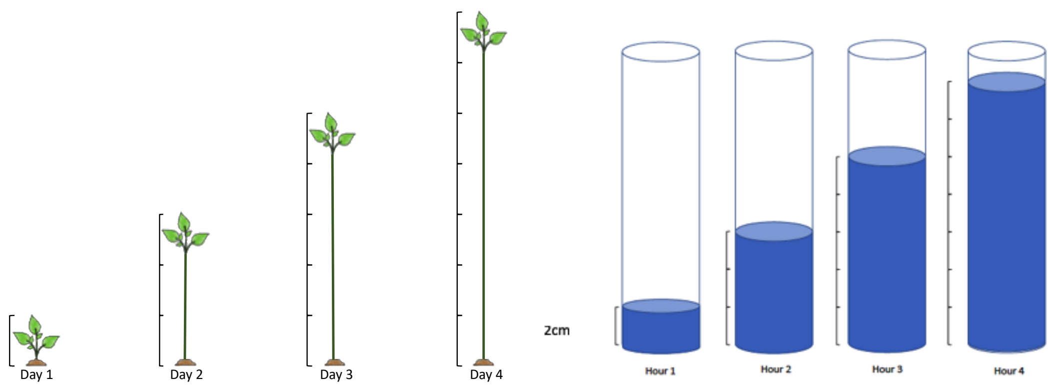

Figure 1: Growing Plant (Version A) and Container Filling with Rain Water (Version B)

A second difference between teachers was the time spent on specific quadratic functions topics. For example, Ms. Henry spent the most lessons on factoring quadratics expressions, real-world problems modelled with quadratic functions and graphing quadratic functions (see Table 1). In contrast, Mr. Anderson spent the most lessons on solving quadratic equations.

Table 1: Breakdown of Lessons Devoted to Each Quadratic Functions Topic

Teacher	Lesson Topic	# of Lessons
Ms. Henry	Lessons on factoring (factoring diamonds, difference of squares, leading coefficient≠1)	4
	Lessons on real-world problems	4
	Lessons on graphing (x/y-intercept, max/min, axis of symmetry)	3
	Lessons on solving by graphing	2

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

	Lessons on solving by factoring	2
	Lesson on standard, vertex and x-intercept equation forms	1
Mr. Anderson	Lessons on solving (square root both sides, complete the square, quadratic formula)	5
	Lessons on graphing (x/y-intercepts, max/min, axis of symmetry, translations/dilations)	2
	Lesson on solving by factoring	1
	Lesson on factoring	1
	Lesson on real-world problems	1
	Lesson on standard, vertex and x-intercept equation forms	1

Data Set

Our data set was comprised of the pre-/post-tests, video-recorded pre-/post-interviews (4 students per class), video-recorded classroom observations and observation field notes.

Data Analysis

Analysis of the data was conducted in three stages: (a) analysis of the pre-/post-test responses and interviews for students who participated in interviews, (b) analysis of the pre-/post-test responses for remaining students, and (c) analysis of fieldnotes from the classroom observations.

Analysis of pre-/post-test responses and interviews for interviewed students. For stage 1 of the analysis, the first author compared the pre-/post-test responses and the pre-/post-interviews for the 4 students from each class that were interviewed. The unit of analysis was each sub-part of each question. From these comparisons, the first author developed an initial set of codes with *grounded theory* (Strauss & Corbin, 1994). Initial codes and supporting evidence were presented to the second and third authors for feedback on the validity of the codes. Once agreement was reached about the interviewed students, the authors proceeded to the next step in the analysis.

Analysis of pre-/post-test responses for the remaining students. Each author took one-third of the remaining tests and again compared sub-parts on one test for each student to the associated sub-part on the other test, noting whenever a change in reasoning fit one of the existing codes. During this analysis, we blinded ourselves to which tests were pre-tests and which were post-tests. The initial codes were in some cases insufficient to capture changes in students' views of functions. In those cases, the coder either refined an existing code or created a new code. Once, each author had coded their dataset, we met in pairs to discuss changes in reasoning that had been identified, to check for agreement on whether the evidence that a change had occurred was compelling and the right code had been applied. Whenever a pair of coders failed to reach consensus on a response, it was flagged and discussed by all three authors until consensus was reached. As such, we continued to refine our grounded theory with a *constant comparison approach* (Strauss & Corbin, 1994). Once all coded responses had been discussed in pairs or by all three authors, and the codes had stabilized, we recoded all the coded responses to ensure that the refined codes fit the entire data set.

Analysis of classroom observations. Analysis of the classroom observations is ongoing. Thus far, the first author has summarized the field notes on a spreadsheet, and identified differences between the two classes in terms of numbers of lessons devoted to specific quadratic functions topic (see Table 1). In future analyses efforts, the three authors will divide up the recordings of the lessons and identify particular episodes in which potential connections to the changes in reasoning observed in the pre-/post-tests exist. Episodes will be transcribed and analyzed to identify the interactions, visual representations, gestures, etc., that could account for to the observed changes in reasoning identified in the pre-/post-test responses.

Results

Analysis of the pre- and post-tests revealed that, in a number of cases, students' views of functions had changed in terms of an action versus a process view. In some cases, the changes in views were similar across the two classes, whereas in other cases, they differed. Next, we show how students' views of functions changed on each of the three sub-parts for the picture problem.

Reasoning with a Build-up Process or a Repeated Build-up Process

Our first finding was that several students changed the way they reasoned on the first sub-part of the pictorial problem, which asked "*Explain in words how to find the height of the plant on day 17*" (Version A) or "*Explain in words how to find the total amount of rainfall if the storm lasts for 11 hours*" (Version B). One reasoning strategy was to repeatedly add the amount of growth, one day or one hour at a time, while simultaneously keeping track of the days or hours, until the height on the desired day or hour was attained. Kaput and West (1994) called reasoning about linear functions in this way a *build-up process*. The second strategy was to multiply the rate of growth by the number of days or hours. Kaput and West (1994) called this an *abbreviated build-up process*. In other words, the abbreviated build-up process required one step, while the build-up strategy required multiple steps. A third strategy was to add a combination of *given* heights and times to find a desired height, such as finding the height on day 11 by doubling the given height on day 4, and adding the given height on day 3. Since this third strategy also required multiple steps, we included this strategy as a build-up process.

Frequency counts supporting this claim. A change from *build-up* to *abbreviated build-up* process was observed for 6 of Ms. Henry's 24 students and 7 of Mr. Anderson's 33 students (see Table 1). A change in the other direction was less common: only 3 of Ms. Henry's students and 2 of Mr. Anderson's students. The remaining students maintained one strategy on both tests or did not provide sufficient responses for us to determine if their reasoning had changed. Note that the more common change from build-up to abbreviated build-up process was similar across classes.

Table 2: Change Involving Build-up Process and Abbreviated Build-up Process

Teacher	Students per class	From build-up to abbreviated build-up	From abbreviated build-up to build-up	Maintained build-up: Maintained abbreviated build-up
Henry	24	6	3	12:2
Anderson	33	7	2	8:3

Change interpreted in terms of action versus process view of functions. We interpreted a change from reasoning with a build-up process to reasoning with an abbreviated build-up process as a possible shift away from an action view towards a process view of functions, because with a shift to an abbreviated build-up process, students reasoned as if they knew, without calculating each separate change in height, that all the changes in height would be the same and that they could simply multiply by how many changes in height there were. This aligns with Asiala et al. (1997), who described the process view as "it is not necessary to perform the operations, but to only think about them being performed" (p. 8).

Reasoning about Independent and Dependent Variables

Our second finding was that several students changed their way of reasoning on the second sub-part of the pictorial problem, which asked "*Can you find the day* [independent variable] *the plant was measured if you were given the height* [dependent variable]*? If yes, explain how. If no, explain why not*" (Version A), or "*Can you find the hour* [independent variable] *the rain water was measured if given the height* [dependent variable]*? If yes, explain how. If no, explain why not*" (Version B). Students who exhibited this change, either reasoned on the pre-test *it is not possible* to use the dependent variable to find the corresponding value of the independent variable and on the post-test reasoned *it is possible*, or vice versa. Note that *it is possible* is correct because the independent variable in a linear function can be found (i.e., $x = (y - \mathrm{b}) / \mathrm{m}$).

Frequency counts supporting this claim. This change was more common for Ms. Henry's students than for Mr. Anderson's students, by a ratio of 9:2 (see Table 2). Additionally, Ms. Henry's students who exhibited this change were almost evenly split on changing from *it is* to *it is not possible* (4 students) versus changing from *it is not* to *it is possible* (5 students). In contrast, all of Mr. Anderson's students who exhibited this change, went from *it is not* to *it is possible* (2 students). The remaining students maintained the same reasoning on both tests or did not provide sufficient responses for us to determine if their reasoning had changed.

Table 3: Change Involving Finding Independent Variable from Dependent Variable

Teacher	Students per class	From *not possible* to *possible*	From *possible* to *not possible*	Maintained *possible*: Maintained *not possible*
Henry	24	5	4	5:1
Anderson	33	2	0	15:1

Change interpreted in terms of action versus process view of functions. We interpreted a change in reasoning from *it is not* to *it is possible*, as a possible shift toward a process view of functions and a change in the other direction as a possible shift toward an action view of functions. This interpretation is based on Asiala et al.'s (1997) characterization of a process view of functions as when a person can "reverse the steps of the transformation" (p. 7).

Reasoning about Specific or General Intervals of Change

Our third finding was that several students changed their reasoning on the third sub-part of the pictorial problem. Version A asked:

> You have to leave the plant in your office over the weekend. You did not measure the plant for 2.5 days. The plant grows at the same rate the whole time. How much did the plant grow in the 2.5 days you were gone? Show any work that helped you decide.

Version B asked:

> You fall asleep while watching TV. You did not measure the rain water for 3.5 hours. It rained the whole time at the same rate. How much rainwater was collected during the 3.5 hours that you were sleeping? Show any work that helped you decide.

Students either reasoned about *specific* or *general* intervals on this problem. When they reasoned about specific intervals, they either found the height of the plant or rainwater on day 2.5 or at hour 3.5 or the height 2.5 days or 3.5 hours after the last height depicted in the picture. When students reasoned about a general interval, they found the amount the plant would grow or the rainwater would rise over *any* 2.5-day or 3.5-hour interval in general. Notice that reasoning about general intervals is correct because, for each version of the problem, the amount of change in the independent variable is for a general interval, not a specific interval.

Frequency counts supporting this claim. Ms. Henry's and Mr. Anderson's classes showed opposite (and mixed) trends in changes in reasoning (see Table 4). A greater number of Ms. Henry's students went from reasoning about a *specific interval* to reasoning about a *general interval* (7 students), than vice versa (3 students). In contrast, a smaller number of Mr. Anderson's students went from reasoning about a specific interval to reasoning about a general interval (3 students), than vice versa (8 students).

Change interpreted in terms of action versus process view of functions. We interpreted reasoning with *general intervals* as more consistent with a *process view* and reasoning with *specific intervals* as more consistent with an *action view*. Our rationale was that to reason about any general 2.5-day or 3.5-hour interval, an individual would need to think about changes in height across all 2.5-day or 3.5-hour intervals in general, without individually calculating changes in heights for all those intervals. This aligns with Asiala et al.'s (1997) description of the process view as not needing to perform all the operations to think about the results of operations.

Table 4: Change Involving Reasoning about Specific or General Intervals

Teacher	Students per class	From *specific* to *general*	From *general* to *specific*	Maintained *specific*: Maintained *general*
Henry	24	7	3	2:1
Anderson	33	5	8	5:3

Discussion

To summarize our results, we saw one type of change in reasoning on each of the three sub-parts for the pictorial problem. This suggests that backward transfer effects by new learning about quadratic functions on prior ways of reasoning about linear functions may be a fairly frequent occurrence. However, the effects were more varied than we anticipated: they occurred in both classes or in one class only, and in the same direction or in opposite directions. We think these results suggest that backward transfer effects, in real classrooms and with teachers who are not purposefully trying to produce these effects, may be somewhat messy.

Messy results for backward transfer effects are significant because they suggest backward transfer may be difficult for researchers to detect unless deliberately tuned to them. Our messy results may also help explain why teachers may be unaware of backward transfer effects in their students.

To add to the messiness, our findings suggest backward transfer effects can be either productive or unproductive. The evidence of productive backward transfer effects is significant because it suggests there are aspects of learning about quadratic functions instruction that could be emphasized to further enhance *productive* influences on students' views of linear functions. For example, the finding that a number of students changed productively from a build-up strategy to an abbreviated build-up strategy suggests that, with further emphasis, even more students could be supported to change along this dimension (e.g., such as by exploring how students could engage in a kind of abbreviated build-up strategy in a quadratic context).

The findings of unproductive backward transfer effects are also significant because they suggest there are aspects of learning about quadratic functions that could be emphasized to inhibit *unproductive* influences on students' views of linear functions. For example, the finding that a number of Mr. Anderson's students went from reasoning with a general interval to reasoning with a specific interval suggests that researchers and educators should look for ways to emphasize reasoning with general intervals to inhibit or even eliminate this unproductive backward transfer effect. Our future research on the topic of backward transfer in the context of linear functions conceptions and quadratic functions instruction will test some of these ideas.

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INTERACTING WITH DYNAMIC COMPUTER ACTIVITIES IMPACTS COLLEGE ALGEBRA STUDENTS' MATH ATTITUDES AND PERFORMANCE

Heather Lynn Johnson
University of Colorado Denver
heather.johnson@ucdenver.edu

Amber Gardner
University of Colorado Denver
amber.gardner@ucdenver.edu

Amy Smith
University of Colorado Denver
amy.l2.smith@ucdenver.edu

Gary Olson
University of Colorado Denver
gary.olson@ucdenver.edu

Xin Wang
RMC Research
wang@rmcres.com

Opportunities matter when it comes to students' math attitudes and performance. Over two semesters, across treatment sections of college algebra students, we implemented a set of dynamic computer activities linking animations and graphs. Across all sections, we administered a fully online survey of students' attitudes toward math. Using mixed methods, we analyzed students' attitudes toward math and their performance on a course final exam. At the end of each semester, we found statistically significant differences between treatment and comparison students' perceived competence toward math. Furthermore, treatment students outperformed comparison students on the course final exam, with statistically significant differences on an item linked to the dynamic computer activities. When students have opportunities to interact with dynamic computer activities, it can impact their math attitudes and course performance.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Post-Secondary Education; Research Methods; Technology

Opportunities matter when it comes to students' math attitudes and performance. Ideally, College Algebra courses would be a place where students could gain opportunities to develop their mathematical competence. However, too often students experience College Algebra courses as gatekeepers rather than opportunity makers (e.g., Gordon, 2008; Herriott & Dunbar, 2009). To address this opportunity problem with College Algebra, we investigate the question: How might College Algebra students' opportunities to interact with dynamic computer activities, linking animations and graphs, impact their math attitudes and course performance?

Strategic uses of educational technology, particularly those that involve opportunities for students to interact with others to discuss mathematical ideas, show promise for promoting math learning for underserved students (Kitchen & Berk, 2016). In an exploratory study, researchers implemented dynamic computer activities in a middle school in a high poverty neighborhood, serving primarily students of color (Schorr & Goldin, 2008). During the intervention, researchers found that students demonstrated both sophisticated mathematical reasoning and positive affect toward math, including emotions such as elation (Schorr & Goldin, 2008). We were inspired by this link between the opportunities for reasoning afforded by educational technology and positive affect toward math, and we aimed to investigate such relationships on a larger scale.

We focus on an aspect of students' mathematical affect—their attitudes toward math (Ding, Pepin, & Jones, 2015; Di Martino & Zan, 2010; Pepin, 2011; Zan, Brown, Evans, & Hannula, 2006). Following Di Martino and Zan (2010), we view students' attitudes as multidimensional, encompassing three interrelated dimensions: emotional disposition toward mathematics, perceived competence toward mathematics, and a vision of what mathematics is (p. 44). For example, students may like math because it makes them think (emotional disposition), may consider themselves as capable of doing math when they practice (perceived competence), and they may view math to be difficult, but fun (vision of mathematics).

Researchers have found emotional disposition and perceived competence to be important dimensions when it comes to university students' persistence in mathematics courses (Bressoud, Carlson, Mesa & Rasmussen, 2013; Ellis, Fosdick, & Rasmussen, 2016). In a large study of university Calculus I students, Bressoud and colleagues (2013) found that students' confidence and enjoyment of math *decreased* by the end of Calculus I, despite the fact that the majority of students completing the survey at the end of the course demonstrated success in the course, as measured by a final grade of A, B, or C. Given the findings from students' experiences in Calculus I, we were particularly interested in courses that served as prerequisites for Calculus I, such as College Algebra.

The study we report is part of a larger research project, for which we designed an intervention providing College Algebra students with opportunities to interact with a set of dynamic computer activities (see Johnson, McClintock, Kalir, & Olson, 2018). We designed the activities to promote students' covariational reasoning, a key form of reasoning that can engender students' conceptions of rate and function (Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Thompson & Carlson, 2017). In the larger research project, we have three aims: to measure students' covariational reasoning (see Johnson, Kalir, Olson, Gardner, Smith, & Wang, 2018), to promote students' positive attitudes toward math, and to promote students' outcomes in College Algebra. Here, we focus on the latter two aims. Using mixed methods, we demonstrate that our intervention resulted in differences in students' attitudes toward math and in students' course outcomes, as measured by a final exam.

Theoretical and Conceptual Underpinnings

Designing Dynamic Computer Activities to Promote Students' Covariational Reasoning

We integrated different theoretical perspectives to design dynamic computer activities to promote students' covariational reasoning. Each of the dynamic computer activities linked computer animations with dynamic graphs. In particular, we integrated Thompson's theory of quantitative reasoning (Thompson, 1994; 2002) with Marton's variation theory (Marton, 2015; Kullberg, Kempe, & Marton, 2017). Drawing on Thompson's theory, we problematized students' conceptions of attributes represented in the tasks (e.g., distance, height). We theorized how students might conceive of task attributes as capable of varying and possible to measure. Drawing on Marton's theory, we problematized students' discernment of different aspects of the tasks (e.g., axes of a Cartesian graph). We theorized what students might discern as we designed differences (e.g., Cartesian graphs with the same attributes represented on different axes) against a background of invariance (e.g., a situation involving a toy car moving along a track.)

Our Perspective on Students' Attitudes Toward Math

With our perspective on students' attitudes toward math, we intend to move beyond McLeod's (1992) categories of beliefs, attitudes, and emotions as comprising distinct components of mathematical affect. We ground our perspective on students' attitudes toward math in the work of scholars aiming to explicate the multidimensionality of students' attitudes toward math (Di Martino & Zan, 2010; Zan et al., 2006). From our perspective, students' math attitudes are not distinct from their emotions. Rather, students' attitudes toward math include students' emotions, as well as other dimensions. Accordingly, we adopt Di Martino and Zan's (2010) multidimensional perspective on students' attitudes toward math. That is, students' attitudes toward math comprise three interrelated dimensions: their emotional disposition toward math, their perceived competence toward math, and their vision of what mathematics is.

Method

The research we report is part of a broader research project taking place across all face to face sections of College Algebra at a public university situated in the downtown area of a large city in the midwestern US. The university serves large proportions of students who identify as students of color and are the first generation of their family to attend college. In fall 2017, 59% of incoming freshmen identified as students of color, and 51% of all freshmen were first generation college students.

Intervention: Dynamic Computer Activities

At the university where we conducted this study, College Algebra is divided in to recitation and lecture components. Each section of College Algebra includes both recitation and lecture components, with a recitation occurring before each lecture. Students always have different recitation and lecture instructors. Typically, recitation instructors are graduate students, and the instructors are faculty members. We implemented a set of dynamic computer activities across selected recitations. In spring 2018, we were still developing some of the activities. Hence, in spring 2018 we implemented five dynamic computer activities across two different recitations. In fall 2018, we implemented seven dynamic computer activities across three different recitations.

To increase students' access to the dynamic computer activities, we developed them on a freely available platform—Desmos—in collaboration with Meyer, the chief academic officer of Desmos. As typical with Desmos, each of our dynamic computer activities involves a series of screens that students move through. There are five main components to each activity. First, students watch a video that depicts a moving object, such as a toy car moving along a track, along with a description of attributes on which the activity will focus (e.g., total distance traveled and distance from a stationary object.) Second, students represent variation in each attribute by moving dynamic segments along vertical and horizontal axes of a Cartesian plane. This design choice was an effort to operationalize Thompson's (2002) discussion of students' use of fingers as tools to represent variation. Third, students sketch a single graph representing a relationship between both attributes. In each of the second and third components, students have opportunities to get computer feedback on their work, a hallmark of Desmos activities. Fourth, students repeat the second and third components for a new Cartesian graph with attributes on different axes. This design choice was inspired in part by tasks developed by Moore and colleagues (e.g., Moore, Silverman, Paoletti, & LaForest, 2014). Fifth, students answer a reflection question which asks them to make sense of another student's reasoning.

We designed the Desmos activities to promote students' covariational reasoning, as well as their conceptions of graphs as representing relationships between attributes that are capable of varying and possible to measure. Through the reflection questions, we aimed to promote students' sense making, rather than rushing to judgments (Johnson, Olson, Gardner, & Smith, 2018). We conjectured that opportunities to engage in covariational reasoning could promote students' productive attitudes toward math and successful outcomes in college algebra.

Treatment and Comparison Groups

Spring 2018 was our first semester implementing the dynamic computer activities. In spring 2018, there were nine sections of College Algebra. One section was a treatment section, and the other eight sections were comparison sections. We selected the treatment section, because the lecture instructor is a co-principal investigator on our larger project, and was willing to implement the Desmos activities before we rolled them out to a larger number of sections.

In fall 2018, there were 13 sections of College Algebra. Prior to the beginning of the semester, we invited recitation instructors to participate in a semester long professional development (PD). The PD had three main aims, to provide recitation instructors opportunities to (1) learn who their students are as humans; (2) to engage in covariational reasoning; and (3) to implement the Desmos activities to promote their students' covariational reasoning. We accepted all recitation instructors who volunteered to participate, resulting in six recitation instructors teaching a total of 10 sections. Hence, in fall 2018, 10 sections were treatment sections and three sections were comparison sections.

Attitude Study

To investigate students' attitudes toward math, we adapted methods used by Pepin and colleagues (Ding et al., 2015; Pepin, 2011). We created an online survey shown in Table 1. The first three questions are the same as those from Pepin's (2011) survey. In addition, we added two questions specific to students' attitudes toward graphs, rather than math in general, because graphs were specific to our intervention.

Table 1: The Attitude Survey

Attitude Survey Questions
I like/dislike math because __________
I can/ cannot do math because __________
Mathematics is __________
I like/dislike graphs because __________
I can/ cannot make sense of graphs because __________

During spring 2018 and fall 2018, we administered the attitude survey four times: Once at the beginning of the semester (in week 2 or 3), and once at the end of the semester (in week 13). To ensure that technology was working and to answer any questions, a research team member was present at each administration of the survey.

Two qualitatively code the attitude survey data, a subset of our team built a coding rubric based on preliminary analysis of pilot data collected in fall 2017. We had initially planned to code students' responses into three categories, the same way as did Ding et al. (2015): Like/Can, Dislike/Cannot, and Neutral/Mixed. However, as we discussed responses from the pilot data set, we determined that we needed two additional categories—Ambiguous and Detached—to capture the scope of students' responses. In our view, these mutually exclusive categories do not position themselves along a line, with positive and negative being at opposite ends of the continuum. Rather, the categories begin and end in a more knotted way, as shown in Figure 1.

Figure 1: Coding for Complexity in Students' Attitudes Toward Math

Table 2 includes sample responses from our data set, with two examples for each of the codes. Responses coded as *Positive* included statements of like/can, while responses coded as *Negative* included statements of dislike/cannot. All of the Positive/Negative responses shown in Table 2 include "because" clauses, in which students explain their response. Not all students included these clauses, which we did not require. We did not code differently when students used a clause to qualify their response. Responses coded as *Mixed* included specific statements of both like/dislike or can/cannot. Often these statements included the word "but" or "however" to indicate the juxtaposition of positive and negative attitudes. Responses coded as *Ambiguous* could cross multiple attitudes. For example, students may state that they like or dislike math because it is challenging. Responses coded as *Detached* separated the mathematics from the humans engaging in mathematical activity. Typically, students responding this way described about math or graphs as things that are "out there," rather than products of their activity.

Table 2: Sample Attitude Survey Responses from our Data Set

Attitude Survey Codes	Example 1	Example 2
Positive: Like/Can	I like math because it challenges me to keep trying and learn more.	I can make sense of graphs because of practice.
Negative: Dislike/Cannot	I dislike mathematics because it is too stressful and complicated.	I cannot make sense of graphs because they don't make as much sense as equations do to me.
Mixed	I only like math when I understand it. When I understand it, I enjoy it. But most of the time I feel like I'm lost. So sometimes I dislike math.	I can make sense of graphs because I know how some functions move by heart. I cannot make sense of graphs because I do not know how all functions move by heart.

Ambiguous	It is challenging.	I look at the points and read info given.
Detached	Math is the universal language.	They (graphs) are just a visual representation of inputs and outputs.

To qualitatively code the attitude survey data, two graduate students served as coders. First, they received training with coding rubric by participating in meetings to discuss the Fall 2017 pilot data set. Next, they independently coded responses, identified disagreements, and then calibrated the disagreements via discussion, consulting with an expert coder if disagreements could not be resolved.

To quantitatively analyze the attitude survey data, we conducted chi square analysis using percent responses in each coding category. In each of spring 2018 and fall 2018, we used the following groups: Treatment (Pre) vs. Comparison (Pre); Treatment (Post) vs. Comparison (Post); Treatment (Pre) vs. Treatment (Post). We examined the data for statistically significant results, then we developed explanations to account for those results.

Outcomes Study: Final Exam

We collected data from students' performance on the common final exam, both in terms of letter grade (ABCDF) and raw score. In addition, we collected item level data for a multiple choice covariation item that we included on the final exam. In the covariation item, students selected a graph to represent a situation involving a relationship between variables. Some of the graphs were unconventional, like the graphs in the dynamic computer activities with which the treatment students interacted. We scored the covariation item as correct/incorrect, with correct responses receiving a score of 1 and incorrect responses receiving a score of 0.

To quantitatively analyze the final exam data, we also conducted chi square analysis using percent responses in each coding category. In each of spring 2018 and fall 2018, we used the following groups: Treatment (ABC letter grade) vs. Comparison (ABC letter grade); Treatment (raw score) vs. Comparison (raw score); Treatment (covariation item) vs. Comparison (covariation item). As we did for the attitude survey data, we examined the final exam data for statistically significant results, then developed explanations to account for those results.

Results

We organize the results into sections devoted to the attitude study and the outcomes study. In each section, we report results by semester: Spring 2018 and Fall 2018.

Attitude Study

At the beginning of both semesters, we found statistically significant differences between students' perceived competence in treatment and control groups, with treatment groups entering with more negative perceptions of their competence. By the end of each semester, each group demonstrated more positive perceptions of their competence, and we no longer found statistically significant difference between treatment and control groups. In spring 2018, we found statistically significant results when analyzing students' perceptions of their competence with graphs. In fall 2018, we found statistically significant results when analyzing students' perceptions of their competence with math writ large.

In the next subsections, we include a number of tables to report results of quantitative data analysis. In tables 3-6, the comparison group is shown in the top row, and the treatment group is shown in the bottom row. In table 7, the treatment group is shown in both rows. In tables 3-7, we report results in terms of percentages of student responses coded in each category. In the narrative, we provide specific numbers for the treatment and comparison groups.

Spring 2018: Attitude. In Spring 2018 the comparison and treatment groups demonstrated statistically significant differences in their perceived competence toward graphs. In the first administration of the attitude survey, the treatment group (n=26) demonstrated a more negative perceived competence toward graphs than did the comparison group (n=112). Table 3 shows the percentages of students' responses coded in each category.

Table 3: Spring 2018 Comparison Pre (top) vs. Treatment Pre (bottom)

	Like/Can	Dislike/Cannot	Mixed	Ambiguous	Detached	X^2, p
I can/ cannot make sense of graphs because __________	55.4%	15.2%	8.9%	13.4%	7.1%	13.83* p=0.010
	34.6%	38.5%	3.8%	3.8%	19.2%	

By the end of spring 2018, there were no statistically significant differences in students' perceived competence toward graphs between the treatment group (n=19) and the comparison group (n=77). Both groups saw increases in the number of students who demonstrated more positive perceived competencies toward math. Table 4 shows the percentages of students' responses coded in each category.

Table 4: Spring 2018 Comparison Post (top) vs. Treatment Post (bottom)

	Like/Can	Dislike/Cannot	Mixed	Ambiguous	Detached	X^2
I can/ cannot make sense of graphs because __________	58.4%	10.4%	11.7%	11.7%	7.8%	5.23
	57.9%	26.3%	0.0%	10.5%	5.3%	

Fall 2018: Attitude. In Fall 2018 the comparison and treatment groups demonstrated statistically significant differences in their perceived competence toward math. In the first administration of the attitude survey, the treatment group (n=251) demonstrated a more negative perceived competence toward math than did the comparison group (n=64). Table 5 shows the percentages of students' responses coded in each category.

Table 5: Fall 2018 Comparison Pre (top) vs. Treatment Pre (bottom)

	Like/Can	Dislike/Cannot	Mixed	Ambiguous	Detached	X^2, p
I can/ cannot do mathematics because __________	57.8%	9.4%	17.2%	10.9%	4.7%	13.86** p=0.008
	54.6%	21.1%	19.9%	2.8%	1.6%	

By the end of fall 2018, there were no statistically significant differences in students' perceived competence toward math between the treatment group (n=204) and the comparison group (n=45). As was the case for spring 2018, both groups saw increases in the number of students who demonstrated more positive perceived competencies toward math. Table 6 shows the percentages of students' responses coded in each category.

Table 6: Fall 2018 Comparison Post (top) vs. Treatment Post (bottom)

	Like/Can	Dislike/Cannot	Mixed	Ambiguous	Detached	X^2
I can/ cannot do mathematics because __________	66.7%	11.1%	11.1%	11.1%	0.0%	5.44
	60.3%	24.0%	10.8%	4.9%	0.0%	

Because of the size of the treatment group in fall 2018, we were able to compare differences between the beginning and end of the semester. Within the treatment group, there were statistically significant differences in perceived competence toward math from the beginning of the semester (n=251) to the end of the semester (n=204). Table 7 shows the percentages of students' responses coded in each category.

Table 7: Fall 2018 Treatment Pre (top) vs. Treatment Post (bottom)

	Like/Can	Dislike/Cannot	Mixed	Ambiguous	Detached	X^2, p
I can/ cannot do mathematics because __________	54.6%	21.1%	19.9%	2.8%	1.6%	11.60* p=0.021
	60.3%	24.0%	10.8%	4.9%	0.0%	

Outcomes Study: Final Exam

In spring and fall 2018, there were no statistically significant differences between the number of students who passed the final exam (ABC) in the treatment or comparison groups. In both spring 2018 and fall 2018, the treatment group outperformed the comparison group. In spring 2018, the final exam passing rate for the treatment group was 70.8% and for the comparison group was 65.0%. In fall 2018, the final exam passing rate for the treatment group was 60.8% and for the comparison group was 59.4%.

The analysis of final exam raw scores revealed similar findings to the final exam passing rates. In both spring 2018 and fall 2018, the treatment group outperformed the comparison group. In spring 2018, the treatment students outperformed the comparison students by 8 points (153.8 vs 145.8). In fall 2018, the treatment students also outperformed the comparison students, but only by 2.2 points (143.8 vs 141.5). The score differences were not statistically significant in either semester.

We found statistically significant differences in students' performance on the final exam covariation item that was linked to the dynamic computer activities. In both spring 2018 and fall 2018, the treatment group outperformed the comparison group. The p value in spring 2018 (p=0.000) was stronger than the p value for fall 2018 (p=0.013).

Table 7: Final Exam Item: Comparison vs. Treatment

Semester	Group	Mean Score	Standard Deviation
Spring 2018	Comparison (n=143)	0.31	0.46

Spring 2018	Treatment (n=25)	0.84***	0.37
Fall 2018	Comparison (n=71)	0.28	0.5
Fall 2018	Treatment (n=258)	0.43*	0.5

Discussion

We provided evidence to support our claim that students' interaction with dynamic computer activities impacted their attitudes toward math and their performance on the course final exam. Our intervention impacted a particular dimension of students' attitudes toward math—their perceived mathematical competence, which has been shown to be an important dimension impacting students' persistence in mathematics courses, such as Calculus I (e.g., Bressoud et al., 2013). We implemented this study in conjunction with an earlier, National Science Foundation funded project promoting students' active learning in College Algebra. Hence, we are encouraged that comparison students also reported more positive perceptions of their competence in graphs and math, respectively.

There are differences in the numbers of students responding to the attitude surveys, with numbers declining from the administration at the beginning of the semester (pre) to the administration at the end of each semester (post). Student attrition was a main cause for the differences. The students responding at the beginning of the semester represent all those students who began College Algebra. The students responding at the end of the semester represent those students who continued to persist in the course. Hence, our pre and post groups are not exactly the same student population.

Students have complex attitudes toward math. Langer-Osuna and Nasir (2016) have called for researchers to develop methods that acknowledge the humanity of students' experiences. By coding students' responses to allow for that complexity—to extend beyond a continuum of positive or negative in students' affect, we have responded to the call. Furthermore, our results suggest that Schorr & Goldin's (2008) findings are applicable on a broader scale, to university students as well as middle grades students. That is, opportunities for reasoning afforded by interactions with educational technology can promote students' positive attitudes toward math.

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SECONDARY MATHEMATICS TEACHERS' EFFORTS TO ENGAGE STUDENTS THROUGH ACADEMIC AND SOCIAL SUPPORT

Amanda Jansen
University of Delaware
jansen@udel.edu

James A. Middleton
Arizona State University
jimbo@asu.edu

Adi Wiezel
Arizona State University
awiezel1@asu.edu

Catherine Cullicott
Arizona State University
ccullico@asu.edu

Xiaoxue Zhang
University of Delaware
veraz@udel.edu

Gabriel Tarr
Arizona State University
Gabriel.Tarr@asu.edu

Kelly Curtis
University of Delaware
kbcurtis@udel.edu

Teachers' efforts to support students, both academically and socially, can play a role in how high school students productively engage with mathematics in the moment. To examine the connection between teacher support and student engagement, we conducted an exploratory mixed-methods study combining data from 20 high school classroom observations with student self-reports taken during the observed activity. Our findings indicate that when teachers provide academic support to their students during a lesson, they are also likely to provide social support. Higher teacher support of both kinds correlates with higher student self-efficacy, as well as social and cognitive engagement. Investigating relationships between observations of teaching and students' self-reports of engagement in-the-moment is a potentially revealing approach for uncovering engaging instructional strategies in secondary mathematics classrooms.

Keywords: High School Education; Affect, Emotion, Beliefs, and Attitudes; Classroom Discourse

Supporting secondary students' engagement with mathematics is a problem that can be addressed through instructional design. Typically, students' self-efficacy, enjoyment, and sense of the utility of mathematics decreases as they move through middle school and into high school, and the longer students are in high school, the more disengaged they tend to become (e.g., Chouinard & Roy, 2008). In this study, we explore the ways in which teachers' attempts to support students, both academically and socially, can play a role in how high school students productively engage with mathematics in the moment.

Mathematics Engagement

Engagement in school is students' expression of affect, beliefs about themselves, sense of belonging, and observable behaviors in the school setting (Jimerson, Campos, & Greif, 2003). Engagement is thus a complex meta-construct that simultaneously accounts for cognitive, affective, and socio-behavioral dimensions (Fredricks, Blumenfeld, & Paris, 2004). More specifically, mathematics engagement is the interactive relationship between students and the subject matter. It is manifested in the moment through expressions of behavior and experiences of emotion and cognitive activity, and it is constructed through opportunities to do mathematics, as situated in both current and past experiences (c.f., Middleton, Jansen, & Goldin, 2017).

For students to learn mathematics, they must be engaged. For example, in a study of almost 4,000 middle school and high school students in Western Pennsylvania, researchers found that higher levels of cognitive, behavioral, emotional, and social engagement predicted students' course grades in mathematics (Wang, Fredricks, Yea, Hofkens, & Linn, 2016). According to Greene (2015), it is well-established in prior research that motivation constructs such as students' self-efficacy support students' engagement in ways that lead to learning. To understand how instruction can support learning, it is important to understand how instruction supports students' motivation and engagement. To do so, we must account how students experience engagement and how teachers' practice creates a context for students to engage.

Students' engagement is malleable and situated. Engagement is influenced by teachers' instructional practices in the moment and by the classroom climate (Anderson, Hamilton, & Hattie, 2004). Mathematical work, social interactions, and identity enactments differ from classroom to classroom. Understanding how teaching influences engagement requires attending to students' experiences, both collectively and subjectively, as they attend to important social and psychological features of their learning environment (Fraser, 1989).

Potentially Engaging Mathematics Teaching Practices

Mathematics instruction is considered to be stronger when teachers provide students with both *social support* for working together on content and *academic support* for accessing rigorous mathematical content (Shernoff et. al., 2016). Academic and social support can take a variety of forms. Academic support may include opportunities for sense-making and reasoning (Stein, Grover, & Henningsen, 1996), opportunities to make conceptual connections (Hiebert & Lefevre, 1986), pressing students to explain their thinking (Engle & Conant, 2002; Kazemi & Stipek, 2001), providing students with specific and detailed feedback (Stipek, Salmon, Givvin, Kazemi, Saxe, & MacGyvers, 1998), and opportunities to solve mathematics tasks in context (Koedinger & Nathan, 2004), or some combination of these. Social support may include motivational discourse with a focus on learning, positive affect, and encouragement of collaboration with peers (Turner, Midgley, Meyer, Gheen, Anderman, Kang, & Patrick, 2002), positioning students as competent (Cohen & Lotan, 1995; Gresalfi, Martin, Hand, & Greeno, 2009), accountability practices in the classroom (Horn, 2017), or some combination of these.

It is worth investigating whether and how these productive classroom practices support students' engagement at the high school level. Many of these prior studies were conducted in middle grades or upper elementary grades, and not all of these prior analyses of strong mathematics instruction were empirically linked to students' engagement. Thus, the research question that guides this study is: *What ways of providing academic and social support during instruction lead to productive engagement in-the-moment in secondary mathematics classrooms?*

Methods

This exploratory, mixed-methods study was conducted using the first year of data (pilot data) from a three-year NSF-funded study. In Spring 2018, we observed lessons and surveyed students in on-grade level ninth-grade mathematics classes in two states (one in the Southwestern region of the United States and one in the Mid-Atlantic region). We chose these locations because schools in these areas take different curricular approaches: integrated mathematics (Mid-Atlantic) and topics-based courses (Southwest). Mid-Atlantic courses were titled Integrated Math 1 or Integrated Math 2. Southwest courses were Algebra I or Geometry. The three Mid-Atlantic schools implemented a block schedule with approximately 90-minute class periods. In the Southwest, the class periods were approximately 50 minutes long.

We gathered data from six schools – three from each state. In the Mid-Atlantic, the schools' demographics ranged from 12-34% low income, 25-60% white, 27-47% Black, and 6-21% Latinx. In the Southwest, the schools' demographics ranged from 76-94% low income, 1-6% white, 1-16% Black, and 77-96% Latinx.

Teachers were recruited by soliciting nominations of teachers from district curriculum supervisors and the mathematics coaches. We invited nominated teachers to participate in the study. The nine participating teachers averaged 10.6 years of teaching (min = 1 year, max = 26 years). All had completed a Master's degree or were in progress of doing so. One of the teachers in the Mid-Atlantic identified as Black, two Southwest teachers identified as Latinx, and one Southwest teacher identified as Asian. The rest of the participating teachers identified as White.

Teacher-selected, Potentially Engaging Activities

We observed 20 lessons. Lessons were selected by asking teachers to identify a lesson with an activity that they conjectured would be engaging for their students. The teachers provided written rationales for their conjectures about why this activity would engage their students. We video recorded the entire lesson, but we focused our analysis on these teacher-selected episodes. In the Mid-Atlantic, we analyzed 12 lesson episodes (four lessons from each of three teachers) and in the Southwest, we analyzed eight lesson episodes (two lessons from two teachers, one lesson from four other teachers).

These observations were paired with Experience Sampling Method (ESM) surveys (c.f., Shernoff, Ruzek, & Sinha, 2017) for the teacher-selected episode. Administering the ESM immediately after the teacher-selected activity allowed us to capture students' in-the-moment interpretations of their experiences during that activity. Each item was ranged on a Likert Scale from 1 to 5 points.

- *Interest* – 3-items: "I enjoyed the activity I was just working on," The activity I was just working on was personally relevant to me," and "I think the topic covered in the activity I was just working on is interesting" (α=.783).
- *Cognitive Engagement* – 3-items: "How hard were you trying during the activity you were just working on?", "I was on task during the activity I was just working on," and "How hard were you concentrating on the activity you were just working on?" (α = .782).
- *Perceived Instrumentality* – 3-items: "I will use what I learned from the activity I was just working on when I grow up," "I will use what I learned from the activity I was just working on in future courses," "How I performed on the activity I was just working on will affect my future success" (α = .758).
- *Self-Efficacy* – 3-items: "Rate how much you understand the math covered in the activity you were working on," "I felt successful in the activity I was just working on," and "I felt challenged by the activity I was just working on" (reverse-coded) (α =.548).
- *Social Engagement* – 4-items: "I built on others' ideas during the activity I was just working on," "I felt like my contribution was respected during this activity I was just working on," "I felt supported by my teacher in the activity I was just working on," and "I had the opportunity to ask questions during the activity I was just working on" (α = .709).

Data Analysis

Our unit of analysis for coding observations was an episode consisting of a teacher-selected, potentially engaging activity. We analyzed episodes surrounding this activity using rubrics (ranging on a 0 to 3-point scale) developed to document instructional practices which have the potential to support secondary students' engagement, as well as the prevalence and quality of those practices. Seven specific rubrics assessed the *academic support* that teachers provided and seven specific rubrics assessed *social support* that teachers provided:

Academic Support – Seven specific rubrics: (1) Sense-making and reasoning, (2) Connections and coherence, (3) Pressing students to explain, (4) Contexts of tasks, (5) Mathematical correctness, (6) Mathematical language precision, (7) Feedback.
Social Support – Seven specific rubrics: (1) Whole-class discussion, (2) Small group work, (3) Status-raising, (4) Motivational discourse, (5) Enthusiasm about mathematics, (6) Attention to students' lives, (7) Accountability and high expectations.

When coding, we applied the rubrics to 10-minute increments for each episode. At least two coders were assigned to each episode, and coders met to reconcile and resolve all coding disagreements. To assign an overall score for each rubric to an episode, we calculated the resolved rubric means across the 10-minute increments. To determine an overall academic support score for an episode, we took the mean of the mean resolved scores across the seven relevant rubrics across the 10-minute increments. We used this same procedure to create an overall social support score. To examine relationships between teaching practices and students' engagement in the moment, we began by computing Pearson correlations between the overall academic and social support rubric scores and the ESM scale scores across the 20 classes (316 students had complete data). Next, we examined the correlations between the specific observation rubrics, and the mean ESM scale scores for each class . Using these exploratory analyses, we selected a lesson that showed strong support practices, both academic and social, and high student engagement, as indicated by ESM responses. We then conducted an in-depth qualitative analysis of instruction during this lesson to explore what instruction looked like during a lesson when students reported high levels of productive engagement.

Results

Our analyses revealed that teachers who strongly enacted opportunities to provide academic support also strongly enacted opportunities to provide social support (see Table 1 for correlations), and both types of support correlated with student self-efficacy, though less so with other engagement variables. Although one might assume that a secondary teacher might be better at providing either academic support or social support rather than both, we found that most of these teachers tended to be either strong or weak at both. When these teachers provided strong opportunities for students to be supported academically, students reported experiencing higher self-efficacy and more productive social engagement. Similarly, when these teachers provided opportunities for students to be supported socially, students also reported higher self-efficacy.

Among the engagement scales, students' reports of their social engagement correlated with their cognitive engagement, their interest, and self-efficacy in the observed activity. Overall, these results suggest that when teachers supported their students along academic and social dimensions, students felt more confident, which related to their cognitive engagement and social engagement during the lesson.

Table 1: Relationships Between Teaching and Student Engagement (n=20 classes)

	Observation Scores (Teaching)		Experience Sampling Method Survey (Student Self-Reports: In-the-moment Engagement)			
	Academic Support	**Social Support**	**Interest**	**Cognitive Engagement**	**Perceived Instrumentality**	**Social Engagement**
Social Support	.766**	---				
Interest	0.096	0.112	---			
Cognitive Engagement	0.385	0.356	.579**	---		
Perceived Instrumentality	0.028	0.011	.602**	0.184	---	
Social Engagement	.453*	0.44	.620**	.895**	0.199	---
Self-Efficacy	.591**	.559*	0.074	.505*	-0.02	.566**

** Correlation is significant at the 0.01 level (2-tailed).
* Correlation is significant at the 0.05 level (2-tailed).

Relationships Between Specific Observation Rubrics and Student Engagement

Table 2 displays relationships between the specific observation rubrics applied to each lesson and self-reports from students in those class periods on ESM scales. For academic support, efforts to support students' use and understanding of mathematical language significantly correlated with self-efficacy and social engagement, while feedback significantly correlated with self-efficacy. For social support, status raising significantly correlated with self-efficacy. Students appeared to feel more confident when teachers positioned students as competent in specific ways (status raising), helped students understand mathematical language, and gave students more detailed feedback focused on concepts. Students also participated more by sharing their thinking during class (social engagement), often by negotiating meaning, when the teacher developed mathematical terminology (mathematical language).

Table 2: Significant Relationships Between Specific Observation Rubrics and ESM Scales

	Specific Observation Rubrics		
	Academic Support: Mathematical Language	**Academic Support: Feedback**	**Social Support: Status Raising**
Self-Efficacy	0.647**	0.559*	0.784**
Social Engagement	0.495*	0.164	0.376

** Correlation is significant at the 0.01 level (2-tailed).
* Correlation is significant at the 0.05 level (2-tailed).

When these teachers' instruction was observed to have higher mean scores for academic and social support appeared, students appeared to have opportunities to engage productively and feel confident. However, results from the specific observation rubrics (7 academic support rubrics and 7 social support rubrics) did not indicate many significant correlations with students' motivation and engagement on the ESM. These results suggest supporting engagement involves a more complex interaction of observed support variables than a single rubric can detect for a few observations. Academic and social support behaviors may interact in ways that are not easily categorized by our specific rubrics for academic and social support. Moreover, students' reports of their engagement may not be highly related to specific teaching support behaviors, but they may be related to more general patterns (hence the strong correlations of mean academic support and social support to our ESM scales). In other words, our ESM scales may detect general feelings about the class sessions and not make fine distinctions that relate to specific observed support behaviors.

Teaching that Supports Productive Engagement

It is unsurprising that secondary students have better opportunities to engage productively when teachers provide both academic and social support, but it is worth exploring what such support looks like in practice. Correlational analyses indicated that specific forms of academic support productively engaged students: opportunities for students to learn about precise mathematical language and receive specific feedback about concepts. A specific form of social support appeared to productively support engagement: teachers' efforts to explicitly raise students' statuses.

However, it essential to investigate these instructional practices qualitatively to seek insights about enacting them. Our analyses suggest that when these teachers enacted specific academic supports that correlated with engagement (mathematical language and feedback), teachers also supported students' engagement by engaging them in sense-making and reasoning, making connections, and pressing students to explain, as these forms of academic support appeared to also take place when discussing mathematical language and when the teacher gave feedback. Additional social supports that qualitatively appeared to co-occur with status raising were having whole-class discussions that were collaborative and using motivational discourse with students.

We illustrate these features of instruction that appear to support productive engagement among secondary students by describing an episode from the third observation of Kathy's teaching in the Mid-Atlantic region. This episode of teaching was rated as having some of the highest academic and social support scores in the sample (2.07 and 2.00, respectively, out of 3) and Kathy's students reported some of the highest self-efficacy and cognitive engagement scores on the ESM in the sample (4.33 and 4.11, respectively, out of 5).

Kathy selected a card sorting task as a potentially engaging activity. Each card had a type of representation of systems of linear *equations* or systems of linear *inequalities*. The representations were symbolic, graphical, or in story problem form. The activity began with students working in groups for about nine minutes to sort the cards and compare how they were similar and different. They were expected to identify distinguishing features of systems of linear equations and systems of linear inequalities and record these ideas on sticky notes. After working in groups, the teacher led the students in a four-minute whole class discussion about the cards. Below, we share some examples of this first round of whole-class discussion during this activity.

> Kathy: (Rings bell.) Bring in back in 5, 4, 3… Here's where we're going next: You have to make some decisions about which ones [cards] were inequalities and which ones were

equations. Those decisions lead to this question: How are systems of equations and inequalities the same or different? There's something you had to think about and you used that information to make your decisions in your groups. Now before I have you go and summarize that on your post-its, if you haven't done so yet, let's go ahead and make sure that we understand all eight of these correctly.

On a SmartBoard, Kathy wrote an E for equations and an I for inequalities on the projections of the cards as students identified them using "shout it out" (choral class responses). Kathy and her students had interactions that centered around their ideas. For example:

Student 1: Inequations.
Kathy: Did you say inequations? Inequations. It's one or the other. It's supposed to be an inequality. [Student 2] tell us why this is an inequality [symbolic representation].
Student 2: It doesn't have y-equals.
Kathy: Doesn't have that equals, y-equals. Which makes this one what, [Student 2]? [points at a different set of two equations] Equation or inequality?
Student 2: Equations.

This example illustrates that the purpose of the lesson was focused on making sense of mathematical language (inequalities and equations) and concepts (similarities and differences between systems of equations and inequalities). Students were expected to make connections between multiple representations, as they encountered each type of system as symbols, graphs, and story problems. The teacher then gave feedback using revoicing and redirecting. Students were pressed to explain ("tell us why this is an inequality"). This discussion included other similar moments.

Teacher: [reading the problem on the SmartBoard] Two CDs and 4 DVDs cost $40.
Student 3: Equation.
Teacher: Whoa, it's like jeopardy! You didn't let me finish. 3 CDs and 5 DVDs cost $55. Why an equation here?
Student 4: Why a what?
Student 3: Because it's giving $40, right?
Student 4: Because there's no budget.
Student 5: And the other, $55.
Teacher: It's giving an exact amount for a certain number of items. It's not a budget, it's not a limit or restriction. It's exactly equal.

In this interaction, Kathy's feedback revoiced the students ("not a budget") as an example of a constraint that could be a signal for an inequality rather than an equation, which helps position that contribution as valuable. She also gave more detailed conceptual feedback in response to the multiple student contributions about how the representation aligned with being an inequality, in contrast to a shorter evaluative statement or a statement about procedures.

Then, students had another short amount time to work in groups (four minutes) and reflect upon what they identified as important features in systems of linear equations versus linear inequalities. They wrote these reflections on post-its. The teacher brought the students back together as a whole class for two minutes to debrief and highlight the features on the post-its that groups put on a blackboard. Below is an excerpt of how Kathy debriefed the group work.

Kathy: All right stop what you are doing and look at me. You are looking, I appreciate that. [Student 6], open your eyes, pick your head up. These are ideas from your class. I will condense them so there are no repeating ideas. Here are some ideas that your class came up with. Inequalities have the symbols less than or greater than, less than and equal to, greater than and equal to. A system of equations equals something. How many solutions are there to a system of equations? [Student 7] was holding up one finger. If we're talking a linear system with straight lines, how many times can they cross in a graph?

Students: Once

Kathy: Just once. That may change when we talk about things that are not linear, maybe graphs that curve, but for what we're seeing, one solution. This same idea, one has equal signs and the other doesn't. I'll put it in the middle. Oh, here's a new idea! You only shade on the graph when you have an inequality. this one is about shading. An inequality has a feasible region. Does anyone know what a feasible region is? [Student 8], say that again?

Student 8: The shaded part.

Kathy: The shaded part on your graph. It shows all of your solutions.

Motivational discourse was present when the teacher expressed excitement about students' new ideas ("Oh, here's a new idea!") and when she expressed appreciation for students when they paid attention. There was an explicit opportunity to understand mathematical language, at least to some degree, around terms such as "feasible region" and systems of equations and inequalities. There were also attempts to credit students with productive thinking anonymously (reading off what groups wrote on post-its) and by calling attention to students who were calling out productive ideas [students 7 and 8], which could potentially serve to raise students' statuses. Students were also pressed to explain features of systems of inequalities and systems of equations in during the previous classroom discussion and in groups in writing on the post-its. Finally, Kathy gave students feedback on their thinking by reflecting back what she noticed as similarities and differences in groups' responses. (Analysts observed a limitation on mathematical accuracy: linear systems can also have infinite solutions or no solutions.)

This description of a classroom episode illustrates some of the ways that Kathy enacted teaching that aligned with students' self-reports of productive engagement during this activity. Kathy supported students academically through opportunities to make sense of concepts and make connections among representations, and through opportunities for students to negotiate meaning about mathematical language. She pressed students to explain their reasoning and gave them feedback to specifically support understanding of mathematical concepts. Kathy also supported students socially through motivational discourse and status-raising efforts, particularly in terms of how she appreciated students' thinking when they shared ideas and elevated their ideas through revoicing them. Students were asked to explain multiple times throughout the activity, which supported the development of meanings of language and concepts and provided opportunities to position more students' ideas as valuable, to raise their statuses. Thus, for Kathy, both of these two aspects together – the academic and the social support she provided – appear to bolster the mathematical engagement of her students.

Conclusions

Secondary mathematics students in this study reported higher levels of engagement, particularly higher self-efficacy and social engagement, when they had opportunities to negotiate meaning about mathematical language, received specific feedback targeted toward making sense of concepts, and when teachers made the effort to raise students' status. This is an exploratory study; our future analyses will include more observations and ESM measures across an academic year, allowing for more sophisticated statistical analyses. Nevertheless, our preliminary results suggest that investigating the relationships between observations of teaching and students' self-reports of engagement in-the-moment is a potentially revealing approach for uncovering engaging instructional strategies in secondary mathematics classrooms.

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RETHINKING MATHEMATICS CLASSROOM PARTICIPATION: STUDENT AND TEACHER PERSPECTIVES

José Manuel Martínez
Texas State University
josemartinez@txstate.edu

Participation in mathematics classrooms has received considerable attention in previous research. Scholars have emphasized researchers' and teachers' perspectives on what it means to participate. I draw on sociocultural theories of learning to explore both students' and a teacher's perspectives on participation. I describe a participatory research study where a Spanish immersion third-grade teacher and I collaborated to elicit students' perspectives on mathematics class participation. Data generation included multi-method focus groups with the students, and an interview with the teacher. Drawing on social semiotics as an analytical framework, findings suggest that there were two connotations of participation in this classroom: While the teacher's connotation characterized participation mainly as quantifiable, observable student talk attributable to individual children, students' connotation characterized participation mainly as qualifiable, amorphous, diverse and focused on the social. I argue that developing mathematics teaching that is inclusive of diverse ways of participating involves learning about students' perspectives on participation.

Keywords: Classroom Discourse, Equity and Diversity

Research on participation in mathematics classrooms has primarily studied the role of the teacher in eliciting student talk. Few studies have focused on how students actually participate, starting with what they have to say about their own participation (Dallimore, Hertenstein, & Platt, 2004). The tendency to overlook students' perspectives in favor of teachers' and researchers' perspectives is particularly marked in studies involving elementary school children (Groundwater-Smith, Dockett, & Bottrell, 2014). This study addresses this issue by considering both students' and a teacher's perspectives. Learning about insiders' (teacher and students) meanings associated with participation was a first step for the teacher to support this class to become receptive of mathematical ideas that emerged in unanticipated ways.

The purpose of this study was to question taken for granted ideas about participation in mathematics classrooms, whose perspectives we acknowledge, and whose we overlook. I report on an exploration of students' and a teacher's participation-related perspectives, in the context of a participatory research study in a third-grade Spanish immersion mathematics classroom. I ask the following research question: What does it mean to participate in a mathematics classroom according to students and to the teacher? I argue that developing mathematics teaching that is inclusive of diverse ways of participating involves learning about students' perspectives on participation. Grounding research and teaching efforts on students' perspectives challenges the imposition of adult-generated visions of what it means to participate, and it helps conceptualizing participation in ways that are consistent with and inclusive of students' classroom experiences.

Literature Review

Participation is nearly equated with student talk in mathematics education research. Many studies use the terms participation and student talk interchangeably (For a few examples see

Brown, 2017; Franke et al., 2015; Jansen, 2006). For example, when studying students' participation in mathematics classrooms Jansen (2006) focused on students "talking about mathematics" (p. 412). Not surprisingly, previous research has characterized the teacher's role regarding participation mainly in terms of eliciting student talk. Specifically, teachers strive to uniformly distribute talk time among students, disrupting the tendency to attend to mathematical ideas uttered by a few dominant students. As Webb et al. (2014) put it, "Teachers can set ground rules and guidelines for desired participation, promoting greater student talk" (p. 81). These norms influence student talk by signaling to students the desirable ways to add to discussions.

Little is known about how teachers become aware of and draw on students' multiple ways of participating in mathematical activity. In this study, I consider the teacher's role not only in eliciting predetermined forms of student talk but also in learning about unanticipated ways of participating. As a first step toward challenging preconceived notions about participation, I focus on students' perspectives on participation and raising the teacher's perspective to awareness.

Theoretical Framework

I draw on sociocultural theory to define participation as an evolving social experience of engaging in meaningful shared activity, that combines doing, talking, thinking, feeling, and belonging (Rogoff et al., 2003; Vygotsky, 1978; Wenger, 1998). Participation is a situated phenomenon in which the individual and the social interact as members of a bounded social community, such as a classroom, coordinate the pursuit of a shared enterprise. It is situated because the context influences both the activities that are considered meaningful and the ways of becoming part of an activity (Thomas, Whybrow, & Scharber, 2012). Participation involves an interaction between the individual and the social because, rather than being a static set of practices, members of a community permanently negotiate what it means to participate (Wenger, 1998). As the community develops over time, the activities in which members engage and how they participate in these activities evolve, too. Different members may introduce novel activities or ways of engaging in activity. When an individual's ways of participating are legitimized and influence how others participate, the member experiences a sense of belonging.

This conceptualization has implications regarding the role of the teacher. What counts as participation varies from classroom to classroom (Anderson, 1998). Even within each classroom, teachers' and students' meanings of what counts as participation may differ (Mafra Goulart & Roth, 2006). Teachers may decide to focus on helping students participate a certain way. Although this approach has the potential of helping students expand the ways in which they participate, it can also exclude children whose preferred ways of participating are overlooked. In contrast, when the classroom allows multiple ways of participating to influence activity, the community expands the ways in which students make sense of ideas and more students experience a sense of belonging in mathematical activity (Dallimore et al., 2004). In this study, I consider the role of the teacher expanding her own conceptualization of participation by learning about students' ways of participating.

Methodology

Participatory Research

Participatory research seeks to challenge power hierarchies, including the positioning of the researcher as an expert that comes to the site to execute a predetermined agenda (Fals-Borda, 1987). Participatory approaches do not regard the researcher as the producer of knowledge and the researched as the object of analysis and the recipient of knowledge. Instead, participatory researchers move from working *on* or *for* participants to working *with* them (Hansen, Ramstead, Richer, Smith, & Stratton, 2001). Accordingly, participatory research is based on the needs of the community and it focuses on the process of working together to gain understandings and initiate transformations of particular situations. In this study, the issue of participation emerged during my prolonged collaboration with the teacher, as we both became aware and were puzzled by students' differential participation. The teacher and I collaborated in making sense of the ideas about participation that coexisted in this classroom.

Sites and Participants

Located in the Midwest of the United States, this school offered a Spanish language immersion program. Instruction was in Spanish all school day (including mathematics class), except for one hour taught in English. There were 21 students in this third-grade classroom: 15 female students (one Latina, one Asian American, two African American, 11 White) and six male students (two Latino, one African American, three White). All students seemed comfortable sustaining conversations, as well as following directions and discussions in Spanish. The teacher, Señora Abad (all names are pseudonyms) was a Spanish-English bilingual Latina, with four years of teaching experience, all in this school.

Consistent with principles of participatory research, I position myself as a research-participant in this study. I am a Latino Spanish native speaker. I was a teacher for 7 years. As part of my involvement with this classroom, I visited the school for three years, collaborating with Señora Abad on my research. I observed this specific third-grade and supported students during small group work throughout the school year.

Data Sources

I conducted multi-methods focus groups with students to discuss participation in general and in math class specifically. I conducted six focus groups with three or four students in each group. Consistent with tenets from participatory research, the structure of these focus groups was flexible: First, I reiterated that participation was voluntary and that students could choose whether to join a focus group or not. I communicated to all students that I would share their responses with the teacher. Second, students chose their own groups, in an attempt to help the children feel comfortable sharing their ideas. Third, students could choose to leave the group at any moment and join another group later. Fourth, students could choose to speak in English or in Spanish at any point. Fifth, students had the option of choosing among multiple methods to share their ideas.

I initially asked students to think about what it means to participate and how they participated in math class. Each student individually decided whether to draw or write about these ideas, or to think and then share during the conversation. Paper and markers were available. Then, each student could decide whether to share the ideas with the rest of the group or not. All students shared their drawings and writings and explained what they represented. I asked follow-up questions to clarify points or to ask for specific examples. In all groups, students frequently addressed each other, adding to what others said or proposing alternative ideas. Each focus group was about 30-minutes long. I audio recorded and transcribed each group and I collected students' sheets.

I also conducted an interview with the teacher. This 45-minute semi-structured interview took place after all the focus groups, and the teacher and I discussed her ideas about participation in her mathematics class. I audio recorded and transcribed the interview.

Analytical Framework

Principles from social semiotics (Halliday, 1978; van Leeuwen, 2005) resonate with this study's focus on a shift away from overvaluing predetermined ways of participating. Social semiotics argues that when individuals communicate, in addition to externalizing personal understandings, they also intend to achieve effects in their community (Morgan, 2006), which is consistent with sociocultural theory's considerations of participation as involving a sense of social involvement. Social semiotics challenges the assumption that pre-given, static meanings reside in specific semiotic resources. Acknowledging the influence of social contexts and interactions in meaning making processes, social semiotics contends that specific practices have different meanings in different contexts, and the culturally situated meanings of those practices can be the focus of social semiotics analyses (van Leeuwen, 2005). In this study, I focus on the meanings and practices associated with participation in this mathematics classroom.

Connotations. I focus on this analytical tool from social semiotics to make sense of the students' and the teacher's responses. Connotations are variations within the culturally shared meanings, values and practices of a social group. The context and the perspectives of different participants influence these variations. When the connotations that different members of a group hold about a practice are sufficiently complementary or similar, the coming together in joint activity is more likely and more inclusive than when these connotations are incompatible. For example, if a teacher's connotation of participation involved students sitting quietly and raising their hands to respond to the teacher's questions and most students shared this connotation, it would seem as if there is cohesion and ease within the group. At the same time, however, the few students whose connotations differed could potentially be ignored. In the presence of this incompatibility, elementary school children tend to conform, and adults' perspectives tend to prevail (van Leeuwen, 2005).

Data Analysis

Consistent with the participatory methodology, the teacher and I collaborated analyzing focus group and interview responses to develop agreed-upon interpretations of the connotations of participation that coexisted in this classroom. I annotated each line of the transcripts, focusing on nuances in characterizations of what it means to participate. I attended to word choice, the use of pronouns, and expressions of emotion or intensity. Analysis of these language cues helped "link micro-analysis of texts to various forms of social analysis of practices" (Fairclough, 2013, p. 7).

I also annotated statements where students agreed, disagreed or built on others' responses to identify convergence and divergence of perspectives on participation among students. I repeated this process, refining annotations. Then, I conducted a thematic analysis to identify emerging themes in the focus groups and in the teacher interview. I shared these tentative themes with the teacher, and we analyzed excerpts of responses from the focus groups that I selected. I selected three excerpts from different focus groups that included references to the main themes that I had identified and that were representative of the entire data set. The teacher confirmed most of my interpretations and she helped me strengthened them by contextualizing some of the responses based on what she knew about the students.

Findings

Analyses suggest that Students characterized participation mainly as qualifiable, amorphous, social and diverse. The teacher, on the other hand, characterized participation mainly as quantifiable, observable, individual student talk.

Students' Connotation

I illustrate the interpretation of students' connotation of participation mainly as qualifiable, amorphous, diverse and focused on the social with a vignette from the second focus group I conducted. Three students were part of this group: Stacey, Rose and Jimmy. Stacey mentioned she participated in gym class watching what the teacher did and then trying to do the same. I followed up asking how they participated in math class.

1	Author	So, I have a question for the group. How do you participate in math class?
2	Stacey	Like, what?
3	Author	You just talked about how you participate in gym, so now I'm wondering how
4		you participate in math class.
5	Stacey	But, like, what? Like in charlas de números [number talks], or when Señora Abad
6		is explaining something, or when we work with our table?
7	Author	Is it different?
8	Jimmy	It's different but it's the same. It's the same because sometimes what we're doing
9		doesn't matter. Like, I love to ask questions all the time.
10	Rose	And it's different at our tables. I like participating because I feel part of the team.
11	Stacey	I like helping others because I can help everyone feel part of the team.
12	Jimmy	And also sometimes you can participate too much because you are not really
13		helping but telling others 'this is the answer.'
14	Rose	Yeah. You have to think, and you have to listen first.
15	Jimmy	And you can say like 'I think we can try this way because I draw it here and I
16		think it works. What do you think?'

Participation as qualifiable. In all focus groups, the idea of qualifying participation emerged. That is the case in the vignette when Jimmy stated: "sometimes you can participate too much because you are not really helping but telling others 'this is the answer'" (lines 12-13), to which Rose agreed. Later in the conversation, Stacey articulated a similar idea. She was referring to a whole class discussion in math class earlier that day when the teacher mentioned it was hard for her to understand because several students were talking at the same time. Stacey stated: "sometimes, like what they were just doing is over-participating cause they shouldn't be talking, and they still were, so that's like kind of participating, but, yeah." In this case, Stacey qualifies a classroom event as involving over-participation.

Amorphous, perceived participation. Although students mentioned observable aspects of participation, they also described a non-observable sense of participating. In the vignette, all three students related ways of participating with something they loved or liked. Jimmy, for example, said: "I *love* asking questions all the time." Similarly, Rose said: "I like participating because I *feel* part of the team" (line 10). Rose related participation with her feeling part of the team. Stacey agreed, adding that she *liked* to "help everyone feel part of the team" (line 11).

Multiple ways. In all focus groups, students mentioned diverse ways of participating. In the vignette, for example, when I asked how students participated in math class, Stacey helped the group consider diversity of participation within math class. Specifically, she focused on diverse ways of participating related to types of activities, including number talks, teacher explanations, and small group work (lines 5-6). Group members added other ways of participating that involved student talk, such as telling the answer to a problem (line 13), and some ways that did not necessarily involve student talk, such as thinking, listening, and drawing (lines 14-15). Additional ways of participating that students mentioned in other focus groups included watching, writing, reading, and using specific materials (e.g. rulers and base ten blocks).

Students' comments did not seem to favor one way of participating over others or to subordinate one way of participating as ancillary to others. For example, students did not mention listening or observing as ways of participating that would ultimately facilitate student talk. Instead, students mentioned these ways of participating as alternative and complementary.

Two ways of participating were salient in all focus groups. One way was helping. In the vignette, Stacey introduced this idea when she mentioned she liked helping others (line 11), and Jimmy and Rose expanded on this by characterizing ways to help and ways that are "not really helping" (line 12). Another salient way of participating that emerged in focus groups was asking questions. Although asking questions involves student talk, it is not a kind of talk where students explain an idea or justify a response. This kind of talk elicits others' ideas and expresses curiosity, confusion, or the need for help. In the vignette, Jimmy offered an example of this way of participating when he said he loved asking questions all the time (line 9) when referring to ways of participating that may occur regardless of the type of mathematical activity.

Relating the individual and the social. Students' connotation of participation tended to consider the interaction between the individual and the social. In the vignette, students mentioned individual preferences related to participation. Students, however, related their individual participation preferences with social interactions. For example, Jimmy mentioned he loved asking questions (line 9) when he was explaining that his awareness of the context of the activity influenced how he participated. Similarly, when Rose mentioned her liking feeling part of the team (line 10), she was simultaneously recognizing the sense of membership to a social group as an aspect of her participation. When Stacey stated she liked helping (line 11), she signaled her involvement with those who she helped as part of this way of participating. Finally, students' characterization of participation as qualifiable implied that what the teacher did or how some students participated influenced others' participation. That is the case when Jimmy said "you can participate too much" by telling others the answer to a problem (line 12).

The Teacher's Connotation

The following vignette illustrates the interpretation of Señora Abad's connotation of participation mainly as quantifiable, observable student talk attributable to individual students. This vignette is part of Señora Abad's response when I asked her what she did to help students participate in math class. Giving an example of the strategies she used, she mentioned the cup where she had popsicle sticks, with the name of one student written on each stick. She mentioned she would randomly draw one stick and call on the student whose name was written on that stick to answer a question or share an idea. Señora Abad shared that besides helping her avoid calling the same student multiple times, this strategy signaled to students everybody needed to be ready.

> I use them [the popsicles] because then they [the students] know anybody can participate. Not only those who always raise their hand. It doesn't always work because they can pass if they don't want to participate, and then I draw another name. But I would like to see everybody participating [in math class]. And there are different forms of participating, it doesn't have to be the same for everyone, but [I'd like to see] everyone participating. Because sometimes some [students] don't participate. I know that sometimes Daniel is going to be doing something else and then he won't be able to say anything when I ask him a question. Or Emi. She doesn't like going to the board and telling us about her strategies. So, it's difficult.

Absence or presence of participation. Señora Abad seemed to characterize participation in terms of a binary: at any given moment, a student either is or is not participating. In this excerpt, this presence or absence of participation refers to whether students shared an idea or answered a question when the teacher called on them. In line 1, she talked about how when she told students she would use the sticks "then they [the students] know anybody can participate." Not everybody would actually participate, but since she randomly selected the name of the student who would participate, all students needed to be prepared. She reinforced the notion of presence or absence of participation when she said that when she called on students, they could pass "if they don't want to participate" (lines 2-3). According to this characterization, students could choose between participation and non-participation. If a student accepted Señora Abad's invitation and contributed an idea, that student chose to participate. In contrast, the student could decline and, therefore, choose to not participate. Señora Abad made this contrast more explicit when she stated: "sometimes some [students] don't participate" (line 6). This comment also suggests that for Señora Abad the status of participating or not participating was not static and changed over time. The use of the sticks strategy relates to the teacher's desire "to see everybody participating" (lines 3-4). This statement further supports the interpretation of Señora Abad's characterization of participation in terms of some students falling into the participating side of the binary and others falling into the non-participating side.

Participation as observable. Señora Abad's connotation focused on observable aspects of participation. That is the case, for example, when she expressed her wish to "*see* everybody participating [in math class]" (lines 3-4). Her characterization referred to "different *forms* of participating" (line 4), suggesting participation that takes a form. Participation seems to be embodied in specific student behaviors accessible to the teacher through direct observation. Specifically, in this vignette Señora Abad referred to participation that takes the form of hand raising and going to the board. The absence or presence of participation relates to its observable nature, in Señora Abad's connotation. According to her comments, whether students were or were not participating was something that she could see. Students' enactment of a specific form of participation would be visible to Señora Abad. Conversely, the absence of such forms would mean that there would be no participation to observe.

Overemphasis on student talk. Señora Abad explicitly acknowledged her perception that different students could participate in different forms when she stated that "there are different forms of participating, it doesn't have to be the same for everyone" (lines 4-5). Señora Abad, however, emphasized student talk during the interview. In several of her comments she suggested that when students talked about mathematical ideas, they participated, and when they did not talk, they did not participate. For example, in the vignette she mentioned that Daniel sometimes got distracted "and then he won't be able to *say* anything" (line 7). Similarly, she talked about Emi not liking "*telling* us about her strategies" (line 8). Talking about mathematical ideas is a form of participation that Señora Abad valued, and students could expect Señora Abad to encourage them to talk. The vignette depicts students who participated as the ones that showed willingness to talk by raising their hands (line 2), and those who did not participate as those who passed when Señora Abad called on them (lines 2-3). The strategies that Señora Abad described communicated to the class that she valued student talk, and they related her idea of absence and presence of participation to absence and presence of talk.

Foregrounding the individual. Señora Abad's connotation of participation tended to foreground the individual over the social. In the vignette, Señora Abad's comments suggest that she attributed participation and non-participation to individual students at any given moment. She provided the example of Daniel (lines 6-7), stating that he tended to not participate when she asked a question. Furthermore, Señora Abad claimed that there was a reason for Daniel not to participate: his getting distracted doing something not related to the class discussion. Similarly, Señora Abad shared that Emi tended to not participate when it required explaining something in front of the class. In the vignette, Señora Abad hinted at Emi's aversion to being put on the spotlight as the reason for her non-participation.

Discussion

In this study, I answered the question: What does it mean to participate in a mathematics classroom according to students and to the teacher? Findings suggest that there were marked differences in the students' and the teacher's perspectives. While students characterized participation mainly as qualifiable, amorphous, social and diverse, the teacher characterized participation mainly as quantifiable, observable, individual student talk. Students' connotation of participation challenges implicit definitions of student talk frequently found in the mathematics education literature. Students' connotation resonates with tenets from sociocultural theory. This suggests the need to examine the theoretical consistencies and inconsistencies in the definitions of participation that researchers implicitly or explicitly use. Methodologically, these findings suggest the need to consider not only adults' but also children's perspectives on participation.

Some aspects of this study's findings reinforce previous research on participation. Specifically, the idea of helping as part of participation that emerged during the focus groups echoes the ideas of students in other contexts (Civil & Planas, 2004; Jansen, 2006). This indicates that some students' ideas about participation may transcend context and be relevant in a variety of classrooms. Simultaneously, findings suggest the need to consider context-specific meanings of participation. For example, unlike other studies, these findings invite a reconsideration of the goal of observing equal participation, understood as the amount of student talk. Students' perspectives, especially when contrasted to the teacher's, suggest that an alternative goal may be to learn about and embrace students' different ways and rates of participation. This way, the class may consider diverse ideas and ways of becoming part of mathematical activity, instead of overemphasizing spoken ideas proposed by talkative students.

Rather than arguing against student talk or against teachers' efforts to support student talk, I argue for teachers and researchers to consider additional ways in which students engage with mathematical activity. Findings from this study suggest that teachers interested in participation may need to consider more than only observable indicators in individual students. Instead, teachers may need to consider a complex network of both social and individual, observable and perceived, quantifiable and qualifiable aspects. In turn, drawing on strategies that help students participate in practices they prefer (such as helping others or asking questions), may foster their exploration of mathematical ideas.

My intention was not to present either connotation of participation as good or bad, but to understand the nuances and contrasts between the two. Moreover, I acknowledge that using focus groups to elicit students' connotation may have favored the emergence of a composite of connotations, but there may be differences within students' connotations. Teachers awareness of their own connotation of participation and learning about their students' is a first step in transforming their practice toward becoming inclusive of diverse ways of participating.

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CONNECTIONS BETWEEN FEEDBACK AND STUDENT HAPPINESS AND ENGAGEMENT IN HIGH ACHIEVEMENT CLASSROOMS

Monica Anthony
University of Maryland
mrd@umd.edu

Blake Turner
University of Maryland
bclark37@umd.edu

Pamela Callahan
University of Maryland
pcallaha@umd.edu

Casey Archer
University of Maryland
cjarcher@umd.edu

Researchers have investigated how teacher feedback contributes to students' learning, yet there is insufficient research into the connection between teacher feedback and student happiness, which is linked to engagement, socio-emotional well-being, and ultimately academic achievement. We coded lesson transcripts from elementary teachers to determine emergent patterns between types of teacher feedback and average student self-reports of happiness in mathematics. We found teachers to employ specific feedback more than general feedback. In addition, feedback for effort and ability accounted for a small percentage of feedback. Findings suggest that student happiness is linked to specific combinations of feedback dimensions.

Keywords: Affect, Emotion, Beliefs and Attitudes; Instructional activities and practices

The role of teacher feedback in the classroom is both instructional and affective. That is, teacher feedback is multi-dimensional, serving differing purposes. Teachers employ verbal feedback in evaluation of student responses and as a classroom management tool, such as evaluating student behavior (Brophy, 1981; Floress & Beschta, 2018). While learners desire feedback, not all feedback benefits academic performance (Black & Wiliam, 2009; Hattie & Timperley, 2007). In general, researchers agree that for feedback to be considered effective, it must be specific about the learner's performance or behavior (Black & Wiliam, 2009; Burnett & Mandel, 2010; Hattie & Timperley, 2007). However, in Burnett and Mandel (2010), the most prevalent type of feedback documented was general praise, in which the teacher makes a general statement but is not explicit regarding the relationship between a student's action and the awarded praise. Like other studies (e.g., Floress & Beschta, 2018), teachers rarely provide behavior-specific praise. Comparable to our cited literature, a large portion of research on teacher feedback focuses specifically on praise, a subcategory of feedback which goes beyond evaluating for correctness by expressing approval or assigning worth (Brophy, 1981). Interestingly, praise does not correlate with student academic achievement gains (Brophy, 1981). Thus, the types of feedback described as "effective" for supporting student learning in research literature may not be found in classroom interactions.

Classroom feedback and praise not only influences students' learning and behavior, but also influences students' orientation to learning. Mueller and Dweck (1998) found that praising students for ability rather than effort impacted students' orientations towards learning. Students praised for ability focused on the performance of themselves (and others), while students praised for effort adopted a mastery-oriented mindset. Students' orientations towards learning impact their disposition towards learning, especially when encountering failure. In short, praising students for their intelligence communicates that "they can measure how smart they are from how well they do" (Mueller & Dweck, 1998, p. 43). Students' dispositions are also influenced by their preferences for feedback. A preliminary review of the research shows that elementary students prefer effort feedback to ability feedback (Burnett & Mandel, 2010), and prefer private praise to being spotlighted (Burnett, 2001). Thus, feedback type and student feedback preferences impact students' dispositions towards learning.

In this study, we ask: *Controlling for teachers' success at raising test scores, are there associations between the types of verbal feedback provided by teachers and measures of student self-reported engagement and happiness?* To answer this question, we examined lesson transcripts of twelve teachers who were effective at raising test scores but were either in the top or bottom quartile for student self-reported happiness (survey questions shown in Table 1). The twelve teachers were selected from a larger dataset where students were randomly assigned to classroom teachers. Given the teachers' success at improving students' academic performance, it is reasonable to suspect that teacher feedback satisfies the markers of effective feedback. However, the extremes in student-reported happiness that there may be differences in how often and what type(s) of feedback teachers provide.

Table 1: Student Survey Items Measuring Happiness in Class

This math class is a happy place for me to be.
Being in this math class makes me feel sad or angry.
The things we have done in math this year are interesting.
Because of this teacher, I am learning to love math.
I enjoy math class this year.

From Blazar & Kraft (2017)

Our study extends the current research by examining instances of both academic and behavioral feedback, including corrective feedback. Previous studies report that general academic praise is not predictive of a positive classroom environment, but they have not examined the influence of social or behavioral feedback (Burnett & Mandel, 2010). Other studies have examined the use of praise as a classroom management tool, monitoring academic and behavioral praise but not documenting instances of feedback for correction (Floress, Jenkins, Reinke & McKown, 2018). Our research focuses specifically on feedback and praise during mathematics lessons because students' mathematical dispositions are linked to how they engage with the subject (Gresalfi & Cobb, 2006). We posit that students' perceived happiness in class is a component of their disposition towards the content. Thus, our proposed research has the potential to uncover links between feedback, praise, and students' self-reported engagement with and enjoyment of mathematics.

Conceptual Framework

We developed our conceptual framework based on current and seminal research on feedback and praise. Following Hattie and Timperley (2007), we define feedback as "information provided by an agent (e.g., teacher) regarding aspects of one's performance or understanding" (p. 81). As a component of instruction, feedback can be both informative and educative, in that feedback can merely evaluate (denote correctness) or can provide additional information that moves learning forward (Black & Wiliam, 2009). As shown in the table, there are multiple facets, or components, of teacher to student feedback. Our framework posits five dichotomies: behavioral or academic, affirmation or criticism, specific or general, process or product, and ability or effort. As noted earlier, different types of feedback vary in their effectiveness for supporting student learning and disposition (Hattie & Timperley, 2007; Mueller & Dweck, 1998). Academic feedback addresses students' understanding of the content or task, while behavioral feedback includes other non-academic classroom behaviors (Brophy, 1981). For example, "I can tell Student A is ready to learn because she is sitting quietly," is a specific affirmation of Student A's behavior. Whether academic or behavioral, feedback is either affirmative or corrective (evaluative). In our framework, praise is a subcategory of affirmation. Brophy (1981) defines praise as "to commend the worth of or to express approval or admiration" (p. 5). According to Brophy, praise expresses positive teacher affect, providing feedback beyond affirmation or evaluation. Feedback will also be categorized as general or specific. General feedback consists of "any nonspecific verbalization or gesture that expresses a favorable judgment on an activity, product, or attribute of the student" (Floress, Jenkins, Reinke & McKown, 2018, p. 414). By contrast, specific feedback is explicitly linked to "an activity, product, or attribute of the student(s)" (Floress, Jenkins, Reinke & McKown, 2018, p. 414). Thus, "good job" is general feedback, while, "I like the way you are organizing your work on your paper" is specific feedback.

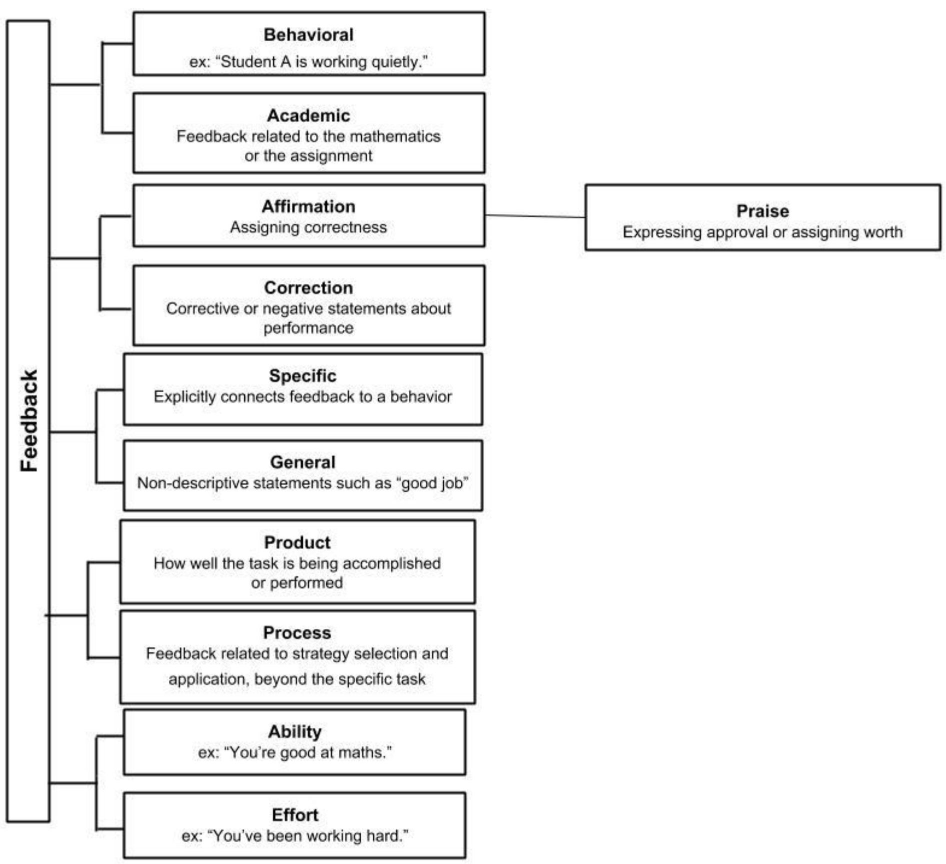

Figure 1: Conceptual Framework: Categories of Teacher-student Feedback

Not all feedback can be assigned to categories on the final two dichotomies: product or process, and ability or effort. Feedback about the product is typically corrective feedback about how well a task is being performed (Hattie & Timperley, 2007). Process feedback addresses the underlying processes of the specific task, potentially linking processes across tasks, or relating to students' error-analysis (Hattie & Timperley, 2007). Consequently, behavioral feedback, such as correcting a student for speaking too loudly, is often not linked to the specific mathematical task, and thus would not be categorized as process or product feedback. Finally, ability feedback is related to intelligence or personal attributes, while effort feedback is characterized by the use of the word "try" or a similar term which emphasizes that the student is putting forth effort (Floress & Beschta, 2018). General feedback, such as telling a student "Good job" for providing a correct response, would not be categorized as effort or ability feedback.

Methods

The primary focus of data analysis is to investigate if teacher feedback (type, frequency, etc.) is associated with student self-reported engagement and happiness.

Data

The transcripts analyzed in this study are from a dataset from the National Center for Teacher Effectiveness (NCTE) (as described in Blazar & Kraft, 2017). The teachers in this study are either 4th or 5th grade comprehensive teachers, who self-selected the lessons to be recorded. The recorded lessons were subsequently transcribed. As part of a larger research study, we first identified teachers who are effective at raising students' test scores. Then, we selected a subset of twelve such teachers who are either particularly effective or ineffective in making their classrooms happy and engaging places for students (as self-reported by students on end-of-year surveys). Six teachers score in the upper quartile of student happiness and engagement ratings while six teachers score in the bottom quartile. For each teacher we have three mathematics lesson transcripts, for a total of 36 lessons. The lessons ranged in length from 33 minutes to 79 minutes with an average length of 52.6 minutes.

Analysis

We coded lesson transcripts in NVivo. Our conceptual framework serves as our coding scheme, with each evidence of teacher feedback coded as behavioral or academic, affirmation or criticism, praise, general or specific, process or product, and ability or effort. For example, the statement "Not seven. Negative seven" was coded as academic, correction, specific, and product. Comparisons across codes, such as instances of affirmation versus criticism, aim to provide evidence for classroom climate and measures of student happiness. The lesson transcripts do not include descriptions, such as non-verbal gestures or how students are configured. Thus, our analysis only examines verbal feedback, which is presumed to be public.

The authors established the codebook, as defined by our conceptual framework, then completed independent coding on their assigned transcripts. Two randomly assigned researchers coded each lesson transcript. During independent coding, the authors regularly met to renorm and examine problematic codes. The authors examined conflicting codes (such as feedback being coded as both academic and behavioral) and were able to reach consensus. Ambiguous feedback where categories of feedback were influenced by potential student interpretation was brought up for review and ultimately excluded from analysis. Examples of ambiguous or interpretable feedback include instances where teachers praise a student, or group of students, in a manner that admonishes the other students ("Student G's group figured it out, so what seems to be the hold-up for this group?").

Limitations. We only have access to lesson transcripts, which do not include video recordings of the lessons, or descriptions of teachers' or students' paralinguistic and nonverbal behaviors. Consequently, we cannot make conclusions about the credibility of teachers' praise or feedback, or how it is received by the students (Brophy, 1981). Another limitation of the data is that measures are collected at the classroom level. Students differ in their preferences for praise and feedback (Brophy, 1981; Burnett, 2001; Burnett & Mandel, 2010), which in turn may influence their self-reported feelings of happiness and engagement. However, we are unable to link specific classroom instances with students' self-reports - we can only examine larger classroom trends.

Findings

We identified a total of 1588 instances of feedback across twelve teachers and thirty-six lessons. Table 2 shows instances and rates of overall feedback and affirmative (positive) feedback per teacher per lesson. Given our strict definition of praise, we are comparing our more inclusive category of affirmative feedback with other authors' reports of teacher praise, which tends to be more encompassing.

Teachers' rate of affirmative feedback (instances per hour) are within the ranges of rate of praise of 4th and 5th grade teachers, as reported by Floress et al (2018). Floress et al (2018) observed an average rate of 22.5 praise statements per hour for 4th grade teachers and an average of 30.9 praise statements per hour for 5th grade teachers. In our data, teachers with high measures of student self-reported happiness and engagement are within these averages, with 24 affirmative statements per hour, while teachers with low measures are about the same rate, with an average of 39.5 statements per hour. Overall rates of feedback are higher for teachers with low measures of happiness and engagement (m = 58.5 statements per hour), compared to those with high measures (m = 43 statements per hour).

Table 2. Amount and Rates of Feedback Per Teacher and Lesson

	High measures of happiness	Low measures of happiness
Total instances of feedback	721	867
Rate of feedback*	43	58.5
Percent of affirmative feedback	56	68.6
Rate of affirmative feedback*	24	39.5

*Calculated in terms of feedback per hour.

Other studies found teachers to provide more general than specific feedback and praise (Burnett & Mandel, 2010; Floress et al, 2018). However, both groups of teachers provided specific feedback more often than general feedback. Specific feedback accounts for 77% of all feedback for teachers with high measures of happiness and engagement and 83.7% of all feedback for teachers with low measures. Feedback for effort and ability accounted for a small percentage of feedback, across all teachers. Together, effort and ability related feedback accounted for only 6% of total feedback for teachers with high measures and 7% for teachers with low measures.

Table 3: Corrective Feedback Counts and Percentages

	Teachers with high measures of student happiness	Teachers with low measures of student happiness
Total corrective feedback	318 (44.1%)	268 (30.9%)
Academic corrective feedback	129 (40.6%)	164 (61.2%)
Behavioral corrective feedback	189 (59.4%)	104 (38.8%)

Number of instances of feedback with percent of total feedback.

Our study extends previous research on teacher-student feedback by examining corrective feedback (Table 3). Nearly half of all feedback provided by teachers with high measures of student happiness is corrective feedback. As shown in Table 3, teachers with high measures of happiness gave more behaviorally-focused corrective feedback while teachers with low measures provided significantly more academically-focused corrective feedback than behaviorally-focused corrective feedback.

Discussion

Our analysis captures the natural rates of feedback for twelve teachers, six of whom scored highly on measures of happiness and engagement and six who scored low, all who are successful at raising students' test scores. Our results show that teachers with low measures of happiness and engagement give more verbal feedback overall. Both groups of teachers have high rates of feedback, tend to give more affirmative than corrective feedback, provide specific feedback, and have low instances of effort and ability feedback.

Burnett (2001) found that 4th and 5th grade students desire teacher praise more than any other group of students under investigation, "indicating that this is a phase of development where students are looking for reassurance and recognition from their teachers" (p. 21). Yet, in our analysis the highest instances of affirmative feedback come from teachers with low measures of student happiness. Thus, the existence of affirmative feedback alone is insufficient to bolster student happiness and engagement in mathematics. Hence, the content of the feedback must be examined more closely. There were observed differences in the types of specific affirmations awarded to students. For example, teachers with low measures of engagement had a total of 278 instances of specific affirmations (including praise), 51 (10.7%) of which addressed students' processes and strategies, compared to teachers with high measures, who had 281 instances of specific affirmations, 53 (19%) of which addressed students' processes and strategies. Furthermore, there are differences in the process and strategies that are affirmed by teachers. For instance, students in low engagement classrooms were praised for helping a peer ("Good. That's what a partner should do."), or for following an algorithm ("I like how you wrote the clusters in there, not the products.”). In contrast, students in high engagement classrooms were affirmed for their higher-order thinking (“I like the connections that are being made. You’re connecting these fractions to division and your multiplication, correct?”). Surface level process feedback (such as successful use of an algorithm) may support student confidence, but feedback related to underlying processes supports students in transferring strategies across tasks (Hattie & Timperley, 2007). Corrective feedback also influences students’ disposition. Teachers with high measures of engagement provided surface level corrections to processes, while teachers with low measures were more likely to reject student ideas related to processes (“I don’t think this is a good idea. Let me show you how I would do it, okay?”). We posit that students may enjoy mathematics more when they are supported in making connections across tasks, and when their thinking is “taken up” by the teacher.

Our proposition is further supported by the differences in effort feedback provided by teachers. Teachers with high measures of happiness and engagement affirmed students for asking questions and for collaborative work, and corrected students for passivity. In contrast, teachers with low measures of happiness and engagement affirmed students for following procedure, being organized, and following along, and corrected students for rushing through tasks and “being messy.” In both sets of classrooms, students are provided feedback about how they are engaging with the content. Students’ interest in or affiliation with mathematics is linked to the ways in which they engage with the content (Gresalfi & Cobb, 2006). The feedback from teachers with high measures of happiness indicates that students are expected and encouraged to be actively engaged with the tasks, ask questions, and work together - all of which we presume to have a positive effect on students’ disposition towards mathematics. Conversely, the feedback provided by teachers with low measures of happiness and engagement reinforces the concept of mathematics as rule-bound and highly structured (Gresalfi & Cobb, 2006).

In addition to the analysis of teacher to student feedback, this study also presents a conceptual framework to investigate a more layered view of teacher feedback. However, our framework did have limitations. Since our definition of feedback was limited to teacher statements that were distinctly evaluative, we did not code instances of teachers revoicing or posing questions in response to a student. While coding transcripts, we did see teachers responding to students in these ways. In fact, there appeared to be an abundance of parroting as revoicing, where the teacher repeats the student response verbatim. Further research should include examining the connections between additional forms of teacher feedback (parroting, revoicing, and responding with a question), and students self-reported happiness and engagement in mathematics.

Conclusion

This study adds to the growing body of literature around teachers’ classroom interactions with students and the influence of those interactions on student engagement and disposition. Our findings provide a new aspect of classroom interactions and feedback by combining student survey results with transcript review. Using this approach, we found that both teachers with high and low measures of student happiness use specific feedback in their classrooms more frequently than general feedback. Contrary to popular discussions about ‘growth mindset’ classroom practices, our analysis did not find evidence that students with teachers who are effective at raising test scores are providing students with high levels for feedback on their effort or ability regardless of the classroom’s designation as either a high or low engagement setting.

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BOTTLE FILLING TASK REASONING: A COMPARISON OF MATCHING VERSUS CONSTRUCTED STUDENT RESPONSES

Dominique Banner
Boise State University
dominiquebanner@boisestate.edu

Laurie Overman Cavey
Boise State University
lauriecavey@boisestate.edu

In this paper, we compare the levels of reasoning elicited during the completion of three versions of a bottle filling task: high school level matching; middle school level matching and constructed response. The goal of the tasks was to make visible secondary students covariational reasoning methods. Video of students completing the task while explaining their reasoning during one-on-one interviews were analyzed. Analysis demonstrated a wide range of reasoning when provided a matching version with a greater incidence of accuracy with students who exhibited lower levels of reasoning. Conversely, the constructed response task demonstrated higher levels of reasoning more consistently with decreased accuracy. Implications for assessment are discussed.

Keywords: Algebra and Algebraic Thinking, High School Education, Middle School Education

In this study, video from a larger study in which students completed modified versions of the well-known bottle filling task (Thompson & Carlson, 2017) while explaining their thinking were analyzed (Cavey et al., in press). The solutions and reasoning methods of four groups of students were analyzed for three separate versions of the task: a high school matching, middle school matching, and a constructed response in which the students were asked to draw their own graphs to represent the given situations. None of the versions provided any measures or scales. Across all groups, students were encouraged to verbally explain their reasoning after completing the task.

Assessing student understanding in mathematics from students' written work can be difficult. Students can be successful at completing a task without understanding the targeted learning goal or understand the targeted learning goal but be unsuccessful in the task for an unrelated reason. Although, in mathematics, there is usually one correct answer, it is also important to recognize, and acknowledge, student success in their thought processes. Relatedly, there has been a recent shift in standardized test structures from entirely multiple choice to include some constructed response items (e.g. the addition of "grid-in" questions on the SAT). This shift in testing structure, along with our interest in supporting secondary student's covariational reasoning, motivated us to examine the student reasoning elicited across different types of tasks.

Purpose

In this study, student work on three versions of a bottle filling task, two matching and one constructed response, were compared with the purpose of identifying aspects of the covariational and graphical reasoning students appeared to use and to compare the results between the groups. The questions which guided our analysis are:

- How do the aspects of covariational reasoning exhibited by students compare across the three tasks?
- Do students complete the task more accurately in one format over the other?

In this paper, we describe our methods, results, and analysis of the student work. We also discuss the implications for teaching and student assessment.

Theoretical Framework

This study is informed by research concerning covariational reasoning and assessment of students' mathematical knowledge. The intent of the bottle filling task is to elicit covariational reasoning by asking students to imagine the dynamically changing scenario of a bottle filling with a liquid and think about how the relationship between the height and volume of liquid is represented graphically.

Covariational Reasoning

Covariational reasoning involves coordinating how two quantities vary together and is essential to the understanding of functions (Thompson & Carlson, 2017). Various versions of the bottle filling task have been used in studies designed to assess covariational reasoning. For example, Carlson, Jacobs, Coe, Larsen and Hsu (2002) used a version of the bottle filling task with college undergraduates in which students were asked to create a graph of the height as a function of volume. In addition, Johnson (2012, 2013) used a version in which volume was shown as a function of height and asked high school students to sketch a picture of a bottle which would generate the given graph. The researchers in these studies also developed frameworks for analyzing student covariational reasoning. As the students in our study consist of both middle school and high school level students and we were interested in how the student understanding was translated to a graph, these frameworks informed the creation of our framework, but did not completely align with our purposes.

Assessment

Assessment is an integral component to teaching and learning. Assessment provides teachers and students information regarding a student's level of comprehension. Student work can be assessed: during whole class instruction, small groups, or one-on-one; through verbal or written interaction; and can take the form of formal or summative assessment. Information gathered from assessment can be used to inform instruction, group students, or develop interventions.

Black and Wiliam (2009) discuss the structure of formative assessment in which the teacher presents a task, the learner responds, and the teacher then acts in response to the student. To provide the most appropriate response, it is essential for the teacher to understand the student's reasoning as well as possible. However, as Black and Wiliam (2009) point out, this can be a very difficult task for teachers as it can be impossible to know entirely what a student is thinking.

One method for enhancing teachers' ability to understand a student's reasoning is for the teacher to request the student articulate their thought process, prompting the student to expand the description of their reasoning where necessary and to reflect on their own thinking. Black and Wiliam (2009) cite work by Shayer and Adey (2002) who found that articulating and reflecting on one's own thought is essential to the learning process.

Teachers must determine the types of tasks necessary to elicit student reasoning and questions to create this rich learning environment. Chaoui (2011) cites research regarding the differences in cognitive demand of multiple choice questions compared to constructed response, but also states there can be great difference in the level of cognitive demand required in the way a multiple choice question is constructed. Chaoui (2011) also cites several research studies showing similar performance on multiple choice questions compared to constructed response. This study looks at these types of comparisons specific to the bottle filling task.

Methods

Participants

This study involved the analysis of one-on-one interviews, obtained from a larger study, of secondary students completing one of three versions of a bottle filling task. Thirty-five students completed the task in the first round of interviews (15 high school and 20 middle school) and 16 in the second (9 high school and 7 middle school). It is important to note, high school students in the first round were enrolled in a Calculus class while those in the second round were enrolled in lower level mathematics courses. As we are comparing the reasoning and accuracy levels for each version of the assessment, we do not consider this an issue.

Student Task

During the first round of interviews, students were provided a matching version of a bottle filling task. All versions the bottle filling task used of were adapted from a task created by Swan (1985). The task pictured 3 bottles with different shapes: an evaporating flask, an ink bottle, and a bucket. Two versions of the matching task were developed, one for high school students, which provided 5 graph options, and one for middle school students, which provided 3 graph options. Each version depicted height as a function of volume. Students were also provided the option of creating their own graph.

In the second round of interviews, the bottle filling task was redesigned as a constructed response problem. There were no longer two separate versions, the same three shaped bottles were used, and students were asked to create a graph for each. To assist the students in creating their graph, partial planes were provided with two perpendicular rays representing the first quadrant of a graph. There were no markings on the axes or grid in the plane to indicate any type of scale. The axes were labeled to indicate volume as a function of height. The reason for this change was due to the student's ability to conflate volume with time, and thereby arriving at a correct answer without demonstrating the desired level of covariational reasoning.

Data Analysis

To compare student reasoning and accuracy, videos and transcripts of the interviews were reviewed and coded. Table 1 shows the framework developed for coding student reasoning elicited across all versions of the task:

Table 1: Student Reasoning Coding Framework

Level	Description
1	Recognizes that both quantities increase or provides no explanation related to volume or height (E.g. student graphs a straight line)
2	Makes connections between the shape of the bottle and how one quantity (either volume or height) changes (E.g. Verbally (and with gestures))
3	Coordinates the changes in one quantity with characteristics of a graphical representation (E.g. student explains how one quantity changes over time and translates their ideas to a graph)
4	Makes connections between the shape of the bottle and how volume and height change together (E.g. The student can verbally explain (or show with gestures) how the quantities change together)
5	Coordinates changes in volume and height with characteristics of a graphical representation (E.g. in addition to being able to explain, the student can also translate those ideas to a graph)

While reviewing the student interview videotapes, it was determined that students would occasionally employ differing levels of reasoning for different bottles. Therefore, student work and reasoning for each bottle were coded individually.

For the purpose of coding student accuracy, three different scoring frameworks were implemented, depending on the version of the task. A score of 0 was given for incorrect responses on all versions. For the middle school version of the matching task, students were given a score of 1 for a correct response. The high school version of the matching task was scored as follows: Evaporating Flask, 1 for a correct response; Ink Bottle, 1 for selecting option A and 2 for selecting option D; and Bucket, 1 for selecting option E and 2 for selecting option C. The task was scored in this manner due to the similarities of options A and D (continuous graphs containing 3 sections), as well as options C and E (smooth curves with differing concavity). The constructed response version of the task was scored as follows: 1 for a graph than contained a correct feature (e.g. number of changes in the graph, concavity, etc.) and 2 for a correct graph.

Subsequent to coding student reasoning elicited and the accuracy of their work, a ratio chart was completed cross-referencing the reasoning and accuracy scores. Relative frequencies of each cell were color coded by percentile group (1-9%, 10-19%, etc.) and compared, looking for trends. Trends within individual students were also analyzed.

Results

Through our analysis of the descriptor code ratio charts, it was found students completing both matching versions of the task had a greater incidence of accurately completing the task with lower level reasoning skills. In addition, there was a higher incidence of accurately completing the task for those who exhibited higher levels of reasoning. Students completing the constructed response were more apt to employ higher levels of reasoning: middle school students were much more likely, high school students only slightly more likely (possibly due to the discrepancy between the first round of high school students being enrolled in Calculus). Finally, students completing the constructed response did not show the same level of accuracy as those completing the matching.

During the evaluation of individual students for trends, it was also found, while completing the matching version of the task, students were more likely to employ different levels of reasoning for the variously shaped bottles. Whereas, those students completing the constructed response were more likely to consistently employ the same level of reasoning throughout.

Discussion and Conclusion

Results indicate matching and constructed versions of a mathematical task, such as the bottle filling task, elicit different aspects of student knowledge. Depending on the purpose of the task for the teacher, one version may be more useful over another. For example, a matching version could be more useful as a pre-test to determine prior knowledge on a topic; where constructed response could be more useful to elicit higher levels of reasoning. However, teachers must be mindful of the propensity of a student to perform higher on matching, or lower on constructed response, than their actual ability. To counteract this possibility, teachers may also wish to construct questions aimed at prompting students to articulate their reasoning.

It is important to note the limitations of this study. The groups studied were small and consisted of students who volunteered for a study with a different focus. Although all available videos were included in the study, it was not meant as a representative sample. Therefore, the research conducted here is meant only as an exploratory study.

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DOES PARTICIPANT SELECTION SKEW MATHEMATICS EDUCATION RESEARCH FINDINGS? CONSIDERING QUANTITATIVE REASONING RESEARCH

William Zahner
San Diego State University
bzahner@sdsu.edu

Quantitative reasoning (QR) is a set of mathematical reasoning practices rooted in conceptualizing problem situations in terms of quantities and quantitative relationships. QR research building on Thompson's (1993) framework has widely influenced mathematics education research and teaching. In this research report I (a) introduce a rationale for including more diverse research samples as participants in QR research, (b) present a brief systematic review of the literature showing a lack of diversity in research participants in extant QR studies, and (c) discuss potential benefits of broadening the diversity of the participants in QR research.

Keywords: Equity, Research Methods, Algebra and Algebraic Thinking

In a recent commentary, the Research Committee of the National Council of Teachers of Mathematics (NCTM) issued a call for collective action toward equity across the field of mathematics education research (Aguirre et al., 2017). One fundamental mechanism for incorporating an equity stance in mathematics education research is acknowledging the sociocultural diversity of students in school mathematics classrooms and designing studies that are responsive to student diversity. Yet, there is a well-documented lack of diversity, in general, among participants in social science research (Henrich, Heine, & Norenzayan, 2010). Many research-based claims about human cognition are based on research that was conducted mainly with White, educated participants, and Henrich et al. (2010) show that the reasoning of White, educated research participants may be quite unusual when put into global context. Taken together, the NCTM Research Committee's stance and Henrich et al.'s (2010) review prompt reflection on the current state of mathematics education research.

In my recent work, conducted in collaboration with a team of researchers, we[1] have examined the quantitative reasoning (QR) of linguistically diverse groups of middle school students, including many students classified as English learners (Els[2]). To ground our work, we reviewed extant research on QR. Attempting to use insights from QR research has forced us to consider whether and how learning to engage in QR might be different for ELs. In this brief report I present a motivation for investigating QR among students learning the language of teaching and learning. Next, I present a brief review of the peer-reviewed journal articles on QR, focusing on the specific issue of research participant selection. Finally, I close with a discussion of what might be gained by broadening the pool of participants in QR research, and consider implications for other areas of mathematics education research.

QR and Language

Quantitative Reasoning is a form of mathematical reasoning that is rooted in making sense of problem situations. QR as conceptualized by Thompson (1993, 2011) and Thompson and Smith (2008) is a process that involves identifying quantities in a situation and then building and reasoning with quantitative relationships. Thompson and Smith (2008) describe how QR often starts with a student *imagining* a problem situation, identifying initial quantities, and then iteratively developing new quantities. For example, a student may solve a word problem about a race with a head start by imagining herself in the race catching up to another runner. Then she might focus on the rate at which the distance to the runner ahead closes. Smith and Thompson (2008) argue that reasoning quantitatively (rather than applying formulas such as D=RT) can serve as a gateway to algebraic reasoning and that this form of reasoning connects to student experience:

> Quantitative reasoning *draws heavily on everyday experience.* The basic approach… depends on the reasoner projecting herself into the situation…. Quantitatively oriented solutions tend to vary more widely than algebraic solutions to the same problem, primarily because they are *grounded in how students conceive of situations,* and there is *tremendous range in these conceptions* (Smith & Thompson, 2008 p. 105, emphasis added).

Research on students' QR has led to innovations in mathematics education, including the development of learning trajectories (Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016) and technological tools (Roschelle, Kaput, & Stroup, 2000). This research has also influenced curriculum and standards documents. In particular, QR research appears to be one of the primary sources for the second standard for mathematical practice (SMP2) in the Common Core State Standards, "Reason Abstractly and Quantitatively" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). However, it bears noting that Thompson (2011) describes a framework for QR that delves into significantly more detail than the text of SMP2.

What is the Relationship between QR and Language?

The quotation from Thompson and Smith highlights that QR builds from student experience. Typical descriptions of QR start with students interpreting problem situations (usually presented in text), creating abstract models and representations of quantities and quantitative relationships (usually equations, diagrams, tables, or texts), manipulating these models and representations to produce proposed solutions, and then communicating the proposed solution and checking its reasonableness (coordinating texts). Throughout this QR process, students engage in disciplinary literacy practices (Moschkovich 2015). This engagement can be challenging for all students, and it can be especially challenging for ELs when mathematical problems are set in contexts that may be unfamiliar (Martiniello, 2008; Zahner, Milbourne, & Wynn, 2018).

The linguistic complexity of QR, together with researchers' typical focus on students' verbalizations may lead researchers to select participants whose verbal communication is easy for the researchers to understand. For example, Ellis (2007) described criteria for selecting case study participants: "Those who had average-to-high grades and could articulate their thinking were desirable as interview participants because they were likely to have developed powerful generalizations in class that they could subsequently explain and discuss in the interview" (p. 447). Ellis's transparency about these selection criteria helps readers interpret the results of the study. As will be seen below, Ellis (2007) is an exception in the QR literature because Ellis made these participant selection criteria transparent. However, salient to our interest in the reasoning of ELs, this selection process excludes students who are not able to articulate their thinking in a way comprehensible to a researcher but who may provide additional insights into QR.

A Brief Review of Participants in QR Research

To understand this issue better I conducted a brief systematic review of QR research focused on participant selection criteria. The research literature on quantitative reasoning is distributed across journal articles, conference proceedings, book chapters, and monographs. In order to capture a snapshot of the literature, I focused on empirical papers appearing in peer-reviewed journals that have used Thompson's (1993, 2011) framework for QR. I conducted a search of the ERIC Database using the search string "quantitative reason*", restricting the results to peer reviewed studies. The initial search yielded 171 potential articles. I noted that the *Journal of Mathematical Behavior* is not completely indexed in ERIC. Since many QR studies have appeared in *JMB* (e.g., Lobato & Siebert, 2002), I also searched *JMB* individually on the journal website. This yielded 31 additional articles for consideration (with some repeats on the original ERIC list). I and a research assistant then reviewed the titles and abstracts to remove duplicates and identify whether articles were about quantitative reasoning and using the framework from Thompson (1993). (A number of articles were about "quantitative reasoning" courses or tests that had no connection to Thompson's framework). At this stage we also noted there were a number of studies related to newer constructs (e.g. covariational reasoning) or learning trajectories (e.g., a trajectory for exponential functions) that were derived from Thompson's QR work. For the sake of focus we narrowed our attention to studies that primarily relied or developed on the constructs of QR. After applying the exclusion criteria, I narrowed the review to 21 studies. Next, each article was analyzed to identify the study design, number of participants, age of participants, and demographic information provided about participants. Studies were also coded according to primary method used and whether demographic information was provided about the participants. Table 1 contains a summary of the coding.

Table 1: Summary of Methods and Participant Demographics for QR Studies

Method of Study	No Mention of Demographics	International Setting / Comparison	Demographics Described	Explicitly Sampled for Diversity
Interview	7	1	1	
Teaching Experiment	6		3	
Observation			1	
Other		1	1	
Total	13	2	6	0

Thirteen of 21 studies in this review did not describe demographic information about participants. In these 13 studies, case-study students were typically described by grade level, and gendered with a pseudonym, but no additional demographic information was provided. Case selection criteria often highlighted students' salient or interesting forms of reasoning, as well as students' willingness to participate in research. Two studies were conducted in international settings. These studies did not include additional demographic information about participants other than nationality. Of the six studies that included descriptions of participant demographics, all but two described the demographics of the school but not the participants in the study. Details on participant demographics were often vague. For example, Olive and Caglayan (2008) write "The 24 students were between 13 and 14 years old and had been placed in the algebra class based on their success in 7th-grade mathematics. The students were racially, socially, and economically diverse, with an approximately equal distribution of gender" (p. 273). The previously discussed paper by Ellis (2007) was unusual as Ellis included information about participants' racial/ethnic identity and English learner status. Based on the review of QR study methods and participants, I concluded that none of the QR studies in this review explicitly selected culturally or linguistically diverse participants to investigate social, linguistic, or cultural features of QR, with the exception of the one international comparison study (Thompson, Hatfield, Yoon, Joshua, & Byerley, 2017).

Discussion

As described in the introduction, I chose to do this review of QR research in support of an ongoing collaborative project in which we are investigating QR among English learners. One emergent finding in that work is EL groups engaged in significant effort to make sense of the problem context, and they did fewer iterations of QR than non-EL groups in similar conditions. One possible implication of this result is that if educators are designing lessons to promote QR for linguistically diverse students, then they need to consider whether all students have adequate access to the problem context. Current trends in participant selection of case study students for QR research may warp researchers' view of QR and may bias research to recognize only those students whose ideas are typically heard and validated in mathematics classrooms.

The results presented here are not intended to impugn QR research or the people who are doing this work. It is possible, for example, that researchers doing QR research did not include student demographics in case study descriptions to avoid the appearance of stereotyping. Additionally, it is likely that the issues related to participant selection identified in this paper are not unique to QR research, and would be found in reviews of other areas of mathematics education research. Nonetheless, omitting participant demographics is problematic because, paralleling the broader phenomenon of selection bias in social science research (Henrich, Heine, & Norenzayan, 2010), there is a danger that many "research-based" results in mathematics education are based on a small and skewed sample of students--typically White middle-class students whose verbalizations are easy for researchers to interpret.

Moving forward, to respond adequately to the NCTM Research Committee's call (Aguirre et al., 2017), there is an urgent need for mathematics education researchers to develop consistent ways for describing the demographics of research samples, as well as principled rationale for participant selection. This need to report demographics is especially important in domains of research like QR, where there is a preponderance of small *n* in-depth qualitative case studies. Moreover, in the name of both promoting equity and high-quality science, the field of mathematics education must engage in continued dialogue about the overarching issue or research participant selection. Some questions for consideration are, What principles should guide the selection of research participants? How do we ensure that our "research-based" results are, in fact, derived from research that considers the full diversity of students enrolled in our mathematics classrooms? These and more questions must be addressed as we develop a more robust and equitable mathematics education research enterprise.

Endnotes

1 This Research Report was solo authored but reports on work done in a collaborative project. Tracy Noble, Teresa Lara-Meloy, HeeJoon Kim, and Phil Vahey all contributed to the conceptualization of the project. The project was Funded by NSF Grant #1534626. Any opinions, findings, and conclusions or recommendations are those of the authors and do not necessarily reflect the views of the National Science Foundation.

2 The term English learners is limited because it 1) lacks of acknowledgement of students' emerging multilingual status 2) hides the diversity of students under the category ELs, 3) downplays the variation in the criteria used to classify students as ELs, from state to state and even across school districts within a given state (Sireci & Faulkner-Bond, 2015). Nonetheless, the category English Learner (EL) identifies a group of students who have historically not received adequate attention from the research community. For this reason, I will use the category EL for the emerging multilingual students (Hopewell & Escamilla, 2014) described in this paper, while also acknowledging the category's limitations.

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"THEN WE GO LIKE… THIS": MATHEMATICAL ACTIVITY AND COLLABORATION IN A TINKERING ENVIRONMENT

Ruben Echevarria
Northwestern University
rubenechevarria2015@u.northwestern.edu

This study aims to push beyond narrow and normative characterizations of mathematics by using Bishop's (1991) definitions of mathematical activity to identify "designing" in the context of a tinkering environment, and to better understand what "designing" accomplished. Tinkering environments focus on interdisciplinary science, technology, engineering, and mathematics (STEM) learning and emphasize creating, inventing, and iterating projects. I analyzed video from a tinkering environment that served nondominant youth and focused on a group of girls who participated in "designing." I identified two different actions of designing, which I call creating "durable" and "ephemeral" representations and found that the group used designing as a way to communicate and coordinate their contributions.

Keywords: Equity and Justice, Using Representations, Communication, Informal Education.

Introduction

Students from nondominant communities have an extensive history with deficit labeling (Medin & Bang, 2014; Valencia, 2012; Yosso, 2002) and their mathematical practices routinely go unnoticed in schools (Appelbaum & Stathopoulou, 2016; Nasir, Rosebery, Warren, & Lee, 2006). Therefore, I argue for using a wider epistemological lens that looks beyond how schools define mathematics as a way to position nondominant communities as resources and producers of knowledge. Although there may still be risks of reproducing a narrow definition of what counts as science, technology, engineering, and mathematics (STEM) and who is considered successful in STEM in ways that exclude nondominant communities, tinkering environments are a possible context where more expansive definitions may be possible because they invite and encourage multiple pathways and solutions. As a result, participants are more likely to create their own systems of value based on what they find useful, such as invented strategies and different mathematical and collaboration practices that are not valued in schools. I understand tinkering as a playful, experimental, and iterative disposition towards design and making, one "often replete with the utterance and practice of *what if, could be, maybe, perhaps, let's try it out*, etc." (Vossoughi, Escudé, Kong, & Hooper, 2013, p. 4). When the kinds of utterances mentioned above are allowed to be pursued, it may support students' learning by creating a context for "reassessing their goals, exploring new paths, and imagining new possibilities" (Resnick & Rosenbaum, 2013, p.164). In theory, tinkering activities are designed to support multiple pathways and solutions (Vossoughi et al., 2013), in part through enabling a "bottom-up" approach where goals and approaches are emergent and negotiated in real-time (Resnick & Rosenbaum, 2013). Of particular interest is the claim that tinkering is "exactly how real science and engineering are done" (Martinez & Stager, 2013, p. 41).

Conceptual Framework and Research Questions

Bishop (1991) argues that mathematics is a product of culture and that all cultures generate mathematics. However, "the face of mathematics is White; the people most credited with 'inventing' or 'discovering' it are the Greeks and others of European descent" (Gutiérrez, 2015, p. 270). There are direct implications of this narrative in mathematics education as reflected in how "girls, Blacks, Latin@s, American Indians, recent immigrants, English learners, and students who come from working class families" are overrepresented in developmental courses, underrepresented in advanced courses, and underperform on standardized tests (Gutiérrez, 2015, p. 272). As a result, the educability of those students is questioned, arising from and often leading to deficit labeling (Valencia, 2012) because mathematics is used as a proxy for intelligence and those that are unsuccessful in school mathematics are seen as less intelligent (Gutiérrez, 2015) or even "disposable" (Pais, 2011, p. 218).

The belief that mathematics is free or absent of culture has also been debunked because culture shapes the form of the mathematics (D'Ambrosio, 1991). In fact, mathematics has been co-constructed through cross-cultural contact between at least Egypt, Greece, Mesopotamia, India, the Arab world, China, the Mayan empire, and Europe (Ascher, 2002; Harris, 1991; Joseph, 2010; Powell & Frankenstein, 1997). Mathematics as "free of culture" or "universal" holds that the rules of mathematics apply everywhere, but does not, for example, question the motivation behind mathematical ideas (Bishop, 1991). In particular, Bishop (1991) claims that there are six universal forms of mathematical practices: counting, locating, measuring, designing, playing, and explaining. However, Bishop warns against interpreting his use of the word "universal" as a claim that everyone, for example, counts the same way. Therefore, using Bishop's wider epistemological lens of mathematical activity, I decided to focus on "designing" because although multiple mathematical practices were enacted in the tinkering environment, the tinkering activity I focused on lends itself to designing which Bishop (1991) defines as

> Creating a shape or a design for an object or for any part of one's spatial environment. It may involve making the object, as a copyable 'template', or drawing it in some conventional way. The object can be designed for technological or spiritual use and 'shape' is a fundamental geometrical concept. (p. 33)

In this study, I draw on Bishop's definition of designing to investigate the following research questions: What does designing look like in the context of a tinkering environment? What did designing accomplish in a tinkering activity?

Methods

This study draws from a larger ethnographic dataset spanning three years of research carried out by Shirin Vossoughi and Meg Escudé at the Tinkering Afterschool Program (TAP), a collaboration with Boys and Girls Club in California that functioned as a drop-in space for youth (K-5). The program predominantly served African American, Latinx and Asian American youth. I drew on interaction analysis (Erickson, 1992; Jordan & Henderson, 1995) to analyze a video clip (1m17s) of two youth, Tania and Stefanie (pseudonyms), who worked on a tinkering activity called Marble Machines. The objective of Marble Machines was to work in pairs, create an obstacle course for a marble, and produce a story about the course from the marble's perspective.

After watching the video, I tagged the stretch of interaction as "designing" as defined by Bishop. Although other mathematical practices were used during the stretch of interaction, I focused on designing for this brief report because Tania and Stefanie engaged in designing as a way to coordinate their collaboration by creating representations to communicate, contest, and invent ideas. I segmented the video for this analysis into four episodes reflecting shifts in activity directions (Matusov, 1998) (Figure 1). In the first episode, Tania and Stefanie worked on deciding the placement of the plastic tube on the pegboard. Then, they worked on deciding the placement of the wooden plank below the plastic tube. Next, they worked on deciding how to block the marble. The final episode is presented as a culmination of all the episodes where Stefanie drew a representation of all their designing from previous episodes. Noting that the youth created varied representations across episodes, I developed grounded codes (Charmaz, 2006) to identify two different actions of designing, which I call creating "ephemeral" and "durable" representations. Creating a durable representation is an action that is physically available for reference or iteration. Creating an ephemeral representation is unlike creating a durable representation in the sense that ephemeral representations as actions disappear from the physical space but can also be remembered and used for reference or iteration.

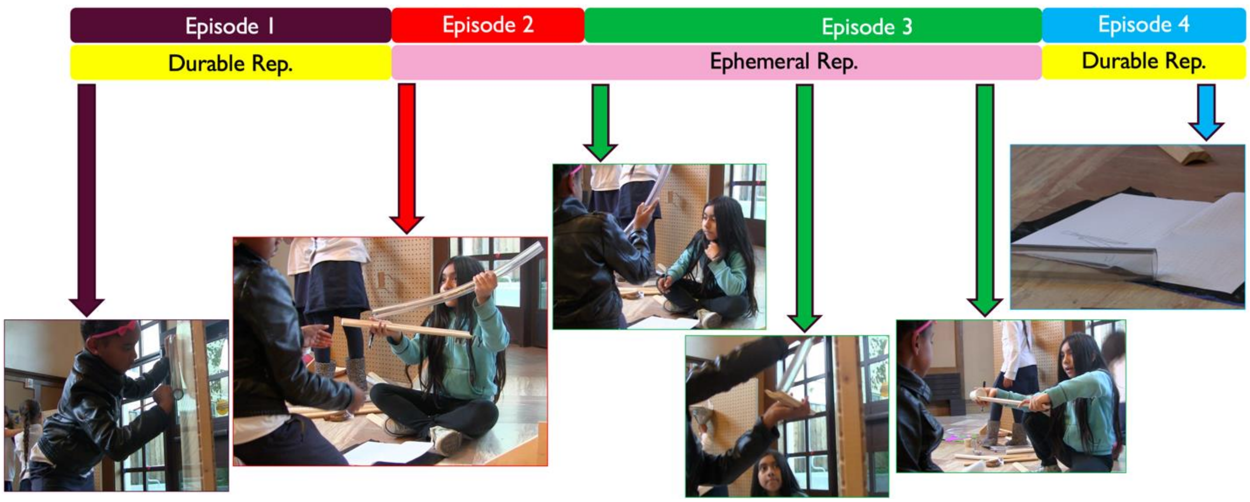

Figure 19: All Four Episodes are Presented Chronologically and Show Actions of Creating Durable and Ephemeral Representations. Tania (left) and Stefanie (right) in Episode Two

For this brief report, I will only elaborate on Episode Two to describe what "designing" looked like and what "designing" accomplished for Tania and Stefanie.

Analysis

To provide a brief overview of the first two episodes, Tania and Stefanie agreed on the placement of the plastic tube in Episode One. Immediately after, they worked on deciding how to add a wooden plank to the design. In Episode Two, Tania and Stefanie decided to put a wooden plank below the plastic tube to catch the marble. When Stefanie sat down, she grabbed a wooden plank and positioned the wooden plank and plastic tube using her hands to show how a marble would drop through the tube and feed into the wooden plank. She created a representation with her hands and the materials and held them in the air to show Tania, and waited for her approval (Figure 1, Episode 2). The episode ends with Tania agreeing and Stefanie handing the materials to Tania.

In Episode Two Stefanie created an ephemeral representation that helped coordinate ideas with Tania. Episode Two began with Stefanie gathering the wooden plank from the floor and the plastic tube from Tania's hand as she said, 'Then, then we go like, then we go like this." It is worth noting that Stefanie could have explained her idea without the materials in her hand, but decided to grab them as she started talking. When Stefanie had the materials in her hands, she started to create the ephemeral representation, and at the same time she said, "Then we go like…

this" to show how to possibly incorporate the wooden plank in the design. Stefanie finished creating her ephemeral representation when she said "this" after pausing briefly to complete her idea and she held her hands still and shook her head up and down more than twice. When Stefanie held her hands still, she created a "copyable template" for her and Tania to think with and remember. I interpret the head nod as an invitation for Tania to give a "yes" or "no" answer, but preferably a "yes" answer. Tania tilted her head and examined the ephemeral representation for nearly two seconds before saying "Yeah." I interpret Tania's pause before saying "Yeah" as an indication that she is contributing to the design and not merely agreeing with the idea for the sake of agreeing with Stefanie. When Tania agreed, Stefanie handed the materials over to Tania in such a way that resulted in Tania holding the materials the same way Stefanie held them, but a mirrored version. This transferring of materials is significant because it extended the life of the ephemeral representation to a new person. Tania then became the author and she used the materials to create an added ephemeral representation to illustrate a concern in Episode Three.

Across all of the episodes (including the other three not discussed here) analysis documented that Tania and Stefanie both created durable and ephemeral representations to coordinate their contributions. Creating durable representations may serve as a way to finalize ideas, such as Tania's drawing in the first episode and Stefanie's drawing in the last episode, and creating ephemeral representations may serve as a way to test ideas. It does appear that ephemeral representations become actors in the interactions in a different way than durable representations because Tania and Stefanie created ephemeral representations to share and contest ideas, but created durable representations less often and only after they reached an agreement. Overall, Tania and Stephanie demonstrated a level of expertise in communication and it is evident in how they collaborated smoothly moving to and from activity directions, even when ideas were contested, and how both of their ideas were represented in the final version of their project.

Significance of the Study

The case study presented is an example of how mathematical practices can be made visible to educators and researchers and raises questions of what other mathematical practices we can see in tinkering environments using Bishop's (1991) definitions of mathematical activity. I was able to make visible the skills that Tania and Stefanie engaged in to communicate, contest, and invent ideas. Without this study, their deep collaboration may have only been attributed to their friendship, erasing the thinking and mathematical activity they both engaged in based on the ubiquity of normative and more narrow ways of recognizing mathematics. One possible implication for educators is to learn how to recognize and acknowledge the mathematical practices that exist in a space and leverage such practices in teaching and designing activities. For example, schools that emphasize the process of working out solutions in their classrooms could benefit from learning to see and value the actions of creating ephemeral representations as well as durable representations. I could imagine a school that would evaluate Tania's and Stefanie's performance based solely on their drawing in Episode Four because it is challenging to grade ephemeral representations as depicted in Episodes Two and Three. Perhaps, as educators, we can find ways to go beyond emphasizing process to our students and learn to see their expertise in different processes. More broadly, my aim is that others will look beyond the stereotypical imagery of mathematics, using my methodology or their own, to learn when, what, and how mathematics emerge in their spaces as a way to see nondominant communities as resources and producers of knowledge. This wider epistemological lens invites new approaches to research on learning and teaching of mathematics, and is a step in repairing the damage caused by researchers that imposed and validated deficit framings on nondominant communities.

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EMMA'S COUNTING AND COORDINATION OF UNITS: SUPPORING SENSE MAKING FOR STUDENTS WITH WORKING MEMORY DIFFERENCES

Jessica H. Hunt, Ph.D.
North Carolina State University

Juanita Silva, Ph.D.
Texas State University

We investigated how one elementary child with working memory differences made sense of number as a composite unit and progressed in her reasoning. We analyzed eight teaching experiment sessions using ongoing and retrospective analysis, where we uncovered four shifts in the child's thinking of number over time that suggests an association between the child's participatory knowledge and the extent to which she enacted activity met her goals for solving the problem more than her current "knowing".

Keywords: Learning disability, Working memory, Number knowledge, Constructivism

Number reasoning is one of the most challenging areas in mathematics for children with learning disabilities (LD) (National Center for Educational Statistics, 2009; National Mathematics Advisory Panel, 2008) and a fundamental area to develop (National Mathematics Advisory Panel, 2008). Yet, research on these children's flexible cognitive structures (e.g., Olive, 2001) is rarely if at all utilized as a basis for instructional intervention. We address a gap in relation to what is known in mathematics education concerning how children without LDs develop their number knowledge and how children with LDs advance their knowledge in the same learning environment. We examine the following research question: (a) How does the child's working memory and prior mathematical knowledge interplay with the child's learning of number?

Conceptual Framework: Small Environments

"Small environments" situates learning within children's adaptive activity in place of remediation (see Hunt & Silva, in press). Children advance their cognition by noticing and reflecting upon their actions such that they become coordinated around a central idea (Piaget, 1972; von Glasersfeld, 1995). *Noticing* begins when the child becomes aware of what happens as a result of her actions as she works through pertinent situations. *Reflection* supports the child to begin to anticipate coordinated actions and coordinate them so that she can draw upon them in future situations.

Prior research explains that children with working memory differences experience certain difficulties when constructing new knowledge that is not yet abstracted (Hord, Tzur, Xin, Si, Kenney, & Woodward, 2016). We hypothesize this occurs because the child does not see the need to adapt her reasoning as she is noticing and reflecting on her new actions. The child makes evaluations regarding how new actions help her solve the problem and may discard noticing and reflecting in favor of past compensation aids for learning differences (e.g., automatizing "learned facts" as opposed to forming groups of tens; Piaget, 1972). This is a possible explanation as for why children with working memory differences may not transition toward anticipatory understanding.

Much of the intervention research in special education argues to remove the mathematical activity from the child in instruction once she becomes "stuck" (e.g., Gersten, Chard, Jayanthi, Baker, Morphy, & Flojo, 2009). Yet, we argue that, by necessity, as opposed to delivering knowledge onto the child and measuring her response to teacher, teaching must become responsive to the child's activity and use the child's shifting noticing and reflection as a resource to decision making (Jeltova, Birney, Fredine, Jarvin, Sternberg, & Grigorenko, 2011; Tzur, Johnson, McClintock, & Risley, 2012). Thus, our framework is based on the teacher's instructional decisions to promote the child to remain engaged in noticing and reflecting upon her own reasoning.

Methods

This study utilized a constructivist teaching experiment design (Steffe & Thompson, 2000). Emma, the sole participant, was a 10-year-old child enrolled in an elementary school in the southwest portion of the United States. Inclusion criteria for the study were: (a) having an individualized education program goals in mathematics, (b) a cognitively defined label of learning disability (LD) with working memory as the dominant cognitive factor, (c) identification through clinical interview data that the child used notions of two as a usable unit, and (d) identification by the classroom teacher as not benefiting from supplemental, small group instruction from a textbook or supplemental curriculum over at least a two year period.

We used a trajectory from mathematics education research to plan tasks for Emma (e.g., Olive, 2001). Learning situations were organized into two areas: (a) a linear number board game (Tzur & Lambert, 2011) and (b) a contextualized number game where students converted back and forth between "ones" and "composite units" (Simon, personal communication). We positioned teaching as indirect and a response to student's unique knowing and learning (e.g., Butterworth, Varma, & Laurillard, 2011; Compton, Fuchs, Fuchs, Lambert, & Hamlett, 2012; Mazzocco & Devlin, 2008; Murphy, Mazzocco, Hanich, & Early, 2007; Vukovic, 2012). We define *responsiveness* as the researcher-teacher revisiting *her own behavior* to better align teaching to the child's prior and current knowing. The planned teaching included three primary moves. The first was changing the learning situation in some way (context, constraints, representation, the task itself) to match the child's actions and/or create a need for adaptation.

The second was orienting the child to notice to and discuss how *she* was using representations or strategies. The last was to promote the child' noticing and reflection on why the new activity was more meaningful than how she was, in that moment, operating.

Data were gathered using video cameras to capture student responses when engaging in the tasks throughout the eight sessions. Two teacher researchers formed a research team and met before and after each session to examine the student's understanding, redesign tasks for the next session (conceptual analysis), and afterwards to examine the student's understanding after data were gathered (retrospective analysis). Prior to each meeting, the research team developed weekly field notes, transcripts, and coded these data. Meetings focused on (un)common responses from Emma and the nature of the codes (e.g., open codes, categorized codes).

To depict how meaning became negotiated with the teacher, we considered evidence of bidirectional (child ⟵⟶ teacher) communications (verbal and gesture-based) that seemed to occur and coincide with Emma's conceptual shifts. Specifically, we used in-vivo coding as an initial start to this inquiry (Saldaña, 2015). Next, we applied the codes in a more standard way to the text utilizing three main questions: (a) what is the context of the situation? (b) what are the child and teacher saying and doing? and (c) what might the teacher's actions and statements take for granted about the child's knowing and her reasoning (Charmaz, 2011; also see Corbin & Strauss, 1990). We used these questions to consider how the teaching moves in each instance served to maintain, extend, or, at times, hinder the child's way of reasoning. The analysis yielded emergent evidence (Creswell, 2007) of responsiveness done by the teacher and ways in which the teaching moves enabled the way Emma made sense of the mathematics.

Select Results

During all phases of Emma's learning, evidence of both learning differences as well as past compensations the child made to support her working memory differences constrained her reasoning to some degree. However, Emma could (and did) advance her reasoning. This report shows some examples of Emma's progression with number concepts. Full details of the child's advancements will be shared in session.

Challenges with Finger Patterns and Larger Addends

In one board game problem, Emma quickly responded with "14" when trying to figure out the sum of three and 11. This prompted a suggestion from the researcher-teacher to explain her solution by using her fingers and counting from three. Soon, she began counting from three, raising a finger for each time she counted on, yet miscounted when she crossed the number ten, finally arriving at 15 as an answer as opposed to 14. This miscounting prevented Emma from determining "how many" by coordinating her counting with addends larger than ten; this activity continued across several problems. The next excerpt begins in the first problem in session five, just after the child had rolled a three and a 13. The child quickly said "16". The researcher asks the child to justify her reasoning using her counting. As opposed to coordinating her counting through figurative activity, Emma brought forward her procedural knowledge.

Excerpt 2: 3 + 13 (Session 5)

E: Like 3…4, 5, 6…16. And then the one.

R: You said, 3…4, 5, 6 and the one. What's the one? E: The one is the… [pauses; frowns].

R: Can you show me what you mean? [hands child a paper and pen] E: So it was…13 plus 3 is… [writes 13 + 3 = 16 long form].

R: [points to the one in 13] So this is the one? E: That's the one in the 13.

R: But is that like, ONE [shows one finger]? E: Yeah only one.

R: So, if this is one, how come I can't just count one more…seven? E: Because…three doesn't have a one.

R: Suppose you had started counting from this three [points to the three underneath 13 and references third space on the game board]? How would you do that? [removes paper and game board]

E: [puts up hand] 3…, 4 [raises one finger], 5 [raises 2nd finger], 6 [3rd finger], 7 [4th finger], … 8 [5th finger], 9 [6th finger], 10 [7th finger], 11 [8th finger], 12 [9th finger], 13 [10th finger; stares at ten fingers and pauses for 5 seconds].

R: 13…. How many have you counted so far? E: Ten.

R: How many more do you need to count?

E: I think… [sticks out lower lip; pauses for 3 seconds]. I think…. [frowns, looks down].

R: [grabs a paper and pen] So you started at three [writes three and an empty number line] …and you did 4 [makes hop on number line], 5 [makes hop on number line], 6 [makes hop on number line], 7 [makes hop on number line], 8 [makes hop on number line], 9 [makes hop on number line], 10 [makes hop on number line], 11 [makes hop on number line], 12 [makes hop on number line], 13 [makes hop on number line]. Then you stopped and did this [holds up all ten fingers and wiggles them, covers up the number line].

E: Oh! 13… 14 [raises a finger], 15 [raises a finger], 16 [raises a finger]. I need three more. I needed three more to get 16.
R: How did you know it was three more? E: *Because my answer got me 16.*
R: OK. I wonder if there is a way to figure it out without having to know the answer first. E: [shrugs]

Emma begins this problem by using an algorithm to find the total. She names the one in 13 as "one" as opposed to "10". Yet, when pressed to use her figurative counting, Emma notices thirteen as ten and some more, ceasing her activity. When Emma could not explain the meaning of the "one" in 13, the researcher-teacher realized the child needed a way to relate her algorithmic knowledge to her emerging, coordinated counting. To do so, she introduced a counterargument - "how come I can't just count one more ...seven"- to entice the child to engage in her own misconception about the "one" in 13. Emma said, "Three doesn't have a one in it". In the moment, the response puzzled the researcher-teacher, so she asked Emma to begin counting from the three. This proved to be a fortunate teaching decision, as it restored the child's counting and coordinating activity and also facilitated the child to notice "10" in her figurative counting. Yet, the child struggled to coordinate the "10" on her fingers with the "10" and "three" that comprise the second addend. Realizing this may be due to working memory, the researcher-teacher introduces a second representation—the open number line—to simulate the child's counting. The partial re-showing and supplementation of Emma's previous activity seem to remind her of her counting. Knowing that the answer to the problem was 16, Emma counted up three more.

Emma made a connection, although she was not yet aware of it, between "13", "10" and "three" in her figurative counting. Yet, she did not link this activity to the problem solution. As opposed to decomposing the second addend into 10 and three, reflecting upon and coordinating her figurative counting, and arriving at the solution of 16, the child *counts up to* a known answer that she previously arrived at procedurally. Arguably, Emma used this reasoning to compensate for demands on working memory during the coordination of counting. Yet, we argue that Emma's use of procedures to arrive at a total kept her counting actions *sequential* as opposed to *coordinated*. Over the next several problems, Emma continued to (a) use her fingers to hold a start value for counting, (b) figure out the stop value through an algorithm, and (c) used the known stop value to count up to using her fingers to decompose the second addend and keep track of the count. The activity continued despite our several attempts to ask for alternate ways in which to consider the problem ("I wonder if there is a way to count on without having to know the answer first?"). This use of algorithms to first figure out the answer and then count up to it eclipsed Emma's propensity to notice and reflect upon her figurative activity to coordinate a double count (3... 4 [that's 1], 5 [2], 6 [3] ... 16 [13, or ten and three]). As opposed to an awareness of two number sequences at once, Emma kept the two number sequences separate when considering addends larger than 10 whose result crosses decades.

Yet, later in the sessions, Emma advanced her reasoning in a problem involving the addition of 13 and 14. We include this important shift in the child's reasoning in session.

Conclusion and Significance

The implicit intervention methods used in this study involved explicitness, yet not in ways reported in much of the special education literature and policy (e.g., Gersten et al., 2009). When we first worked with Emma, she evidenced premature counting, suggesting that her learned strategies and procedures replaced noticing, reflection, reasoning and sense making, and (later) abstracting number knowledge. Yet, for Emma, a child who has differences in her working memory, the inclusion of timely tasks, interactions, and questioning were critical in supporting her noticing of and reflecting on her own activity to promote learning.

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DISCOURSE AND STUDENT BELIEFS WITHIN A STANDARDS-BASED MIDDLE SCHOOL MATHEMATICS CLASSROOM

Jennifer Ericson
University of Massachusetts Amherst
jericson@umass.edu

Sandra Madden
University of Massachusetts Amherst
smadden@umass.edu

Student participation in high-quality discourse promotes greater opportunities to learn and may lead to higher mathematics achievement. Despite implementation of standards-based mathematics instruction [SBMI], students may not participate in discussions due to their mathematics related beliefs. This study examined students' participation in small-group inquiry in relation to mathematics beliefs. Data consisted of survey responses, classroom observations, and discourse analysis based upon an established discourse taxonomy. Findings revealed consistencies and inconsistencies between observed discourse practices and stated beliefs and point to the need to consider beliefs to support higher-quality discourse.

Keywords: Beliefs, Classroom Discourse, Instructional Practices

Student participation in mathematical discourse—talking, writing, listening, agreeing and/or disagreeing, and gesturing about mathematics—is considered integral to learning mathematics (NCTM, 2014). The degree to which high-quality mathematics discourse occurs in a classroom, including justification of reasoning, predicting, relating to prior knowledge, challenging the thinking or reasoning of others, and generalizing, is a positive predictor of middle-school and high-school students' achievement (Weaver, Dick, & Rigelman, 2005). Despite implementation of pedagogical practices and classroom environments that support high-quality discourse, students may participate in discourse with varying degrees of engagement leading to unequal opportunities to learn mathematics (Esmonde & Langer-Osuna, 2013). Student beliefs about mathematics can impact their discourse engagement. If students believe argumentation or justification is not a part of mathematics or that the teacher is the locus of knowledge, they may not participate in whole class discourse (Hoffman, 2004; Jansen, 2008); nonetheless, there have been few studies on the impact of students' beliefs on small-group discourse (e.g., Esmonde & Langer-Osuna, 2013).

Mathematics-related beliefs [MRB] are the implicit and explicit beliefs one holds true that have an important influence on mathematics learning and problem-solving (De Corte, Op't Eynde, & Verschaffel, 2002). Studies of MRB include beliefs about the nature of mathematics and mathematical learning, the self in the context of mathematics learning and problem solving, mathematics teaching and the social context of mathematics learning and problem solving, and epistemological beliefs (De Corte et al., 2002). These beliefs, often shaped by and situated in classroom practices (De Corte, Verschaffel, & Depaepe, 2008; Greene, Muis, & Pieschl, 2010; Maltese & Tai, 2010; McGregor, 2014; Muis, Franco, & Gieus, 2011; Schoenfeld, 1989), influence how students engage with and conceptualize mathematics as well as continue in their study of mathematics and related sciences (Maltese & Tai, 2010; Moakler & Kim, 2014).

MRB can affect students' academic performance directly and indirectly (Cano, 2005; Trautwein & Ludtke, 2007). Beliefs that result in negative or neutral consequences are considered non-availing and beliefs that result in positive consequences are considered availing (Muis, 2004). When students believe they are capable of learning mathematics through perseverance (an availing belief), they are more likely to employ productive cognitive and behavioral learning strategies (Paulsen & Feldman, 1999; Schommer-Aikins & Duell, 2013) and persist in solving challenging problems (Moakler & Kim, 2014). Students who believe learning is the acquisition of knowledgde imparted by the teacher (a non-availing belief) view public sharing of their thinking as risky for fear of being wrong (Hoffman, 2004).

Research has demonstrated that students' experiences with standards-based mathematics instruction, those practices of learning mathematics that support all learners (NCTM, 2000; 2014), can have a positive impact on students' beliefs about what it means to do mathematics and who can do mathematics (e. g., Boaler, 2006; De Corte et al., 2008; McGregor, 2014; Muis et al., 2011), ultimately having the potential for improving student achievement in and affect towards mathematics (Schoenfeld, 1989; Schommer-Aikins, Duell, & Hutter, 2005). The purpose of this study was to examine the relationship between students' MRB and their small-group discourse within a standards-based mathematics classroom.

Methods

This report draws on a subset of data from a larger mixed methods study that examined students' MRB and their small-group discourse participation during the second quarter of 7th grade. Students were in their second year of this NSF-funded curriculum and it was the teacher's sixth year teaching this curriculum to 7th-graders at this school. The participating class consisted of 22 students (girls = 9; boys = 13) and routinely employed rotating structured roles of reporter, recorder, and project manager to guide small-group inquiry. Classroom norms included ensuring all group members understood the reasoning behind the procedures and equitable airtime. This report focuses on one triad (1 girl, 2 boys) situated within this class.

Data Collection

Students were administered the Conceptions of Mathematics Inventory-Revised (Briley, Thompson, & Iran-Nejad, 2009) survey at the beginning of the study. The 38-item survey was based on a 6-point Likert scale. Focus triads were observed daily over a two-week period beginning with the formation of new groups and the introduction of a new unit within the curriculum. The triads were observed during small-group work and whole-class discussions. Detailed field notes and audio recording of each lesson were created. Each participants' discourse was coded as either low-quality or high-quality (Weaver et al., 2005) and a daily percentage of high-quality discourse for each triad member was calculated. To triangulate findings, each member of the triad participated in a short, semi-structured interview at the end of the project. Student interviews focused on student confirmation of the observer's perceptions of the discourse within the group work as well as student confirmation of survey results.

Results

We examined whether there was a difference in the quality of small-group discourse in relationship to students' mathematics related beliefs. Participants in this setting held availing beliefs across most of the survey questions. The focus triad held similar beliefs to their classmates. The triad's survey results for one of the latent variables of the CMI-R, "Doing, validating, and learning mathematics" [DVLM], and specific indicators within that latent variable as well as their small-group discourse analysis results will be presented.

Survey Results

All three participants held slightly availing to availing beliefs about DVLM, indicating their overall beliefs about the learning and doing of mathematics support learning. Differences within the triad for specific indicators related to the importance of conceptual understanding, sense-making, and the locus of mathematical authority in the classroom were identified.

Sarah. Sarah has the most availing beliefs in her group for the beliefs about DVLM (DVLM = 4.563). She believes her work on mathematics problems needs to make sense and to understand why an answer is correct. She disagrees that it is possible to make good grades in mathematics by following procedures without understanding the steps. However, Sarah holds some non-availing beliefs. She believes that learning computational skills is more important than learning to problem-solve. Sarah also believes that she can only learn mathematics when someone shows her how to work a problem and that she can only find out if her answer is wrong is if it is different from the textbook or the teacher.

Aaron. Aaron holds availing beliefs about DVLM (DVLM = 4.375). Making sense in mathematics of his classmates' answers and his own is important to Aaron. He values conceptual understanding. It is not enough for Aaron to follow an algorithm without understanding why it works. He strongly disagrees that mathematics problems must be solved quickly or not at all. Although Aaron agrees that it is usually possible to solve a mathematics problem and that he and a classmate can work to solve a problem together when they do not initially agree on the answer, he believes the locus of authority and knowledge validation resides with the teacher or textbook.

Marcus. Marcus has slightly availing beliefs about the DVLM (DVLM = 4.063). He indicates he might look to his group or friends to determine if the answer was correct rather than relying on the teacher's validation of his answer. It is important to Marcus to understand why an answer is correct in mathematics and that it makes sense to him. However, he also strongly believes that he can earn good grades in mathematics simply by following the procedures. He also believes he can only learn mathematics when someone shows him how to work a problem.

Small-Group Discourse Analysis

Analysis of the study small-group's discourse revealed about 79% of discourse within the small group was on the low-quality end of the discourse taxonomy, mostly answering a direct question or making a simple statement or assertion. Group members frequently stated their answers, with explanations occurring when there was a request for clarification or a challenge by another group member. A small sample of Day 1 low-quality discourse about which two teams are farthest apart in Jeopardy:

Sarah: "KIA and SB"
Aaron: "Why?"
Sarah: "I don't know…"
Aaron: "-300 plus 150 is -150"
Sarah and Marcus copy the algorithm.
Aaron: "So, RS and SB are 150 points apart"
Marcus: "So RS and KIA are 600 apart?"
Aaron: "No 650"

No further exploration as to why Marcus arrived at an incorrect answer ensued, nor did the group check in with Sarah about her understanding. Group members rarely posed questions to each other to further their understanding or to challenge the thinking of each other.

Generalizations and predications came towards the end of the unit. For example, when Sarah was explaining her understanding of the relationship between adding and subtracting positive and negative integers, she utilized a number line to visualize "When you start with a negative

number and you add a positive number, you go that way (pointing right) towards the positive but when you add a negative number, you go that way (pointing left) towards the more negative." For □ – (-3) = -4, Sarah states, "We know that a negative minus a negative would go up (points to the right on a number line), so the □ has to be smaller than -4".
By the end of the unit, there was more emphasis on understanding. By Day 9, the group now pauses and retraces steps to make sure everyone understands and agrees.

Sarah: "Wait…. why is D1 $^{-}2$?"
Marcus: "Because it ended with 2."
Sarah: "Wait. Go back to D1. 3 negatives plus 5 positives should be $^{+}2$. But you said $^{-}2$?"
Aaron: "No it is $^{+}2$."
Sarah: "Oh ok. Just wanted to be sure we agreed."

Initially, Sarah did not participate much in small group discourse. After believing her answers to be incorrect, she completely withdrew from discussion and put her head down. Over time, Sarah became a more active group member. Marcus began the unit engaged in discourse but focused on the correctness of answers. After presenting incorrect answers to the class and falling behind due to absences, Marcus withdrew from the group and often worked independently. Aaron remained consistent in his participation in group discourse. Regardless of the daily role assigned to him, he often functioned as the project manager and directed the group. Most days he participated in discourse more than either of the other two group members. When Sarah or Marcus had a question, Aaron was usually the one to provide an explanation or justification of the reasoning, relegating to him the role of authority in the group. From the beginning and throughout the entire unit, Aaron made certain that if he had a clarifying question it was heard and addressed by the group.

Summary and Conclusion

A comparison of the study groups' responses to the survey and classroom observations provided for a comparison of the participants' espoused and enacted beliefs. There appeared to be both inconsistencies and consistencies between observed discourse practices and stated MRB.

All students agreed with the importance of explaining work and understanding it conceptually (DVLM), but their discourse was devoid of explanations unless challenged by a groupmate. Providing explanations for how the problem was solved was not a group norm, although it was an established classroom norm. Sarah, with the highest score for the DVLM latent variable, asked for the most explanations and often expressed, verbally and with gestures, her frustration when she did not understand procedures. Marcus, with the lowest DVLM latent variable score, often accepted Aaron's solutions without questioning. He strongly believed it was possible to succeed in mathematics by merely memorizing procedures without conceptually understanding the topic. Individual group members' acceptance or frustration with explanations appear to be in line with their beliefs about DVLM. In particular, their beliefs about needing to understand the concept and not just memorize the information stood out.

All students self-reported that analyzing mistakes is important, yet the observations and interviews suggested that each one of them felt making a mistake was indicative of being less capable than the other group members (Hoffman, 2004). Observed beliefs may be a function of external pressures: grades, parental expectations, teacher expectations, or peer expectations.

Results of this study are subject to limitations yet hold significant signals for future work. The size, scope, and duration of the study limits the generalizability of the findings across settings. Further research is warranted to expand the study across several variables - to multiple small groups, multiple classrooms, and multiple schools in different community contexts, as well as expanding the length of time over which the study is conducted, as different groups and different classroom norms may have different discursive patterns. Better understanding students' beliefs and how those beliefs may impact participation in small-group discourse is necessary to support students' opportunities to learn and potentially promote persistence in STEM pathways.

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NARRATIVES OF HYPER-ACCELERATION OF ALGEBRA I: OPPOSITIONS IN PERSPECTIVES ON OPPORTUNITIES TO LEARN

Terrie McLaughlin Galanti
George Mason University
tgalanti@gmu.edu

Hyper-acceleration of Algebra I is an emergent marker of smartness in K-12 mathematics which presumes a greater opportunity to learn (OTL) on advanced mathematics pathways toward STEM careers. This dissertation study elicited the longitudinal perspectives of 12 recent graduates from one International Baccalaureate high school who studied Algebra I in Grade 7. Critical events analysis and temporal resequencing of participant identity claims revealed augmented OTL with a focus on conceptual understanding and diminished OTL with a focus on social positioning. These preliminary findings add to extant literature on acceleration of Algebra I and contribute to critical conversations about what it means to be "good" at mathematics.

Keywords: Algebra and algebraic thinking; Affect, emotion, beliefs, and attitudes

The acceleration of formal algebra instruction to Grade 8 has expanded access to high school calculus with the expected affordance of improving readiness for STEM undergraduate study (Clotfelter, Ladd, & Vigdor, 2015; Rickles, 2013). However, an unintended and controversial trend toward *hyper-acceleration* has emerged in parallel with increased access to formal algebra in middle school. School divisions are now offering high school Algebra I courses in as early as 6th grade to an increasing number of students who seek to distinguish themselves from their peers and to increase their competitiveness for admission to top universities (Loveless, 2013; Domina, Hanselman, Huang, & McEachin, 2016).

Objective of Study

Prior research on acceleration of Algebra I has reported positive outcomes in terms of grades and standardized test scores (Gamoran & Hannigan, 2000; Ma, 2005; Smith, 1996; Spielhagen, 2006). However, these objective measures cannot explicate the subjective beliefs that students construct about smartness and success in mathematics. These beliefs can impact students' development of conceptual understandings in preparation for calculus (Bressoud, Camp, & Teague, 2012), persistence in advanced mathematics, and evolving mathematics identities of competence. The exploration of the lived experiences of hyper-accelerated students can describe a situative opportunity to learn (OTL) beyond quantitative research outcomes. This research can challenge presumptions that further acceleration of secondary courses provides more rigorous and relevant mathematics instruction for high-achieving students.

Narrative inquiry "frames the human experience" (Webster & Mertova, 2007, p. 14) and captures "the range of meanings which shape decisions" (Maynes, Pierce, & Laslett, 2008, p. 25). Through interpretative inquiry, this research retells the stories of 12 recent graduates from one International Baccalaureate (IB) high school who participated in a series of dissertation focus groups. Their collective reflections on their individual mathematical journeys in middle and high school offer new insights on appropriate acceleration in secondary mathematics.

Theoretical Framework

While OTL is often used to reframe the achievement gap discourse and to highlight how marginalized students lack experiences with high expectations and appropriately challenging curricula (e.g. Flores, 2007; Schmidt, Cogan, Hoang, & McKnight, 2011), I argue that it can also be a critical component of characterizing hyper-accelerated Algebra I experiences from a sociocultural perspective. Mathematically promising students who accelerate Algebra I should experience an augmented OTL as productive engagement which transcends the ability to perform or replicate procedures. Yet narrow constructions of mathematics ability and achievement (Louie, 2017) which often characterize students' experiences in secondary courses can compromise opportunities to engage with mathematics in meaningful ways (Boaler, 1997; Gutiérrez, 2012). A situative perspective on OTL (Greeno & Gresalfi, 2008) integrates the content and sociocultural experiences of students within figured mathematical worlds (Boaler & Greeno, 2000; Holland, Lachicotte, Skinner, & Cain, 1998; Horn, 2008). Students, teachers, and parents expect that Grade 7 Algebra I offers an educational advantage (Lucas, 2001). It may instead diminish OTL when success in mathematics becomes social positioning (Hatt, 2012) without an expectation of struggle. I adopt this situative perspective on OTL to answer the following research question: How do recent graduates from one socioeconomically diverse suburban IB high school narrate their quality of learning after hyper-acceleration of Algebra I?

Mode of Inquiry

Using an interpretative phenomenological paradigm (Kim, 2016), I constructed stories of OTL as retrospective narratives of achievement and identity. As a former electrical engineer and a veteran high school mathematics educator at West Valley High School (WVHS), I leveraged my professional knowledge of the IB curriculum and my relationships with the community to construct a rich data set of lived experiences revealing the essence of hyper-acceleration as a privileged educational phenomenon. I sought to build profound understandings of ways in which individuals were socially and culturally positioned within mathematics course trajectories. My subjectivist worldview reimagined OTL as an important intertwining of the pursuit of conceptual understandings and evolving student perspectives as productive doers of advanced mathematics.

Context

I conducted a series of five focus groups with recent graduates of WVHS, a mid-Atlantic public high school with diverse population of over 2500 students and broad Algebra I acceleration policies. IB programs are often placed in mixed-income or disadvantaged neighborhoods with a vision of breaking down barriers to rigorous educational opportunities for traditionally underrepresented students, yet the students in the WVHS IB courses are almost exclusively White and Asian. The IB curriculum is especially appropriate for describing OTL in hyper-accelerated contexts because it emphasizes connected understandings across multiple topics and offers two course pathways to calculus with differing rigor.

Participants

The 12 graduates from WVHS described seven different mathematics high school course trajectories after an honors sequence of Algebra I, Geometry, and Algebra II in Grades 7-9. These trajectories are depicted in Table 1. Shaded cells represent a "deceleration" from the most rigorous two-year IB sequence. With these deceleration choices, students may have studied similar mathematics content for two years, or they may have selected a less rigorous pathway. Blank cells represent the choice not to enroll in a mathematics course during Grade 12.

Data Analysis

An iterative sequence of critical events analysis and thematic analysis offered a rich hermeneutic cycle of moving between transcriptions. These cyclical description of the phenomenon built the reliability of the analysis. In the first stage of analysis, I identified critical, like, and other events (Webster & Mertova, 2007) followed by implicit, explicit, and tacit identity claims (Dennis, 2018) within each of the focus group transcripts. In the second stage of analysis, I sequenced transcript excerpts to create cohesive chronological wholes (Riessman, 2008) spanning elementary school, middle school, high school, and college. In the third phase of analysis, I coded identity claims within each of the reordered transcript excerpts to capture specific experiences which augmented or diminished meaningful OTL. Sampling strategies, peer scrutiny, triangulation of data across focus groups, and researcher reflective commentary promote confidence that the multiple realities of hyper-accelerated mathematics study are historical truths (Polkinghorne, 2007) from the perspectives of the participants.

Table 1: High School Mathematics Course Pathways

Participants	Grade 10	Grade 11	Grade 12
Claire	Precalculus	IB Standard Precalculus	IB Standard Calculus
Ashley	IB Standard Precalculus	IB Higher Precalculus	IB Standard Calculus
Elizabeth	IB Standard Precalculus	IB Standard Calculus	
Jessica	IB Higher Precalculus	IB Standard Calculus	IB Higher Calculus
Samantha, Peter, Jonathan	IB Standard Precalculus	IB Higher Precalculus	IB Higher Calculus
Robert	IB Higher Precalculus	IB Higher Calculus	
Alex, Mindy, Thomas, Nadine	IB Higher Precalculus	IB Higher Calculus	Multivariable Calculus

Results

Confirming Identities of Smartness – Selection for Grade 7 Algebra I

Evolving beliefs about smartness in mathematics emerged in the participants' telling of their elementary, secondary, and college mathematical worlds. The selection for Algebra I in Grade 7 was the critical event that confirmed identities of smartness in mathematics which had been established by selection for gifted academics in elementary school. Each of the participants vividly recalled taking the Iowa Algebra Readiness Assessment (IARA) in Grade 6. Failure to qualify for Algebra I in Grade 7 was a threat to their identities of smartness. "That was a whole big thing…if you don't get this then you are looked down upon." (Elizabeth, STEM major).

For many of the participants, this mathematical gate was not perceived as a choice nor as a metric of intrinsic motivation and enjoyment. "I don't remember thinking ever, 'Oh, do I want to?' It was more like I didn't want to be the one person not doing this." (Robert, non-STEM major). The obligation to maintain this identity of smartness was further constructed socially within the Grade 7 Algebra I classroom. Alex, a computer science major, retrospectively questioned the level of competition that he felt in Algebra I. "I think a lot of people were trying to prove how smart they were. They were all crazy stressed about their futures."

In contrast, Jonathan (non-STEM major) described an intrinsic desire to learn more in Grade 7 Algebra I with his anticipation of a more challenging setting to experience mathematics. He remembered scoring one point above the minimum qualifying IARA percentile.

> I'm pretty sure I had to have a conversation with my teacher saying, "Oh, you barely got it. Are you sure you want to do this?" And I was like, "Of course I do," because I felt like math was one of my stronger subjects and it was the most interesting subject for me.

Shifting Identities of Smartness – Intrinsic and Extrinsic Valuations of Mathematics

Identities of smartness were challenged for many of the participants in subsequent mathematics courses. Jonathan "grew a bit disinterested" in Algebra II when he perceived expectations were lowered, while others articulated an augmented OTL with emerging views of struggle as productive and mathematics as relevant. Elizabeth (STEM major) recalled enjoying mathematics when her teachers "derived the formula and explained why it made sense and where it came from". Other saw themselves as creators of mathematics when they could "play with numbers" (Samantha, non-STEM major) and model societal problems (Robert, STEM major).

Evolving identities reflected oppositions in perspectives on OTL that changed with time. "As I got older and mathematics got harder, it was definitely more about understanding the concepts rather than getting A's on tests" (Claire, non-STEM major). Pressures to maintain socially constructed identities of smartness often interfered with "full understanding" (Samantha). Fears of getting a wrong answer or asking stupid questions were extrinsic threats to smartness which diminished OTL. "The whole notion of going afterwards to ask questions and be vulnerable to need to ask for help was always hard for me" (Jessica, non-STEM major). Teacher or peer influences which gave students permission to decelerate mathematics pathways or to take risks in front of one another fostered an intrinsic valuation of mathematics learning.

> I came from feeling like there had to be one right way all the time until maybe halfway through high school. I'd have a mental block when I didn't get that problem right. It made such a big difference for me to finally be like, oh, I can try different things. (Ashley, STEM major)

Contrasting valuations of mathematics also emerged in retrospective consideration of OTL with Grade 7 Algebra I. Nadine's persistence on the most accelerated path reflected an extrinsic valuation of mathematics. "It just seemed logical. I've never taken math in college and I'm never going to. Multivariable calculus gave me the credit and I don't need math." In contrast, Mindy, a mathematics major, regretted that she had not pursued a deeper understanding of concept of function in her early algebra courses. "I think I struggled with that because I just wanted to get the right answer and then I started caring kind of late." She also articulated a diminished OTL in the perceived motivations of her teachers. "They're not preparing people to be mathematicians in high school. They're preparing people to pass tests and to not hate math."

Discussion

With emerging conceptions of the race to high school calculus as "often misguided" (National Council of Teachers of Mathematics [NCTM], 2018, p. 19), there is a need for additional research on the experiences of hyper-accelerated students. Persistence in advanced mathematics has a largely unexplored relationship to quality of learning in accelerated courses. Contextual challenges to identities when students ascribe smartness to correctness and competition may negatively impact students' identity formation as productive doers of mathematics.

The initial categorizations of augmenting and diminishing OTL characterized evolving identities of smartness over the six years of middle school and high school. These themes will be further explored in an additional layer of temporal narrative analysis using individual participants' mathematics trajectories. As a public policy major, Samantha reflected on hyper-acceleration as a "commodity you should be taking" within a mathematics hierarchy. This sociopolitical reality is redefining the ways in which our students build and use mathematics as a resource. Its implications for equitable access to meaningful OTL across the spectrum of demonstrated mathematical readiness should not be ignored.

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VISUAL REPRESENTATIONS OF UNDERGRADUATE STUDENTS' EXPERIENCES OF UNIVERSITY MATHEMATICS DEPARTMENTS

Jennifer Hall
Monash University
jennifer.hall@monash.edu

Travis Robinson
Monash University
travis.robinson@monash.edu

Jennifer Flegg
University of Melbourne
jennifer.flegg@unimelb.edu.au

Jane Wilkinson
Monash University
jane.wilkinson@monash.edu

Declining numbers of undergraduate students, particularly women, graduating in mathematics impact national innovation. In this paper, we report on findings from an Australian study about undergraduate students' conceptions of the supports and challenges that they experience in mathematics departments. Specifically, we discuss findings from our analysis of the visual representations (photographs) provided by the participants and discussed in focus group interviews. We provide insight into students' lived experiences – related to gender issues and peer support in particular – that may contribute to attrition from mathematics.

Keywords: Undergraduate-Level Mathematics; Affect, Emotion, Beliefs, and Attitudes; Research Methods; Gender and Sexuality

In many countries, concerns have been raised regarding the lack of participation of students in university-level STEM education due to a dearth of skilled professionals to meet the needs of a world that is "becoming increasingly technological and significantly more mathematical" (Australian Academy of Science [AAS], 2016, p. 37). In Australia, a very small proportion (0.4%) of students enrol in university degrees in the mathematical sciences (AAS, 2016). Consequently, several organizations (e.g., AAS, 2016; Australian Mathematical Sciences Institute [AMSI], 2017) have stressed the need for increased participation in tertiary mathematical sciences. Women are a minority in these programs, and women's proportion of the enrolments has declined in recent years (AAS, 2016; AMSI, 2017).

There is a paucity of research in Australia on the lack of participation, particularly by women and gender minorities, in undergraduate mathematics degree programs. The ways that students experience mathematics vary due to the gendered constructs placed on mathematical knowledge (e.g., Ernest, 1998). Thus, understanding how students of all genders experience tertiary mathematics is crucial in deconstructing this kind of gendered knowledge.

Here, we report on a study in which we address issues of student experience in mathematics degree programs, via a multimodal methodology, photovoice, combined with individual interviews. We focus on findings from our analysis of the photographs provided by the participants to represent the supportive and challenging aspects of their mathematics departments.

Theoretical Framework

This study is framed by a feminist and social constructivist epistemological stance (e.g., Butler, 1999; Fosnot, 2005). We view knowledge, including that pertaining to mathematics, as a human construction that is gendered and culturally, socially, and historically situated. Regarding this study, we apply this lens to the students' experiences in mathematics degree programs by viewing their learning as “both a process of active individual construction and a process of enculturation into the mathematical practices of the wider society” (Cobb, 1994, p. 13).

We have adopted Piatek-Jimenez’s (2015) socio-cultural perspective, in which participation in mathematics is viewed as a choice. We conceive of our participants as having agency in their educational and vocational pathways. This lens allows us to understand the many factors that contribute to students’ decisions to study mathematics at the university level.

Objectives

The research project is a comparative case study of first- and third-year students in the mathematics departments at two prestigious Australian universities, with a focus on gendered aspects of students’ experiences (Note: Undergraduate mathematics degrees in Australia are typically three years in duration). The aims of this project are: (1) to understand the experiences of undergraduate students enrolled in mathematics degree programs, in order to explore how mathematics departments support or challenge them, and (2) to examine how gender may be playing a role in students’ experiences of studying mathematics at the undergraduate level.

Our project is guided by the following research questions:

1. What are mathematics majors’ experiences of university mathematics departments?
    a. What aspects of the departments do students find supportive?
    b. What aspects of the departments do students find challenging?
    c. How do students’ experiences in the mathematics departments influence their career aspirations?
2. Are there differences in experiences by:
    a. Gender?
    b. Year level?
    c. Institution?

To address these questions, we are using qualitative research methodologies, as outlined next.

Methodology

We first provide an overview of the study’s methodology – comparative case study and photovoice. Then, we discuss the data sources, participants, and analysis methods for the study.

Comparative Case Study

As a case study, our research involves “the study of an issue explored through one or more cases within a bounded system” (Creswell, 2007, p. 73). Our project is an instrumental case study, as we are focusing on a broader issue, of which the case is representative, and a collective case study, also known as multiple case study design (Stake, 1995, 2005). The broader issue is the differential experiences and participation by gender and year level in studying university mathematics, which we explored using a modified version of photovoice (Wang & Burris, 1997).

Photovoice

Photovoice involves participants taking photographs relevant to their lives to “promote critical dialogue and knowledge about important community issues through […] discussion of photographs” (Wang & Burris, 1997, p. 370). The use of photovoice has grown exponentially in the past few years, yet we only know of three such mathematics education studies (Chao, 2012; Harkness & Stallworth, 2013; Tan & Lim, 2010); none were at the post-secondary level.

We began with individual semi-structured interviews about the participants’ educational experiences in mathematics. Then, per the photovoice process (Wang & Burris, 1997), participants took photographs to represent the supportive and challenging aspects of the mathematics department; these photographs were shared in focus group interviews. Supported by the interview facilitator, participants discussed themes that they saw across the photographs.

Participants

Participants were recruited from two comparable, highly ranked Australian universities (herein University X and University Y), both with large mathematics departments. Participants were recruited through classroom presentations, mass emails, posters, and snowball sampling. This resulted in 26 participants: 14 from University X and 12 from University Y. All participants have completed individual interviews, plus three focus groups have been completed.

We focus on findings related to the photographs that the participants provided about supports and challenges that they faced in their mathematics department. Information about the focus group participants and their photographs is provided in Table 1. We view gender as a non-binary social construct, so we asked the participants to identify their genders. However, all participants provided binary genders, so the gender data are separated into “men” and “women” categories.

Table 1: Focus Group Participant and Photograph Information

Name	Focus Group	Institution	Year Level	Gender	“Support” Photos	“Mixed” Photos	“Challenge” Photos
Ang-Yin	1	University X	1st	Woman	3	0	3
Bing	1	University X	1st	Man	0	0	1
Chelsea	1	University X	1st	Woman	3	0	3
Darius	2	University X	3rd	Man	2	0	0
Estelle	2	University X	3rd	Woman	2	0	4
Fiona	2	University X	3rd	Woman	5	0	1
Georgia	3	University Y	3rd	Woman	4	1	4
Hélène	3	University Y	3rd	Woman	0	0	1
Ivan	3	University Y	3rd	Man	1	0	2
Total number of photos					20	1	19

The photographs were evenly split between supports and challenges when considering the dataset as a whole, as well as by year level and gender. However, the women provided far more photographs than did the men (average of 5.7 vs. 2.0 photos per person, respectively).

Analysis

First, using emergent coding methods (Creswell, 2014), we analyzed the photographs in the context of the focus group interviews to retain the participants’ explanations. Second, using content analysis methods (Riffe, Lacy, & Fico, 2014), we further analyzed the photographs to provide detail and description that may not have been evident in the focus group interview videos. In our analyses, we were attentive to differences by gender, year level, and institution.

Results

We discuss two themes that arose from our analysis of the photographs: (1) Gender Issues and (2) Peer Support. Additional themes (e.g., teaching) will be shared at the conference.

Gender Issues

Consistent with the individual interviews, gender issues were only raised in the focus group interviews by the third-year women. While some issues were expected, such as gender imbalances, other, subtler forms of gender discrimination were raised, such as the unsanitary state of the women's bathrooms in the mathematics department building at University Y.

Figure 1: Photograph of a Woman's Bathroom Stall in the Mathematics Department Building at University Y (provided by Georgia)

Georgia and Hélène (3rd year, University Y) discussed how the women's bathrooms were poorly maintained, while Ivan confirmed this was not the case for the men's bathrooms. The women often had to line up to use the bathrooms, and sometimes went to other buildings instead. Such "everyday" microaggressions make mathematics departments uninviting places for women.

Peer Support

Several students discussed the value of peer support, both academically and socially. Many first-year participants, particularly international students, were surprised that their mathematics classes featured group work, with students collaborating to solve problems, as shown in Figure 2.

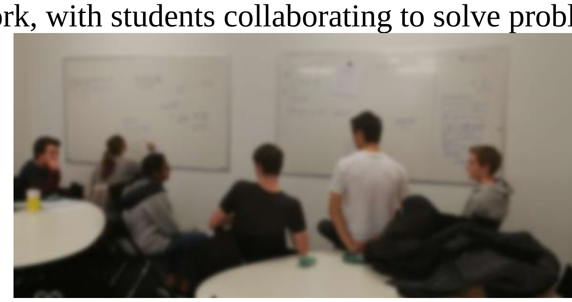

Figure 2: Photograph of Group Work in a First-year Mathematics Class at University X (provided by Bing)

Students at both institutions reported that it was useful to share ideas and learn from each other: "This helped me to […] like the subject better and also just see how other people – be able to, like, combine and collaborate I think to find the solution, which was really fun" (Ang-Yin, 1st year, University X). Students valued student mathematics societies where they had the opportunity to befriend other mathematics students. Hélène (3rd year, University Y) noted, "Subjects where I've got friends, I do better in than subjects when I don't." Third-year students at both institutions expressed a desire for more social spaces to encourage a sense of community.

Discussion and Conclusions

The experiences relayed in the focus group interviews through the participants' photographs highlight issues affecting students enrolled in undergraduate mathematics degree programs, such as those relating to gender issues and learning resources. The participants communicated a desire for a sense of community – socially and academically – within their mathematics departments.

Photovoice is a novel methodology in mathematics education, particularly at the university level. By sharing our experiences with other mathematics education researchers at the

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conference, we may assist colleagues in expanding their methodological repertoires.

Our findings will inform practice at the participating universities and hopefully increase retention of mathematics majors. This study contributes to a greater understanding of the mathematics "pipeline" – namely the supports and challenges faced by mathematics students.

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TEACHER-STUDENT RELATIONSHIP AND MATHEMATICAL PROBLEM SOLVING IN CHINA: A MEDIATION MODEL

Da Zhou
Beijing Normal University
zhoudadepict@163.com

Xiaofeng Du
Beijing Normal University
dxfice@163.com

Jian Liu
Beijing Normal University
ncctlj@126.com

While previous studies have demonstrated that teacher-student relationship influenced students' school learning, few studies compared the mechanism which teacher-student relationship influenced mathematical problem solving in the urban and rural regions. Structural equation modelling and Multi-group latent variable modelling were used to determine the effect that the teacher-student relationship influences mathematical problem solving by the mediation effects of motivation factors in the urban and rural regions. The results showed that (1) interest and self-efficacy can significantly mediate the link between teacher-student relationship and mathematical problem solving in the urban and rural regions; (2) the mediating effects of interest and self-efficacy are significantly better in the rural region than in the urban region, except for one path.

Keywords: Teacher-student relationship, Mathematical problem solving, Motivation factors, Urban-rural difference

Purposes of the Study

In the early 21st century, China began to implement *the Newest National Mathematics Curriculum Reform* and issued *the Full-time Obligatory Education Mathematics Curriculum Standards (Experimental Version),* which encouraged teachers to change their role from leaders to organizers, guides, and collaborators of mathematics learning. Teaching not only focus on knowledge spreading, but also focus on cultivation of abilities such as problem solving, reasoning and arguments etc. The approaches of developing abilities were also beginning to focus on the motivational factors such as interest and self-efficacy etc. Thus, exploring the relationship between teacher-student relationship and mathematics abilities by the motivational factors is important for teachers and students. In addition, China has been experiencing a huge change since 1978 (the starting point of reform-and-opening policy). However, the huge change also brought some problems. Urban-rural difference has become one of the most important problems (Sicular, Yue, Gustafsson, & Shi, 2007). Therefore, the study will set out to examine whether the effect of teacher-student relationship on mathematical problem solving is mediated by interest and self-efficacy in the urban and rural regions.

Theoretical Framework

Classroom environment contributes to the development of students' motivation, engagement in learning and further student academic performance (Ma, Du, Hau, & Liu, 2018; Wubbels & Brekelmans, 2005). Teacher‑student relationship, as the most important aspect of classroom environment, has been associated with student learning outcome (Wentzel, 2010; Barile, Donohue, Anthony, Baker, Weaver, & Henrich, 2012). Some studies found that supportive teacher-student relationship, based on attachment theory (Bowlby, 1980), had a positive correlation with students' mathematics achievement in Grades 1 through 9 (Hamre & Pianta, 2001; Ang, 2005; Valiente, Lemery-Chalfant, Swanson, & Re, 2008; Roorda, Koomen, Spit, &

Oort, 2011). On the basis of self-determination theory, there were also studies that indicated motivational factors played a crucial role between teacher-student relationship and learning outcome (Deci & Ryan, 2012; Pajares & Miller, 1994; Roorda et al., 2011; Kikas & Mägi, 2017). As motivation factors, self-efficacy and interest generally displayed consistent positive associations with mathematics achievement in most countries through PISA data (Schunk, 1989; Chiu, Zeng, 2008). Meanwhile, some studies have also shown that gender is an important factor affecting mathematical problem solving (Gallagher & De Lisi, 1994; Casey, Nuttall, & Pezaris, 2001), so gender need to be controlled in the study. Based on previous studies, we found that most studies just focus on general academic achievement instead of a specific mathematics ability. Thus, exploring the relationship between teacher-student relationship and mathematical problem solving, the mediation effect of self-efficacy and interest, will expand extant research. In addition, combined with the Chinese social context, further exploring the urban-rural difference in the following mediation model is also very necessary for the development of education. The mediation model as follows:

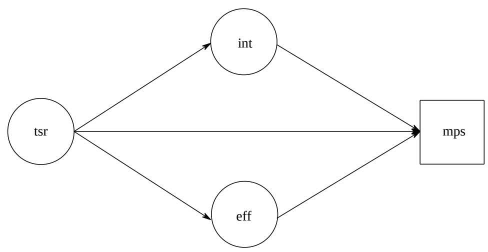

Figure 1: Theoretical Framework of the Present Study

Notes: tsr = teacher-student relationship; int = interest; eff = self-efficacy; mps = mathematics problem solving.

Based on the purposes and the theoretical framework of the present study, three research questions need to be explored. Q1: Does self-efficacy and interest significantly mediate the relation between teacher-student relationship and mathematical problem solving in the urban and rural regions? Q2: Are there differences each path coefficient by the model comparison?

Methods

Participants and Procedures

Participants are 1762 eight graders from 11 lower secondary schools of XX city. We chose participants using Conditional sampling methods for ensuring reasonable sample distribution between male and female, urban region and rural region. Males are 883 (50.1%), and females are 879 (49.9%). The number of students in the urban region is 972 (55.2%), and the number of students in the rural region is 790 (44.8%).

Students were invited to complete a self-report paper questionnaire including teacher-student relationship, interest, and self-efficacy. Except for the questionnaire survey, students were also invited to complete a mathematical problem solving test. The validity and reliability of all instruments are good.

Data Sources and Analysis

The data of this study were drawn from *the Program of Regional Education Assessment* implemented in Chinese XX city, in October, 2017. This study offers two evidence-based models to explore the mediation effects of interest and self-efficacy between teacher-student relationship and mathematical problem solving in the urban and rural regions. Model A and Model B present the mechanism that teacher-student relationship influences mathematical problem solving in the urban and rural regions respectively. According to proposed models, data analyses in this study were performed by three steps. Firstly, confirmatory factor analysis was conducted to provide evidence for psychometric properties of items within each factor construct (they won't be presented in this paper due to the limited words). Secondly, structural equation modelling was generated to assess the effects of independent (exogenous) latent variables on dependent (endogenous) latent variables in the urban and rural regions. Third, multi-group latent variable modelling was used to compare with differences between different models.

Results

Structural Equation Modelling Analysis

Structural Equation modelling was specified and analyzed using Mplus7.11. The standardized path coefficients of the two models are depicted in figure 2, and figure 3 respectively.

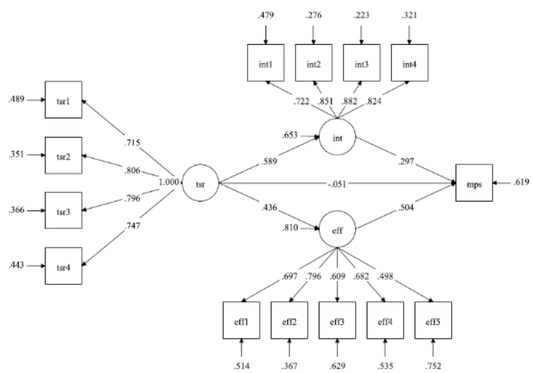

Figure 2: Model A (urban region)

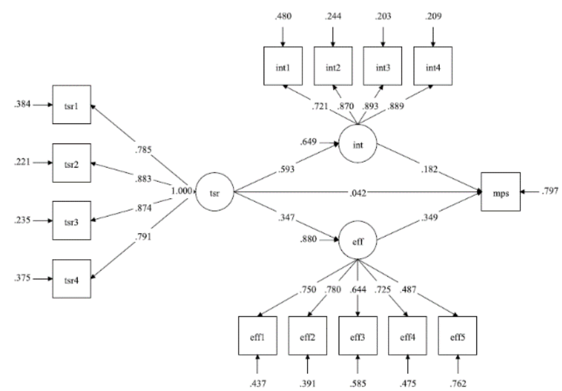

Figure 3: Model B (rural region)

The Model A fitted the data adequately with CFI = .959, TLI = .948, RMSEA = .066, SRMR = .073, $\chi 2$ (73, 1762) = 382.449, p<.001. The Model B also fitted the data adequately with CFI = .956, TLI = .945, RMSEA = .062, SRMR = .071, $\chi 2$ (73, 1762) = 297.014, p<.001. In addition, Model A and Model B shows that teacher-student relationship significantly predicts interest and self-efficacy in the urban and rural regions, and interest and self-efficacy exert significant effect on mathematical problem solving in the urban and rural regions, but teacher-student relationship has no significantly direct effect on mathematical problem solving in the urban and rural regions. In further analysis of mediation effect of interest and self-efficacy, bootstrapping procedures were used to generate a 95% confidence interval (CI 95%) with 1000 samples from the original data (N = 1762). The CI 95% of the indirect effects estimations does not have zero, indicating that the mediation effect will be statistically significant in the urban and rural regions. The results address Q1.

Multiple-group Latent Variable Analysis

Urban-rural difference in each path was examined by means of multi-group latent variable modelling. Table 1 presented the results. Constraining the factor loading corroborated metric invariance (M1 vs M2: p = 0.841), indicated that the measurement models could be equal in the urban and rural regions, and the structural models differed between the urban region and the rural region. For further exploring the difference of structure models in the urban and rural regions, this procedure was conducted from M2 to M7. The results showed that (1) M2 vs M3 ($p < 0.01$) indicated the model A and model B had significant fit-difference; (2) M3 vs M4 ($p= 0.286$) indicated there was not significantly difference, from teacher-student relationship to interest, in the urban and rural regions; (3) M3 vs M5 ($p < 0.01$) indicated M3 fitted significantly worse than M5; (4) M3 vs M6 ($p < 0.01$) indicated M3 fitted significantly worse than M6; (5) M3vs M7 ($p < 0.01$) indicated M3 fitted significantly worse than M7. The results address Q2.

Table 1: Invariance Test Statistics

Model	$\chi2$	df	CFI	RMSEA	$\triangle\chi2$	$\triangle$df
M1: Configural model	679.463	146	0.958	0.064		
M2: Weak invariance model	685.153	156	0.958	0.062	5.69	10
M3: All the Paths in the structure model were set free across region	714.934	161	0.956	0.062	29.781**	5
M4: Path from *tsr* to *int* was set free with constraining the other four ones to be equal across regions	713.794	160	0.956	0.063	1.14	1
M5: Path from *tsr* to *eff* was set free with constraining the other four ones to be equal across regions	706.190	160	0.957	0.062	8.744**	1
M6: Path from *int* to *mps* was set free with constraining the other four ones to be equal across regions	703.200	160	0.957	0.062	11.734**	1
M7: Path from *eff* to *mps* was set free with constraining the other four ones to be equal across regions	700.044	160	0.957	0.062	14.89**	1

*Notes: tsr = teacher-student relationship; int = interest; eff = self-efficacy; mps = mathematics problem solving. *p < .05, **p < .01.*

Discussion

According to the results of structural equation modelling analysis, we can find that interest and self-efficacy are important mediation factors between teacher-student relationship and mathematical problem solving in the urban and rural regions. The result is consistent with the theory of curriculum reform and previous some studies (Jerome, Hamre, & Pianta, 2009; Schunk, 1989; Spark, 2014). Attachment theory and self-determination theory provided theoretical support for model A and model B (Bowlby, 1980; Bandura, 2002). The difference between this study and previous studies is that this study focuses on students from urban and rural regions in the Chinese curriculum reform context. Thus, the results will further extend current research, and provide some implication for teachers on how to improve student mathematical problem solving in China.

Based on the result of multiple-group latent variable analysis, we can find that the mediating effects of interest and self-efficacy are significantly better in the rural region than in the urban region, except for the path from teacher-student relationships to interest. Combined with the Chinese education context, the reason for this phenomenon need to be explained. Parents in the urban region pay more attention to students' mathematics learning, so they usually give students more extra resources for helping students learn mathematics such as online mathematics curriculum, tutorial mathematics class and so on (Liu & Wang, 2018). However, parents in the rural region pay less attention to their children than those in the urban region (Tian & Yuan, 2010). Thus, the influence of schools and teachers on students is particularly important in the rural region. The results make an important contribution to the literature by showing that the difference of each path coefficients in the urban and rural regions, and provide some implications for teachers to develop rural education and urban education together.

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STRATEGIES USED BY STUDENTS WITH LEARNING DISABILITIES FOR REASONING ABOUT SLOPE

Anna F. DeJarnette
University of Cincinnati
anna.dejarnette@uc.edu

Casey Hord
University of Cincinnati
casey.hord@uc.edu

Students with learning disabilities (LD), like other learners, show a range of resources and strategies for reasoning about complex concepts in mathematics. This study comes from a project in which a group of five ninth-grade students with LD participated in a once-weekly tutoring program with university pre-service teachers. We asked, what strategies did students use to reason about slope? Students drew upon knowledge of concepts related to constant covariation when given the opportunity. This study suggests that students with LD have rich conceptual knowledge that can be leveraged to improve their success in Algebra.

Keywords: Algebra and algebraic thinking, Classroom discourse, Students with learning disabilities

The concept of slope presents a significant challenge in the teaching and learning of linear functions. On one hand, slope is a complex concept, referring to a single quantity that describes a pattern of constant covariation among two distinct, but related, quantities in a linear relationship (Lamon, 1995). On the other hand, it is possible for students to work somewhat fluently with slope on "rise over run" procedural tasks without attending to the conceptual underpinnings of slope (DeJarnette, Marita, & Hord, 2019; Zahner, 2015). Students with learning disabilities (LD) may be more likely to receive instruction that gives more focus to the correct application of procedures and less focus to underlying concepts (Foegen & Dougherty, 2017). Given the high stakes of passing high school algebra, such efforts seem well-intentioned towards supporting struggling students to be successful. However, there is an opportunity to create more meaningful learning opportunities for all students, and students with LD in particular, by attending to and building upon the conceptual knowledge that they apply to tasks about linear functions.

With this study we posed the question, what strategies do students use to reason about slope across different types of tasks related to linear functions? The purpose of this study is to use knowledge of students' strengths to suggest ways in which future instruction can privilege those strengths. In particular, this study supports the claim that procedural fluency is not a pre-requisite for building upon students' conceptual understanding of slope, and in fact making connections to concepts can support students in completing procedural tasks.

Research Perspectives

Attention to slope as a composite quantity is part of a broader perspective known as a covariational approach to the teaching and learning of functions (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1994; Saldanha & Thompson, 1998). A covariational approach to functions is a contrast to a "correspondence" approach, through which functions are treated as relationships between individual inputs and outputs (Confrey & Smith, 1994). More generally, the covariational approach is part of an effort to develop students' understanding of foundational concepts such as variable, function, and rate of change in ways that can be applied through increasingly advanced levels of mathematics (Thompson & Carlson, 2017).

Many recent efforts to support students' conceptual understanding of slope align with a covariational perspective. Slope as a ratio is a "complex composite unit" (Lamon, 1995), a single quantity representing a multiplicative relationship between two distinct quantities. To work fluently with linear functions, including making interpretations or predictions about what linear relationships represent, students need to act on this ratio as a single quantity (Ellis, 2007). In practice, however, students often treat numerical representations of slope as two distinct quantities representing horizontal and vertical change along a graph (DeJarnette et al., 2019; Lobato, Ellis, & Muñoz, 2003; Zahner, 2015). Such conceptions can be sufficient for solving tasks related to calculating slope given a pair of points or a graph of a line. However, these conceptions tend to leave out foundational ideas, such as how slope describes the steepness or direction of a graph. Knowledge of the concept of slope can serve as a resource for students on a variety of tasks, including correcting errors when procedures are mis-applied.

Methods

This study comes from a broader research effort to train university pre-service teachers to be effective tutors for students with LD taking Algebra 1. Five ninth-grade students from a large suburban high school agreed to participate in the project, after being selected by their math teacher as students who would likely appreciate and benefit from 1-1 tutoring. The students all were identified by their schools as having LD. The students were enrolled in the first year of a two-year Algebra 1 sequence at the school. The tutors were recruited for the project from a mathematics teaching methods course taken by pre-service teachers in the special education program and the middle childhood program at the university. Four of the tutors were pre-service special education teachers; one tutor was a pre-service middle-grades teacher. The tutors were recruited based on high achievement and engagement in their methods course.

Beginning in December 2018, students met with the tutors for 1-1 tutoring one time per week during their usual math class. There were two tutoring sessions from which we drew data for this study. The first was a day on which students had an assignment where they needed to solve systems of linear equations by graphing (Figure 1). The second was a day on which students worked on a task for which they compared slopes of six different lines. We selected these two sessions for analysis because the tasks elicited the use of slope concepts and procedures.

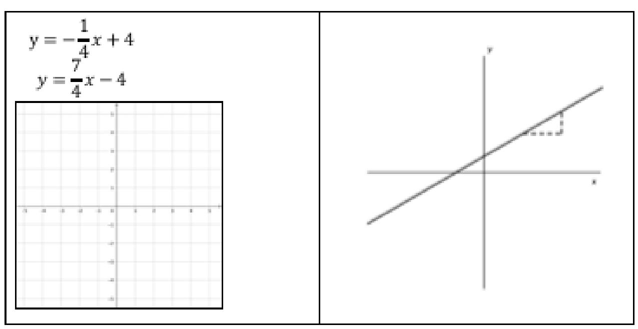

Figure 1: Tasks from the Sessions, Systems of Equations (left) and Comparing Slope (right)

All of the tutoring sessions were recorded with document cameras that captured students' papers as well as the students' and tutors' hands as they worked. We analyzed seven tutoring videos—four videos from the "systems of equations" session and three videos from the "comparing slopes" session. We applied a constant comparison analysis (Strauss & Corbin, 1998), using the video recordings and partial transcripts of the sessions, to look for patterns in the strategies that students used for calculating, graphing, or interpreting slope. The first author viewed each video in its entirety, separating the tutoring sessions into "segments" to highlight key moments in students' and tutors' interactions and constructing transcripts of notable excerpts. After writing a descriptive narrative of each of the relevant segments, the first author used these descriptions to identify emerging themes to characterize students' strategies. The first author shared these emerging themes, as well as the relevant video data, with the second author as well as with a research assistant to refine our interpretation of students' work.

Results

We identified four different types of strategies that students applied to reason about slope.

Strategy 1: Flexible Counting "Rise Over Run"

For graphing lines that had been written in the form $y=mx+b$, students had learned a process of first graphing the y-intercept, followed by one or two more points on the line. One student in particular, Ben, showed a great deal of flexibility in how he applied the "rise over run" technique to identify points on the line. For example, one task on the worksheet required him to plot a line with a slope of 1/2; after plotting the y-intercept of the line Ben audibly counted, "one, one-two; one, one-two" while moving his pencil sequentially up and to the right. With this counting Ben first accounted for the vertical change in the graph and then accounted for horizontal change. The next task required him to graph a line with a slope of two. In this case Ben used the same audible counting pattern, "one, one-two; one, one-two," but this time he accounted for the horizontal change between points first, and then the vertical change. Moreover, although his audible counting was identical to the previous task, Ben moved to the right and down—and, then, to the left and up—on this task to account for the negative slope. Throughout the worksheet, Ben switched back and forth between whether he counted vertical change or horizontal change first, and he did so with few errors.

Ben's flexibility in how he counted "rise over run" is significant because it challenges the assumption that students should apply the same procedure in the same way for every task in order to avoid errors. Our tutors recalled mnemonics that teachers had used (e.g., "place the ladder before you climb it") to reinforce a specific order of graphing procedures. Cases like Ben's illustrate students' flexibility in applying procedures to represent constant covariation.

Strategy 2: Extending Slope Patterns

Another area of strength that students showed in applying linear reasoning was to establish a pattern for the distance between the y-intercept and another point on the graph, and then to extend that pattern to locate other points on the line. For example, Katy needed to graph the line $y=(-1/2)x+3$. After plotting a point at (0, 3), she moved up and to the right to plot a second point at (1,5). She then seemed to interpret the visual relationship between the two points she had plotted, and she transposed that relationship to plot a third point at (-1,-2) to create a line with constant slope. Katy solution here began with an error stemming from apparent confusion about the fraction of -1/2 should be used to determine the slope of the line. However, once she defined a line by plotting two points, Katy applied knowledge of constant rate of change to condense the procedure for locating points on the graph. Although students had spent substantial time developing procedures for counting over and up (or down) between points, examples like this illustrated how students applied broader conceptual knowledge of the shapes of linear graphs.

Strategy 3: Distinguishing Between Positive and Negative Slope

When students worked on the "comparing slopes" tasks (without numerical information), all three of the students from whom we collected data correctly identified which of the six lines had positive slope and which had negative slope. But students often made errors in the directions they needed to move horizontally and vertically to construct a graph with negative slope, even when they had reminders in place. One example of this came from the work of Mia. When working on the "systems of equations" tasks, she consistently wrote each case of a negative slope in two ways on her paper. For example, if given the line $y=(-2/3)x+b$, Mia made a note of $m=-2/3=2/-3$, to remind herself that she could move two places down and three to the right or two places up and three to the left. However, even with this notation and the explanation of why she used it, Mia made several errors in the direction she needed to move along the graph.

Around halfway into the lesson, Mia's tutor had a brief conversation about how lines with positive slope would increase from left to right, and lines with negative slope would decrease from left to right. This concept was familiar to the student, and it seemed to spark her memory that she could use knowledge of the general direction of a graph to construct lines, rather than relying on instructions for what direction to move and when. Several minutes later in the session Mia needed to graph the line $y=(-1/7)x+7$, and her tutor briefly noted the negative slope. Mia reacted with some excitement, noting, "so it's gonna go this way," indicating with her pencil that the graph would move down and to the right, and she made no errors when graphing the line.

Strategy 4: Attending to Relative Horizontal and Vertical Change

When students worked on the "comparing slopes" activity, they attended to the relationship between the horizontal and vertical change without any numbers or scale provided (except to note that the axes were scaled in a 1:1 ratio). When Ben encountered the question of whether the graph include in Figure 1 had slope greater than, less than, or equal to, one, he observed the triangle included with the graph and noted that the horizontal leg of the triangle looked twice as long as the vertical leg. In the next graph he noted that the horizontal and vertical legs of the triangle were equal in length. Ben showed a sophisticated level of reasoning here in attending to the multiplicative relationship between the horizontal and vertical change of the line. This multiplicative reasoning is essential towards conceiving of slope as a composite unit.

Discussion and Conclusion

Prior research has recorded how challenging it is for all students to move beyond procedures for calculating slope towards understanding of slope as a single quantity representing constant covariation (Ellis, 2007; DeJarnette et al., 2019; Lobato et al., 2003; Zahner, 2015). We observed several instances of students with LD applying sophisticated aspects of covariational reasoning—including knowledge of constant covariation and the multiplicative relationship between horizontal and vertical change—in addition to flexible application of procedures. It is illuminating that procedural fluency is not a pre-requisite for the development and application of conceptual understanding, even for students with LD who are struggling in Algebra. Students' knowledge of constant rate of change—and how they interpreted this concept visually—was a resource that they could draw upon to avoid or correct procedural errors. From this preliminary study it is clear that students might sometimes compartmentalize their understanding of concepts away from their use of procedures. But when carefully examining the strategies students draw upon, it becomes clear that students do a better job of bridging concepts with procedures than is immediately apparent. Students may benefit from instruction that makes these connections—and, especially, the strengths of students' reasoning—more explicit.

Acknowledgments

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HIGH SCHOOL MATHEMATICS TEACHERS' POSITIONING WITHIN THE CULTURE OF BLAME: A CASE STUDY

Ashley Garner
Georgia State University
a.marieplummer@gmail.com

Overemphasis on high-stakes testing in mathematics, particularly in schools with economically disadvantaged students, has led to the formation of a culture of blame (Lau, 2009) and inconsistent instructional practices. However, there are teachers who have been successful at ensuring deep mathematics learning takes place in spite of the demands of high-stakes testing instruction (Ladson-Billings, 2009; Leonard & Martin, 2013). The purpose of this qualitative holistic, multiple-case study was to explore how high school mathematics teachers position themselves within the culture of blame that is manifested in the high-stakes testing environment. Positioning theory was employed to examine data. Findings suggest that teachers' instructional decisions are deeply influenced by high-stakes testing and that misalignment of goals among school stakeholders lead to low teacher retention.

Keywords: High School Education, Teacher Belief, Instructional Vision, Instructional Activities and Practices

Introduction

The purpose of this study was to explore how high school mathematics teachers' position themselves within the culture of blame and to understand the instructional decisions that those mathematics teachers make when under pressure to increase student achievement in high-poverty schools. As such, underpinning my research is the culture of blame which manifests itself in the high-stakes testing culture present since the 1980s and the school reform era (Lau, 2009). The culture of blame is the idea that teacher autonomy and instructional decision-making is lost because of their fear of being blamed for students' low high-stakes testing performance. Instead of trusting in their professional pedagogical abilities, teachers are deferring to their administrators and others outside of the classroom to determine the best instructional practices to implement in their classrooms. I chose to focus on one high school mathematics department to provide a deep description of teacher blame within a school across multiple grade levels and mathematics content areas. I used positioning theory as a lens through which to analyze the participants' narratives with the aim to explore teaching practices that counter teacher blame and even defy the common rhetoric of rampant poor teacher quality in schools (National Commission on Excellence in Education, 1983).

Literature Review

Four major issues exist that perpetuate the culture of teacher blame; instability in the mathematics department due to teacher attrition, poor evaluation systems for teachers, an extreme focus on getting students to pass achievement tests, and the lack of focus on gains and students' successes. After retirement, teacher blame was the highest reason for teacher attrition within five years (Curtis. 2012). Owens (2015) found that teacher attrition was linked most prominently with the number and emphasis of mandated tests. High teacher attrition has been shown to negatively impact student achievement, especially in schools with lower achievement, and particularly in mathematics (Hanushek, Rivkin, & Schiman, 2016; Ronfeldt, Loeb, & Wyckoff, 2013). While the Owens (2015) study is specific to the state of Georgia, the Curtis (2012) study included teachers from various states, showing that the problem of low teacher retention persists nationally.

While several studies have been conducted on the deficit view of underserved communities through counternarratives and counterstories (Gutierréz, 2000; Leonard & Martin, 2013; Tan et al., 2012) those success stories are often only given niche status in academic journals and conferences. The niche status of success stories not connected directly to test scores is indicative of the same level of importance these types of successes receive in school buildings and at the district level.

The Culture of Blame and Instructional Practices

According to Lau (2009), control over the accidental has resulted in governments attempting to predict and then neutralize threats before they act, which thereby prevents harm. Lau (2009) further asserts that this constant search for absolute control over the accidental is referred to as *predictability-preventability fetishism*. The notion of predictability-preventability underpins the culture of blame and perpetuates it in culture, resulting in institutions (specifically healthcare and now in education) adopting a blame culture, in which stakeholders avoid risk for fear of being blamed. Dowman and Mills (2008) used participants from primary and secondary education to explore how teachers made sense of teacher blame in the media, but did not emphasize a specific subject area in their study. By exploring stories and discourses and making sense of them, the sensemaker is in turn creating them. Dowman and Mills (2008) found that four distinct repertoires arose from the interviews with teachers; *defensive, empathetic, cynical,* and *collaborative*. The participants' narrative responses within my study mirrored these four repertoires.

Lattimore (2003) discusses the "pedagogy of poverty" or the "pedagogy of mediocrity" that exists in schools, which includes routine instructional practices that are teacher-centered, rather than student-centered. Instruction covers mathematics at a surface level, but student agency is absent. When efforts are made to engage students, "deficient outcomes are countered by reducing expectations to levels of whatever the student seems willing to do" (Lattimore, 2003, p. 124). Instead of preparing students to be able to use and understand mathematics fluently and address misconceptions that led to poor performance, Lattimore found that high-stakes testing forced the teachers into believing that they should teach solely to prepare the test.

However, the scholarship surrounding teachers of successful mathematics students expressed a common theme of effective teachers of mathematics having and promoting strong mathematics identities for their students (Gutiérrez, 2000; Ladson-Billings, 2009). According to Grootenboer and Ballentyne (2010), the participants made quick decisions with minimal consideration, which suggests that the decisions were made from the teachers' identity or sense of self.

Positioning Theory as a Framework

Positioning theory is defined as "the assignment of fluid 'parts' or 'roles' to speakers in the discursive construction of personal stories that make a person's actions intelligible and relatively determinate as social acts" (Harré & van Langenhove, 1998, p. 17). The focus of positioning theory is how phenomena are constructed through discourse. Individuals can position themselves or be positioned by others.

Method

My purpose was to gain insight into how mathematics teachers cope, how they are sustained in their profession, if and how they push back against the blame culture, and how they alter, if at all, their instructional decisions because of the blame culture. As such, my research question was best studied using an exploratory case study; my case being a high school mathematics department consisting of six teachers over the course of one school year. A holistic, multiple-case approach permitted a global analysis of each case. In this research, I examined the experiences of six teachers in one mathematics department. The unit of analysis, which is defined as the entity being studied, is the high school mathematics teacher. While the teachers within this department share the same principal, assistant principals, and even students in some cases, the teachers' experiences with mathematics, their students, and the culture of blame are all unique. Examining participants through a multiple-case study provided the opportunity for a stronger, more in-depth synthesis regarding the culture of blame and its effect on mathematics instruction.

Procedures

High school mathematics teachers' experiences within the culture of blame were explored through multiple sources, including a survey, a group interview session, individual interviews, documents, student work samples, emails, and other artifacts. I facilitated the two-hour group interview by asking a series of open-ended questions based on the survey responses, and then mediated the discussion to ensure that all voices were heard. Of the six mathematics teacher participants, four were able to participate in the group interview. Semi-structured interviews with each individual in the mathematics department, varying from forty to eighty minutes in duration, took place one week after the group interview. Each interview included an opportunity for clarification questions stemming from the previous group interview as a layer of member checking. To provide data triangulation and increase the validity of the study, other forms of data were collected in the form of artifacts. These data included photographs, formal and informal lesson plans, teacher websites, email correspondence between teachers and administrators, student work, newspaper articles, and social media posts for a more complete representation of their experiences.

Data Analysis

To analyze data, I used inductive analysis of the transcripts and other collected data to locate patterns and commonalities that lead to exploring teacher positions with respect to the culture of blame. After the data were organized, I used the whole-part approach to begin thematic analysis (Vagle, 2014). I then used the *three-dimensional space* approach to restory participants' narratives developed by Connelly and Clandinin (2000). The three dimensions Connelly and Clandinin (2000) referred to are interactions (both personal and between other people), continuity (past experiences of the storyteller), and situation (specific places or spaces in the story). I developed a complete and complex view of the teachers' experiences and thus provided a rich description of the phenomenon of teacher blame and the participants' positions within the phenomenon. I also classified the teachers' narratives according to the four repertoires from Dowman and Mills's (2008) research.

Findings

Teachers in the study fell into two categories as a result of the courses they taught; four teachers were responsible for high-stakes tested courses (Algebra 1 and Geometry) and two taught upper-level courses that did not. The teachers in high-stakes courses felt pressure to teach more directly to the curriculum and reported feeling a loss of creativity and flexibility in their classrooms. Although all six teachers discussed students' need for intensive remediation during the school year, only the teachers in upper-level courses were able to deviate from the curriculum to work with students in their deficit areas without the time constraint and pacing pressure placed on them from the district pacing guide and the building administrators.

Students' Perception of Mathematics

Common among individual interviews was what teachers perceived as students' lack of focus on learning mathematics in their classrooms; multiple teachers discussed the culture of passing and the culture of cheating prevalent at DuBois High School. Perseverance came through as a common theme among four of the six teachers. A significant percentage of students at DuBois High School had failed multiple high-stakes tests in the past and brought that defeated attitude to their current classroom. Teachers said that students learn for the immediate benefit of taking a test, rather than the basis for mathematics in future years and beyond in their lives.

High-Stakes Testing

One finding pertaining to testing was that of pass-rate currency from administrators. One teacher who had a higher than average pass rate in Algebra 1 the previous year spoke of having more flexibility in lesson planning from administrators. As a whole, teachers discussed this same flexibility that they felt was favoritism among DuBois High teachers in other tested areas that were traditionally more successful at producing high achievement scores on high-stakes tests. All teachers agreed that they were less stressed when teaching non-tested courses and could focus more on helping students see connections between mathematics and their lives. Teachers also spoke of passing the pressure and anxiety that they felt onto their students, who then pushed back against being measured by testing.

Student Learning Goals and Instructional Decision Making

A common theme among participants was their creating lessons that make connections among topics both across and within content areas. The geometry teachers created connections by repositioning standards within the given curriculum guides to save time and capitalize on students' memories. These connections were eventually accepted by and changed for every teacher in the district. Algebra teachers focused on connecting concepts to real-world situations and problems to access students' funds of knowledge. However, depth of teaching was presented as an issue because of time lost to preparing for the looming high-stakes tests in the form of building-level common assessments and district benchmarks. Differentiation, according to the DuBois High School teachers, was necessary, but difficult because of the overwhelming number of students who needed daily one-on-one instruction and support. Teachers reporting feeling that they catered to lower-performing students to the detriment of students who needed enrichment. In the quest to have students experience success, one participant stated that they lowered their standards and chose assignments requiring a lower depth of knowledge. All teachers discussed how they negotiated their instructional decisions, including how they routinely balanced teaching for the high-stakes test and teaching for a deeper understanding of mathematics.

Conclusion

At the base of this research study is the culture of blame which, according to Dowman and Mills (2008), results from the human need to explain and rationalize the accidental or unexplainable. The culture of blame at DuBois High School encompasses blaming mathematics teachers for high-stakes test scores, dealing with students who cheat leniently, which perpetuates a culture of cheating, and promoting a culture of passing students despite deficient student performance because of the potential effect on the School Performance Index. The culture of blame not only causes teachers to feel undervalued and overwhelmed, but hinders students from getting the highest-quality mathematics education possible and maximizing their growth potential each year. The culture of blame is leading to the dehumanization of students, by assigning their value according to their high-stakes test scores, and the deprofessionalism of teachers. Both of these issues can be rectified through decreasing the importance of high-stakes testing.

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EXPLORING APPLICATIONS OF SCHOOL MATHEMATICS: STUDENTS' PERCEPTIONS OF INFORMAL LEARNING EXPERIENCES

Thomas Roberts
Bowling Green State University
otrober@bgsu.edu

Christa Jackson
Iowa State University
jacksonc@iastate.edu

Margaret Mohr-Schroeder
University of Kentucky
m.mohr@uky.edu

Sarah B. Bush
University of Central Florida
Sarah.Bush@ucf.edu

Cathrine Maiorca
California State University, Long Beach
Cathrine.Maiorca@csulb.edu

Ashley Delaney
Iowa State University
delaneya@iastate.edu

Informal STEM learning experiences can provide students with opportunities to apply their knowledge of mathematics. The purpose of this study was to explore middle grades students' perceptions of mathematics after participating in a week long informal STEM learning experience. Using a situated learning lens, we analyzed 254 student interview transcripts collected from informal STEM learning experiences at three different universities in the United States. The informal learning environment (a) positively influenced students' view of mathematics and (b) exposed students to applications of mathematics. Using authentic STEM experiences engaged students in the process of mathematics through real world applications.
Keywords: Equity and Diversity, Informal Education, Middle School Education

Mathematics is often a gatekeeper for more advanced STEM study (Crisp, Nora, & Taggart, 2009; Martin, Gholson, & Leonard, 2015). This begins in elementary school as elementary school mathematics often falls short of preparing students for more advanced coursework (Coxon, Dohrman, & Nadler, 2018). Generally, mathematics instruction often occurs through whole class instruction (Tate, 1995) where students listen to teachers describe the standard algorithm before individually practicing the algorithm at nauseum. In fact, lower quality mathematics instruction is more likely to be the norm for students attending majority minority schools and low-income schools (National Council of Teachers of Mathematics [NCTM], 2014). With underwhelming experiences, many students decide before eighth grade mathematics and more broadly, STEM courses, are not in their futures (Boaler & Greeno, 2000; PCAST, 2010). However, informal STEM learning experiences can provide context for and extend students' learning in formal STEM learning environments (Roberts et al., 2018). For students learning mathematics, concepts are more meaningful when they are situated within real-world problems in a problem-based context that focus on the Standards for Mathematical Practice (SMPs; NGA Center for Best Practices and CCSSO, 2010). Informal learning environments allow for real-world modeling examples (Martin, 2004; Meredith, 2010), help students better understand concepts (Popovic & Lederman, 2015), and encourage students to interact with peers (Barker et al., 2014) and others. Moreover, participation in informal learning environments increases students' interest in STEM (Mohr-Schroeder et al., 2014; Baran et al., 2016). Given the need to spark students' interest in STEM and to address limitations many students experience in formal mathematics, we used a situated learning perspective to answer the following research question: How does participation in an informal STEM learning environment influence middle school students' perceptions of mathematics?

Theoretical Framework

We used situated learning theory to explore how students' participation in an informal STEM learning environment influenced their perceptions of mathematics. Learners have the opportunity to acquire knowledge due to the interactions between them and the environment (Kirshner & Whitson, 1997; Lave & Wenger, 1991), which are mediated by social interactions. By interacting with each other and the environment while participating in authentic activities (Brown et al., 1989), learning is contextualized (Hung, Lee, & Kim, 2012). In a STEM learning environment, learning occurs in a community of practice where "learning is authentic and relevant, therefore representative of … an actual STEM practice" (Kelley & Knowles, 2016, p. 4).

Data Collection & Analysis

Participants in this study were 254 students who attended the STEM summer informal learning experience at the University of Kentucky, Iowa State University, and California State University, Long Beach. The summer learning experience is open to all incoming fifth through eighth grade students, and targets underrepresented populations in STEM fields (i.e., females, Black, Hispanic/Latinx, American Indians or Alaska Natives, and Native Hawaiians or Other Pacific Islands [National Science Foundation, 2017]).

Data were collected from semi-structured interviews designed to explore students "lived experiences" (Van Manen, 1990, p. 9). Students participated in a five-minute, audio recorded interview during the last two days of the summer informal learning experience. The interviewer also took notes to conduct member checks during and at the end of each interview. The majority (78%) of students interviewed were from underrepresented populations in STEM.

We used an inductive approach to analyze the data and managed data through reduction, organization, and connection (Lecompte, 2000). One author used initial coding to develop preliminary codes (Saldaña, 2016), which were then refined and used to code an initial set of interview transcripts. Four authors coded the interviews. Inter rater and intra rater reliability standards were set at 90% agreement (James et al., 1993). All four authors exceeded the 90% threshold on intra- and inter-rater reliability. After initial coding, four of the authors conducted second cycle coding. The team used pattern coding to highlight common themes and divergent cases (Saldaña, 2016; Delamont, 1992). After initial themes were identified, the entire team reviewed themes and supporting data to add content validity.

Findings

Two primary themes emerged from the data. The informal learning environment (a) influenced how students view mathematics and (b) allowed for the exploration of applications of mathematics, particularly in robotics.

View of Mathematics

Students compared their mathematics experiences in the informal STEM learning environment with their mathematics experiences in a formal setting. Jaden explained, "in school they just teach you one way to do it, but here they're teaching me other ways" (Interview, 2016). Laurie noted, "it's showing us different ways to see things other than the way the teacher teaches. Because I have a hard time understanding things and I feel like STEM camp's giving us a way to understand" (Interview, 2016). These "different ways" often focused on active ways of problem solving. For example, Mia explained, "STEM camp is like helping me be a better problem solver and like helping me think outside the box," which helps "in the classroom to develop ideas" (Interview, 2016). Similarly, Rosa said her experience will help her in mathematics class because the interactive STEM learning experiences taught her “new ways how to think, and new ways how to put things together” (Interview, 2017). These new, different ways to approach problems came about because the STEM learning experiences gave the students “different scenarios to work with” which helped them “look at the problem in different ways not just in the same ordinary way” (Interview, 2016). The creativity students associated with problem solving in mathematics contests was markedly different from the “one way to do it” associated with school. As Lexie said, “you get to learn about different like stuff like without actually doing all the equations and stuff. It’s learning different ways how to do like math” (Interview, 2016). Thus, the focus was how to use mathematics and not just solving an equation.

Students also emphasized the importance of problem solving when discussing how the informal STEM learning experience will help them in mathematics classes. For example, Jerel reflected, “I think it prepares me good [sic] because it can help me problem solve and everything because math you kinda have to problem solve and STEM Camp…will help me think through problems, be patient especially” (Interview, 2016). Similarly, Mya noted the importance of “trial and error and what they taught there gives you more patience for math if you can’t figure out the problem you just have to try again” (Interview, 2015). Michael added, “it’s teaching me about the trial and error so that we’re not just gonna get it the first time. We have to try” (Interview, 2016). Without referencing the Standards of Mathematical Practice (SMPs), students consistently voiced the essence of the first SMP, make sense of problems and persevere in solving them. The students’ responses demonstrate how the informal learning environment’s emphasis on the practices over specific content translates to students’ perceptions of STEM. Students’ responses highlight what they perceive to be a different way of doing mathematics than the “one way to do” mathematics in school. This relates to the way students experience mathematics in the informal STEM learning environment and the second theme: applications of mathematics.

Applications of Mathematics

Students identified ways mathematics applied to real world topics. Some students connected their experiences with mathematics to the future or to real life. For example, Alex said “it’s teaching you like real life situations, where like they don’t necessarily teach you math but they teach you like…where you can use this math type of thing” (Interview, 2016). Jared suggested he could use his experiences in college, “We learned a little bit of calculus, which I’m probably gonna learn that in college, but I think it’s preparing me for the real world” (Interview, 2016). One way this was evident was in the use of mathematics in STEM careers. Erin focused specifically on engineering as she explained:

> ...engineering also focuses on um, it focuses on math like how if you measure a plane, if you measure the length or width of a plane, it shows you like the length and width, like base times height, length times width, stuff like that. It helps you with like the width and then you can do the math well. Because in order to build a plane or to do math, so it shows you different ways to do math problems while doing fun things. (Interview, 2015)

Javier had a similar perspective on how mathematics relates to careers. He explained, “It’s going to teach me how to do certain different art like 3D art, so it prepares me for what I want to be when I grow up” (Interview, 2017). For these students, mathematics was relevant for their future. The informal STEM learning experiences helped provide context for when and how mathematics can be used outside of school.

In addition to students mentioning the real world and careers, many students found applications of mathematics in their experiences with robotics. As Janel explained, she learned "other things that could involve math instead of just regular problems like multiplication and things like that like other things you can use math like in robotics and programming" (Interview, 2016). Some students saw the application of specific mathematics content. For example, Darius explained how in programming "you have to use decimals and fractions…you have to coordinate how far you need to go…the multiplication [to] figure all that out" (Interview, 2015). Many students noted how "there's like a lot of measurement involved" (Interview, 2015). Maria elaborated:

> I guess there's a lot of math when you program robots. You have to know the exact number of times your robot's wheels have to rotate so it can go forward. If not, if you miscalculate the whole thing can go bad, as I was experiencing. And you have to know the angle of the turn. (Interview, 2015)

The experiences exemplified real life applications of mathematics through programming.

Students also commented on the importance of precision when programming. Cesar noted, "especially for robotics you have to basically figure out whether it's like 0.6 you have to go forward or 0.65 or like you have to be good at math to be able to do robotics because programming is like precise" (Interview, 2016). Jesse similarly reflected, "The robotics are preparing me for math classes because it's showing me how to be exact, how to build one piece at a time, which is like math" (Interview, 2017). These students voiced another SMP–attend to precision. The mathematical applications in programming required students to be precise to achieve their goals. These students connected the precision required in programming to the precision needed when doing mathematics. As Chris noted, "we've had to work out problems especially in robotics. If something goes too far we need to add something else or try something new…[Just] because we're not actually do[ing] math in that, there's a lot of math required" (Interview, 2016). Thus, the application of mathematics in robotics allowed students to make connections to when they would use mathematics content and processes.

Discussion and Conclusion

The STEM learning environment positively influenced students' views of mathematics and provided access to authentic STEM learning experiences that showcased a variety of real world applications of mathematics. All students do not receive access to this type of engagement with mathematics because constraints and pressures placed on formal settings (e.g., lack of funding, accountability measures, incessant testing) prevents many schools from implementing different curricular options (Meyers et al., 2013; Weis et al., 2015). Students from underrepresented populations tend to feel these negative side effects at a higher rate than others (Coxon, Dohrman, & Nadler, 2018; NCTM, 2014). Thus, one implication from this study is the importance of access to high quality learning environments, such as those provided in the informal STEM learning environment, where students can explore applications of mathematics. The ways in which students engage in mathematics are also important. Students repeatedly voiced the essence of many SMPs, such as persevering in problem solving and attending to precision. By engaging in these practices, students connected mathematics with their projects. They were not doing mathematics problems; instead, they were actually doing the process of mathematics. In the informal learning environment, students engaged in the process of doing mathematics, instead of focusing on thoughtlessly practicing the algorithms of mathematics.

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DEVELOPMENT OF PROFESSIONAL VISION IN MATHEMATICS TRANSFER STUDENTS AT A FOUR-YEAR RESEARCH UNIVERSITY

Sarah A. Roberts
UC Santa Barbara
sroberts@education.ucsb.edu

Krista Lucas
UC Santa Barbara
klucas@ucsb.edu

Hannali Pajela
UC Santa Barbara
pajela@education.ucsb.edu

While past research has focused on understanding the overall experiences of transfer students, little work has focused specifically on mathematics majors. This study explores the development of professional vision among mathematics transfer students, drawing from focus groups with 21 participants enrolled in a bundle of three transitional mathematics courses concurrently at a four-year research university. Overall, the experiences of these students in the bundle of courses exposed the importance of community, competence aligned with the structure of mathematics, development as a mathematics student, and resources for understanding future pathways.

Keywords: transfer students, professional vision, college mathematics, HSI

Currently we have very little, if any, research about how to support mathematics majors as they make the transition from two-year institutions to four-year institutions of higher education. While there is more general scholarship about transfer students (e.g., Melguizo, Kienzl, & Alfonso, 2011), there is a need to understand this pipeline for a number of reasons. First, this pipeline includes underrepresented minorities, low income, and first generation college students, who are more often than not underrepresented in the STEM fields. If we understand how to support transfer mathematics students, we likely also support our understanding of those who are traditionally underrepresented, potentially expanding the field and pipeline of mathematics majors and graduates. Second, at this university, and likely at other universities, we have seen a high level of attrition of transfer students from the mathematics major. This leads to a need for research to understand how to retain students in the major and whether the courses suggested in this study support students' retention. There is a clear need to better understand mathematics transfer students, their needs, and those who work to support their success.

Framing

We use Lave and Wenger's (1991) *legitimate peripheral participation* and Goodwin's (1994) *professional vision* as a conceptual framework to consider mathematics transfer students' development along the novice to expert continuum of professional vision. In particular, our research question is: In what ways do community college transfer students develop the professional vision for what it means to be a four-year college mathematics major and to have a career in mathematics?

Legitimate Peripheral Participation

To do this work, we start with Lave and Wenger's (1991) legitimate peripheral participation. Their work draws on apprenticeships and the active participation in a community of knowledge. Legitimate peripheral participation is the "way to speak about relations between newcomers and old-timers, and about activities, identities, artifacts, and communities of knowledge and practice. It concerns the process by which newcomers become a part of a community of practice" (Lave & Wenger, 1991, p. 29). An individual is not just "'receiving' a body of factual knowledge" (Lave & Wenger, 1991, p. 33) in this community; instead, they are actively engaged with it, mutually constituted, and an active agent. Lave and Wenger note that "[t]here is no place in a community of practice designated 'the periphery'" (Lave & Wenger, 1991, p. 36); therefore, peripheral participation is full participation. Legitimate peripheral participation allows for individuals to know what it means to be part of a particular community, such as a four-year mathematics department or the mathematics community more broadly, or what it means to support a student in such a community. We are particularly interested in the interaction between novices and experts. Access to experts provides newcomers with the opportunity to be "apprenticed into the learning of knowledge and skills" (Lave & Wenger, 1991, p. 145).

Professional Vision

Such novice-expert interactions as those previously described are organized for the development of professional vision. Goodwin (1994) notes, "[c]entral to the social and cognitive organization of a profession is its ability to shape events in the domain of its scrutiny into the phenomenal objects around which the discourse of the profession is organized" (p. 626). Goodwin explains that each profession has its own ability to see.

Goodwin (1994) defines professional vision using three characteristics: 1) it is "perspectival, lodged within specific social entities, and unevenly allocated" (p. 626); 2) it is not just a mental process but it is instead "accomplished through the competent deployment of a complex of situated practices in a relevant setting" (p. 626); and 3) "insofar as these practices are lodged within specific communities, they must be learned" (p. 627).

Methods

Twenty-one undergraduate transfer students, nine females and 12 males, from a Hispanic-serving four-year research university in California participated in this study. Eighteen of the students were mathematical science majors and three were applied mathematics majors. In Fall 2018, these students took three mathematics courses as a bundle: 1) special topics in mathematics (an academic and career advising course), 2) group studies in mathematics (a course to develop problem solving and the expectations for upper level mathematics courses), and 3) transition to higher mathematics. This third course was a prerequisite for many mathematics courses that follow, and it was described as an introduction to the elements of propositional logic, techniques of mathematical proof, and fundamental mathematical structures, including sets, functions, relations, and other topics.

We conducted and audio-recorded focus groups with three groups of transfer mathematics students, split evenly between three facilitators. We used these semi-structured focus groups (Yin, 2016) to understand how participants developed professional vision around being a mathematics major and mathematics professional. The interview covered the following: background information, three course bundle, being a transfer student, ways of thinking and doing mathematics, and preparation for a career in mathematics.

We used open coding (Strauss & Corbin, 1990) in our first round of coding, noting themes across the focus groups. We then applied four themes generated from this first round of coding: community, proofs and problem solving, being a student, and future as a mathematician, to code the corpus of data. We then looked within and across each set of coded data and research question, looking for consistencies and inconsistencies.

Findings

We found that the transfer students in the bundle of three mathematics courses in our study were very positive overall about their experiences. The students highlighted four ways that these courses helped them develop professional vision as mathematics students, future mathematics professionals, or both. These courses: 1) cultivated community among the transfer students; 2) created mathematical competence that was different from their prior mathematics learning experiences, particularly around proof and problem solving; 3) developed facility with studying and how to be four-year mathematics student; and 4) provided resources for mapping a future as a future mathematics professional. We will now discuss each of these findings, drawing on insights from the transfer students from these courses (named using pseudonyms below).

These courses helped to create community among the students. In all three focus groups, students mentioned the importance of the presence of community as a result of being in these courses. This community came about from working together on problems, being in all the same classes, and having what students described as a “comfortable” environment (Raul, FG2, 59). Because all the same students were in all three courses, they saw each other regularly, almost forcing community throughout the term. Additionally, because they were all transfers, many of the students lived together in transfer housing, increasing the likelihood that the students would run into each other there, as well. Kacey explained, “It felt like we were going through it together, and we can definitely help each other out” (FG1, 89-90). Sasha reinforced this feeling, “I think having the same people around you definitely was beneficial….Like, it makes you really want to go to class” (FG1, 102-103, 111). This collaboration helped students feel like they were part of the larger mathematics department and part of their smaller cohort of mathematics transfer students. Additionally, Maggie noted importantly, “I feel like a mathematician needs to know how to collaborate with others, too” (FG3, 540). The transfer students were developing professional vision around what it meant to do mathematics, not just singularly and alone, but as a collaborative, comfortable process with their classmates.

The second key finding about how this course bundle supported transfer mathematics students’ development of professional vision was that students felt that the courses created mathematical competence that was different from their previous mathematical experiences, particularly around proofs and problem solving. Two of the three courses in the bundle focused on developing students’ facility with how to approach and write proofs. Multiple students noted that this work with proofs was the most useful coursework throughout the term. Li noted, “I didn’t really know how to write proofs until the bundle of classes” (FG3, 195). Students revealed that these courses made them starkly aware of how different their four-year university mathematics courses were compared to their community college and high school mathematics courses. Antonio explained, “Studying for upper division is a lot different from our community colleges, where we would study formulas and just know where to plug stuff in” (FG2, 276-278). The transfer students repeatedly shared an importance for understanding the mathematics they were learning.

Our third finding is about the transfer students’ study skills and their development as four-year university mathematics students. The transfer students reported that the bundle of courses systematically supported this maturation. Because a mathematics department advisor taught one of the transfer student bundle courses, the students noted that it was helpful to “build a connection” with her (FG3, 233). The advisor also required students to attend five office hours for their courses, instilling in them a practice of what mathematics students should do. Raul explained, “it was a little scary, but after the first time, I kind of got the hang of it…I feel like for the next quarter that’s going to be a lot easier” (FG2, 190-194). The students expanded their understanding of the purpose of office hours over time, as well. Javier shared:

> I realized I don't have to go to office hours with a specific question and can just sit there and hear what people have to ask or just, like, talk to the professors about their research or what I should do with my year here before applying to grad school. (FG2, 201-204)

In their courses, transfer students also learned how to study. All instructors, with one in particular, spent time with the students discussing how to study for exams. Bastian shared that he learned what to study and how to study. The final aspect of becoming a mathematics student was learning about research at a four-year university, as Javier mentioned in his visit to office hours. Because they came to the university later than students who enter as freshman, the transfer students illustrated the challenges associated with getting to know and getting involved with professors' research, which often provides avenues to graduate school. Randy reported, "We don't have as much time as freshman to get involved into research before, like, applying into graduate programs" (FG2, 369-371). Raul shared a similar sentiment, "I wish there was a way to introduce us to, like, faculty professors that are working on research" (FG2, 434-436), because it would make it easier for transfer students to gain access to research opportunities. Several students noted that it felt difficult to gain access to research opportunities, even with the support of the bundle of courses.

The final findings we share are related to the resources transfer students reported for mapping their futures as mathematics professionals. Overall, the transfer students expressed satisfaction with the resources the advisor provided in the course she ran, as well as the multitude of resources she and the department offered outside of class. During class, there were panels with current graduate students and students currently applying to graduate school, as panels on careers in mathematics. Sophia noted, "It seems like the department just really cares and wants us to go to grad school or wants us to get a job right out of school" (FG1, 502-503). Even with this support, similar to comments about visiting their professors about research opportunities, the transfer students noted that they felt behind in the process. Keith shared, "it's still early on, so it's hard to know what we are interested in, because there are some math concepts we haven't learned" (FG1, 569-570). Additionally, students felt crunched for time. Michael explained, "we only have two more years…you kind of start thinking about your future now since we only have, like, one summer for an internship" (FG1, 554-557), and it was hard to know whether to go to graduate school, get a job, or do research with a professor. Multiple students mentioned this short timeline of having to make a decision about their futures and having limited time to make connections with professors to do research and/or have opportunities to do internships in the field. They described being behind before they got to the university.

Discussion & Conclusions

Transfer students in this bundle of courses developed several aspects of professional vision. The first aspect was a community to work with, which could help not only with their mathematics professional vision but also provide comfort at a new university, hopefully keeping them in the mathematics major and at the university. These transfer students also saw themselves differently *as* mathematics students and saw what mathematics students *do* differently. The clear role of problem solving and proof in upper-level mathematics work and their increased facility with this work was evident. To cross the threshold into a professional career, the courses provided some clear supports and requirements to build professional vision, like forced office hours attendance and career panels. But students still felt uncertain because of their late admittance, especially with research and internships. It may be that community college and four-year partnerships are needed earlier to ensure that students feel that they are not behind in these processes.

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INTENSIFYING FLUENCY INTERVENTIONS FOR MIDDLE SCHOOL STUDENTS WITH MATHEMATICS LEARNING DISABILITIES

Nicole B. Tucker-Whitaker
The Ohio State University
tucker.289@osu.edu

Students with executive function (EF) disorders, including ADD, ADHD, and dyslexia, are prevalent in our nation's public schools. Too often, students with learning differences have had a lack of exposure to complex academic language considered to be a critical component for the conceptualization and learning of mathematics. Additionally, there is a gap in research and understanding of how academic language might mediate building mathematical understanding for students who have EF disorders. This project is a small part of a larger ethnographic study and seeks to investigate what role modeling academic language might play in differentiating the fluency component of a curriculum that is being adopted nationally. Teacher modeling in the form of think-aloud protocols, feedback and reinforcement, and Detect, Practice, Repair (DPR) strategies are investigated as initial intensifications for the intervention.

Keywords: Middle School Education, Modeling, Number Concepts and Operations, Special Education

Introduction

Rationale and Purpose

A primary goal for students according to the Principles and Standards for School Mathematics is the ability to fluently and naturally speak using mathematical language as a "powerful tool...to foster the learning of mathematics" (NCTM, 2000, p.128). These process standards call for an increased role of language in the development of mathematical conceptual learning. "Communicating is a broad goal in mathematics learning that provides means to develop and share understanding" (Rubenstein, 2007, p.206). Thus the study of language as one of many learning tools enacted in the classroom, "of the ways in which it is employed in the classroom, and of the classroom as a social system, has been given considerable and increasing attention" (Austin & Howson, 1979, p. 174). An examination of how the development of academic language can be used and integrated into remediation techniques may be an initial step in aiding students with mathematics deficits to achieve mathematics procedural proficiency. This teacher-researcher teaching experiment seeks to understand the possibilities for academic language mediating the development of student procedural understanding and enculturating students into the community of mathematicians.

Perspectives

Numerous studies indicate that not all practice opportunities are equally effective in fluency building (Codding, Burn & Lukito, 2011). Research has found that "adding demonstration to standard drill and practice procedures resulted in better outcomes for students with learning disabilities, and … that students with low fluency benefited more from a treatment including modeling than one without." (Codding, et al., 2011, pp.43-44). However, feedback and reinforcement may be particularly important when fluency-building activities are in the form of timed activities (Duhon, House, Hastings, Poncy & Solomon, 2015). In addition to learning difficulties, many special education students have also been exposed to years of deficit model teaching, often languishing under the judgment and ridicule of both teachers and students alike. Therefore it is not just feedback that must be provided to the student about their academic performance, but more specifically, positive and academic self-confidence building feedback that incorporates the language of mathematics which may act as an external motivator for students to persevere. Duhon, et al. defines performance feedback as “a set of procedures in which students are given detailed information about some aspect of their academic performance such as accuracy or fluency.” (2015, p. 77). When such performance feedback is couched in the language of mathematics, it may then also serve the purpose of aiding students to move toward a position of “legitimate peripheral participation” on their way toward becoming a full participant in the community of practice of mathematicians (Lave & Wenger, 1991). Thus, academic language used by the teacher may serve numerous purposes simultaneously. For this research, modeling is defined as “an expert’s performing a task so that the students can observe and build a conceptual model of the processes that are required to accomplish it” (Collins, Brown & Holum, 1991, p. 13). Modeling, in the form of teacher think aloud, specifically using the complex academic language of mathematics, makes the invisible visible. Collins, Brown & Holum term this “cognitive apprenticeship… a model of instruction that works to make thinking visible” (1991, p. 1). Because “concurrent reasoning takes place in verbal form in working memory” (Cowan, 2017, p. 3), making the invisible thought processes visible through language may be a significant scaffold for students with executive function disorders to begin to develop structures to organize and retrieve the large quantities of verbally encoded procedural knowledge that may be required for success in mathematics.

Method

Participants

The research was conducted at a small, midwestern, independent school for students with learning differences. The larger study began in December 2017 when the teacher-researcher was made aware that she would be “looping” a cohort of students. From the original group of 2017-2018 seventh graders (n = 16), eight students continued on to eighth grade. Of the eight students having the teacher-researcher as both their seventh- and eighth-grade mathematics teacher, two are adopted (25%), eight have been diagnosed with ADHD (100%), two have been diagnosed with an anxiety disorder (25%), one has been diagnosed with a math learning disability (12.5%), one has been diagnosed with expressive-receptive language disorder (12.5%), four have been diagnosed with dyslexia (50%), one has been diagnosed with dyscalculia (12.5%), one has been diagnosed with a conversion disorder (12.5%), one has been diagnosed with auditory processing disorder (12.5%), and two have been diagnosed with dysgraphia (25%); many students have multiple diagnoses. The entire class participated in the intervention and intensification of intervention. The student featured in this brief report is a reasonably intuitive mathematics learner who experienced anxiety that often prevented meaningful engagement with learning throughout much of fifth- and sixth-grade, and like many students, has multiple diagnosis including dyslexia and ADHD.

Materials

The focus of this investigation is one aspect of the Eureka Math curriculum: Fluency Support for the development of procedural automaticity. All fluency materials come directly from the Eureka Math curriculum.

Task Identification

Eureka Math fluency support speed drills were matched to learners needs in order to provide intervention at an appropriate level. Determining where procedural supports are required is of primary importance to appropriately match the intervention with the skill to be remediated (Burns, M., 2011). Prior to the beginning of the intervention intensification, fluency support drills from Eureka grades three through six were administered to determine proficiency. These trials were all administered using the prescribed Eureka Math protocol.

Procedures

To capture both the intervention intensification as well as gauge success of the intervention intensification, all class periods where fluency sprints were administered were video-recorded. Since intensifying an intervention often begins with the adding the least intensive, least intrusive treatment component first, followed by additional components until an effective program is developed (Duhon, et al., 2015; Codding et al., 2011), the first layer was a combined intensification of teacher-researcher think aloud modeling and explicit feedback and reinforcement both utilizing academic language. The fluency drills were administered as follows: Trial A, two minutes, completed independently and individually. Upon completion the teacher-researcher modeled, using think-aloud protocol with academic language, how she thought about the fluency trial working to make the invisible thought processes visible through language. Students were encouraged to self-correct and ask questions during the think aloud before Trial B. Following the two minute Trial B, answers for the fluency drill were put up and students were asked to correct their work looking for progress between Trials A and B. The teacher-researcher circulated the room, viewing each student's work, giving specific feedback on progress and cues for improvement using academic language.

The second layer of intervention intensification, DPR, was implemented over three weeks for a total of seven sessions, consecutive to the first layer of intensification. Following a partial Trial A think aloud with initial feedback, students were asked to detect errors in the remainder of their work and make corrections. After two to five minutes of silent work, the teacher-researcher put the answers for the sprint not covered in the think aloud on the board. Students then checked their work one more time to ensure that they detected and repaired all problems. While students were working, the teacher-researcher circulated the room, answering questions and providing additional support to specific students as necessary. Once the process was complete, students completed Trial B as above, looking for progress and receiving feedback and reinforcement.

Data Analysis

The first step in analyzing the data was to view and code video recordings. The identification of a pattern of student behavior, which occurred repeatedly, with more frequency, and by more students as the intervention continued, determined specific sections of video to be transcribed. Specifically, students began asking to lead the think-aloud modeling portion of the procedure. As students were allowed to lead the think-aloud, being required to do so using specific and detailed academic language as had been modeled by the teacher-researcher, the rate of student requests to lead the think-aloud increased as did the number of students requesting to lead the think-alouds. Each of the student-led think-alouds were transcribed for further analysis.

Data analysis is informed by Nagy & Townsend who posit that it is not merely the imitation of academic vocabulary that is a significant indicator of developing conceptual understanding in mathematics but rather it is the ability to generate discourse within the mathematical register for the purposes of clearly communicating mathematical ideas (2012; Corson, 1997; O'Halloran, 1998). Changes in the uptake and use of academic language may also indicate the degree to which students are becoming enculturated into the community of mathematicians, along with growth in academic confidence, and may be indicated by increased willingness to share academic thinking aloud (Bloome, Puro, & Theodorou, 1989).

Results

Preliminary Findings

This study is still in data collection and analysis stages, however, a pattern of student behavior of interest has been identified. As students became familiar with the fluency practice protocol and teacher modeling, they began to request to lead the think aloud portion of the intervention. Following is a brief excerpt from just one of the many student demonstrations:

Student (S): Can I do the think aloud?
Teacher/Researcher (T/R): Let me start and you can take over
T/R (begins the process): We should recognize that these are operations with integers
S: Okay, so, with this it's like adding positives but they are negatives
S: It's getting more negative

When the student-led think aloud stalls, other students chime in to encourage the process and get the presenting student to give more detail about what is "going on in his head." This pattern of student interaction serves to demonstrate one way in which language may be mediating a mathematics procedural fluency intervention while simultaneously mediating student movement toward a position of "legitimate peripheral participation" (Lave & Wenger, 1991).

Discussion

From a sociolinguistic ethnographic perspective, attention must be given to the way in which students are "establishing a public system of cultural meanings and significance" (Bloome, Puro, & Theodorou, 1989, p. 269) for the engaging with, and doing of, mathematics within the classroom, with a focus on "how classrooms function to promote the enculturation of new members" (Bloome, Puro, & Theodorou, 1989, p. 270). Thus, modeling the language of mathematics to support fluency sprints through think-alouds allowed the teacher-researcher to verbalize the thought processes, and thus make visible the invisible, that a mathematician may engage in when completing the types of problems on the sprint. If we view the think-aloud process from the sociocultural perspective, "the development of a social language is a process of being socialized or acculturated into a specific community of practice, such as 'academic' social practice in school mathematics" (Huang & Normandia, 2007, p. 296; Gee, 2001, p.13). Student requests to begin to participate in the community of mathematicians being modeled by the teacher-researcher may then be viewed as a request to engage in the construction of knowledge through social interaction in the classroom. In this way, using the language and logic structures of mathematics to do the work of engaging students in the culture of mathematics and academic thinking, a third space was opened for students to engage in the practice of verbalizing their academic thinking helping them to move from apprenticeship toward a position of "legitimate peripheral participation" on their way toward becoming a full participant in the community of practice of mathematicians (Lave & Wenger, 1991). Such opportunities are critical to create a cognitive connection between vernacular language, mathematics vocabulary, and symbols used to describe mathematical concepts that are represented in the computations and procedures that may be performed in fluency sprints (Miller, 1993).

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WHAT MATTERS FOR INVESTIGATING IDENTITY IN MATHEMATICAL LEARNING SETTINGS: POSITIONS, FUNCTIONS AND INDIVIDUALIZED REPERTOIRES

Fady El Chidiac
University of California, Berkeley
fchidiac@berkeley.edu

In the last two decades, research on identity has been multiplying and building on various theories and methodologies from different disciplines. The task of providing a comprehensive framework of identity for mathematics learning has been arduous. Based on a broad literature review, I highlight three indispensable components for a robust mathematical identity research: positions framed culturally and discursively, functions/goals set by participants, and individualized semiotic/behavioral repertoires. I leverage this framework to shed light on a puzzling comparative case of two undergraduate women learning number theory through small-group work.

Keywords: Identity, Research Methods, Gender, and Post-Secondary Education.

Since the end of the twentieth century, a multitude of studies have been employing the construct of identity to shed light on the learning and teaching of mathematics (for reviews of this scholarship see Darragh, 2016; Hand & Gresalfi, 2015; Langer-Osuna & Esmonde, 2017; Radovic, Black, Williams, & Salas, 2018). These studies have been building on various theoretical approaches within and across many disciplines, such as psychology, sociology, anthropology, psychoanalysis and linguistics. They investigated various units of analyses pertaining to individuals, societies, discourses and learning environments. Consequently, the increasing complexity of the construct of identity in mathematics educational research is hindering the advancement of the field and communication across researchers.

In this paper, I lay out three indispensable components, namely positions, functions, and individualized repertoires (see Table 1), to which any research on learners' identities must attend to claim robustness. The three components were distilled from a broad literature review encompassing multiple theoretical approaches (Interested readers may consult El Chidiac (2018) for more details). After outlining the three components of learning identities next, I will illustrate their indispensability through a comparative case study of two women learning number theory by working with four other men in small groups over a semester. The learning identities of the two focal women, Bettie and Melissa, developed drastically in two different ways, which could not be explained without attending to the three aforementioned components.

Table 1: Three Components that Matter for Investigating Mathematics-learners' Identities

	Positions framed culturally or discursively	**Functions/goals**	**Individualized semiotic/behavioral repertoires**
Time relevance	Current behaviors/acts	Near and far future	Past experiences
Theoretical approaches	Social, cultural, discourse and pragmatics	Social and cognitive	Psychoanalytic, psychology and socio-historical

Positions

Positions are *behaviors and acts framed culturally or discursively in a learning activity,* such as a solitary study, small-group work, whole-class discussion, and outside of school project. They can be behaviors performed by students, such as searching the internet for an answer to a mathematical problem. They can also be acts done to participants, such as assigning a grade to a student's work. The component of position is concerned with students' behaviors in the *hic et nunc* of a learning activity. The meanings of behaviors and acts are mediated by cultural and discursive frames, commonly called storylines (Davis & Harre, 1990), figured worlds (Holland, Lachicotte, Skinner, & Cain, 1998), community of practice (Wenger, 1998), big 'D' discourses (Gee, 1999), and meta-rules (Sfard, 2007).

Functions/Goals

I define a *function as a goal set to be achieved or a state unintentionally achieved by enacting a position in a cultural activity*. Functions are concerned about the effect that a position can have beyond the *hic et nunc* of its animation. They may pertain to different timescales; they can be or aim at being achieved immediately, shortly or in a while. For example, the function of explaining mathematical ideas to peers can be immediate, as in improving the shared understanding of group members, short-term, as in preparing for the exam by testing one's own mathematical understanding with peers, or long-term, as in rehearsing being a teacher for a future career. The relationship between functions and learners' identities have been established in the work of Martin (2000), Nasir (2002), and Sfard & Prusak (2005). What we thrive to become in the near or far futures are part of who we are now.

Individualized Semiotic/Behavioral Repertoires

The component of individualized repertoires attends to students' histories that bear on their animation of positions and functions. It covers the *frames of mind and entrenched habits that students have individualized through their life experiences*. People construct and negotiate their understandings of self, others and the world through the mediation of semiotic repertoires, such as metaphors (Lakoff & Johnson, 1980) and *l'imaginaire* (Lacan, 2006), which they have made their own over their life experiences. People also individualize embodied habits of dealing with and negotiating different ecologies (Bourdieu, 1990). Individualization is the process by which *what we saw, heard, knew, and did in the social realms of our past become who we are now* (Copjec, 2015; de Certeau, 2011; Erickson, 2004; Martin, 2000). What count for this component are the repertoires, not as set socially, linguistically, and culturally, but rather as idiosyncratically processed by participants.

Methods

Activity and Focal Participants

Data reported was collected from a semester-long number theory class at a Northern California University. The class met for one hour ten minutes twice a week. Professor Hoffmann, the instructor, used group work as the predominant form of teaching in class. He gave students worksheets of theorems and problems to solve in class with their groups. A weekly homework, consisting of selected problems from the worksheets, was assigned to be submitted individually. It was assessed and returned to students. Students took a midterm and final exams. For the purpose of illustrating the centrality of the aforementioned three components of identity (see Table 1), I report a comparative case study (Yin, 2013) of two focal students: Melissa and Bettie. Both of them participated with four men in classroom groupwork and reported low self-efficacy at the start of the class. Melissa was majoring in Mathematics for teaching and Bettie in

Mathematics for liberal arts.

Data Sources

Starting from the second week of class, all group sessions of participating groups were *videotaped*. Participants submitted *individual memos* after every group session and took three types of interviews. First, they participated in *early individual interviews*, where they were asked about their motivations for majoring in mathematics, feelings about the mathematics discipline, history with mathematics classes in high school and college, experience with group work in classes, and first impressions about current group members. They also participated in *SCNI interviews* (Stimulated Construction of Narratives about Interactions; see El Chidiac, 2017). In the SCNI interviews, participants watched a video of their recent group session and commented on their social interactions. The SCNI interviews were conducted individually and within twenty four hours of the end of class. Lastly, they participated in *individual interviews by the end of the semester*, where they were asked about their understanding of number theory, confidence in the material, the changes throughout the semester in their learning methods and behavior in group work, and the roles their groupmates tended to play.

Data Analysis

I tracked the enactment of ten active and passive positions for each participating student through the videotaped group sessions (for details see El Chidiac, 2018, pp.34–35). Then I documented all functions and individualized repertoires pertaining to these positions as reported by participants in their interviews.

Illustrative Comparative Case

Melissa's and Bettie's Positions

Melissa and Bettie lacked confidence in their abilities to construct and understand mathematical proofs, a pre-requisite skill for the course. In classroom, they worked with peers whom they considered "smarter" than them. However, their modes of participation in group work differed, mainly with respect to two positions (see Figure 1). While Bettie sustained the same participation pattern throughout the course, Melissa enacted active participation in the second and thrid months of the course. Surprisingly, the participation patterns in classroom groupwork did not explain Bettie's and Melissa's learning gain. Despite her active participation in groupwork, Melissa's scores were low (45%) on the midterm and continued to drop (20%) on the final exam. Contrary to Melissa, Bettie improved her scores from the midterm (53%) to the final (61%) exams. Additionally, Bettie invested in her learning by regularly joining a study group outside classroom, which Melissa joined only once.

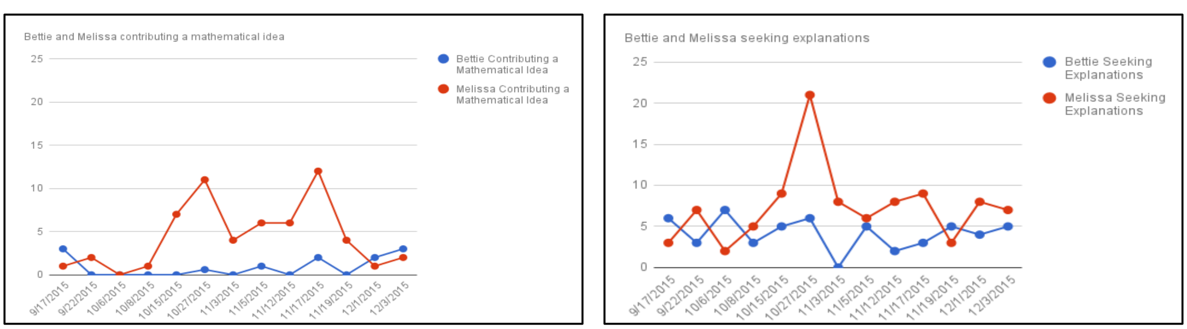

Figure 1: The number of instances when Bettie (Blue) and Melissa (Red) contributed mathematical ideas to group (left graph) and sought mathematical explanations from

groupmates (right graph) in the group sessions of classes attended by both of them.

Melissa's and Bettie's Functions/Goals

Melissa and Bettie set two opposite functions of their participations in classroom group work. Melissa animated her positions in groupwork to stimulate her groupmates' brains instead of hers, whereas Bettie intended to enhance her mathematical understanding. The following reported evidence was typical.

Melissa remained a passive learner despite her active participation in classroom group work. During her early interview (on 9/17), Melissa described her participation in classroom groupwork as follows.

> Melissa: more than trying to figure out what's going on I'm trying to help my friends who know better . like give them my ideas of what I might be thinking to help them put put it together because once they put it together then they can explain to me what's going on. […] And so they [Tom, Robert and Emil] are kind of like the three main brains […] if they stop talking I just kind of like ask a question to get their brains going again because I mean they know way more than I do.

On the contrary, Bettie primarily intended to foster her mathematical knowledge through her participation in classroom. During the SCNI interview on 10/15, Bettie commented on her participation in classroom groupwork as follows.

> Bettie: when I hear people [groupmates] talking and I don't understand I just zone them out because it confuses me more […] I just keep looking on my own. […] That's basically all I do when I'm in class. I just listen to what [my groupmates] are saying and look at the book . cause if I don't understand it then . when they're like talking […] I just zone people out . until I look at it myself because . otherwise it just confuses me more.

Melissa's and Bettie's Individualized Repertoires

Melissa invested in classroom groupwork (see Figure 1), while Bettie joined a study group outside classroom. Functions alone could not explain Bettie's and Melissa's differential investment. However, their individualized repertoires could.

Melissa was passionate about dancing yet decided to become a middle-school mathematics teacher. For Melissa, teaching was a long-term reliable career to provide for a living, whereas a dancing career would last only as long as her body is agile. In the exit interview on 12/02, she confessed that the multiple dance rehearsals did not leave her time to study number theory outside classroom. She thus focused on classroom to learn it.

On the contrary, based on her schooling experience, Bettie considered herself "really good at math [arithmetic] and sucks at everything else" (Early interview on 9/22). Sustaining a positive self-concept in mathematics was existential for her. After month through the course, Bettie realized that her learning in classroom was not sufficient to improve her scores on the homework. Wanting to replicate a past tutoring experience that helped her succeed the modern algebra course, Bettie asked a groupmate, Ted, to tutor her. Instead of providing one-on-one tutoring, Ted invited all groupmates to study together in the library. In the study group, Bettie had the opportunity to go over the classroom materials a second time with her group after reviewing them at home between the class and study group sessions.

Conclusion

The comparative case study of Melissa and Bettie, two undergraduate women majoring in mathematics, highlighted the insufficiency of attending solely to their positions in order to understand their learning trajectories. The analyses of the functions they set for their participation in groupwork and the individualized repertoires they actuated were necessary to explain their discrepant learning trajectories.

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MATHEMATICAL IDENTITIES AND GENDERED INTERACTIONS IN AN 8TH GRADE CLASSROOM

Bailey Braaten
The Ohio State University
braaten.7@osu.edu

This research project examines how gendered interactions in an 8th grade mathematics classroom can influence students' mathematical identity formation. Microethnographic methods were used to collect and interpret the data within this study. Preliminary results of this study indicate that gendered interactions have an influence on female students' mathematical identity formation, particularly for those who have weak mathematical identities. This study has implications for understanding the social and cultural influence of gender on mathematical identity formation for female students.

Keywords: Classroom Discourse, Gender and Sexuality, Middle School Education

Introduction

As the science, technology, engineering, and mathematics (STEM) job market continues to grow at a rapid pace, especially jobs related computer science and mathematics, there is still a major divide between male and female participation in these jobs (U.S. Department of Labor, 2017; Fayer, Lacey, & Watson, 2017). A foundational aspect of students who pursue STEM majors is their experiences in middle and high school mathematics (Shapiro & Sax, 2011). Mathematics courses have been viewed as a critical filter for participation in STEM fields (Damarin, 2008), and a foundational time in deciding to continue mathematics studies occurs for students in 8th and 9th grade (Erchick, 2001). An essential component to how students relate to mathematics is based on students' mathematics experiences and mathematical identity (Bishop, 2012). If we are to better understand the rationale for the lack of female participation in STEM careers, particularly computer science and mathematics, it is crucial to understand female students' experience in mathematics in 8th and 9th grade, how their mathematical identities are constructed with in the classroom space, and how gendered interactions influence mathematical identity construction. This study proposes to examine how students' mathematical identities are constructed through gendered interactions in an 8th grade mathematics classroom.

Background

For the purposes of this study, *identity* is operationalized as a personal characteristic that is socially constructed through relationships and interactions that are bound by specific contexts and also informed by an individual's personal narratives (Anderson, 2007; Boaler, 2002; Bucholtz & Hall, 2005; Davies & Harre, 1990; Duff, 2002; Mendick, 2005; Nasir & de Royston, 2013; Solomon, Lawson & Croft, 2011). Following this definition of *identity*, *mathematical identity* is defined as a personal characteristic that is socially constructed through students' interactions and relationships that are bound by a mathematics context and informed by their own personal mathematical narratives (Anderson, 2007; Bishop, 2012; Boaler, 2002; Cobb, Gresalfi, & Hodge, 2009).

In this study, *gender* is operationalized as a socially constructed characteristic associated with individuals that is broader than male and female, and is also of consequence in mathematical identity construction (Damarin & Erchick, 2010). Particularly, gender can be examined using the theory of hegemonic masculinity, in which the culturally dominant form of masculinity devalues all other expressions of gender (Connell, 1997). According to Burton (2008), the epistemology of mathematics is masculine, white, and classed. Both the masculine epistemology of mathematics (Burton, 2008) and hegemonic masculinity (Connell, 1997) create an environment in many mathematics spaces that influence how students are able to construct their mathematical identities. Gender is examined through the social construction of the participant students within this study, who in most cases operationalize gender within a binary male-female structure.

While not all mathematics contexts are constrained to a classroom environment, the cases examined within this study are bound in particular to an 8^{th} grade mathematics classroom. Thus, this study attends to mathematical identities that students construct within the bounds of their 8^{th} grade mathematics classroom, and how gendered interactions play a mediating role in mathematical identity construction.

Research Design and Methods

This study examines a set of interactions from within the course of a two-year ethnographic study of students' mathematical identities over the course of 8^{th} and 9^{th} grade mathematics classes. In this study, gendered interactions during the 8^{th} grade year of mathematics six focal students were examined. Focal students were selected based on student interactions and participation during classroom observation, teacher perception of student mathematical achievement, and student mathematical narrative given within interviews. The six focal students were selected to represent a range of experiences within each of these criteria. In conducting this research, a microethnographic perspective was taken to analyze these interactions, and how they influence students' mathematical identities.

The location for this research project is an 8^{th} grade mathematic classroom in a large suburban school district in the Mid-Western United States. The data collected for this study were 50 classroom observations throughout the school year with field notes and video recordings of each classroom observation. In addition to the classroom observations, each focal student was interviewed twice during the school year, once during the fall semester and once during the spring semester, about their mathematical identities.

The collected data was analyzed using a coding scheme to identify when gender was a salient part of classroom interactions for the students, and when students were actively constructing their mathematical identities around these gendered interactions. Transcripts were generated for the selected interactions, and these transcripts were then analyzed further using Wortham and Reyes' (2015) narrative analysis framework.

Findings

Within the observations, several instances where gendered interactions intersected with students' mathematical identity construction were identified. During these interactions the female student responses were of particular interest, often times challenging gender norms or gendered expectations. The following excerpt exemplifies these types of gendered interactions. The students in this class were working on a project creating a poster that illustrates a linear relationship.

Richie: This is easy. (Pointing to the assignment instructions.)
Cayla: I can't draw.
Richie: You literally just have to do that (Pointing to the assignment instructions.) Just write little flowers and show growth.
Cayla: Just because I'm a girl doesn't mean I'm feminine and have to draw flowers on my paper.
Richie: Yeah, it does.
Kyle: No, it doesn't.
Cayla: No, I don't even like flowers.
Richie: Then draw a bicycle.
Cayla: What do bicycles have to do with slope and y-intercept?

In this interaction, Cayla interprets Richie's suggestion that she draws flowers as an implication of how a traditionally feminine person could represent their linear relationship. Cayla actively rejects this assertion from Richie, and is supported by a male classmate in this rejection of traditional femininity. In addition to this, she continues to assert her own mathematical identity in re-focusing the conversation on how Richie's next suggestion of using a bicycle is related to linear functions. In this example, Cayla has a strong mathematical identity that she is able to assert, as well as the confidence to disrupt gendered expectations.

Another example occurred when students were arranged in small groups for instruction. The teacher was working on small group instruction with Cayla, Pete, Ayman, James, Kyle, and Paris. In this classroom event, the group was reviewing how to simplify expressions using the properties of exponents. The teacher had several mathematical exercises projected onto the white board, at which point James and Pete immediately went to the board to begin writing their solutions to two of the problems, and were quickly joined by Ayman and Kyle. Cayla had been absent the previous day, but asked the teacher if she could go to the board and attempt a problem by saying "Can I go up there too? Or is this just for the boys?" Cayla is disrupting hegemonic masculinity by asserting her own place at the board with the boys. This particular event continued with Pete telling Paris that she needed to go to the board to solve a problem. Paris was instead solving the same problems on her own on her worksheet. When called out by Pete, Paris refuses to go to the board. Pete is wielding his hegemonic masculinity to police Paris' behavior of not going to the board. Paris' refusal is a disruption of the masculine epistemology of mathematics through rejecting the notion that participating in public displays of mathematical knowing is essential to mathematics.

The interactions examined illustrate how two female students demonstrate their own mathematical identities in different ways that work to disrupt hegemonic masculinity and the masculine epistemology of mathematics. The focus of this study is to examine instances such as these where female students disrupt hegemonic masculinity and the masculine epistemology of mathematics in asserting their own mathematical identities. These instances provide insight into the powerful work that female students engage in when disrupting social constructs, even in seemingly small moments of classroom interactions.

Conclusion

The implications of this study relate to a better understanding of female experiences in STEM majors and careers in which they have underrepresented (U.S. Department of Labor, Bureau of Labor Statistics, 2017). The experiences of young women developing their mathematical identities in environments where hegemonic masculinity and masculine epistemology are dominant cultural forces are important to examine. The moment to moment classroom interactions in which young women disrupt or attempt to disrupt these forces are critical to their mathematical identity development. Examining these moments allow for insight into how mathematical identity development may shape these female students' future choices to pursue majors and careers within STEM.

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READING AND MATH GO SIDE-BY-SIDE: STUDENTS EXPLORE THE COMPLEXITIES BETWEEN TWO CORE DISCIPLINES

Gregory Beaudine
Michigan State University
gbb@msu.edu

Mathematics and reading have shared a long history in schooling, where both have regularly served as a core discipline. Despite the strong correlation researchers have observed between mathematics and reading, schooling has separated the disciplines into sterile spaces where mathematics is largely absent in reading spaces, and vice-versa. The text below explores the responses of middle school mathematics students as they are asked about their feelings toward mathematics, toward reading, and how reading and mathematics relate. The responses identify the importance of both disciplines, though reading was deemed more leisurely. Additionally, these students expressed a relationship between reading and mathematics that focuses on word problems, directions, and definitions.

Keywords: Middle School Education, Classroom Discourse, Instructional Activities and Practice, Affect, Emotion, Beliefs, and Attitudes

Against suggests either opposition or adjacent items. This project takes up the latter definition, to recognize the ways reading and mathematics relate to each other with regard to student learning and achievement (Grimm, 2008). Past research has demonstrated a strong correlation between reading and mathematics in early elementary school, and the achievement of students (e.g., Duncan et al., 2007; Grimm, 2008). Additionally, mathematics and reading are the two most tested subjects in public schooling (U.S. Congress, 2002, 2015), and both reading and mathematics are heavily utilized in other subject areas. The two disciplines could be viewed as sitting “against” one another, touching, related. It is in this sense that I view reading’s influence, or lack thereof, on mathematics to be a “new horizon,” worthy of exploration.

If it is accepted that “literacy involves the integration of listening, speaking, reading, writing, and critical thinking and includes the cultural knowledge that enables a speaker, writer, or reader to recognize and use language appropriate to different social situations” (Gibbons, 2009, p. 7), then mathematical literacy involves an individual’s focus on the same five components, through the lens of mathematics. In other words, learning to become mathematically literate includes learning to listen, speak, read, write, and think like a mathematician. The study, described below, explores a single question: What do middle school students believe about the relationship between mathematics and reading?

Literature

While much of the work around mathematical literacy tends to focus on discourse and writing (e.g., Herbel-Eisenmann et al., 2015; Sigley & Wilkinson, 2015; Moschkovich & Zahner, 2018), a subset of this work attends to reading in mathematics. This study sits as a single component of a larger line of research that aims to improve upon the ways students read text in mathematics spaces. Reading in mathematics, as identified by Adams (2003), pertained mathematical text that utilized numerals, words, and other symbols to communicate mathematical ideas. A decade of research expanded this idea to include graphical representations within the mathematics texts that readers must also interpret (Hillman, 2014). While it appears that mathematicians read a variety of styles of written text, “American students do little reading of published mathematics texts, either in or out of class” (Adams, Pegg, & Case, 2015, p. 499).

Recent years have brought several studies that relate to mathematical reading (e.g., Doerr & Temple, 2016; Shepherd & van de Sande, 2014). These studies tend to focus on the expert in the room, whether it be a professor of mathematical or a K-12 classroom teacher, and rarely attend to the reading done by students of mathematics. While Shepherd and van de Sande did explore the ways graduate and undergraduate students read mathematics, “the perspective of [elementary and secondary] students are recognized as valuable, but not often queried or considered” (Zheng, Arada, Niiya, & Warschauer, 2014, p. 279), and yet we know little about the ways K-12 students read mathematical texts, or the relationship students see between reading and mathematics.

Method

Ten middle school students were kind enough to sit through a “deliberately nonstandardized” (Ginsberg, 1997) clinical-style interview, designed to explore how these participants view reading, mathematics, and reading in mathematics. The semi-structured nature of the interview allowed the researcher flexibility to shift the line of discussion when the opportunity arose. The students were each asked three questions: How do you feel about mathematics? How do you feel about reading? What role do you see reading play within mathematics?

The ten students who participated in this study range from 6th grade to 8th grade, and were all enrolled in either 8th Grade Mathematics or Algebra 1 (Table 1). These students shared a range of opinions about mathematics and reading, which will be explored below. All ten students attend the same suburban middle school, and they were selected from four separate classes – two 8th Grade Mathematics and two Algebra 1 classrooms.

Table 1: Study Participants

Pseudonym	Gender	Grade Level
Melanie	F	8
Saed	M	8
Eric	M	8
Junhai	M	6
Mark	M	6
Sammy	F	8
Caroline	F	8
Ophelia	F	8
Zeta	F	8
Zahera	F	8

The one-on-one interviews were video recorded and averaged 27 minutes. Each interview was completed in two segments. The first consisted of a read- and work-aloud, where participants engaged with standardized test questions. The second section explored the participants’ feelings toward reading, mathematics, and reading in mathematics. The interview transcripts and videos were paired and examined through a discourse analysis lens, an appropriate lens for one interested in the “examination of speech, writing, and other signs” (Wortham & Reyes, 2015, p. 01). In this process, the top three words most often spoken within each question (Table 2) were revisited and themes were established from the ways the participants used these words.

Table 2: Top 3 Words

	First	Second	Third
How do you feel about mathematics?	math (15) mathematics (4)	like (16) liked (1)	understand (9) understanding (2)
How do you feel about reading?	reading (17) read (8)	like (18) unlike (1)	really (11)
What role do you see reading play within mathematics?	reading (16) read (10)	math (16) mathematics (2)	like (18)

Results

The words above are not surprising, considering the participants were asked about reading and mathematics, and each showed up over 35 times across all ten interviews. The intrigue stems from the ways words were used, and by whom. The remainder of this paper will focus on three root-words – like, really, and understand – and the phrases in which they were used.

Like

After *reading* and *math*, *like* was the most commonly used root with over 50 usages across all three interviews. Five students said "I like math" or "I really like math." Zeta repeated this phrase three times in four sentences, then proceeded to explain "I don't like doing problems … [but] I like math, I like discussing about it." When asked about reading, Zeta stated "I like it very much and it's awesome." Awesome, an adjective not used by any student about mathematics. Participants also stated they "enjoy" or "love" reading, without a qualifying statement, whereas the highest praise offered about mathematics was "I really like math" (Eric) or "I enjoy it. It's okay" (Melanie).

When discussing reading in mathematics, *like* was most commonly used to direct attention to something. Caroline, for example, suggested "If you don't know how to read, then you can't really do math because math is about problem solving and questions like these [word problems]." This is an interesting comment when compared to Zeta, who suggested "I like story problems but I feel like story problems are way different from math." In one interview, word problems qualify as reading mathematics, in another, they do not.

Really

In 19 moments, *really* was used by these participants to accentuate their beliefs. Two of the 19 came in the mathematics discussion, where Eric "really likes mathematics" and Ophelia must "really try" understand her mathematics work. When discussing reading, the participants collectively used *really* 11 times. In most cases, these participants did so to describe the kind of text they enjoy reading (e.g., "I really like graphic novels." or "I really like fantasy."). Mark described struggles with reading math problems, stating that they get "confusing if it gives you things you really don't need." Zahera "really loves reading" and Sammy likes reading because "it's really easy and really fast."

When discussing the relationship between reading and mathematics, these participants used *really* four times, to make two completely different claims. Melanie believes "reading is a really big part of Math," a perspective echoed by Caroline who said reading is important, because without reading "you can't really solve the problem and you can't understand what the problem means." Junhai and Mark both recognize minor roles for reading in mathematics, but when asked about a relationship between reading and mathematics, they both close with "not really."

Understand

Understand or *understanding* was used a total of 19 times across the three questions, with 11 coming when participants were talking about their feelings of mathematics. In this mathematics conversation, Ophelia used nine of them. She suggested that mathematics is good and "fun when you understand … [math is] not something that I can just look at and just understand … [but] it's something that I can understand if I really try or want to." Saed reported that reading helps him understand the mathematics, a perspective shared by other participants. Junhai, for example, suggested "if I was better at reading, I might understand the problem better." Across all discussion, the idea of understanding was only applied to mathematics. None of the participants discussed a need to understand reading.

Discussion

The project and results described above sit as a small part of a larger line of inquiry. The pursuit outlined above was to gather information about how students view the relationship between reading and mathematics. Specifically, these students spoke of reading and mathematics quite differently. Reading was identified to be more leisurely and enjoyable for the participants. Mathematics, conversely, seemed to be more about a search for understanding – a goal helped by better reading abilities.

When exploring the perceived relationship between reading and mathematics, the responses were really mixed. The participants in this study have differing views as to what extent reading plays a role in mathematics, if at all. Broadly speaking, researchers have discussed the variety of forms used by mathematical texts (e.g., Adams, 2003; Hillman, 2014). These ten students, when asked about the role of reading in mathematics, largely focused on written prose (e.g., word problems, definitions, and directions) as opposed to the symbolic and graphical representations found in equations, graphs, and diagrams.

The qualitative data used for this study stems from three questions within a 30-minute interview exploring the mathematical reading of middle school students. The student responses came toward the end of their session, just before they went to, or returned to, class. Even with the limited time, their responses suggest a promising line of study, relating to the way students view reading in mathematics. With a better understanding of what students view as reading and an added exploration on how reading is instructed within mathematics spaces, we might find more effective strategies for teaching middle school mathematics.

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THE USE OF PARADOXES FOR UNDERSTANDING AND REVEALING STUDENTS' MISCONCEPTIONS IN PROBABILITY AND STATISTICS

Vladislav Kokushkin
Virginia Tech
vladkok@vt.edu

Jesse L.M. Wilkins
Virginia Tech
wilkins@vt.edu

Keywords: Probability, Undergraduate-Level Mathematics.

There is a substantial body of research confirming that students often have misconceptions about probability (Anway & Bennett, 2004; Kahneman, Slovic & Tversky, 1982; Konold, 1991). These misconceptions impede students' ability to judge in situations of uncertainty and master fundamental concepts related to probability. Paradoxes may play an important role in the teaching and learning of probability and statistics (Shaughnessy, 1997; Movshovitz-Hadar & Hadaas, 1990; Wilensky, 1995; Sowey, 2001; Leviatan, 2002; Klymchuk & Kachapova, 2012). The purpose of this study is to provide evidence that carefully selected paradoxes can help to understand and reveal undergraduate students' probabilistic and statistical misconceptions. More specifically, we address the following research question: *How can classical probabilistic paradoxes be used for revealing and understanding undergraduate students' misconceptions in probability and statistics?*

Tversky and Kahneman (1982) described different heuristics that are employed to assess students' thinking under uncertainty. The basic assumption underlying this discussion is that a great deal of human judgement involving probability is based on heuristics. In this study, we focus on the representativeness heuristic, referring to students' tendency to erroneously think that samples which resemble the population distribution are more probable than samples which do not. This approach to the judgement of probability leads to numerous misconceptions, such as "insensitivity to prior probability of outcomes," "insensitivity to sample size," "misconceptions of chance," "insensitivity to predictability," "illusion of validity," and "misconceptions of regression" (1982, 4-9).

The study involved four undergraduate students of different majors and ages. Three of the students were first-year females and one a senior male. The rationale for involving undergraduate students is that they are familiar enough with the concept of probability and have some knowledge of making judgments under uncertainty, but often have numerous misconceptions in probability and statistics. The participants were involved in a semi-structured clinical interview, lasting approximately 60 minutes. Participants were presented with four probability tasks and asked to solve them. The four probability tasks each involved a different classical paradox: Galton's paradox, Simpson's paradox, the Monty Hall paradox, and the St. Petersburg paradox (for discussion of the paradoxes see Grimmett & Stirzaker [2004] and Székely [1986]).

The findings of this study support the hypothesis that classical paradoxes can play an important role in revealing and understanding students' different misconceptions in probability and statistics. By engaging with the four paradoxes, the students demonstrated the effect of three misconceptions associated with their judgement under uncertainty. More specifically, the Galton's and the Monty Hall paradoxes revealed an "insensitivity to prior probability of outcomes"; Simpson's paradox turned out to be useful for understanding students' "insensitivity to sample size"; and the St. Petersburg paradox revealed the gambler's fallacy in students' reasoning (which is an indication of a "misconception of chance"). Future research could examine other classical paradoxes that may reveal additional misconceptions within different heuristics and how these too could be useful for instruction of probabilistic thinking.

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ACADEMIC PERFORMANCE AND ATTITUDES TOWARD MATHEMATICS WITH AN ADAPTIVE HYPERMEDIA SYSTEM

José Antonio Juárez López
Benemérita Universidad Autónoma de Puebla
jajul@fcfm.buap.mx

Gerardo Rocha Feregrino
Benemérita Universidad Autónoma de Puebla
grferegrino@gmail.com

Keywords: Attitudes toward mathematics, Undergraduate-Level Mathematics, Mathematical competencies

Low academic performance and negative attitudes toward mathematics are persistent problems among students in the first undergraduate semesters, and this leads to a deficient understanding of problems in their areas of study and inadequate application of solution strategies. In this investigation we evaluated the impact of adaptive learning strategy through an Adaptive Hypermedia System (AHS) in the development of three mathematical competencies: use of Symbolic Language, to model mathematical problems and to solve problems through mathematical reasoning. In addition, we measured the impact on attitudes toward mathematics and toward computer-taught mathematics of the first semester students on a private university in Mexico. Through a quasi-experimental design with two groups, we tried to answer the questions: What is the impact of an AHS in the mathematical competencies' development, what is the impact in the attitudes toward mathematics and what is the impact in attitudes toward computer-taught mathematics? To measure mathematical competencies, a pretest-posttest were applied in both control and experimental groups (158 students). For measuring attitudes, a modified version of the Ursini, Sánchez, & Orendain, (2004) scale was used. For the statistical analysis, we used the U-Mann-Whitney test.

The analysis of the pretest-posttest showed that the experimental group exposed to the use of AHD during the course achieved significantly higher performance than the control group in the competence *modeling situations through the appropriate use of mathematical expressions*. However, the levels reached in the competencies *to solve problems through mathematical reasoning* and *use of symbolic language* were statistically equivalent in both groups. About changes in attitudes toward mathematics, results showed that although both groups initially had similarity in attitudes toward mathematics, at the end of treatment, the experimental group showed a positive difference with respect to the control group.

Among the most important findings, it was observed that using AHS had a positive impact on the development of mathematical competencies as well as attitudes toward mathematics. However, the impact on attitudes towards mathematics taught by computer was slightly negative. It is necessary to continue investigating the effect of the AHS in the teaching-learning of mathematics in order to reinforce these findings.

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AFFORDANCES OF CONNECTING MATHEMATICAL REPRESENTATIONS IN COMPUTATIONAL SETTINGS

Sylvia Valdés-Fernández
Oregon State University
valdesfs@oregonstate.edu

José Fernández
Oregon State University
fernanj2@oregonstate.edu

Elise Lockwood
Oregon State University
elise.lockwood@oregonstate.edu

Constructing, interpreting, and connecting various mathematical representations is an important mathematical practice, as it can help students develop mathematical conceptions. We present findings from a study in which a pair of students solved combinatorics problems in a computational setting using Python programming. We present a case of the students' use of representations in this setting, highlighting ways in which the computational setting offered unique opportunities for connections among multiple representations.

Keywords: Using Representations; Undergraduate-level Mathematics, Technology

Introduction and Motivation

The notion of computational thinking (CT), a way of thinking that one uses to formulate a problem in such a way that a computer could effectively carry it out (Wing, 2014), has gained prominence among computer scientists and STEM educators. We present results from a study in which students solved "counting" problems using Python programming to explore such problems. We focus on ways in which students engaged in computational thinking to develop and make connections among mathematical representations. Students often have partial understandings of concepts, so connecting multiple representations allows them to develop more complete mathematical conceptions (Greeno & Hall, 1997). Thus connecting representations is highly valued as a means for students to deepen their understanding (eg. Pape & Tchoshanov, 2001; Stein, Engle, Smith, & Hughes, 2008). We address the following research question: *In what ways can representations that students develop and use in a computational setting help them deepen their understanding of combinatorial concepts?*

Methods

Students worked at a computer together and coded in the the PyCharm environment on tasks that covered a variety of counting problems. We asked the students to explain their work and predict the output of their code. We qualitatively coded the data for several types of mathematical representations, and analyzed the data for observable student behaviors, such as verbal explanation, gazing, pointing and gesturing as evidence of a connection.

Results

We identified five representations: computer codes, outputs, lists, tree diagrams, and expressions. For example, one pair answered the question "*Write a program to list all possible outcomes of flipping a coin 7 times."* They related code output (Fig. 1) and a tree diagram (Fig. 2), saying, "this [tree diagram] is what we talked about with the third column [of output] like it's splitting into four." We explore how CT afforded such connections.

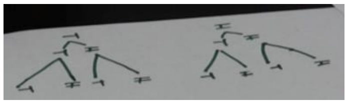

Figure 1: Partial Output

```
heads heads heads tails tails tails tails
tails heads heads tails tails tails tails
heads tails heads tails tails tails tails
tails tails heads tails tails tails tails
heads heads tails tails tails tails tails
tails heads tails tails tails tails tails
heads tails tails tails tails tails tails
tails tails tails tails tails tails tails
128
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Figure 2: Tree Diagram

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OPPORTUNITY TO LEARN: DIFFERENT TYPES OF TASKS AND MATHEMATICAL LITERACY

Jihyun Hwang
University of Iowa
jihyun-hwang@uiowa.edu

Yeajin Ham
University of Iowa
yeajin-ham@uiowa.edu

Keywords: Opportunity to learn, Mathematics tasks, Mathematical literacy, PISA 2012

The purpose of this exploratory research is to model the relationships among students' perceived opportunities to learn (OTLs; Carroll, 1963) with different types of tasks, students' perceived control, and students' mathematical literacy. Specifically, we examined how the OTLs facilitated with different mathematics tasks are related with mathematical literacy, and the role of perceived control in the relation between different task types and mathematical literacy.

We collected the PISA 2012 data of 1,649 Korean students. The structural equation modeling (SEM) approach was applied to the database using the R package *lavaan.survey* (Oberski, 2016). In the hypothesized model for the SEM analysis, latent variables representing four different types of tasks were considered: algebraic word problem, procedural tasks, pure mathematics reasoning, and applied mathematics reasoning (see OECD, n.d., pp. 37–41). OTLs with each type of tasks were measured through students' responses to the survey questions – how often they have encountered each type of tasks in their mathematics lessons and in the tests. We considered other two latent variables in the model: Internal perceived control (e.g. 'whether or not I do well in mathematics is completely up to me') and external perceived control (e.g. 'if I had different teachers, I would try harder in mathematics'). Using all these 6 latent variables, we investigated the relationships among OTLs with different tasks, perceived control, and mathematical literacy.

The results showed that the four types of tasks had different relationships with mathematical literacy. Specifically, for the three task types (algebraic word problems, procedural tasks, and pure mathematics reasoning tasks), the more frequent engagement with those tasks were linked with higher mathematical literacy score (β's are approximately 0.1, $p<0.05$). Contrarily, in the case the applied mathematical reasoning tasks, the more frequent engagement was linked with students' mathematical literacy scores ($\beta = 0.16$, $p < 0.01$). This implies not only that all of the four different types had varying relationships with mathematical literacy, but also that applied and pure mathematics reasoning tasks have different relations with mathematics literacy, even though the both are categorized as higher-order thinking. This could be because of differences between cognitive processes required to solve pure and applied mathematics reasonings tasks.

The results also showed that OTLs with the procedural tasks were likely to increase mathematical literacy indirectly through internal perceived control ($\beta = 0.103$, $p<0.001$). Engaging in the applied reasoning tasks is positively related to external perceived control ($\beta = 0.08$, $p<0.1$), but no relationship was observed between mathematical literacy and external perceived control. Considering cognitive demands of tasks, we speculate that students are likely to have repeated succeeds to solve procedural tasks, which may contribute to shaping internal perceived control.

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LATINX LEARNERS AND TRANSLANGUAGING IN A MATHEMATICS CLUB

Craig Willey
Indiana Univ.-Purdue Univ. Indianapolis
cjwilley@iupui.edu

Hector Morales, Jr.
Northeastern Illinois University
hmorale3@neiu.edu

Keywords: Latinx Bilinguals, Translanguaging, After-school, Activity Systems

This paper reports on a study designed to showcase Latinx bilingual children's linguistic and cultural resources for learning mathematics in an after-school mathematics club. Specifically, we examine the design of the activity system (Engstrom, 1999) and social interactions therein through a translanguaging perspective in which students leverage their language and culture to engage in mathematical learning. Our primary objective is to highlight some of the triumphs and struggles of bilingual children as they expand communicative practices and mathematical resources via interactions with facilitators and communication with a math wizard, *El Maga*.

Theoretical Underpinning and Context of Study

Our research was conducted in an after-school club in a large, urban context with a school population was 99.4% Latinx and 68% English learners. Sociocultural theory guided the design of the club, which was adapted after *The Fifth Dimension* (Cole, 2006) and *La Clase Mágic*a (Vásquez, 2003), and aimed to provide Latinx students experiences doing nonremedial math in an environment that challenged traditional classroom experiences through bi-literacy. We draw on translanguaging to reconceptualize bilingualism as a set of liberating and empowering communicative practices, capable of transforming learning, that goes beyond students' transition to the dominate school language (Cenoz, 2017; García, 2017; MacSwan, 2017).

Thirteen students, five undergraduate Latinx facilitators, four graduate student facilitators, and five parents participated. Ethnographic methods were employed to capture the meaning participants made of engaging in mathematical activity through a translanguaging approach. Twenty sessions were video-recorded. Facilitators' descriptive and reflective fieldnotes captured participants' interactions as well as student work and correspondence with *El Maga*.

Findings and Conclusions

First, language resources (e.g., L1) were observed to be used automatically. A pedagogy intentionally designed to build L1 to support L2 was needed. Second, asking students who were already used to the exclusion of their home language in school, to reappropriate features of their home language in an after-school setting proved challenging and took time. In addition, bilingual facilitators often expressed discomfort with doing math in Spanish. Persistence and explicit support from project leaders were key to maximizing opportunities for children to use Spanish. We also found that when activities in the after-school involved parents and community, L1 was more readily used by students and reclaimed for academic purposes.

The combination of foregrounding Spanish and incorporating families had a direct impact on how mathematical norms are socially (re)constructed, and ultimately how bilingual Latinxs see themselves as competent math learners. Furthermore, a consistent effort to redefine the role of Spanish in mathematical activity is critical to easing the discomfort, or unnaturalness, that has been built up historically and serves to cast math as something to be done in English only.

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ADDRESSING THE MATH ANXIETY OF A STUDENT WITH A LEARNING DISABILITY DOING ALGEBRA

Casey Hord
University of Cincinnati
casey.hord@uc.edu

Anna F. DeJarnette
University of Cincinnati
anna.dejarnette@uc.edu

Keywords: Learning Disabilities, Math Anxiety, Algebra, Gestures, Strategic Questioning

To access many post-secondary educational and occupational opportunities, students with learning disabilities (LD) have to succeed in gatekeeper mathematics courses such as Algebra I (Achieve, 2015; Ysseldyke et al., 2004). While students with LD are often quite capable with academics at this level, these students often experience difficulties with memory and processing (e.g., working memory) that can contribute to difficulties with higher level mathematics (Marita & Hord, 2017; Swanson-Beebe Frankenberger, 2004). Math anxiety (i.e., a detrimental, emotional response to mathematics which can have a negative effect on mathematical performance) is a common challenge facing many students with LD and can exacerbate these students' issues with working memory (i.e., the processing, storing, and integration of multiple pieces of information) (Ashcraft & Krause, 2007; Baddeley, 2003; Nelson & Harwood, 2011).

Researchers have implemented interventions involving strategic use of visual representations—often in the form of gestures or diagrams—to support students' working memory as they engaged in mathematics that was likely to present challenges related to working memory (e.g., Hord et al., 2016; van Garderen, 2007). The use of these visuals can help students with LD to offload information from short-term memory (e.g., storing information in visuals), allowing students to focus attentional resources on critical thinking (Risko & Dunn, 2010; Xin, Jitendra, & Deatline-Buchman, 2005). In an exploratory study, Hord and colleagues (2018) described how a teacher addressed the needs of a student with LD and a high level of math anxiety; the teacher provided a combination of visual representations and emotional support at key moments of challenge while still pushing the student toward higher levels of understanding.

To further explore this topic, the researchers in this study conducted a qualitative case study of ten sessions of a tutor working on Algebra I content in a one-on-one setting with a student with a LD with high levels of math anxiety. His math anxiety was described in his school records (e.g., concerns of his father about his son's anxiety limiting his academic performance) and demonstrated by the student during tutoring sessions (e.g., frequent comments about his worries about finishing his work on time and avoidance behaviors such as changing the conversation to topics unrelated to the task at hand when he started to struggle with math). The tutor worked with the student on Algebra I topics such as systems of equations and comparing the slopes of two lines. To gauge the level of challenge she could present to the student, the tutor carefully considered the level of rapport and trust she had built with the student while monitoring the student's signs of fluctuating levels of math anxiety, current mathematical knowledge, and potential difficulties with working memory (Hackenberg, 2010; Hord et al., 2018). This poster will summarize our analysis of the interactions between the tutor's perceptions of the student's anxiety, the tutor's decisions about how to support the student, and the student's progress towards communicating his understanding verbally and in writing.

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PRESERVICE TEACHER ANXIETY: A CROSS-CULTURAL STUDY

Heidi B. Hansen
Bemidji State University
hhansen@bemidjistate.edu

Keywords: Teacher Education-Preservice; Affect, Emotion, Beliefs and Attitudes

The Study: Examining Anxiety Differences Across University Type

Mathematics anxiety has been studied in a wide variety of settings (Dowker, Sarkar, & Looi, 2016), but not necessarily across cultural boundaries in preservice teachers. To answer the question about whether differences exist in preservice teacher anxiety levels among different types of universities in the US and abroad, this small-scale exploratory study examined students at three universities: one US public, one US private, and one Scandinavian.

The Research Questions

The questions the author was attempting to answer with colleagues at the three universities were as follows:

- To what extent do preservice teachers at public, private, and foreign universities differ with respect to their levels of math anxiety?
- To what extent does preservice teachers' math anxiety correlate with their skill levels in mathematics, and is there a difference among the university types?
- To what extent are the variables, age, gender, past mathematics experience, and grade level teaching preference related to math anxiety in the different universities?

The Tools and Subjects

The author selected three instruments: 1) a demographic survey to identify possible factors related to the students' anxiety level, such as age, gender, math background, desired grade level for teaching, and year in college; 2) the 9 question Abbreviated Math Anxiety Scale (AMAS) designed and validated by Hopko and colleagues (Hopko, Mahadevan, Bare, & Hunt, 2003); and 3) a 15 question math quiz designed to assess math skill level (Johnson & Kuennen, 2006).

Students from three small universities (one public, one private and one foreign) were studied. An $n = 97$ was obtained from the preservice teaching courses. Students had no significant pedagogy background. Some had taken a liberal education or problem-solving course. The participants numbered 21 males and 76 females.

The Results

Results as tested by ANOVA indicated no difference among the three university populations in math anxiety levels as measured by the AMAS. Math skill levels were found to be higher at the private university. There was a significant negative correlation between math anxiety and quiz scores, controlling for math background and grade level preference ($r = -0.329$, $p < 0.05$). Without controlling for any variable, quiz score correlated negatively with anxiety score (Spearman's Rho = -0.315, $p = 0.02$). Grade level preference was related to anxiety (Spearman's Rho = -0.259, $p = 0.010$): the higher the grade level preference, the lower the anxiety score. In a surprising finding, math background correlated negatively with grade level preference: the more math courses the students had taken in high school and college, the lower the grade level they preferred to teach. Gender and age were not significantly related to anxiety or skill level in this study.

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WHAT DOES MATHEMATICAL ENGAGEMENT MEAN TO STUDENTS?

Kwaku Edusei
University of Delaware
kedusei@udel.edu

Amanda Jansen
University of Delaware
jansen@udel.edu

Amanda Mirzaei
University of Delaware
alrusso@udel.edu

Keywords: Affect, Emotion, Beliefs and Attitudes; High School Education; Instructional Activities and Practices

The purpose of this study is to investigate how ninth-grade students define what it means to be engaged with mathematics and the classroom factors they report as impacting their engagement. Student engagement has a strong influence on student achievement (Finn, 1993). Specifically, self-regulation of effort has shown to improve achievement with mathematics among high school students (León, J., Núñez, J. L., & Liew, 2015). Thus, it is important to understand the many different, and specific ways in which students come to be engaged in a classroom setting. However, teachers' or adults' perspectives on engagement likely different from students' perspectives. It is important to understand what mathematical engagement means to *students*. A premise underlying this work is that awareness of students' definitions of engagement can inform teaching practice in ways that can lead to higher engagement with mathematics among students. Two research questions guide this study: (1) How do high school students define being engaged with mathematics? (2) What classroom features to high school students report as impacting their engagement with mathematics?

Ninth-grade students (N = 39) were interviewed for this study. Interviewees were selected from surveys (N = 256) in which reported their typical levels of self-efficacy, interest, and other aspects of their engagement with mathematics. Interviewees represented various profiles in the sample, as determined by cluster analysis. Interviewers were one-on-one interviews or in focus groups of two to four students. Open coding was used to develop themes to categorize student conceptions of engagement and the factors students reported that impacted their engagement.

Five themes were found for student definitions of engagement: understanding (cognitive), fun/interesting (affective), focused/attentive (cognitive), willpower & effort (behavioral), and relevancy (utility & relevance). Seven themes or features of the learning environment were found to have impacted student engagement in both positive and negative respects. These themes are: teaching style, understanding (cognitive), peer support (social), math topic, type of activity, sleep/rest and attentiveness, and competition (performance goals).

This study will help teachers gain an understanding of how students view what it means to be engaged in mathematics and the features of the mathematics classroom that impact their engagement. These findings provide insight to help teachers develop and facilitate classrooms in which there is higher student engagement.

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STUDENTS' PERCEPTIONS OF FEEDBACK IN UNDERGRADUATE MATHEMATICS

Evan Czysz
Univ. of Northern Colorado
evan.czysz@unco.edu

Jenna Krieschel
Univ. of Northern Colorado
jenna.krieschel@unco.edu

Adam Ruff
Univ. of Northern Colorado
adam.ruff@unco.edu

Keywords: Affect, Emotion, Beliefs, and Attitudes; Undergraduate-Level Mathematics

To be beneficial to students, feedback should be accessible, robust, unbiased, unambiguous, encouraging, timely, iterative, and unique (Cardella, Diefes-Dux, Oliver & Verleger, 2009) or should answer the questions: "Where am I going?", "How am I going?", and "Where to next?" (Hattie & Timperley, 2007). However, there are a lot of unknowns about student use of feedback. Some feedback may come too late to help (To, 2016) or students may lack the knowledge of how to use the feedback on their own (Jonsson, 2012; Diefes-Dux, et al., 2012; Van der Kleij, Adie & Cumming, 2017). Which form of feedback is best and under what circumstances is relatively unknown (Jonsson, 2012).

Traditional feedback in a mathematics classroom is written feedback given by the instructor which is often ineffective and not beneficial for students (e.g. Attali, 2015; Harks, Rakoczy, Hattie, Besser, & Klieme, 2014; Hattie & Timperley, 2007). Students frequently find the feedback they receive to be unsatisfying (e.g. Nicol, 2010; Ferguson, 2009; Poulos & Mahony 2008; Landers & Reinholz, 2015) and instructors often give feedback that cannot be acted upon or vague and general comments, both of which are received poorly by students (Rodgers, Horvath, Jung, Fry & Diefes-Dux, 2015; Jonsson, 2012). Prior studies have focused on what feedback instructors should give. However, even if instructors give the kind of feedback that can help students bridge the learning gap, this feedback is useless if students are not motivated to use the feedback they are given.

We conducted hour-long semi-structured interviews with undergraduate students enrolled in Calculus I or Mathematics for Elementary Teachers. Interviews focused on students' perceptions of the feedback they received and their motivations for using or not using the feedback. Despite the diversity in student backgrounds and goals, the interviewees had very similar views on the aspects of feedback they preferred yet diverged on their motivations for making use of that feedback.

Using open coding and thematic analysis we found types of feedback that students are motivated to use. The students in our study clearly indicated a preference for feedback targeted at specific mistake points, positive feedback directed at specific components of the work, and feedback that provided hints as opposed to providing correct solutions. Together these findings indicate that these students are interested in feedback that helps them to correct their work and perform better on future assignments.

Even if students receive the type of feedback they prefer, they may not be motivated to use that feedback. For the students in this study grades and time were significant factors in students' choices around using feedback. Considerations about future use of the knowledge also seemed to play a role, especially for the two students planning careers in education. Our study considered only a small number of students. More research is needed in this area to better understand the mix of factors that determine if a student uses feedback that is provided.

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INVESTIGATING PARTICIPATION AND MATHEMATICS IDENTITIES OF DEVELOPMENTAL MATHEMATICS STUDENTS

Francis Nzuki
Stockton University
francis.nzuki@stockton.edu

Keywords: Developmental Math, Mathematics Identity, Equity

One of the significant challenges facing higher education is narrowing the educational attainment gap between students who are academically prepared and those who are not. Although the intention of developmental education is to help support underprepared students in achieving academic success, there have been disagreements among researchers on the effectiveness of achieving this goal (Goudas & Boylan, 2012). On one hand, developmental mathematics has the capability of providing the impetus that can propel students to their overall academic success. On the other hand, the long road the students have to go through in completing mathematics requirements causes many to give up before they can finish the sequence of courses (Rosin, 2012).

This study examines how developmental students' general and mathematical experiences help to shape mathematical identities they develop and how these identities in turn hinder or enhance their successful participation in mathematics. Also examined are the factors that influence students' mathematics identities after taking a developmental mathematics course. To this end, the following research questions guided this study of first year students taking a developmental mathematics course at a mid-sized, urban public university:

1. What mathematics identities are held by students taking developmental mathematics?
2. What factors shape students' mathematical identities after taking a developmental mathematics course?

Data was collected using pre-post surveys and semi-structured interviews. The analysis reported here is based on the data from the survey instrument. The statements in the survey were grouped into five aspects of mathematics identity; self-concept, self-efficacy, motivation, and anxiety, and value of mathematics. Qualitative data from the open-ended items of the instrument was systematically analyzed using grounded theory to uncover patterns and trends in participants' responses while descriptive statistics was calculated for the quantifiable portions of the surveys. Items were compared both within surveys and across surveys to identify correlations and trends, as well as to support qualitative themes.

Analysis revealed that differences of the overall mean scores of all five aspects of mathematics identity between females and males were not statistically significant. Further, students scored the lowest on self-concept while the highest score was on their perception of the value or importance of math in their lives. As Klinger (2004) pointed out there are many students who do not particularly enjoy mathematics and report a disliking for the subject (negative affect), even though they still respect the utility and importance of math in their future lives and careers. Also, the study revealed that students' self-efficacy and self-concept increased significantly over time.

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THE STORIES OF AFRICAN AMERICAN MALES LEARNING MATHEMATICS USING SOCIAL JUSTICE

Audrea Bankston
Georgia State University, Graduate Student
abanskton4@student.gsu.edu

Keywords: Teaching for Social Justice, Critical Race Theory, Stories, African American Males

Current discourse surrounding African American males in K-12 mathematics is often shown in a deficit manner (Martin, 2012). African American males are often placed in lower level mathematics courses or do not have access to rigorous courses such as Pre-Calculus (Thompson & Lewis, 2005). Since the discourse surrounding African American males is predominately negative, more research is needed to document mathematical success surrounding African American males (Jett, 2011).

Researchers such as Malloy (2009), Mathew (2009) and Tate (1995) suggest relating instruction to the lives of students. Teaching for social justice is one type of critical pedagogy that is related to students' lives and their communities (Gutstein & Peterson, 2013). Teaching mathematics for social justice allows African American students to engage in inquiry learning, work with peers and to discuss mathematics, which are characteristics that contribute to the success of learning mathematics (Malloy, 2009; Ladson Billings, 1995).

Modeling Gutstein's (2003) implementation of teaching for social justice with a Latino population, this proposed study will capture the experiences through stories of African American males engaged in learning mathematics for social justice situated in an informal setting. This proposed study intends to capture the stories using open-ended interviews as well as surveys recording their mathematical experiences before, during, and after the instruction.

A critical race theoretical perspective will guide this proposed study. This proposed study will focus on storytelling, which is one of the major tenets of critical race theory (Ladson-Billings & Tate, 1995). Stories from marginalized groups can give different perspectives from the stories of the oppressors (Tate, 1997; Delgado & Stefancic, 2012; Dixson & Rousseau, 2006). In this proposed study, allowing African American male students to share their stories will possibly provide insight into their 'realities' of learning mathematics using social justice.

Data from this proposed study will provide a viewpoint about the mathematical experiences of African American males using their own stories. In conducting this proposed study, academia and the public would have information documenting their experiences, which could encourage implementing a critical pedagogy such as teaching for social justice to teach mathematics to students of color.

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GIRLS, LEADER, FOUNDER: INFLUENCES OF AN AFTERSCHOOL STEM PROGRAM FOR GIRLS

Sue Ellen Richardson
Purdue University
richa114@purdue.edu

Lili Zhou
Purdue University
zhou756@purdue.edu

Girls begin to lose interest and confidence in STEM fields in middle and high school, even though they perform as well as boys (Hill, Corbett & Rose, 2010). Research suggests that informal learning programs can provide female role models, hands-on activities, collaborative learning, and single-sex environments can help girls build identities and confidence in STEM fields. Many previous studies used quantitative instruments to measure girls' interest, engagement, and confidence in STEM fields; fewer studies utilize participants' voices to understand these attributes. Extant research has focused on girls' interest or identity during STEM programs or post-course or career selection, but less often on how the informal STEM learning shapes their life experiences in the long term. Even though the number of girls who study STEM in K-12 schools is increasing, the number of women who become scientists and engineers is not increasing at the expected rate. Many women who earned STEM degrees either did not pursue a career in STEM fields or left STEM fields early in their career. Therefore, understanding retention issues is critical for women's participation in STEM fields.

The mission of the *Girls Excelling in Math and Science* (*GEMS*) program is to increase girls' interest in STEM by providing integrated activities that engage girls in the fun and wonder of these fields. In this study, participants' narrative descriptions of their experiences in *GEMS* were used to understand the long-term impact of this informal learning programs. Through understanding participants' (i.e., founder, leader and girls) stories, we expected to learn more about how the earlier STEM experiences impact females later career and life choice. The research questions include:

1. How do *GEMS* participants remember and describe their experiences? In what ways did participation in *GEMS* shape their personal and professional experiences?
2. What are the perceptions of the *GEMS* founder and a leader about their experiences in developing and implementing the program?

Because the purpose of this study is to understand participants' experiences, our framework includes Dewey's (1938, 1998) *experience* construct, particularly *continuity*, which indicates that a current experience is influenced by past experiences and will influence future experiences. We also used Lave and Wenger's (1991) Community of Practice framework to understand participants' experiences from a community perspective.

We used narrative inquiry as both phenomena under study and method of study to understand the general construct of continuity in our participants' lives and personal experiences (Clandinin & Connelly, 2000). Original *GEMS* participants, a leader, and the founder completed questionnaires about their experiences with *GEMS* clubs. Some participants, the leader, and founder also completed semi-structured interviews. Initial narrative analysis of the founder's interviews, indicate that, although the founder's original motivation for starting *GEMS* was from a mom perspective, her story indicated that after twenty years of running *GEMS*, her role has grown far beyond a mom: she has integrated roles as a teacher, an educator, a STEM leader, and

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most importantly to her, an advocate for girls' learning and empowerment. The findings will be utilized to comprehensively improve the strength of *GEMS* and expand learning approaches in an effort to scale-up STEM learning participation for girls.

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Chapter 13:
Teaching and Classroom Practice

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

CONVERSATIONS ACROSS THE DECADES: LEARNING AFRICAN AMERICAN PEDAGOGICAL EXCELLENCE FROM BLACK MATHEMATICAL "DREAMKEEPERS"

Jenice L. View
George Mason University
jview@gmu.edu

Toya Jones Frank
George Mason University
tfrank4@gmu.edu

Asia Williams
George Mason University
awillid@masonlive.gmu.edu

Examining the Trajectories of Black Mathematics Teachers: Learning From the Past, Drawing on the Present, and Defining Goals for the Future is a three year-long research study funded by the National Science Foundation. Our study was designed to reflect on the dearth of Black mathematics teachers, including problems with recruiting and retaining Black mathematics teachers. The research study mitigates some of the oversights in the history of mathematics education. The first phase of the study was a large-scale survey of more than 500 currently practicing Black mathematics teachers. Simultaneously, the research team began conducting oral histories of retired Black mathematics teachers. In the second phase of the project, there will be focus group with currently practicing Black mathematics teachers. Ultimately, the objective of the research is to leverage the epistemological wealth of Black teachers, particularly through an historical lens, to guide contemporaneous teacher education policy and practice. This paper focuses on preliminary findings from the oral histories part of this multifaceted research study.

Objectives

Examining the Trajectories of Black Mathematics Teachers: Learning From the Past, Drawing on the Present, and Defining Goals for the Future is a three year-long research study funded by the National Science Foundation. Our study was designed to reflect on the dearth of Black mathematics teachers, including problems with recruiting and retaining Black mathematics teachers. The research study mitigates some of the oversights in the history of mathematics education. The first phase of the study was a large-scale survey of more than 500 currently practicing Black mathematics teachers. Simultaneously, the research team began conducting oral histories of retired Black mathematics teachers. In the second phase of the project, there will be focus group with currently practicing Black mathematics teachers. Ultimately, the objective of the research is to leverage the epistemological wealth of Black teachers, particularly through an historical lens, to guide contemporaneous teacher education policy and practice. This paper focuses on preliminary findings from the oral histories part of this multifaceted research study.

Education theorist Popkewitz (1987) noted that what is *not included* in the history of education is just as telling as what gets to be told. Toward the centering of race, Milner (2006) offers the charge that educators and researchers should look to "Black teachers' experiences and success both pre- and post-desegregation for insights about how all teachers can deepen and broaden their knowledge and understanding to better meet the needs and situations of students at present, particularly among Black students" (p. 90).

Henry (2006) highlighted the powerful nature of oral history in collecting the stories of marginalized groups, as this methodology honors the generative power of spoken word, which has been noted as central to Black peoples' collective way of knowing. We argue that our mixed methods research study contributes to the Critical Race Theory (CRT) literature through its linkage of the historical counternarratives of retired Black mathematics teachers with current

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educational policies for racially diversifying STEM instruction (National Math and Science Initiative, 2016; U.S. Department of Education, 2016). These counternarratives in the form of oral histories help fill the gap left by many texts that claim to be comprehensive. One such text, published by the National Council for Teachers of Mathematics and commonly used by professors of education, is *A History of School Mathematics* (Stanic & Kilpatrick, 2003). Our oral histories intend to offer a meaningful historical foundation for understanding the contemporary racialized experiences and practice of Black mathematics educators.

Two components of Critical Race Theory (CRT) shape our research – racial realism and counternarratives. Racial realism is an integral component of critical race theory (Bell, 2005). It accepts that we live in a society in which racism has been internalized and institutionalized to the point of being an essential and inherently functioning component, in this case an intransigent, intractable component of educational institutions, including the ways that whiteness appears in mathematics education (Battey & Levya, 2016).

In response to the racial realism that dominates educational practice and outcomes, we look to history for counternarratives (Solorzano & Yosso, 2002), or evidence of equity-centered pedagogies that focused on teaching excellence for well-rounded student achievement, particularly for Black children and particularly in the discipline of mathematics. These pedagogies include but are not limited to culturally relevant teaching (Irvine, 1989; Ladson-Billings, 2009) and African American pedagogical excellence (Acosta, Foster, & Houchen, 2018; Siddle-Walker, 2001).

We were curious about the extent to which retired Black mathematics teachers – facing the intractability of racism in a discipline that falsely purports to be race-neutral and objective – challenged and overcame systemic barriers during the historical period of formally racially segregated schools, well into the mid-1970s. We wondered whether they manifested African American Pedagogical Excellence (Acosta et al., 2018). In short, we wondered whether these teachers demonstrated the characteristics described in Ladson-Billings' seminal work *The Dreamkeepers: Successful Teachers of African American Children* (2009). Just as there is evidence of enduring institutional racism in schools – a legacy of the history of segregated schooling -we queried whether there was there evidence of enduring teaching ideologies and common behaviors that historically helped students achieve academic success overall and mathematics success specifically while maintaining a positive identity as African Americans (Ladson-Billings, 2009). Our research team explores how the past and the present communicate with one another to shape educational policies.

Figure 1. A Conceptual Framework of African American Pedagogical Excellence

Ideology	Beliefs	Instructional Practices
• Political clarity • Racial uplift • Afirmative view of African American culture • Ethics of care • Oppositional consciousness	• High student intellectual potential • All children can learn • Teacher accountability for student learning • Positive views of families and communities	• Insistence • Connectedness • Interdepent learning community • Collective success • Positive views of families and communities • Curriculum relevant to students' lives

Figure 1. Adapted from "Why seek the living among the dead? African American pedagogical excellence: Exemplar practice for teacher education," by M. M. Acosta, M. Foster and D. F. Houchen, 2018, *Journal of Teacher Education, 69* (40), p. 342. Copyright 2018 by American Association of Colleges for Teacher Education.

Figure 1: A Conceptual Framework of African American Pedagogical Excellence

Participants

Collecting histories of retired Black math teachers is time-sensitive, as these educators who have experiences spanning decades, are aging. To date, our team has conducted 13 oral histories. A total of 20 fulfills the deliverables for our NSF grant. The oral history participants taught mathematics in Atlanta, Washington, DC, and Maryland schools between the 1950s and the 2000s. Their ages ranged from mid-60s to late-80's. The average teaching experience was 40 years, meaning a cumulative total of over 400 years of teaching mathematics to Black children. And all participants were women.

Methods and Data Sources

In qualitative research, generating findings to explore research questions typically involves making direct field observations, conducting in-depth interviews, and collecting artifacts such as written documents (Patton, 2002). Oral history is one method of qualitative research methodology that gathers thick description from information-rich participants, contextualized within specific historical moments across space and time (Llewellyn & Ng-A-Fook, 2017; Patton, 2002). In-depth interviews, digital video recording, as well as both document and artifact analysis were the primary forms of data collection for our research study.

Due to the age and stamina of our participants we conducted one pre-interview by telephone and one follow-up interview. The first interview encouraged participants to recollect and arrange their teaching experiences. The second interview explored their experience in detail and the context in which it happened, as well as an exploration of the meaning that their experience holds for the participant. This second interview happened in person and was professionally video recorded. During this second interview, many of the participants shared artifacts.

Qualitative data analysis is an iterative process that begins at the design of the research study. As a team, we made amendments to the protocols as we conducted pre-interviews by phone and face-to-face professional videos. At the same time, we began the process of crafting initial codes and using the codes to create categories and sub-categories for understanding the first interview and shaping the second interview. We housed these codes in Excel. By using Rev.com, our research team was able to analyze transcripts while watching the video recording. The next step in our data analysis was archiving clips of videos with time stamps related to the codes that were initially created.

Following this process was a close examination of the written transcripts. All transcripts were both printed and became digitized copies when added to a Qualitative Data Analysis Software. These digitized transcripts afforded the research team a way to analyze the same documents individually and together during team meetings. We continued to add additional codes or modify existing codes for each individual participant. It was during this time that our analysis incorporated participants' artifacts such as yearbooks and curricula unit plans. This allowed the team to get a more holistic view of the participants.

The team worked as a cohesive unit to discuss difference and similarities among codes and categories to develop consensus and address potential validity threats in our analysis. As we continue the iterative process of coding, we are beginning to construct a narrative of each participant. While constructing narratives for each participant, we repeat the process of coding and categorizing *across* participants to gain a collective narrative of the experiences of retired Black mathematics teachers. In order to deepen our understanding of the participants' life experiences, we also began analyzing historical artifacts such as school board records, mathematics curricula, and records from the Miner Teachers' College. This process will continue with the additional interviews.

Our analysis of these transcripts and digital video oral histories of retired Black mathematics teachers empowers us to "address historical harms" (Llewellyn & Ng-A-Fook, 2017, p. 5) or the absences demonstrated in the history of mathematics education textbooks, as well as to offer "intergenerational dialogues" (p. 7) to transform "masculinist approaches for socially constructing knowledge in STEM…to (re)orient ways of knowing away from disembodied, objective truths and toward an embrace of positionality, inquiry, and social context." (p. 8).

Results

We knew from the start of our study that finding retired Black mathematics teachers would be a challenge. Our participants were recruited from our personal and professional networks as well as using snowball sampling. Even as such, our research team become keenly aware that all retired mathematics teachers that we interviewed were female. As such, we began to develop theories to understand this phenomenon. We will continue to search for male retired mathematics teachers and anticipate that doing so will bring a different dimension to our research study.

To date, the experiences of retired Black teachers of mathematics suggest alignment with the ideology, beliefs, and instructional practices expressed by African American Pedagogical Excellence (Acosta et al., 2018), as adapted to the context of each teacher. as well as alignment with Ladson-Billings' (2009) descriptions of successful and culturally relevant classroom interactions, which we have labeled "Dreamkeeper" codes. She describes instructional methods for eliciting *student competence* (p. 134), building student confidence through *instructional scaffolding* (p. 135), creating a collective classroom *focus on instruction* (p. 135), an approach to education that extends and contextualizes *student knowledges* (p. 136), and teachers having *deep knowledge* of both the students and the subject matter (p. 140). These transcripts also suggest codes for *institutional racism* (such as *teacher tracking*), *interpersonal microaggressions* (such as *parental vetting* and *peer disdain*), *racialized teacher professional development* and *mentoring*.

For example, one teacher expressed *oppositional consciousness* in a late 1960s setting with Black administrative leadership by insisting that James Brown's song, *Say it Loud - I'm Black and I'm Proud* was a governing ideology for her classroom presence and, by extension, her mathematics instructional practices.

An example of interpersonal microaggressions included a teacher stating: One [colleague] even told me several times, he said, 'you just don't have the experience to teach math'...He was a [white] navy man, been in the navy for years and retired. I don't know what he meant by that, but he kept telling me that [*peer disdain*]…and the white parents would pick their teachers for their students…and didn't want their kids in my class [*parental vetting*]. In response, she persisted with an *ethic of care* and *high student intellectual expectations* that eventually earned the respect of colleagues and parents.

Several teachers who taught in multiracial school settings recalled *teacher tracking* where they were initially assigned to teach "consumer math," or "the Black kids' math," and introductory algebra, rather than Algebra 1, Algebra 2, Geometry, or "intensive math" that was offered to white students. To counteract these professional slights, these retired Black math teachers described the ways they used planning time, departmental meetings, after school clubs, grant-seeking and professional conferences to both design instruction to improve student content knowledge and to provide one another opportunities to advance their professional skills. Forming *interdependent learning communities* with an eye on *collective success,* these teachers offered one another the *mentoring* and *racialized teacher professional development* that was lacking in their schools and districts.

Despite the barriers of racism, the teachers described student-teacher relationships and classroom environments that speak to several of the Dreamkeeper and AAPE codes. One extended quote is an example:

> I kind of enjoyed working with [the really tough kids] because I enjoyed the results that I was getting, enjoyed the kids coming to me and saying, ' I want be in Mrs. L's class because she *teaches*!'…I think [they meant] that I presented lessons in a way that they understood it, they could comprehend it…I had this thing always that I would tell them that my classroom is not a quiet room…we don't have holy quietness in this room. We're going to talk, we are going to talk about the work. We are going to learn the language of mathematics.

This quote has been coded as follows: "I kind of enjoyed working with [the really tough kids] because I enjoyed the results that I was getting [*student knowledges]*, enjoyed the kids coming to me and saying, 'I wanna be in Mrs. L's class because she *teaches*!' [*student competence*; *focus on instruction*]…I think [they meant] that I presented lessons in a way that they understood it, they could comprehend it [*student knowledges; scaffolding*]… I had this thing always that I would tell them that my classroom is not a quiet room…we don't have holy quietness in this room. We're going to talk, we are going to talk about the work. We are going to learn the language of mathematics [*deep knowledge; student knowledges*]."

The individual and collective narratives that emerge from the coded transcripts illuminate their own stories about their K-12 and undergraduate learning, and their experiences of teaching mathematics in schools pre- and post-desegregation. Ultimately, these narratives will be compared with data from other quantitative and qualitative components of our research study that focus on the experiences of contemporary Black teachers of mathematics. The "dialogue" between past and present will help shape an understanding of both change and continuity over time.

The research is ongoing and we expect to have the concluded all of the oral history interviews by September 2019.

Scholarly and Policy Significance

Ladson-Billings (1998) stated that, "adopting and adapting CRT as a framework for educational equity means that we will have to expose racism in education *and* propose radical solutions for addressing it" (p. 22). Teacher education for pre-service teachers and teacher professional development for in-service teachers, particularly in mathematics education, stand to benefit from the insights gained from the oral histories of retired Black mathematics teachers. The stories shed light on the origins and persistence of what Ladson-Billings refers to as the "education debt" (2006) as a direct counter to the language of the "achievement gap" in mathematics acquisition and numeracy. While the oral histories to date suggest that not all 20th century Black children, particularly in rural areas, had access to formal instruction in higher mathematics, their experience of numeracy stands in stark contrast with the lower level, rote mathematics that is typical in most 21st century majority-Black classrooms (Clark, Badertscher, & Napp, 2013; Gillen, 2014; Neil, 2015).

Teacher education placement offices may attempt to mitigate the intractability of the hiring discrimination that D'Amico, Pawlewicz, Earley, & McGeehan (2017) document and which finds its origin in some of the histories of retired Black mathematics teachers. The wisdom of seasoned Black mathematics teachers, made available on the project website in digital form, can lead to greater reflexivity for in-service teachers and to metaphorical cross-generational "conversations" with novice teachers in teacher education programs. Without this knowledge, teacher education and professional development programs will continue to lack sufficient data for reforming their curricula and pedagogy to recruit and retain high-quality Black math teachers who have the resilience and readiness to meet 21st century needs (Battey et al., 2018). Teacher education programs that talk about "equity" and "diversity" while explicitly ignoring how the history of race and racism impact the praxis of former and current Black mathematics teachers serve no one (Cook, 2015).

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EXAMINING THE TENSIONS BETWEEN HIGH QUALITY DISCOURSE AND SHARING MATHEMATICAL AUTHORITY WITH STUDENTS

Patrick Sullivan
Missouri State University
patricksullivan@missouristate.edu

Current reform efforts challenge teachers to create more student-centered classrooms focused on high quality classroom discourse (NCTM, 2014). There are difficulties, however, teachers face in bringing this vision to fruition. Over the past three years, we have worked with a group of 7-12 teachers supporting their efforts to implement high quality classroom discourse. Although their espoused beliefs aligned with our vision of high-quality discourse, their enacted practices did not align with those espoused beliefs. Further analysis suggested that many of these challenges are related to sharing mathematical authority with their students. We intend to share a set of "Look Fors" to build awareness of features of sharing mathematical authority that might be inhibiting the quality of teachers' classroom discourse. A set of questions will also be shared to help teachers decide upon a plan of action to overcome those challenges.

Keywords: Classroom Discourse, Teacher Education-Inservice/Professional Development, Mathematical Knowledge for Teaching

Purpose of the Study

Over the past three years practicing middle and secondary mathematics teachers (n = 16) from small, rural school districts (75% from high school needs school districts) participated in a grant project to improve the quality of classroom discourse in their classroom. During the project, teachers engaged in over 300 hours of professional development focused on Effective Mathematics Teaching Practices (NCTM, 2014) that supported high quality classroom discourse. Emphasized practices included: 1) implementing tasks that promote reasoning and problem solving, 2) facilitating meaningful mathematical discourse, and 3) posing purposeful questions. In spite of our best efforts to support their growth in these areas, teachers faced challenges in improving the quality of their classroom discourse. The quote from one teacher captures the emotional toll of these challenges:

> "I am really trying to change the way I teach, I really am, but I am so frustrated! I turn my students loose with a task, but our classroom discussions seem to get bogged down! The students are frustrated, I am frustrated! So, I end up going back to my old way of teaching: telling them how to do everything!"

We sought to better understand the nature of these challenges. Analyzing teaching episodes, we explored four questions: 1) Did teachers' espoused beliefs align with their practices, 2) Were there elements of their specific elements of their classroom discourse that reflected growth, 3) What role did sharing mathematical authority play in teachers' ability to implement high quality classroom discourse, 4) As facilitators of professional development, what are the additional tools that we need to provide teachers with so that they can navigate the challenges of implementing high quality classroom discourse?

Perspective

Teacher Beliefs

Guskey (2002) suggests a sequential model of teacher change moving from the professional development experiences to the teachers' classroom practices, to changes in student learning outcomes, and lastly, to changes in teachers' beliefs and attitudes (p. 383). Using Guskey's model, it was our view that if we were able to quantify changes in teachers' beliefs aligned with our desired instructional practices then we would expect to see those beliefs enacted in their instructional practices. The REU (Research Experience for Undergraduates) Beliefs Instrument (O'Hanlon et al., 2015) seemed to be an appropriate instrument to measure these changes because it assessed the three domains of beliefs (teaching, student learning, and personal learning) that we believed were central to teachers' decision-making about their instructional practices.

High Quality Discourse

The Instructional Quality Assessment (IQA) Classroom Observation Tool (Boston, 2012) was used as a tool to quantitatively measure elements of what we viewed as high quality classroom discourse. It also served as a shared lens for teachers to reflect upon the nature of their classroom discourse. The IQA consists of two components, academic rigor and accountable talk. Each component has five aspects each of which has an accompanying rubric. These elements include 1) implementing cognitively demanding tasks (AR1 and AR2); 2) holding students accountable for their thinking (AR3, AT4, AT5); 3) asking academically relevant questions (AR-Q); 4) linking mathematical contributions (AT2, AT3); and 5) whole-class engagement (AT1).

Table 1: Instructional Quality Assessment Rubrics

Academic Rigor	Accountable Talk
AR1: Potential of the Task	AT1: Participation
AR2: Implementation of the Task	AT2: Teacher's Linking
AR3: Student Discussion After Task	AT3: Students' Linking
AR-Q: Questioning	AT4: Asking (Teacher Press)
AR-X: Mathematical Residue	AT5: Providing (Student Response)

Mathematical Authority

It was our hypothesis that a necessary condition of high-quality classroom discourse, as described by the elements of the IQA, was teachers' capacity to share mathematical authority with their students. Sharing mathematical authority requires teachers to listen to students' thinking, process and act upon potentially unplanned, and sometimes unfamiliar, mathematical statements. It also requires teachers to press students for explanations, and ask questions, in the moment, to move students' thinking forward. Sharing mathematical authority is not about who is in charge of the classroom but about who gets to decide which tools to use to solve a problem and who determines the correctness of mathematical contributions (Gresalfi, Martin, Hand, & Greeno, 2009; Hiebert et al, 1997). It is also about the nature in which students' mathematical reasoning and sense-making is valued and affirmed by the teacher during classroom discourse. Sharing mathematical authority is about the opportunities teachers give students to share ideas, evaluate others' ideas (Harel & Sowder, 2007) , and provide justification for their reasoning and sense-making (Hufford-Ackles, Fuson, & Sherin, 2004).

Methods and Analysis

A mixed-method design was used to capture both quantitative and qualitative aspects of teachers' classroom discourse. At the beginning of the professional development experience, teachers were asked to complete the REU (Research Experience for Undergraduates) Beliefs Instrument (O'Hanlon et al., 2015). During the first year, teachers engaged in a series of professional development experiences (over 100 hours) to understand the Cognitive Demand Framework (Stein, Smith, Henningsen & Silver, 2000) and the rubrics of the Instructional Quality Assessment. During the first year, several teachers agreed to have teaching episodes video-recorded. With the support of our coaching and using the IQA, the teaching episodes were evaluated by their peers and feedback was provided. This process supported two goals: 1) to create a professional learning culture in which participants were comfortable sharing episodes of their teaching with peers and willing to receive constructive feedback from their peers, 2) to use these episodes to develop consistency in rubric ratings.

Each of the IQA rubrics are scaled from 0 to 4 with 0 being the lowest rating and 4 being the highest rating. The rubrics of the IQA focus on two dimensions of high-quality classroom discourse: academic rigor and accountable talk. For example, one of the rubrics of the IQA related to academic rigor *is Potential of the Task.* At the highest ratings, students are engaged in a task that involves complex non-algorithmic thinking or applying a broad general procedure that is closely connected to mathematical concepts. In order to be considered a 4-rating the task must explicitly prompt for evidence of student thinking. At the lower ratings students are either engaging in no mathematical activity or memorizing rules, formulae or definitions. The *Potential of the Task* is strictly about what the teacher puts in front of students to do. Likewise, one of the rubrics related to student accountability during classroom discourse is *Asking (Teachers' Press)*. The higher ratings of the rubric correspond with the teacher consistently asking students to provide evidence of their contributions (i.e. press for conceptual explanations) or to explain their reasoning. At the lower ratings there is no discussion or efforts to ask students to provide evidence for their contributions (Boston, 2012). It is important to note that during the first year, each teacher was visited at least once by a researcher, and their teaching episode was scored using the IQA.

During Year 2 and Year 3, each teacher was provided with a SWIVL robot and iPad to record lessons. Teachers were given the autonomy to record and share lessons that they believed best represented their growth in classroom discourse. These lessons were shared with researchers via the cloud and scored using the IQA rubrics. Each teacher who agreed to share lesson recordings was scored at least once. These episodes were rated by two or more facilitators who were trained in scoring the IQA. Due to scheduling conflicts, the facilitators were unable to develop inter-rater reliability. As a result, the ratings for the cohort (n = 16) on each rubric for the last two years were averaged. Each teacher was also provided with scoring on each rubric for the episodes that were shared. Written constructive feedback corresponding to each rating was also provided.

Teachers continued to participate in professional development experience (over 100 hours each year). These experiences involved deepening the content knowledge (i.e., geometry and data analysis) and continuing to improve the quality of classroom discourse. These experiences also involved small groups analyzing teaching episodes using the IQA and providing constructive feedback to their peers. Teacher were assigned a "teacher buddy" to share additional teaching episodes and to provide feedback aligned with the rubrics of the IQA.

At the end of the third-year professional development experiences, the REU (Research Experience for Undergraduates) Beliefs Instrument was given to teachers again. Teacher responses were averaged and Cohen-D effect sizes were computed to measure the significance of changes in teachers' beliefs from the beginning to the end of the professional development experiences.

In the post analysis of our data, after analyzing changes in teachers' beliefs and the IQA rubric ratings of teaching episodes, we took a Grounded Theory approach (Corbin & Strauss, 1996) to understand the role of mathematical authority. Teachers had strong initial beliefs related to elements of sharing mathematical authority with students, and the strengths of those beliefs over the three years of professional were either sustained (see table 2, questions 19 and 21) or advanced (see table 2, question 6 and 14). We wanted to understand whether those beliefs were present in teachers' practices and, if so, what were the challenges teachers faced in sharing mathematical authority with students?

In our fine-grained analysis of a few teaching episodes, IQA ratings (3 or higher) in the rubrics related to task potential (AR1) and questioning (AR-Q), we noticed that sharing mathematical authority was not as prevalent in the actual teaching episodes as teacher beliefs would suggest. Also, we noticed that, in the instances in which teachers did attempt to share mathematical authority with students, there were different challenges that they faced in continuing to share the mathematical authority with students. These instances were analyzed to categorize the nature the classroom activities that either had the potential to or resulted in the sharing of mathematical authority with students. Themes that emerged were: 1) a teacher pressed a student to further explain their reasoning, 2) a teacher asked a question to advance students' reasoning (Bill & Smith, 2008), 3) a student generated a conjecture, 4) a student asked a question, or 5) the validity of mathematical contribution by a student needed to be established.

Results

Teacher Beliefs

Items from the REU (Research Experience for Undergraduates) Beliefs Instrument (O'Hanlon et al., 2015) related to elements of classroom discourse are shared. Bolded item numbers are reverse scored and those with large effect sizes (Cohen's D > 0.60) are starred.

Table 2: Pre and Post Significant Changes in Beliefs (|ES| > 0.60)

#	Question	Pre	Post	Effect Size
6	During class discussions, students should analyze and critique another students' work.	$\bar{x} = 3.80$ $s = 1.23$	$\bar{x} = 4.40$ $s = 0.60$	0.68*
11	The teacher should demonstrate how to solve mathematical problems before the students are allowed to solve problems.	$\bar{x} = 3.00$ $s = 0.82$	$\bar{x} = 4.05$ $s = 0.51$	1.84*
14	During class discussions, the teacher should be the authority in terms of whether a student's mathematical conjecture or justification is correct.	$\bar{x} = 2.60$ $s = 1.07$	$\bar{x} = 3.50$ $s = 0.89$	0.91*

19	Teachers should provide opportunities for students to critique mathematical arguments, and discuss their own conjectures.	$\bar{x} = \mathbf{4.50}$ $s = \mathbf{0.52}$	$\bar{x} = \mathbf{4.50}$ $s = \mathbf{0.53}$	0.00
21	During class discussions, students should play a role in determining whether mathematical justifications are valid.	$\bar{x} = \mathbf{4.50}$ $s = \mathbf{0.52}$	$\bar{x} = \mathbf{4.50}$ $s = \mathbf{0.52}$	0.00
31	Struggling with mathematical concepts is detrimental to understanding.	$\bar{x} = \mathbf{2.80}$ $s = \mathbf{1.55}$	$\bar{x} = \mathbf{3.94}$ $s = \mathbf{1.12}$	0.84*
32	Group discussions often lead to tangents, or incorrect mathematics, and should be limited in their use.	$\bar{x} = \mathbf{3.70}$ $s = 0.82$	$\bar{x} = \mathbf{4.25}$ $s = 0.45$	0.83*
49	The teacher should provide verification for mathematical arguments, rather than expecting students to do so.	$\bar{x} = \mathbf{3.50}$ $s = \mathbf{0.97}$	$\bar{x} = \mathbf{3.81}$ $s = \mathbf{0.75}$	0.36

Quality of Classroom Discourse

These results suggest that the cohort's beliefs about classroom discourse either moved, or were already consistent, in the direction we espoused. The average ratings for Years 2 and 3 are shown below. A two-sample unequal variances t-test was conducted to determine whether there was a statistically significant change in average cohort ratings (n = 16) on the rubrics of the IQA.

Table 3: Comparing Year 2 and Year 3 IQA Cohort Ratings

	Year 2	**Year 3**	**P-Value**
Academic Rigor			
R1: Task Potential	2.63	2.97	0.25
R2: Task Implementation	1.91	2.22	0.32
R3: Student Discussion	1.69	1.91	0.45
AR-Q: Questioning	1.84	1.78	0.84
AR-X: Mathematical Residue	1.85	2.07	0.46
Accountable Talk			
AT1: Participation	2.02	2.56	0.049*
AT2: Teachers' Linking	1.68	1.94	0.32
AT3: Students' Linking	1.63	1.27	0.19
AT4: Asking (Teacher Press)	1.82	1.94	0.68
AT5: Providing (Student Response)	1.66	1.81	0.61

These results indicate that the only statistically significant ($\alpha < 0.05$) change in the ratings occurred in Participation (AT1) rubric, the percentage of students participating in teacher-facilitated discussion. It is important to note that the numerical rating indicate an average of between 50% and 75% of students participating in class discussions.

Mathematical Authority

Although teachers expressed strong beliefs about sharing mathematical authority with students (see table 2), we identified very few teaching episodes in which teachers shared mathematical authority with their students. Analyzing the specific instances in which teachers did share mathematical authority with students, we attempted to categorize teacher actions that supported those efforts, or hypothesized about the nature of the challenges they faced in as they shared mathematical authority with their students. From this analysis, a set of guiding questions were generated to enable us to further reflect upon teacher actions that supported the sharing of mathematical authority with students.

1. Are students given an opportunity to choose the tools, or understandings, they want to use to make sense of the problem or task?
2. Are students deciding the mathematical correctness of an answer or a students' contribution to the discussion?
3. Are students given an opportunity to share their reasoning and sense-making with the class?
    a. If so, in what forum? (e.g. group discussions, whole class, etc.)
    b. Are these contributions valued by other members of the class? By the teacher?
4. Is the teacher asking pressing/assessing questions to better understand their students' reasoning and sense-making?
5. Is the teacher asking questions to advance students' understanding of their ideas?
6. Is the teacher giving their students opportunities to explore their own conjectures?
    a. If so, do my students have access to the necessary tools to do so?
    b. If so, does the teacher value these experiences? How is the teacher making this evident to students?
    c. If so, what demands is this placing on the teachers' knowledge (e.g. mathematical, pedagogical content knowledge, etc.) to meaningfully orchestrate this experience?

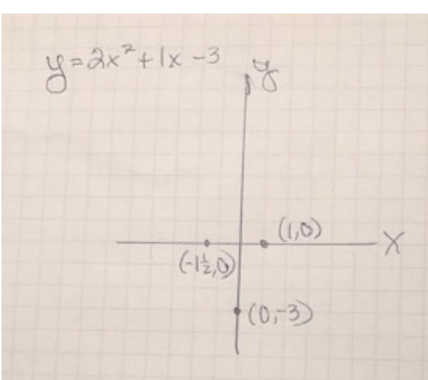
Discussion

It seems important to share a few, brief teaching episodes that exemplify the role of sharing mathematical authority with students in high-quality discourse. The episodes also illuminate a few of the demands sharing mathematical authority with students places on teachers.

Mrs. Barnes Classroom

Mrs. Barnes (pseudo-name) gave the following task to her 10th grade students, "Graph $y = 2x^2 + 1x - 3$." Working in groups, students were provided graph paper, no technology, and no guidance. Most groups were able to locate the zeros and y-intercept, but most struggled to find the location of the vertex.

After a few minutes, one of the groups *shared a mathematical contribution* about the relationship between the zeros of the function and the location of the axis of symmetry (see fig. 1, lines 50-53). The exchange between Mrs. Barnes and the group is shown in Figure 1.



S1: Okay so we found the zeros, but we really don't know where to go from there. I mean we kind of guessed on the vertex but we don't really know.
T: You guessed the vertex…
S2: The axis of symmetry is right here.
T: Axis of symmetry is at the y-axis. Do you think it is right there at x equals zero? Why?
S1: No, because it is uneven. You have one at one and one at negative one and a half.
T: Could you find the middle between those two points?
S1: Yeah
T: Could you find the middle between those two points.
S1: We did, it's negative one-fourth. That is where we tried to put our point. We just don't really know what we are doing.
S2: Okay, so we used the negative three and went down three and then did that negative one-fourth, we think.
T: Okay, so do you think the negative one-fourth should be higher or lower than that negative three?

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* St Louis, MO: University of Missouri.

Figure 1: Small Group Discussion of the Location of the Axis of Symmetry

During the small group discussion there are multiple ways in which Mrs. Barnes shared the mathematical authority with her students. She asked the group to *clarify a statement*, "we kind of guessed on the vertex" to better understand the meaning of <u>their</u> words (see fig. 1, lines 46-48). She also asked students to *provide a justification for a conjecture* she inferred from the location of their finger on the graph, "Axis of symmetry is at the y-axis. Do you think it is right there at *x* equals zero? Why?" (lines 50-51). Also, instead of giving the students the answer, she *asked a question to advance students' understanding of their ideas* related to the possible location of the vertex (lines 60-61), "So do you think the negative one-fourth should be higher or lower than the negative three? "[referring to the location of the vertex].

However, during the whole class discussion Mrs. Barnes struggled to continuing sharing the mathematical authority with her students (see fig. 2).

T: What if I can't immediately look at that and know it? Is there a way for me to find the axis of symmetry just by doing some math?
S1: Can't you just add both of them and divide by two?
T: You mean your intercepts? Yeah, we could average them. We could average the intercepts. We could also take negative b over 2a. Do you remember we did the quadratic formula and we started with negative b plus or minus you got that big long square root and then it was 2a. Did you know that the number on the outside of the radical, not the discriminant, the number to the outside, actually procedures the axis of symmetry?

Figure 2: Excerpt of Mrs. Barnes Whole Class Discussion

In reflecting upon the teaching episode, Mrs. Barnes expressed frustration that she was unable to do more with the conjecture shared by students, "the axis of symmetry of symmetry is halfway between the two x-intercepts" (see fig. 1, lines 50-53). She indicated that the conjecture was unanticipated, and that it challenged her because she did not know, in the moment, what to do to further advance students' reasoning and sense-making." She recognized that she dismissed the group's contribution by stating "Is there a way for me to find the axis of symmetry just by doing some math?" (see fig. 2, lines 242-243). She also stated that this group was actually "doing mathematics" based on their engagement in multiple Standards of Mathematical Practice (CCSSI, 2010). It is interesting to note that even in the midst of Mrs. Barnes explanation, the student, desiring to retake the mathematical authority exclaimed, "Can't you just add both of them and divide by two?" (line 244).

Mr. Pho's Classroom

Unlike Mrs. Barnes' classroom episode, in which there were productive moments of sharing the mathematical authority with students, Mr. Pho's classroom episode did not have those same moments. Mr. Pho (a pseudo-name) gave students the following task, *A virus is doubling every 30 minutes, if the initial number of virus present is 2000, how long will it take before there are 3 million virus? 10 million virus? Write an equation to model the situation.* After a lack of meaningful small group discourse, Mr. Pho started a class discussion in which the following dialogue occurred.

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*meeting of the North Americ ... ology of Mathematics
Education.* St Louis, MO: U

T: What unit are we on?
S1: Exponents
T: So you know it has something to do with exponents.
S2: B squared…B squared. I don't know.
T: We are looking at exponential functions. [T goes to back of class and grabs a student-made poster that involves characteristics of exponential functions.]
T: We are looking at exponential functions.
S3: Okay, prime factors.
T: Exponential functions
S4 Hey, can I see that again!
S5: a times b, whatever that is supposed to mean.
T: Yes
S6: Didn't you explain this one once like it was time times growth?
T: I am giving you a hint. [Holding poster in front of the student.]
S3: It is how many times it duplicates.
S7: Multiply it.
T: Exponential functions
S3: Two-thousand
S4: What's the multiplier
S2: Three

Figure 3: Excerpt of Mr. Pho's Whole-Class Discussion

In reflecting upon the teaching episode with Mr. Pho, we hypothesized as to the nature of the challenges he faced in sharing the mathematical authority with students, and potential strategies to overcome those challenges. Mr. Pho recognized that he lowered the cognitive demand of the task, and reclaimed mathematical authority when he stated, "what unit are we on? (Line 46)". He stated, however, that, in the moment, he could not think of a question to advance students' reasoning and sense-making without imposing his mathematical authority. It was our hypothesis that students did not seem to be able to access the necessary tools to reason and make sense of the task. By *asking an advancing question*, such as "Could you make a table to make sense of the situation?" or "How could you find the number of virus after 30 minutes, 1 hour, etc.?" , Mr. Pho may have been able to continue to share the mathematical authority with students. Thus, letting students' reasoning and sense-making drive the discussion instead of Mr. Pho's mathematical topic hint. We also hypothesized that, asking students, as part of the initial prompt, to "write an equation", instead of giving students an opportunity to make sense of the task may have hindered their ability to assume the mathematical authority. Giving students an opportunity to read the stem of the initial prompt and ask him clarifying questions, such as "What does it mean to double every 30 minutes?", might have been enough for students to feel that they had the permission, and necessary reasoning tools, to assume the mathematical authority.

Conclusion

As the teaching episodes of Mrs. Barnes and Mr. Pho illustrate, teachers' capacity to implement high quality discourse is closely connected to their ability to share mathematical authority with their students. While teachers may have strong beliefs that support sharing mathematical authority with students, there are significant demands placed on teachers in enacting instructional practices that support those beliefs. Sharing mathematical authority with students requires the teacher to not only listen and make sense of student contributions, but to, in the moment, act upon what was heard and understood. Effectively acting upon student contributions requires teachers to determine the nature and validity of student reasoning, and ask questions to advance their current understanding. In other instances, sharing mathematical authority with students challenges teachers to identify instructional barriers (e.g., unfamiliarity with the task, lack of reasoning tools) that are hindering students ability to assume the mathematical authority. Future teacher training sessions that involve improving the quality of classroom discourse needs to include: 1) opportunities to develop an awareness of the nature of the challenges associated with sharing mathematical authority with students, and 2) sharing of specific strategies that support teachers efforts to share the mathematical authority with students.

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USING A DISCURSIVE FRAMEWORK TO ANALYZE GEOMETRIC LEARNING AND INSTRUCTION

Peter Wiles
Eastern Illinois University
pswiles@eiu.edu

Rick Anderson
Eastern Illinois University
rdanderson@eiu.edu

In this study we applied a discursive perspective of learning (Sfard, 2008) to a sequence of 21 geometry mini-lessons taught in a fourth grade classroom. From this perspective, learning is defined as changes in mathematical discourse. We first characterize and then compare discourse from the beginning and the end of the mini-lesson sequence. We identify shifts in the discourse that occurred during the sequence. We then discuss how the characteristics of the students' geometric discourse informed task design and instruction. This perspective provided a useful means for linking instruction to student learning in an operationalized manner.

Keywords: Classroom Discourse, Geometry and Geometrical and Spatial Thinking, Instructional Activities and Practices, Learning Theories

While geometry is an important branch of mathematics, U.S. students' geometry achievement lags behind achievement in other areas of mathematics (Mullis, et al., 2016). At the same time, scholars have observed that the geometry instruction students receive is often lacking (Sarama & Clements, 2009). Much of the early research on geometry learning (e.g., Burger & Shaughnessy, 1986; van Hiele, 1959) has stemmed from a cognitive perspective, while more recent research has shifted to a participationist perspective (Lave & Wenger, 1991; Sfard, 1998; Wenger, 1998), specifically, a discursive perspective (Sfard, 2007, 2008; Sfard & Lavie, 2005). This shift results in advantages, both analytic and practical (Sinclair, Cirillo, & De Villiers, 2017). From an analytic perspective, a discursive theory operationalizes learning and does not require researchers to make inferences about unseen cognitive processes. From a practical point of view, the theoretical framework can provide suggestions for task design and implementation to inform instruction.

This project aimed to analyze the discourse about geometric shapes and properties that developed in a fourth grade classroom across a series of 21 geometry mini-lessons. We use a discursive framework (Sfard, 2008) to (1) characterize learning and (2) identify ways instructional changes were made in response to students' discourse. We focus on the following questions:

- What is the impact of geometry mini-lessons on students' geometric learning?
- How does a discursive theory of learning inform instruction?

Discursive Theory of Learning

According to Sfard (2007), "the different types of communication that bring some people together while excluding some others are called discourses" (p. 571). *Communication* is the patterned activity where the action of one individual is followed by the action of another individual and "*thinking* is an individualized version of (interpersonal) communicating" (Sfard, 2008, p. 81). Used in this way, the notion of discourse embodies more than the communicative features of talk, but positions one in a larger community of practice. Sfard (2008) identifies two important types of discourses: colloquial and literate. Colloquial discourses are those that arise in one's everyday experience, while literate mathematical discourses are those that make use of standard mathematical terminology and symbolism recognized by the broader mathematical community. Sfard notes that *commognitive conflic*t "arises when communication occurs across incommensurable discourses" (p. 296). Much of schooling involves this intersection of discourses, with students moving from the colloquial discourse to the literate discourse. From this perspective, *learning* is defined as a lasting change in discourse.

In Sfard's framework, four features distinguish mathematical discourse: word use, visual mediators, narratives, and routines. *Word use* consists of the various terms and words that are unique to mathematical discourse. The meaning that is conveyed by these words, however, may differ depending on the discourse community that one is acting in. *Visual mediators* represent the symbolic and visual artifacts that are used as the basis for mathematical communication. As with words, the way that we attend to these visual mediators depend on the discourse. *Narratives* embody the ways that we describe, engage with, and identify relationships between mathematical objects. Narratives can be endorsed or rejected within the discourse. For instance, a possible narrative may be that all squares are rectangles. This narrative may be rejected by students in a discourse community based on the notion that a rectangle should have two long sides and two short sides. Finally, *routines* represent the patterns of activity found in a discourse. For instance, a routine for identifying a shape in one discourse community might involve matching the shape to a set of canonical shapes. In another, it might involve measuring side lengths and angles. Similarly a routine for comparing shapes in one discourse community could focus on their size and orientation, while in another, the routine might focus on their geometric properties.

Much of the research that has applied a discursive framework to geometry learning has focused on narrow instructional segments or small populations. Some of these studies have included students. For example, Sinclair and Moss (2012) analyzed one 30-minute lesson on creating and transforming triangles with a group of 11 kindergarten students. A pair of bilingual high school calculus students exploring in a dynamic geometry environment were the participants in a study by Ng (2015). Other studies have teacher participants. Pre- and post-tests were administered to 63 prospective elementary and middle school teachers in a study by Wang and Kinzel (2014). They report on two participants' geometric discourse but do not link to instruction. Sinclair and Yurita (2008) analyze the discourse of a high school geometry teacher, comparing static and dynamic environments. A goal of our project was to expand beyond the scope of these studies and explore the shifts in geometric discourse for a whole classroom over instructional episodes spanning several weeks. Further, we wanted to investigate how the discursive framework could also inform instruction.

Methodology

A teaching experiment methodology (Steffe & Thompson, 2000) was adopted for this study. As teacher-researchers in the project, we developed and implemented 21 geometry mini-lessons over the course of 11 weeks in a fourth grade classroom in a small, midwestern town. Approximately 22 students were in each class session. The mini-lessons occurred before students received geometry instruction in the regular fourth-grade math class. For each mini-lesson, one researcher acted as the instructor, while the other was an observer.

The geometry mini-lessons followed a format similar to that of *number talks* (e.g., Humphreys & Parker, 2015). Like number talks, these geometry mini-lessons were 15-20 minutes long and focused on geometric shapes and relationships (as opposed to number and computation). The teacher solicited ideas from several students and facilitated the subsequent discussion connecting these ideas. The mini-lesson tasks were designed to allow for multiple entry points and a range of student responses. We discuss three types of tasks:

- Quick Image (Wheatley, 2007). Students were shown a figure for 2-3 seconds and asked to draw it. Students described how they saw the figure and how they knew what to draw. This was repeated with three other related figures. The task concluded with a discussion of the similarities and differences of the figures.
- Which One is not Like the Others? (Danielson, 2016). Students were presented with four shapes and asked: Which one is not like the others? Any of the four shapes could be chosen as not like the others. Students discussed their reasons for their choice.
- Guess My Shape. The class was presented with a collection of shapes, one of which had been secretly selected by the teacher. The students asked yes/no questions that the teacher answered. The teacher led a discussion of the consequences of a yes or no answer to the questions before answering. The instructor strategically selected questions to answer based on the goals for the mini-lesson.

The data sources for this report come from video recordings, copies of students' written work, and field notes of each mini-lesson, along with notes from teacher-researcher post-lesson debriefing. Two video cameras captured the lesson from different angles in the classroom. When viewing video of the sessions in ATLAS.ti, each student utterance was identified and coded in two ways. First, codes were created to identify the geometric properties and shapes under consideration in the utterance. Then the utterance was coded for features of discourse, namely word use, routines, narratives, and visual mediators (Sfard, 2008). Any words related to the geometric properties and shapes, whether colloquial or literate, were coded. Instances where students compared or identified shapes were coded with the specific routines they used. Following Sfard, we coded students' descriptions of shapes as narratives.

We then looked at all the lessons and identified the main geometric topics addressed, e.g. angles, parallel lines, congruence. We examined each task and identified the potential geometric topics to be discussed by students. We looked for the emergence of these topics in the classroom discourse and linked together sessions that addressed the same topic. This gave us a mapping of the geometric topics that were addressed during the 21 mini-lessons.

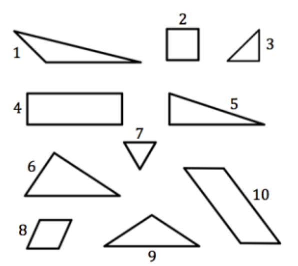
Changes in Discourse over Time

To identify learning that occurred during the sequence of mini-lessons, we characterize the discourse of two lessons, one occurring near the beginning (mini-lesson 4) and one near the end (mini-lesson 18) of the 21-lesson sequence. Sfard (2007) noted that, "because mathematical discourse learned in school is a modification of children's everyday discourses, learning mathematics may be seen as transforming these spontaneously learned colloquial discourses rather than building new ones from scratch" (p 573). As such, we focused on the characteristics of the students' use of words, visual mediators, narratives, and routines in both mini-lessons, and examined how these shifted. Both mini-lessons used the Guess My Shape task structure, so the nature of the mathematical activity was similar.

Early Discourse

The fourth mini-lesson in the instructional sequence featured a collection of triangles and quadrilaterals (Fig. 1a). Notably, the shapes included right, acute, and obtuse triangles, as well as several shapes with different configurations of congruent sides. The students used these shapes as visual mediators to establish routines for categorizing or differentiating between shapes. The routines could be identified primarily through the questions that they suggested, as well as the accompanying discussion surrounding those questions. Figure 1b shows the questions generated by the students during the lesson.

Many of these questions demonstrated that students' routines for distinguishing between two shapes were tied primarily to the shapes' visual characteristics such as size and orientation. Only two questions focused on using geometric properties as a basis for distinguishing shapes, namely side congruence or number of sides. Even the use of the terms "rectangular" and "triangular" rather than "a rectangle" or "a triangle" highlight a focus on visual characteristics rather than identifying the shape as part of a class of shapes.



(a)

1. Is it slanted or up?
2. Is it upside down?
3. Is it long?
4. Is it short?
5. Is it on its side or flat?
6. Does it have four sides?
7. Is it rectangular?
8. Is it triangular?
9. Are the sides equal?

(b)

Figure 1: Early Discourse Guess My Shape Task and Student-generated Questions

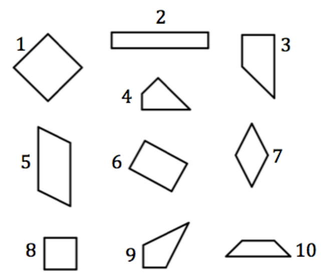
The questions and the discussion surrounding them heavily featured colloquial word use such as *slanted*, *up*, *flat*, *upright*, *upside down*, *laying*, and *corner*, words that are not well defined in the literate geometric discourse. Some words, which do have specific meaning in the literate discourse, were used in alternative ways. For example, a student referred to a shape being "straight" as a way to differentiate it from shapes she considered to be "slanted." Similarly, the sides were considered "equal" rather than "congruent." A rhombus (Fig. 1, shape 8) was referenced as a "little square" and a parallelogram (shape 10) was referred to as "rectangular."

Several competing narratives regarding the characteristics of the shapes under consideration emerged during the discussion. In the end, the students were left considering shapes 1, 5, and 6. The students were considering the question, "is it up"? When asked to show with their fingers how many of the shapes were "up," the responses were roughly equally distributed between 1, 2, 3, and 5. Some students only considered the isosceles triangle (shape 9) as being "up." Others included shapes that had vertical sides (e.g., shapes 1 and 5). Generally, the students did not clearly communicate their descriptions (narratives) of shapes that were "up" in a manner where it could be subjected to endorsement and agreement by the group. After the questions with geometric properties (e.g., number of sides and congruence) were answered, there was no clear resolution of the task due to the differences in the discourse. The words and narratives students were using made it difficult for them to determine which shapes to eliminate. In the end, the teacher had to reveal to the students which shape was selected.

Later Discourse

The eighteenth mini-lesson occurred approximately six weeks after the aforementioned mini-lesson. The quadrilaterals in the Guess My Shape task (Fig. 2a) were purposefully selected and oriented to serve as visual mediators for the students' geometric discourse. As with the previous mini-lesson, the shapes we selected gave students an opportunity to focus on both visual characteristics, such as size and orientation, as well as geometric properties, such as presence of right angles, parallel sides, and congruent sides.

The discursive routines students used to distinguish between shapes is evident in the questions they posed (Fig. 2b). Except for the final two, each question featured a distinguishing geometric property. The question, "Is the shape congruent?" was later clarified by the student to mean, "Does the shape have congruent sides?" The final two questions came at the end of the mini-lesson after all the shapes had been eliminated except for the two squares (shapes 1 & 8). Since both squares shared all salient geometric properties, the students' routines naturally reverted to using orientation and size, both visual characteristics.



(a)

1. Does it have two sets of parallel sides?
2. Does it have a right angle?
3. Is the shape congruent?
4. Does the shape have congruent sides?
5. Is it a rectangle?
6. Does it have obtuse angles?
7. Are all four sides congruent?
8. Is it big?
9. Is it tilted?

(b)

Figure 2: Later Discourse Guess My Shape Task and Student-generated Questions

The narratives generated in this mini-lesson emphasized geometric properties, and were communicated in ways that could be endorsed by other students and the broader mathematical community. Late in the lesson, the instructor answered yes to the question, "Is it a rectangle?" When asked to consider which shapes should be eliminated from consideration, a few competing narratives surfaced. One student indicated a rectangle needed to have "even" sides, while another indicated that a rectangle needed to have four right angles. The latter narrative was endorsed when the instructor asked whether shape 9 (kite) should be eliminated from consideration. Several students agreed, with one stating that even though it had a right angle, it should be eliminated, "since it [didn't] have four [right angles]." This discussion came to a head when students were asked if shape 1 (square) should be eliminated. After checking that the shape did, indeed, have four right angles, the students were reminded that the question answered was "Is it a rectangle?" At this point, some students proclaimed that the shape was a square, not a rectangle. Others, however, argued that it should remain since a rectangle has four right angles. In the end, the class was asked to raise their hands to indicate if they felt the shape should still be considered, and all but one student agreed. It is important to note that there is little evidence to suggest that this constitutes endorsing the narrative *all squares are rectangles*. In fact, some students' hesitation based on naming the shape "square" rather than "rectangle" suggests that they have not. This episode does suggest that the students were comfortable endorsing the narrative *rectangles have four right angles*.

Comparing the Discourse

We see a stark contrast in the discourse between the two mini-lessons. This change is learning (Sfard, 2008). To begin, there is a difference in the questions the students asked to identify the shapes (Fig. 1b & 2b). Although the shapes were similar in each mini-lesson, students focused on orientation and visual appearance in the early mini-lesson; whereas, they were mostly attuned to properties of the shapes in the later mini-lesson. This suggests their routines for comparing shapes changed, i.e., students learned to focus on the properties of the shapes rather than the appearance of the shapes.

Word use also changed markedly from the early to the later mini-lesson. The early discourse was filled with colloquial and sometimes ambiguous words when referring to the shapes. The only words students used from the literate geometric discourse were the names of specific shapes (e.g., rectangle, rhombus, square) and these were not always used in conventional ways. For example, one student referred to a rhombus as a "slanted square." A hallmark of the students' later discourse was the frequent use of geometric words recognized by the mathematical community. Only a few utterances featured colloquial word use. The discussion was replete with geometric terms such as *parallel*, *congruent*, *obtuse angle*, and *right angle*.

We can also see changes in the apparent student narratives for the geometric shapes. In the early discourse, for example, students described a rectangle as have 4 sides, two of them longer than the other. In the later discourse, students had come to describe rectangles as having 4 right angles. This change of narrative set the stage for future consideration of the narrative: *A square is a rectangle*. Class inclusion is the prevailing narrative in the broader mathematical community although it proves challenging for students (De Villiers, 1994).

Sfard's discursive framework proved useful in identifying students' geometric learning across the span of several mini-lessons. As students had opportunities to discuss geometric shapes, the words they used to describe shapes, the routines they used to compare shapes, and the descriptions (narratives) about shapes all changed. In sum, the students' everyday discourse was transformed to be closer to that of the literate discourse of geometric shapes. This did not happen in isolation, however, but was influenced by the instruction they received. As teacher-researchers, we were uniquely positioned having one foot in the students' discourse and the other in the discourse of the mathematics community.

Initiating Changes in Discourse through Instruction

We now turn our focus to show how the discursive framework allowed us to analyze the existing discourse and how it informed the instructional decisions for the subsequent mini-lesson. We consistently monitored the discourse in each mini-lesson and designed the subsequent mini-lesson to modify the students' discourse in specific ways to shift it closer to the discourse of the mathematics community.

The content goals for grade 4 geometry (CCSSI, 2010) include, among other things, identifying parallel lines in two-dimensional figures and classifying two-dimensional figures based on the presence or absence of parallel lines. After the first six mini-lessons, students had not yet noticed or discussed, either informally or formally, parallel lines.

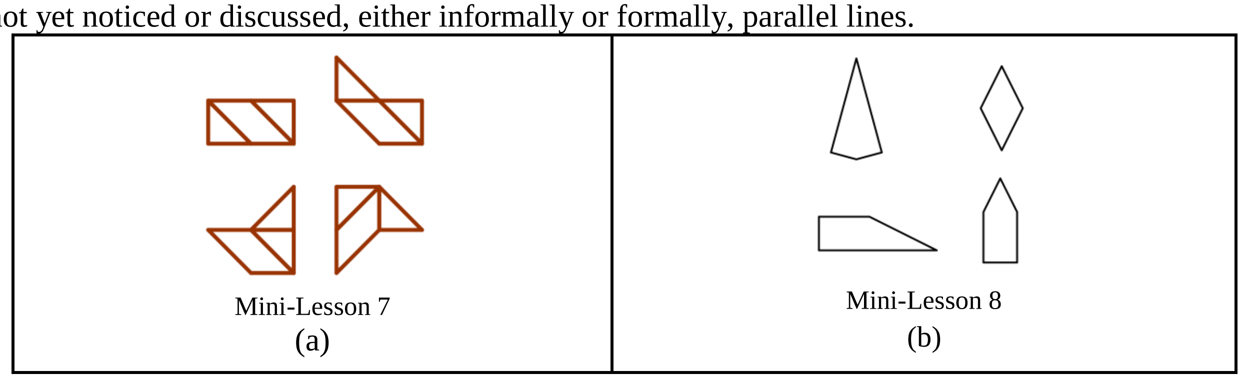


Figure 3: Four Tasks Used in Mini-Lessons

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

The task for the mini-lesson 7 provided opportunities for students to talk about parallel lines due to the figures chosen for the task. The Quick Image task (Fig. 3a) featured four different arrangements and orientations of parallelograms and right triangles. The teacher projected the first image (upper left) for the students to draw. Then, the teacher asked the students to describe what they drew. Students described the figure as having a sideways slanted square, a rectangle, a slanted rectangle, and a square. Some students commented on the diagonal lines but after reviewing their work, we noticed many students had drawn two diagonal lines while some had drawn three.

As we progressed sequentially through the images, a student noticed a parallelogram in each figure. The teacher asked the class what made that shape a parallelogram. The only verbal response was that it had four sides, although one student gestured the shape, indicating a routine of identifying a shape by matching it to a visual prototype. The teacher continued by mentioning "parallel" was part of the word *parallelogram*. "What does parallel mean?" the teacher asked. Students responses included, "the two shorter sides and the two longer sides are the same," "same size," and "side by side."

By the end of the discussion, it was clear that students did not share a narrative for parallel lines. Moreover, the students did not spontaneously use the word parallel, as this exchange was initiated by the teacher. The student-used words (side by side, same size) are not part of the literate mathematical discourse for parallel. Also, students' routines for classifying the parallelogram focused on congruence rather than parallel sides. This, then, informed their narratives about the meaning of parallel, conflating it with that of congruence.

In an attempt to change the students' discourse (i.e., learning) about parallel lines, we used the discursive framework to design and implement mini-lesson 8. We used a task where students would decide Which One is not Like the Others? We selected shapes that had the potential to encourage students to identify and discuss parallel lines. In particular, we selected three shapes that had parallel sides and one that did not (Fig. 3b). Other properties that we included were symmetry, right angles, and number of sides. Furthermore, we wanted to introduce a narrative: *parallel lines go in the same direction*. To this end, we had two long, thin wood dowels to position over the sides of the projected shapes to further illustrate that the sides go in the same direction. We specifically chose to focus on direction rather than the alternative narrative that *parallel lines do not intersect*. We wanted to avoid the possibility of introducing a narrative that conflated segments with lines. Two segments that do not intersect are not necessarily parallel. For example the left and right sides of the right trapezoid do not intersect and they are not parallel.

When we implemented this task in the classroom, students described ways in which the right trapezoid ("The only one pointing to the side."), the pentagon ("The only one with 5 sides."), and the rhombus ("It has two spikes [acute angles].") were different from the other shapes. At this point in the mini-lesson, the students have not provided any reasons for why the kite was different from the other shapes. The teacher had students recall the parallelogram in the previous mini-lesson. He introduced the dowels as a way to focus on the direction the sides were going and introduced the description of parallel lines as going in the same direction. Together the class revisited each shape and checked to see if any had parallel sides. Since the dowels extended the sides, it was clear that parallel sides did not intersect, but, more important, they had the same slope (go in the same direction). After checking the shapes, the mini-lesson ended with a student noting that the kite was the "different one because all of the other ones have parallel sides," to which no one disagreed.

Although new words, visual mediators (dowels), narratives, and routines for determining parallel lines were introduced by the teacher, they became part of the students' discourse. As we saw in the discussion of the later mini-lesson, the students' routine for comparing shapes included noticing the presence or absence of parallel lines.

Discussion

As we have shown, Sfard's discursive framework was useful for analyzing the impact of the mini-lesson sequence on students' geometric learning, defined here as a change in discourse. By operationalizing learning in this way, we can see "the expansion of the existing discourse, attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives" (Sfard, 2007, p. 573). Taken together, these characteristics are pieces of a larger puzzle. Using the words rectangle and parallelogram was not always indicative that students could identify distinguishing properties (e.g., right or acute angles) when comparing them. Likewise, students were not necessarily willing to call a square a rectangle even though they acknowledge they both had four right angles.

Just as learning is a change in discourse, instruction can be viewed as catalyzing changes in the discourse. Sfard (2008) argued, "proactive participation of the expert interlocutors is critical to the success of learning" (p. 605). One of the most important roles of the teacher is to serve as the source of commognitive conflict where the encounter between the teacher's and the students' discourses necessitates a change in discourse. Commognitive conflict arises in the interaction between two discourses that do not operate under the same discursive rules. We can observe this commognitive conflict taking shape in the above episodes surrounding the norms that govern how shapes are described or differentiated. In the instructor's discourse, the means for discussing shapes was through identifying the geometric properties of the shapes, while the students allowed for the use of visual characterizations such as size and orientation. It was not enough to just introduce the properties into the students' discourse, the goal was for the students to similarly focus on geometric properties when examining shapes. In order for this shift in discourse to occur, it is important that the students agree that this shift is both necessary and advantageous.

The tasks outlined above were chosen because they had specific objectives (e.g., guess the mystery shape, find the shape that does not belong) that were more effectively resolved through the use of geometric properties than visual characteristics. As we saw in mini-lessons 4 and 8, the students' existing discursive routines acted as a hindrance to accomplishing this objective. In lesson four, the teacher had to eventually tell the students the mystery shape since their questions could not adequately distinguish the shapes. Similarly, the students could not tell why the kite did not belong without attending to parallel sides. The introduction of the word parallel, the routine of checking the direction of the sides with the dowels, and the narrative of parallel lines going in the same direction served as important tools to successfully resolve the task.

As teacher-researchers investigating elementary school students' geometry learning, we found great utility for Sfard's discursive theory in both unpacking the nature of the learning that took place, as well as being explicit about the instructional actions to support learning. By viewing learning as changing discourse, and attending to the characteristics of the students' discourse, we believe that teachers could be supported to become more explicit about their instructional goals.

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COMPONENTS OF HIGH-QUALITY MATHEMATICS CLASSROOMS: ATTENDING TO LEARNING OPPORTUNITIES FOR ENGLISH LANGUAGE LEARNERS

M. Alejandra Sorto
Texas State University
sorto@txstate.edu

Kathleen Melhuish
Texas State University
melhuish@txstate.edu

Eva Thanheiser
Portland State University
evat@pdx.edu

Katty Zied
Texas State University
kattyzied@txstate.edu

Christina Koehne
Texas State University
crz7@txstate.edu

Amanda Sugimoto
Portland State University
asugimo2@pdx.edu

Autum Pham
Portland State University
autpham@pdx.edu

Simon Byeonguk Han
Portland State University
byeonguk@pdx.edu

Sharon Strickland
Texas State University
strickland@txstate.edu

In response to the call for research on integrating best general practices in teaching with those that promote equity and access, we present a two-part study focused on instructional strategies that may remove learning barriers for English Language Learners[1]*. We theoretical developed and empirically explored supplemental components for traditional quality of instruction measures (MQI, Hill, 2014, Math Habits Tool, Melhuish & Thanheiser, 2017). We share results from a quantitative study empirically verifying the effect of suggested ELL-focused instructional strategies (Chval & Chávez, 2011) on ELL learning via the creation of an additional MQI dimension. Based on these results, we then provide theoretical operationalizations of these strategies to integrate into the student-and-teacher interaction tool: the Math Habits Tool (Melhuish & Thanheiser, 2017) as means to concretize these strategies for researchers and practitioners.*

Keywords: Classroom Discourse, Instructional Activities and Practices, Inclusive Education

In general, the mathematics education field has endorsed certain images of classrooms that are student-driven and supported by instructional practices that serve to focus on student thinking and support students in engagement in productive discussion (e.g., Jacobs & Spangler, 2017). Recent work, however, has revealed that simply engaging in such practices without attention to issues of equity and access can produce classrooms where not all students have opportunity to learn (e.g., Johnson, Andrews-Larsen, Keene, Melhuish, Keller, & Fortune, in press). As a result, researchers have called for attention to equitable teaching that can support the engagement of all students in mathematical practices (Bartell, Wager, Edwards, Battey, Foote, & Spencer, 2017).

Our work addresses this call in the context of English Language Learners (ELLs.) This growing population (National Academies of Sciences, Engineering, and Medicine, 2018) frequently incur barriers to fully engage in learning opportunities in the classroom. While there is literature about teaching practices that may support this population (Barwell, Moschkovich, & Setati Phakeng, 2017), little work has been done at-scale to empirically explore what instructional strategies may support these students. We share results from a quantitative study identifying the effect of suggested instructional strategies on the mathematical achievement gains of middle grades ELLs. We pair this work with a theoretical, qualitative analysis serving to operationalize these results in a way that is congruent with other best practices for instruction

(Melhuish & Thanheiser, 2017), and can serve as grounds for teachers to concretely work on their practice.

Literature Background

There is general consensus in the mathematics education literature that high quality mathematics classrooms contain certain attributes. These classrooms are ones in which student voices are heard and orchestrated, and student thinking is leveraged as the means to move instruction forward (e.g., Ball, 1993; Jacobs & Spangler, 2017; Nasir, & Cobb, 2006; Schoenfeld, 2011; Turner, Dominguez, Maldonado, & Empson, 2013). However, teachers hold an essential role in orchestrating opportunities that encourage and support such forms of student participation to promote active student learning and engagement (Franke, Kazemi, & Battey, 2007; Gresalfi, 2009; Jacobs & Spangler, 2017). Students are not just passive receivers of mathematics, but active participants in sense-making including through the use of justifying and generalizing mathematics (e.g., Boaler & Staples, 2008). Such qualities have influenced a number of quality of instruction tools including the summative Mathematical Quality of Instruction (MQI) instrument (Hill, 2014) and the formative Math Habits Tool (Melhuish & Thanheiser, 2017).

In alignment with this research, standards reform movements have also increasingly called on students to use language in a variety of ways to increase their mathematical understandings, e.g., explaining their thinking or discussing connections between multiple representations (National Governor's Association Center for Best Practices & Council of Chief School Officers, 2010). However, merely providing tasks and discussion support focused on mathematical practices may conceal the complexities of students in the classroom (Bartell et al., 2017). In particular, students such as ELLs, may encounter barriers to full participation and engagement if they are not able to access the mathematical discourse and content in the classroom (Moschkovich, 2002, 2007).

Scholars have explored pedagogical strategies to bypass such barriers, including: (1) exposing students to mathematical content and instruction in multiple modalities, e.g., gestures, mathematical representations, and manipulatives, and (2) exploring the meanings and multiple meanings of words used in the mathematical register (Bartell et al., 2017; Barwell et al., 2017; Campbell et al., 2007; Moschkovich, 2002, 2007; Shein, 2012; Turner et al., 2013). These strategies are designed to develop students' understanding of mathematical content while simultaneously acquiring higher levels of language proficiency (de Araujo et al., 2018; Bartell et al., 2017; Barwell, et al., 2017; Campbell et al., 2007; Moschkovich, 2002, 2007; Turner et al., 2013). In order to provide students with opportunity to learn, instructors need to attend to both general best practices in instruction, but also to the particular needs of their students.

Theoretical Orientation

In our work, we take the viewpoint that mathematically productive classrooms are ones in which each and every student has access to high quality mathematics. High quality mathematics is mathematics in which students regularly have opportunities to make sense of big mathematical ideas via opportunity to justify, generalize, and leverage mathematical structure (e.g., Boaler & Staples, 2008). Further, these classrooms leverage high cognitive demands tasks (e.g., Stein & Smith, 1996), and support students in productively engaging in mathematical discourse (e.g. Jacobs & Spangler, 2017), in order to develop mathematical habits of mind (Cuoco, Goldenberg, Mark, 1996). Such classrooms reflect a micro-community where teachers and students interact in

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

tandem to develop norms around mathematical activity (Takeuchi, 2016; Staples, 2007; Yackel & Cobb, 1996).

In order to operationalize these interactions, we identify core classroom components in the Mathematically Productive Classroom Framework: *mathematically productive teaching routines* that are extended routines enacted by teachers (such as selecting and sequencing student ideas), *catalytic teaching habits* are in-the-moment teaching moves to support students in mathematical reasoning (such as prompting a student to justify), and *mathematical habits of mind* and *interactions* which are ways that students engage with mathematics and each other around mathematics (habits such as justifying, using representations, or engaging in critique and debate) (Melhuish & Thanheiser, 2017). Through this lens, mathematically productive classrooms can be analyzed through a series of connections across these interaction types. However, we acknowledge that the foci of such analysis may obscure the complexities of attending to mathematical access where different members of a classroom community bring individual lenses, backgrounds, and knowledge to engage with mathematics.

As such, we have aimed to complement traditional quality of instruction frameworks with nuanced attention to the qualities of instruction that can promote mathematical learning and engagement for students from linguistically diverse backgrounds. In particular, we focus on the discursive moves that provide students, and specifically ELLs, with *opportunity to learn* (Gresalfi, 2009; Jackson et al., 2013; Takeuchi, 2016). As noted in the recent commentary by Cai, Morris, Hohensee, Hwang, Robison, and Hiebert (2017), one of the most robust results from the research literature is that students learn best when provided opportunity to learn. Identifying moves that can support learning opportunities is imperative. As an initial grounds of analysis, we leverage the elements of Chval and Chávez's (2011) research-based instructional strategies to support ELLs. Our adaptions of these strategies can be found in Table 1.

Table 1: Instructional Strategies for Linguistically Diverse Classrooms (ISLD)

Instructional Strategy	Description
Connections of mathematics with students' life experiences	Teachers reference the mathematics found in daily life by students.
Connections of mathematics with language	Teachers reinforce a mathematical representation with its meaning.
Meaning and multiple meanings of words	Teachers and students explore the meaning of mathematical words and objects through speech and other forms of expression.
Use of visual aids or support	Teacher supplements instruction with powerful visual media that enhance comprehension of mathematical concepts.
Record of written essential ideas and concepts on board	Teacher makes careful and conscientious use of the board or any visual display media, and students have access to pertinent information throughout instruction.
Discussion of students' mathematical writing	Teachers use student work as an instructional tool and point of discussion.

See Sorto and Bower (2017)

Methods

We leverage several video-banks of lessons from grades K-8 to explore the integration of access to learning opportunities in partnership with instruments such as MQI and the MHT. In this report, we share results from two phases of our research: (1) empirically establishing the impact of the instructional strategies for linguistically diverse classrooms through adjoining a qualities of linguistically diverse classroom dimension to the standard MQI instrument and (2) theoretically developing teaching routines, catalytic teaching habits, and student habits of mind/interactions to operationalize what this looks like in the classroom.

To address the research aim in phase 1, we share the results of coding 99 mathematic lessons from a sample of 34 sixth-eighth mathematic teachers and their 4,522 students representing all of the 11 middle schools in a large district in the southwest United States. The majority of the teachers (75%) taught in classrooms with at least half of students classified by the school district as ELLs. Lessons were coded using the MQI and the augmented new dimension, ISLD, with each 7.5-minute segment assigned a code of Not Present, Low, Medium, and High (0 – 3). All videos were coded by two coders independently (average alpha=0.824) with disagreements settled via discussion.

To measure the effect of the set of strategies, as a whole, on teachers' student learning gains and in particular on their ELL students, a multivariate statistical analysis was conducted. We hypothesized that the presence and quality of implementation of the routines, teachers' knowledge, and general quality of instruction, might impact student achievement differently depending on students' language status. We developed the following Hierarchical Linear Model (HLM) in which the outcome variable of student achievement, Y, for the ith student of teacher n in school j was seen as a function of a vector of students' background variables, $\boldsymbol{X}$, and individual teacher quality measures, T:

$$Y_{inj} = \beta_X' \boldsymbol{X}_i + \beta_T T_n + (u_n, u_j) + \varepsilon_i$$

The model includes fixed effects at the school level, u_j, and random effects at the teacher level, u_n. Student achievement was measured as a standardized gain score by using the difference between their state test result at the end of the study year and the previous year's result. The data were pooled across all three middle school grades, and the control variables for students included measures for current grade, grade repetition status, economic status ("disadvantaged" is the label adopted by the state to designate students who qualify for free and reduced-price meals), and their baseline (prior year) mathematics test score. Because of potential collinearity the modes were estimated separately (one by one) for each individual teacher quality measure.

To address the research aim in phase 2, we leverage the empirical results to both identify existing components of the Mathematically Productive Classroom Framework and develop new components to best account for the types of teacher and student interactions that may provide access to mathematical learning opportunities. This process involved theoretically analyzing the existing framework, operationalizing new habits and routines, then testing and refining these habits and routines through qualitative analysis of videos. The goal of this theoretical contribution is to provide a testable theory of in-the-moment interactions, and provide a means to traverse the holistic research results in order to concretize the work in an actionable way.

Results

In this section, we share first the empirical evidence around instructional strategies to support ELLs. We then put these results in communication with the Mathematical Productive Classroom Framework to concretize the teaching work involved.

Phase 1: Empirical Evidence of Effectiveness for ELLs

Table 2 presents the results of a side-by-side comparison of the variables in separate regression models restricted to ELL and Non-ELL.[2] Table 3 provides a breakdown of the individual elements (measured on a 0-3 scale). The augmented MQI dimension, *Instructional Strategies for Linguistically Diverse Classrooms*, was only a significant variable in the ELL student model validating the hypothesized instructional strategies support ELL students above and beyond *Mathematical Quality of Instruction*. We unpack these results further below.

Table 2: Teacher Knowledge and Quality of Instruction Effect Sizes (Standard Deviations of Gains in Mathematics' Achievement Scores) by ELL/non-ELL Students

Independent Variable	ELL	Non-ELL
Mathematical Knowledge for Teaching	0.03 (0.60)	0.30* (2.40)
Instructional Strategies for Linguistically Diverse Classrooms	0.10* (2.36)	0.16 (1.43)
Mathematical Quality of Instruction	0.07† (1.87)	0.25**(2.74)

† $p < .10$; * $p < .05$; ** $p < .01$

Note: t-statistics in parentheses

Table 3: Mean and Standard Deviation of Teaching Strategies for Linguistically Diverse Classrooms

	Mean (0-3)	Standard Deviation
Connections of mathematics with students' life experiences	0.28	0.27
Connections of mathematics with language	1.32	0.41
Meaning and multiple meanings of words	1.16	0.40
Use of visual aids or support	0.70	0.65
Record of written essential ideas/concepts on the board	2.32	0.39
Discussion of students' mathematical writing	0.97	0.51

As expected, the variable of MQI is positive and significant for both samples, but effects were much larger in the non-ELL sample. If we take the MQI measure as a proxy for 'good teaching', this result may imply that the common phrase 'good teaching benefits *all* students' is supported by this data, with the caveat that 'it more greatly benefits the non-ELL'. It further suggests that the gap of learning opportunities still remains and that teachers' pedagogical actions and moves may not reach students in an equitable manner. Additionally, the measure of MKT was only significant for non-ELL, and with a strong effect size. On the contrary, the only variable that was positive and statistically significant for ELLs, but not for non-ELL, was the overall measure of the instructional strategies for linguistically diverse classrooms (see Table 1 for the list of strategies). An increase of 1 point on the 0-3 point scale corresponds to an estimated gain of 0.1 standard deviation in achievement score. This result may be interpreted as a necessary aspect of instruction (going 'beyond good teaching') to help close the learning opportunity gap. These results imply that effective teachers of ELLs need to provide quality mathematics instruction in general, and that they also need to incorporate teaching routines that attend to providing access to learning opportunities, in particular reinforcing the connections between mathematics representations and their meaning, exploring the meanings of mathematical words and objects through speech and other forms of expression, and making conscientious use of the board or any visual display media so that students have access to pertinent information throughout instruction.

Furthermore, we also verified that the overall score for the augmented dimension of the MQI, *Instructional Strategies for Linguistically Diverse Classrooms* was significantly and positively associated with the overall score of MQI (r = 0.592; p = 0.0002) and with the Mathematical Knowledge for Teaching (MKT) survey (r = 0.311; p = 0.0776). These results reflect that anticipated relationships with outside variables hold, providing validity evidence for the measure.

Phase 2: Theoretically Deconstructing Instructional Strategies into Teacher and Student Interactions

In this section, we share examples of how we have adapted and augmented the Mathematically Productive Classroom Framework to operationalize the above aspects of classrooms. We focus on teaching routines that promote learning opportunities through engagement with mathematical tasks and contexts (*Providing access to mathematical tasks and terminology*), and through engagement with one another's mathematical ideas (*Working with public records of students' mathematical thinking*). The first routine aligns with the instructional strategy *Discussion of Students' Mathematical Writing* from the prior section. The second routine incorporates components of the instructional strategy *Meaning and Multiple Meanings of Words*. The following two excerpts provide brief exchanges to illustrate the interconnected nature of teacher and student interactions, and how specific interactions can facilitate potential learning opportunities. In particular, we focus on key triangles of an extended routines into (*Access to Learning Opportunity* (ALO) routines), embedded with a catalytic teaching habit to support students in a particular habit of mind or interaction (see Figure 1).

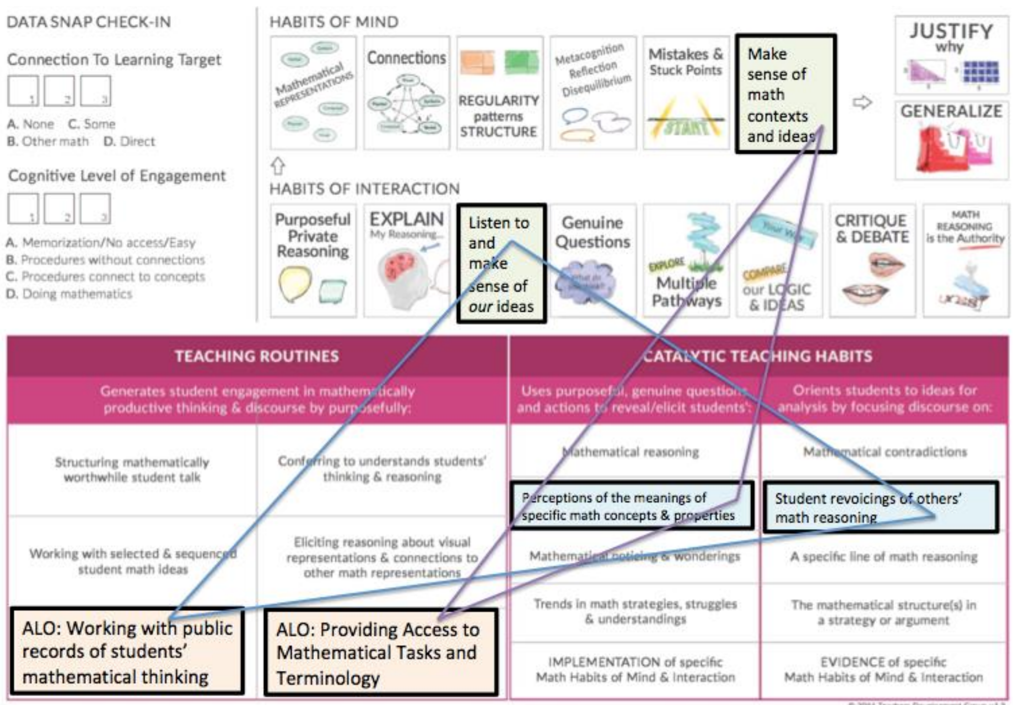

Figure 1: Mathematically Productive Classroom Framework -Connection Between Teaching Routines, Catalytic Teaching Habits, and Student Habits of Mind and Interactions

In this first clip, a pair of 7th grade students from a middle school located in the US southern border is engaged in a task related to the circumference of a circle. However, one of the partners' (identified as ELL by the district) is becoming familiar with words that describe the task.

T: (to S1) Can you explain to her what is circumference? S1: Umm Sure. …	Routine: Access to Learning Opportunities: Providing Access to Mathematical Tasks	CTH: Uses purposeful genuine questions and actions to reveal/elicit students' perceptions of the meanings of specific math concepts & properties.
S1: Well *(Spanish) Circumference is like the area of a circle.* T: *(Spanish) The distance.* S1: *(Spanish) The distance.* T: [inaudible] good. *(Spanish) Did you understand?*		Math Habit: Make sense of mathematical contexts and ideas
S2: *(Spanish) Yes.* T: (to S2) *(Spanish) Show me the circumference of the circle? What did she do?*		CTH: Orients students to ideas for analysis by focusing discourse on student revoicing of others' math reasoning
S2: *(Spanish) [inaudible]* (tracing the circle)		Math Habits: Make sense of mathematical contexts and ideas, Reason with mathematical representations

Figure 2: Student Teacher Interaction on Circumference Analyzed with the Mathematically Productive Classroom Framework

This excerpt (Figure 2) illustrates explicit attention to the mathematical meaning of terms where the teacher spends extended attention (that goes beyond the scope of the brief excerpt above) having students engage with the mathematical context. Within this routine, the teacher prompts for the perception of the meaning of a specific mathematical idea (circumference) (CTH), a student makes sense of this idea (habit of mind), the teacher prompts for revoicing (CTH), and the second student makes sense and reasons with a representation (habit of mind) to illustrate circumference.

P: …yeah- it's greater than- cause it's above the half of 42.	Routine: Access to Learning Opportunities: Working with public records of students' mathematical thinking Terminology	Math Habit: Explain my reasoning
T: So, that was a little bit- that was a lot of information for us. I'm going to ask N to revoice, but before N revoices I'm going to ask one other person to revoice. I want a total of two students to revoice P's thinking. So if what P did made sense to you, you can talk about it... Let's have J revoice, and then N, I'm going to ask you to revoice after J. So J, if you need to stand up here, go on. I actually do want you to, so we can hear you better.		CTH: Orients students to ideas for analysis by focusing discourse on student revoicing of others' math reasoning
J: She said she divided 42 divided by two and she got 21. And since 24 is greater than 21, than it's over- the half. It's greater than half.		Math Habit: Listen to and make sense of our ideas

Figure 3: Student Teacher Interaction on Fraction Comparison Analyzed with the Mathematically Productive Classroom Framework

The second excerpt (Figure 3) comes from a fourth-grade classroom where the teacher has students share their strategies publicly for comparing 24/42 to ½, providing a written record of their ideas. We see Student P's conclusion in the excerpt (preceded by a long explanation). The teacher then leverages a revoicing prompt to engage students in analyzing and making sense of the idea proposed by student P. J then makes sense of P's thinking, through revoicing. As in the prior exchange, teachers and students engaged in interactions within the larger routine, this time the routine was working with public records of students' mathematical thinking' By prompting for revoicing, the teacher engaged in a CTH that catalyzed students in listening and making sense of mathematical ideas (Math Habit of Interaction.)

Due to page limit constraints, we provided shortened excerpts to highlight the primary routines supporting ALO including the pre-existing public records routine, and the newly developed task-focused routine. High quality mathematics classrooms do more than just provide contexts for students to engage in valued mathematical activity (such as justifying and generalizing), but also attend to establishing common ground (e.g., Staples, 2007) where students have opportunities to access the mathematics embedded in tasks and in one another's ideas. We can make sense of classroom interactions through attention to the ways in which routines, catalytic teaching habits, and math habits of mind and interaction co-occur.

Discussion

The work presented above stems from a major effort to bring knowledge of teaching to support ELLs in communication with standard mechanisms of best practices in mathematics teaching. As such, we presented dual prongs of work focused first on empirically establishing that literature-suggested supports do in fact support ELLs in their mathematical learning, and second on putting such work in communication with a framework designed to support researchers in parsing student-teacher interactions and support teachers formatively as they grow in their practice.

We posit that research focused on equitable teaching practices must be both empirically grounded, and operationalized in a way to bridge the research practice divide. Through cycles of empirical exploration, theoretical operationalization, and refinement, we have worked to develop actionable teacher routines and catalytic teaching habits that may remove barriers providing access for ELLs' learning opportunities. We acknowledge that our work is limited to the particular lenses we have adopted and certainly cannot attend to all aspects of mathematical productive classrooms, nor are we exhaustive in the teaching practices to support ELLs. Rather, we focus our work in the context of mathematical reasoning first and what types of instructional routines may engage students deeply in mathematics regardless of language background. We see our work as having implications for research (in the creation of operationalized in-the-moment interaction patterns that can be explored in relation to important outcome variables) and for practice (in the creation of actionable routines and catalytic teaching habits that can be incorporated into practice.)

Endnotes

[1]We use this terminology to align with majority of literature; however, we acknowledge this terminology privileges the English language and has potential deficit connotations.

[2]We choose to only report on selected variables, for effect size on other variables related to teacher preparation and education see Sorto, Wilson, & White (2018).

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PASS THE *IDEA*: FOLLOWING BILINGUAL MATHEMATICS DISCUSSIONS

Gladys Krause
William and Mary
ghkrause@wm.edu

Melissa Adams-Corral
The Ohio State University
adams.2153@buckeyemail.osu.edu

Luz A. Maldonado
Texas State University
l.maldonado@txstate.edu

In this study we document how a mathematical idea, shared by a student, changes the development of the class discussion. Our work characterizes the flexible and open dynamics in mathematics instruction that a teacher adopts to give her students control of their own mathematical ideas. Our work argues that this flexibility is necessary for teachers to engage in instructional practices that respond to students' mathematical thinking. To demonstrate the teacher's flexibility we use the analogy of soccer, relying on the literature of Game Sense coaching approach, which suggests that the coach (the teacher in our context) is positioned to manipulate the practice environment with the purpose of facilitating learning.

Keywords: Elementary Education, Equity & Justice, Standards, Teacher Knowledge

Our article describes a typical day in Ms. Amaris' bilingual second-grade classroom in a southern U.S. public school. We intend to present the characteristics of a fluid and productive mathematics class discussion. Our analysis focuses on how one idea can change the trajectory of an entire class' discussion and the ways that teachers take up new ideas that students propose. We identify the origins of mathematical ideas within a collective discussion and show an idea continuing to develop during an exploratory, one-on-one conversation between Ms. Amaris and a student. Eventually, this conversation becomes the key to expanding the mathematical ideas that the whole class discussed. In order to better explain how one student's mathematical idea can spread throughout a classroom, we have decided to use an analogy that connects to our Latinx culture: soccer. We view the teacher as a coach, with a general plan for the class' discussion—in this case the match. But, as any true soccer fan knows, a soccer match is rich in possibilities and the potential for surprise. In this case we will see how students begin to toss around a more complex idea, passing the ball back and forth. Gabriel, the focal student in our analysis, launches the ball — his mathematical idea— and Ms. Amaris, seeing where her students have taken the game, decides to follow their ideas, allowing the students to control the direction of their class' discussion. This case allows us to see how a mathematics class can support the natural development of mathematical ideas by positioning students as the stars on the field.

In this class, 21 students learned mathematics without any restrictions on their language practices. In other words, students were asked to share ideas in either English or Spanish, based on their preference. Additionally, the students' culture was an important component in daily instruction. In previous investigations in Ms. Amaris' classroom we have argued that allowing students to express their mathematical ideas, even when they were incomplete or could be interpreted as incorrect, allowed for the creation of a space where mathematical ideas could flow and where a teacher could help develop ever stronger mathematical understandings (Maldonado, Krause & Adams, 2018; Carpenter; Fennema; Franke; Levi & Empson, 2015). We have also argued that, for a teacher to support students as they develop stronger conceptual understandings, both the teacher and the students need to understand one another clearly. For this reason, the use of language plays a central role in the practice of teaching and learning mathematics (Maldonado et al., 2018). Based on our prior work, along with the analogy of a soccer match, we will present how Ms. Amaris' students were able to promote a mathematical idea, encouraged by their teacher, making visible how mathematical ideas can be formed collectively through rapid interchanges that could be difficult to trace in the moment. In this class, while discussions began with a set plan, the coach was always ready for a change of tack in order to better meet the game's dynamics. Ms. Amaris was not only focused on the individual passes from player to player, she also had a view of the entire field of play. In this way, through her questioning, she was able to guide students and their emerging ideas until they were finalized as a deep mathematical idea or a rich connection (and, following our analogy, until they ended in a winning goal!). The experience of seeing ideas pop up helped us identify the role the teacher plays in helping her class to solidify their mathematical ideas. Specifically, our analysis presents an answer to the questions: *What does mathematics instruction look like when students' emergent ideas are the central focus? How can teachers support students to develop these ideas?*

Our article contributes to the literature by demonstrating how mathematical ideas develop in a bilingual classroom in the U.S. This article focuses on describing how a teacher observes and participates in order to facilitate productive classroom discussions, where students' mathematical ideas serve to build the class' ever deeper mathematical understanding. We argue that ideas are produced collectively, where individual cognition is embedded within a collective cognition in which all the members of a classroom community participate. This can be seen when one idea, as it is passed to Gabriel, develops in ways that could support the entire class' growing mathematical understandings.

Theoretical Framework

A mathematics class where students have the opportunity to share, justify, connect and extend ideas has been the center of educational reforms in the US in recent years (NCTM, 2014). As a response to this focus, many researchers have conducted studies on how mathematical discussions positively impact students (Moschkovich, 2018). These same studies have shown that including students' mathematical ideas in classroom discussions is a complex task. At the same time, other studies have indicated that students are not benefited equally by discussions. For example, Moschkovich (2018) found that discussions can serve to strengthen certain students' positioning as experts, allowing and even naturalizing, their domination of the class' discussions because those students who are identified as using more advanced strategies tend to share more often. This ensures that students labeled English Language Learners are further disadvantaged due to their minimal participation in class discussions (Adler, 1997; Secada & De La Cruz, 1996).

There are studies that have demonstrated the contributions of bilingual students and the significant role they can play in supporting an entire class' learning (e.g. Turner & Celedón-Pattichis, 2011; Turner, Dominguez, Maldonado & Empson, 2013). Regardless, given the complexity of managing a culturally and linguistically diverse classroom, questions remain around what teachers must do in order to orchestrate productive discussions around rich mathematical content, particularly as bilingual and multicultural contexts become increasingly common across the U.S. Our work is an effort to contribute to an understanding of what needs to take place in bilingual mathematics classroom through an example of how this can be done.

We intend to establish a clear methodology by delineating what Ms. Amaris' practice looked like. In order to do so, we have decided to employ a theoretical framework common in the literature on sports known as the Game Sense coaching approach (GSA). In GSA, the coach uses the game, or the format of the game, as the starting point and continuing focus during training sessions. With GSA, the coach supports players' development of stronger decision-making techniques on the field by encouraging players to understand the game and acquire tactical knowledge and techniques through authentic play (Australian Sports Commission, 1996). GSA argues that the use of games in practice provides players greater opportunities to test out ideas and propose strategies that can be developed through team discussions (Evans & Light, 2008; Light & Evans, 2010). The literature on GSA suggests that the coach should be positioned as a facilitator (Light & Evans, 2010). As a facilitator, the coach guides players to make decisions and solve problems as they arise during games. This "athlete-centered" stance gives players more options and a greater sense of control in the game. The "coach as educator" (Jones, 2006) can adjust the context and conditions during practice in order to facilitate greater learning (Light, 2013). Some work in education has also suggested that the practices of coaches can support teachers (Duncan-Andrade, 2010). Duncan-Andrade (2010) argues that coaches often employ motivation tactics and relationship building in ways that can also benefit the work of teachers.

If we think of lesson planning as a form of preparation for the game, in which there is more than one way to approach the goal, then we would want to avoid linear progressions, instead focusing on the complexity of the field of play—the classroom—and the abundance of possibilities at any given time. It is precisely this abundance that makes it impossible for a teacher to determine all the possible directions that a discussion might take. At the same time, it also would be counter-productive to focus only on what interests the teacher and ignoring the desires of students. For this reason, the role that the teacher assumes is so important—she can read and react to what occurs instantaneously, changing plans to fit the dynamic she encounters (Duncan-Andrade, 2010). We know that mathematics learning is complex and cannot be limited to one sole pathway or trajectory (Empson, 2011). It requires teachers to accept that they are not in control of where the game is headed. Just like in a soccer match, the coach and the players must react to and follow the game in the moment.

Methods

Data

We have worked closely with Ms. Amaris and her students during two consecutive years and have collected extensive data demonstrating her work in the classroom. The data presented in this article is focused on one conversation between Ms. Amaris and Gabriel, one of her 2nd graders. This conversation was video-recorded and transcribed for analysis.

Problem Solving and Number Talks

The episodes from the class that we present in our analysis come from a discussion that generated in response to the problem in Figure 1 and the two equations in Figure 2.

Ms. Amaris typically begins math lessons with a Number Talk (Parrish, 2014; Bray & Maldonado, 2018). This lesson she began with a problem that connected to the class' study of the crisis affecting the water supply in Flint, Michigan (Figure 1). The problem used the context of the class' fundraiser, organized by the students, and asked them to think of how they could get from $38 to $60. After following a lesson sequence common in CGI classrooms (Carpenter, et al., 2015), the class moved to a Number Talk. Ms. Amaris' class had done Number Talks since the beginning of the school year. The students were familiar with this routine, and knew that these discussions focused on their thinking when solving equations and the strategies they used. Ms. Amaris' original plan to see how students used place value knowledge to subtract (Carpenter, et al., 2015).

Analysis

Our analysis involved not only the consideration of what Ms. Amaris and Gabriel said and did, but also of the situation and context within which they were situated. This meant that we included what other students said and did. We identified specific points in the conversation to determine our unit of analysis (Jacobs & Morita, 2002). At times, this unit consisted of an individual comment or question from Ms. Amaris. At other times it included a sequence of comments. Our analysis began with the initial problem (Figure 1). There we identified the mathematical ideas that emerged as well as the linguistic practices of all those involved in the discussion.

Last week we had 38 dollars in the jar of money for Flint. If at the end of this week we have 60 dollars, how much money would we have raised during the week?

Figure 1: Problem to Start Class

62 – 38 = □	65 – 38 = □

Figure 2: Number Talk

Results and Discussion

The Number Talk began with a discussion of the equation *60 - 38 =* □ which emerged from the problem in Figure 1. The purpose of posing this equation was to continue engaging students in using their knowledge of Base-10 and place value in order to compose and decompose numbers and find ways to 'take 8 from 0.' Ms. Amaris starts off asking Emilio for his idea on how to solve the equation and he states that the answer is *22*. Ms. Amaris writes the answer and insists on an explanation for why that was the answer, asking: "¿Cómo sabían que la respuesta era 22?" Many students shared different explanations. As an example:

> Diana: *60 minus 30 equals 30. 8 plus 10 equals, wait, 8 plus 2 equals 10. So then, 8 minus, I mean, 30 minus 8 equals 22.*
> MA: *Okay, so 60 menos, 30 es 30. Luego al 30 le quitas 8 y te quedan 22* [Ms. Amaris scribed on chart paper while repeating in Spanish what Diana shared in English]

After sharing strategies that resulted in *22*, Ms. Amaris continues the Number Talk by writing the following equation: *62 – 38 =* □ (Figura 2). This equation also prompted various strategies. For example:

> Juan Luis: *It's 22, just that you take more, 2 more, 20.*
> MA: *Okay, some people are saying that. Let's see if there's some other ideas, too.*
> Diana: *24.*
> MA: *Ah, I hear two ideas now.*
> Diana: *'Cuz last time we added 71 and the answer was 20, it was, the answer was 22, wait, 24.*
> MA: *I'm gonna put up the two answers that I've heard. I heard Juan Luis say 20. I heard Diana say 24. If you have another idea, go ahead and call it out.*

Ms. Amaris continued the discussion asking for explanations of why they thought the answer might be 20.

MA: *Okay, so háblame de como sacaron 20.*
Kellys: *La respuesta no más de 60 – 38, fue 22 yo no más le quité los 2 del 22 y después el cero.* (The answer from 60 – 38 was 22 and I just took 2 from 22 and then the zero.)
MA: *Okay, voy a escribir lo que me explicaste. 60 – 38 son 22* (OK, I'll write what you explained to me, 60 – 38 was 22) [Ms. Amaris writes 60 - 38 = 22] *le quitaste 2 más. ¿Por qué le quitaste 2 más?* (You took away 2 more, why did you take away 2 more?)
Diana: *It's 20. I thought of it in my head.*
MA: *Why? ¿Por qué le quitas otros 2?* (Why did you take away another 2)
Kellys: *Le quité el 2 y después puse el 60, le quité el 60 el cero del 60* (I took the 2 and then I put the 60, I took the 60 the zero from the 60)
Hugo: *It's gone past.*
MA: *Gone past what?*
Gina: *The 22. If you take away the 2 from the 22.*
MA: *Why?*
Diana: *El sesenta también se puede quitar.* (You can also take away the 60)
SA: *Okay, Carlos is jumping out of his seat. What are you thinking?*
Carlos: *24, because if the 60 minus 38 equals 22, and then you, on that 60- And then, you added 2 more, 'cuz 62.*

The conversation starts with the search for making sense of why the answer is *20*, but the explanations end up justifying why the answer is *24* or *20*. In this case, many students are starting to divert the direction of the game, from place value strategies to pre-algebraic thinking, but they still don't manage to fully shift the course of the discussion Ms. Amaris continues by asking "why one or the other?" Finally, she proposes: "Should we try and prove it?" and the whole class sets to work. Ms. Amaris suggests that they might think of using tens and ones, which they had used the day before. In this moment, students shared many ideas about how to represent *60* using groups of *10* and how to represent the ones, then taking away until finding the answer *24*. After modeling, they decide to prove their answer by adding *38* and *24* to see if it gave them *62*.

Now Ms. Amaris turns to the last equation: *65 – 38* = □. During the discussion of this equation, the game takes the unplanned turn in the direction first suggested by Juan Luis.

Gabriel: *It's just like 62 minus 38, but you add 3 more.*
MA: *3 more where?*
Gabriel: *To the 62.*
MA: *Okay. Gabriel, go ahead and tell us what you're thinking.*
Gabriel: *It's 30-something. It's just like 62 minus 38, but you add 3 more to the 62 and then you should have 65.*
MA: *Okay, so to this one I added 3 and I got the 65. I'm still taking away 38. What do I do to the 24?* [Sra. Amaris escribe la idea de Gabriel (Figure 3)]
Kellys: *24 take away 4?*

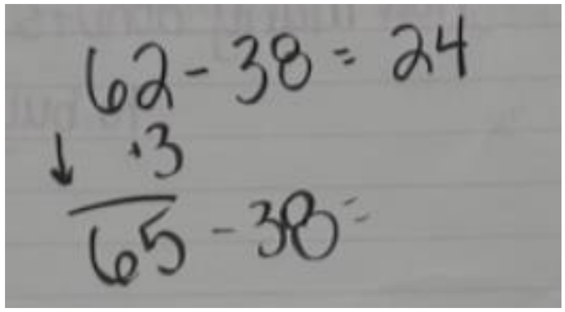

Figure 3: Gabriel's Strategy as Represented by Ms. Amaris

Ms. Amaris' notation (Figure 3) and her questions show she had every intention of getting students to notice how what is done to one side of an equation must also be done to the other. But, the conversation again took another turn and the idea was sidetracked momentarily (think of a ball kicked off course). So, Ms. Amaris decided to give students more time and the freedom to work at their desks in order to find the answer. Meanwhile, Ms. Amaris made her way to Gabriel hoping to better understand his thinking

Gabriel: *If it's 65 minus 38, 'cuz you're taking away 38 and there's only 5 in the 65, and then… And if you're taking away 38 and then you, um, take away the 8 from the 30, then you take away 5, there's gonna be 2 more in the 8.*
MA: *So, ¿lo que te está como atorando es la idea de quitarle 8 cuando solo hay 5 en las unidades?* (So, what's bothering you is the idea of taking away 8 when you only have 5 ones?)
Gabriel: *Hm…*
MA: *Oh, okay. Pues ¿cómo lo haríamos?¿Hay algún otro lugar donde le podríamos quitar?* (Well, how could we do it? Is there somewhere else where we could take from?)
Gabriel: [Nodded, pointing at a stack of ten cubes]
MA: *Ah! Okay, vamos a quitarle este 8.* (Oh! Ok, let's take away that 8, then)
Gabriel: *two, four, six, eight.*
MA: *Okay, ¿qué nos quedó?* (Ok, so what do we have left?)
Gabriel: *10, 20, 30, 40, 50. 50. 50, 55, 56, 57. 57.*
MA: *Okay, diez, veinte, treinta, cuarenta, cincuenta, cincuenta y uno, cincuenta y dos, cincuenta y tres, cincuenta y cuatro, cincuenta y cinco, cincuenta y seis, cincuenta y siete. So, lo que nosotros encontramos es que, para quitarle el ocho, tuvimos que entrarnos a uno de los dieces. ¿Qué te faltó quitar? Porque sólo le quitaste 8.* (Okay, ten, twenty, thirty, forty, fifty, fifty-one, fifty-two, fifty-three, fifty-four, fifty-five, fifty-six, fift-seven. So, what we found was that to take way the 8 we had to get inside one of the tens.What's left to take away? Because you've only taken away 8 so far.)
Gabriel: *30 ...*
MA: *¿Ves una manera mas fácil de quitar 30?* (Do you see an easy way to take away 30?)
Gabriel: *Should I just take away these?* [Pointing to 3 groups of 10 cubes]
MA: *That sounds easier, right?*
Gabriel: *Yeah.*
MA: *How many did you take away?*
Gabriel: *30. The answer is twenty-seven.*

The mathematical details in Gabriel’s strategy are important. He takes *8* from a *10* and then takes *30* more from the *5* tens that were left. This flexibility, in both his thinking and his sense of number (eg. seeing *65* as a group of *6* groups of ten and *5* ones) allowed him to see how he could take *8* from *10* instead of *5*, which had at first struck him as being difficult. Once this difficulty was noted and tackled with Ms. Amaris’ support, she kept talking with Gabriel. At this point, she suggests that they go and look at the equation that the class had been trying to solve (Figure 3).

> SA: *¿Hay alguna relación entre 24 y 27? So, tú me habías dicho: 62 más 3 es 65. ¿Qué es 24, si le sumamos otros 3?* (Is there some relationship between 24 and 27? So, you had said: 62 plus 3 is 65. What’s 24, if we add another 3?)
> Gabriel: *27*
> SA: *27. So, si hubiéramos ido con tu idea de hacer como una balanza entre lo que teníamos y lo que acabamos de hacer… Tu dijiste, aquí le sumamos 3, pues aquí también le podemos sumar 3. Y eso te dio la respuesta que sacaste, ¿no?* (So, if we had kept going with your idea of making like a balance between what we had before and what we just did…You said, here we added 3. Well, here we can also add 2. And that gave you the same answer that you just found, right?)

Ms. Amaris was able to identify an opportunity to return to the initial idea that Gabriel had shared. At first, he thought of $62 - 38 = \square$ and how, if you added *3* to *62* you would get *65*, but then he got stuck and was not able to continue. Ms. Amaris saw the opportunity to broaden the scope of his thinking by returning to his initial idea after he had solved the problem. In Figure 3, we see how Ms. Amaris uses the idea of a scale to find an equilibrium between both sides of the equation that Gabriel had attempted to solve. Ms. Amaris tried to support Gabriel in recognizing that the problem the class had, $62 – 38=\square$, was very nearly the problem they were attempting now, $65 – 38=\square$. In keeping with the techniques of GSA, the coach uses the game itself during training sessions in order to develop the technical understanding necessary to better control the game’s dynamics. In this case, Ms. Amaris saw that Gabriel could take the discussion further than where she had planned for it to go, but it didn’t quite get there. So, once she had sent students back to their desks, Ms. Amaris improvised a strategy, where her focus was to develop Gabriel’s idea giving him the opportunity to prepare to share with the class in future discussions.

These moments from a portion of a class session are just one example of how Ms. Amaris used an open and fluid dynamic to cede control of the discussion to her students, allowing them to determine the ideas that would be shared and how they might grow. In this example, we see how the idea of the balance starts with Juan Luis, Kellys and Gina, is picked up by Carlos and, eventually, is passed along to Gabriel. The idea never belonged to only one student, nor were Juan Luis or Kellys’ initial attempts a failure. As the idea was generated collectively, it continued to grow as more students participated. By stopping to focus on Gabriel, Ms. Amaris responded to the ideas that emerged from the group, shifting from her initial focus on base-10 concepts to fundamental concepts of pre-algebra.

Conclusions

Throughout this article we show Ms. Amaris’ flexibility in determining the content of her class discussions, an example that we feel justifies an argument for curricula, including grade level standards, to be similarly flexible. For example, Ms. Amaris’ initial goal was basically regrouping, using base-10 concepts and place value. However, Juan Luis, Kellys, Carlos and Gabriel's idea took the class to a new discussion on "balancing" an equation. In allowing the students to guide the direction of the discussion, letting them determine which plays to follow, we can see how Ms. Amaris' class not only met the initial objective of solving the equation, but also went further than what their teacher had planned. The idea of "balancing" the equation goes far beyond the curricular standards established as content to cover in second grade. According to Principles and Standards for School Mathematics (NCTM, 2000) this idea only begins to develop in third grade. Additionally, in classrooms with linguistically diverse students, who are traditionally silenced during mathematical discussions (Adler, 1997; Secada & De La Cruz, 1996), it is of utmost importance that teachers recognize that the control they impose on students' mathematical ideas and on the language in which ideas can be shared also impose mathematical silence. In other words, the more control we place on what is welcome in our discussions, the more ideas may be left out.

To conclude, based on our work this far, we pose the following questions for our field to consider in future investigations: Who decides what is taught each day in any given mathematics classroom? If we truly wish to promote the centering of students' mathematical ideas in mathematics instruction, what role do those ideas play in deciding what should be taught each day? And, if we are working to see our Latinx students positioned as true knowers of mathematics, how can we ensure that their classrooms are really theirs to direct and inspire? Perhaps, to start, we can play a little soccer, as a model of a collective dynamic that inspires through cooperation.

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PÁSAME LA *IDEA*: SIGUIENDO DISCUSIONES MATEMÁTICAS BILINGÜES

Gladys Krause
William and Mary
ghkrause@wm.edu

Melissa Adams-Corral
The Ohio State University
adams.2153@buckeyemail.osu.edu

Luz A. Maldonado
Texas State University
l.maldonado@txstate.edu

En este estudio documentamos cómo una idea matemática, compartida por un estudiante, cambia el desarrollo de la discusión de la clase. Nuestro trabajo caracteriza la dinámica flexible y abierta en la instrucción de matemáticas que una maestra adopta para dar a sus estudiantes control de sus propias ideas matemáticas. Nuestro trabajo argumenta que esta flexibilidad es necesaria para que los maestros se involucren en prácticas de instrucción que responden al pensamiento matemático de los estudiantes. Para demostrar la flexibilidad de la maestra usamos la analogía del fútbol, apoyándonos en la literatura de Enfoque Comprensivo del Sentido de Juego, la cual sugiere que el entrenador (el maestro en nuestro contexto) se posicione como un facilitador para manipular el entorno de práctica con el propósito de facilitar el aprendizaje.

Keywords: Educación Primaria, Equidad y Justicia, Estándares, Conocimiento del Profesor

Nuestro artículo describe una investigación desarrollada en un salón de segundo grado bilingüe en una escuela pública en el sur de los Estados Unidos. El objetivo de nuestro trabajo es delinear las características de una discusión productiva y elocuente durante la instrucción de matemáticas. Principalmente nos enfocamos en el análisis de cómo una idea puede cambiar el desarrollo de una discusión, y cómo los maestros pueden seguir las ideas nuevas que proponen sus estudiantes. Nuestro artículo identifica el origen de las ideas matemáticas durante una discusión colectiva, y cómo las ideas continúan desarrollándose en una pequeña conversación entre la maestra y un estudiante. Eventualmente esta conversación se convierte en la clave para expandir las ideas matemáticas discutidas por la clase. Para intentar explicar cómo la idea de un solo estudiante se propaga por todo el salón, hemos decidido usar una analogía muy propia a nuestra cultura Latina, el fútbol. Así la maestra, como una entrenadora técnica, tiene un plan general para la discusión — en este caso, el partido — pero, como los aficionados bien conocen, en un partido de fútbol lo que sobra son posibilidades. En este caso, veremos como los estudiantes proponen una idea más compleja, pasando la idea como la pelota. Gabriel, el estudiante en nuestro análisis, tira la pelota — su idea matemática — y Sra. Amaris, la maestra en nuestro análisis, viendo adónde sus estudiantes han llevado el partido, decide seguir esta idea, dándoles libertad a los estudiantes de tener control del partido. En este caso, lo que pudimos ver es cómo una clase de matemáticas apoya el desarrollo natural de las ideas matemáticas posicionando a los mismos estudiantes como las estrellas del campo.

En esta clase 21 estudiantes aprenden matemáticas sin restricciones de lenguaje. La cultura de los estudiantes es un componente importante en su aprendizaje diario. En investigaciones anteriores, basados en el aula de la Sra. Amaris, hemos argumentado que el permitir que los estudiantes expresen sus ideas matemáticas, incluso cuando aún no están completas o son temporalmente incorrectas, permite que se abra un espacio para que las ideas matemáticas fluyan y también se abra un espacio para que la maestra pueda ayudar a construir un conocimiento aún más sólido de las matemáticas (Maldonado, Krause & Adams, 2018; Carpenter; Fennema; Franke; Levi & Empson, 2015). Hemos argumentado también que esta construcción de conocimiento se hace posible cuando la comunicación entre la maestra y los estudiantes es clara (para los dos). Por lo tanto, el uso del lenguaje juega un papel central en el aprendizaje y la enseñanza de las matemáticas (Maldonado et al., 2018). Usando como base nuestro trabajo anterior, y usando la analogía del fútbol, presentamos aquí como los estudiantes de la Sra. Amaris son capaces de poner en marcha una idea matemática que la Sra. Amaris apoya con ánimo, fomentando y haciendo visible cómo las ideas matemáticas se forman de manera colectiva a través de intercambios rápidos y a veces difíciles de trazar. En esta clase, aunque las discusiones inician con un plan trazado previamente, la directora técnica está lista para cambiar su estrategia en cualquier instante siguiendo la corriente del partido. La Sra. Amaris no solamente se enfoca en los pasos de jugador a jugador, sino tiene la vista en la cancha entera. De esta manera, por medio de sus preguntas, guía a los estudiantes con sus ideas matemáticas emergentes hasta que finalizan en una idea o conexión matemática profunda (y tal vez, siguiendo nuestra analogía, podríamos decir, ¡hasta que terminan en un victorioso gol!). La experiencia de ver surgir las ideas nos ayuda a identificar el papel que juega la maestra en ayudar a la clase a solidificar la idea matemática. Específicamente nuestro análisis presenta una respuesta a las preguntas: *¿Cómo se ve la instrucción de matemáticas donde las ideas emergentes de los estudiantes determinan el enfoque de la clase? ¿Cómo pueden los maestros apoyar la evolución de estas ideas?*

Nuestro artículo aporta a la literatura demostrando cómo se desarrollan las ideas matemáticas en un salón bilingüe, específicamente en el contexto educativo de los Estados Unidos. Nuestro artículo se enfoca en describir cómo una maestra observa y participa para facilitar discusiones productivas, en dónde las ideas matemáticas de los estudiantes se usan para construir una idea matemática más avanzada. En argumentar que estas ideas se producen de manera colectiva, argumentamos que la cognición individual es parte de una cognición colectiva en que participan y comparten todos los miembros del salón. Esto se puede ver cuando una idea, pasada a Gabriel, se convierte en la idea que servirá para apoyar a la clase entera.

Marco Teórico

El intercambio de ideas durante la clase de matemáticas dónde los estudiantes tienen oportunidades de compartir, justificar, conectar y extender ideas ha sido el centro de reformas educativas en los últimos años en los Estados Unidos (NCTM, 2014). Siguiendo este enfoque varios investigadores han realizado estudios sobre discusiones matemáticas demostrando un impacto positivo en el aprendizaje de los estudiantes (Moschkovich, 2018). Estos mismos estudios han demostrado que incluir las ideas matemáticas de los estudiantes a través de la discusión en clase de matemáticas no es una práctica fácil. Al mismo tiempo otros estudios han indicado que no todos los estudiantes se benefician de igual manera durante las discusiones en la clase de matemáticas. Por ejemplo, Moschkovich (2018) encontró que muchas veces estas discusiones sólo consiguen afianzar la posición de expertos con la que algunos estudiantes se identifican y usan a diario para dominar las discusiones. También aquellos estudiantes que se identifican por usar estrategias avanzadas tienden a compartir sus estrategias más frecuentemente. Estas desventajas son aún más marcadas en aquellos estudiantes cuya primera lengua no es inglés porque tienden a participar de una manera mínima en estas discusiones (Adler, 1997; Secada & De La Cruz, 1996).

Muy pocos estudios se han enfocado en demostrar las contribuciones de estudiantes bilingües y sus aportes significativos al aprendizaje de toda la clase (por ejemplo Turner & Celedón-Pattichis, 2011; Turner, Dominguez, Maldonado & Empson, 2013). Sin embargo, dada la complejidad de manejar un salón de clase con una amplia diversidad cultural y lingüística, todavía existen preguntas sobre cómo, y qué se requiere de las maestras para, orquestar una discusión productiva y rica en contenido matemático, especialmente en el contexto bilingüe y multicultural característico de los Estados Unidos. Nuestro trabajo es un esfuerzo para contribuir a entender qué se necesita y para dar un ejemplo de cómo se puede hacer.

En un intento por establecer una metodología clara que delinee cómo se ve la práctica de la Sra. Amaris, hemos decido usar un marco teórico común en la literatura deportiva conocida como Enfoque Comprensivo del Sentido de Juego (Game Sense coaching approach). En GSA (por su sigla en inglés) el entrenador usa el juego o la forma de juego como el punto de partida y el enfoque continuo en la sesión de entrenamiento. Además, a través de GSA el entrenador anima a los jugadores a comprender el juego y a adquirir conciencia táctica y técnica dentro del contexto del juego para ayudar a desarrollar una mejor toma de decisiones cuando juegan (Australian Sports Commission, 1996). GSA argumenta que el uso de esta práctica permite que los jugadores tengan más oportunidades de probar ideas y proponer estrategias que se desarrollan mediante la discusión entre ellos y el entrenador (Evans & Light, 2008; Light & Evans, 2010). La literatura de GSA sugiere que el entrenador se posicione como un facilitador (Light & Evans, 2010). Como facilitador, el entrenador guía al jugador en la toma de decisiones para resolver los problemas del juego. Esta postura educativa "centrada en el atleta" hace que los jugadores tengan más opciones y control sobre el juego. El entrenador como educador (Jones, 2006) puede manipular el entorno de práctica para estructurar y facilitar el aprendizaje (Light, 2013). La literatura en educación también tiene precedente en ver a la maestra como una entrenadora (Duncan-Andrade, 2010). Duncan-Andrade (2010) argumenta que la enseñanza requiere tácticas de motivación que deben ser parte de la formación de cualquier entrenador técnico.

Si vemos la planeación de lecciones como una preparación para el juego, en donde no hay una sola manera de llegar al arco, en lugar de enfocarnos en un progreso lineal, podremos ver la complejidad del campo — el salón de clase — y la abundancia de posibilidades. Por esta abundancia, precisamente, no es posible para una maestra determinar todos los caminos que una discusión puede tomar. Al mismo tiempo no sería muy productivo enfocarse sólo en lo que le interesa a la maestra. Por esta razón el papel de la maestra es fundamental. Ella puede leer y reaccionar a lo que sucede instantáneamente, cambiando de plan de acuerdo con la dinámica que encuentra (Duncan-Andrade, 2010). El aprendizaje matemático es complejo, y no se puede limitar a un solo camino o trayectoria (Empson, 2011). Requiere que la maestra acepte el no controlar la dirección del juego. Igual con un partido de fútbol, el entrenador y los jugadores deben reaccionar y seguir el juego en el momento.

Métodos

Datos

Durante dos años consecutivos hemos trabajado muy de cerca con la Sra. Amaris y sus alumnos y hemos recolectado datos extensos de su trabajo en el aula. El trabajo que presentamos en este artículo se centra en un pasaje de una conversación entre la Sra. Amaris y Gabriel, uno de sus estudiantes del 2do grado. El pasaje fue grabado en video y transcrito para su análisis.

Problemas y Conversaciones Numéricas

Los episodios de la clase que presentaremos en nuestro análisis vienen de la discusión que generaron el problema en la Figura 1 y las dos ecuaciones en la Figura 2.

La Sra. Amaris suele comenzar sus lecciones de matemáticas con una conversación numérica (Number Talks en inglés) (Parrish, 2014; Bray & Maldonado, 2018). En esta lección comenzó la clase con un problema basado en la crisis de contaminación de agua en Flint, Michigan (Figura 1). El problema usó el contexto de un recaudo de fondos que los niños organizaron y pidió que los niños pensaran en cómo llegar de $38 a $60. Después de seguir una secuencia de lección común en clases de CGI (Carpenter, et al., 2015), pasaron a una conversación numérica.

La semana pasada recolectamos 38 dólares en la jarra para mandar dinero a Flint. Si después de una semana tenemos 60 dólares almacenados, ¿cuánto dinero recolectamos durante la semana?

Figura 1: Problema para iniciar clase

62 – 38 = □	65 – 38 = □

Figura 2: Conversaciones Numéricas discutidas después del problema inicial

La conversación numérica es una práctica pedagógica que la Sra. Amaris ha usado desde el principio del año escolar. Los estudiantes ya familiarizados con esta rutina sabían que estas discusiones se centraban en la manera de pensar acerca de ecuaciones y las estrategias utilizadas para resolverlas. Estas ecuaciones se seleccionaron con el propósito inicial de la Sra. Amaris de ver como los niños usan valor posicional (Carpenter, et al., 2015).

Análisis

Nuestro análisis involucró no sólo una consideración de lo que Sra. Amaris y Gabriel dijeron e hicieron, sino también la situación y el contexto, incluyendo lo que todos los otros niños dijeron e hicieron. Identificamos puntos específicos en la conversación para determinar nuestra unidad de análisis (Jacobs & Morita, 2002). A veces la unidad consistía en el comentario o pregunta individual de la Sra. Amaris, y otras veces incluía una secuencia de comentarios y preguntas. Nuestro análisis comenzó con el problema inicial (Figura 1). Ahí identificamos las ideas matemáticas que surgieron y las prácticas lingüísticas de todos los involucrados en la discusión.

Resultados y Discusión

La conversación numérica empieza con la discusión de la ecuación *60 - 38 =* □ del problema en la Figura 1. El propósito de la ecuación era continuar trabajando en la instrucción de Base-10 y valor posicional, de tal manera que los estudiantes pudieran componer y descomponer los números para "quitar 8 al 0". La Sra. Amaris comenzó preguntándole a Emilio sus ideas para resolver la ecuación y él dijo que la respuesta era *22*. La Sra. Amaris escribe la respuesta e insiste en una explicación acerca de por qué esa respuesta: "¿Cómo sabían que la respuesta era 22?" Varios estudiantes compartieron explicaciones diferentes. Por ejemplo:

> Diana: *60 minus 30 equals 30. 8 plus 10 equals, wait, 8 plus 2 equals 10. So then, 8 minus, I mean, 30 minus 8 equals 22.*
> SA: *Okay, so 60 menos, 30 es 30. Luego al 30 le quitas 8 y te quedan 22* [La Sra. Amaris escribió todo en el tablero a medida que repitió en español lo que Diana dijo en inglés]

Después de compartir las estrategias que resultan en *22*, la Sra. Amaris pasó a escribir una ecuación, *62 – 38 =* □ (Figura 2), para continuar con las conversaciones numéricas. De la misma manera esta ecuación provocó varias ideas. Por ejemplo:

> Juan Luis: *It's 22, just that you take more, 2 more, 20.*
> SA: *Okay, some people are saying that. Let's see if there's some other ideas, too.*
> Diana: *24.*
> SA: *Ah, I hear two ideas now.*
> Diana: *'Cuz last time we added 71 and the answer was 20, it was, the answer was 22, wait, 24.*
> SA: *I'm gonna put up the two answers that I've heard. I heard Juan Luis say 20. I heard Diana say 24. If you have another idea, go ahead and call it out.*

La Sra. Amaris continua la discusión pidiendo explicaciones sobre ¿Por qué *20*?

> SA: *Okay, so háblame de como sacaron 20.*
> Kellys: *La respuesta no más de 60 – 38, fue 22 yo no más le quité los 2 del 22 y después el cero.*
> SA: *Okay, voy a escribir lo que me explicaste. 60 – 38 son 22* [La Sra. Amaris escribe 60 - 38 = 22] *le quitaste 2 más. ¿Por qué le quitaste 2 más?*
> Diana: *It's 20. I thought of it in my head.*
> SA: *Why? ¿Por qué le quitas otros 2?*
> Kellys: *Le quité el 2 y después puse el 60, le quité el 60 el cero del 60*
> Hugo: *It's gone past.*
> SA: *Gone past what?*
> Gina: *The 22. If you take away the 2 from the 22.*
> SA: *Why?*
> Diana: *El sesenta también se puede quitar.*
> SA: *Okay, Carlos is jumping out of his seat. What are you thinking?*
> Carlos: *24, because if the 60 minus 38 equals 22, and then you, on that 60- And then, you added 2 more, 'cuz 62.*

La conversación inicia con la búsqueda de ¿por qué es *20*? y las explicaciones terminan justificando porque la respuesta es *24* ó *20*. En este caso, Carlos y Kellys están empezando a dirigir el partido en una dirección inesperada, pero todavía no logran cambiar la discusión. La Sra. Amaris continúa preguntando ¿por qué la una o la otra? Luego termina por proponer: "Should we try and prove it?" Y la clase entera se embarca en la tarea. La Sra. Amaris propone que usen las decenas y unidades de las que habían hablado el día anterior. En ese momento los estudiantes comparten varias ideas sobre cómo representar *60* usando grupos de *10* y cómo representar las unidades y quitando hasta encontrar la respuesta 24. Después de modelarlo lo deciden comprobar con una suma de *38* y *24* para ver si les salía el *62*.

Finalmente La Sra. Amaris propone la última ecuación *65 – 38* = □. Durante la discusión de esta ecuación el partido toma un giro fuera de lo planeado, va en la dirección inicialmente sugerida por Juan Luis.

Gabriel: *It's just like 62 minus 38, but you add 3 more.*
SA: *3 more where?*
Gabriel: *To the 62.*
SA: *Okay. Gabriel, go ahead and tell us what you're thinking.*
Gabriel: *It's 30-something. It's just like 62 minus 38, but you add 3 more to the 62 and then you should have 65.*
SA: *Okay, so to this one I added 3 and I got the 65. I'm still taking away 38. What do I do to the 24?* [Sra. Amaris escribe la idea de Gabriel (Figure 3)]
Kellys: *24 take away 4?*

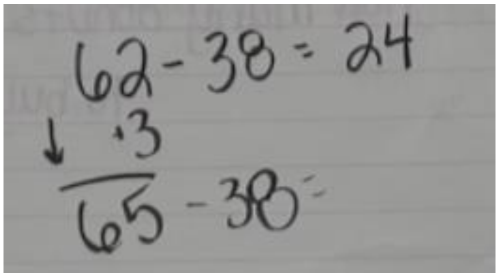

Figura 3: Idea de Gabriel representada por la Sra. Amaris

La notación de la Sra. Amaris (Figura 3) y sus preguntas tenían toda la intención de hacer que los estudiantes pudieran ver que lo mismo que se hace a un lado de la ecuación se debe hacer al otro lado. Pero la conversación tomó un giro diferente, la pelota se desvió. La Sra. Amaris decidió darles más tiempo y la libertad de trabajar en sus escritorios para encontrar la respuesta. Mientras eso pasaba, la Sra. Amaris se acerca a Gabriel tratando de entender un poco más su manera de pensar.

Gabriel: *If it's 65 minus 38, 'cuz you're taking away 38 and there's only 5 in the 65, and then… And if you're taking away 38 and then you, um, take away the 8 from the 30, then you take away 5, there's gonna be 2 more in the 8.*
SA: *So, ¿lo que te está como atorando es la idea de quitarle 8 cuando solo hay 5 en las unidades?*
Gabriel: *Hm…*

SA: *Oh, okay. Pues ¿cómo lo haríamos?¿Hay algún otro lugar donde le podríamos quitar?*
Gabriel: [Asintió, apuntando a un grupo de 10 cubos]
SA: *Ah! Okay, vamos a quitarle este 8.*
Gabriel: *two, four, six, eight.*
SA: *Okay, ¿qué nos quedó?*
Gabriel: *10, 20, 30, 40, 50. 50. 50, 55, 56, 57. 57.*
SA: *Okay, diez, veinte, treinta, cuarenta, cincuenta, cincuenta y uno, cincuenta y dos, cincuenta y tres, cincuenta y cuatro, cincuenta y cinco, cincuenta y seis, cincuenta y siete. So, lo que nosotros encontramos es que, para quitarle el ocho, tuvimos que entrarnos a uno de los dieces. ¿Qué te faltó quitar? Porque sólo le quitaste 8.*
Gabriel: *30 ...*
SA: *¿Ves una manera mas fácil de quitar 30?*
Gabriel: *Should I just take away these?* [Apuntando a 3 grupos de 10 cubos]
SA: *That sounds easier, right?*
Gabriel: *Yeah.*
SA: *How many did you take away?*
Gabriel: *30. The answer is twenty-seven.*

Los detalles matemáticos en la estrategia de Gabriel son importantes. Quita *8* de un *10*, y luego quita *30* más de las *5* decenas que quedaron. La flexibilidad en su forma de pensar y su sentido numérico (es decir, ver a *65* cómo un grupo de *6* decenas y *5* unidades) le permite ver cómo quitar *8* de *10* en lugar de *5*, lo que él había calificado como difícil. Una vez que esta dificultad fue notada y abordada por Sra. Amaris, ella continuó hablando con Gabriel. Esta vez ella propone ir y ver la ecuación que estaban resolviendo antes como clase (Figura 3).

SA:*¿Hay alguna relación entre 24 y 27? So, tú me habías dicho: 62 más 3 es 65. ¿Qué es 24, si le sumamos otros 3?*
Gabriel: *27*
SA: *27. So, si hubiéramos ido con tu idea de hacer como una balanza entre lo que teníamos y lo que acabamos de hacer… Tu dijiste, aquí le sumamos 3, pues aquí también le podemos sumar 3. Y eso te dio la respuesta que sacaste, ¿no?*

La Sra. Amaris pudo identificar una oportunidad para volver a las ideas iniciales compartidas por Gabriel. Al principio pensó en $62 - 38 = \square$ y que si sumaba *3* a *62* obtendría *65*, pero luego se atoró y no pudo continuar. La Sra. Amaris vio la oportunidad de ampliar su pensamiento volviendo a su idea inicial después de haber resuelto el problema. En la Figura 3 vemos como la Sra. Amaris usa la idea de una escala para equilibrar ambos lados de la ecuación en la que Gabriel estaba pensando. La Sra. Amaris trataba de ayudar a Gabriel a darse cuenta de que el problema que él realmente resolvió, $62 – 38=\square$, era casi el problema que quería resolver, $65 – 38=\square$. Como en las técnicas de GSA, el entrenador usa el juego durante entrenamientos para desarrollar la conciencia táctica necesaria para controlar mejor la dinámica del juego. En este caso la Sra. Amaris vio que Gabriel podría llevar a la discusión de la clase una idea más allá de lo que ella había planeado, pero no se logró. Pero al enviar a los niños a trabajar en sus mesas, la Sra. Amaris improvisó una estrategia, en dónde su enfoque era desarrollar la idea de Gabriel y darle la oportunidad de prepararse para compartirla con la clase en futuras discusiones.

Este fragmento de la clase es un ejemplo de como la Sra. Amaris usa una dinámica flexible y abierta para ir pasándoles el control a sus estudiantes, a que ellos determinen y controlen las ideas que logran fluir y ser compartidas. En este ejemplo podemos ver la idea del balance empezando con Juan Luis, Kellys y Gina y luego pasando a Gabriel. Esta idea no le pertenece a un solo estudiante, es una idea generada en grupo que continúa creciendo. Al parar y enfocarse en Gabriel, la Sra. Amaris respondió a las ideas que surgían (como grupo), cambiando el enfoque inicial, base-10, para discutir conceptos básicos de pre-algebra.

Conclusiones

A través de nuestro artículo hemos demostrado como la flexibilidad de la Sra. Amaris para determinar el contenido de la clase justifica el argumento de que el currículo debe ser flexible de la misma manera. Por ejemplo, el punto inicial de la Sra. Amaris era básicamente reagrupar, usando conceptos de Base-10 y valor posicional. Sin embargo las ideas de Juan Luis, Kellys, Carlos y Gabriel los llevó también a hablar sobre "balancear" una ecuación. En permitir que los estudiantes guiaran la conversación, dejándolos determinar las jugadas a seguir, pudimos ver que la clase de matemáticas de la Sra. Amaris no sólo cumplió con el objetivo inicial de solucionar la ecuación, pero también llevó la discusión aún más allá de lo que la Sra. Amaris había planeado. La idea de "balancear" la ecuación va mucho más allá de lo que los estándares curriculares establecen como contenido que se debe cubrir en segundo grado. De acuerdo con Principles and Standards for School Mathemtics (NCTM, 2000) esta idea empieza a desarrollarse en el tercer grado.

Adicionalmente, en salones de clase con estudiantes lingüísticamente diversos, quienes tradicionalmente se ven silenciados en discusiones matemáticas (Adler, 1997; Secada & De La Cruz, 1996), es de mayor importancia que las maestras reconozcan que el control que imponen a las ideas matemáticas de los estudiantes, y que el idioma en que las ideas son compartidas impone igualmente un silencio matemático.

Para finalizar, basado en el trabajo que hemos expuesto hasta este punto, queremos dejar algunas preguntas al campo de investigación de educación de matemáticas como punto de partida para futuras investigaciones: ¿Quién determina qué se debe enseñar cada día en el aula de clase? Si en realidad estamos proponiendo usar las ideas matemáticas de los estudiantes durante la instrucción de matemáticas, ¿qué papel juegan las ideas matemáticas de los estudiantes en lo que se debe enseñar diariamente? Si estamos trabajando verdaderamente para ver a nuestros estudiantes Latinx posicionándose como verdaderos conocedores matemáticos, ¿cómo podemos asegurar que los salones de clase sean suyos para dirigir e inspirar? A lo mejor, para empezar, podemos jugar al fútbol, pero cómo un modelo de dinámica colectivo que promueva cooperación e inspiración.

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CO-CONSTRUCTING "QUIET" THROUGH PEER INTERACTIONS: UNDERSTANDING "QUIET" PARTICIPATION IN A SMALL-GROUP MATH TASK

Heather Fink
University of California, Berkeley
hfink@berkeley.edu

To disrupt patterns of marginalization that play out through interactions in math classrooms, teachers need to identify and address inequities in student participation, both in terms of participation outcomes and processes. In this study, I take an expansive view of participation and examine how the "quiet" participation of one 9th grade student is co-constructed through small-group interactions during an Algebra task. Analysis reveals three features of the group's interactions that fostered the co-construction of Becca's "quiet" participation: 1. Becca was positioned as a non-contributing silent beneficiary of learning, 2. Becca's contributions received less support than her peers', 3. Disagreement with Becca was softer than with John. Findings suggest that the perceived issue of low verbal production did not reside within Becca, but rather was the result of inequitable participation processes that played out through peer interactions.

Keywords: Classroom Discourse, Equity and Diversity

Educational inequities come in all shapes and sizes and are enacted through classroom interactions in many different ways. Institutional level inequities connected to race, gender, and socioeconomics undoubtedly shape students' mathematical experiences (Esmonde & Langer-Osuna, 2013; Herbel-Eisenmann, Choppin, Wagner & Pimm, 2011). While teachers alone cannot fix these historical injustices, they *can* disrupt patterns of marginalization by supporting more equitable interactions among students (Boaler & Staples, 2008). In this paper, I focus on equity as it relates to student participation in a math task. I examine how the participation of one student, labeled as "quiet" by her math teacher, is co-constructed through group interactions.

Labeling students based on behavioral patterns is common and done often with good intentions. However, labeling can be harmful if behavioral tendencies are seen as inherent characteristics of students (Gutierrez & Rogoff, 2003). By accepting that some student voices will inevitably take up more space than others, we run the risk of endorsing different participation expectations for different students, leading to inequitable learning opportunities (Hand, 2012). In addition, labels can position certain students as being deficient or inferior, since the labels themselves are typically not neutral (McDermott, 2010). Being "quiet" in a math class is often associated with less-than-desirable traits, such as being timid, less confident, and less knowledgeable. Instead, if we acknowledge that "quiet" tendencies are a byproduct of human interactions within learning environments (Cole, 1998; Lave, 1996), we can then address participation disparities through interactional interventions. The goal is not to *fix* individual students, but rather to support more equitable interactions among students. To do this, we need a better understanding of how students labeled as "quiet" participate through interaction, and we must be careful not to assume all "quiet" students share the same backgrounds or have the same strengths or needs (Gutierrez & Rogoff, 2003).

This study explores the interactions of four 9th grade students during a small-group math task. I pay particular attention to Becca, a student described as "quiet" by her math teacher. Research questions are: 1) *What did participation look like for a "quiet" student working on a small-group math task?* 2) *How was "quiet" participation co-constructed through peer interactions?*

Theoretical Framework

My work draws on sociocultural and situated theories that claim learning happens through participation in cultural activities (Lave & Wenger, 1991; Rogoff, 1990; Vygotsky, 1978; Wenger, 1998), considering language and discursive practices as central to developmental processes (Lerman, 2001). In this study, my perspectives on equity, learning, and participation are based on those described by Esmonde (2009). Esmonde asserts that "in the context of cooperative group work and participation in mathematical practices, equity can be defined as a fair distribution of opportunities to learn or opportunities to participate" (p. 1010). She views mathematical learning as referencing both the development of content-related understandings *and* the development of productive positional identities. Esmonde takes a broad view of participation, noting that "Although talk is a valued form of participation in many mathematics classrooms, there may be other valuable forms of participation that are less visible" (p. 1033).

One key part of combatting participation inequities is understanding how opportunities for participation are constructed and taken up (or not) through classroom interactions, since equity is both a *goal* and a *process* (Martin, 2003). If opportunities for participation are unfair, then it is reasonable to assume participation (the process through which students learn) will be inequitable as well, since a person's participation in learning activities is a function of the opportunities that person is given to participate (Gresalfi, Martin, Hand, & Greeno, 2009). And those opportunities for participation are shaped by the roles and responsibilities that a student is assigned through acts of positioning (van Langenhove & Harré, 1999). Equitable learning processes require that each and every student be positioned as a competent learner and doer of mathematics who has ideas worth sharing and from whom her peers can learn. Students positioned with competence and authority have more opportunities to participate in consequential and influential ways during student interactions, and therefore, have better access to opportunities for rich mathematical learning in terms of content and identity development (Cohen & Lotan, 1995; Engle, Langer-Osuna, & McKinney de Royston, 2014; Gresalfi et al., 2009; Langer-Osuna, 2011).

Methods

Data Collection

Participants. The group consisted of two girls (Becca & Paloma) and two boys (John & Kyle) from an Algebra 1 class in an urban, public high school. Names were changed. Becca was identified as a focal student based on her being described by her math teacher as "really quiet." The students' math teacher taught the lesson and shared post-lesson reflections. This class used CPM curriculum (cpm.org) and worked in groups on a daily basis.

Task. I designed the task and lesson plan for this lesson with the goal of eliciting productive mathematical participation from all group members. The math task, Searching for Sequences, addressed content related to linear growth patterns. Task materials include 16 pattern cards and a playing board, shown in Figure 1. The image below shows the focal group's final task solution.

Figure 1: Searching for Sequences Task - Pattern Cards and Playing Board

At the start of the task, four pattern cards were dealt to each student and kept hidden. Students then took turns placing one card at a time on the board, trying to create 3-card pattern sequences. Students were instructed to justify card placements by explaining patterns they saw. Cards were designed to allow for multiple correct solutions.

Video. Using an external microphone, video recorded participants' speech, gestures, body movements, and eye gaze. The video was 15 minutes long, the time it took to complete the task.

Data Analysis

Video was transcribed for speech, card placements, and salient gestures then divided into talk turns. A *talk turn* was uninterrupted speech by one person. If two people spoke simultaneously, each person was assigned a separate talk turn with overlapping timestamps. Drawing on previous research (e.g., Reinholz and Shah, 2018), I coded two forms of verbal participation: explanations and questions. I also coded one task-specific form of non-verbal participation: card placements. Details of the coding scheme are organized by form of participation and described below.

Questioning. I started by identifying every question that was asked based on speaker's word choice and intonation. I then coded who the question was *asked by* and *answered by*. *Answered by* included any verbal response to the question asked, even if the response was "I don't know." If more than one person responded to a question, multiple people were coded as *answered by*. If no one responded verbally to a question, the *answered by* was coded as "no one". I then tracked to whom questions were directed. The content of the question, students' body positions and eye gaze, and the sequence of talk and action were used to determine *directed to*. If a question was asked without a clear target (e.g., the asker was looking at the board and the content of the question did not indicate who was expected to respond), *directed to* was coded as "everyone."

Sense-making. Initially, I coded sense-making participation based only on explanations of pattern growth. *Explaining growth* was when a student talked about how a pattern was growing / shrinking or described how another figure in the pattern sequence might look. Growth explanations were then attributed to particular sequences to determine who explained which sequences. I then flagged non-verbal indicators (i.e. suggesting the 2^{nd} or 3^{rd} card in a sequence) and non-explanatory verbal indicators (i.e. exclamations) for each sequence. *Suggesting 2^{nd} / 3^{rd} card* was assigned if a student made a verbal suggestion for or placed the 2^{nd} or 3^{rd} card in a mathematically correct pattern sequence. *Sense-making exclaiming* was assigned if a student gave verbal indication that she agreed with a 2^{nd} or 3^{rd} card placement or understood a completed sequence (e.g., "Yeah, that makes sense" or "Oh!" after a 3^{rd} card was placed). These additional indicators assume sense-making happened internally even though the details were not shared verbally. *Explaining growth* is the clearest indicator of sense-making since we had access to how students were thinking about patterns. *Suggesting 2^{nd}/3^{rd} card* is a moderate indicator, since a mathematically correct placement was made, but we did not have access to exactly how students were thinking about patterns. *Sense-making exclaiming* is the most speculative indicator, since nothing is known about how or what students were making sense of with regard to the patterns.

Card placement. For each card placement, I captured the card image, the starting and ending card locations, and the person placing the card. A *card placement* was when a pattern card was placed on the board from someone's hand, a card was moved from one location on the board to another, or a card was removed from the board. For each person, I totaled the number of placements made and the number of unique cards placed. A *unique card* was a pattern card that had not yet been placed by that student. For example, if a student placed a card then on her next turn moved that same card to a new location, that process would count as two placements and one unique card. I then determined if someone other than the student placing the card had suggested where to place it. If so, I coded it as a *placement suggestion*. I also coded how each student responded to the placements made by her peers. *Positive response* was coded with clear approval of placement, either verbal (e.g., "Yeah, that makes sense") or non-verbal (e.g., head nod). *Negative response* was coded with clear disapproval, either verbal (e.g., "No, I don't think that goes there" or non-verbal (e.g., head shake no). *Neutral response* was coded if no clear approval or disapproval was given but the student seemed to know the placement had been made based on eye gaze, body position, or subsequent talk. *Unaware response* was coded with no clear evidence the student knew the placement had been made.

Findings

Findings are divided into two sections and organized by research question: 1) *What did participation look like for a "quiet" student working on a small-group math task?* 2) *How was "quiet" participation co-constructed through peer interactions?*

Becca's "Quiet" Participation

Table 1 summarizes key participation metrics for each of the four students and the teacher. Students worked for 15 minutes on the task. The teacher joined for 90 seconds of that time.

Table 1: Participation Metrics by Person

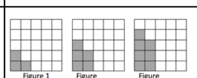
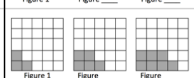
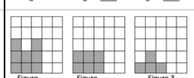
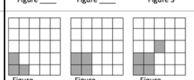
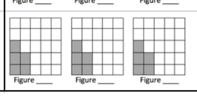
	Talk Turns		Words Spoken		Words	# of Questions		# of Growth	# of Card Placements	
	#	%	#	%	per Turn	Asked by	Answered by	Explanations	Total	Unique Cards
Becca	**32**	**11%**	**98**	**5%**	**3.1**	**7**	**14**	**0**	**11**	**5**
John	111	37%	806	45%	7.3	20	32	2	20	12
Kyle	49	16%	240	13%	4.9	10	14	5	8	5
Paloma	100	33%	504	28%	5.0	21	17	5	8	5
Teacher	8	3%	141	8%	17.6	5	0	0	3	2
TOTAL	300	100%	1789	100%	6.0	63	77	12	50	n/a

Note. Some questions received multiple responses, hence the number *answered by* exceeds the number *asked by*.

Becca spoke the least out of the four students in the focal group. She took the fewest talk turns (11%, 32 turns), spoke the fewest words (5%, 98 words), and had the shortest talk turns (avg. 3.1 words/turn). John and Paloma led the group with roughly the same number of talk turns (111 and 100 turns respectively), although John spoke the most words (45%, 806 words) and had the longest talk turns (avg. 7.3 words/turn). Becca also asked the fewest questions (7 questions) but answered the same number of questions as Kyle (14 questions) and just three fewer than Paloma (17 questions). John dominated the answering of questions by responding to twice as many questions as everyone else (32 questions). Becca made 11 card placements but did not offer any verbal explanations of growth patterns she saw. John made about twice as many placements as everyone else (20 placements) and offered two explanations. Kyle and Paloma made the fewest card placements (8 placements each) but gave the most explanations (5 explanations each). We also see that Becca, Kyle, and Paloma each made placements with 5 unique cards, meaning they each touched the four cards originally dealt to them plus one additional card. In contrast, John made placements involving 12 unique cards, meaning he touched his four original cards plus eight more, accounting for 75% of all pattern cards.

After considering these metrics, much is still unknown about Becca's mathematical sense-making, a key part of her learning trajectory. We do not yet know if Becca's 11 placements were based on her own mathematical reasoning, someone else's, or no mathematical reasoning at all. If we use student explanations as an indicator of learning (e.g., Hatano, 1993), we see no evidence of sense-making by Becca. However, if we broaden our view of sense-making to include contextual non-verbal indicators (i.e., suggesting 2nd/3rd card in a sequence) and non-explanatory verbal indicators (i.e., sense-making exclaiming), there is evidence of Becca's sense-making for five out of the group's six sequences. Even though the final task solution included only four sequences, the group completed six mathematically correct sequences in the process of reaching their solution. Students formed two sequences that were later dismantled in favor of using the cards for other sequences. Table 2 shows the six pattern sequences the group formed in chronological order. The shaded blocks on the right represent the clearest sense-making indicator by person by sequence. The darker the shading, the clearer the indicator. For example, Kyle and Paloma's dark boxes in the top row of the table indicate they both explained growth for this first sequence. John's lightly shaded box indicates he made a sense-making exclamation, and Becca's unshaded box indicates no evidence of sense-making by Becca for this first sequence.

Table 2: Students' Mathematical Sense-Making by Sequence

				Mathematical Sense-Making							
		Time of Pattern…		Indicated by…				Enacting by…			
Order	Pattern Sequence	Completion	Dismantling	Explaining Growth	Suggesting 2nd Card	Suggesting 3rd Card	Sense-making Exclaiming	Becca	John	Kyle	Paloma
1		4:59	13:56	Kyle & Paloma	Kyle	Kyle	John & Paloma				
2		7:18	n/a FINAL FOUR	John	Kyle	John	Becca & Paloma				
3		7:33	20:10	No one	Becca	John	John & Paloma				
4		11:41	n/a FINAL FOUR	Kyle & John	John	John	Becca, Kyle, & Paloma				
5		13:46	n/a FINAL FOUR	Kyle & Paloma	Kyle	Becca, Kyle, Paloma	Becca & Paloma				
6		14:41	n/a FINAL FOUR	Paloma	John & Paloma	John & Paloma	Becca, John, Paloma				

Even though Becca did not explain any growth patterns, her 2nd and 3rd card suggestions for two different sequences, indicate she was indeed making some sense of those growth patterns. Becca made sense-making exclamations for three additional sequences. For example, Becca said, "Yeah, that makes sense," just after John placed the 3rd card in the fourth sequence and Kyle explained the pattern he saw. Considering all sense-making indicators together, we see evidence

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

that Becca made sense of at least five out of the six sequences, the same number as John and one more than Kyle. Paloma was the only student who indicated sense-making for all six sequences.

Participation metrics indicate that Becca was indeed the "quietest" student in the group, with Kyle coming in a close second. Becca spoke least often, said the fewest words, asked the fewest questions, and offered no verbal explanations of the patterns she saw. However, her card placement data were comparable to those of Kyle and Paloma, and a broader view of mathematical sense-making participation indicated Becca made sense of five out the six sequences the group created, the same number as John and one more than Kyle.

Co-Constructing "Quiet" Through Interactions

Three features of the group's interactions fostered co-construction of Becca's "quiet" participation: 1. Becca was positioned as a non-contributing, silent beneficiary of learning 2. Becca's contributions received less support than her peers', 3. Disagreement with Becca was *softer* than with John.

Becca was positioned as a non-contributing, silent beneficiary of learning. This task challenged all four students; they looked to each other for help and guidance. Well, they looked first to John for help, and then to Kyle and Paloma, but never to Becca. Out of 63 questions asked in this group, 26 were directed to everyone in the group and 37 questions were directed to a particular person. Out of the 37, only two were directed specifically to Becca. When students were deciding who should take the first turn based on birthdays, John asked Becca, "What day of December are you?" Later when they were determining whose turn was next, Paloma asked Becca, "You don't have any more cards?" Both questions elicited specific information from Becca related to task facilitation not to mathematics. Neither question required more than a one-word response. No one asked Becca for help, ideas, or approval, which therefore positioned Becca as a beneficiary of the group's learning as opposed to a contributor to it.

Throughout the process of task completion, Becca did much more listening than talking. By listening to her peers' explanations and not offering justifications for the placements she made or explaining patterns she saw, Becca reinforced her role as a non-contributing beneficiary. However, Becca answered 9 of the questions directed to everyone in the group, the same number as Paloma, three more than Kyle, and just four fewer than John. Becca also had the highest ratio of questions *answered by* compared to questions *directed to* (14:2), meaning she answered seven times as many questions as were directed to her. Becca's peers answered 1.5 to 2.5 times as many. Overall, interactions between Becca and her peers positioned Becca as a non-contributing, "quiet," beneficiary of the group's learning, though her willingness to answer questions directed to everyone or to people other than herself contradicted this positioning in subtle ways.

Becca's contributions received less support than her peers'. As an active member of the group, Becca's contributions included question asking, question answering, and card placements. However, many of these contributions received less support from her group than those made by her peers, as revealed by the responses Becca received to her card placements (Table 3).

Table 3: Card Placement Responses from Peers

	Total Placements	Responses from Peers	Positive		Negative		Neutral		Unaware	
			#	%	#	%	#	%	#	%
Becca	11	33	5	15%	3	9%	7	21%	18	55%
John	20	60	26	43%	1	2%	32	53%	1	2%
Kyle	8	24	11	46%	0	0%	12	50%	1	4%
Paloma	8	24	10	42%	0	0%	14	58%	0	0%

Note. **Responses** from **Peers = 3 *** Total Placements

John, Kyle, and Paloma had similar distributions of responses across categories, all very different from Becca. Over half the time, Becca's peers seemed unaware of her placements. Six out of Becca's 11 card placements, occurred while her teammates were talking with one another and making other card placements, resulting in the 18 unaware responses. Looking closer at Becca's placements that *were* noticed, we find that all five positive responses were in response to two placements that were made by Becca but suggested by John. For one placement, John pointed to a location on the board and described how the next pattern card should look, asking, "We don't have one that has three slots, or do we?" Becca responded, "Yeah," as she placed the card described by John in the location suggested by John. Becca's other positive placement occurred after John shared an idea for a new 3-card pattern sequence that required dismantling one of the previously created sequences. Becca helped enact John's suggested sequence by placing a specific card in the location John had suggested. John and Paloma responded positively to both of these placements made by Becca; Kyle gave a neutral response to the first and positive response to the second, accounting for the five positive responses. Becca received no positive responses for placement decisions she made on her own. In addition, she was the only student to make a placement that received negative response from everyone, described in the next section. Interactions within the group allowed many of Becca's contributions to be invisible and others unvalidated, perpetuating Becca's "quiet" participation.

Becca's silent, often subtle movements were not enough to attract the attention of or elicit validation from her peers. She made 10 out of her 11 card placements without saying a single word. The one time she spoke was to ask John, "This one here?" to clarify the placement he had suggested she make. Despite lack of support, Becca continued to engage with her peers' ideas throughout the task. She even made several verbal attempts to propel the group's learning forward. At one point, Becca said, "Let's try it," in response to a suggestion by John, and another time she said, "Show us what you mean," prompting Paloma to describe a pattern in more detail. Becca contributed to her group's collective work, though much of her work went unrecognized.

Disagreement with Becca was *softer* than with John. Two card placements during the 15 minutes of work time received negative responses from peers. One was made by Becca and the other by John. The interactions surrounding these two placements were quite different and led to different outcomes. The challenge to Becca's placement was relatively *soft*, and her card remained in the location she had chosen. The challenge to John's placement was *firm*, and his card was removed from the location he had initially chosen.

The interaction around Becca's placement began when John pointed to her and said, "Now it's your turn." Becca looked down at her cards. Paloma and John pointed to the same card on the board as Paloma suggested, "Maybe that goes there," pointing for a moment to another location. At the same time, John said, "Move this one, move this one, move this one to where you think it goes." Becca looked up to see them pointing. Becca put her finger on the card John was still pointing to. John responded, "Yeah, that one." Becca slid the card to a new location on the board, a different location from the one suggested by Paloma. Surprised by the move, Paloma and John turned to each other and smiled. Becca watched. Paloma said, "I think it goes-" and stopped. Still smiling, John turned back to the board and said, "Alright." Becca put her hand to her mouth, smiling slightly, and looked at the board. Kyle then said, "Ah, probably this one goes there," pointing to the location Paloma had suggested. Paloma responded, "Yeah." Becca whispered the words, "I don't know," audible only after close examination of the video. Paloma softened her response by saying, “Maybe. I mean, you never know.” This interaction was *soft* in that Becca’s peers did not push too hard; their actions and words were gentle. They made their opinions known but did not put Becca on the spot by asking her about her placement decision. It was as if they all believed she was wrong, but did not want to make her feel bad, so they just let it go.

The interaction around John’s negative response placement occurred immediately after Becca’s. Five seconds after Paloma’s “you never know” comment, John said, “I think this one goes over there,” as he slid a different card from one location on the board to another. John pulled his hand away from the card, looked down at the two cards left in his hand and continued, “Cuz I have a card here.” Kyle then rejected John’s placement by saying, “No, I have a card that goes here.” As Kyle said this, he slid John’s card back to its previous location. For a moment, John put his fingers back on the card, saying “Nnnnno.” Kyle said again, “I have a card that goes here,” pointing to the now empty location where John’s card used to be. John took his hand away and looking directly at Kyle’s eyes asked, “Are you sure?” Kyle looked back at John and said confidently, “Yeah.” John then continued his turn by placing an unrelated card. This interaction was *firm* in that Kyle challenged John’s placement confidently and directly. Kyle moved John’s card back to its previous location without waiting for approval and despite John’s initial attempt to stop him. Ultimately, John accepted Kyle’s rejection of his placement and moved on.

Becca’s peers did not challenge her in the way that Kyle challenged John. They treated Becca as if she was fragile, sensitive, and less knowledgeable than they. These traits often go hand-in-hand with the label of “quiet” student. In response, Becca did not offer any defense of her placement even though it was mathematically sensible. She appeared timid and unsure of her decision, fulfilling the students’ expectations of her. However, she could have moved her card to the location suggested by her peers, but she did not. Instead, she rejected the suggestion and left the card where it was. Becca stood her ground and exercised some agency, albeit silently.

Discussion

Teachers and researchers agree that in discourse-heavy classrooms student talk is important for student success (e.g. Barron, 2003; Engle & Conant, 2002; Langer-Osuna, 2011), but I argue there is more for us to learn if we expand our participation lens to include non-verbal indicators and underlying interactional mechanisms, as opposed to just isolated participation metrics. An expansive view of participation allowed me to look beyond Becca’s lack of talk to understand more about a) Becca’s mathematical participation b) the co-construction of “quiet” participation through interactions c) how inequitable opportunities for participation unfolded, and d) what might be done to support more equitable participation in the future. In addition, I argue that caution needs to be taken when labeling students based on easily observable participation metrics, because this non-neutral, often gendered and racialized process can be detrimental to students’ learning experiences. In the case of Becca, her “quiet” label encompassed more than simply an expectation for limited verbal production. Becca and Kyle’s “talk” participation in the task was similar in that they both spoke very little compared to John and Paloma. However, the *opportunities* they had for various types of participation were considerably different due to how they were positioned by their peers. Becca was positioned as an insecure, less knowledgeable, non-contributor to the collective learning; Kyle was positioned as a confident, knowledgeable contributor. Coincidentally (or not), Becca and Kyle were described differently by their teacher. Becca was described as a student who is “really quiet,” “very shy,” and “uncomfortable being in a group.” Kyle was described as a student who is “independent” and “does not like working in groups." Although both descriptions set low expectations for verbal participation, the underlying assumptions about students' self-beliefs and competences are quite different.

Students perceived to be "quiet" are an understudied group of young people who have strengths that are going unrecognized and potential that is going unrealized. I fear "quiet" students' contributions are going unnoticed and their participation is misunderstood. While I cannot claim this is true for *all* "quiet" students, I can claim this was true for Becca, a non-white, female student. I fear that these oversights and misunderstandings are happening in racialized and gendered ways that are perpetuating the status quo, allowing marginalized students to be further marginalized in our math classrooms. Our quest for more equitable educational outcomes through more equitable means, requires more in-depth study into the learning experiences of students who are perceived as "quiet" by their teachers, parents, peers, and/or themselves.

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USING INTERPERSONAL DISCOURSE IN SMALL GROUP DEVELOPMENT OF MATHEMATICAL ARGUMENTS

Shande King
The University of Tennessee
nking5@vols.utk.edu

Tye Campbell
The University of Alabama
tgcampbell1@crimson.ua.edu

The development of proofs and argumentation is one of the major standards for mathematical practices in K-12 education that researchers and practitioners alike are continuing to improve. Further, the use of discourse is considered essential in the learning of mathematical concepts at these levels. However, K-12 educators continue to confound how best to utilize student interpersonal discourse to advance the development of mathematical arguments. This qualitative study examines the nature of student discourse in small-group interactions as students create collective arguments based on mathematical evidence. In examining the patterns of discourse in small groups, the study concludes the effectiveness of various types of discourse in peer-to-peer interaction as students develop more analytical thoughts through the support of their discourse with one another in creating proofs and arguments.

Keywords: Argumentation, Reasoning and Proof, Classroom Discourse

The ability for students to develop their own mathematical arguments at the K-12 level remains a priority in mathematics education. Scholars agree that developing mathematical arguments ensures true understanding when one can convince oneself and others of one’s mathematical explanation (Ellis, 2007, p. 195; Cáceres, Nussbaum, Marroquín, Gleisner, & Marquínez, 2017, p. 356, Stylianou, 2013, p. 23). Additionally, the eight mathematical practice standards in which students should consistently engage include the development of mathematical arguments and the critique of others’ arguments (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). A major factor in creating the space for students to develop and critique mathematical arguments with one another is the dynamic of social interaction in the mathematics classroom. Student participation in a mathematical learning community in a classroom is dependent on the culture. Civil and Hunter (2015) found it was necessary to have an open atmosphere that allows social talk and humor, so that students feel comfortable to share, make mistake, and engage in dialogue about the mathematical learning. To further the mathematical learning, Rojas-Drummond and Zapata (2004) state that this opened the door for students to engage in exploratory talk, where they felt free to examine their own opinions, observations, and explanations of the mathematics. This interpersonal discourse allowed students to engage in healthy and open dialogue focused on the mathematics.

Purpose of the Study

The connection between student interpersonal dialogue and mathematical understanding remains the area that researchers have yet to fully explore. The mathematics education community seems to agree upon and have substantial research on the importance of mathematical argument development at the K-12 level (Brown, 2017; Byrne, 2013; Yee, Boyle, Ko, & Bleiler-Baxter, 2017). However, a dearth of research about the interpersonal dialogue and its relationship to the development of mathematical arguments at the K-12 level persists. This study analyzes student discourse and the mathematical arguments developed. Further, the project will not only discover research-based facts for the elements of and the reasoning for social discourse in student interactions with the mathematics, but also the pedagogical approaches that K-12 mathematics teachers can take towards implementation in regular practices in their classrooms. Finally, as a result of collaborative discourse, students will begin to grow in their autonomous thoughts about the mathematics and thus develop their own valid mathematical arguments.

Theoretical Framework

The situated learning lens is used in this study to understand how students learn. Using this perspective, we seek to understand learning in the context in which it happens. Particularly, this research study emphasizes social participation of students in the small group discussions, so they consider social norms and the culture of the classroom as they participate in their argument development. All of these factors impact their learning in the K-12 setting (Anderson, Greeno, Reder, & Simon, 2000). Learning mathematics in the context of the standards of mathematical practice is a collaborative process, and it should be studied within the contexts in which it is occurring, particularly for K-12 education. This project study of small group discourse in collective argumentation requires participation in the mathematical learning community. The community can influence the discourse that supports student learning.

Further, with the emphasis on discourse of the students, this research takes on a discursive framework, narrowing in on the elements of discussion that students use with one another in their collaborative small groups as they discuss, argue, and critique mathematical concepts with their own ideas of mathematical facts and evidence.

Finally, both the situated learning and discursive framework fall within a greater idea of social constructivism, in which student learning is formed through the interactions that students have with the content in the classroom settings. The foundational learning theory of constructivism provides the necessary lens to look at a deeper, more refined look of learning in this research in the overlapping ideas of the situated learning lens and discursive learning framework.

Review of the Literature

Student creation of mathematical arguments has been established as a primary focus in K-12 mathematics classrooms by national organizations and the recent creation of national standards for mathematical practices. The National Council of Teachers of Mathematics (1989) recommended that reasoning and proof should regularly be incorporated in K-12 classrooms, which was emphasized again in 2010 by the Common Core State Standards (CCSS) publication of eight mathematical practice standards in which students should consistently engage (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Although argumentation has been established as a recommendation for K-12 mathematics, classroom teachers still lack in fostering student development of valid mathematical arguments in a way that satisfies these standards.

Argumentation in the Development of Student Learning

Though argumentation has consistently been a recommended practice of K-12 mathematics, students are generally unable to produce valid arguments (Stylianou, Blanton, & Knuth, 2010), and current methods of teaching proofs and arguments are largely inaccessible to K-12 students (e.g. Karunakaran, Freeburn, Konuk, & Arbaugh, 2014). In several K-12 mathematics classrooms, students learn to create arguments simply by watching the teacher's approach and attempting to recreate the same process. This type of instruction leads students to believe that the teacher is the supreme authority of an argument in terms of what is acceptable and what is not. Thus, students develop an authoritarian proof scheme (Harel & Sowder, 1998), wherein the process of developing a proof becomes a computational exercise to find the specific solution for which the teacher is searching.

In contrast to the traditional teaching approach to teaching mathematical arguments, encouraging collaboration and allowing students to engage in the process of proving can improve students' proof development (e.g. Brown, 2017; Byrne, 2013; Yee, Boyle, Ko, & Bleiler-Baxter, 2017). Collective argumentation, as one example, allows students to engage in a process where they collaborate to create arguments and come to agreement on the arguments that can be accepted within a community. Students create arguments within a group and present their work to the class, helping students gain authority (Stein, Engle, Smith, & Hughes, 2008) in their work and participate in authentic mathematical community.

The Role of Collaboration in the Mathematics Classroom

Research has shown that mathematical communication within a classroom community is crucial for the development of students' reasoning and mathematical understanding (Alrø & Skovsmose, 2003; Forman, 2003). Lampert and Cobb (2003) argue that by providing students the opportunity to discuss their ideas with others can develop their mathematic reasoning more readily. According to Howe and her colleagues, the most successful instances of collaboration occur when students propose and defend their ideas and when they explain their reasoning to each other (Howe et al, 2007). Further, Howe et al. (2007) discovered that collaboration was most productive when the teacher offered little intervention and allowed students to exercise their own authority in solving the mathematical tasks proposed to them.

Teacher Use of Mathematical Arguments

To discover more about the role of the teacher in fostering a community that encourage collaboration to develop student learning via mathematical arguments, Mercer (2008) builds upon the research of Howe et al. to explain that the teacher's role should be one where he or she guides the students in creating mathematical arguments. In this capacity, the teacher assists the students as they learn to collaborate effectively and utilize "exploratory talk" as a cultural and psychological tool to contribute to their development of reasoning (Mercer, 2008; Mercer, Daws, Wegerif, & Sams, 2004). Exploratory talk is the idea where partners engage critically but constructively with each other's ideas. Their statements and suggestions are considered jointly, as they challenge and counter-challenge, requiring justification and alternative hypotheses (Mercer, Daws, Wegerif, & Sams, 2004). Exploratory talk holds students accountable to reasoning.

To create an atmosphere where exploratory talk is commonplace in the mathematics classroom, Brown (2017) encourages teacher participation in a way that guides and pushes student thinking, as they listen and observe the activities of students in their small groups, as in the aforementioned collective argumentation model. The observation of activities can then inspire students by then challenging them to engage in different types of representations, explanations, and justifications (Brown, 2017) than what they had previously created. Brown (2017) continues that the teacher can do this by asking questions about representations, adding to the representations, or even by providing his or her own personal representation. The active role of the teacher can create an environment where students are not only accountable to developing viable mathematical arguments, but they also are inspired to actively engage in them as they are challenged to create new representations of the mathematics.

Discourse in Collaboration of Mathematical Arguments in a Mathematical Learning Community

Discourse in the classroom is dependent upon the social setting of the classroom and can have multiple meanings, involving more than language (Gee, 1996; Moschkovich, 2007). Moschkovic (2007) contends that discourse also involves representations and behaviors, which involves collaboration about arguments and proofs in the context of mathematics classrooms. The discourse of a mathematics classroom is important to note, then, because the language, representations, and behaviors in a class because the teacher and the students may have different interpretations to meanings and focus of attention.

During the act of collaboration in a mathematics class, collective mathematical understanding may take place when students work together on one mathematical task (Martin, Towers, & Pirie, 2006). This collective understanding requires the social context of the learning environment, and it cannot be described by looking at the actions of the individual learners. Through the process of working jointly on a problem, problem-solving leads students to share ideas and their ways of solving, so individual understanding becomes shared. Teachers should establish a classroom environment that encourages this type of discourse where students will jointly partake in a discourse that transforms individual student thinking about mathematics due to the collective understanding that takes place via student language, patterns, behavior, and interactions with the mathematics as well as each other. This study pursues the nature of the small group discussions and the class-wide, whole-group consensus in collective argumentation, hoping to clarify the elements of these discussions that most encourage student learning through the development of mathematical arguments.

Modes of Inquiry

Research Design

The research is qualitative by nature, as audio recorded conversations of the individual small groups for each day of the instructional sequence were transcribed verbatim to allow for coding to occur. We use open and a priori coding to discover patterns concerning student discourse, particularly the types of discourse used that led to mathematical breakthroughs or insights into developing their mathematical arguments. By gaining insight into the patterns of discourse, we see the necessity of interpersonal discourse of the mathematics so that students can be able to defend and justify their critical thinking as others critique their reasoning, as encouraged by the third standard for mathematical practices. Because of the theoretical grounding in a situated learning lens via social constructivism, a qualitative study of the discourse in the small groups will open doors to understanding different factors influencing valid argument development. The findings will lead to discovery of what elements of the discussion best support autonomous student formation of mathematically sounds arguments, as well the practices that teachers can use to best support this environment in a K-12 mathematics classroom.

Methods

Our study took place during the 2018-2019 school year at a private middle school in the Southeast. The participants included 44 eighth-grade students enrolled in one of four different Pre-Algebra classes. The students were all taught by the same teacher, and they were accustomed to collaborating with one another while working on mathematical tasks. For the purposes of our study, we co-taught each of the four class periods for three days of instruction (55-minute class periods), which made up the entire teaching series. On each day of instruction, students worked on one task (see Figure 1) where they were to create a collective mathematical argument. Each class discussed the criteria to create a valid mathematical argument, and the criteria was written on a whiteboard to ensure that it was visible to each member of the class.

Within each class, students were placed into groups of three or four (14 groups in total), and we used Brown's (2017) key-word format to organize our time. Students first spent four minutes creating an argument on their own, then they spent twenty minutes creating a collective argument within their groups. After creating collective arguments, groups shared their arguments with the class to be validated based on the criteria from the beginning of the teaching series.

Task 1

The sum of any three consecutive integers is divisible by three. Is this conjecture true? Write an argument for why or why not.

Task 2

Conjecture: *The sum of two odd numbers is even.* Is this conjecture true? Write an argument for why or why not.

Task 3

How many triplet primes of the form *p, p+2, p+4* are there? Write an argument for how you know.

Figure 1: Tasks

Data Collection

Three types of data were collected: student written work (both in class as collective argumentation groups and homework assignments), audio recordings of the group discussions, and field notes written. Individual written arguments (during the four minutes of individual think time) were not collected, as they were not included in the analysis of the data. The arguments created by each group on each task are the primary source of data collection. Groups were tracked to allow comparisons to be made between each task. Observational field notes were taken during the instructional sequence to record any relevant remarks made by students in both small group and whole group discussions.

Data Analysis

To determine how students' arguments developed as a result of the discourse in collective argumentation, a coding system was developed. Students' collective arguments as evidenced by the audio recordings and written work were coded according to Stylianides and Stylianides (2009) framework. Their framework consists of five hierarchical levels for judging the sophistication of a mathematical argument. Students create *non-genuine arguments* (Level 1) when they commit little effort. *Empirical arguments* (Level 2) are arguments that rely on examples as warrants for a mathematical claim. An *unsuccessful attempt at a valid general argument* (Level 3) is an argument that uses general warrants but contains flaws. A *valid general argument but not a proof* (Level 4) is an argument that uses a deductive chain to argue a claim but uses warrants that are not accepted by the mathematical community. A *proof* (Level 5) is a deductive argument that uses warrants accepted by the mathematical community.

Beyond the arguments themselves, we transcribed and studied the small group discussions to account for different types of discourse that impacted student thinking, particularly as they moved to more advanced levels of arguments. We discovered which patterns of explanation, interjection, questioning and defense, for example, best led students to understanding mathematics and advancement of mathematical arguments. Further, we begin to understand what the classroom environment requires for this collective argumentation and collaborative discussion to foster advanced mathematical proofs and arguments. This will develop findings in pedagogy to inspire further research and practitioner models to encourage small group discourse

Data Sample

When split into groups of 3 students (or less depending on number of students in the class), students were audio-recorded by group. One specific group developed a mathematical induction proof orally in explaining his reasoning to his group:

> So, you start with 0,1,2 as your base one. And then, what you do to bring it up to 1,2,3, is you add one to each of the integers. So, when you're adding one to each of the integers, you're really just adding three. So, that's why it's divisible by three. And then, that's why it's divisible by three every single time.

This oral argument aligns with a proof by mathematical induction. The speaker explains that the base case (0+1+2) is divisible by three. Then, he explains adding one to each integer will gives the next case (e.g. 1+2+3). He explains why this new case is divisibly by three by stating "you're really just adding three" appealing to the property that adding three to a number already divisible by three creates a new number that is also divisible by three. His group asked for further clarification, to which the speaker further explained his reasoning.

- Speaker 2: But I do feel like in some instances it won't be right.
- Speaker 1: No, it will be right every time because you're just adding three every time.
- Speaker 2: But, why? We need facts and evidence.
- Speaker 1: Because if you have three consecutive numbers, if you go up one for each of them, you just add three. And if the base one does—is divisible by three that means all the other ones will be divisible by three. And in this instance the base one is divisible by three.

In being able to provide a rationale to a problem, particularly in response to a question about the validity of the response, the speaker thoughtfully engages in personal discourse to promote a valid argument and its reasoning.

Results

Table 1 shows the frequency of the types of arguments that were created by each group across the three tasks. The data suggest an overall increase in the level of argument created by the groups over the three tasks. There was a steady decrease in the use of empirical arguments; however, students had less success in creating a proof on task 3 than on task 2. This is likely due to the difficulty of task 3 in comparison to task 2, and the expectation of creating a conjecture before attempting to write an argument. Even so, the data suggests that by task 3, students understood that an argument that uses examples (empirical argument) was not viable in their mathematical setting.

To understand how each group progressed across the tasks, we tracked their level of argument from task 1 to task 3. Table 2 shows the level of argument for each of the fourteen groups (listed A-N) on task 1 and task 3 and the difference in level of argument. Most groups increased in their level of argument while few groups decreased or remained the same. To determine whether there was a statistically significant difference in level of argument from task 1 to task 3, a Wilcoxon-signed rank test was used. There was no statistically significant difference in the level of argument the groups created from task 1 to task 3 (T=47.5, p=0.178). The sample size, lack of variability in the levels of argument, and ceiling effects of not including baseline data likely contributed to the non-significant test statistic. Still, the descriptives illustrate that groups generally made progress in the sophistication of their collective arguments from task 1 to task 3.

Table 1: Level of Argument by Task

Task	**Level 1: Non-genuine Argument**	**Level 2: Empirical Argument**	**Level 3: Unsuccessful Attempt at Valid General Argument**	**Level 4: Valid General Argument but Not a Proof**	**Level 5: Proof**
Task 1	2	8	2	0	2
Task 2	1	5	2	2	4
Task 3	3	1	5	4	1

Table 2: Difference in Level of Argument from Task 1 to Task 3

	Groups													
Tasks	*A*	*B*	*C*	*D*	*E*	F	G	H	I	J	K	L	M	N
Task 1	1	2	5	5	1	3	2	2	3	2	2	2	2	2
Task 3	1	3	3	4	1	4	1	4	4	3	3	2	5	3
Difference	0	-1	2	1	0	-1	1	-2	-1	-1	-1	0	-3	-1

Discussion

Preliminary quantitative results of this study suggest that the eighth-grade students benefited from engaging in interpersonal discourse via collective argumentation. Communication among peers provided access to the standards for a viable mathematical argument, thus making it more likely that they will have success in creating their own arguments. Further, engaging in collective argumentation gives students opportunities to critique and validate others' work which places them in a position to learn directly from their peers, as supported by the review of the literature. Our study was unique in its application of the aforementioned supports for middle grade students, and it agrees with findings from previous research that students benefit from interpersonal discourse in creating mathematical arguments (Rojas-Drummond & Zapata, 2004). However, more research should be devoted to how students use interpersonal discourse to create increasingly sophisticated arguments. Furthermore, scholars should test the supports mentioned in this paper in other settings such as primary and secondary schools. The preliminary results of this paper is promising in providing further promotion of and support for K-12 mathematical argument development.

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MAINTAINING THE MATHEMATICAL FOCUS OF WHOLE-CLASS DISCUSSIONS: DILEMMAS AND INSTRUCTIONAL DECISIONS

Nicholas J. Gilbertson
Winona State University
ngilbertson@winona.edu

As teachers shift their practice from traditional student-to-teacher interaction patterns to collaborative discussions around rich mathematics, they often encounter instructional challenges. One such challenge is deciding when to pursue interesting and productive ideas that run contrary to the particular mathematical goal of the lesson. In this research summary, I report results from a study involving one experienced teacher's instructional decisions and how she maintained attention to the lesson-specific content goal given possible alternative pathways.

Keywords: Classroom Discourse, Instructional Activities and Practices

Over the past several decades professional mathematics teaching organizations and mathematics education research has described the potential benefits of classroom discourse for supporting students in meaningfully understanding mathematics. This attention to student thinking and discussions have important implications for students' opportunities to access mathematical ideas and their views of what it means to be mathematically competent. As such, many sets of instructional techniques have developed to support teachers in facilitating rich mathematics discussions. These foci have varied from identifying cognitively demanding tasks (e.g. Stein & Smith, 1998), developing instructional practice (Smith & Stein, 2011), the use of various talk moves (Chapin, O'Connor & Anderson, 2009), and attention to equity (Khisty & Chval, 2012), to name a few. Many of these approaches share the common goal of creating opportunities for equitable access to meaningful mathematics.

At the same time, it has been well documented that many U.S. teachers still use direct instruction approaches by asking simple-fact based questions, receiving responses from students, and then providing feedback or evaluation (Cazden, 1988; Chapin, O'Connor & Anderson, 2009; Franke, Kazemi & Battey, 2007; Sahin & Kulm, 2008; Stigler & Hiebert, 1999). Perhaps one reason for this more traditional approach is that by limiting student talk, there is a clearer pathway forward in a lesson, providing less opportunity for a student question or insight to potentially deviate the trajectory of the lesson. Additionally, researchers have contended that facilitating classroom discussions where students are collaboratively engaged around rigorous mathematical content is quite challenging (e.g. Franke, Kazemi, & Battey, 2007; Lampert & Cobb, 2003; Lampert et al., 2013; Tyminski, Zambak, Drake & Land, 2013), especially when teachers have multiple goals they are attempting to satisfy (Sleep, 2012). Lampert and colleagues (2010) for example, argued that, "Maintaining a coherent mathematical learning agenda while encouraging student talk about mathematics is perhaps the most challenging aspect of ambitious teaching" (p. 131).

As such, facilitating discussions has the potential to create instructional dilemmas as teachers grapple with advancing the mathematical agenda of the lesson while balancing possibly competing instructional goals such as developing student engagement in mathematical practice, supporting long-term content goals (e.g. understanding linear relationships), and establishing a mathematical community within the classroom. Because facilitating discussions around rich mathematical tasks requires a different set of skills, this raises the question, as to what barriers still persist in supporting teachers with facilitating these types of mathematics discussions and how one might go about mitigating these barriers.

One approach to studying this phenomenon would be to study teachers who typically use direct instruction but are open to the idea of facilitating discussions. This could provide insight as to the reasons why these teachers have not yet taken the first step towards a shift in their instruction. In the study described in this article, I take a different approach, focusing on teachers who are well beyond their first step in facilitating discussions. The middle school teacher described in this research summary is an experienced teacher (having taught at other grade levels), is philosophically aligned with the value of student talk, and she used the *Connected Mathematics* (Lappan et al., 2005; Lappan et al., 2014) middle grades curriculum – a textbook series designed to engage students in thinking through student discourse.

The results described in this summary attend to instructional decisions during whole-class discussions where the decisions took place at a potential crossroads in the lesson, where the teacher could have chosen to pursue alternate goal-types but was able to maintain her attention to the lesson-specific content goals through her instructional choices. The results described in this research summary focus on this teacher's decisions when she was presented with situations which could have deviated her instruction from lesson-specific content goals as she instead chose to attend to broad content goals, mathematical practice goals, or goals for establishing a mathematical community.

Purpose

Broadly, the purpose of this study was to better understand the potential barriers and inherent complexity in teachers' decision-making while facilitating classroom discussions. More specifically, the results given in this proposal describe pivotal moments in lessons, where teachers chose to maintain the lesson-specific content goal instead of deviating and focusing on a different type of instructional goal. The hope is that by better understanding how experienced teachers navigated the complexity of attending to the lesson-specific content goal(s), that this might better inform the field as to the potential deviations that teachers are likely to encounter while facilitating whole-class discussions. In some ways, these potential deviations might be viewed negatively as they move the focus away from the mathematical goal within the lesson; however, the reason these posed actual dilemmas for teachers is because they are important aspects of the long-term development of students as thinkers, reasoners, and doers of mathematics. Thus, it is through understanding how experienced teachers supported these other goals while still maintaining focus on the lesson-specific content goal, that we might better understand how to support novice teaching practice in facilitating discussions.

Theoretical Framework

The guiding theoretical perspective for this study was *situated cognition* (Greeno, 1989). The situated perspective places strong emphasis on the interaction between individuals and their environment. As Greeno (1989) stated, "Cognition, including thinking, knowing, and learning, can be considered as a relation involving an agent in a situation, rather than as an activity in an individual's mind" (p. 134). In this study, teacher's activity and decisions were understood in the context of their activity in their classrooms. I attended to teachers' interactions with their students over multiple lessons, observing behavior, hypothesizing about their decisions, and later interviewing them about possible decisions. This approach differs from simply interviewing teachers about a simulated classroom or asking teachers to state what decisions they would make in a theoretical situation in their own classroom. The approach I used in this study aimed to provide a more authentic view of teachers' decision-making as a cognitive activity necessarily embedded in their environment.

Additionally, I investigated the nature of teachers' decisions, by using the theoretical construct of *professional obligations* (Herbst & Chazan, 2011) which states that teachers' decisions are a function of their obligation to (a) students as individuals, (b) in developing students interpersonally in the classroom, (c) in representing the school as an institution and (d) in representing the discipline of mathematics. This construct helped in better understanding the nature of instructional decisions when teachers encountered instructional dilemmas, defined as a conflict between one of four instructional goal types (lesson-specific content goals, broad content goals, mathematical practice goals, and goals for establishing a mathematical community). A teacher's professional obligations differ from their goals in that goal(s) are about accomplishing a task (e.g., supporting students in effectively communicating their thinking with the class), whereas the teacher's professional obligation(s) describe who or what they feel obliged to satisfy (e.g., the individual student, the school, etc.) during the discussion.

Methods

To better understand the nature of teachers' instructional goals and identify the moments where potential conflicts occurred I analyzed data from a variety of sources, including an initial interview with each of the three grade-seven teachers, a baseline lesson observation, lesson plans for three rounds of at least three consecutive lessons per round, post-lesson interviews, and video-stimulated interviews at the conclusion of each round. To identify points at which goals may have been potentially in conflict with one another, I first conducted a preliminary analysis to identify teachers' instructional goals and determined moments when any consistent patterns of instruction deviated from what was observed. Across the three teachers in the study, I identified goals in all sources of the data. The instructional goal is defined as what the teacher hopes to accomplish as a result of enacting a whole-class discussion. Most goal statements occurred during interviews, where teachers were asked directly about their goals such as, "…were there like a few big things you really wanted to come out of [this lesson]?" or "Were there other goals, or other things you wanted…"

During the initial round of data analysis, I identified goal statements where the teacher either explicitly stated their goal or described their intended learning outcome. For example, during Ms. Mitchell's post-lesson interview (Round 2, Observation 2) she stated "…my goal as I told you was for them to get a better sense of what a sample space is." This is an example when the teacher explicitly identified her instructional goal. As a second example, Mr. Sandberg stated his goal in the form of an expected learning outcome, "…they need to be able to understand y-intercept, what that looks like on a table, graph, and equation" (post-lesson interview, Round 1, Observation 2).

After identifying goal statements, I coded goals as to whether the goal was primarily focused on (a) lesson-specific mathematical content, (b) mathematical content that was beyond the scope of an individual lesson: (long-term or unit-level), (c) mathematical practice goals (including goals attending to communication and dispositions) or (d) goals focused on being a member of a mathematical community. All teachers had cases of all four types of goals.

This first category, *lesson-specific content goals*, focused on mathematical content to be learned in a particular lesson. The second category *broad content goals*, included unit-level content goals such as understanding linear relationships, a topic taught over several lessons. This category also included long-term content goals dealt with goals that went beyond the unit, such as learning goals that connected to the high school curriculum. This type of goal is strongly related to Ball's (1993) notion of teaching towards the "mathematical horizon".

Mathematical practice goals are consistent with NCTM's (2000) Process Standards and the Common Core's Standards of Mathematical Practice (NGA-CCSSO, 2010). This goal type is related to the first two goal types in that they focus on mathematics, but they differ in that they focus not on content, but on types of mathematical practices, such as justification and critiquing other students' arguments. The fourth category, *goals for establishing a mathematical community* included statements about developing a classroom that would support shared exploration and expectations of having students to learn in a social environment.

It is worth noting, that I do not make the claim that teachers are always aware of their goals or, if they are aware of them, that they are able to articulate them clearly. In many cases, teachers may have instructional goals that are somewhat unknown to them (e.g., a novice teacher knowing what they should be teaching a particular activity without a clear reason why they ought to be doing it). A teacher may be acting consistent with a goal, without knowing what their goal is. Some teachers, for example, might simply have a goal to make sure the class runs smoothly, and maintaining order is the most important aspect of their instruction with very few mathematical goals. I take as an assumption that the nature of teaching falls in the category of "goal-oriented activity," but I am cautious in not assuming that any discussion necessarily includes all of the four types of goals.

Results

In this section, I describe the results specifically pertaining to instances where Ms. Mitchell's goals actively competed with lesson-specific content goals. To illustrate these instances, I describe one instance of each of the three types of conflict between lesson-specific content goals and the other three goal types. While all three teachers had at least one conflict between the lesson-specific content goal and each of the other three goal-types, I provide data only on Ms. Mitchell as she had the most collective instances of conflicts between lesson-specific content goals and the other three types.

As an overview of her instructional approach, Ms. Mitchell could be best characterized as a teacher who thinks deeply about her teaching practice, as she often reflected on her instructional decisions in relation to developing content, practice, and dispositional goals for her students. More than once, she described the importance of social inequities and her role in teaching mathematics as a way to provide opportunities for students to grow both personally and academically. As such, she often found herself in various instructional dilemmas where lesson-specific-content goals potentially conflicted with other goal types.

In one of Ms. Mitchell's lessons, a student was describing the y-intercept as the "starting point" on the graph, and during the post-lesson interview she described how uncomfortable that made her feel. "When I first saw that [the y-intercept] I was like, oh, I don't like that one bit, 'cuz in high school I had to fight that for three years. It's not a starting point." Ms. Mitchell allowed the student to talk about the y-intercept as a starting point as they were just beginning to understand this important point across representations. This exchange highlighted Ms. Mitchell's attention to long-term content goals (broad-content), specifically related to her experience as a high school teacher.

Ms. Mitchell also frequently mentioned Common Core standards (NGA-CCSSO, 2010) at the beginning of class – both content and practice – and her use of CCSSM practice standards was a theme throughout her instruction both in describing her decisions and in making these standards explicit to students during class time. In one interview she mentioned wanting students to engage in repeated reasoning and thinking about structure. Other practices she mentioned included having students provide evidence to support a claim and persevering when trying to solve a problem. These statements reflected mathematical practice goals.

Ms. Mitchell also talked about the importance of protecting students' think time, and what students can learn from each other in groups (establishing a mathematical community). She wanted her students to be able to interact with each other directly in the discussion and for her role to be minimized in the discussion. In Ms. Mitchell's classroom this involved one lesson where she explicitly changed the typical norms of interaction taking a less prominent role in the classroom discussion allowing for greater student-to-student interaction.

Case 1: Lesson-Specific Content Goals vs. Broad Mathematical Goals

In this first lesson, students were analyzing their previous day's work from a problem involving the fundraising graphs for a walkathon by three fictitious students (Figure 1). The problem in the textbook (Moving Straight Ahead, p. 8, *CMP2*) gave the fundraising as follows: Leanne \$10 regardless of distance; Gilberto \$2 per km; Alana \$5 plus \$0.50 per km. The lesson-specific content goal of this lesson was for students to make and recognize linear tables, equations and graphs from contexts.

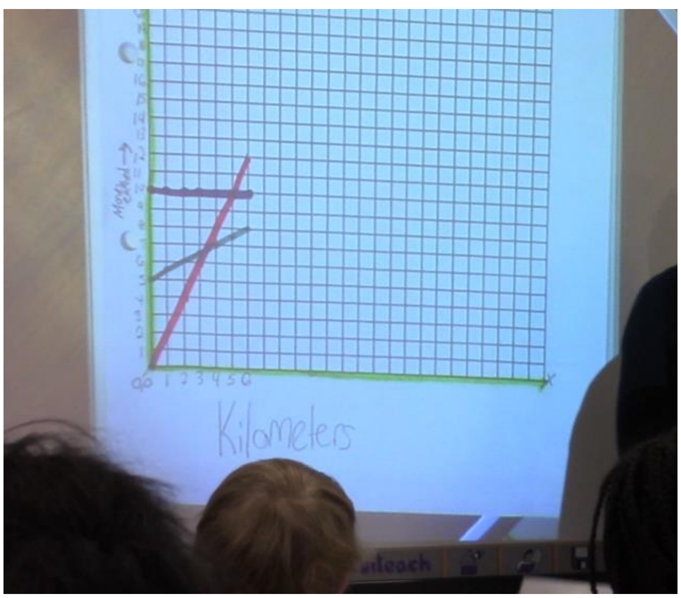

Figure 1: Ms. Mitchell Displays Walkathon Results

During the class discussion, Ms. Mitchell pressed a student to explain why the graphs were linear, after the student noticed the consistent shape in the three graphs. To contrast linear and non-linear graphs, Ms. Mitchell asked the student to draw a graph of a non-linear relationship on the board. Whereas the student may have simply drawn a non-linear function (e.g., a quadratic or exponential relationship), the student instead drew a graph that was not even a function (Figure 2). Given this strange graph, a different student asked, "Could a graph actually go backwards like that? Like could it go like this way and then eventually go back that way again?" This question focused on an important relationship in understanding a mathematical function. At this point, Ms. Mitchell was faced with a decision of whether to (a) pursue this important mathematical insight, which was beyond the scope of the lesson, but was potentially productive in supporting students' thinking about functional relationships, or (b) refocus attention to the lesson-specific content goal.

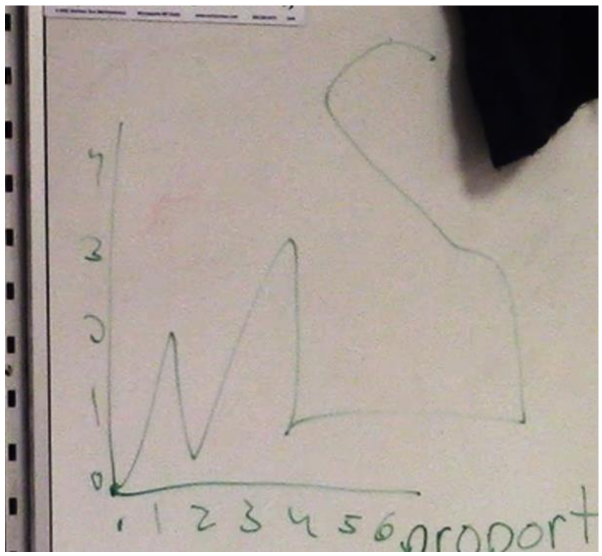

Figure 2: A Student Draws a Graph of a Non-linear Function

Ms. Mitchell could have moved on from this student's thought, minimizing the opportunity for disruption. Her decision was to write a note on the side of the board and to validate the importance of the student's insight. "I'm going to write that right here Charlie because that's outside what we want to talk about today. But that's an extremely important question; could a graph actually go backwards? And we need to talk about that."

During the video-stimulated interview after the first round of observations, Ms. Mitchell commented on this clip:

> Charlie's question was so awesome as it usually is. That is an example where I don't want to shut down the conversation or, well I do want to shut down the conversation because the level of thinking it would take based on what they had had by one-two would be too great…I do want to push a little bit…Now that thing about going back on itself I still have that written down and we will come [back] to that.

Although Ms. Mitchell's decision was to move on from the student's question, she did this only after valorizing the student's question. Ms. Mitchell could have chosen to pursue the student's idea directly, essentially moving away from her lesson goals for the discussion. Instead, she refocused the discussion in line with the lesson goals. By supporting the student's thinking and publicly acknowledging its importance, she continued to make explicit the importance of thinking and questioning in her classroom.

Case 2: Lesson-Specific Content Goals vs. Goals for Establishing a Mathematical Community

This second case occurred as students explored a problem on compound probability. In this lesson, students rolled two six-sided dice and found the product of the two values. They did several trials. One student won if the product was even; the other student won if the product was odd. This game would not be considered a “fair game”, because the likelihood of getting an even product is greater than an odd product. The lesson-specific content goal for this lesson was to develop an initial understanding of whether a game is fair or not.

During the discussion, Ms. Mitchell asked the class if they thought the game was fair. One student, Jonathon, said the game was fair because there are 3 odd outcomes and 3 even outcomes per die. This is incorrect since the outcome is based on the products of the two values, not just the individual values appearing. Ms. Mitchell was faced with how to appropriately respond to the student given that this particular student often has difficulties in the class. On the one hand, Ms. Mitchell needed to advance lesson-specific content goals, helping students see the inherent unfairness to this game. On the other hand, if her response was to be consistent with trying to establish a mathematical community, she would need to be careful about how she responded to a student who typically has difficulties in the class.

One choice would have been to press students to see why the claim was incorrect. She also could have simply stated that Jonathon was incorrect, or called on a different student in the class to explain why he was incorrect. Instead, Ms. Mitchell supported Jonathon’s thinking, and acknowledged how he was reasonable (in some way) in thinking that the game could be fair. The decision Ms. Mitchell made when faced with an incorrect student solution was to attend to the lesson-specific content goals and goals for establishing a mathematical community. During the discussion she had a student (Josh) sitting close to Jonathon talk to him about the odd number times odd number case, and later asked for other students’ thinking about the problem. She described this scenario during the post-lesson interview, describing her thought process during this moment in the lesson.

> Jonathon, you know, he flat out fails; he doesn't have a whole lot of perseverance intellectually. He thinks of things sometimes really in strange ways, but I was happy that he was participating, and is in fact that half the numbers are even and half the numbers are odd. So I wanted to, you know, honor that he said, and then, but also try to build on that and then I felt like Josh kind of jumped ahead and wasn't real clear about odd times odd. So rather than just go there, and have him or someone explain how he knew that, I felt like the whole group would kind of get lost, so I just stated it very simply after he said it so people could think about it. Seemed to me there was still a question about the odd times odd. And they were all just kind of letting it go… (Ms. Mitchell, video-stimulated interview, May 11, 2015)

Case 3: Lesson-Specific Content Goals vs. Mathematical Practice Goals

A common dilemma for Ms. Mitchell was the need to meaningfully engage students in mathematical practice while also attending to lesson-specific content goals. This dilemma occurred across several lessons. Primarily, Ms. Mitchell worried about whether her students were developing their abilities in independent problem solving. During one lesson, students were learning how variability affects the mean and median of a data set, in relation to large clusters (the lesson-specific content goal). Throughout the lesson, there were several opportunities for Ms. Mitchell to focus on mathematical practice goals, but she continued to focus on the lesson-specific content goals due to the pressure of a standardized department final at the end of the month. She felt frustrated about having to choose one over the other given the content coverage expectations.

> Well there's a dilemma, because on the one hand I would love to have all the ideas for them to have access to; on the other hand there's the bigger idea of the mathematical practices, like what do I do when I don't know what to do?...And, so I've got this treadmill keeping us moving toward the end at the same time as I'm trying to incorporate as much of the thinking stuff that I can. (Ms. Mitchell, post-lesson interview, May 19, 2015)

She later referred directly to how this dilemma impacted that day's lesson and the previous day's lesson, wondering if students saw the mathematical purpose of what they were doing. She also worried about the long-term impact of her instructional decisions to focus on the lesson-specific content goals in order to satisfy a common department final at the sacrifice of developing students' self-efficacy in problem solving. In contrast to the two previous cases, where Ms. Mitchell was able to navigate the potential conflict between lesson-specific content goals and other instructional goals, this case highlights that not all potential conflicts end in a productive resolution even for an experienced teacher. The institutional constraints of expectations for content coverage, standardized assessments, and time constraints was sufficiently pervasive to cause challenges for Ms. Mitchell.

Conclusion

During whole-class discussions, teachers are faced with decisions related to the types of questions they ask students, which student ideas they choose to highlight, and how deeply to pursue a student response. As teachers facilitate discussions, the dilemmas they encounter and the decisions they make send messages (explicitly or implicitly) to students about what is valued as thinkers and doers of mathematics. These messages are important because as Jansen (2006, 2008) pointed out, students participate in discussions for various reasons, yet student participation is a necessary component of teaching using whole-class discussions.

As was seen in the three cases provided in this research summary, Ms. Mitchell's experience in facilitating discussions and her orientation to balancing several competing instructional goals allowed her to make decisions that maintained the focus on the lesson-specific content goals. Much like other approaches to supporting teachers in facilitating discussions which include sets of instructional practices (e.g. Smith & Stein, 2011) and talk moves (e.g., Chapin, O'Connor, & Anderson, 2009) this study highlights that there are particular sets of technical skills needed to make in-the-moment decisions which likely impacted the immediate and long-term experiences and opportunities for students to learn mathematics via classroom discussions. In the findings reported in this summary, Ms. Mitchell showed particular techniques for how to navigate potential situations as they appeared in the course of an open dialogue with her students. These particular techniques might not all be obvious or clear to teachers who are shifting their practice from more traditional interaction approaches to a more open discussion format.

This study has the potential to further complexify existing sets of instructional techniques as well as to develop its own set of categories of situations a teacher may potentially encounter while facilitating discussions, such as, responding to an incorrect answer, responding to a question that moves beyond the scope of the lesson, or responding to an answer that other students do not understand.

As one such example, one of Smith and Stein's (2011) *Five Practices* is anticipating student thinking. In the findings in this study, one might consider particular likely scenarios, such as the Case 1 situation, where a student raised an interesting mathematical question that is relevant to the lesson but would likely move the focus of the discussion elsewhere. The teacher could anticipate this particular situation and consider how they would generally approach a similar scenario. The teacher could consider under what conditions it would be appropriate to deviate from the lesson goal, or alternatively, how one refocuses the class on the more immediate lesson goal while still attending to the productiveness of the question and the student's engagement. By identifying and naming these types of situations, teachers that are shifting their practice towards increasing the quality of student talk in their classrooms may be better positioned to make in-the-moment decisions that allow them to maintain their lesson-specific content goal while also addressing other valuable instructional goals.

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AGAINST THE EPISTEMIC HORIZON TOWARD POWERFUL DISCOURSE

Sunghwan Byun
Michigan State University
byunsun3@msu.edu

In mathematics classrooms, teachers are often the authority as a knower of mathematics. To support students to develop their identity as doers of mathematics, teachers need to decenter the authority and to place students in a more powerful position as autonomous thinkers. Building on the work of Herbel-Eisenmann and colleagues (2013), this paper aims to expand the notion of powerful discourse from a conversation analytic approach. By examining the fine-grain epistemic characteristics of interactional sequences, I argue for the significance of the interactional knowledge domain, in which positioning of students as more or less knowledgeable takes place. Further, I discuss the need for examining the discursive dimension of knowledge and discursive practices that can redistribute the epistemic authority in mathematics classrooms.

Keywords: Classroom Discourse, Positioning, Conversation Analysis

The social turn in mathematics education (Lerman, 2000) widened the view on mathematics teaching. Beyond transmitting mathematical knowledge to students, teachers need to attend to the development of student identity as doers of mathematics. Through social interactions, students' identities are constructed and reconstructed over time (Hand & Gresalfi, 2015); hence, the attention to interpersonal aspects of social interactions became crucial. One way to attend to the interpersonal aspect of classroom interactions is through the lens of powerful discourse (Herbel-Eisenmann, Steele, & Cirillo, 2013), which entails purposeful positioning of students during mathematical discussions to support their identity development. The goal of this paper is to offer a concrete way to examine powerful discourse through a fine-grain analysis of social interaction. Drawing on conversation analysis (CA), I identify important aspects of interactions related to the nature of knowledge and teacher authority. I begin with a discussion on powerful discourse.

Powerful Discourse in Mathematics Classrooms

As a framework for mathematics teachers and researchers to examine discursive practices in a mathematics classroom, Herbel-Eisenmann and colleagues (2013) offered two theoretical lenses: productive discourse and powerful discourse. Productive discourse is for widening students' access to mathematical language and ideas, whereas powerful discourse attends to the social and interpersonal functions of language in discourse. Although these two lenses may overlap and interact each other, this paper mainly focuses on powerful discourse since it is closely related to the social interactions and relationships among students and teachers—a significant part of the social turn.

Drawing on positioning theory (Harré & van Langenhove, 1999), Herbel-Eisenmann and colleagues (2013) highlighted the significance of the linguistic functions that position a person or a thing (e.g., student, textbook, mathematics) to a particular position (e.g., a "smart" student). Note that "positioning" is an active doing of language, in contrast to the static meaning of "a position." For discourse to be powerful for student learning, both people (e.g., teachers and peers) and social and material environments (e.g., textbook, and policy) can position a student as a doer of mathematics, a legitimate participant in mathematical activity. Wagner and Herbel-

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

Eisenmann (2013) argued for purposeful positioning of students, teachers, and textbooks during instructions to reorganize the authority structure (i.e., who or what decides what is true or good) within a mathematics classroom. This deliberate form of positioning can be a tool for teachers to support identity development of students, especially those with marginalized social markers (e.g., students with disabilities, students of color, girls, emergent bilingual learners). Through this mode of positioning, teachers can promote a more equitable learning environment.

This idea of supporting a student's identity development by disrupting the visible social asymmetry within a classroom is an interest of many equity-oriented researchers in mathematics education. For instance, one of the major discussions in complex instruction is "assigning competence" (Featherstone et al., 2011, Ch. 6). In a small group setting, by deliberately drawing "public attention to a given student's intellectual contribution to a group's problem-solving efforts" (p. 90), a teacher can encourage the student and other peers to see the given student as a valuable and intelligent member of the group. Thus, assigning competence is one of the many ways a deliberate positioning by a teacher can take place in the mathematics classroom. Through the lens of powerful discourse, we can see many ways, sometimes subtle and difficult-to-notice ways, that positioning could happen in a classroom. One of the strengths of the positioning theory is its conceptual flexibility, which allows an analyst to apply the theory in a broad range of interactional contexts with a variety of analytic tools (Herbel-Eisenmann, Wagner, Johnson, Suh, & Figueras, 2015).

Building on the work of Herbel-Eisenmann and colleagues (2013), this paper expands the notion of powerful discourse from a conversation analytic approach. Positioning can happen at multiple levels within the range of "scales" (Herbel-Eisenmann et al., 2015, p. 193). The conversation analysis presented in this paper allows for the finest-grain analysis (i.e., "utterance" level with a typical duration of 10^0–10^1 seconds). This paper highlights that this type of fine-grain analysis is necessary to understand powerful discourse in the context of moment-by-moment interaction in which a teacher and students are engaged in sequences of acting and reacting. Furthermore, Herbel-Eisenmann and colleagues (2015) stated that mathematics education researchers often make claims about positioning without much attention to communication acts. This paper illustrates how the application of CA can address the common pitfall by grounding the discussion of positioning on the temporal progression of the sequence of actions. The following section discusses CA and its key constructs related to powerful discourse in more depth.

Epistemic Dimensions in Conversation Analysis

The primary focus of CA is to understand normative practices of conducting social interactions, in particular, talk-in-interaction. As the term "talk-in-interaction" refers, CA attends to forms of talk (e.g., lexical, syntactic, speech features) and other visible social cues (e.g., gaze) in interactional contexts. The context includes not only when, where, and by whom the utterance was made but more importantly the temporal sequence of talk-in-interaction. As Heritage (1984b) articulated, "any speaker's communicative action is doubly contextual in being both *context-shaped* and *context-renewing*" (p. 242, emphasis original). In other words, an utterance is examined based on the context shaped by the prior interactions—notably, the immediately preceding action. The same utterance also reshapes the context for the following action. Both before- and after-utterances are, therefore, considered to infer action-performing aspects of a given utterance.

Influenced by ethnomethodology (see, for more information, Ingram, 2018), CA lays its epistemological grounding on what *participants* orient to (i.e., treat something as relevant to the conversation at hand) and on how both speaker and recipient treat the target utterance, which offers a window into participants' view on conducting moment-by-moment interaction. CA scholars argue that mainstream social science researchers too often impose a presupposed social model to explain a system of social interaction (Schegloff, 1997; Speer, 2004). On the contrary, the focus of *ethno*-methodology is on revealing participants' (i.e., ethno-) methods to design an utterance as a social action (e.g., advising, offering) and to recognize the utterance as such by the recipient. The social norms, from the view of CA, are not seen as theoretical rules that govern interactions among participants. Normative practices of social conduct are rather accomplished by the participants through routinized, seen—but unnoticed—procedures, which is the main interest of CA.

Epistemic Dimensions in Conversation and Epistemic Engine

This paper focuses on a particular topic in CA called *epistemic dimensions*—the "dimensions of knowledge that interactants treat as salient in and for conversation, particularly with respect to asymmetries" (Stivers, Mondada, & Steensig, 2011, p. 9). Epistemic dimensions include but not limited to: epistemic status (relative status as more or less knowledgeable based on social relationships); epistemic stance (moment-by-moment expression of being more or less knowledgeable); epistemic access (relative access to knowledge); and epistemic authority (relative authority of knowledge). Initiated by Goodwin (1979) and Heritage (1984a), CA scholars have been discussing the significance of participants' relative statuses as more or less knowledgeable in a particular knowledge domain. For instance, Heritage (2012b) coined the term, "epistemic engine," which is a normative social force for participants to engage in a sequence of interaction when information imbalance between participants is acknowledged. The sequence closes when the participants acknowledge that the imbalance is equalized for a practical purpose. Notably, CA does not concern what participants actually know in their minds. The focus is on how participants display themselves and treat the other participants as relatively more or less knowledgeable and on how such visible actions function as an epistemic engine, which shapes the normative patterns in interaction.

According to Heritage (1984a), the epistemic dimensions are an important concern for participants *in situ* because participants' epistemic positioning, called "epistemic status" (Heritage, 2012a), is a crucial part of the interactional context for participants as they design and interpret each other's actions in everyday conversations. Heritage (2012a) illustrated the differing action-performing nature of particular morphosyntax and intonation depending on the epistemic status of participants. For instance, the declarative syntax with falling intonation is often aligned with a speaker in a knowing position (K+) *conveying* new information to a recipient in an unknowing position (K–). A speaker in a K– position, however, can use the same morphosyntax and intonation to *request* information from a recipient. In an interview setting, for instance, when an interviewer (K–) says, "You have a child," despite the declarative syntax and falling intonation, the interviewee (K+) would respond as a request for information (e.g., "Yes, I have a son.") rather than an assertion from the interviewer. In other words, participants rely on their relative epistemic statuses to determine what the utterance performs and to project ways they can respond.

Heritage's (1984a; 2012a) argument above brings up an important issue when teachers and researchers examine the action-performing nature of discourses in a mathematics classroom. Solely focusing on textual and speech features of talk-in-interaction may overlook the epistemic dimensions of the conversational exchange, an important source for participants to interpret and respond to social actions *in situ*. To attend to the epistemic dimensions, analysts need to consider the knowledge domain in which the sequence of interaction is situated. The epistemic status is not static—the same participants may occupy different epistemic statuses depending on the knowledge domain in which participants are engaged. In a classroom setting, for instance, the teacher is often a sole K+ individual during a mathematical discussion (Ball 1993, Lampert 1990). However, when students share out what they did over the last weekend, students can be positioned as K+ in such a knowledge domain. This "epistemic asymmetry" (Heritage & Sefi, 1992) between the teacher and students is shaped by the institutional context of the mathematics classroom, in which a teacher with required educational credentials teaches and students learn school mathematics. This normative institutional setup thus creates an inherent epistemic asymmetry with K+ teachers and K– students.

Epistemic Dimensions in Mathematics Classroom Interactions

To illustrate the significance of knowledge domains and epistemic stance during moment-by-moment interactions, I first compare two cases from the study by Ingram (2012) in a secondary classroom setting in the U.K. I place a particular focus on teacher-initiated question-response sequences to discuss the contrasting epistemic characteristics of the sequences. Second, I turn my attention to two extracts from an article by Ball (1993), which is widely read and known for "Sean's number" in mathematics education. I apply the epistemic lens to offer an insight into the important discursive characteristics embedded in the discussions in Ball's (1993) study.

Epistemic Dimensions in Question-Response Sequences

The following two extracts share a common characteristic—they both consist of teacher-initiated question-response sequences, what Mehan (1979) called Initiation-Response-Evaluation sequence (IRE). In Extract 1 below, Simon is the teacher, and other speakers are students.

Extract 1 (Ingram, 2012, p. 110)

```
Simon:   … what was the highest number of days absence.
A:       eight
Simon:   it was eight. And what was the lowest number of
         days absent.
B:       zero or one I don't know
Simon:   you don't know. ok someone else then, what's the
         lowest number of days pe- someone was absent.
         George.
         2.5)
Alex:    zero?
         (0.9)
George:  zero.
Simon:   that was Alex talking I want to hear it from you.
         look at the table, what was the lowest number of
         days that someone had absent.
George:  zero.
Simon:   it is zero, because ↑twenty people had no days off.
```

Figure 1: Extract 1

The epistemic stances, which Simon and the students display, align with the epistemic statuses of K+ teacher and K– students. For instance, in the third turn position (i.e., the turn after the question and response turns) of the question-response sequence (line 330), Simon repeats the answer and confirms that the answer is correct. The action performed by the third turn is twofold. First, it closes the question-response sequence as a "sequence closing third" (Schegloff, 2007), which indicates that Simon has received a preferred response from a student. Second, Simon reasserts his K+ position. In this IRE sequence, the teacher holds a K+ position with K– students, and the third turn publicly displays that Simon already knew the answer to the question he posed, reaffirming his K+ position.

Simon's epistemic authority is also evident based on how the students treat the question-response sequence. Students orient to his epistemic authority with their downgrading epistemic stance. Note, for instance, that Alex offered his answer "zero?" (line 337) with rising intonation (noted by "?") thus downgrading the certainty of his answer. Moreover, students offer only short answers (e.g., "eight," "zero") without offering any account for their answers. This lack of accountability (i.e., the social norm that requires the speaker to offer an account for their response) can be explained by the epistemic statuses among the K+ teacher and K– students. The epistemic engine, the normative social force, works so that information flows from K+ position to K– position. When students are positioned in a K– position, students offering additional account works against such normative social force. To further argue for this point, I illustrate a contrasting case in the following extract, in which Richard is the teacher.

Extract 2 (Ingram, 2012, p. 115)

```
561   Richard: what do you understand by the idea of (.) proof.
562            mathematical proof. p r double o f.
563            (0.9)
564            what do you understand by that (.) concept, that
565            idea. Maybe say one thing about it (.) then let
566            somebody else say something else. Um:: hands going
567            up. Alex.
568   Alex:    you can't prove anything apart from maths because
569            it's all point of view.
570   Richard: oh I see
571            (0.7)
572            um: can you give an example or something
573   Alex:    um (there's different kinds of things) everybody's
574            eyes might be slightly different, you can't tell
575            (.) because it's like, (.) you see different
576            shades ((inaudible)) the eye could be different.
577   Richard: oh so when you look at your red thing there,
578            somebody might (.) see it differently.
579   Alex:    yeah
580   Richard: I see, whereas mathematically? What are you saying
581            about maths that's different?
582   Alex:    it's because the maths deals with absolute
583            (substances) like numbers, you can't be
584            ((inaudible)) can you.
585   Richard: ah: ok that's very interesting. very good. …
```

Figure 2: Extract 2

Similar to the case of Simon, we see Richard also practices his process-authority (Oyler, 1996). He selects who speaks next (e.g., line 567), and he controls the topic of the discussion by initiating a series of question-response sequences. The epistemic statuses are, however, shaped differently compared to the earlier case of Simon.

In the first question-response sequence, after Richard selects Alex as a next speaker (line 567), Alex offers his response with an account—note the use of "because" in line 568. Richard responds to Alex with "oh," a "change-of-state token" (Heritage, 1984a), which indicates that Richard's epistemic position changed from K– to K+. In other words, Richard displays that he now knows Alex's understanding of proof, which he did not know before the conversational exchange. The following "I see" also reaffirms that his question is a "genuine question," not what Searle (1969) called a "test question." This third turn closes the sequence, and Richard initiates another sequence (line 572) by requesting an example. This second question can be categorized as "probing a student's thinking" (Herbel-Eisenmann et al., 2013), but also there are important yet subtle epistemic stances that Richard and Alex display. Alex offers his response with an account—note "because" (line 575) once again and then Richards responds with his "oh" (line 577) with a subsequent statement. Despite its declarative syntax, we see Alex orients to Richard's statement as not an *assertion* but a *request for confirmation*, to which Alex responds with his approval of "yeah" (line 579).

The teacher and student, in this case, are positioned in contrasting epistemic statuses compared to the case of Simon. Richard is in K– position, wanting to know how his students understand the concept of proof, and Alex is in K+ position with his exclusive epistemic access to his own understanding of proof. In this epistemic terrain, the epistemic engine naturally works for the flow of information from K+ student to K– teacher. In other words, as a knower of his own understanding of proof, Alex is under the normative social force for explaining his knowledge to Richard and other students, which is evidenced by Alex's use of "because" and additional accounts that he provides after his responses. Thus, the question-response sequences above position Alex in a more powerful position with his epistemic authority.

Richard's case also shows how teachers' epistemic authority is tied to the notion of mathematics. In line 580, Richard focuses his question on the *mathematical* meaning of the proof by invoking "mathematically" and "maths." Alex shifts his epistemic stance by downgrading his certainty about his answer with a tag question, "… can you." (line 584). Alex's downgraded epistemic stance shows that, for his idea to be "mathematical," it requires approval from an external authority like a teacher. This particular sequence serves as an example of a teacher' identity as a legitimate bearer of "official knowledge" (Apple, 1993).

Examining the epistemic dimensions in the above extracts shows that epistemic statuses are shaped differently *relative to the knowledge domain* that teacher and students dwell during the question-response sequence. On one hand, when the question is targeted at the mathematical understanding that students hold in their mind, the students are positioned in K+ position. On the other hand, when the question is targeted at the mathematics defined by the institution, the teacher occupies K+ position.

Epistemic Dimensions in Reform-based Teaching

I now turn to two extracts from Ball's (1993) study to situate the current discussion on knowledge domains and positioning in the broader discussion of reform-based teaching. CA can reveal the routinized, seen—but unnoticed—procedures. In the case of Ball's study, by applying the lens of epistemic dimensions to her data, it may explain what Ball (1993) described as an abstract goal of reform-based teaching as more tangible discursive practices. In her paper, Ball (1993) discussed her dilemma as a teacher between hearing what students know now and supporting them as they transcend their present understanding. Her paper in part illustrates how she *hears* students' understanding, and I examine her actions through the lens of epistemic dimensions and positioning. In Extract 3 below, Mei offers her understanding of Sean's number, which can be both even and odd.

Extract 3 (Ball, 1993, p. 386)

Mei: I think I know what he is saying ... is that it's, see.
I think what he's saying is that you have three groups
of two. And three is an odd number and an even number.
T: Is that what you are saying, Sean?
Sean: Yeah.

Figure 3: Extract 3

After Mei explains her understanding of what Sean is trying to convey, the teacher asks Sean to confirm if that is what he is saying (line 304). Sean also orients to the teacher's question as seeking confirmation, and he gives his approval of "Yeah." Asking Sean to confirm is a discursive move that positions him as a K+ individual, but there is more to show in this scene. In this sequence of seeking and offering confirmation, the participants are engaged within the knowledge domain of *Sean's* understanding. That is, the goal of the exchange is not to know how school mathematics defines even and odd numbers, but rather, the participants' concern in the interaction is how Sean understands even and odd numbers. In this knowledge domain, Sean holds the primary right and authority as a knower of his own thinking, which positions Sean as K+ individual. This epistemic terrain aligns with Mei's downgrading epistemic stance indicated by her use of "I think …" (lines 301-302) and the teacher's confirmation-seeking. I discuss another example in the following extract of how epistemic status can be reshaped in interactions.

Extract 4 (Ball, 1993, p. 390)

Mei: So when we put two in each group in order to make
one because it's below zero.
T: I don't understand this part— put two in each group
in order to make 1.
Mei: If we take six and add six to it, we get twelve above
zero, but it's below zero, so— and three plus three is
six, so we add three more to the six above zero.

Figure 4: Extract 4

Starting from line 401, Mei explains her way of understanding 6 + (–6), and then the teacher indicated that she does not understand a part of her explanation. As a response, Mei expands her explanation. Here, we see the visible working of the epistemic engine. The teacher did not ask a question, but she only indicated her K– position of unknowing what Mei explained earlier. The publicly displayed epistemic status was enough for Mei to expand her explanation. This move is similar to what Herbel-Eisenmann (2000) observed from one teacher, named Karla, who often voiced her own confusion (e.g., "You lost me ….”). Through the epistemic lens, Karla publicly displayed her K– position and pushed her student to offer further explanation. As Drew (2005) suggested, we can see the state of confusion as an interactional resource generated *for* an interaction. Thus, teachers downgrading their epistemic status in a particular knowledge domain, to which students have primary access, is a powerful pedagogical move. This way, teachers can unleash the power of the epistemic engine and allow information flow from a K+ student.

Discussions and Implications

Through the lens of epistemic dimensions, I have discussed how positioning can happen in a sequence of interaction based on the knowledge domain in which participants engage and epistemic stances that participants display. In this section, I make connections between the epistemic dimensions in conversations and important discussions on knowledge and authority in the mathematics classroom.

The examination of discursive practices related to epistemic dimensions expands the notion of knowledge in the mathematics classroom. Often in mathematics education, the discussion of knowledge is limited to what individuals have in their mind. As Barwell (2013) critiqued, this individualistic approach overlooks the discursive dimension of mathematics knowledge embedded in social interactions. Barwell (2013), for instance, noticed that displays of teacher knowledge are "often inter-related with constructions of students as *not* knowing" (p. 605, emphasis original). His observation aligns with the relative nature of epistemic status that I discussed above, and I argued that powerful teaching is sometimes teachers displaying themselves as not knowing (i.e., downgrading epistemic status). That is to say, expanding the discussion from what knowledge a teacher *has* to how a teacher *performs* knowledge can further our understanding of the nature of knowledge situated in the interpersonal context.

The epistemic dimensions in conversation offer a framework to understand the interactions between the discursive dimension of knowledge and teacher authority in a more nuanced way. For instance, in her discussion on teacher authority, Herbel-Eisenmann (2000) illustrated how the tag questions (e.g., … isn't it?) used by one teacher made his corrective course of actions appear more polite and less authoritative. Through the epistemic lens, Herbel-Eisenmann's (2000) insight shows the graded nature of epistemic asymmetry. The teacher's use of tag questions downgraded his epistemic stance; yet, staying in the same knowledge domain, the teacher continued to maintain a K+ position. This case shows that teachers' deliberate effort to downgrade their epistemic status—without moving the discussion to another knowledge domain—has limited effect on reshaping epistemic status.

The significance of the knowledge domain to reshape the epistemic asymmetry between a teacher and students points to the need for moving across multiple knowledge domains during a mathematical discussion. Such as the case of Sean's number, focusing the discussion on how students understand a mathematical idea is one way to distribute epistemic authority to students based on their exclusive epistemic access to their minds. As Greeno (1991) stated, valuing students' thinking and reasoning encourages "positive epistemological beliefs and attitudes that are favorable to participation" (p. 211). The epistemic dimensions of interaction offer a lens to see what "valuing" students' thinking and reasoning looks and sounds like *in practice* and its fine-grain interactional characteristics. Based on the significance of the knowledge domain, the "valuing" requires moving the target of the conversational inquiry from official mathematics to the mathematics in students' mind. Furthermore, there are other knowledge domains to which students have relative epistemic access. For instance, students' cultural knowledge (Ladson-Billings, 1995) and community knowledge (Civil, 2006) can serve the same purpose, especially when the teacher's cultural and communal backgrounds do not overlap with those of students. These knowledge domains are great interactional resources to position students in a K+ position with epistemic authority.

Lastly, the distinction that I presented between teachers' content-authority and process-authority (Oyler, 1996) requires further discussions. This paper does not argue for an anti-authority stance. Rather, it argues for a more nuanced, multi-dimensional understanding of teacher authority and better use of authority during discussions for student learning. For instance, to reshape the epistemic terrain during the discussion, teachers need to be in authority to control the topic of discussion so that the teacher can intentionally steer the discussion toward different knowledge domains. Moreover, to accomplish more equitable participation among students, teachers may need to intentionally distribute the speakership among students with a particular social marker (e.g., race, gender) or prior history of participation. Without teachers' enactment of process-authority, the participation patterns in classroom discussions would likely continue to marginalize a particular group of students and remain dominated by a small number of students. Epistemic authority, a content dimension of authority, however, needs to be displaced from the teacher to allow students to engage in a mathematical activity as legitimate doers of mathematics rather than mere receivers of knowledge. The powerful discourse that I addressed in this paper concerns this epistemic authority.

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JUMPING INTO MODELING: ELEMENTARY MATHEMATICAL MODELING WITH SCHOOL AND COMMUNITY CONTEXTS

Erin E. Turner
University of Arizona
eturner@email.arizona.edu

Julia Aguirre
University of Washington-Tacoma
jaguirre@uw.edu

Amy Roth McDuffie
Washington State University
mcduffie@wsu.edu

Mary Q. Foote
Queens College-CUNY
mary.foote@qc.cuny.edu

Mathematical modeling is a high-leverage topic, critical STEM education and civic engagement. This study investigates culturally responsive, school and community-based approaches that support mathematical modeling with elementary students. Specifically, we analyzed two modeling lessons in one fifth grade classroom, with a focus on how students drew upon their experiences and funds of knowledge as they engaged in the mathematical modeling process. Our findings illustrate different ways that students' experiences and situational knowledge informed and guided their modeling activity, including the quantities that they deemed relevant, and how they interpreted and refined their solutions. We also attend to the teacher's role in supporting such connections. Implications for mathematics teacher educators and research are included.

Keywords: Elementary School Education, Equity and justice, Modeling, Instructional activities and practices

Mathematical modeling is a high-leverage topic, critical for participation in STEM education and civic engagement (Aguirre, Anhalt, Cortez, Turner & Simic Muller, in press). Unlike typical textbook word problems that often require students to disregard realistic considerations (Greer, 1997; Verschaffel, De Corte & Borghart, 1997), modeling tasks invite students to consider real-world contexts, as well as real-world solutions. Students engage in a cyclical process of: (a) analyzing situations; (b) constructing models that represent the situation, based on information and assumptions; (c) using models to perform operations and reason about results; (d) validating or revising models; and (e) reporting conclusions (CCSSM, 2010; Lesh & Doerr, 2003).

Although mathematical modeling has a well-established research base in secondary and undergraduate education (Doerr & Tripp, 1999; Gainsburg, 2006), it has been underemphasized and under-supported at the elementary level, with a few notable exceptions (e.g., Carlson, Wickstrom, Burroughs & Fulton, 2018; English, 2006; Suh, Matson, Seshaiyar, 2017). This may be due to limited attention to modeling in elementary teacher preparation, and an absence of mathematical modeling tasks in elementary curricula (Burkhardt, 2006). However, research conducted with elementary grade students and teachers demonstrates that mathematical modeling *is accessible* to children in elementary grades (GAIME, 2016), including students with limited prior experience with modeling (Chan, 2009; English, 2006), and students from a diverse range of mathematical and cultural backgrounds (Turner et al., 2009).

Mathematical Modeling with Cultural and Community Contexts

Research suggests that culturally responsive, community-based approaches to teaching mathematics have added benefits, particularly for students from underrepresented groups

(Aguirre & Zavala, 2013; Civil, 2007; Ladson-Billings, 2009; Lipka et al., 2005; Turner et al., 2008). Grounding mathematics in meaningful contexts that connect to students' experiences can enhance student engagement and learning, and encourage students to draw upon situational knowledge and real-world considerations, instead of "cutting bonds with reality" (Bahmaei, 2011). Moreover, these connections help students understand how mathematics matters in personal and socially meaningful contexts (Anhalt, Cortez & Smith, 2017).

We build on this prior research to investigate culturally responsive, school and community-based approaches that support mathematical modeling with elementary students. Specifically, we collaborated with elementary teachers to develop modeling lessons that build on students' mathematical thinking (Carpenter et al., 1998) *and* their community-based knowledge and experiences (Civil, 2007). As teachers enact tasks in classrooms, a focus of our research has been on how students' contextual knowledge and experiences inform and support their modeling activity. Specifically, we focused on these research questions: How do students draw upon their experiences and funds of knowledge as they engage in the *mathematical modeling process*? How do teachers support these connections, and how do they shape students' modeling activity? We explored these foci via an analysis of modeling lessons in one fifth grade classroom (Mr. H).

The Mathematical Modeling Process

Figure 1 depicts the mathematical modeling process we have used with teachers and students. This model is informed by prior research (Anhalt, Cortez, & Bennett, 2018), and the modeling cycle included in the CCSSM's (2010) standard. The arrows represent what we have found to be frequent pathways through the modeling process, but others are possible.

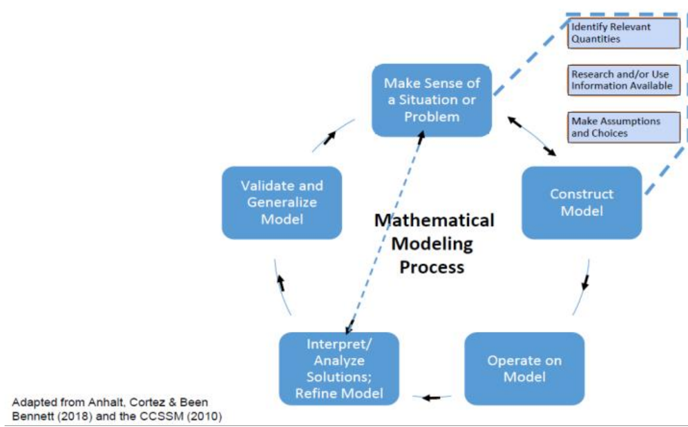

Figure 1: Mathematical Modeling Process for Elementary Students and Teachers

Phase 1, Make Sense of a Situation or Problem

Students begin the mathematical modeling process by making sense of a situation or problem. They consider questions such as: What do I know about this situation? What experiences have I had related to the situation? What additional information do I need?

Phase 2, Construct a Model

Next, students consider what quantities are relevant and important, and how those quantities relate to one another. They also consider what information is provided, what might need to be collected, and what they will need to assume or decide. These considerations are connected to the context, so as students construct a model, they continue to make sense of the situation.

Phase 3, Operate on Model

Next, students create a solution for the problem, perform computations, and check for precision both in their results and in their labels and/or explanation of their work.

Phase 4, Interpret/Analyze Solutions and Refine Model

Students then interpret their solutions in relation to the original situation. They ask whether their solution makes sense based on their experiences and knowledge about the context, draw conclusions about what the solution(s) imply, and refine and revise their model (if needed).

Phase 5, Validate and Generalize Model

This final phase in the mathematical modeling process involves validating and generalizing the model so that it is reusable and allows for applications to similar scenarios.

Study Context and Participants

As part of the broader project, approximately 30 teachers from two districts participated in 30-hour summer institutes focused on *teaching mathematical modeling in grades 3-5.* During the institutes, teachers engaged in modeling tasks developed by the research team, explored key phases of mathematical modeling, and discussed critical features of meaningful, relevant tasks.

During the academic year, teachers met on a monthly basis to plan, discuss, and reflect on modeling tasks enacted in their classrooms. Twice per year, researchers presented tasks and related lesson materials (photos, handouts, lesson launch slides) designed by members of the research team. These tasks addressed content-standards reflected in district quarterly curriculum maps, and connected to relevant contexts in schools and communities. All teachers were invited to enact these tasks, to facilitate analysis of a common lesson across different contexts.

Focal Teacher: Mr. H

We selected Mr. H's class because Mr. H enacted multiple mathematical modeling tasks across the school year, including tasks that connected to different school and family contexts. Mr. H is a White 5th grade teacher with 13 years of teaching experience. In his class of 27 students, 6 students were new learners of English. The racial/ethnic demographics mirrored the school with 30% Latinx, 19% White, 15% African American, 14% 2 or more races, 9% Asian, 3% Native Hawaiian/Pacific Islander, 1% American Indian/Alaskan Native. His classroom included a typical range of student backgrounds in mathematics.

Focal Modeling Lessons

We analyzed two mathematical modeling lessons in Mr. H's classroom, *Abuelo's Birthday* from the beginning of the year, and *Upcycling Jump Ropes* from the end of the year.

Abuelo's birthday. This task adapted from Aguirre and Zavala (2013) presented a realistic scenario about four grandchildren who want to share the cost of a birthday gift for their grandfather. Using information about each grandchild, students generate a "fair" plan for sharing the costs, and explain how their plan could be used in similar situations.

It is Sr. Aguirre's 70th Birthday. Four of his grandchildren want to buy him a gift. They found a photo printer on sale for $119.99. They want to buy him the printer to print family photos.

- Alex, a 9th grader, earns between $15 and $20 each week from babysitting jobs.
- Sam, a 6th grader, earns $10 each week taking care of a neighbor's pets.
- Elena, a 4th grader, earns about $5 each week doing odd jobs for an aunt.
- Jaden, a 1st grader, has no weekly job but has saved $8 in her piggy bank

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

One of the grandchildren says that they should split the cost of the printer among them and each pay the same amount. Another grandchild says that it is not fair and they should each pay different amounts. Help the children make a plan to share costs in a fair way to buy the gift.

- Your plan should work in other situations where family members want to share costs fairly.

Figure 2: Abuelo's Birthday Task

Upcycling jump ropes. This task focused on upcycling plastic bags to make jump ropes. This task was inspired by students' interest in environmental issues, including recycling and upcycling to reduce waste. Students designed a set of jump ropes for their school, calculated the number of plastic bags needed to make the jump ropes in the set, and then explained how their model could be used in other situations. The task materials included a video that showed how to braid plastic bags to make ropes, and the number of bags needed per foot of rope (3 bags/foot).

We selected these two modeling lessons because they were designed to facilitate different kinds of connections to students' experiences (i.e., to family practices in the Abuelo's birthday task, and to environmental concerns and school-based play practices in the Jump Rope task).

We want to make a jump rope set to be used in gym class. A set contains jump ropes of different lengths. How many plastic bags will be needed? Your plan for making the jump rope set must show:

- How you know that you will have enough jump ropes for gym class, without a lot of extra
- How many plastic bags you will need
- How others could use your plan to make jump rope sets for their school

Figure 3: Upcycling Jump Ropes task

Methods

Case Study Design

Using a qualitative case study design (Creswell, 2013; Stake, 1995; Yin, 2013), we considered lesson as a case, and the students who participated in the lesson as the primary focus of study. Case study research lends itself to "how and why questions" regarding social phenomena, especially questions that require "extensive and in-depth description" (Yin, 2013, p. 4). Correspondingly, we examined patterns within and across the two lessons (Stake, 2006).

Data Sources

Data sources included video-recorded observations of mathematics lessons, post-observation teacher interviews, teacher reflections on lesson enactments during a subsequent teach study group, student work, and other lesson artifacts (e.g., images of board work). The two lessons analyzed in this study ranged in length from 1.5 to 2 hours. When video-recording, we followed the teacher to capture instructional decisions and moves. All interviews were audio-recorded and transcribed. Videos were selectively transcribed with a focus on pivotal teacher questions and prompts, and examples of students' thinking and experiences that they brought to the task.

Data Analysis and Analytical Framework

Through multiple and iterative cycles of analysis, we developed preliminary categories based on themes identified in the literature related to modeling and connecting to students' experiences and funds of knowledge, and emerging themes evidenced in our data. These included: teacher moves to elicit and/or connect to students' experiences; ways students' experiences/knowledge connected with each phase of the modeling cycle; ways that connections to students' experiences supported sense-making. We engaged in iterative cycles of sorting data under these categories and writing analytic memos to identify and refine themes (Miles, Huberman, & Saldaña, 2013). To achieve *interpretive convergence*, multiple researchers developed and reviewed the memos. While viewing the data through the lens of the phases of the modeling cycle, we generated a *narrative compilation* (Creswell, 2013) of preliminary findings across the two lessons.

Findings

We structure our findings by the first four phases of the mathematical modeling process. (We address Phase 5 in the discussion.) In each section, we draw on examples from lessons analyzed.

Phase 1: Making Sense of the Situation or Problem

In both lessons, students drew upon understandings and experiences, from family and community, and school activities, to make sense of the task context. Mr. H also shared his own stories relevant to the context, as way to mirror the kinds of connections students might make.

Connections to family practices sharing costs in Abuelo's birthday task. During the launch of the Abuelo's birthday task, Mr. H began by narrating a personal story about eating out at a restaurant with a friend, and determining a fair way to split the cost.

> Mr. H: This happened to me last night actually. I went out to dinner with a friend, and then we ordered all this food, and we had to think about how we were going to pay. … Is it fair that one person pays for all it? I make more than my friend, so should I pay for all of it, because I make more?
> Students: No!
> Mr. H: She ordered more, so she should pay for all the things that she got, and I pay for my things? [Students call out both Yes! And No!]
> Mr. H: So I want you guys to think, what situations with your family, friends or siblings, have you been involved in where you had to share costs for something? … Can you talk about that for a minute?

Students then contributed their own family stories related to sharing costs such as splitting the cost of a new video game system evenly among siblings, with one sibling contributing the extra funds to cover the tax, or collaborating with cousins and an uncle to pool the money needed to purchase a family television set, with everyone contributing as much as they could.

Connections to play activities in the Jump Rope task. Mr. H also used personal stories to invite students to share experiences in the Upcycling Jump Ropes task. For example, he described his own attempts at specific jump rope styles.

> Mr. H: Yesterday I tried to use this [pointing to a jump rope] in some ways I was successful and some ways I was not as successful. ... Maybe it's [the jump rope] not a good fit for me. … If I had the longer one I would have been able to do the cross over thing [gesturing by crossing his hands in front of him to cross the rope] and that would have been kind of cool.

This prompted students to discuss different jump rope activities (i.e., "double dutch" versus "normal jumping"), and their ideas about height and rope length, noting that jump ropes need to be "different lengths, because … there are people of different sizes [in our class]."

Across the two lessons we found that Mr. H used his own stories, coupled with targeted probes to elicit students' experiences related to the task context, and to connect those experiences to key features of the task, such as quantities that would be important in building a model.

Phases 2 and 3: Constructing a Model and Operating on a Model

As students continued to make sense of the situations, they identified relevant quantities, and discussed information that was known, or that they needed to decide or assume. While we found similar patterns in both lessons, we focus here on *Abuelo's Birthday*, highlighting students' reasoning about quantities and relationships that were central to their model-building. Given that students' often iterated between constructing and relating quantities (Phase 2) and operating on those quantities (Phase 3), we describe both phases in this section.

Reasoning about the role of tax. As students in Mr. H's class discussed how the cost of the printer should be shared among the grandchildren, they considered the role of sales tax in the total price. Several groups assumed that tax would be approximately 5 dollars, because, as one student explained, "usually if you buy something from the grocery store, it's [tax] usually underneath $5." Other students drew on experiences shopping at dollar stores to reason about the tax. In the excerpt below Students 1 and 3 suggest that Jaden (the youngest sibling) pay the tax using his savings, because tax "is never over 5 dollars." Student 2 disagrees, drawing on experiences at the dollar store to explain that tax is charged at a rate of 10 cents per dollar.

Student 1: The littlest one [Jaden] can just pay tax.
Student 2: Eight dollars wouldn't be enough for tax [Jaden has $8 saved.]
Student 3: The tax is never gonna be like over 5 dollars.
Student 2: Tax is always over 5 dollars! Tax is ten cents, ten cents on a dollar. That means it would be 10 dollars [estimating the tax on the printer].
Student 1: But if you are [only] buying one thing-
Student 2: Listen. 1 dollar is 1 dollar and 10 cents, 2 dollars is 2 dollars and 20 cents.
Tax is 10 cents [per dollar]! Do you guys ever like go to the dollar store? You buy [one] thing and have to give them one dollar and 10 cents. …
Student 1: Tax is usually like under 5 dollars. Like at the grocery store. …
Student 4: We can add the tax after we do everything else. That's what people usually do.

Ultimately, this group decided to follow Student 4's suggestion to focus first on how the four grandchildren should split the cost of the printer (without tax) and "add the tax after." Yet their conversation evidences that they leveraged their experiences with tax charged at different stores to reason and make decisions about key quantities in their model.

Deciding how to share costs "fairly". Most students decided that the youngest sibling should only pay the tax; yet they made different decisions about "fair" plans for the other siblings. Some assumed that each sibling should pay the same amount, "that wouldn't be fair if 1 kid paid more." Others reasoned that the older siblings should pay different amounts, proportional to their earnings. Figure 4 displays one group's solution - the older three siblings each contribute their weekly earnings for four weeks, which added to Jaden's savings ($8), gives $148 total.

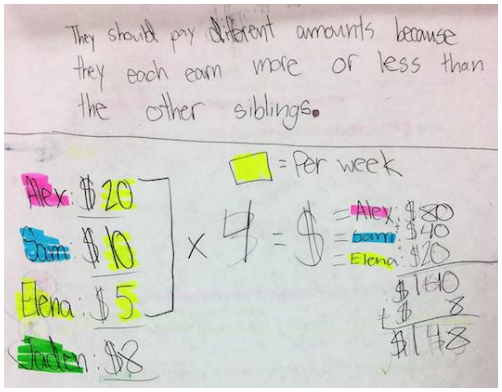

Figure 4: Equal Time Model for Sharing Costs. Each Sibling Contributes Earnings for 4 Weeks

In the discussion that follows, students explain both the quantities they deemed important (in this case, each sibling's weekly earnings) and the way that they related and operated on those quantities in their model (multiplying each weekly earnings by a set number of weeks).

> Student 1: we got what they all made in a week, except for Jaden. And we multiplied it by 4 weeks. And we got these [total earned per sibling in 4 weeks] and we added them …
> Mr. H: Why did you pick 4 weeks to multiply by?) Do you remember?
> Student 2: Because when we did 3 weeks we had less than what we needed, so then we added another week. … Alex paid $80, Sam paid $40, and Elena paid $20 and Jaden paid $8.
> Mr. H: Why did you think that was fair? What was your thinking? … Was there a reason, though, that you let them pay different amounts?
> Student 3: Well because each person, well, Alex made the most. So he would pay like most of it. Well, like Elena, and Sam would pay some of it too. ..
> Student 4: A question I have is … since the printer was $120, but when I look at your paper, it said they paid $148 if you add everything together.
> Student 3: We will just have extra.
> Student 5: They could use the extra money they had to pay tax.

Students drew on shopping experiences, including understandings about tax, as well as perspectives on "fair" ways to share costs to relate and operate on quantities in their models. While some students proposed models based on equally dividing the cost among the three older siblings, other students drew on sensibilities about fairness and sharing in families to reason that fair does not always mean equal. Yet all models were based on the assumption that all four siblings should contribute to the gift, something that students reasoned was important.

Phases 4: Interpret/Analyze Solution, and Refine Model

After students operated on their models to determine the amount paid by each sibling (in Abuelo's Birthday) or the number of bags needed to make the jump ropes in their set (in Jump Ropes), they analyzed their solutions, and in some instances, refinements were needed. While we found similar patterns in both lessons, we focus here on *Upcycling Jump Ropes*, with attention to how students' experiences informed this phase of their modeling activity.

In some instances, students' analysis of their emerging models led to revisions in how they defined an appropriate set of jump ropes for their target audience. For example, several groups of students proposed models based on grouping students by height, and then providing each group of students with a set of appropriate length ropes. As one student explained, “We came up with the idea that we could split people into different categories based on their height because some people are different sizes.” Often students also included one or more long jump ropes in the set, to allow for group jumping activities. Figure 5 below display such a solution.

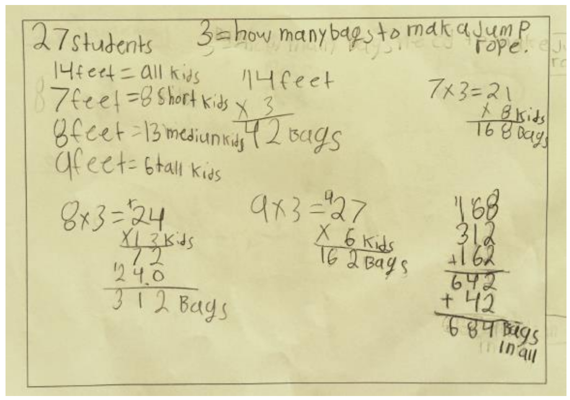

Figure 5: Mr. H’s Students Operate on a Model Based on Grouping Students by Height

As this group explained their model to the class, they highlighted the ropes of different lengths, and how they operated on those quantities to find the total number of bags needed (i.e., “we thought if one foot is 3 bags, we did 3 times 7 is 21 [bags for one 7-foot rope], 21 times 8 is 168 bags [for eight 7-foot ropes], etc.” Yet because they included only one 14-foot rope in their set, a classmate (Student 1) asked them to consider whether that configuration would allow for double-dutch style jumping (which requires two 14 foot jump ropes).

Student 1: For double-dutch you have to have two [jump ropes].
Mr. H: Oh, for double-dutch you have to have two? (to group of students who shared their model) Did you mean actual double-dutch or big jumping?
Student 2: [We meant] double dutch.
Student 3: I just want to jump rope.
Mr. H: If actual double-dutch was a requirement of our class, would that change our outcome? What would we need [to change]? … So if you are doing double-dutch what do we have to do for the bags for the 14 foot [rope] if we are doing double-dutch?
Student 2: We have to double it [double the number of bags]
Mr. H: We could double it to get a double dutch rope [to have enough for two ropes].

This exchange highlights how students’ prior experience and understandings related to jumping rope informed their analysis and interpretation of their own solutions (and those of peers), as well as the teacher’s roles in prompting students to consider refinements. We found that as students analyzed, interpreted and refined quantities (e.g., the number of jump ropes of a given length in a set), they often refined ideas generated in *Phase 2*, and these refinements were informed by the knowledge and experiences students brought to the task.

Discussion

Across the two lessons, we found that students' (and in some instances the teacher's) experiences and understandings were pivotal to their engagement in each phase of mathematical modeling. Students leveraged their experiences to identify important quantities and relationships, to make assumptions, to analyze and interpret the reasonableness of their solutions, and to revise their models when needed. Students' reliance on their own experiences and sense making highlights the potential of relevant mathematical modeling lessons to support students' empowerment as mathematical learners, as well as their real-world reasoning (Bahmaei, 2011). Perhaps the salience of student experience was a reflection of the relevance of the tasks. A well-chosen context supports students in developing "informal, highly context-specific models and solving strategies," (Doorman & Gravemeijer, 2009). Yet we also suspect that Mr. H's frequent invitations for students to share their experiences played a key role. Consistent with other research that has noted the positive impact of honoring students' ideas (Ladson-Billings, 2009), we found that students responded to teachers' invitations with interest, and that these discussions provided opportunities for students to consider and respond to the perspectives of others.

The most challenging phase of the modeling process was generalizing. This is in part due to time constraints; after an extended period of work students were left with minimal time to explore how their models could be used by others. Though not taken up, we see Mr. H's invitation for students to consider how their models could work in other similar situations as an initial step. Future research should investigate how to support generalization in elementary mathematical modeling, both through task development and lesson implementation.

Implications and Conclusion

Our findings have important implications for mathematics modeling instruction and research. Given the salience of children's funds of knowledge across all phases of the modelling process, teachers should explicitly elicit students' experiences and perspectives, and position these experiences as resources to support meaningful engagement in mathematical modeling. A detailed analysis of how specific teacher moves support connections to students' experiences during modeling lessons would be a productive focus for future research.

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FLIPPED INSTRUCTION IN ALGEBRA 1: IS IT AN OLD IDEA IN NEW CLOTHES?

Wenmin Zhao
University of Missouri
wenminzhao@mail.missouri.edu

Jessica Kamuru
University of Missouri
jgc7vc@mail.missouri.edu

Samuel Otten
University of Missouri
ottensa@missouri.edu

Zandra de Araujo
University of Missouri
dearaujoz@missouri.edu

What are the substantive differences between flipped and non-flipped instruction? This study examined the instruction of two teachers who have worked together within the same school using the same Algebra 1 curriculum for years. One teacher flipped his instruction (creating lecture videos assigned as homework), while the other teacher continued with non-flipped instruction. Data from classroom observations were analyzed qualitatively using the Flipped Mathematics Instruction Framework. Results show that although there were clear differences in the format of flipped and non-flipped lessons, there were also substantial similarities with regard to features of instruction (e.g., procedural mathematical development, teacher authority, and tasks with low cognitive demand). Our analysis indicates that flipped instruction is not necessarily an innovative model when compared with non-flipped.

Keywords: Flipped instruction, Instructional activities and practices, Algebra, Technology

Introduction

In the past decade, mathematics teachers have increasingly adopted or at least tried flipped instruction in their classrooms (Smith, 2014). *Flipped instruction* is an instructional model in which a teacher assigns videos or other types of multimedia to be viewed as homework, which frees up in-class time for other purposes such as practice problems or collaboration (Bergmann & Sams, 2012). With the increased availability of electronic devices accessible to students, flipped instruction holds promise in terms of allowing students access to lectures at their own pace (and also retroactive access), as well as allowing the teacher to use more of the classroom time for activities that are more collaborative than lecture-based.

Flipped instruction is often regarded as innovative due to the use of video technology or the fact that it appears on the surface to be different than the traditional lecture model of mathematics teaching. Some advocates of flipped instruction (e.g., Bergmann & Sams, 2012) believe that its innovative nature may lead to more favorable student outcomes. However, it is still unknown whether flipped instruction as implemented by secondary mathematics teachers is substantively different than non-flipped instruction. Therefore, in this study, we examined the similarities and differences between the instruction of two teachers, one employing flipped instruction and the other not, who are from the same mathematics team within the same school using the same curriculum for lessons on Exponential Rules. More specifically, in this paper we use the Flipped Mathematics Instruction Framework (Otten, de Araujo, & Sherman, 2018) to answer the following research question:

In what ways are these flipped and non-flipped lesson implementations similar and different with regard to activity formats, duration, instructional quality characteristics, and interactivity characteristics?

Literature Review

Many studies of flipped instruction have focused on its impact on various learning outcomes, presuming that flipped instruction is different in an important way from non-flipped instruction. Some of the studies revealed that secondary students in a flipped section have higher learning gains on their pre- and post-tests than those in a traditional section (e.g., Bhagat, Chang, & Chang, 2016; Charles-Ogan & Williams, 2015), whereas others have shown no differences in learning gains between the flipped and non-flipped sections (e.g., Clark, 2015; DeSantis, Van Curen, Putsch, & Metzger, 2015). In either case, there was insufficient detail with regard to what was happening in the classroom. The authors did not report on the substantive differences between flipped and non-flipped instruction, nor did they draw inferences about which features of the instruction contributed to learning outcomes.

Some studies (e.g., Maciejewski, 2018; Rudd et al., 2017) included classroom observations of the in-class activities but did so generally. For example, Maciejewski (2018) evaluated the effectiveness of a flipped undergraduate calculus course. Students in the non-flipped sections spent more time listening to the instructor, while students in the flipped sections spent more time on individual or group work. In another study, Rudd and colleagues (2017) reported that elementary students in a flipped mathematics section had many opportunities in class to explain what they had learned at home and to solve real-life problems. However, these studies only recorded the types of activities and teacher actions without considering more detailed quality indicators such as conceptual development and interactivity.

In the present study, we look more closely at the lesson implementation in flipped and non-flipped instruction and we add to the growing research base of flipped instruction in secondary mathematics by focusing on Algebra 1. This mathematical setting is important because Algebra predicts the future success of students (Williams, 2011) and mathematical concepts introduced in Algebra are critical for mathematics learning in future mathematics courses (Carraher & Schiemann, 2007).

Theoretical Framework

The framework (Figure 1) of this study draws upon existing frameworks (e.g., Remillard & Heck, 2014; Stein, Grover, & Henningsen, 1996), observation instruments (e.g., MQI, M-Scan), and advice from experts in different fields (e.g., educational technology, mathematics education). Our framework, which is built for a lesson-level scope, allows us to distinguish the different roles that students participate either when viewing the video/multimedia at home or when taking part in in-class activities (Otten et al., 2018). The *in-class* phase captures what occurs during class time, including the *whole-class* and *non-whole-class* activity formats. The *at-home* phase captures the expected activities of students outside the classroom and includes any videos, multimedia, or traditional homework problem sets assigned to students. Within each phase, we focused on certain aspects of the quality of implementation.

We used our framework to develop a classroom observation protocol (Zhao, Han, Kamuru, de Araujo, & Otten, 2018) to capture the general lesson characteristics in terms of focus (what is to be learned), rationale (why is it to be learned), and flow between activities (how the learning activities link together or not). We further distinguished two main in-class components of a lesson, *whole-class* and *non-whole-class* discourse. The *whole-class* discourse is when everyone in the class is expected to be attending to the public discourse regardless of whether it is the teacher (e.g., lecture) or another student (e.g., classroom discussion or student presentation) who is speaking. Within the *whole-class* discourse, we noted the quality of instruction regarding the mathematical development of ideas, integration of mathematical representations, absence of unmitigated errors, and connections to past/future mathematics content. We also focused on the interactivity during the *whole-class* discourse, namely, the exhibited mathematical authority, student public involvement, the sharing/collaborative nature of discourse (Staples & Colonis, 2007), and video/media involvement.

The *non-whole-class* discourse component of our protocol captured time spent when students were expected to work either in groups or independently. We further specified the extent of peer talk, students' use of video/multimedia (e.g., if students watched an instructional video while they were working in class), teacher circulation, and cognitive demand of the tasks. In addition, our framework allowed us to analyze the level of behavioral engagement during the *whole-class* and *non-whole-class* discourse.

For the *at-home* phase, the key video characteristics were examined along three aspects: instructional quality, multimedia design, and interactivity. Instructional quality was the same as described above. The multimedia design was analyzed according to Clark and Mayers' (2008) six digital material design principles: multimedia, modality, contiguity, redundancy, coherence, and personalization. Another aspect, interactivity, documented the interactive elements of the video, such as embedded questions, discussion boards, or virtual manipulatives. More details can be found from Otten, Zhao, de Araujo, and Sherman (in press). Due to the space restriction, the findings of the *flow* and *multimedia design* in this study will not be reported in this paper.

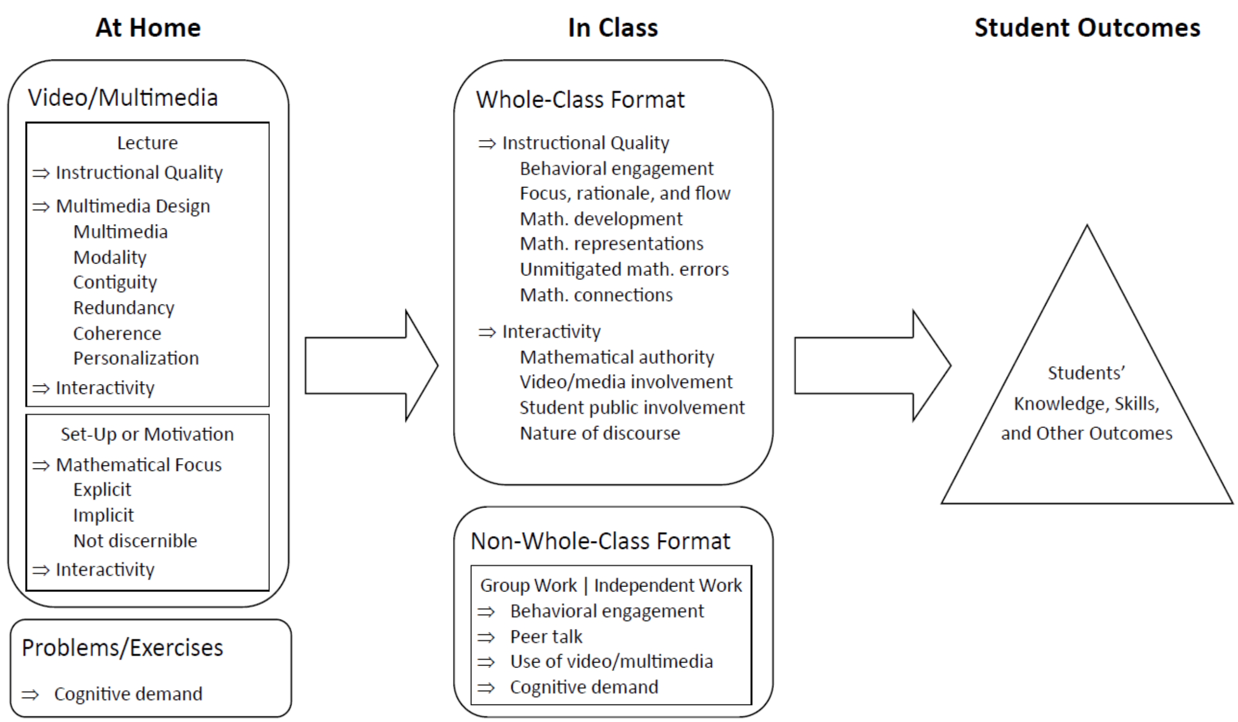

Figure 1: Framework for Flipped Mathematics Instruction (Otten et al., 2018)

Mode of Inquiry

Sample and Setting

The two teacher participants, Mike and Kristen, were both Algebra 1 teachers working at the same public high school in a rural area of Missouri. They worked on the same mathematics team and shared the same curriculum. Mike has a Bachelor of Science in Mathematics. He had eight years of teaching experience and had been using flipped instruction in Algebra 1 for two years. Mike flipped more than 75% of his lessons but still incorporated non-flipped homework (e.g., problem sets) or sometimes no homework. Mike's class had 22 students who knew the class was flipped when they enrolled, though that may not have been the primary motivation in choosing his class. The other teacher, Kristen, had a Bachelor of Science in Secondary Mathematics Education and nine years of mathematics teaching experience. She had 23 students in her class and was *not* flipping her instruction.

Data Collection and Analysis

The primary data sources for this study were classroom observations and lesson artifacts (e.g., video homework and in-class worksheets). Two researchers observed the classroom instruction three times during a semester using our observation protocol. In this study, we focused on the data from the first classroom observation and examined both teachers' lessons on the rules for operating with expressions that involve exponents (Missouri Learning Standards–A1.NQ.A.1) which is an important topic in Algebra 1. The length of Kristen's lesson was 47 minutes, and the length of Mike's lesson was 47 minutes with an additional seven minutes and three seconds of lecture video assigned to be watched before class. Mike produced the video using a digital pen and a tablet, with an audio voice-over from himself (Figure 2).

Based on the observation protocol, we analyzed the field notes from the classroom observation and coded the interactive features as well as the quality of classroom instruction and the instructional video. A third rater reconciled disagreements between the two raters.

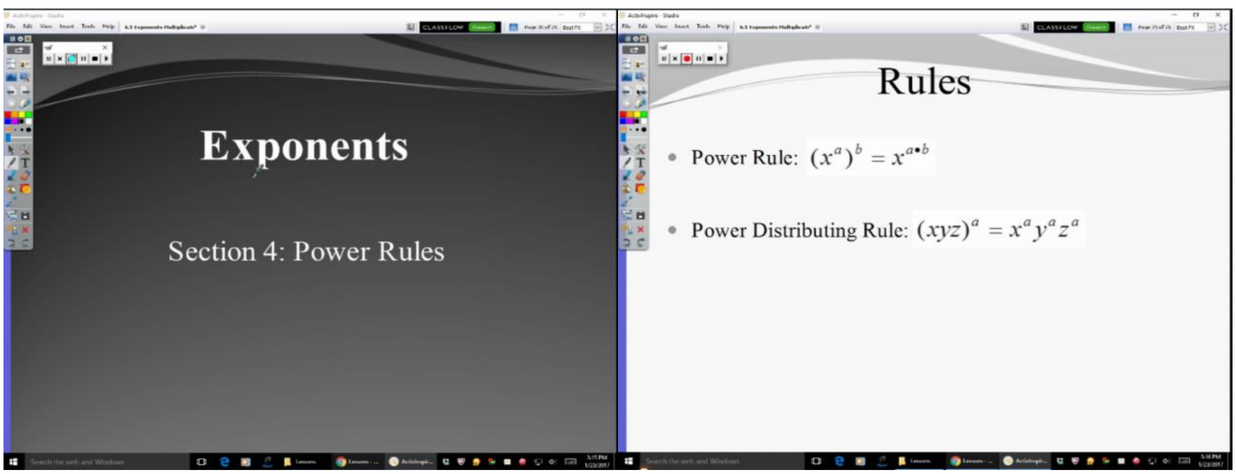

Figure 2: The Lesson Focus on Mike's Lecture Video

Findings

In this section, in our comparison of the two teachers' lessons, we share the research results focusing on the differences and similarities in the (1) lecture video and whole-class discourse, and the (2) non-whole-class discourse phases of the flipped and non-flipped lessons with regard to the instructional quality, interactivity, cognitive demand of the tasks, and student engagement.

Video and Whole-Class-Discourse Phases of the Lessons

Both lessons started with a lecture. Mike's lecture took the form of a video assigned as homework before class, with a brief follow-up in class (i.e., Mike used a whole-class discourse format to go over a few key ideas from the video). Kristen's lecture began when students entered her classroom. Beyond the obvious distinction in the modalities of the lessons (i.e., at-home video vs. in-class lecture), we captured the differences and similarities in terms of the time allocation of in-class activities, lesson focus, rationale, mathematical development, mathematical errors, mathematical connections, use of multiple representations, and nature of discourse.

Differences. As previously mentioned, both lessons began in different physical locations and modalities of interaction. The lecture portion of Mike's lesson started with his seven-minute video. He then spent another eight minutes of class in a whole-class discourse format, for a total of 15 minutes as the first portion of the lesson. A unique piece of Mike's lesson was that his initial whole-class discourse involved several verbal references back to the video, to which students still had access. In contrast, Kristen's lecture was live and started in-person in the classroom. She spent 24 minutes (rather than 15) addressing the class on the content of the lesson and, unlike Mike, she wove this whole-class discourse with brief instances of independent work time (i.e., students tried an example after a worked example).

Similarities. Although there were differences in time allotment, modalities, and spaces of instruction, the teachers had many similarities in their lesson enactments. First, both teachers started their lessons with **content delivery**, which included **an explicit lesson focus** but did not include a **rationale** for why it should be learned. For example, in Mike's lecture video, he started the lesson with the **explicit focus**, "Power Rules," on the screen (Figure 2). Mike mentioned that the lesson would involve learning how to multiply exponents and how to apply power rules and power distributing rules. When the lesson continued into the in-class time, Mike was also clear about the **lesson focus**; he explicitly mentioned that students would continue to learn the properties of exponents. However, in neither the at-home nor in-class phases did Mike give students **a rationale for learning** the rules. Similarly, near the beginning of Kristen's lesson, she displayed a PowerPoint slide that explicitly introduced the **focus of the lesson** and displayed the definition of "Quotient of Powers." Like Mike, however, she too gave **no additional rationale**, such as a real-world application or an expanded understanding of mathematical operations, for why students should learn the lesson.

In terms of the mathematical instructional quality, the lectures were also similar. Both had a **procedural emphasis** with regard to mathematical development. For example, in the video, Mike began by showing the power-to-a-power rule and "power distributing" rule. He then applied the rules to solve problems in which he explicitly directed students to "distribute" the "four" or the "two" as an exponent onto all of the interior terms (Figure 3). Mike did not give reasons for why the rule works. Similarly, Kristen showed the students $\frac{a^m}{a^p} = a^{m-p}$ on the board and worked through three examples. She offered no further explanation other than the procedure that she presented. At no time during the observed lesson did Kristen offer the students a conceptual model to understand the procedures in the lesson.

For both lectures, neither Mike nor Kristen committed any **mathematical errors**. Both teachers referred briefly to the previous lesson, yet neither gave substantial emphasis to the **mathematical connections** between the lessons. For example, Mike mentioned that the new lesson was connected to the previous lesson which only included multiplication of exponents with same bases. However, he did not explicitly explain how the new lesson developed from the previous one. In comparison, Kristen gave her students procedural problems which were a review of the previous lesson as a warm-up before class began. She went over those problems and reminded students of the power rules that they had already covered (e.g., multiplication rule). Then she started her new lesson without any conceptual connection between the previous lesson and the new lesson. In both lessons, the teachers only solved the problems symbolically without attending to other types of **representations** (e.g., tables and graphs) to help students understand the Exponential Rules.

The nature of discourse of both lectures was "sharing" (Staples & Colonis, 2007) because ideas were generally conveyed from the teacher as a source of mathematical authority to the students as receivers. Students were not oriented to each other's thinking or encouraged to connect, extend or critique ideas. Most students only answered in one-word answers or small phrases. For instance, Kristen wrote an equation on the board, $\frac{1c^4d^4f^3}{2c^2d^4f^2}$, then she asked, "What can you tell me about the *d*'s?" The students chorally responded, "Cancel out." We found that leading questions with fixed answers dominated most of both Kristen's and Mike's classroom discourse. The teachers were the **authority** over the mathematics knowledge in both lessons. Kristen and Mike, rather than the classroom community or students, determined the validity of a student's strategy. Overall, **students' public involvement** in both classes was low. Few students asked questions related to the procedures that the teachers worked on the board or responded to the teachers' prompts.

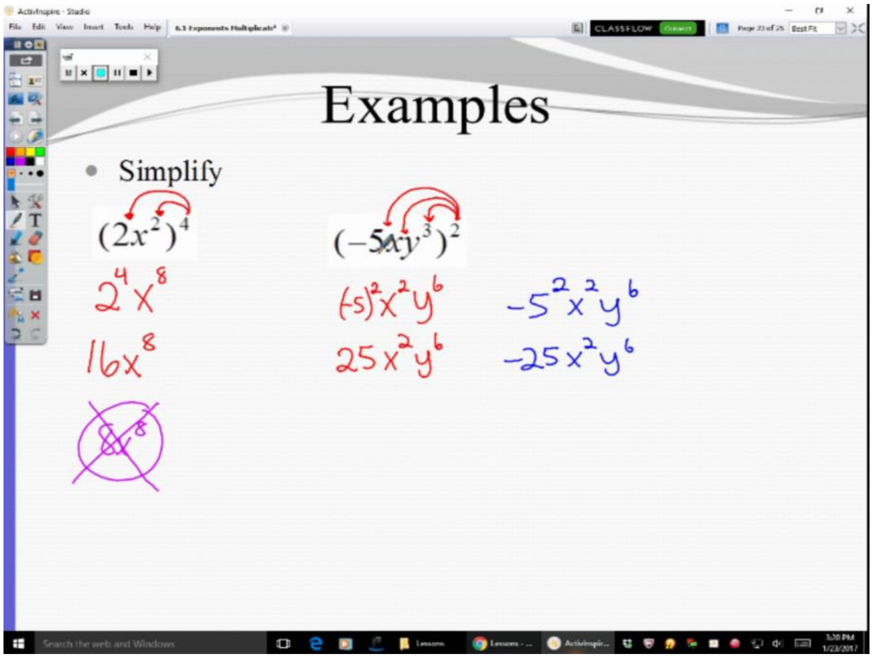

Figure 3: Two Examples in Mike's Lecture Video

Non-Whole-Class-Discourse Phases of the Lessons

For both Mike and Kristen, the lecture or content delivery portions of the lessons were followed by a substantial amount of non-whole-class-discourse (i.e., independent) work time. During the independent time, the teachers were available for questions or even purposely attended to certain students, however, the expectation was for students to work on their own. During this portion of both flipped and non-flipped lessons, we examined three lesson characteristics: the length of the independent work time, cognitive demand of the practice problems, and student engagement.

Differences. The two lessons differed in the amount of time that each teacher allotted for independent work. Because Mike's flipped lesson moved seven minutes of content delivery to the homework assignment, he gave substantially more in-class time for students to work on the practice problems (36 minutes out of 54, or 66.7%). Kristen's lesson, on the other hand, 20 minutes out of 47, or 42.6% was independent work comprising practice after the lecture was completed. Some of Kristen's students finished all the problems during that time, but others were left to finish additional problems at home.

Similarities. It is worth noting that both lessons were similar in the fact that they followed the content delivery portion by then providing students with independent work time for practice problems. The lessons were also similar in that the **cognitive demand** of the practice problems was low. In both Mike and Kristen's classes, they passed out a worksheet with the prompt "Simplify. Your answer should contain only positive exponents." Problems such as $(2m^3)^3$and $(-4n^4)^{-1}$were included in Mike's worksheet. Problems such as $\left(-\frac{vu^4}{2u}\right)^3$and $\frac{(-2x^3y^4)^4}{-2x^4}$were included in Kristen's worksheet. We considered these problems as *procedures without connections* since the main intent of the problems was to produce the correct answer without conceptual connections to the procedure beyond what their teachers had already demonstrated. There was also no explanation or justification required.

The levels of **student engagement** were similar in both lessons. During the students' independent work time, the behavioral engagement was mixed (i.e., many students on task for a majority of the time but also many students off task for at least a substantial portion of the time). In both classes, there were periods where the vast majority of students were working on the practice problems, but in Kristen's class, some were slow to get started and talked to their peers about non-mathematics related topics. In Mike's class, and to a lesser extent in Kristen's class, as some students completed their worksheet, they started to chat with their classmates about non-mathematics related content.

Figure 4 depicts a summary of the two lessons examined in this article.

Conclusion and Implications

The major objective of this study was to compare and contrast the instructional implementations of a flipped and a non-flipped lesson by teachers from the same school using the same curriculum. Many people continue to uphold flipped instruction as an innovative instructional method, which allows teachers to make better use of their in-class time to create more collaborative activities leading to a deeper conceptual understanding of the content. However, as is evident from our analysis, this is not necessarily the case.

Although the modalities of delivery (i.e., in person vs. via video) and the time allocation between the two lessons were different, we found similarities in the instructional quality, interactivity, cognitive demand of the practice problems, and student engagement of both lessons. The flipped instruction, as we examined in this paper, did not move beyond the "video lecture at home and homework in the classroom" model of instruction and in this way remained similar to the non-flipped instruction. Thus, in this case, it seems that merely flipping the spaces and modalities through which content is delivered does not necessarily change the nature of instruction. We invite researchers to be more cautious in crediting flipped instruction as an innovative model of mathematics teaching. Our framework allows us to gain insights into the nuances of similar lessons taught in both flipped and non-flipped classrooms. In reporting future findings of flipped classrooms, we encourage researchers to consider the importance of reporting the details of actual classroom instruction.

Acknowledgments

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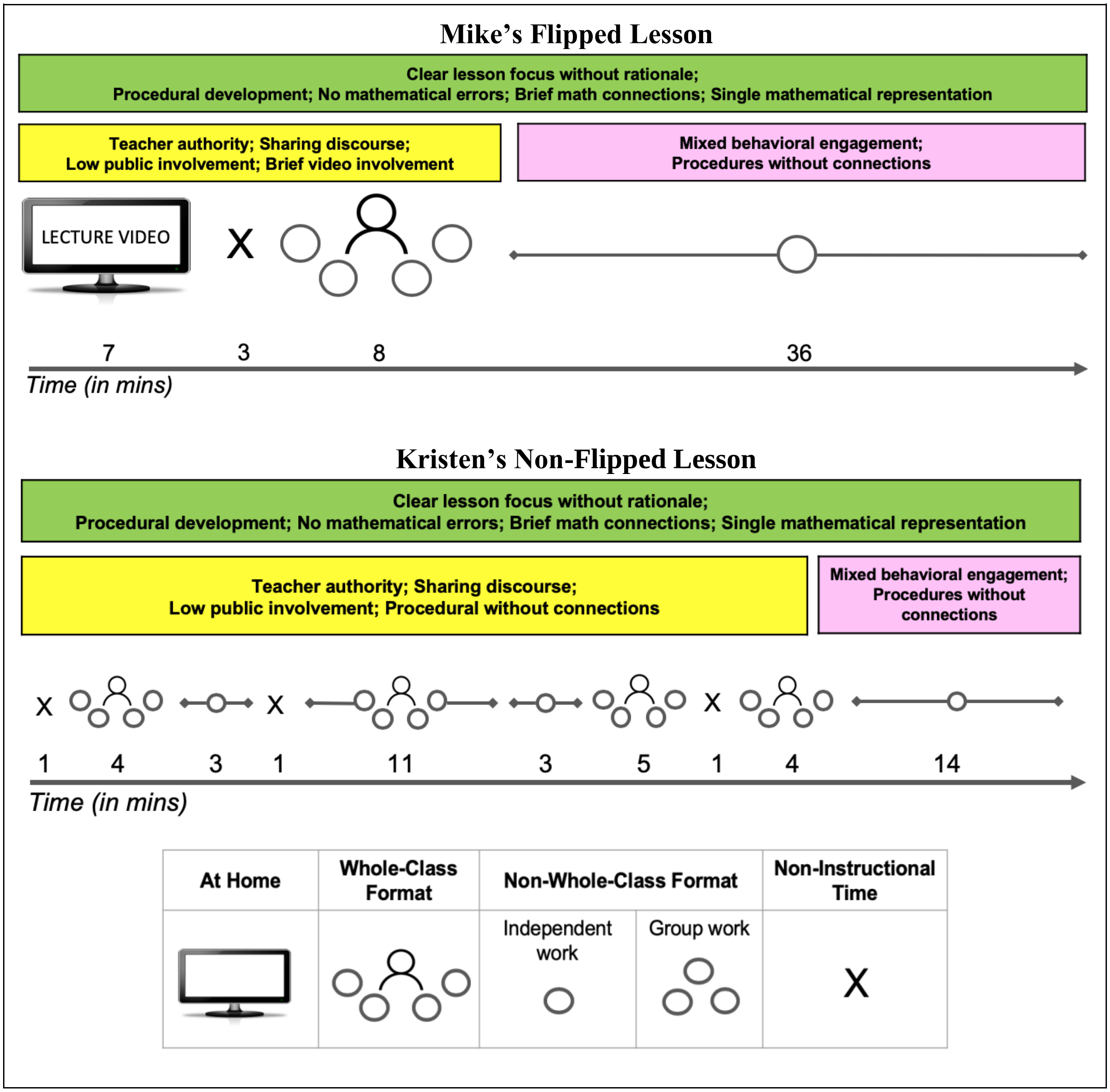

Figure 4: Comparison of the Flipped and Non-Flipped Lesson Implementations

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CRITERIOS DE SELECCIÓN DE RECURSOS DIGITALES PARA LA ENSEÑANZA DE LA GEOMETRÍA EN PRIMARIA

Marisol Santacruz-Rodríguez
Cinvestav
mary21391@gmail.com

Ana Isabel Sacristán
Cinvestav
asacrist@cinvestav.mx

La selección de recursos para la enseñanza es una actividad desafiante en el trabajo de los profesores. Determinar qué recursos usar requiere de conocimientos profesionales específicos que orientan la acción del profesor. En este sentido, presentamos algunos resultados de un estudio que utiliza la Aproximación Documental de la Didáctica para analizar el trabajo documental de un grupo de profesores de primaria al seleccionar recursos digitales para la enseñanza de la geometría en quinto grado. Para inferir los esquemas que orientaron su selección de recursos, se llevó a cabo una investigación reflexiva, utilizando también una técnica de introspección. El análisis obtenido permitió inferir invariantes operatorias en el proceso de selección de los maestros, las cuales categorizamos como "criterios de selección". Ejemplificamos todo esto a través del análisis de datos de dos profesores.

Palabras claves: Actividades y Prácticas de Enseñanza, Tecnología

Introducción

La selección de recursos es una parte fundamental del trabajo documental del profesor (Gueudet & Trouche, 2009), la cual involucra conocimientos profesionales mediante los cuales los profesores identifican recursos y los usan en clase. En este artículo presentamos resultados de un proyecto que buscaba analizar la selección de recursos digitales para la enseñanza de la geometría por profesores de primaria en Colombia. Dicho proyecto tomaba, como marco teórico y metodológico, la Aproximación Documental de la Didáctica propuesta por Gueudet y Trouche (2009, 2012).

En el proyecto se estudió el trabajo de cinco profesores de primaria para sus clases de geometría. En Santacruz y Sacristán (2016) ya habíamos reportado datos de una primera fase de investigación donde se infirieron las rutas de selección y orquestación de recursos digitales que siguieron dos de los profesores participantes. Ahora presentamos datos de una segunda fase del estudio, analizando los casos de dos profesores de quinto grado para inferir y categorizar sus conocimientos profesionales que orientaron sus procesos de selección de recursos en "criterios de selección" (considerados invariantes operatorias en el sentido de Vergnaud, 1998).

Marco teórico

La Aproximación Documental de la Didáctica o ADD (Gueudet &Trouche, 2009, 2012), consiste en un enfoque teórico y metodológico que se ocupa del estudio del trabajo documental de los profesores. El trabajo documental es el conjunto de actividades que realiza el profesor en interacción con recursos; incluye actividades como la selección, diseño, adaptación, y uso de recursos, así como el trabajo colaborativo del profesor con colegas y sus reflexiones sobre su trabajo.

Una de las ideas centrales de la ADD es el concepto de recurso y sus usos. Adler (2000) considera que un recurso es todo aquello que da sentido al trabajo del profesor; y contempla que los recursos pueden ser: materiales, humanos y culturales. Otro concepto clave en la ADD, es el de documento (por ende se habla de trabajo documental), el cual no es algo que se le proporciona al profesor, sino que éste lo construye por medio de procesos de génesis documental; es decir, un documento se construye a partir de los recursos disponibles mediante un proceso extenso y continuo a lo largo del tiempo. Gueudet y Trouche (2009) consideran que un documento está constituido por uno o varios recursos junto con sus respectivos esquemas de utilización.

Los esquemas de utilización de los recursos corresponden a estructuras cognitivas del profesor que orientan el uso que él hace de ellos (Gueudet & Trouche, 2009). Vergnaud (1998, citado por Gueudet & Trouche, 2009) señala que los esquemas dependen de la situación, de la intención y competencia del sujeto, por tanto, no son algo predefinido. Los componentes de los esquemas son: metas y sub-metas; anticipaciones; reglas de acción, invariantes operatorias (que corresponden a los conocimientos contenidos en el esquema y son de dos tipos: concepto-en-acto y teorema-en-acto) y posibilidades de inferencia (Vergnaud, 1998). En nuestro estudio inferimos los aspectos arriba señalados mediante una técnica de introspección descrita más abajo.

Metodología

Como se dijo, en nuestro estudio analizamos la selección de recursos de cinco profesores de primaria en Colombia. Para su participación, se tomó en cuenta: su participación voluntaria, su interés en la enseñanza de la geometría y el uso, en su práctica docente, de recursos digitales.

El desarrollo de nuestro estudio se fundamentó en la investigación reflexiva de Gueudet y Trouche (2012), mediante la cual se fomenta, en entrevistas, que los profesores lleven a cabo una mirada retrospectiva de su práctica, para estudiar su trabajo documental. En el estudio empezamos observando y analizando la planeación de clases de los profesores participantes, su bitácora (registro) de clase, y, sus clases (*in situ*). Posteriormente, en entrevistas con cada profesor, se aplicó una “técnica de introspección” (ver abajo), para promover su reflexión y la elaboración de representaciones de su ruta de selección de los recursos para sus clases. Los datos recabados se analizaron para inferir los componentes de los esquemas (en términos de Vergnaud, 1998) que orientaron la selección de recursos por cada profesor, principalmente sus invariantes operatorias o criterios de selección.

La técnica de introspección

Esta técnica integra aspectos de la investigación reflexiva (Gueudet & Trouche, 2012) y de la investigación introspectiva (Vollstedt, 2015). Consiste en estimular la introspección para evocar conocimientos profesionales de los profesores a través de “recuerdos estimulados” (Calderhead, 1981). Dicha técnica la diseñamos para que el profesor evoque (*a posteriori*) sus pensamientos y acciones en un momento o actividad específica; la usamos para indagar sobre lo que llamamos la “ruta-recorrida” por el profesor en su selección de recursos. La técnica se aplicó durante entrevistas donde se presentaba a cada profesor información de su práctica (extractos de videos, de fotos, de las planeaciones de clase, etc. –generalmente una semana después), y se le pedía que representara sus procesos de selección de recursos mediante mapas o diagramas, junto con explicaciones verbales. Los datos que obtuvimos a través de nuestra técnica de introspección fueron transcritos, segmentados, codificados y categorizados para inferir los esquemas, particularmente las invariantes operatorias (los criterios de selección) de estos esquemas.

Estudios de caso de dos profesores: Sonia y Pedro

Sonia y su selección de recursos, basada en las necesidades de sus estudiantes

Aquí, analizamos el tipo de recursos que Sonia seleccionó para su clase y propuso a sus estudiantes respecto a un tema específico: la estimación y medición de longitudes. Sonia es una profesora normalista, licenciada en enseñanza de las matemáticas, con once años de experiencia como maestra de primaria; trabaja en una escuela pública que atiende población vulnerable.

Identificamos que Sonia ha ido reuniendo una colección de recursos para la enseñanza que incluye: recursos curriculares, textos escolares con sus correspondientes guías del profesor, hojas de trabajo para estudiantes (que ella misma diseña o adapta), y presentaciones de diapositivas (que incluyen imágenes, hipervínculos a sitios web o a videos).

Sonia, en la primera clase que observamos, propuso a sus estudiantes algunas actividades de estimación de longitudes mediante comparaciones de líneas con igual longitud (aunque de distinta forma); posteriormente, introdujo el uso de uso de la cinta métrica pidiendo a los niños que agregaran marcas en una tira de papel. En la siguiente clase, Sonia decidió introducir algunas ideas del sistema métrico decimal, para lo cual decidió usar un recurso digital –un *applet* en GeoGebra (https://www.geogebra.org/m/qYHjHDAf)– para que los estudiantes observaran la medición de longitudes usando una regla graduada virtual. En una de las sesiones de planeación de Sonia de sus clases, observamos cómo esta maestra realizó una búsqueda extensa en un repositorio de recursos de GeoGebra, encontrando finalmente ese *applet*. Para profundizar en la manera en que Sonia seleccionó ese recurso, utilizamos nuestra técnica de introspección.

Para la elaboración de su "ruta-recorrida", le propusimos a Sonia representar su proceso de selección de ese *applet*. Le proporcionamos a Sonia el video de la sesión en la cual ella buscó el recurso por Internet, lo seleccionó y lo adaptó para su clase.

En su mapa de su ruta-recorrida es notorio el interés de Sonia por organizar sus acciones en bloques de actividades consecutivas, tomando en cuenta: las orientaciones curriculares, qué ha hecho en clases pasadas (registrado en su bitácora de clases, que ella llama "parcelador"), y los libros de texto. Para Sonia es importante ubicar su selección dando seguimiento a sus clases previas y tomando en cuenta hacia dónde quiere avanzar con sus estudiantes. Más aún, ella considera que el uso de recursos digitales puede servir para atender las dificultades de estudiantes con bajo desempeño escolar.

La situación de selección de Sonia se caracterizó por su búsqueda de un recurso digital que complementara un trabajo que ya venía realizando en clase y que se articulara con la hoja de trabajo para los estudiantes, ya prevista. De allí que sus anticipaciones se relacionaran con el repositorio en el cual buscó el recurso, en términos de la calidad ergonómica (el diseño del recurso) y didáctica (la situación que el recurso le propone a los estudiantes).

Las invariantes operatorias) que inferimos como parte del esquema de Sonia para seleccionar recursos digitales para su clase, se centran en el papel de la estimación, de la equivalencia, el concepto de unidad de medida, y el uso de geometría dinámica. Dichas invariantes muestran su interés por focalizar su enseñanza en aspectos que ella considera centrales y que van más allá de aprender fórmulas. Por ejemplo, Sonia tomaba en cuenta criterios de orden didáctico-matemático cuando señalaba: *"De nada sirve que los niños se aprendan las tablas de equivalencia del sistema métrico si eso no tiene sentido para ellos. […] Usando el* applet*, los niños pueden estimar, medir, mover, hacer cosas diferentes, pero en una actividad controlada que no sea mover 'por mover puntos', sino que sea guiada y que les quede claro que están estudiando."*

Pedro y su proceso de selección de recursos para diseñar en la clase de geometría

Pedro es normalista, licenciado en matemáticas con énfasis en computación, y experiencia docente de 9 años (en primaria y secundaria); los últimos 4 años ha trabajado como profesor de matemáticas (en la escuela participante en el estudio) en grados cuarto y quinto. A este profesor le hicimos seguimiento durante 2-3 meses cada año, en dos años escolares.

Para que sus estudiantes de quinto grado aprendieran sobre las magnitudes, su medida y medición, Pedro propuso a los niños un proyecto donde diseñaran y construyeran una casa; de esa manera, tenían que realizar mediciones y trabajar en colaboración. El proyecto involucró trabajo con papel, lápiz, reglas y compás y el uso de un recurso digital: Sweet Home 3D (http://www.sweethome3d.com/) el cual incluye herramientas diversas para el diseño de una casa (e.g., herramientas para hacer bocetos, para medir, para modificar lo construido, etc.).

En una de las clases observadas, la tarea era que los alumnos reprodujeran y modificaran la escala de un plano de una casa que habían trazado previamente en sus cuadernos, y construyeran un primer boceto de su diseño usando el recurso digital. Las principales ideas matemáticas involucradas en esta actividad eran el cambio de escala y la medición de longitudes (de las paredes y otras partes de la casa).

El mapa de la ruta-recorrida por Pedro para seleccionar su recurso digital, tuvo varias versiones a lo largo del tiempo. En la última versión de su mapa hay varios elementos a tener en cuenta y que Pedro enfatizó al explicar su ruta: Pedro resalta, en primer lugar, la necesidad de contar con criterios didácticos y matemáticos claros a la hora de escoger un recurso para la clase. Consideramos a esos criterios de selección como invariantes operatorias del esquema de Pedro. Pedro escoge el recurso dependiendo de si puede adaptarse a sus objetivos: en este caso considerando la disponibilidad de herramientas para el diseño y la medición, y luego definiendo otros criterios (e.g. de orden ergonómico: de libre acceso, "agradable", "motivante"). También pone su énfasis en el trabajo colaborativo. Por ejemplo, Pedro tomó en cuenta criterios de orden didáctico-matemático y ergonómico cuando señaló: "*Busco una herramienta que sea útil para muchas cosas, porque estamos trabajando en un proyecto de varias semanas. […] La idea es que, al trabajar con Sweet Home, los niños pueden ver qué cosas se pueden medir y qué cosas no se pueden explorar, y que prueben diferentes [herramientas].*"

Los criterios de selección de Sonia y Pedro

Sonia se ubica en una situación de selección "local", dado que escoge recursos específicos para una clase particular; mientras que la situación de selección de Pedro es "global", en el sentido de que escoge un recurso para promover un proyecto que involucra muchas sesiones de clase. Los principales criterios de selección de recursos de Sonia, fueron de orden didáctico-cognitivo: la visualización, la aplicación de las equivalencias y la interacción estudiantes-recursos. Sus criterios de selección fueron bastante específicos respecto a la demanda cognitiva del recurso y del currículo. En el caso de Pedro, sus criterios se centran en diseñar varios tipos de tareas para sus estudiantes, tener muchas posibilidades para tareas matemáticas y exploraciones, y permitir el trabajo colaborativo de sus estudiantes; de hecho, poder utilizar el recurso de múltiples maneras, era uno de los principales criterios de selección de Pedro para el recurso.

Conclusiones

Así pues, en este trabajo buscamos inferir los conocimientos y esquemas de los profesores participantes, específicamente sus criterios, que orientan su trabajo documental y su proceso de selección de recursos. Para esto, el uso de nuestra técnica de introspección permitió recabar datos del trabajo documental de cada profesor y analizarlos de acuerdo el contexto de cada profesor. En particular, nuestro intento por inferir y categorizar los conocimientos profesionales de los profesores nos permitió reconocer la variedad de conocimientos involucrados, en particular las invariantes operatorias de los esquemas de cada profesor, identificadas como criterios de selección. Los criterios de selección de los dos profesores presentados aquí, fueron categorizados en tres tipos: (i) Criterios de orden didáctico-matemático (los cuales corresponden al uso didáctico del recurso y el tipo de tareas matemáticas que se pueden desarrollar a través de él). (ii) Criterios de orden cognitivo-curricular (los cuales se refieren a la demanda cognitiva del recurso y las orientaciones curriculares que el profesor tiene en cuenta para su selección). (iii) Criterios de orden ergonómico: los cuales contemplan aspectos técnicos y de funcionamiento del recurso.

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CRITERIA OF DIGITAL RESOURCE SELECTION FOR THE TEACHING OF GEOMETRY IN PRIMARY SCHOOL

Marisol Santacruz-Rodríguez
Cinvestav
mary21391@gmail.com

Ana Isabel Sacristán
Cinvestav
asacrist@cinvestav.mx

The selection of resources for teaching is often a challenging task in teacher's practice. Determining what resources to use, requires professional knowledge that guides the teacher's action. It is in that sense that we present here some results from a study based on the Documentational Approach to Didactics, which focuses on analyzing primary-school teachers' documentation work when selecting digital resources for their teaching of geometry in fifth-grade. In order to infer the schemes that guided their selection of resources, we carried out a reflective investigation and used, as well, an introspective technique. Through the analysis of the data, we were able to identify certain operational invariants in each teacher's resource selection process, which we classified in "selection criteria". We illustrate the above, through the analysis of data from two teachers.

Keywords: Instructional Activities and Practices, Technology.

Introduction

The selection of resources is a fundamental part of a teacher's documentation work (Gueudet & Trouche, 2009), involving professional knowledge through which teachers identify resources and use them in class. In this paper, we present results derived from a project that sought to analyze the selection of digital resources for the teaching of geometry by primary-school teachers in Colombia. That project used, as a theoretical and methodological framework, the Documentational Approach to Didactics proposed by Gueudet and Trouche (2009, 2012).

We studied, in that project, the work of five primary-school teachers in their geometry classes. In Santacruz and Sacristán (2016) we had already reported data from a first research phase, presenting the selection paths and orchestration of digital resources that two of the participating teachers followed. We now present data from a second phase of the study, analyzing the cases of two fifth-grade teachers, and infer and categorize their professional knowledge that guided their resource selection processes into "selection criteria" (considered as operational invariants in Vergnaud's, 1998, terms).

Theoretical Framework

The Documentational Approach to Didactics or DAD (Gueudet & Trouche, 2009, 2012), consists of a theoretical and methodological approach that deals with the study of the documentation work of teachers. The documentation work is the set of activities that the teacher performs in interaction with resources; it includes activities such as the selection, design, adaptation, and use of resources, as well as the collaborative work of the teacher with colleagues, and his/her reflections on his/her work.

One of the central ideas of ADD is the concept of resource and its uses. Adler (2000) explains that a resource is everything that gives meaning to the teacher's work; she considers that the resources can be: material, human and cultural. Another key concept in ADD is that of a document (which is why we talk of documentation work), which is not something that is provided to the teacher, but that the teacher constructs through a process of documentational genesis; that is, a document is constructed over time from available resources through an extensive and continuous process. Gueudet and Trouche (2009) consider that a document is constituted by one or more resources, together with their respective usage schemes.

The schemes of resource use correspond to cognitive structures of the teacher that guide the use he makes of them (Gueudet & Trouche, 2009). Vergnaud (1998, cited by Gueudet & Trouche, 2009) points out that the schemes depend on the situation, the intention and mathematical competences of the subject and, therefore, are not predefined. The components of the schemes are: goals and sub-goals; anticipations; rules of action, operational invariants (that correspond to the knowledge contained in the scheme and are of two types: concept-in-act and theorem-in-act) and possibilities of inference (Vergnaud, 1998). In our study, we inferred the above components through the use of an introspective technique described below.

Methodology

As mentioned above, in our study we analyzed the selection of resources of five elementary teachers in Colombia. For their participation in our study, we took into account: their voluntary participation, their interest in the teaching of geometry and their use, in their teaching practice, of digital resources.

Our study was based on the reflective investigation proposed by Gueudet and Trouche (2012), through which, in order to study teachers' documentation work, they are encouraged, in interviews, to carry out a retrospective reflection of their practice. In our study we began by observing and analyzing, the participating teachers' class planning, their class logs, and their classes (*in situ*). Subsequently, in interviews with each teacher, an "introspective technique" was applied (see below), to promote their reflection and the elaboration of representations of their paths of selection of resources for their classes. The data collected was analyzed to infer the components of the schemes (in Vergnaud's terms, 1998) that guided the selection of resources by each teacher, mainly the operational invariants or selection criteria.

The Introspective Technique

This technique integrates aspects of Gueudet and Trouche's (2012) reflective investigation and of Vollstedt's (2015) introspective research. It consists of a "stimulated recall" (Calderhead, 1981) in order to evoke teachers' professional knowledge. This technique is designed for the teacher to evoke (*a posteriori*) his/her thoughts and actions at a specific time or in a specific activity; we use it to investigate the route or path followed by the teacher in his selection of resources. The technique was applied during interviews where each teacher was presented with information from their practice (extracts of videos, photos, class planning, etc. –usually a week later), and asked to represent his/her resources selection processes, through maps or diagrams, along with verbal explanations. The data we obtained through the use of that introspective technique, was transcribed, segmented, coded and categorized, in order to infer the schemes particularly the operational invariants (the selection criteria).

Case Studies of Two Teachers: Sonia and Pedro

Sonia and Her Selection of Resources, Based on the Needs of Her Students

We analyze here the types of resources that Sonia selected and that she proposed to her students for her class on a specific topic: the estimation and measurement of lengths. Sonia is a primary-school teacher, with a bachelor's degree in mathematics teaching, with eleven years of experience; she works in a public school with a low-income student population.

We identified that Sonia has been gathering a collection of resources for teaching that includes: curricular resources, textbooks with corresponding teacher's guides, worksheets for students (which she designs or adapts), and slide presentations (including images, and hyperlinks to websites or videos).

In the first class that we observed, Sonia proposed length estimation activities to her students for comparing lines of equal length (but of different shapes); subsequently, she introduced the use of the measuring tape, asking her students to add marks on a strip of paper. In the next class, Sonia introduced some concepts related to the metric system, for which she opted to use a digital resource –a GeoGebra applet (https://www.geogebra.org/m/qYHjHDAf)– for the students to observe the measurement of lengths using a virtual graduated ruler. In one of Sonia's planning sessions of her classes, we observed how this teacher carried out an extensive search in a GeoGebra repository, finally finding that applet. To deepen our understanding of the way in which Sonia selected this resource, we used our introspective technique.

For determining the selection path followed by Sonia, we proposed to her that she write or draw a representation (a map) of her selection process for that applet. For that, we provided her with a video of her planning session in which she searched for the resource online, selected it and adapted it for her class.

In her selection process map, we observed Sonia's interest in organizing her actions in blocks of consecutive activities, taking into account: the curricular guidelines, what she has done in past classes (registered in her class log), and textbooks. For Sonia it is important to follow up on her previous classes and to take into account where she wants to go with her students, in order to make her resource selection. Furthermore, she considers that the use of digital resources can help students with low school performance.

Sonia's selection situation was characterized by her search for a digital resource that would complement a work she had already done in class and that was articulated with the worksheet for students, already planned. Hence, his anticipations were related to the repository in which he sought the resource, in terms of ergonomic quality (the design of the resource) and didactic (the situation that the resource offers to students).

The operational invariants that we infer in Sonia's scheme for selecting digital resources for her class, center on the role of estimation, of equivalence, the concept of a measuring unit, and the use of dynamic geometry. These invariants show her interest in focusing her teaching on aspects that she considers central and that go beyond the simple learning of formulas. For example, Sonia considered didactical-mathematical criteria, when she said: "*It is useless for children to learn the equivalence tables of the metric system if that does not make sense to them. [...] Using the applet, children can estimate, measure, move [objects], do different things, but in a controlled activity that is not just the moving of 'moving points', but is guided and it is clear to them what they are studying."*

Pedro and His Process of Selection of Resources to Design in the Geometry Class

Pedro has a bachelor's degree in mathematics with an emphasis in computer science, and a nine-year teaching experience (in elementary and high school); for the past four years he has worked as an elementary mathematics teacher (at the school participating in the study) in grades four and five. We followed this teacher for 2-3 months each year, in two school years.

In order for his fifth grade students to learn about magnitudes and their measurement, Pedro proposed to the children a project where they would design and build a house; in that way, students had to take measurements and work collaboratively. The project involved work with paper-and-pencil, rulers and compass and the use of a digital resource: Sweet Home 3D (http://www.sweethome3d.com/), which includes diverse tools for the design of a house (e.g., tools for sketching, for measuring, for modifying what was already built, etc.).

In one of the observed classes, the task was for the students to reproduce and modify the scale of a house's blueprint that they had previously drawn in their notebooks, and to create a first draft of their design, using the digital resource. The main mathematical ideas involved in this activity were the change of scale and the measurement of lengths (of the walls and of other parts of the house).

The map of the selection path followed by Pedro of his digital resource, had several versions throughout the time. In the latest version of his map, there are several elements to consider and that Peter emphasized when explaining his selection path: Pedro highlights, first of all, the need to have clear didactic and mathematical criteria when choosing a resource for the class. We consider these selection criteria as operational invariants of Pedro's scheme. Pedro chooses the resource depending on whether he can adapt the resource to his class-goals: in this case, he considered the resource's affordances for designing and measuring, and then defined other criteria (e.g., ergonomic criteria: a resource that is freely available, "pleasant", "motivating", etc.). He also placed emphasis on collaborative work. For example, Pedro took into account didactical-mathematical and ergonomic criteria when he said: "*I am looking for a tool that is useful for many things, because we are working on a project that lasts several weeks. [...] The idea is that, by working with Sweet Home, children can see what things can be measured and what things cannot be explored, and that they try different [Sweet Home tools]. "*

The Selection Criteria of Sonia and Pedro

We consider that Sonia’s selection was “local”, because she chose a specific resource for a particular class; while Pedro's selection was "global", in the sense that he chose a resource to promote a project that involved many class sessions. The main selection criteria of Sonia's resource were didactic-cognitive: the visualization, the application of equivalences and the student-resource interaction. These selection criteria were quite specific regarding the cognitive demand of the resource and of the curriculum. In the case of Pedro, his criteria are centered on designing various types of tasks for his students, having many possibilities for mathematical explorations, and for his students to work collaboratively; in fact, being able to use the resource in multiple ways was one of Pedro's main selection criteria for the resource.

Conclusions

In our work we sought to infer the knowledge and schemes of the participating teachers, specifically the criteria that guided their documentation work and their process of selection of resources. For this, the use of our introspective technique allowed us to collect data from the documentation work of each teacher and analyze it according to the context of each teacher. In particular, in our attempt to infer and categorize the teachers’ professional knowledge we were able to recognize the different knowledge involved, in particular the operational invariants of each teacher's schemes, identified as selection criteria. We categorized the selection criteria of the two teachers discussed in this paper, into three types: (i) Didactic-mathematical criteria (which correspond to the didactic use of the resource and the type of mathematical tasks that can be developed using it). (ii) Cognitive-curricular criteria (which refer to the cognitive demand of the resource and the curricular guidelines that the teacher takes into account for the selection). (iii) Ergonomic criteria (which includes the technical and operational affordances of the resource).

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LANGUAGE AND MATHEMATICS: QUESTIONING STRATEGIES IN A DUAL LANGUAGE BILINGUAL EDUCATION CLASSROOM

Gladys Krause
William & Mary
ghkrause@wm.edu

Katherine Barko-Alva
William & Mary
kbarkoalva@wm.edu

This study presents an emerging framework of teaching moves for teaching mathematics in a DLBE classroom. Our preliminary findings indicate how the teacher in our study uses language during mathematics instruction to a) support the development of conceptual understanding, b) create opportunities for cross-linguistic connections, and c) create opportunities to support bilingual students' linguistic and mathematical understanding.

Keywords: Curriculum, Elementary School Education, Classroom Discourse, Equity and Diversity

Introduction

In this research report, we present an emerging framework of teaching moves for teaching mathematics in a DLBE classroom. We use examples from the linguistic interactions enacted by a 4th-grade teacher, Ms. Lucía, with her students during mathematics instruction. We identify salient elements throughout her mathematics instruction that provide insight into how Ms. Lucía approaches questioning strategies: 1) to elicit bilingual learners' responses during class discussion, and 2) to explore how she uses Spanish while formulating questions during mathematics instruction.

Our present work is guided by two theoretical frameworks. First, we use Moschkovich's (2015) framework of academic literacy in mathematics that suggests language learning goes beyond simply learning new vocabulary. Second, we build on Bunch's (2013) definition of pedagogical language knowledge (PLK). This particular viewpoint provides insight into how teachers should negotiate language within the context of a particular academic discipline. Supported by these frameworks, our study is guided by the following research question: how does a dual language teacher use Spanish to implement questioning strategies designed to teach problem solving strategies to (Spanish-English) bilingual learners?

We approach this question by documenting the use of Spanish and English during instruction that centers on teaching mathematics based on students' mathematical thinking. In addition, we attempt to explore the balance between language and content instruction in a DLBE classroom. Documenting the elements of this interaction is a key step in promoting the development of teaching expertise in dual language classrooms. We seek to maintain rigorous mathematics content by ensuring the development of bilingualism among students.

A Word on Terminology

The following terms will be used throughout this research report: *bilingual* students, classrooms, and teachers. By using the term *bilingual,* we refer to those teachers and students who speak English and Spanish and those who are actively engaged in learning English or Spanish as a second language (i.e., both simultaneous and sequential bilinguals). In the present work we adopt the term Dual Language Bilingual Education (DLBE) as noted in Menken's

(2017) work. We will refer to DLBE classrooms as those intentionally serving students who are learning academic content in two languages: in this particular case, Spanish and English.

Conceptual Framework

For decades, research in mathematics education has documented best practices for learning mathematics; understanding children's mathematical thinking; focusing on the process of learning mathematics, of problem solving in mathematics, and of discussing mathematical ideas as revealed through children's strategies (Carpenter, Fennema, Peterson & Carey, 1988; Boaler & Staples, 2008; Carpenter; Fennema; Franke; Levi & Empson, 2015). The exchange of ideas about problems, solution strategies, and mathematical connections closely aligns with the mathematics education reform view of what it means to do and learn mathematics (NCTM, 2014). Despite growing evidence that discussions during mathematics instruction can support students' learning, counterbalancing literature suggests that all students may not benefit in the same way from discussion-oriented instruction (Turner, Dominguez, Empson, Maldonado, 2013). Studies by Civil & Planas (2011) have shown that students who are less familiar with the language of instruction, culture, and classroom norms are often marginalized or invisible during classroom discussions. Our work presents a window into a mathematical discussion in a DLBE classroom.

Our purpose is to identify the potential benefits of discussion for linguistically and culturally diverse students. In addition, we attempt to analyze instructional moves tailored to provide ample opportunities to capitalize on students' thinking and linguistic repertoires. Recent work by mathematics educators across the country adds to the perspective of our work by incorporating culture, language, and social differences into mathematics instruction (see Aguirre, Mayfield-Ingram, & Martin, 2013; Turner & Celedón-Pattichis, 2011; Turner, Dominguez, Maldonado, & Empson, 2013; Maldonado & Krause, 2017; Maldonado, Krause, & Adams, 2018). These perspectives impact lesson development, mitigating the overemphasis on learning English (Gándara, et al., 2010), and the narrow focus within the teaching and learning of mathematics in bilingual contexts which tends to underline, for the most part, supporting bilingual students as they solve word problems and tackle standardized tests, without fully recognizing the linguistic complexity of such endeavors (see de Oliveira, 2012; Martiniello & Wolf, 2012). Our work contributes to the literature by offering an understanding of the language practices a bilingual teacher enacts in the context of teaching and learning mathematics.

Methods

This case study (Guba & Lincoln, 1981) facilitates a deep exploration and a robust description of the teacher's and students' academic language negotiation in a DLBE classroom within the context of mathematics. We conducted classroom observations during mathematics instruction in Spanish, debriefed after each observation with Ms. Lucía, and collected students' artifacts. We then analyzed Ms. Lucía's teaching practices to explore how she approached her instructional decisions.

Data and Participants

Our 4th grade DLBE classroom is located in the Tidewater region of Virginia. Libertad Elementary School (pseudonym) is currently a PK-4 elementary Title I and Title III school with approximately 300 students. Ms. Lucía's DLBE classroom has 14 students. Eight students are native English speakers, six native Spanish speakers. Both groups of students intermingle throughout the day, receiving academic instruction in all content areas 50% of the time in English and 50% in Spanish.

Analysis

Using Charmaz's (2006) Constructivist Grounded Theory we began by transcribing each debriefing session and classroom observation. We read the transcripts to gain insight into the data, and we identified words and phrases that resonated while reading the data. We then wrote memos to examine our thoughts and identified patterns in the data. We conducted an initial coding phase which involved identifying segments of the data with short phrases that represented what was happening in the segment. After discussing and deciding on more simplified codes for the segments, we moved to a focused, selective coding phase to separate, sort, and synthesize our segments. Finally, we synthesized all of this information to develop profiles that reflected teaching moves. We double-coded the data, with discussion and resolution of discrepancies

Tasks

Ms. Lucía started the class with a discussion about interpreting decimal numbers using a 10-by-10 grid (see Figure 1). The task consisted of writing the decimal represented in the figure. After the class discussion, they worked on the second problem (see Figure 2). Ms. Lucía translated the problem into Spanish from a problem found in the pacing guides for 4th grade.

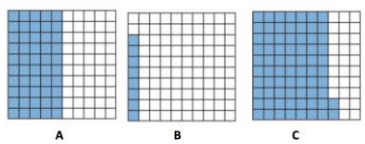

Figure 1: 10-by-10 Grids

En el primer día de clases, el profesor puso 1 centavo en un frasco. En el segundo día de clases el profesor puso 2 centavos en el frasco. En el tercer día el profesor puso 4 centavos en el frasco. Todos los días el professor puso el doble de monedas en el frasco del día anterior ¿Cuántos centavos habría en el frasco en el quinto día de clases? [On the first day of class a teacher placed 1 penny in a jar. On the second day the teacher placed 2 pennies in the jar. On the third day the teacher placed 4 pennies in the jar. Every day the teacher placed twice the number of pennies in the jar than the day before. How many pennies would be in the jar on the fifth day of classes?]

Figure 2: Word Problem Used in Problem Solving Lesson

Findings and Discussion

Through a systematic analysis of Ms. Lucía's instructional practices, and the language she used during her questioning strategies in Spanish, three themes emerged.

Incorporating Language to Increase Mathematical Conceptual Understanding

While attempting to tease out the definition of *doble* [double] when unpacking the word problem (Figure 2), Ms. Lucía utters the following question: "¿Qué es *el doble*?" [what is *double*?]. One student was quick to provide the corresponding English: 'double'. Ms. Lucía then asked the following question: "¿Me pueden dar un ejemplo?" [can you give me an example?]. Some students provided examples such as: "dos más dos" [two plus two] or "ocho es el doble de cuatro" [eight is twice four]. Ms. Lucía acknowledged students' responses and continued to unpack the problem. Ms. Lucía was asking students to provide the conceptual definition of the word 'double'. While it is clear students understand the concept of double, a specific definition was not procured.

In a DLBE classroom, teachers are tasked with teaching to increase students' access to the language and content. This teaching skill seems to be particularly developed as teachers become more comfortable discovering the nuances of language, and how it is used in academic contexts (de jong & Barko-Alva, 2015). We argue that a purposeful shift in how questions are posed in math classrooms could foster the integration of language and content instruction. For instance, Ms. Lucía could have asked, "¿Quién me puede dar el significado de la palabra *doble*?" [who can provide the meaning of the word 'double'?]. Thus, by incorporating *meaning* into the question, the teacher is creating opportunities to further explore verbs such as: to 'define', to 'identify', and to 'describe', which will likely reinforce students' skills in defining specific academic concepts.

Incorporating Language to Create Opportunities for Cross-linguistic Connections

The word "frasco" [jar] was featured in the word problem. She asked, "¿Qué es frasco?" [what is 'jar'?]. Students proceeded to provide the common English equivalent "jar." In this particular interaction, the question could have been reframed to signal number and gender. For instance, "¿Qué significa frasco?" [what does 'jar' mean?] could have created a teaching opportunity to briefly discuss that the noun "el frasco" [jar] marking its gender and number (i.e., singular and masculine). Ms. Lucía could have then identified connections between the similarities and differences in how Spanish and English signal the use number and gender.

Incorporating Language to Create Interdisciplinary Connections

As Ms. Lucía continued to unpack the problem, she asked, "¿Qué sabes?" [what do you know?]. Had she reframed this question as "¿Qué información nos da el problema?" Or "¿Qué evidencia nos da el problema?" [what information is the problem providing/what evidence is the problem giving us?], Ms. Lucía could have strategically used words such as 'information' and 'evidence' to highlight cross-disciplinary connections as well as increase student access to cognates. While working on interpreting the decimals represented in the 10-by-10 grids (Figure 1), Ms. Lucía addressed students by asking them the following question, "¿Qué es A?" [what is A?]. This question is quite open-ended (a possible answer would be 'Una letra.' [a letter (of the alphabet)]). Ms. Lucía could have followed up by specifying the intended sense of the question with more refined phrasing: A is the label of a grid, and it represents a specific decimal number. Thus, shifting the question to, "¿Cuál es el decimal representado en el diagrama A?" [what is the decimal represented in grid A?] introduces students to the verb "representar" [to represent] and to the noun "grid" [el diagrama], which are found across different academic disciplines (e.g., math, science, social studies, and language arts), thus not only increasing students' access to common academic collocations, but arguably more importantly, also modeling the importance of refining one's use of language to more closely fit the thoughts expressed. We conjecture that these categories of teaching moves can vary depending on the part of the lesson the teacher is working on. For instance, in the unpacking part of the lesson, the teacher is focusing more on making sure students understand the context of the problem, and for this reason a focus on vocabulary might be more prominent than during the part of the lesson where the teacher is sharing specific strategies for solving a problem.

Conclusions

The emergent themes highlighted in this work provide concrete examples of how Ms. Lucía could shift her language practices to promote math instruction. As illustrated here, these questioning strategies could be strategically crafted and formulated to further support dynamic language practices designed to increase students' linguistic repertoires while learning Spanish within the academic context of mathematics (Bunch, 2013). We argue that limiting the complexity of the language teachers use while asking questions during mathematics instruction does not necessarily foster an active bilingual learning environment. We encourage DLBE teachers to plan their lessons including multiple opportunities to integrate language and content instruction as bilingual learners actively negotiate mathematical and linguistic concepts.

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INTRODUCING AN "OPPORTUNITIES TO IDENTIFY" FRAMEWORK

Emma C. Gargroetzi
Stanford University
egroetzi@stanford.edu

Just as the learning of mathematics requires that students have opportunities to learn, it also requires that children have opportunities to see themselves and be seen by others as mathematical doers and knowers. This paper proposes a framework for identifying teacher-introduced opportunities to identify through the examination of 1) instructional tasks, 2) participation structures and 3) storylines. The framework was developed and is illuminated through a year-long ethnographic case study of Ms. Wong's classroom, a place where students had opportunities to identify (a) as people with dignity and competence, (b) who have mathematical ideas worthy of consideration, (c) and mathematical lives worthy of exploration, and (d) whose participation is vital to their communities.

Keywords: Equity and diversity, Instructional activities and practices, Identity

Just as the learning of mathematics requires the availability of opportunities to learn (Walkowiak, Pinter, & Berry, 2017), it also requires that children have opportunities to see themselves and be seen by others as mathematical doers and knowers. Attention to mathematics identity has been named a core component of any approach to mathematics education that centers equity (Gutiérrez, 2013; Martin, 2012), of particular importance for students from minoritized groups whose recognition as mathematical knowers and creators has been historically and systematically denied (Tate, 1994; Delpit 2012). While a handful of "opportunities to learn" frameworks have begun to incorporate the importance of attending to student identities (see Gresalfi & Cobb, 2006), this work has yet to be centered in its own right as a way of examining classroom practice. This paper offers a framework for considering a construct that I name *opportunities to identify*. I propose a framework that includes examination of 1) instructional tasks, 2) participation structures and 3) storylines, as they are explicitly introduced by the classroom teacher. The paper takes a positional approach to identity, making use of an ethnographic case study of one high school classroom to illustrate how the teacher introduced student opportunities to identify as (a) people with dignity and competence, (b) who have mathematical ideas worthy of consideration and (c) mathematical lives worthy of exploration and, (d) whose participation is of value to their communities.

Identity as Positioning

The paper approaches identity as positional (Davies & Harre, 1990; Holland, 2001). Positioning theory conceptualizes identity as a process of negotiation, enactment and inhabitance of available social positions that are made available through existing structures and storylines. A positional approach to identity affords for analytical distinction between *positional resources* – the material, relational, and ideational resources which make certain identities available (Nasir & Cooks, 2009), and *acts of positioning* – the moment-to-moment interaction through which an individual's position within an existing *storyline* becomes salient (Davies & Harre, 1990). These storylines are recognized bundles of associated rights, duties, and obligations that can be invoked implicitly or explicitly through language, actions or structures. The "Opportunities to Identify" framework offered here focuses solely on positional resources introduced by the classroom teacher: instructional tasks, participation structures and explicit storylines.

From Opportunities to Learn to Opportunities to Identify

Opportunities to learn (OTL) frameworks developed over the last fifty years (McDonnell, 1995) as a way to shift measures of student learning from a narrow focus on assessment-based outcomes to a more holistic focus on classroom practice. In mathematics education, Walkowiak, Pinter and Berry (2017) offer an OTL framework focused on four classroom features: 1) the teacher's mathematical knowledge for teaching, 2) time, 3) tasks, and 4) talk. In another OTL framework that builds on Ladson-Billings's (1994) tenets of culturally relevant teaching, Gresalfi and Cobb (2006) name four crucial behaviors of teaching math for equity: 1) holding high regard for students, 2) believing that all students can succeed, 3) encouraging students to make connections between their multiple communities and identities, and 4) using instructional practices that validate and draw on students' mathematical meaning making.

Building off these two OTL frameworks, I propose a related framework for illuminating teacher-introduced *opportunities to identify* – the opportunities that students are offered for coming to see themselves as doers and knowers of mathematics, in the context of a larger community of other such doers and knowers. I propose that opportunities to identify introduced by the teacher can be traced through three classroom features: 1) instructional tasks, 2) participation structures, and 3) the storylines a teacher explicitly introduces or interrupts. While all classroom communities are co-constructed through the interaction of students and teachers, this paper focuses only on the opportunities to identify that are introduced by the teacher.

Methods: The Case of Ms. Wong's Classroom

The case of Ms. Wong's Integrated Math 1 (IM1) class comes from a year-long ethnographic study of the mathematics identity pathways available to, and negotiated by, students at one predominantly Latinx-serving California high school. Ms. Wong was selected as a focal teacher because her teaching practice was recognized as largely fulfilling the tenets of both OTL frameworks mentioned above.

Development of the "Opportunities to Identify" framework emerged from coding of field notes, lesson plans and other classroom artifacts from over 80 days (~130 hours) of classroom observation, and was elaborated and refined through analytic memos focused on the classroom practices and available student identities in Ms. Wong's classroom. The classroom was hypothesized to be a space where students were consistently offered opportunities to identify as (a) *people with dignity and competence, (b) who have mathematical ideas worthy of consideration and (c) mathematical lives worthy of exploration, and (d) whose participation is vital to their communities*.

The "Opportunities to Identify" framework was developed in conversation with existing literature in order to facilitate ongoing analysis of student uptake, notably not included here. Presented here are only a small set of illuminating examples from a much larger data set of Ms. Wong's teaching practice.

The Opportunities to Identify Introduced by Ms. Wong

Ms. Wong's practice offers an illustration of one model for how teachers introduce opportunities to identify in their classrooms. Through the instructional tasks, participation structures and storylines that she introduces, Ms. Wong offers students opportunities to identify as (a) people with dignity and competence, (b) who have mathematical ideas worthy of consideration, and (c) mathematical lives worthy of exploration, and (d) whose participation is vital to their communities. Table 1 summarizes the ways in which this occurred.

Table 1: Positional Resources and "Opportunities to Identify"

Resources	"Opportunities to Identify"
Instructional Tasks	Tasks have multiple solutions and/or solution pathways Tasks examine student contexts and experiences
Participation Structures	Students move between individual, pair, group, and whole class structures Students have time to prepare, revise and refine ideas Multiple strategies and solutions are considered publicly Students evaluate the mathematical reasonableness of shared ideas
Storylines	Participation is vital to one's own learning and the learning of others Everyone has something to teach and something to learn Smartness is not restricted to a limited few: everyone has competence

Instructional Tasks

Curriculum and pedagogical materials are common features examined through OTL frameworks (e.g. Walkowiak, Pinter, & Berry, 2017). Research suggests that open-ended tasks with multiple potential solutions or solution pathways offer more opportunities for students to construct mathematical meaning and make connections between concepts (Smith & Stein, 1989). In Ms. Wong's classroom, tasks often have multiple solution pathways and all pathways are treated as worthy of consideration, whether it be through a directive to share individual thinking with a partner, or through the solicitation of multiple representations for whole class consideration. The introduction of tasks with multiple entry points and multiple solution pathways provides all students an opportunity to identify people with mathematical ideas. Literature also suggests that opportunities to learn are supported when students can make authentic connections between classroom academic content and their lived experiences (Gresalfi & Cobb, 2006; Ladson-Billings, 1994). Ms. Wong introduced tasks that drew on student lives at various levels. In a unit on systems of equations, students predicted when the Latinx population in California will surpass the non-Latinx population of the state (Fieldnotes 5.08.18). In a statistics unit, students examined responses to two class surveys that they took on participation, one from November, the other from February, and identified trends in participation from their own classroom (Fieldnotes 2.06.19). These tasks and others offered students opportunities to identify as part of a mathematical world with lives that are worthy of mathematical exploration.

Participation Structures

Different participation structures allow for the demonstration of different forms of competence (Philips, 1972). In Ms. Wong's class, participation structures shift frequently between individual, pair, group, and whole class interaction. Students seated in groups of four were consistently given time to consider ideas individually before being asked to share those ideas with a partner or their group. Pairs and groups were then given time to share, revise and make sense of those ideas collaboratively. Groups were frequently asked to share their ideas with the class. Ms. Wong elicited multiple solutions and strategies, soliciting participation from multiple groups, and often times verbally reminded students that all contributions were valuable for mathematical consideration and refining. Students were then asked to consider the differences and connections between the responses, increasingly working toward a more robust and connected understanding of the mathematical concept.

These participation structures communicate that mathematical sense making takes time and requires multiple modes of engagement including both individual and with peers. By providing all students multiple opportunities to prepare and revise their thinking before calling on students to share, Ms. Wong communicates the expectation that everyone has something to share given sufficient time to think and prepare. While individual work provides students opportunities to identify as independent mathematical thinkers, group work and whole class conversations, both of which entail prior preparation, provide students opportunities to identify as people with mathematical ideas worthy of the consideration of others, and vital to the community.

Storylines

The storylines introduced by a teacher include the explicit messages that they provide about who is in the classroom and their rights and obligations within that space. In Ms. Wong's classroom one of the dominant storylines she introduced repeatedly was about participation. She repeatedly emphasized the importance of participation of all students as vital to the whole class's learning including through instruction such as the task described above. On one occasion, the day before a national election, she made a connection between the importance of student in-class participation and the importance of people sharing their voices in order for a democracy to function (Fieldnotes 11.05.18). Ms. Wong's storyline around the importance of the participation of all students offered students the opportunity to identify as people whose participation matters to not only their own learning but also to the learning and wellbeing of others in the community. Storylines about the exceptionality of "smartness" in mathematics are widespread and known to be damaging to all students, but especially to students from social identity groups historically excluded from mathematics (Storage, Horne, Cimpian, & Leslie, 2016). Comments about some students being "smart" came up in Ms. Wong's classroom, like in most American classrooms. Ms. Wong interrupted this storyline when it came up. On one occasion, when one referred to another student as "really smart" Ms. Wong responded firmly, asserting that everyone in the class was both learning and a person others could learn from, and that everyone is equally smart in this class (Fieldnotes 10.03.18). She also offered alternative storylines of dignity and competence. In an activity that Ms. Wong called "Math Identity Rainbows," students were asked to consider and rank their own mathematical competencies and then share how their strengths could support their group (Fieldnotes 1.07.19). Through her explicit rejection of the idea that only some people are smart, and through activities like the Math Identity Rainbows activity, Ms. Wong introduced alternative storylines around competence that offered students opportunities to identify as people with dignity and mathematical competence based on a wide variety of forms of mathematical thinking and engagement.

Discussion and Future Directions

Essential to providing students opportunities to learn mathematics are the opportunities students are provided to identify as doers and knowers of mathematics, and to be recognized by others as such. While some OTL frameworks (Gresalfi & Cobb, 2006; Walkowiak et al, 2017) have this construct buried within them, equitable approaches to mathematics must center the role identity plays in learning. The "Opportunities to Identify" framework provides a way to do so.

Ongoing analysis attends to the teacher-student co-construction of opportunities to identify, including student uptake and creative re-purposing of opportunities introduced by the teacher.

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COGNITIVE INTERVIEWS TO LEARN ABOUT STUDENT STRENGTHS AND PREFERENCES RELATED TO LANGUAGE IN MATHEMATICS TASKS

Jill Neumayer DePiper
Education Development Center
Jdepiper@edc.org

Johannah Nikula
Education Development Center
Jnikula@edc.org

Diana Serrano
Education Development Center
Dserrano@edc.org

Analysis of data from cognitive interviews with 43 sixth grade students who are English Learners (ELs) identified students' strengths with making meaning of mathematics tasks, specific challenges with oral language, and complexity of students' language preferences. We discuss our use of cognitive interviews, and the findings and implications for supporting teacher learning and designing mathematics instruction for ELs.

Keywords: Classroom discourse, equity and diversity, and instructional activities and practices.

Research suggests that teachers need deep understanding of the language strengths and needs of students who are English learners (ELs) *in content courses* (Bunch, 2013). Teachers, however, may be unprepared or unfamiliar with how to support ELs in their mathematics courses (Reeves, 2006; Walqui et al., 2010). Addressing students' language needs in mathematics as well as identifying the language resources that students are bringing to the classroom are critically important given the increasing language and literacy expectations across the mathematics curriculum and the attention to mathematical practices and communication (Bunch, 2012). Deepening teacher understanding of language issues relates to deepening teacher understandings of students who are ELs. Not only is it critical that teachers know their students, they must understand that how they see themselves *in relation to* their students who are ELs impacts students' positioning in class and students opportunities for equitable participation (Yoon, 2008). Teachers' understanding of ELs' cultural and social needs and their active responses to them has been shown to positively influence ELs' classroom participation and positioning (Yoon, 2008). Supporting mathematics teachers to learn more about ELs and to identify students' strengths and needs in the context of mathematics may lead to more learning opportunities for ELs.

Purposes of the Study

Our project is developing and testing an innovative instructional unit designed to support students who are ELs and focused on division of fractions, a key topic in Grade 6 that is foundational to ratio and proportional reasoning and fraction problem solving. Research shows that fraction knowledge gained and consolidated in the middle grades is predictive of high school mathematics achievement (Siegler et al., 2012). Our intervention is hypothesized to support fraction content understanding and problem-solving and communication for students who are ELs. This unit is being developed through an iterative design and research process and it will be tested to understand outcomes for students. The purpose of the cognitive interview sub-study is to investigate: How do students engage with the language in mathematical tasks and what are implications for mathematics teachers and mathematics lessons related to students who are ELs?

Theoretical Framework

Cognitive interviews provide an opportunity for students to think aloud as they work on a task and retrospectively report on what they are thinking after task completion (Branch, 2000)

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

and they have been found to be particularly helpful for learning about ELs' interactions with a task (Johnstone, Bottsford-Miller, & Thompson, 2006). Cognitive interviews are also an approach that can take into account language variation among ELs, which is "critical to ensuring appropriate inclusion of these students" during materials development (Solano-Flores & Prosser, 2010, p. 13). As part of our iterative design process, we conducted cognitive interviews to learn about the language strengths and needs of students who are ELs. These cognitive interviews respond to calls for research using cognitive interviews with ELs (Prosser & Solano-Flores, 2010) and support us to understand the experience of students who are ELs in the math classroom and to refine our materials to minimize any unnecessary linguistic complexity.

Methods and Modes of Inquiry

Cognitive interviews were conducted with 43 sixth grade students designated as ELs by their school—note that we use the phrase "students who are English learners," abbreviated as ELs, to align with ways in which schools label their students, while recognizing that this term can be problematic because it privileges the learning of English rather than language (and other) resources that students bring to the classroom. We interviewed students from six school districts across a New England state. The average percentage of students designated as ELs was 15.7% across participating schools, but all participating students were designated as ELs, and teachers provided available data on students' overall English proficiency and proficiency levels in reading, writing, speaking, and listening.

A researcher worked with each student individually or in a pair (depending on the scheduling of interviews and available researchers), with each interview lasting thirty to sixty minutes. The researcher presented the student with a fraction division task and asked the student to read it aloud. An example task is: "The teacher has 4 cups of ice cream. He would like to give each student a $\frac{1}{2}$ cup serving of ice cream. How many $\frac{1}{2}$ cup servings can the teacher give?" Next, the researcher asked: "Are there any words that you, or other students, may not understand?" Asking about words that other students might not understand allowed students to identify words that are confusing even if they do not want to admit to being confused. The researcher clarified the meaning of any unfamiliar words or phrases, then students worked on solving the task. While the student was working, the researcher asked questions, probing for student understanding of the task and seeking to learn more about a student's problem-solving approach. The interview concluded with questions about the student's language and educational background and preferences such as: "What language(s) do you speak at home?" and "Do you feel more comfortable speaking [that language] than English, a little more comfortable speaking [that language], or almost the same speaking [that language] and English?"

We first analyzed the interview data, including researcher notes and student work, by task. Analyzing all student responses for a particular task, we identified students' struggles and successes with specific words or phrases in that task. Next, we looked across all tasks to see how there were similar or different critical events, such as struggles or successes on tasks or specific challenging words, present across students' work as tasks varied (e.g., Powell, Francisco, & Maher, 2003). In this cycle of analysis, we noted questions and made hypotheses about possible explanations (Corbin & Strauss, 2007). In our analysis of students' responses to the language background and preferences questions, we made note of key ideas, such as language preferences related to context, within a student's response, and then proceeded to look across student responses for themes related to language background and preferences.

Results

Three themes emerged from qualitative analysis of the cognitive interview data.

Making Meaning of Math Tasks Despite Unfamiliar Contextual Words

Analysis of cognitive interview data suggests that students with varying levels of English proficiency could sometimes identify key information in math tasks and make sense of the tasks despite encountering unfamiliar English words related to the context. For example, four students identified "servings" (in the task about servings of ice cream) as a word that someone else may not understand, but these students did not evidence struggling with the word in the context of working on the task and they were able to get started on the task and in some cases accurately solve it. In another example, where the task included the name Maureen, some students could not pronounce the name and mentioned it as unfamiliar, but did not struggle to complete the task.

Oral Language When Reading a Task

The cognitive interview data also indicate that some students mispronounced some words, such as "syrup," when asked to read a task aloud. Many students also struggled to read the fractions in a task aloud correctly; for example, students said, "two over three" instead of "two thirds." By having students read tasks aloud, we were able to identify these oral language challenges that may limit student understanding or participation.

Language Preferences

We found that students have a complex set of language preferences depending on the context and content. Across the 43 participating students, 13 home languages were represented, and Spanish was the most reported home language. Students also ranged in the number of years spent in US public schools, English language proficiency, and home countries. Seventeen of 43 students said they feel more comfortable speaking their home language, while fourteen said they feel equally comfortable speaking their home language and English, and that they speak English at home some of the time. Only a few (n=5) said they prefer to speak English at home although their home language was not English. Yet students acknowledged that their language use is influenced by the people they interact with and the setting (e.g., home, school, etc.). Some said that while they generally feel more comfortable speaking their home language, in school settings they prefer to speak English, and they indicated three main reasons: (1) English is the language of instruction and the only language the teacher speaks; (2) they do not have the vocabulary in their home language to discuss new material; and (3) they want to improve their English.

Discussion

Our interviews were designed to identify student language challenges and support task revisions to support access for all students. The interview findings have practical implications for lessons and for teachers, and also add knowledge about the experience of students who are ELs.

Sociocultural Perspective of Language

Findings about students' fluency with making meaning of unfamiliar words is consistent with other research using a sociocultural perspective of language, where language learning in mathematics is about participation (e.g., Moschkovich, 2002). As students identified words that were names or other proper nouns, they evidenced that they had a process to identify which words were not essential to the problem-solving. Similarly, students were successful in navigating a mathematics task when words, such as "servings," were embedded in full sentences and contexts. Our analysis suggests that while some everyday words were identified as challenging, students were able to get started on the task when these words were embedded in a task structure. This is consistent with Moschkovich (2002), who encourages mathematics teaching and learning to embrace a perspective that is considerate of the variety of resources that students bring to communicate mathematically, rather than focusing more narrowly on the difference between the everyday register and the mathematical register.

Importance of Specific Language Supports

Given our findings, it is important to consider instructional approaches that support students in understanding the language and mathematics in context. Analysis of the cognitive interview data points to the need in this unit (and in other lessons) for modeling language and reading aloud to support student understanding of mathematics concepts and everyday language. Reviewing pronunciation of mathematics terms and everyday language can develop students' skills as readers, listeners, and producers of language. We added teacher instructions to our unit for supporting all students to read fractions, as this may be an area where ELs, as well as students who are low literacy, struggle. We also removed some context words—for example, changing a task about "ribbon" to be about "string"—in response to analysis of student struggles. We did not remove all potentially challenging words because students demonstrated an ability to get started even when faced with unfamiliar words. Instead, we integrated language supports for all students to access each task. We sought to draw on the language resources that students bring to the classroom by providing instructional strategies for students and teachers that provide access to the language context (e.g., around serving ice cream) while supporting students to think through the words that they need and the ones they do not. A focus on the task context and on the overarching content (in this case, fraction division), rather than the specific vocabulary, supports students in building upon the mathematical skills they already possess and in developing their mathematical understanding while they gain English proficiency.

Diversity of Students' Language, Cultural, and Educational Experiences

Our interviews highlighted the diversity of language, cultural, and educational experiences that exist within the EL student subgroup. Students had different levels of comfort speaking in English and different levels of English use at home. Some students who have a home language other than English noted that they may prefer mathematical instruction in English because they are not familiar with mathematical terms or concepts in their home language. It is critical that classroom teachers work to uncover and then to understand this diversity within the EL student subgroup in their own classrooms. Interviews to learn about student language and cultural backgrounds, two critical aspects of a student's background, may support teacher understanding and learning and will inform their assumptions about connections between language and culture. Students' language use may be both context- and content-specific.

Conclusion

The opportunity to interview students while doing a task provided rich insights into students' language use and understanding. Our methods and analysis have identified specific learnings about language use and preferences for ELs and offer insight into how teachers may continue learning about their students. Knowing about ELs' home language, language preferences, and educational background may support teachers in understanding their students and themselves, in designing their instruction, and in positioning themselves in ways that support ELs' equitable participation. Student interviews within teacher education coursework or professional development may provide opportunities to include attention to ELs' strengths, needs, and language preferences and support teacher learning.

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FOSTERING STUDENT AGENCY AND DISTRIBUTED MATHEMATICAL AUTHORITY: SURFACING TEACHER LEARNING AND TEACHER CHANGE

Kathleen Pitvorec
University of Illinois at Chicago
kapitvor@uic.edu

Christopher Fry
University of Illinois at Chicago
cfry3@uic.edu

Few studies provide insights into how teaching practices shift over time as teachers become adept at engaging students in mathematical discourse. This report presents the early stages of a study that through the analysis of field notes, observations, teacher interviews, and teachers' reflections on videotapes of their lessons, looks to understand what shifts teachers make in their practices and how these shifts relate to mathematical discourse in the classroom. Our coding system for analyzing discourse captures the extent to which student ideas function at the center of the mathematical activity (student agency) and the extent to which teachers foster space for students to explore their own understanding of mathematical ideas versus ideas presented by the teacher or text (the distribution of mathematical authority). We believe this study has important implications for understanding teacher learning and teacher change.

Keywords: classroom discourse, teacher professional development, teacher learning

Since the 1980s, national reforms in mathematics teaching have called for increasingly engaging students in mathematical processes and practices (Common Core State Standards Initiative, 2010; Kilpatrick, Swafford, & Findell, 2001; NCTM, 1989, 2000). Assuming a strong link between social interaction and the construction of knowledge (Bruner, 1997) and drawing on a sociocultural framework (Cobb & Yackel, 1996), this movement towards processes and practices has been accompanied by research focused on discourse in elementary school mathematics classrooms. For example, Cobb et al. (1992) explored qualitative differences in the practices of explanation and justification. Wood, Williams, and McNeal (2006) described different interaction patterns they found in classrooms. Hufferd-Ackles, Fuson and Sherin (2004) described developmental trajectories of math talk across classroom communities. More recently, Correnti et al. (2015) went further than simply "documenting and describing" teachers' classroom discourse practices by developing an instrument that they propose might be used to help teachers improve their practices in the context of "orchestrating productive classroom discussions." Beyond cataloguing and describing types or different levels of classroom discourse, few, if any studies provide insights into how teaching practices might shift over time as teachers become more adept at and successful with engaging students in mathematical discourse. This paper presents the early stages of a study designed to investigate shifts in teaching practices and how those shifts correlate to changes in classroom discourse.

The study described here is situated in a larger study focused on providing professional development (PD) around learning trajectories (LT) and formative assessment (FA) with the goal of improving middle school algebra teaching and learning. The study included workshops for deepening content through explorations of a curriculum-based LT (Maloney, Confrey, & Nguyen, 2014). Later, teachers engaged with researchers in reflecting on videotape of their own teaching. The video sessions focused on the development of FA practices. Wiliam and Leahy (2015) recommend FA practices that promote student agency and increased attention to making student thinking visible providing teachers with opportunities to better understand what students

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know and how they know it. The PD supported enhanced classroom discourse such that, student thinking would be surfaced, and students' ideas would become the objects of reflective discourse and collective reflection (Cobb, Boufi, McClain, & Whitenack, 1997). Specifically, this study aims to unpack how teaching practices evolve towards surfacing student thinking and making student thinking central to the mathematical activity, thus enhancing FA opportunities.

Conceptual Framework

This study follows sociocultural theorists who emphasize interactions and the social world as the location for the process of learning (Cobb, 1994; Cobb et al., 1997; Confrey, 1991; Sfard, 1998). A key premise of a sociocultural theory of learning is that interaction is a primary mediating factor for development (Vygotsky, 1978; Wertsch & Stone, 1985). Interactions among people act to develop and move understandings forward. Since dialogue or communication is at the heart of interaction, discourse and communication become key factors for making sense of development (Bakhtin, 1981).

This emphasis on social interactions intersects with recommended FA practices to activate students as resources for each other and as owners of their own learning (Wiliam & Leahy, 2015). Teachers simultaneously support students' engagement in discussions and students becoming owners of their own learning—that is, teachers foster student agency. In an effort to maximize student agency, teachers promote students' mathematical ideas to prominence in discussion as objects of collective reflection (Cobb et al., 1997). For decades, National Council of Teachers of Mathematics documents have called for distributing mathematical authority from its traditional location in textbooks to include and privilege student reasoning (Herbel-Eisenmann, 2007; NCTM, 1991).

Methods

Three sixth-grade mathematics teachers are at the center of the study in this report. One teacher had two years of experience at the beginning of the project. The other two had respectively twelve and fifteen years of experience. All three teachers worked in diverse high-needs schools in suburbs just outside of a large Midwestern city. Class sizes ranged from 18 to 32 over the two years of data collection. Forty-one lessons were videotaped. At least one project researcher observed each of these lessons and took field notes.

In addition to welcoming researchers to observe lessons, each of the three teachers participated in videotaped reflections before and after lessons as well as at least two retrospective video reflections—that is, they watched video of themselves teaching lessons reflecting and commenting on differences they noticed. Based on teachers' reflections as well as researchers notes, we generated a narrative for each teacher about the kinds of instructional changes that took place over the course of two years.

Because whole-class discussions were more consistently accessible in the video than small-group discussions, we selected ten whole-class discussions to analyze. We segmented the selected five- to twenty-minute discussions into turns. We took a grounded theory approach (Strauss & Corbin, 1990) to developing codes aimed at capturing who was doing the intellectual work as well as who was talking to whom (that is, was the student talking to the teacher or to other students). A significant foundation for our early process involved surveying existing mathematics discourse coding systems (e.g., Alexander, 2017; Chi & Wylie, 2014; Correnti et al., 2015; Hennessy et al., 2016; Elizabeth A van Es, Tunney, Goldsmith, & Seago, 2014; Wei, Murphy, & Firetto, 2018; Wood et al., 2006). When possible, we drew on language others used to describe turns. Twenty-two student codes and thirty-three teacher codes emerged from the iterative process of coding. We realized that the discourse nuances captured by our 55 codes were overwhelming for trying to discern patterns in the discourse.

After coding the ten whole-class discussions, we aggregated the codes with respect to where there might be shifts in the discourse activity. We ultimately arrived at a set of six course-grained codes for students and six course-grained codes for teachers. Within the teacher codes we identified shifts from (T1, T2) teachers doing all of the intellectual heavy lifting to (T3, T4) teachers supporting students in limited engagement with the mathematics, and ultimately to (T5, T6) teachers becoming facilitators with student thinking as central to the mathematical activity. On the student side, we identified shifts from (S1, S2) students simply listening to the teacher and responding to the teacher's prompts to (S3, S4) students listening to and responding to each other's mathematical ideas, and finally to (S5, S6) students thinking and problem solving together in the context of a discussion. We conceptualize these course-grained codes as falling on continua. The continuum on the teacher side (see Table 1) captures the degree to which mathematical authority is distributed during discussions—that is, the extent to which teachers make space for students to explore their own understanding of mathematical ideas versus exploring the way the teacher or the textbook present content. The continuum on the student side captures student agency during discussions (see Table 2)—that is, the extent to which student ideas are at the center of the mathematical activity.

Table 1: Continuum of Distribution of Mathematical Authority

Code	Mathematical Reasoning	Interaction Diagram	Examples of Features of interactions
T1	T does intellectual work	T→S	T provides instructions. No S engagement.
T2	T does intellectual work	T→S	Invites S to respond to T prompts.
T3	T does intellectual work	T→S	Setting the stage for S-to-S engagement.
T4	T and S do the mathematical reasoning	T→S	T invites and engages with S contributions
T5	T and S do the mathematical reasoning	T→S↔S	T facilitates in-the-moment S interactions around listening and dialogue.
T6	Ss do the mathematical reasoning (T facilitate S-to-S engagement)	T↓ S⇔S	T facilitates Ss reasoning about and solving problems together.

Table 2: Continuum of Student Agency as Related to Reasoning about Mathematics

Code	Mathematical Reasoning	Interaction Diagram	Examples of Features of Interaction
S1	Reasoning not relevant	S→T	Student is speaking to teacher
S2	Individual reasoning	S→T	Student is speaking to teacher
S3	Individual reasoning	S→(T) S	Share own individual ideas/ explanations with classmates.
S4	Individuals reason and respond to others.	S↔(T) S	Active listening & responding to classmates' ideas
S5	Individuals reason and connect with others.	S↔S	Students engage others in conversations about mathematical ideas.

S6	Students reason together.	S⇔S	Students construct/generate new ideas/ knowledge together.

The interaction diagrams in the tables indicate the directionality of the interaction as well as the speaker (left) and intended audience (right). The diagrammatic configuration in T5 indicates that the teacher presence serves to facilitate student-to-student talk. In T6, the teacher's presence serves to facilitate a productive level of mathematical reasoning in student-to-student talk already in progress. The configuration in S3 indicates that a teacher may facilitate one student talking to another. Although the teacher may be involved in S4, students are talking to each other. Wider arrows in T6 and S6 indicate that students are engaged in knowledge building.

Results

We are still at the first stages of correlating shifts in instructional practices with discourse patterns. We have noted, for example, that there is little to no student-to-student discourse in early videos for any of the three teachers' whole class discussions. Both teacher and student codes are low on our continua. Prompted by professional development experiences, teachers begin to engage in new instructional practices aimed at making student thinking more central and to distribute the mathematical authority. For example, all three teachers institute the five practices for orchestrating discussions (Stein, Engle, Smith, & Hughes, 2008). In their reflections, the teachers comment that the new instructional practices do not seem to result in increased agency. Students are still responding mostly to teacher prompts and not to each other. The discourse analysis aligns with this assessment in that the discourse is almost entirely constituted of lower codes for both teachers and students.

By Year 3 of the study, the researchers and teachers reflected on teachers' own videotaped lessons to unpack possible reasons why implementing new instructional techniques such as the five practices did not seem to provide improved conditions for student-to-student talk. In these early reflective sessions, teachers noticed that during small group exploration time, they (the teachers) were "supporting" students in successfully completing assignments. This support took the form of teachers "stepping in" to small groups to, for example, ask students funneling questions intended to lead students towards solving problems, question students about their strategies, or simply validate student answers. In response to noticing this, teachers intentionally started "stepping out" of small groups and aiming to turn the intellectual work over to the students. New discourse patterns during whole-group discussions accompanied the emergence of the teacher's intentional "stepping out" of small-group discussions. We see the teacher codes build up and the student codes follow. When the teaching codes are low in this context, they are often in service of supporting higher agency for students (for example, interjecting a content question to get them back on track or to remind them of norms for engagement). In addition, teachers exit and enter the discourse as needed. There are places where teacher codes disappear entirely as student thinking becomes the object for further mathematical activity and as student agency increases.

Discussion

Although this is a small case study of three teachers, we believe that it may have significant implications for teacher learning and teacher change. As others have reported, videotape of teaching can be a powerful tool for professional growth (Borko, Jacobs, Eiteljorg, & Pittman, 2008; Geiger, Muir, & Lamb, 2016; Jacobs & Philipp, 2004; Sherin, 2000; Sherin & Han, 2004;

Sherin & Van Es, 2009; Elizabeth A. van Es & Sherin, 2002), but there is scant research involving teachers reflecting on their own classroom video with a focus on how shifts in instructional practices might correlate with changing discourse patterns. We believe the two-part discourse coding system we have developed (for teachers and students) informs our understanding as we study how shifts in practice aimed at distributing mathematical authority and increasing student agency work in concert to provide students with opportunities to own their own learning and to engage with their own and each other's ideas.

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STUDENT-TO-STUDENT RESPONSIVENESS IN MIDDLE-GRADES CLASSROOMS

Hamilton L. Hardison
Texas State University
HHardison@txstate.edu

Julia Przybyla-Kuchek
University of Georgia
juliapk@uga.edu

Jessica Pierson Bishop
Texas State University
jbishop01@txstate.edu

NCTM's Principles to Actions called for teachers to develop environments that support students' engagement with each other and teachers. In this report, we consider how often students discursively engaged with the mathematical thinking of their peers (i.e., student-to-student responsiveness) in whole-class settings across 11 middle-grades classrooms. We consider the prevalence and variability of two types of student-to-student responsiveness: responses requested by the teacher and responses offered spontaneously. Although student-to-student responsiveness was not uncommon, we noted considerable variability in prevalence across classrooms.

Keywords: Classroom Discourse, Responsiveness

Responsiveness to students' mathematical thinking is the extent to which students' mathematical ideas are present, attended to, and used to drive mathematics instruction (Bishop, Hardison, & Przybyla-Kuchek, 2016; in press). Extant literature focusing on how student ideas are taken-up and used in mathematics classrooms has primarily emphasized how *teachers* might respond to student thinking. For example, Leatham, Peterson, Stockero, and Van Zoest (2015) explicated criteria teachers might use to determine when to pursue a particular student idea. Other researchers have outlined various kinds of talk moves teachers can enact to publicize, support, or extend student thinking (e.g., Chapin, O'Connor, & Anderson, 2013; Jacobs & Empson, 2016). We agree teachers' intentions, choices, and actions are important components of characterizing responsiveness to students' mathematical thinking; however, we also recognize that students can (and do) respond to peers' mathematical ideas in classroom discussions. Moreover, NCTM (2014) has recommended for teachers to develop "environments in which students feel secure and confident in *engaging with one another* and with teachers" (p. 115, emphasis added). Yet, there is little empirical research characterizing how students respond to one another in whole-class discussions (see Hufferd-Ackles, Fuson, & Sherin, 2014 for an exception). In this report, we shift our focus from teacher-to-student responsiveness to consider the prevalence of student-to-student responsiveness—instances in which students respond to the mathematical thinking of their peers—in whole-class discussions. The research questions guiding this study were: (a) how prevalent is student-to-student responsiveness during whole-class mathematics discussions in middle-grades classrooms? (b) how often is student-to-student responsiveness spontaneously enacted by students versus prompted by classroom teachers? and (c) how does student-to-student responsiveness vary across classrooms?

Methods

We report on data from a larger study investigating characteristics of mathematics classroom discourse in grades 5–7. Participants included teachers and students in 11 classrooms across 4 U.S. states. Participating teachers were recruited due to reputations for using discussion in their mathematics lessons. All teachers had at least 6 years of teaching experience. In each classroom, we videorecorded and transcribed four lessons from the first curricular unit on fractions (44 total lessons); we chose fractions because it is a foundational middle grades topic. Lesson transcripts and videorecordings were partitioned into segments, which we define as a series of turns of talk with a common focus (e.g., activity or strategy) and a consistent form of participation format (e.g., whole class, small group). In this report, we present an analysis of only those segments that were in whole-class participation format and also contained mathematical talk; we prioritized whole-class format since all individuals were potential discursive participants. These segments were analyzed using the Mathematically Responsive Interactions Framework described below.

Conceptual Framework

Previously (Bishop, Przybyla-Kuchek, & Hardison, 2017; 2019), we developed a framework for characterizing mathematically responsive interactions (Figure 1). The core of the framework comprises two components that holistically describe in-the-moment interactions between teachers and students: (a) students' mathematical contributions and (b) the moves teachers enact in response to these contributions. Additionally, we developed adjunct components to capture the presence of student-to-student responsiveness, as well as beyond-the-moment responsiveness—instances when student ideas originally expressed in previous lessons or segments are revisited. In this report, we limit our focus to instances in which a student mathematical idea was expressed and attended to by a peer within the same segment (i.e., in-the-moment student-to-student responsiveness; see bolded components in Figure 1).

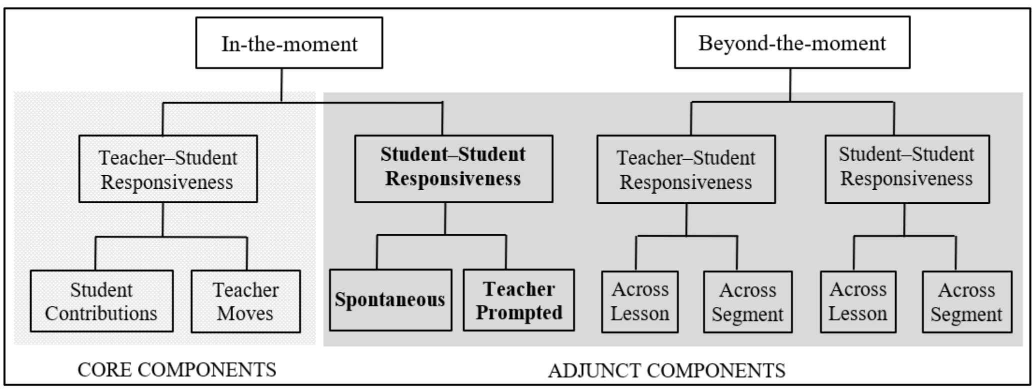

Figure 1: The Mathematically Responsive Interactions Framework

Two kinds of student-to-student responsiveness. To illustrate a distinction in ways students might respond in-the-moment to one another whole-class discussion, consider a portion of a segment from Teacher EK's 5th grade classroom. Prior to the excerpt, Teacher EK asked students to consider whether the equation $3 \div 4 = \frac{3}{4}$ is true or false and to provide a justification. After students considered the equation independently, Teacher EK asked several students to share their reasoning in whole-class discussion including Adam, who had developed two different strategies: one symbolic (represented by equations and discussed below) and one pictorial (partitioning 4 circles into fourths and shading one portion in each circle). In the transcript below, Adam begins by explaining his symbolic strategy. While reading the transcript, consider the ways in which students respond to one another's ideas.

Tchr EK: All right, tell us what you got.

Adam: Um, I think it's true because…I just divided three up into ones and so that I had three ones. And then, I divided each one by, then I divided it by four, one by four equals one

fourth. And then I timesed it by three again. So I pretty much three divided by three then equals one, divided by four, one fourth times three, and then that equals three fourths.
Tchr EK: Very nice, very nice. Can anyone kind of, uh, restate what Adam did in your own words? What did he do? Anyone want to take a chance and kind of tell us what Adam did up there? Rylan?
Rylan: Um, he made it kind of like easier by divided the three into its own, like own little fractions, and he thought it would be easier so he, he drew it first, then he showed how he got the answer.
Adam: Actually, that's actually another way I did it. I did, so I divided each, uh, I wrote down one circle, then I divided each circle into fourths, and then I got three fourths from that too.
Tchr EK: All right. Let's give Adam a round of applause, nice job. Thank you, Adam.

In this excerpt, we noticed two clear instances of student-to-student responsiveness, namely Rylan's and Adam's last turns of talk. Adam's initial explanation of his symbolic strategy in the excerpt above can be summarized as $3 \div 4 = (1 + 1 + 1) \div 4 = 3 \times (1 \div 4) = 3 \times \frac{1}{4} = \frac{3}{4}$. After his explanation, teacher EK invites other students to restate Adam's strategy, and Rylan responds by interpreting Adam's pictorial strategy, which was also visible. Because Rylan's interpretation of Adam's ideas was preceded by Teacher EK's invitation, we characterize Rylan's response to Adam's thinking as an instance of *teacher prompted student-to-student responsiveness*. Without prompting from Teacher EK, Adam responds to Rylan, "Actually, that's actually another way I did it," implicitly rejecting her interpretation of his thinking. Because Adam responded to Rylan's contribution without an invitation from Teacher EK, we characterize this as an instance of *spontaneous student-to-student responsiveness*.

Segment Coding

Across all 11 classrooms, we analyzed 615 whole-class mathematical segments. We coded for the presence of each type (i.e., spontaneous or teacher prompted) of student-to-student responsiveness at the segment level. For example, the segment from Teacher EK's classroom above was coded as containing both teacher prompted and spontaneous student-to-student responsiveness. The presence of a teacher's request for students to engage with a peer's mathematical contribution was necessary but not sufficient to code a segment as containing teacher prompted student-to-student responsiveness; in addition, one or more students actually had to respond to a peer's mathematical contribution to receive the teacher prompted code. Finally, we emphasize that our coding for each type of student-to-student responsiveness was binary at the segment level; therefore, we did not count instances of student-to-student responsiveness within the same segment. Because the number of whole-class mathematical segments per classroom varied, we report the percent of segments per classroom containing different types of student-to-student responsiveness in our results to compare across classrooms.

Results

Looking across all 615 whole-class mathematical segments, we found 35.6% (219 segments) contained some form of in-the-moment student-to-student responsiveness (see last row of Table 1). At large, teacher prompted student-to-student responsiveness was more prevalent than spontaneous student-to-student responsiveness, with these codes occurring in 30.9% and 16.4% of segments, respectively. The co-occurrence of teacher prompted and spontaneous student-to-student responsiveness within the same segment (e.g., Excerpt 1) was common; of the 101 segments containing spontaneous student-to-student responsiveness, 72 also contained teacher prompted student-to-student responsiveness.

Each classroom contained segments in which students responded to peers' mathematical ideas. Across classrooms, the percentage of segments containing student-to-student responsiveness per classroom ranged from 8.9% to 75.4% (Table 1). All classrooms contained instances of teacher prompted student-to-student responsiveness, which ranged from 6.1% to 59.6%. In all but one classroom (SY), the percent of teacher prompted student-to-student responsiveness exceeded the spontaneous counterpart, and, in general, higher percentages of teacher prompted student-to-student responsiveness were associated with higher levels of the spontaneous counterpart ($r^2 = .68$). In 7 of the 11 classrooms, spontaneous student-to-student responsiveness was found in less than 10% of segments. The remaining four classrooms (SY, LE, MA, and AE) accounted for 83% of all 101 segments containing instances of spontaneous student-to-student responsiveness with percentages ranging from 16.1% to 63.2%.

Table 1: Percentage (Frequency) of Segments with S2S Responsiveness by Classroom

Classroom	# of Segments	Either S2S	T Prompt	Spontaneous
SY	57	75.4% (43)	59.6% (34)	63.2% (36)
LE	44	65.9% (29)	59.1% (26)	38.6% (17)
MA	56	50.0% (28)	44.6% (25)	16.1% (9)
AE	75	49.3% (37)	41.3% (31)	29.3% (22)
KM	71	33.8% (24)	31.0% (22)	9.9% (7)
TA	38	39.5% (15)	39.5% (15)	2.6% (1)
EK	44	20.5% (9)	20.5% (9)	4.5% (2)
NY	63	20.6% (13)	19.0% (12)	3.2% (2)
EY	73	16.4% (12)	12.3% (9)	4.1% (3)
SN	49	10.2% (5)	6.1% (3)	4.1% (2)
AY	45	8.9% (4)	8.9% (4)	0.0% (0)
All Classrooms	**615**	**35.6% (219)**	**30.9% (190)**	**16.4% (101)**

Discussion

The results presented in the previous section indicate that it is not uncommon for students to respond to the mathematical ideas of their peers and, in many instances, classroom teachers successfully support them to do so. We were also struck by the variation across classrooms in our study: in some classrooms, student-to-student responsiveness was relatively rare and in other classrooms it occurred in over 50% of the segments. And finally, we note that the presence of spontaneous student-to-student responsiveness may be a particularly powerful indicator of the overall responsiveness in a classroom. Further studies are needed to coordinate how other factors (e.g., classroom norms, authority, etc.) might account for variation in the prevalence of student-to-student responsiveness across classrooms and what practices best support students to spontaneously engage with the reasoning of their peers. Finer-grained analyses are needed to characterize distinctions in the nature of students' responses to their peers' mathematical thinking, as well as to account for the within-segment density of student-to-student responsiveness. Additionally, subsequent analyses are required to examine how student-to-student responsiveness relates to other factors of our framework (e.g., core components). In closing, we note that we analyzed whole-class mathematics discussions in four fractions lessons from each classroom; the prevalence of student-to-student responsiveness may have been different in these classrooms if we had examined other content areas or participation formats.

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CALCULUS TUTOR INTERACTIONS AND DECISION MAKING

Carolyn Johns
The Ohio State University
Johns.125@osu.edu

While most universities offer undergraduate mathematics tutoring, currently is it unknown what occurs during these tutoring sessions or why tutors choose to enact certain practices. Relying on analysis of tutoring sessions and stimulated-recall interviews, we considered what one tutor did during tutoring sessions and factors that influence her decisions regarding how to navigate the interactions. Results indicate the tutor was inclined to ask questions but concern for students' affect may have taken precedence to attempts to further probe student thinking.

Keywords: Post-Secondary Education, Instructional Activities and Practice, Informal Education

In 2010, the Mathematical Association of America performed a survey of calculus students, instructors, and programs, as a step toward understanding the current state of college calculus (Bressoud, Mesa, & Rasmussen, 2015). Undergraduate mathematics tutoring or the presence of a learning center was cited as one of the primary features of institutions with successful calculus programs. In addition, 97% of institutions surveyed in the study offer some form of calculus tutoring with the overwhelming majority offering tutoring by undergraduate tutors. Further, attending tutoring has been shown to positively impact calculus performance as measured by final course grades (Rickard & Mills, 2018). This indicates tutoring plays an important role in undergraduate mathematics education, yet little is known about what occurs in this type of tutoring or the work that these tutors do.

What research exists on tutoring is often content independent, examining the general pedagogical practices of tutors, such as tutor questioning practices (Graesser & Person, 1994). In addition, these studies often examine the practices of 'expert' tutors, most of whom have had formal education training (Lepper & Woolverton, 2002) or teaching experience (Arcavi & Schoenfeld, 1992; Schoenfeld et al., 1992). In addition, studies on tutoring often explore tutor moves in the context of working with a student on a pre-determined topic for a fixed amount of time and number of sessions. Research has not attended to the specific context of undergraduate tutors working in a drop-in tutoring site, the most common form of calculus tutoring (Bressoud et al., 2015).

This report is a part of a larger study in which we attempt to compile and analyze descriptive case studies on tutoring practices in the specific context of drop-in undergraduate Calculus tutoring. In particular, in the larger study we seek to answer the questions: 1) What do tutors do in the course of their interactions with students? and 2) What factors influence the tutors' decisions regarding how to navigate the interactions? This current report offers an analysis of one tutor along the two research questions.

Literature Review

This work is grounded in two conceptual principles: 1) in response to others, humans make decisions based on what they assume is important, drawing from what they notice in the course of interactions (Herbst & Chazan, 2003) and 2) responses humans make in the course of interactions draw on current understandings of the event, such as its circumstances and content, and resources they have at their disposal (Schoenfeld, 2008). Schoenfeld argues teachers' in-the-moment decisions are the result of their resources, goals, and orientations. Resources include knowledge, materials, and interpersonal skills. Goals occur on multiple levels and may be academic or non-academic in nature. Finally, orientations include preferences, values, and beliefs. The data was analyzed with this framework in mind to examine the tutor's references to these factors which influence decisions. We assume tutors make rational decisions as they interact with students and organize the type of experiences they provide for those seeking help. We elaborate in detail in the following section how these points shaped the analysis of data in this study.

We also draw on literature regarding teaching practices that support student learning. In order to create a learning environment that supports students' conceptual understanding, certain social and socio-mathematical norms are accepted as critical to be negotiated by the instructor and student (Cobb, et al. 1992; Cobb & Yackel, 1996; Yackel & Cobb, 1996). Further, scholars have argued that in order to encourage the student to be an active participant and build on students' thinking, instructors need to develop eliciting skills to engage the student in the interaction and draw out the students' thinking. While the student is talking, the instructor needs to listen to the student in order to build an understanding of what the student is thinking (Teuscher, Moore, & Carlson, 2016). Instructors should not be listening for specific answer but instead engage in listening in order to negotiate meaning with the student they are working with (Davis, 1997). Tesucher et al. describe the instructor's cognitive process in listening to the student, "de-centering". In de-centering stance, the instructor builds a model, or way of explaining the student's understanding based on what the student communicates. The instructor then uses this mental model to make in-the-moment instructional decisions on how to proceed.

Methods

The larger study is being conducted using a case study approach with each case being a Calculus I tutor in their first year of tutoring. This report will focus on one these cases, Betty.

Betty is a sophomore working in the Calculus I room in a drop-in tutoring center. Betty is a mathematics major in the education track. Her intended career is as a high school teacher. She previous tutored students in high school but did not feel it impacted her current tutoring practices. When asked how her education courses have influenced her tutoring, she indicated her adolescent development class made her think about how she can be there to guide students but in the end, they need to be able to do it on their own in the future.

Students come to the tutoring room either with questions in mind or to work on their homework and ask questions as they arise. Typically, students ask for help on a specific homework task rather than for help understanding a general concept. Tutors generally help students with their particular task and then move on to the next student. A typical tutoring session may last 5-15 minutes. While tutors may have worked with a particular student in the past, it is more likely they have not worked together before.

As part of her training requirements, Betty recorded three tutoring sessions at four points during her first year of tutoring. A session was defined as an interaction with an individual student, typically on one task. At each of the four points in time, Betty had approximately two weeks to audio record her three sessions and submit the recordings and photographs of the task and written work. Betty collected one set of recordings during her first weeks as a tutor (beginning of September), one set halfway through her first semester (mid-October), at the end of her first semester (late November), and at the beginning of the subsequent semester (late

January). It was left to Betty to select the sessions she recorded and submitted. These recordings and photographs constituted the tutor session data for the study.

In addition, Betty met with the researcher twice to discuss her recorded tutoring sessions. Each meeting was conducted as a stimulated recall interview, having Betty listen to each of her tutoring sessions. At key points while listening to the session, the interviewer stopped the recording to ask Betty to explain what she thought was going on, why she did what she did, or what she was trying to do or get at. This allowed the researcher to better understand Betty's decision making and thought processes rather than making assumptions. In addition, during the interviews, the researcher asked the Betty questions regarding her beliefs about tutoring and learning.

Data Analysis

Analysis of the data occurred in waves. First, Betty's tutoring sessions from September and October were transcribed and indexed to describe the actions of each talking turn. Then these transcripts were open coded to look for patterns of behavior to begin to formulate hypotheses regarding these behaviors and possible rationales. Betty's session transcripts were then marked for passages where these behavior patterns emerged. It was at these marked points in the tutoring sessions that during the stimulated recall interviews, Betty was asked questions regarding her thoughts and behaviors. In addition, the hypotheses for factors influence Betty's behaviors were also addressed through additional interview questions.

After conducting the first interview, the interview transcript was analyzed using a grounded theory approach as we looked for themes which emerged from the data. This phase of the analysis drew on several bodies of literature. Because this was an exploratory study seeking to find baseline data on tutoring, the exact frameworks which may be useful in examining tutoring sessions were not known. Employing grounded theory techniques allowed the researcher to look for aspects of the data which come out as key features of tutoring. However, of particular note was the literature on social norms (Cobb, et al. 1992), decentering (Teuscher, Moore, & Carlson, 2016), and teacher decision making (Schoenfeld, 2011).

During analysis, social and socio-mathematical norms were inferred based on the patterns of interactions. Tutoring sessions were examined for negotiation of social norms such as justifying answers, the student writing, or pushing against the idea of the tutor as the authority on mathematics (Cobb, et al., 1992). In addition, they were examined for negation of socio-mathematical norms such as justifying with reasons why steps work rather than just procedural steps, or drawing of representations to help solve tasks (Yackel and Cobb, 1996). Further, Betty's rationale for their behaviors attempting to negotiate these norms were analyzed. To analyze the data for decentering, Bas Ader and Carlson's (2018) levels of decentering framework was employed. Bas Ader and Carlson's levels of decentering framework provides observable instructor behaviors as indicators they are working at a particular level of decentering.

In light of the analysis of the first interview, an initial model accounting for Betty's thinking and factors influencing her work was constructed. Betty's tutor sessions from November and January were then analyzed and again marked for stimulated recall interview questions. In particular, these sessions and the second interview were examined for data which confirmed or contradicted the existing theory.

Results and Discussion

Betty's first round of recordings from September dealt with analytical limit tasks, and in particular, with each of the three students, at least one task involved an absolute value expression. Overall, Betty's interactions with the students in these sessions can be characterized setting up a solution path she wanted the student to follow. She frequently asked questions to elicit specific information and listened for what she wanted to hear (Davis, 1997). Throughout these sessions, Betty was often at a decentering (Bas Ader and Carlson, 2018) level one or two as she was asking questions to elicit the student thinking but did not attempt to further understand what the students shared or build on their thinking. When the student did not give Betty the answer she desired, she often attempted to redirect them to her planned path. She frequently referred to wanting to "establish" information, assuring that she and the student were on the same page. She assessed the success of the interaction based upon whether the student followed her set path. In addition, Betty rarely asked students to justify her answers or explain why certain procedures worked. What was unique about Betty was her attention to student affect and the ways in which it both constrained and enhanced her tutoring. Although at times she attributed differences in the way she worked with students to the student, she also attributed differences to herself, such as wording her question poorly. When discussing her beliefs surrounding tutoring, learning, and her role as a tutor, one does see alignment between her behaviors and beliefs. For example, Betty understood students need to be able to do the mathematics on their own and discussed her role as being a guide. However, she also talked about her goal being to show them what they can do on their own. This seemed to indicate Betty thought of her role as showing students and guiding them on the current problem, so that it transfers to when they do problems on their own.

Betty's second round of recordings from October showed similar tendencies. However, there were some subtle differences. On the whole, with these related rates task sessions, Betty still looked for the students to take her solution path and have her ways of thinking, however, Betty used more questions in attempt to uncover the students' thinking than in her prior sessions, decentering level two (Bas Ader and Carlson, 2018). In addition, Betty asked the students more regarding if the student understood why procedures worked, rather than immediately explaining why herself. Furthermore, in these sessions, Betty seemed to be implicitly modeling problem solving techniques. In her prior sessions, Betty often referred back to the goal of the task. However, in these related rates tasks, she also organized information by drawing pictures, labeling, and focusing on what information is given in the problem, and what it is asking. Betty described this role of helping students organize information as important for problems such as related rates and optimization.

Data from Betty's 3rd and 4th rounds of recordings, as well as data from her second interview confirmed findings from her earlier data. She continued to press her students to participate and ask for them to carry out steps. She also continued to ask primarily information gathering questions (Boaler & Brodie, 2004) but occasionally asked students if they knew why something was true. Based on these sessions, the amount of time Betty spent explaining and leading the student down her path versus allowing the student to work more independently appeared to be based on her initial impression of the student's understanding.

Results indicate Betty desired for her tutors to be active participants in her sessions. However, her goal of not wanting to harm their confidence appeared to be a primary reason why she did not inquire further into student's incorrect thinking, thus she does not address the student's unique thinking. This indicates that tutor training programs may need to address reconciling concern for student's affect when responding to incorrect responses in pedagogically sophisticated ways. Further, once tutors are asking questions of their students, additional training may be needed to encourage them to ask questions beyond information gathering questions (Boaler & Brodie, 2004). Tutors need to be trained to ask 'why' questions and follow up questions to further understand their student's thinking.

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MIDDLE SCHOOL TEACHERS' BELIEFS ABOUT THE ROLE OF EXAMPLES IN PROVING-RELATED ACTIVITIES

Eric Knuth
University of Texas at Austin
eric.knuth@austin.utexas.edu

Orit Zaslavsky
New York University
orit.zaslavsky@nyu.edu

Rebecca Vinsonhaler
University of Texas at Austin
rkv@austin.utexas.edu

Damon Gaddis
University of Texas at Austin
damongaddis@utexas.edu

Luis Fernandez
University of Texas at Austin
lfmg1992@utexas.edu

Examples play a critical role in mathematical practice. Indeed, the underlying hypothesis guiding this study is that students who are able to strategically think about and productively use examples as they engage in proving-related activities (e.g., developing conjectures, exploring conjectures) will not only gain a deeper understanding of the mathematical content, but also learn to successfully develop proofs. Yet, students typically receive very little, if any, explicit instruction on how to become more deliberate and strategic in their use of examples to support their efforts in proving and learning to prove. The study reported here explores fifty-four middle school mathematics teachers' beliefs about the role examples play in proving-related activities and the instructional practices (or lack thereof) they implement to foster the development of students' productive example use.

Keywords: Instructional Activities and Practices; Reasoning and Proof; Teacher Beliefs

Despite more than two decades of calls to elevate the status and role of proof in school mathematics, students of all ages continue to struggle learning to prove (e.g., Stylianides, Stylianides, & Weber, 2016), and teachers as well struggle to facilitate the development of students' learning to prove (e.g., Bieda, 2010; Bieda, Drwencke, & Picard, 2014; Stylianides, Stylianides, & Shilling-Traina, 2013). Researchers have suggested that students' treatment of examples and, in particular, their overreliance on examples as a means of justification, is a primary source of their difficulties in learning to prove (e.g., Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki, 2012). Yet, examples-based reasoning plays both a foundational and essential role in the development, exploration, and understanding of conjectures, as well as in subsequent attempts to develop proofs of those conjectures. Moreover, we posit that instructional activities designed to help students learn to strategically think about and productively use examples in proving-related activities will facilitate the development of their learning to prove.

The Role of Examples in Proving-related Activities

Examples play a critical role in proving-related activities (e.g., Lakatos, 1976; Polya, 1954). Indeed, the time spent thinking about and analyzing examples can provide not only a deeper understanding of a conjecture, but also insight into the development of a proof. Not surprisingly, Epstein and Levy (1995) contend that "Most mathematicians spend a lot of time thinking about and analyzing particular examples," and they go on to note that "It is probably the case that most significant advances in mathematics have arisen from experimentation with examples" (p. 6). To that end, mathematics education scholars have noted a number of roles and uses of examples related to various aspects of proving (e.g., Alcock & Inglis, 2008; Buchbinder, & Zaslavsky, 2018; Ellis et al., 2012; Iannone et al., 2011; Sandefur, Mason, Stylianides, & Watson, 2013). In our recent work (see Knuth, Zaslavsky, & Ellis, 2017), we extended prior research and developed a comprehensive analytic framework for characterizing and making sense of the roles and uses of examples in the proving-related activities of middle and high school students, undergraduate students, and mathematicians. Although going into detail about the framework is beyond the scope of this paper, the framework illustrates the complexity of example use in proving-related activities and serves as the analytic lens for the study presented here.

Table 1: Example Use Framework

Category	***Sub-categories***		
Purpose of Example (to…)	*Understand what (the conjecture is saying)*	*Understand why (the conjecture is true or false)*	*Explore the truth domain of the conjecture*
	Test the truth of the conjecture	*Refute the conjecture*	*Confirm a belief*
	Convey the claim that the conjecture is true/false	*Develop a new conjecture*	*Convey a general argument*
	Illustrate a representation	*Make sense of a representation*	*Placate the interviewer*
Category	***Sub-categories***		
Criterion for choosing a particular example	*Easy (to work with)*	*First thought of*	*Different from other(s)*
	Minimal case	*Familiar or known example*	*Unusual*
	"Random"	*Boundary case*	*Typical*
Category	***Sub-categories***		
Affordances from example use activity	*Gain insight*	*Produce a justification*	*Lead to new/revised conjecture*
	Generalize	*Learn the limitations of examples*	*Discover an error*
Category	***Sub-categories***		
Strategies employed during example use activity	*Diversity of examples*	*Searching for counterexamples*	*Random feature/pattern search*
	Systematic variation of examples	*Jumping to formality*	*Trying to see the examples through a structural lens*
	Choosing a set of examples with specific properties	*Building formality from examples*	*Informal induction*

We characterize productive example use during proving-related activities as use that helps a learner make progress toward the development of a proof. In particular, we view productive example use as activity that leads to a deeper understanding of a conjecture and the underlying mathematics; leads to insight with regard to the development of a proof; leads to a an awareness of a generalization or structure by using a *generic example* (i.e., a specific example through which one can see the general case); leads to the generation of a counterexample; leads to the development of a new or revised conjecture; or leads to an appreciation for the need for proof. As might be expected, the extent that example use is productive in the proving-related activities of sophisticated learners (i.e., mathematicians) often stands in contrast to the extent that example use is productive in the proving-related activities of less sophisticated learners (i.e., students). Although some secondary students did use examples productively in their proving-related activities (Aricha-Metzer & Zaslavsky, 2017; Ozgur et al, 2017), it was a minority of students; in fact, we posit that students' lack of instructional guidance and experience with using examples productively during proving-related activities is the overarching reason that many students fail to learn from examples.

Research Objective and Questions

The objective of the study was to examine middle school mathematics teachers' views about the role of examples in proving-related activities, and the nature of their instructional practices with regard to example use during proving-related activities. The study was guided by the following research questions: (1) What is the nature of middle school teachers' thinking about and instructional practices related to example-use in proving-related activities? and (2) To what extent do middle school teachers support students with instructional guidance and experience with using examples productively during proving-related activities?

Methods

Participants

Fifty-four Texas middle school mathematics teachers participated in the study; 25 Grade 6 teachers, 22 Grade 7 teachers, and 20 Grade 8 teachers (note that some teachers taught multiple grade levels). Teaching experience was categorized into three levels: 1-3 years of experience (n = 13), 4-10 years of experience (n = 21), and 10+ years of experience (n = 20). Curricula used by the teachers was guided by the Texas Essential Knowledge and Skills (TEKS), and included a variety of curricular programs (e.g., McGraw Hill, Carnegie Learning, Go Math).

Data Collection

Teachers responded to a 26 question Qualtrics on-line survey. The survey included four background questions (e.g., years of teaching experience) and 22 questions related to their views about the role of examples in learning to prove. In particular, question focus ranged from their own instructional practice related to example use during proving-related activities to their views related to expectations for students' example use during proving-related activities. Questions included forced choice responses and open-ended responses; see Table 2 for sample questions.

Table 2: Sample Survey Questions

When developing, exploring, or justifying conjectures, students often use examples. a) To what extent are students strategic in thinking about their use of examples when developing, exploring, or justifying conjectures? *Not very strategic* *Somewhat strategic* *Very strategic* b) What purpose(s) do you think their use of examples serves? c) To what extent do you think students' use o examples when developing and exploring examples helps them learn to develop mathematical proofs? *Very little help* *Somewhat helps* *Significantly helps*
In your own instruction, how often do you explicitly talk with students about how to use or think about examples when developing, exploring, or justifying conjectures? *Rarely* *Occasionally* *Frequently* Please given examples of what you might say to students.
Given the following conjecture: If you add any number of consecutive whole numbers together, will the sum always be a multiple of however many numbers you added up?

a) How likely is it that your students would use examples when exploring this conjecture?
Unlikely *Somewhat likely* *Very likely*
b) Describe a typical student approach to exploring and justifying the conjecture.
c) Assuming your students would use examples, the choices below reflect potential criteria for why middle school students might select examples when exploring and justifying the conjecture. For each choice, select how often you think students use that criterion. [Each choice had the following options: Rarely, Occasionally, & Frequently]
   - to help them understand what the conjecture states or means
   - to check whether the conjecture is true
   - to test a variety of cases in order to see when the conjecture is true (or false)
   - to prove that the conjecture is true (e.g., examples are sufficient as proof)
   - to disprove the conjecture
   - to gain insight about why the conjecture is true
   - to help explain (to someone else) why the conjecture is true
   - to help develop a general argument that the conjecture is true

Data Analysis

Data were analyzed using the examples use framework (Table 1) and the research literature related to proving and example use. Analysis focused primarily on teachers' views about the role examples play in proving-related activities, about students' thinking about and use of examples, about curricular opportunities for engaging students in example use, and about instructional practices related to proving and example use. The analysis resulted in the identification of categories and relationships related to the aforementioned foci as well as the identification of clusters of categories and relationships about example use during proving-related activities.

Results and Discussion

We found that many teachers have limited views of what it means to use examples strategically during proving-related activities; in particular, the teachers did not tend to think students are very strategic in their example use, they tended not to provide explicit instruction designed to help students learn to strategically think about and productively use examples in proving-related activities, and when students do use examples, they thought students primarily use examples to confirm or verify the truth of conjectures. We also found that teachers demonstrated low expectations regarding generic examples, a particularly accessible form of general argument for middle school students (Leron & Zaslavsky, 2013; Zaslavsky, 2018). Specifically, teachers did not expect students to generate generic examples, to understand them, or to be convinced by them (in fact, teachers thought that students would be more convinced by several confirming examples). Excerpts and further details to be provided at the presentation.

Conclusions

Proving is a mathematical practice that is not only critically important to knowing and doing mathematics, but also notoriously difficult for teachers to teach and for students to learn. The perspective underlying this study represents a re-conceptualization of research concerning students' examples-based reasoning, moving from a view of such reasoning as a stumbling block to quickly overcome toward a view of such reasoning as a necessary and critical foundation in learning to prove. Yet, the findings suggest that middle school teachers may have a lack of (or at least limited) awareness and experience with respect to strategic and productive uses of examples during proving-related instructional activities. The findings further suggest the need for both professional development and curricular resources to support teacher efforts to help their students learn to strategically think about and productively use examples.

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AUTHORITY DURING ELEMENTARY MATHEMATICS DISCUSSIONS

Karoline Smucker
The Ohio State University
smucker.37@osu.edu

Keywords: Classroom Discourse, Elementary School Education, Reasoning and Proof

We analyzed interactions in a third grade classroom with the intent to study the ways that different members of the class positioned as math experts during group discussions. Learners took on the role of expert status to explain their reasoning and validate each other. In contrast, the teacher used her position as expert during discussions to challenge and stretch students' mathematical thinking.

The ways in which students justify their thinking, both verbally and in writing, are assumed to be central to effective mathematics learning (Ball & Friel, 1991). The need to engage students in reasoning and justifying ideas appear in both the Common Core State Standards for Mathematical Practice (National Governor's Association Center for Best Practices & Council of Chief State School Officers, 2010) as well as NCTM's Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, 2000). There is agreement that by justifying results and reasoning about ideas students gain a deeper understanding of mathematics as they reflect on both their own contributions and those of their peers. Constructing arguments and explaining them are not individual activities; they must happen in the context of interactions with others to be meaningful (Cobb, Wood, Yackel, & McNeal, 1992). There is also agreement that the teacher plays a critical part in establishing sociomathematical norms that stress reasoning and student autonomy (Cobb & Yackel, 1996; McClain & Cobb, 2001). Of particular interest is gaining a better understanding of interactions that provide a shared space for teachers and learners to actively participate in the construction of knowledge (Wood, Williams, & McNeal, 2006). Observations of a third grade classroom were used to consider this issue. Examining the conversational patterns of the students and the teacher, we considered the locus of authority during the discussion and whether class members used their authority in different ways.

Theoretical Framework

Clarifying and justifying ideas are key components of discourse in mathematics classrooms (Manouchehri & Enderson, 1999). During discussions, the teacher plays a central role in both eliciting student responses and asking for justification. As the most experienced mathematician in the room, the teacher has the responsibility to ask questions and use various interventions to guide the discussion towards specified lesson outcomes (Manouchehri & St. John, 2006). However, it is important that the teacher not be seen as the only expert within the classroom. For meaningful learning to occur, "the intellectual authority in the classroom belongs with the students" (Bennett, 2014, p. 25). Thus, an effective mathematical discussion involves a balance between achieving the teacher's objectives and allowing for student input and discovery.

The term expert references Vygotsky's (1987) concept of the more knowledgeable other; someone who has more knowledge or expertise than the learner. The expert is traditionally an adult, but could also be a peer. In a community of practice approach to learning, members take on various roles and participate in different ways throughout a lesson (Gutiérrez & Rogoff, 2003). Using this framework, “expert” is not a static role in a classroom setting. It allows for both students and teachers to have authority depending on the situation. In addition, classroom authority is influenced by a student’s ability to justify their reasoning. Lampert (1990) observed students take on the role of “more experienced knowers in relation to one another” when involved in whole class discussions that required students to justify their reasoning (p. 42). In these discussions, students were also required to critique the reasoning of others.

Methods

The setting for this study was a third grade classroom at an elementary school in a Midwestern state. The class consisted of a teacher, Ms. Jones, along with 22 students and two support staff (all names are pseudonyms). Data collection and analysis, relying on a naturalistic inquiry method (Armstrong, 2012), focused on how members of the classroom are positioned as experts. Data consisted of field notes and transcripts of classroom interactions. Observations took place during mathematics instruction at least once a week over a six-week period, for a total of 14 hours of observation.

This analysis focused on isolating instances of discourse episodes during which different members of the community acted or were perceived by others as experts. Occasions where students were asked to justify their thinking were noted and tallied. These interactions were analyzed to identify who was seen as an expert and how they used their expert status.

Mehan (1985) emphasizes that turn taking looks different in educational settings than outside of them. Classrooms frequently include patterns where the teacher has both the primary right to speak and the ability to choose who else participates in the discussion (Erickson, 2004). Because of this, moments where students were in control of the discussion were also highlighted as potentially significant in terms of classroom authority.

Findings

Analysis of classroom discussion episodes revealed significant differences in the ways that the teacher used authority as compared to her students. While the students took on the role of resident experts as they expressed, explained or defended ideas, Ms. Jones exercised authority by requesting that students listen to one another, justify results, and/or challenge shared ideas.

Conversational Patterns and Authority

The following vignette from one class discussion illustrates how different members of the classroom community used their authority. Two students, Ashley and Sophie, had just presented their work on a problem involving boxes of doughnuts and are were receiving feedback from their peers. This vignette highlights the differences between a section of discourse involving student control versus one primarily guided by the teacher.

Sophie: James?
James: I like how you organized your boxes.
Sophie: Thanks.
Sophie: Maggie?
Maggie: I like how you put a little mystery box there so it doesn’t tell you what the answer is so people can figure it out on their own.
Ashley: Sarah?
Sarah: I like how your equation was 6+6+6+6
Ms. Jones: Can I ask a follow up question?

Ashley: Yeah
Ms. Jones: Where did you get those 6s and how did you know that there were 4 of them?
Sophie: Boxes of doughnuts
Ms. Jones: The boxes of doughnuts so *(circles one box on whiteboard)* this is like one 6? And this box would be a 6 *(circles another).* Is that true?
Ashley: Yeah
Ms. Jones: And this one is a 6. Hmmm. Why did we need to use 4 6s?

The turn taking in this interaction features several important elements. First, it is noteworthy that the discussion was entirely student-led for the first part of the excerpt. Ashley and Sophie were given charge of the discussion as they solicited feedback from their peers. The structure of this interaction gave Ashley and Sophie a position of authority because the class was discussing their work. In addition, when getting feedback Ashley and Sophie were given the authority to determine who speaks next, a rare occurrence in classroom interactions (Mehan, 1985).

The vignette described above was a common pattern for discussions in the classroom. When students were explaining their work, Ms. Jones stepped to the side and at times even redirected students when they looked to her to take charge. In this mode, Ms. Jones positioned the students as experts during discussions. However, the fact that students still sought assurance from her implies that they saw her as an authority figure even when someone else was leading.

The different ways that Ms. Jones and the students used their opportunities to speak are also significant. Student leadership did not necessarily shape the discussion in mathematically meaningful ways. Note that in the vignette above James, Maggie, and Sarah all gave positive critiques of Ashley's and Sophie's work and explanation. The student-led discourse was characterized by a cycle of compliments which do not appear to be moving the discussion forward mathematically. Most students in the class had gotten the correct answer, so perhaps there was not much to discuss from the students' perspective.

The complimentary dialogue most likely would have continued without Ms. Jones' deliberate intervention. She first asked permission to pose a follow up question. This reinforces the idea the Ashley and Sophie are in charge of the discussion. It is noteworthy that Ms. Jones did not provide an evaluation of whether the students' work was correct; she used her questions to get them to explain their thinking. During this section it appears that Ms. Jones' authority supersedes that of her students; she easily took control using a more traditional initiation-response pattern (Mehan, 1985) to move in a direction which aligned with her lesson goals.

While the previous interaction contained only positive peer feedback, when students had differing answers they were willing to ask questions or challenge them. An example of this is depicted in the vignette below. The discussion concerned how many loaves of banana bread would be needed for the whole class if each loaf has 6 slices.

Paige: I disagree that it is 4 loaves because if there were four loaves then it wouldn't give exactly 21 slices.
Maggie: We were thinking that it would be better to have some extra slices than not enough for everyone.
Paige: I guess I sort of agree, but the problem didn't say anything about having extra.
Ms. Jones: I'm really interested to hear more about what Maggie meant by not having enough – can you say more about that?

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In this instance, both students saw themselves as experts with something to contribute; Paige was willing to dispute an answer which has been presented, and Maggie stood up for her solution after Paige's critique. In addition, it is interesting that, as in the previous example, Ms. Jones' entrance into the conversation was not to evaluate either student as correct, but to push students to think more about the mathematics in the problem.

Importance of Justification

An emphasis on justifying responses and explaining reasoning was a staple of Ms. Jones' practice. The culture reinforced in class was the notion that it is not just a correct solution that is required for a response; one must also be able to explain reasoning. This specific practice defined expert status in the learning community and was one which students exhibited repeatedly as evidenced in the vignette presented earlier in this paper. Regular references to "I agree" or "I disagree" during debates were typical. Students' behaviors were shaped by Ms. Jones' frequent use of phrases that encouraged students to explain their thinking, explain their ideas, rely on "neighbor's" suggestions, and listen carefully to one another. In doing so she emphasized the importance of justification and reasoning in mathematics- endorsing them autonomy in determining legitimacy of ideas shared and at the same time legitimizing their own thinking.

Discussion

Recent standards in mathematics have emphasized the important role of constructing and justifying arguments. In the context of this classroom, justification is prerequisite to mathematical authority during class discussions. However, it appears that expert status is used in different ways by the students and teacher. Analysis suggests that effective discussions where students construct and critique arguments do not happen automatically or easily; they require careful teacher moderation and practice. Because of the preliminary nature of this study, more research is needed to determine if these observations are typical of other educational spaces.

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EXAMINING FEATURES OF ALGEBRA INSTRUCTION IN CLASSROOMS USING A REFORM-ORIENTED CURRICULUM

Erica Litke
University of Delaware
litke@udel.edu

Julien Corven
University of Delaware
jcorven@udel.edu

Drawing on video observations of algebra lessons in classrooms using a curriculum aligned with supporting ambitious instruction, we investigate additional instructional practices that support students' learning opportunities in algebra. We examine instruction on algebraic procedures and the types of connections teachers make within and across algebra content. We find that while identified practices are prevalent across lessons, they are not always enacted in depth. We also find differences in which practices varied more across lessons and which varied across teachers.

Keywords: Instructional Activities and Practices, Algebra and Algebraic Thinking

Contemporary visions of improving mathematics instruction in the U.S. focus on developing teachers' enactment of ambitious instructional practices, positioning students as sense makers, and centering conceptual understanding (National Council for Teachers of Mathematics, 2000; National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). These goals require teachers to provide students opportunities to reason about mathematics, engage in mathematical discussions, solve authentic problems, and engage in productive struggle (e.g., Hiebert & Grouws, 2007). This vision has been supported by the development of curriculum materials that feature meaningful tasks and questions requiring students to engage in mathematical reasoning and justification (Kieran, 2007).

In the domain of algebra, curricula aligned to reform-oriented ideas aim to support teachers in evolving instruction along the lines described above. There are also finer-grained instructional practices that support students' learning opportunities in algebra specifically, and the extent to which these practices feature in more reform-oriented classrooms remains an open question. For example, supporting students' flexibility in algebraic procedures or making connections across algebraic representations supports conceptual understanding in algebra (e.g., Chazan & Yerulshlamy, 2003; Star & Rittle-Johnson, 2009). While curriculum materials may be designed to include these practices, the enacted curriculum often differs from the intended one (Remillard & Heck, 2014). In this study, we explore the extent to which algebra-focused practices are present in classrooms using a reform-focused curriculum. We ask: *To what extent and in what ways do instructional practices that support students' learning opportunities in algebra feature in lessons focused on conceptual understanding? To what extent do particular instructional practices vary between lessons within teachers or across teachers?*

Background and Framing

We take as a framework that not only is the mathematical work in classrooms distinct from general classroom strategies, but that the work of teaching algebra is tied to the specific content taught (Litke, 2019). Any lens brought to observing instruction must select a particular focus—we examine instructional features that offer the potential to support students' learning opportunities of algebra content specifically, regardless of whether they are typically considered elements of ambitious instruction.

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Success in algebra likely demands student fluency in both symbolic representation and manipulation in addition to conceptual understanding (Kieran, 2007). Researchers acknowledge that while conceptual knowledge can lead to the development of procedural fluency, procedural fluency can also aid in the development of conceptual understanding (e.g., Rittle-Johnson, Siegler, & Alibali, 2001). Procedures involving symbolic manipulation feature prominently in algebra (Kieran, 2013). Deep procedural knowledge may in fact be highly beneficial to student understanding of algebraic ideas (Star, 2005). Teachers can support the development of procedural knowledge by connecting procedures to their underlying concepts (Kieran, 2013) or support procedural flexibility through practices such as comparing multiple solution methods or exploring multiple pathways through a procedure (Star et al., 2015).

Connections between and across algebraic ideas may also support students' learning opportunities in algebra. For example, situating the content of an algebra lesson by relating it to other topics in the curriculum may mitigate difficulties students face transitioning between topics (Hiebert & Grouws, 2007). Developing connections between algebraic representations (e.g., graph, table, equation, and context) may help to support understanding of important algebraic relationships (e.g., Chazan & Yerulshlamy, 2003). Finally, connecting abstract algebraic ideas to their concrete or numerical analogs can support students' understanding of algebraic abstraction and promote learning and transfer (Booth, 1988; Booth et al., 2007; Kieran, 2007).

Methods

Data for this study comes from classroom video recorded as part of a project examining the enacted curriculum across teachers implementing a reform-oriented algebra text (Dietiker, Miller, Brakoniecki, & Riling, 2018). The sample includes 29 video recordings of algebra lessons taught by six teachers across two years (3–6 lessons per teacher), each teaching the same three lessons from CPM's *Core Connections: Algebra* textbook. We coded videos using an algebra-focused observational tool—the Quality of Instructional Practice in Algebra (QIPA)—reflecting instructional practices shown in prior research to support students' learning opportunities in algebra (Litke, 2019).

The QIPA is organized into two broad categories. The first captures aspects of instruction that occur during instruction on algebraic procedures and includes *Making Sense of Procedures*, *Supporting Procedural Flexibility*, and *Organization in the Presentation of Procedures*. The second captures the ways in which teachers make connections between and within mathematical ideas and representations and includes *Connecting Across Representations*, *Situating the Mathematics*, and *Connecting Between Numeric and Abstract Algebraic Ideas* (for more information about the QIPA, see Litke, 2015; 2019).

Both authors used the QIPA to independently score all sample lessons, breaking each lesson into 7.5-minute segments and pausing after each segment to record scores on each of the codes listed above. We first scored each segment on whether or not procedures were taught and excluded segments in which no procedures were taught from analyses around the teaching of procedures. Using a rubric, raters independently assigned each segment a score of *not present* (1) to *high* (5) on each element, indicating the extent to which that element characterized the segment and the depth with which it was enacted. We met weekly to reconcile scores. To address the extent to which specific instructional practices feature in this sample, we calculated segment-level frequencies for each code, generating lesson-level scores by averaging scores across segments. To address the extent to which specific practices vary between lessons within teachers and across teachers, we analyzed scores using a cross-classified hierarchical linear model.

Results

Out of 207 segments total, procedures were taught in 98 of them (47%), appearing in 19 of the 29 lessons. Notably, when procedures were taught, they were reasonably well organized with only 12% of segments scoring low, indicating some lack of clarity or mathematical issues. Less attention was given to other important aspects of teaching algebraic procedures. For example, over one third of segments in which procedures were taught scored not present on *Supporting Procedural Flexibility* and another third scored low on this code, indicating only brief attention to elements of flexibility (e.g., noting multiple pathways through a procedure, attending to key decision points in a procedure, or comparing multiple solution methods). In some segments, the attention to flexibility was more sustained, with 27% of segments scoring mid or higher. There was somewhat more attention to giving meaning to algebraic procedures—just over 30% of segments in which procedures were taught scored low on *Making Sense of Procedures* and 34% of segments scored mid. In these segments, teachers connected procedures to underlying concepts or made meaning of solutions generated by procedures, though this happened in passing or relatively briefly. No segments scored high on this code and in only 14% of segments did this feature occur in any depth.

While this distribution seems to indicate that some practices that support students' learning opportunities are only moderately present, scores at the lesson level show these features to be spread out across sample lessons. In Table 1, we show the percent of the 19 lessons in which procedures were taught that had at least one segment scoring at each level. Encouragingly, nearly all lessons included at least one segment in which teachers and students worked to give meaning to algebraic procedures, and almost half of lessons had at least one segment scoring mid/high or high on this code. Over half of lessons also included at least one segment that scored mid or higher on *Supporting Procedural Flexibility*. We note that teaching is a complex endeavor and there may be good reasons *not* to engage in a specific practice at any given moment in a lesson.

Table 1: Percentage of Lessons with at Least One Segment Scoring at Each Score Point

	Low or Above (≥ 2)	Mid or Above (≥ 3)	Mid/High or High (4–5)
Making Sense of Procedures	95	79	47
Supporting Procedural Flexibility	89	53	21
Organization of Procedures	100	100	58
Connecting Across Representations	97	83	21
Situating the Mathematics	86	38	14
Connecting Numeric and Abstract Ideas	90	83	34

Examining the connections codes, we found these practices to be instantiated less frequently and in less depth. For example, 40% of all segments included no work connecting across representations (e.g., graph to equation). When this practice did occur, it happened only briefly or in passing; only 5% of segments scored above mid on this code. However, this practice was spread across more lessons than the segment level distribution may indicate (see Table 1). Across all 29 lessons, the vast majority (83%) had at least one segment that scored mid or higher. Half of segments included no instances of *Connecting Numeric and Abstract Algebraic Ideas*. When these connections occurred, they were largely at the low (17%) or mid (24%) level. In these lessons, teachers made only brief connections between abstract ideas in algebra and their numeric or concrete underpinnings. However, at the lesson level, the majority of lessons had at least one segment that scored at least a mid on this code, and about one third of the lessons included at least one segment that scored at the higher levels. In contrast, situating the mathematics of the lesson within other algebra topics or the broader domain of algebra rarely occurred—71% of segments included no instances of this type of connection and an additional 20% of segments had this practice occur only in passing. Looking across lessons, fewer than half included at least one segment that scored at least mid on this code. Taken together, these scores indicate that, while these types of connections are prevalent across lessons, they are not always instantiated in depth, despite a curricular emphasis on opportunities to engage in connections.

We used intraclass correlations to examine the extent to which variance in segment scores was more attributable to differences between teachers or between lessons. In Table 2, we show the Intra-Teacher Correlation (ITC), reflecting the percent of variance between teachers, and the Intra-Lesson Correlation (ILC), reflecting the percent of variance between the three textbook lessons within teachers. We find differences in which practices seem to be more lesson- than teacher-dependent. For example, the variation in this sample around *Connecting Across Representations* may reflect differences in the lesson content, tasks, or instructions. Indeed, one lesson focused explicitly on multiple representations of quadratic functions, though other lessons also provided opportunities to for connecting representations that were not always taken up. In contrast, in segments in which procedures were taught (n=98), practices appeared more teacher- than lesson-dependent. For example, twice as much of the variance in *Supporting Procedural Flexibility* was attributable to teachers rather than lessons, indicating that, in this sample, the extent to which procedural flexibility is supported varies quite a bit from teacher to teacher.

Table 2: Percent of Variance Attributable to Teachers (ITC) and to Lessons (ILC) by Code

Code	ITC	ILC
Making Sense of Procedures	8.38	0.01
Supporting Procedural Flexibility	22.91	10.15
Organization of Procedures	4.77	0.00
Connecting Across Representations	0.00	12.02
Situating the Mathematics	1.19	0.02
Connecting Numeric and Abstract Ideas	0.01	12.44

Discussion and Conclusion

Results from this study indicate that teachers relying on reform-oriented curricula like the one used here might consider ways to deepen aspects of their instruction that support students' learning of algebra content. While instructional practices that support developing students' procedural knowledge occurred in these lessons, they were not always enacted in depth. In addition, the limited focus on making connections in this sample was particularly surprising given the focus on such connections in the materials themselves. We also find differences in which practices seem to vary among teachers and which seem to be more strongly linked to a particular lesson. It may be that lesson topic or curriculum materials influence the frequency and degree to which some instructional practices are enacted, although we caution generalizing from a small sample of teachers and lessons. A larger-scale study and more detailed qualitative analysis could help us better understand these differences. Findings indicate that these lessons feature instructional practices that support students' learning of algebra content, but the depth of enactment and the differences between enactments on the same lesson demonstrate that additional support may be needed to develop teachers' instruction along these lines.

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ANALYSIS OF MATHEMATICAL REPRESENTATIONS IN A SYNCHRONOUS ONLINE MATHEMATICS CONTENT COURSE

Theresa Wills
George Mason University
twills@gmu.edu

This study investigates the student-created representations in a synchronous online mathematics course. Over a 15-week semester, a select group of mathematics specialists candidates took the course, MATH 613 - Algebra for K-8 Math Specialists using a 100% synchronous online format. Data collection included interactive slides and reflection paper assignments. The findings indicate that the mathematical representations included abstract, pictorial, concrete and dynamic-pictorial. Dynamic-pictorial is defined in this research as pictorial models that use the advantages of technology to move representations on the screen in a way that could not be reasonably replicated using hand held manipulatives. Implications for this research include supporting instructors of math content courses with planning and delivery of online coursework with the expectation of student-created dynamic-pictorial representations. Further research could address the variety of dynamic-pictorial representations to analyze the complexity of these representations.

Significance of Problem on a Personal Level

I am a Professor of Mathematics Education and teach mathematics through conceptual understanding using rich tasks and multiple representations. I struggled with the geographical barrier that restricted access to my courses and wanted to reach out to new communities. I have taught asynchronous online classes, and did not find that they supported my students in areas of engagement, community, mathematical representations and rich mathematical debate. I resorted to creating representations ahead of time and included them in the slides, but these representations lacked engagement because they were not student-created. My goal was to incorporate my knowledge of representations to create a synchronous online course that is modeled as closely as possible to a traditional face-to-face course. In order to do this with as much fidelity as possible, I taught two sections of the same course back to back on the same evening. The 100% synchronous online course was taught on Mondays from 4:30 - 7:10, and the face-to-face course was taught Mondays from 7:20 - 10:00. I attempted to keep my presentation slides as similar as possible, and only changed the wording from “create a slide” to “create a poster” or something similar. This allowed me to maintain similar class experiences.

Broader Educational Significance of the Problem

Universities are implementing more online courses (Yamagata-Lynch, 2014). NCTM (2000) recommends that mathematics courses are taught using contextual examples with a variety of student-created representations in order to enhance their conceptual understanding of the big mathematical idea. Universities need to evaluate their online courses for these representations and the student engagement in these representations. The pedagogical strategies that are identified could be used in all mathematics content courses such that distance learning can be just as engaging as face-to-face learning.

Statement of Research Problem and Question

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

Research Problem

Universities are implementing more online courses (Yamagata-Lynch, 2014) and these courses need to be evaluated for effectiveness (Herrington et al.,2001). In earlier studies, Herrington et al. (2001) found that online courses lack critical learning experiences that are found in face-to-face classes. However, Polly (2016) found that mathematics teacher leaders in online courses were able to attain similar course outcomes. NCTM (2000) recommends that mathematics content courses use a variety of representations, but this can be difficult to experience in an online course.

Research Question

How do mathematics specialist candidates engage in creating multiple mathematical representations in a synchronous online mathematics content course?

Review of the Literature

Higher education has an increasing need to evaluate the process of designing and delivering online learning (Herrington et al., 2001). Education has been slow to pick up on online courses (Polly, 2016). Online courses have an added challenge of engaging students in a virtual environment that has many differences from the traditional face-to-face learning environment. This online environment must be structured to actively engage students in meaningful tasks for effective learning to occur (Simon, 2002). In order to evaluate online courses, Herrington et al. (2001) developed a framework for evaluating online learning settings which focused on incorporating authentic tasks, opportunities for collaboration, learner-centered environments, engagement, and meaningful assessments. This framework can be used across content areas, and will be used to address mathematics, specifically mathematical representations.

It is vital that students have the tools needed to produce, analyse, understand and recognize representations in their world (Lesh & Doerr, 1998). "Having representational fluency, being able to translate a mathematical idea among these representations, strengthens one's strategic competence and conceptual understanding" (Suh & Seshaiyer, 2007, p. 8). Further, NCTM (2000) states that "when students gain access to mathematical representations and the ideas they express and when they can create representations to capture mathematical concepts or relationships, they acquire a set of tools that significantly expand their capacity to model and interpret physical, social, and mathematical phenomena" (NCTM, 2000, p. 4).

In 1991, Kaput stated that researchers are interested in discovering new types of multiplelinked representations (such as how tables link to graphs). Synchronous online course delivery provides an opportunity to research these representations that, without technology, were previously inconceivable (Puentedura, 2006). While these findings show a potential benefit of online courses, further research is needed on various types of courses with a focus on attributes of the course that support student engagement (Dyment, Downing, & Budd, 2013; Polly, 2015). "There is a need for subsequent studies to further examine the influence of specific characteristics of online courses on learning outcomes and learners' perceptions and experiences" (Polly, 2016, p.14).

Method Context

Description of Participants

There were 16 mathematics specialist candidates in the first synchronous online cohort at George Mason University, including female (87%) and male (13%); a combination of White (69%), African American (19%), and Indian American (6%) educators. All of the candidates were employed in K-12 schools, with the majority (81%) in elementary schools (K-5) and 19% who taught at the secondary level (6-12). The candidates represented varied educational settings, including 81% who resided in districts in Virginia, which does not use Common Core State Standards; others were outside of Virginia in Common Core states (13%) or in international schools (6%). Prior to the beginning of the cohort, 25% of the candidates were already working in mathematics specialist positions without formal preparation coursework. The course met weekly on Mondays from 4:30 - 7:10 EST.

Description of the Online Setting

Math specialists candidates were selected for this program knowing that all courses are delivered using a 100% synchronous online delivery method, however there were no technology skill prerequisites. The math specialist program is a 6-semester program that is completed in two and a half years (3 fall, 2 spring, and 1 summer semester). All classes were based in an online classroom environment using Blackboard Collaborate Ultra. This online classroom allowed participants to access audio and video as well as a shared whiteboard, however video was rarely used. Participants continuously talked together as a whole class or in small breakout rooms that limited audio to the small group. Participants were all also engaged simultaneously in Google Slides which was used as a collaborative whiteboard. At any moment, all group members were adding photos of their handwritten work to the slide, adjusting text boxes, and adding screenshots of a virtual manipulative. The slides were frequently overcrowded, but this allowed candidates to see similarities and differences in their representations. Specialists candidates and the instructor held joint ownership over the slides, and participants could add additional slides as needed. Participants easily navigated multiple websites simultaneously.

Data Collection

Google Slides. Since traditional poster paper cannot be used in an online class, the candidates created their "posters" using Google Slides. This online collaborative tool allowed an entire class or small group of candidates to add text boxes, shapes, drawings, pictures, clipart, tables, and many other virtual representations to a shared slide. All candidates and instructors had ownership of all slides, and any participant could add to, or delete materials to the slides.

Reflections assignments. As part of an assignment, all specialists candidates were required to reflect on four of the rich tasks that were completed in class. The rubric for the reflections required that specialist candidates discuss the representations that were created and presented in class and explain their understanding of the representation in detail.

Data Analysis

It has long been established that three common mathematical representations are abstract, representational/pictorial, and concrete (Robold 1978). While these representations are widely referenced, there were some representations in the synchronous online course that didn't fit into either category. Therefore, slides and reflections were analyzed qualitatively using grounded theory. Codes and properties were explored during open coding, and relationships between the codes were identified. Finally, specific properties were selected for codes and all slides and reflections were coded again under these selective codes.

Findings Identification of Representations

Mathematics Specialist Candidates created a plethora of representations. These include hand drawn, virtually sketched, program tools, and virtual manipulatives. The codes identified in this study are the three established categories, abstract, concrete, and pictorial (Robold 1978) and a new category called dynamic-pictorial.

Description of Dynamic-Pictorial

This type of representations was observed in every class, therefore it was common enough to be described and characterized. Dynamic-pictorial refers to a pictorial representation that moved during the discussion.

Using dynamic-pictorial for a large quantity. In one representation, the candidate was able to use the technology to copy and paste the single square into a row of ten squares, and then into an array of over 200 squares. Then, they modeled the distributive property by separating 54 squares away from another quantity. It would be unrealistic to model this problem in a classroom with over 200 square tile manipulatives, and it would be time consuming to model this task in a similar way in a face-to-face classroom environment. If this representation were to be drawn on a traditional poster, there would be no way to move the model. Instead, the presenter might explain the movement or draw a before and after pictorial representation on their poster.

Using dynamic-pictorial for visually connecting. Smith and Stein (2011) discuss the importance of making mathematical connections to various student invented strategies. The synchronous online class showed a unique way of using the technology to connect candidate's strategies. For example, a candidate created a 5x5 array and explained that they could determine the number of tiles on the border of the array by adding the top and bottom row (5 + 5) and the side rows (3+3) to get a total of 15. Another candidate struggled to explain the connection to their strategy and therefore they decided to copy and paste the same 5x5 array such that they were positioned side by side. Then they changed the colors of the top, bottom and side to match their representation. Other candidates continued to copy and paste this array to show a variety of visual strategies. This is fundamentally different from what can be done using a poster in a classroom. While different pictorial representations can be shown side by side, the technology offers a quick color change option that can change rapidly. This simple use of the technology animates the pictorial representation and creates a whole new representation.

The nature of these representations can be described as a mixture of concrete and pictorial. These representations rely on the advantages of technology, and are unique to the synchronous online class setting.

Discussion and Implications Impact on Participants

Classroom teachers are often required to use a combination of virtual and physical or concrete manipulatives (Moyer-Packenham, Salkind, & Bolyard, 2008). Further, Simon (2002) states that students must actively engage in meaningful tasks for effective learning to occur. With the implementation of Google Slides, candidates were able to actively engage in the creation of a variety of mathematical representations. In fact, the candidates demonstrated a variety of ways that they were able to use the virtual manipulative as a dynamic-pictorial model and therefore used the manipulative in a more meaningful and engaging way.

Impact on Education Field

With the increased demand for online teacher education courses in higher education (Yamagata-Lynch, 2014), universities are implementing synchronous online delivery methods. This research can support instructors with suggestions on how to use student-centered representations. Additional research should be done to analyze dynamic-pictorial representations.

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EXPERIENCING EQUIVALENCE WITH GRASPABLE MATH: RESULTS FROM A MIDDLE-SCHOOL STUDY

Katharine Sawrey
Worcester Polytechnic Institute
kbsawrey@wpi.edu

Jenny Yun-Chen Chan
Worcester Polytechnic Institute
jchan2@wpi.edu

Erin Ottmar
Worcester Polytechnic Institute
erottmar@wpi.edu

Taylyn Hulse
Worcester Polytechnic Institute
trhulse@wpi.edu

Understanding equivalence is fundamental to STEM disciplines, but many students struggle with the concept. We present a novel method for students to explore ideas of equivalence where students dynamically transform expressions from initial states to goal states in the web-based app Graspable Math. The structure of the goal state activities implies that any initial and goal state pair represent the same quantity (or varying quantity). We propose that for students, the physical experience of moving algebraic objects and observing how the initial state transforms into the goal state helps generalize notation mechanics. In fall of 2019, we will test this supposition in a randomized control trial of 1500 students in which student performance in pre- and post-tests will be compared to an online problem set control.

Keywords: Algebra and Algebraic Thinking, Technology, Instructional Activities and Practices, Middle School Education

Student misunderstandings or misconceptions about the equivalence and the equals sign have been noted as inhibiting success in upper-level mathematics and other STEM disciplines (Kieran, 2007; Knuth, Stephens, McNeil, & Alibali, 2006; Stephens, Knuth, Blanton, Isler, Gardiner, & Marum, 2013). For example, a common misconception is when students view the equals sign as marking or calling for a computation, such as interpreting "4+1=5" as "four and one makes five." This type of operational understanding of the equals sign has been found to be stable in middle school students (Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007) and is associated with difficulty in equation solving, even when controlled for grade level and standardized mathematics test scores (Knuth, et al., 2006). A robust perspective, in contrast, is what Stephens and colleagues (2013) describe as a *relational-structural* view of the equals sign - understanding that the equals sign expresses an equivalence relation between the two expressions on either side.

Much of the research on students' understanding of equivalence is embedded in work on students' understanding of the equals sign (Blanton, Stephens, Knuth, Gardiner, Isler, 2015; Kieran, 1981, 1992, 2007; Knuth, et al., 2006; Leavy, Hourigan, & McMahon, 2013; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011). As a case in point, Rittle-Johnson and colleagues (2011) talk about the distinction between numerical equivalence - the ability to match sets of objects on the basis of quantity, and mathematical equivalence - "understanding that the values on either side of the equal[s] sign are the same," (Rittle-Johnson et al., 2011, p. 86). Kieran describes this formally as "the symmetric and transitive character of equality," and informally as "the 'left-right equivalence' of the equal[s] sign," (1992, p. 398).

Our work takes a different tactic by having students work through what we call *goal state* activities using the dynamic notation tool, Graspable Math (GM). In GM, algebraic objects (terms, expressions, and equations) operate according to the mechanics of symbolic notation. For example, permissible moves (such as adding $2x$ and $4x$) transform an expression or equation, while impermissible moves (such as adding $2x$ and $4y$) do not. From a perceptual learning perspective, students' experience of moving algebraic objects on the screen, reinforced by the visual feedback of transformed expressions, helps students to generalize notation mechanics and attend to relevant details. In goal state activities, students are presented with two equivalent states: an initial state and an equivalent goal state (See Figure 1). The students' task is to transform the initial expression into the goal state. These transformations can only happen through algebraically permissible actions. Thus, students are finding the road map of equivalence between the initial state and the goal state.

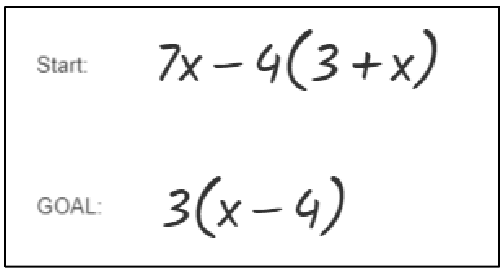

Figure 1: An Example of a Goal State Activity in Graspable Math

In fall of 2019, we will run a randomized control trial of 1500 students in 2 intervention conditions: a GM goal state condition and an online problem set control. Student performance in pre- and post-tests will be compared to explore how GM goal state work might be associated with students' performance on equivalence-related tasks to address the research question, "How does experience with a transformation-based intervention affect student performance on equivalence items compared to a control of traditional online problem set?"

Theoretical Framework

The operational-relational dichotomy in students' perspectives of the equals sign is well-documented in the literature (Blanton et al., 2015; Carpenter, Franke, & Levi, 2003; Kieran, 1992; Knuth et al., 2006; Leavy, Hourigan, & McMahon, 2013; McNeil, Grandau, Knuth, Alibali, Stephens, Krill, 2006). Stephens, et al. (2013) adds nuance to that discussion by differentiating between a *relational-computational* view, where students understand that two sides of the equals sign calculate to the same value, and a *relational-structural* view, where students understand that the equals sign expresses an equivalence relation between the two expressions on either side of the equals sign. This subtle difference is tied to a structural understanding of algebra (Kieran, 2007), where "[students] can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects" (CCSS.Math.Practice.MP7 http://www.corestandards.org/Math/Practice/).

Landy, Allen, and Zednik (2014) proposed that sense making of symbolic notation happens through perceptual and sensorimotor systems. If the capacity for symbolic reasoning is in part the ability to "perceptually group, detect symmetry in, and otherwise perceptually organize symbolic notations," (Landy, Allen, & Zednik, 2014, p. 1), part of the algebraic objectification described above could be perceptual. In other words, since "what students notice mathematically has consequences for their subsequent reasoning" (Lobato, Hohensee, & Rhodehamel, 2013, p. 809), perception of notation is part and parcel of symbolic reasoning (Goldstone, Marghetis, Weitnauer, Ottmar & Landy, 2017). Mathematical fluency, therefore, derives not only from understanding the content, but also a heightened focus on relevant perceptual details. Perceptual learning theory suggests that training one's perceptual and sensorimotor systems in symbolic notation may result in more effective reasoning about the relationships represented by the symbols (Kellman, Massey, & Son, 2010).

Grounded in this theory, GM creates a learning environment where algebraic objects behave according to the mechanics of symbol manipulation in a virtual environment. By making moves on expressions or equations and observing the system response, generalized mathematical properties such as commutativity, distribution, and order of operations, and simplifying expressions or solving equations becomes experiential for students.

For example, imagine a situation where a student is working with the expression "3 + 8 + 4x" (Figure 2). In the first row, the student moves the "8" to the right and drops it past the "4x," experiencing the commutative property by seeing the resulting "3 + 4x + 8." In the second row, the student moves the "8" to the left and drops it on top of the "3." The terms combine, resulting in the expression "11 + 4x." Thus, the student experiences addition. In third row, the student drops the "8" on top of "4x," not a permissible operation since "8 + 4x" is already in simplest form, and the "8" snaps back to its initial location. In each case, the program responds to student actions, and the student has immediate feedback on the impact of their actions. Other gestures beyond selecting and moving allow users to enact most forms of symbolic manipulation including each of the four basic operations, decomposition, distribution, and factoring.

Possible actions on "3+8+4*x*"	Final State in GM	Mathematical Description
move "8" to the right	$3 + 4x + 8$	Commute 8 and 4*x*
move "8" on top of "3"	$11 + 4x$	Add 3 and 8
move "8" on top of "4*x*"	$3 + 8 + 4x$	Not a permissible move ("8+4*x*" shakes)

Figure 2: GM Actions and Response on the Expression "3+8+4*x*"

Research Methods

For the purposes of the 2019 PME North American conference, we will be presenting data from a classroom study to be run in fall of 2019. This study involves a student-level randomized trial of 2 conditions, using the ASSISTments platform (Heffernan & Heffernan, 2014) to maintain condition assignments for each student to either self-paced GM goal state problems or online problem set. The study itself is 3 hours of content over 6 weekly sessions covering the four operations, fractions, and order of operations. Students in the GM tutorials will solve goal state items, while the problem set control will work through similar content items compiled from three open-source mathematics curricula: Engage NY (2014), Utah Math Project (2016), and Illustrative Math (2017). A sample of questions from the control condition is in Figure 3.

Student participants in the pilot will include approximately 1500 middle school students from 50 to 100 classrooms across a large, urban district in the southwestern United States. Participant schools have student populations comprised of 20% to 50% English Language Learners and over

75% identified high needs students, including but not limited to such characteristics of low income, limited English proficiency, and identified learning disabilities. Mathematical performance assessment items are drawn from the Rittle-Johnson, Matthews, Taylor, and McEldoon (2011) and Rittle-Johnson, Star, & Durkin, (2016) assessments of mathematical achievement.

1. Fill in the blank. ____ + $\boldsymbol{g} - \boldsymbol{g} = \boldsymbol{k}$ (adapted from Engage New York)
2. Select **all** the expressions that are equivalent to 4b. a) b+b+b+b b) b+4 c) 2b+2b d) b*b*b*b e) b÷4 (adapted from Illustrative Math)
3. Without solving completely, determine the number of solutions of the equation $3(m-3) = 3m-9$ a) No solution b) one solution c) infinitely many solutions (adapted from Utah Middle School Math Project)

Figure 3: Sample Items from Online Problem Set Control

Approach to Analysis

In line with research question, "How does experience with transformation-based intervention affect student performance on equivalence items compared to a control of traditional online problem set?" we will compare post-performance on mathematical equivalence items from the two conditions. We hypothesize that the immediate, experiential feedback provided in GM heightens student awareness of the mechanics of algebraic notation and is an aide to students' generalizing those mechanics. The primary analysis testing GM for improving math achievement will use linear regressions to estimate mean posttest equivalence scores between students in the GM and the control condition, controlling for pretest scores. We will examine whether GM is more effective for students with lower prior knowledge by testing the condition × pretest interaction. We expect that GM will improve students' knowledge of equivalency more than the control condition, and the effects of condition will emerge in procedural fluency, conceptual understanding, and mathematical flexibility items. This effect is hypothesized to be larger for lower performing students.

Conclusion

Despite major efforts in research, curricula development, and policy, students still struggle with understanding equivalence and the equals sign. Our session presents a novel student experience with equivalence: transforming expressions and equations dynamically and explicitly within a notation environment. Positive results from prior work using a similar arrangement of multiple sessions using a gamified version of GM has shown that it may be effective for decreasing notation errors and improving mathematical understanding (Daigle, Harrison, Braith, Ottmar, Hulse, & Manzo, 2019; Ottmar & Landy, 2017; Ottmar, Landy, Goldstone, Weitnauer, 2015). What will be unique in the analyses is the focus on mathematical equivalence and comparison against a control condition.

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ENACTMENT OF HIGH COGNITIVE DEMAND TASKS ENGAGES STUDENT GROUPS IN HIGHER- AND LOWER-DEMAND COGNITION

Evelyn Gordon
Horizon Research, Inc.
egordon@horizon-research.com

Daniel J. Heck
Horizon Research, Inc.
dheck@horizon-research.com

Jessica Dula
Horizon Research, Inc.
j.dula@horizon-research.com

Courtney Plumley
Horizon Research, Inc.
cnplumley@horizon-research.com

Michelle Blessing
Horizon Research, Inc.
mblessing@horizon-research.com

Productive small-group work begins with the selection and implementation of higher-demand tasks focused on procedures with connections or doing mathematics. Previous research has established factors that result in maintenance and decline of task demands. In developing an instrument for documenting evolution of demands in small groups, we identified two limitations of the Mathematics Task Framework: higher-demand tasks also involve lower-demand elements, and students engage in both higher- and lower-level cognitive work when engaging with higher-demand tasks, without the overall demand necessarily declining. We share a promising instrument for documenting evolution of students' engagement with various task demands.

Keywords: Cognition, Instructional Activities & Practices, Research Methods

Small-group learning environments promote opportunities for conceptual learning and powerful mathematical work (e.g., Mercer, 2005; Veenman, Denessen, vanden Akker, & van der Rijt, 2005). The Peers Engaged as Resources for Learning project studies small-group learning environments in secondary mathematics classrooms, addressing the research question: How is the functioning of small-group learning environments characterized by aspects of mathematics tasks, peer cultures, and mathematics discourse?

Focus of the Study

This report focuses on development of an instrument for analyzing students' cognitive engagement while enacting mathematical tasks in small groups. We share initial findings about the nature of students' engagement as they enact high- and low-level cognitive demand tasks.

Theoretical Framework

We conceptualize the small-group learning environment as comprising three major elements: the mathematics task (Stein, Grover, & Henningsen, 1996), the social peer dynamics among the group members (Hamm & Hoffman, 2016), and the discourse related to the mathematics content (Sztajn, Heck, & Malzahn, 2013). We are broadly interested in what Ryve (2011) distinguishes as Discourse (the culture), viewing the small-group learning environment as a micro-culture in the classroom. It is shaped by and, in turn, shapes these three elements to constitute the opportunities students have to learn mathematics by engaging with content and with one another.

Regarding the implementation of tasks, we ground our work in the literature on cognitive demand of mathematics tasks. Selecting a "group-worthy" task that engages students in high-level mathematics cognition is necessary for productive group work (e.g., Lotan, 2003). However, researchers have found that cognitive demand evolves during classroom enactment from the selected, written task to the teachers' set-up of the task for students, through the students' enactment of the task (e.g., Stein, Grover, & Henningsen, 1996; Warshauer, 2015), and continuing with teachers' conclusion of the task (Otten, 2010). Assessing the cognitive demand of an enacted task therefore requires measurement at multiple points to capture this evolution.

Mathematics Task Analysis Guide

The structure and expectations of written tasks denote a set of demands for what students are intended to do, which may include providing an answer, showing work, justifying a solution method, and/or representing and making meaning of the method and answer. We use Stein and colleagues' Task Analysis Guide (Stein, Smith, Henningsen, & Silver, 2000) to characterize the cognitive demand of a task as written (see Figure 1). The Task Analysis Guide also provided the basis for our characterization of the task as enacted by students in small groups.

Lower-Level Demands	Higher-Level Demands
Memorization	*Procedures with Connections*
• Involves either producing previously learned facts, rules, formulae, or definitions OR committing facts, rules, formulae, or definitions to memory. • …	• Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. …
Procedures without Connections	*Doing Mathematics*
• Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, placement of task. • …	• Requires complex and non-algorithmic thinking (i.e., there is not a predictable, well - rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example). • …
Adapted from Stein, Smith, Henningsen & Silver, 2000	

Figure 1: The Task Analysis Guide (excerpt)

Study of Small Groups in Mathematics Classrooms

Data collection for our broader study of small-group learning environments took place in two waves. In the first wave, we observed lessons that involved small-group work in a naturalistic environment, with no intervention or expectation other than the inclusion of small-group work. In the second wave, teachers consulted with project researchers prior to implementing their lessons. Researchers shared suggestions for making the lesson more "group-worthy," drawing on the three-part framework for the study. Teachers and researchers refined tasks and integrated supports for peer interaction and mathematics discourse. The teachers worked in two different school districts, one municipal and the other rural, in four different middle schools and one high school. Participating classrooms spanned grades 6–9. We observed 50 lessons involving small-group work in classrooms of 11 teachers during the first wave and 8 teachers during the second wave. Seven teachers participated in both waves. Two researchers observed each 90-minute block or 45-minute period, keeping field notes as data. During the lessons, we also audio-recorded the work of small groups of students ranging from pairs to groups of four, with five to fourteen student groups per lesson. A sample of five student group recordings was selected for this instrument development study. To analyze student group recordings, each recording was parsed into 1-minute intervals and researchers applied codes to each interval.

Instrument 1: Factors Associated with Maintenance and Decline of High-Level Demand

Our instrument development began with Stein and Smith's factors associated with the maintenance and decline of high-level cognitive demands (1998, p. 274). We conducted coding trials using two small-group recordings: Group 1 and Group 2. The groups had different teachers and enacted different tasks, which we informally identified as higher- and lower-demand using the Task Analysis Guide. Group 1 experienced a whole class review of formulas for various polygons; afterward, in their small groups, students calculated areas and perimeters of a series of shapes—a lower-level cognitive demand task because the review made evident which procedures students should use. For Group 2, the teacher implemented a high-level cognitive demand task in which students were to minimize the surface area of a box made to contain six identical cylindrical packages, with no arrangement of the cylinders prescribed.

For our study, maintenance and decline of cognitive demand coding did not adequately capture differences in students' experiences enacting these two tasks. Students in Group 1 engaged in lower-level procedures without connections and recall of previously learned information. Students in Group 2 cycled among higher-level non-algorithmic thinking and lower-level procedures without connections and recall of previously learned information. Our coding indicated the original tasks' cognitive demand was maintained in both groups because students enacted the tasks as they were designed. Yet this instrument failed to capture variations and flow in students' engagement with higher- and lower-level demands throughout task enactment.

Instrument 2: Student Engagement in Higher-Level and Lower-Level Cognition

For the second stage of our instrument development, we drafted a rubric based on the Task Analysis Guide to document whether students engaged in higher-level cognition and lower-level cognition during each minute (see Figure 2).

Description	Specific Indicators
Students engage in higher-level mathematical cognition	Students engage in one or more the following: • Complex, non-algorithmic thinking where a procedure is not prescribed or apparent • Seeking and making conceptual connections • Explaining their thinking and reasoning, beyond describing an algorithmic procedure that was used • Pressing one another for justifications, explanations, and meaning, beyond descriptions of an algorithmic procedure • Developing mathematical conjectures or posing new mathematical questions
Students engage in lower-level mathematical cognition	Students engage in one or more of the following: • Applying or explaining algorithmic procedures that are specifically called for or evident based on prior instruction, experience, or placement of the task • Recalling previously learned facts, rules, formulas, or definitions • Routinizing a problematic aspect of the task (e.g., student press the teacher to reduce the complexity of the task by specifying exact procedures or steps to perform)

Figure 2: Student Cognitive Engagement Rubric

Four researchers applied the new rubric to our existing sample and to Group 3, which had enacted a higher-level demand task that was different from Group 2's task. As intended, the new instrument documented differences in the cognitive demand that students experienced throughout the lessons. In addition, for Group 1we generally agreed on the ratings for each minute—students did or did not engage in lower-level mathematical cognition and consistently did not engage in higher-level cognition—and cited the same evidence for applying those ratings. For Groups 2 and 3, which enacted the higher-demand tasks, we found that for some minutes researchers disagreed on the ratings for higher-level or lower-level cognition even though we identified the same evidence to support those ratings. That is, our ratings differed due to researchers' interpretations of what constituted higher-level and lower-level cognition in the context of the higher-level demand task, rather than disagreement about what students were doing.

Instrument 3: Student Cognitive Engagement With Task-Specific Indicators

To develop the third instrument, we documented elements of the task that required higher-level, lower-level, or non-mathematical cognition. To address inconsistencies in our ratings on Instrument 2, we used these task-level elements for decisions about students' cognition. We first conducted a more thorough analysis of the cognitive demands of tasks. We identified likely strategies for tasks and steps students would complete if they used each strategy. We classified each step as memorization, procedures without connections, procedures with connections, or doing mathematics, according to the Task Analysis Guide. Using these task-specific indicators, we rated additional recordings using our Student Cognitive Engagement Rubric.

Results

The Student Cognitive Engagement Rubric has now been applied to a small sample of higher-demand tasks with two important results. The first is that each of the high-demand tasks we have coded includes both higher-demand and lower-demand elements. A second result is that student groups engaged in both higher- and lower-level cognition when enacting these tasks. For example, Group 2 students engaged in higher-level cognition as they figured out how to approach the task and came up with potential arrangements of the cylinders. For each arrangement, students engaged in higher-level cognition as they determined the dimensions of a box to contain the cylinders. They calculated areas using those dimensions, engaging in lower-level cognition by recalling previously learned information and completing procedures without making conceptual connections. Completing, checking, and correcting these calculations sometimes took several minutes, during which there was often no higher-level cognition even though students were actively engaged in work to complete a higher-level demand task. Once the calculations were completed, the students briefly engaged again in higher-level cognition as they considered how to apply their calculations to determine the full surface area of the box. This flow of higher-level to lower-level to higher-level cognitive engagement was characteristic of several strategies identified for addressing higher-demand tasks. In this group, the students engaged in lower-level cognition for several minutes without an overall decline of task demand because the lower-level cognition served the higher-level demands of the task. Consequently, we found that the Student Cognitive Engagement Rubric addressed the limitations of our earlier efforts which failed to capture this evolution of demands in students' task enactment.

Discussion and Conclusion

Our results to date support the importance of selecting and implementing a high cognitive demand task for small-group work, but suggest that teachers and researchers should expect periods of lower-level cognitive engagement during the enactment of such tasks. Groups that enacted lower-demand tasks had few, if any, minutes that included higher-level cognitive engagement, although such tasks allow the possibility for students to do so. Groups that enacted higher-demand tasks tended to have periods of both higher-level and lower-level engagement. The findings suggest that periods of lower-level engagement may occur even when the overall cognitive demand of the task is maintained over the course of a lesson.

Our instrument development work for investigating the evolution of cognitive demand as tasks are enacted by student groups highlights the utility of fully documenting the various cognitive demands of a task in order to support analysis of task enactment.

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AN ELABORATION OF FOUR SUBPRACTICES OF THE TEACHING PRACTICE OF BUILDING ON STUDENT MATHEMATICAL THINKING

Keith R. Leatham
Brigham Young University
kleatham@mathed.byu.edu

Shari L. Stockero
Michigan Tech University
stockero@mtu.edu

Blake E. Peterson
Brigham Young University
peterson@mathed.byu.edu

Laura R. Van Zoest
Western Michigan University
laura.vanzoest@wmich.edu

Alicia M. Heninger
Brigham Young University
aliciah3@byu.edu

Amanda L. Seiwell
Western Michigan University
amanda.l.seiwell@wmich.edu

Carlee E. Hollenbeck
Western Michigan University
carlee.e.hollenbeck@wmich.edu

Joshua M. Ruk
Western Michigan University
joshua.m.ruk@wmich.edu

Keywords: Classroom Discourse, Teacher Education-Inservice/Professional Development

We have conceptualized the whole-class teaching practice of building (Van Zoest, Peterson, Leatham, & Stockero, 2016) as a productive way to take advantage of particularly high-leverage student contributions, which we call Mathematically Significant Pedagogical Opportunities to build on Student Thinking (MOSTs; Leatham, Peterson, Stockero, & Van Zoest, 2015). Building is defined as making student thinking an object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea. The practice is composed of four subpractices: Make Precise, Grapple Toss, Orchestrate, and Make Explicit. From studying teachers’ enactments of the building practice, important aspects of these subpractices and how they are related to one another have emerged. Our poster elaborates on these aspects and relationships.

For example, each subpractice lays the foundation for the subsequent practices. Clarifying the MOST by creating a public record of the student thinking during the Make Precise subpractice facilitates a clear, focused Grapple Toss, and provides a ready scaffold to draw on during the Orchestrate and Make Explicit subpractices. During the Grapple Toss, tossing the MOST to the “right” sized group significantly impacts the ability of the teacher to Orchestrate meaningful, sense-making discussion. Finally, working to keep the MOST in the air and the students’ sense-making focused on the MOST during the Orchestrate subpractice seem to naturally lead to a return to the MOST, supporting a smooth transition from the Orchestrate subpractice into the Make Explicit subpractice.

By elaborating on these subpractices and their coordination, we hope to provide further insights into the complex practice of productively using student mathematical thinking that emerges in whole-class interactions, and thus contribute to efforts (e.g., Grossman et al., 2009; Sleep, 2012) to understand and aid teachers in productively using student mathematical thinking.

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ENSEÑANZA DE LAS MATEMÁTICAS EN ESCUELAS MULTIGRADO Y TELESECUNDARIAS

Erika García Torres
Facultad de Psicología, Universidad Autónoma de Querétaro, México
erika.garcia@uaq.mx

Felipe de Jesús Santiago Flores
Universidad Autónoma de Querétaro
jesus.santiago@usebeq.edu.mx

Gabriela Zepeda Padilla
Universidad Autónoma de Querétaro
gzepeda18@alumnos.uaq.mx

La enseñanza de las matemáticas tiene problemáticas y afronta retos específicos en dos modalidades de la educación básica en México: multigrado y telesecundaria. En este estudio se analizan las estrategias para adaptar el currículo y organizar la enseñanza de las matemáticas en una primaria unitaria y en una telesecundaria, en contextos rurales. A través de dos casos, se muestran elementos de la práctica docente que determinan la actividad matemática de los alumnos. En la docente de primaria unitaria, el diseño de tareas matemáticas generales y de tareas diversificadas por edad, para abordar conceptos matemáticos. En el docente de telesecundaria, el diseño de una situación de naturaleza interdisciplinaria para construir significados en matemáticas y física.

Palabras clave: Actividades y prácticas de enseñanza, currículo, educación primaria, educación secundaria

Introducción

La primaria multigrado y la telesecundaria son dos modalidades de la educación básica en México que se crearon para atender problemas de analfabetismo y cobertura educativa, en población ubicadas en zonas rurales con altos índices de migración y pobreza. Existen dos formas de organización en las primarias multigrado –6 a 12 años–: escuelas unitarias, en donde un profesor atiende a los grupos de primero a sexto grado en un mismo espacio y escuelas con secciones multigrado. Para la educación secundaria –12 a 15 años–, la organización multigrado se ubica en la modalidad de telesecundaria (unitaria y bidocente), cuya característica es que un profesor imparte todas las asignaturas a uno o varios grados escolares. Actualmente 43.2% de las primarias en México son multigrado y 47.7 % de las secundarias son telesecundarias, de las cuales, 20.6% son multigrado (INEE, 2018).

Aunque los datos destacan la presencia importante de estas modalidades en educación básica, el currículo nacional es estandarizado, basado en una concepción homogénea y graduada de los grupos escolares, adecuado para contextos urbanos. Aunado a esto, la especialización que requieren los docentes para la enseñanza multigrado no es prioridad en los programas de actualización o de formación inicial (Galván y Espinosa, 2017).

La enseñanza de las matemáticas en las modalidades de multigrado y telesecundaria presentan problemáticas específicas. Por una parte, atienden a población en zonas rurales, casi siempre en zonas con presencia de población indígena y con altos grados de marginación, por lo que su infraestructura es inadecuada (Juárez, 2017). Las escuelas en muchas ocasiones no cuentan con recursos y materiales básicos como agua, luz, internet y personal de intendencia (Juárez, 2017; Juárez, Vargas y Vera, 2015). Los docentes de ambas modalidades desempeñan funciones adicionales a las de docencia, funciones administrativas y de gestión escolar. Si bien generan estrategias propias de atención para los grupos diversos, no cuentan con una propuesta curricular acorde a sus necesidades y es limitado el uso de materiales educativos. No se cuenta con libros para escuelas unitarias, o para grupos multigrado, tampoco se cuenta con planes de estudio que agrupen los contenidos a aprender si se tienen varios grados en un mismo grupo. Además, la formación inicial de los docentes no se enfoca en los grupos con características multigrado. De este modo, se advierte a nivel curricular y de política educativa, que se invisibiliza en la práctica a la modalidad multigrado.

Pese a este panorama, hay estudios que muestran aportes pedagógicos en estas modalidades. En primaria, Juárez (2017) señala como potencialidades de los docentes, trabajar en un aula diversa y tener autonomía para desarrollar sus actividades, así como mantener contacto estrecho con la comunidad. En telesecundaria, García (2016) evidencia indicadores contextuales que construyen un sentido de pertenencia del docente a la modalidad, generando una autonomía para la toma de decisiones. Como lo advierte Arteaga (2011), existe una invisibilidad de la escuela multigrado en el ámbito de la investigación educativa, pues los estudios sobre el tema son relativamente recientes, sin embargo, algunas investigaciones comienzan a mostrar estudios en esta dirección (Rebolledo y Torres, 2019; Cano e Ibarra, 2018).

En el análisis de las estrategias utilizadas por docentes para la enseñanza de las matemáticas, se muestran las condiciones en que realizan adecuaciones curriculares, para adaptar el trabajo a estas modalidades (Block, Ramírez y Reséndiz, 2015; Block, Carrillo y Reséndiz, 2017). Se observan en estas estrategias, que las actividades al interior del aula están diseñadas por el conocimiento que el profesor pone en uso sobre los alumnos, el currículo, el conocimiento matemático, entre otros. Es decir, las decisiones que toman los docentes se sitúan en sus contextos propios de actuación.

El objetivo de este estudio es caracterizar la enseñanza en primaria multigrado a través de un estudio de caso en una primaria unitaria, sobre las decisiones docentes relacionadas con la enseñanza de las matemáticas. En telesencundaria, se analiza a través de otro estudio de caso, los saberes matemáticos y extra matemáticos que se construyen mediante un diseño de naturaleza interdisciplinaria, como estrategia curricular para la enseñanza de las matemáticas.

Marco teórico

Este estudio se inscribe en una perspectiva sociocultural de la educación matemática. Nos posicionamos con respecto al proceso de enseñanza y aprendizaje de las matemáticas, como un aquel en que intervienen factores sociales y culturales en contextos escolares y extraescolares en diversos ambientes económicos, políticos y multiculturales (Blanco, 2011).

Toma de decisiones docentes en aula multigrado

Arteaga (2011) hace referencia a que el trabajo del docente dentro del aula está regido por saberes que regulan su actividad, los cuales se basan en un tipo particular de conocimiento que frecuentemente, no está sistematizado. Esto implica la solución de problemas que las condiciones de trabajo presentan, además de una reflexión sobre el trabajo diario. Estos saberes y experiencias nutren la práctica docente y con ello las relaciones que existen entre diversas propuestas curriculares. Se reconoce como un referente teórico a la etnográfica, para sustentar el estudio de la complejidad de las prácticas docentes, en las cuales están inmersas también variables institucionales, sociales y personales (Rockwell y Mercado, 2003). Consideramos que estos saberes determinan la toma de decisiones docentes. Algunos autores (Schoenfeld, 2008; Thames y Ball, 2013, citados en Garzón, 2017) han centrado la atención en la toma de decisiones del profesor en los “momentos de enseñanza”, aquellas situaciones de una clase en que emergen oportunidades pedagógicas que posibilitan la transformación del pensamiento matemático del alumno. Stockero y Van Zoest (2013) reconocen que los momentos de enseñanza y la toma de decisiones pueden ser evaluadas a fin de establecer su influencia en el aprendizaje de los alumnos. Las decisiones se caracterizan en los diferentes recursos que usa el docente para lograr su tarea y pueden variar de acuerdo a la organización que use dentro del aula, la forma en que los alumnos se involucren, las estrategias de gestión e intervención, o la manera en que se promueva el trabajo individual y colectivo.

La interdisciplina para la enseñanza de las matemáticas

García (2016) reportó una estrategia utilizada en la práctica de los docentes de telesecundaria, la cual consiste en el uso de situaciones transversales. Las actividades de esta estaretgia pretenden construir significados de varias asignaturas, destacando la manera en que se vinculan. Con este significado que los docentes le otorgan a la transversalidad se corre el riesgo de que los saberes –matemáticos y no matemáticos– se perciban triviales. Velásquez (2009) define a la transversalidad como una estrategia curricular mediante la cual algunos ejes o temas considerados prioritarios impactan todo el currículo; es decir, están presentes en todos los proyectos y actividades que producen los estudiantes, como el medio ambiente, el tiempo, la lengua maternal. Magendzo (1998) considera una concepción reconstruccionista de la transversalidad, la cual está ligada a temas que hacen referencia a los problemas y conflictos de gran trascendencia que se producen en la época actual. Se tratan de atender problemáticas sociales como la corrupción, el abuso infantil, la polución y se entienden como temas trasversales la educación para la paz, la educación moral, la educación vial, la educación del consumidor, la educación sexual, entre otros.

Otra aproximación sobre la vinculación de contenidos se realiza desde la interdisiciplinaridad. De acuerdo Klafki (1998) la enseñanza y el aprendizaje interdisciplinarios en la escuela no es una nueva reforma pedagógica y mostró ventajas en la comprensión de contenidos a través de muchos experimentos de enseñanza interdisciplinarios. Sin embargo, algunas limitantes para definir con precisión las características de una enseñanza que promueva la interdisciplina es la falta de claridad y consenso sobre los conceptos de disciplina, la forma de describir las intervenciones y programas interdisciplinarios, así como la falta de consistencia para identificar y medir los resultados de aprendizaje que (Williams et al., 2016).

En esta investigación se propone documentar los saberes matemáticos y no matemáticos construidos utilizando una estrategia de enseñanza en telesecundaria que vincule contenidos, por lo que nos posicionamos desde la interdisciplina, como una forma de construir significados sobre al menos dos disciplinas, a través de una situación de modelación matemática. La modelación matemática se ha estudiado ampliamente, reconociéndose como un paradigma de enseñanza el uso de los modelos para el aprendizaje de distintos conceptos (Vorhölter, Kaiser y Borromeo, 2014), con el fin de establecer una relación entre las matemáticas y el resto del mundo.

Metodología

Se documenta un estudio de caso en una primaria multigrado unitaria en el municipio de Pinal de Amoles, Querétaro, México. La comunidad tiene alto índice de marginación y los problemas sociales emergentes son altos índices de alcoholismo, drogadicción, vandalismo y migración nacional y a Estados Unidos. La escuela es rural unitaria y se compone de 20 niños: 5 en segundo grado, 6 en tercero, 3 en cuarto, 4 en quinto, 2 en sexto. En cuanto a infraestructura, se cuenta solo con dos aulas, una que se utiliza como biblioteca y sala de medios y la otra, como aula de clase. Se realiza trabajo etnográfico al interior del aula, con la finalidad de llevar registro en audio y video de varias sesiones de clase. Adicionalmente, se realizan entrevistas a la maestra para conocer sus planeaciones, adecuaciones curriculares y materiales utilizados.

La telesecundaria se ubica en un contexto rural en la comunidad de La Laborcilla, en Querétaro, México. En la escuela sólo hay un grupo para cada grado escolar y el grupo de estudio está conformado por 18 estudiantes de tercer grado, sus edades oscilan entre los 14 y 15 años. En esta investigación nos interesa analizar la implementación de una estrategia que nos permita contextualizar contenidos matemáticos y propiciar un tratamiento interdisciplinario. El diseño de la situación interdisciplinaria se fundamenta en el problema de la Separación Ciega de Fuentes (Blind Source Separation-BSS) (Vázquez, Romo, Romo-Vázquez, y Trigueros, 2016). Se realizará un diseño situacional y secuencial, en el que se pongan en juego hipótesis de construcción de conocimiento de cómo tendría que darse el proceso entre las disciplinas: matemáticas y física. Se implementará el diseño y reportarán los resultados del aprendizaje generado sobre saberes matemáticos y saberes no matemáticos.

Discusión y Conclusiones

La docente de la primaria unitaria está a cargo de la escuela, atiende a los alumnos, realiza actividades de dirección, organiza la limpieza de las instalaciones y están en contacto con los padres de familia. Proviene de una institución formadora de maestros y lleva dos años enseñando en multigrado. Atiende de manera simultánea a sus alumnos, organiza y planifica las tareas matemáticas de tal manera que pueda entretejer los contenidos de las asignaturas y grados, evitando planear y dividir los temas por grados. Sus decisiones están centradas en la planeación previa a su intervención en el aula. Consideramos importante visibilizar y sistematizar los saberes que subyacen a las decisiones de la maestra, así como impulsar estudios desde la educación matemática que contribuya a caracterizar los procesos de enseñanza y aprendizaje que ocurren en estas modalidades.

La estrategia curricular de situaciones integradoras transversales vistas desde la perspectiva de la interdisciplina, puede reconceptualizar la práctica de los docentes de telesecundaria, pues para hacer un diseño con estas características, se requiere romper con un paradigma de una vinculación muchas veces limitada de contenidos, hacia un estudio de situaciones de modelación matemática, para dotar de significados a los objetos matemáticos a través de nuevos contextos, en los cuales los discursos de otras disciplinas se mezclan para producir conocimiento.

Se ha destacado la problemática del aprendizaje de las matemáticas en escenarios socioculturales diversos en cuanto a formas de organización escolar. Para muchos niños y jóvenes, la primaria multigrado y la telesecundaria son las únicas opciones de acceso a la educación en esos niveles. Si el currículo nacional es estandarizado, basado en una concepción homogénea y graduada de los grupos escolares y adecuado para ciertos contextos urbanos, no se está promoviendo un acceso equitativo a la educación. El estudio presentando pretende ser un aporte sobre las diferentes estrategias que emplean los docentes de estas modalidades para potenciarlas y seguir proponiendo formas de abordar la construcción de significados de los objetos matemáticos en el aula.

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TEACHING MATHEMATICS IN MULTIGRADE SCHOOLS AND TELESECUNDARY SYSTEM

Mathematics teaching faces specific problems and challenges in two modalities of basic Mexican education: multigrade schools and telesecundary systems. This study analyses the strategies used to adapt and organize the teaching of mathematics in a one-group elementary school and in a telesecundary system in rural contexts. Elements of teaching-practice and the effects on students' mathematics activities are presented in two scenarios. On one hand, the one-group teacher designing general mathematic activities and diversified-age task to teach mathematical concepts. On the other, a telesecundary system teacher designing interdisciplinary situations to construct meanings for mathematics and physics.

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TEACHING FOR UNDERSTANDING OF STRATEGY OR STRUCTURE: A COMPARISON OF TWO REFORM-ORIENTED LESSONS

Jennifer Seat
Middle Tennessee St Univ
jms2dr@mtmail.mtsu.edu

Monica Frideczky
Middle Tennessee St Univ
mrf2r@mtmail.mtsu.edu

Rongjin Huang
Middle Tennessee St Univ
Rongjin.Huang@mtsu.edu

Dovie Kimmins
Middle Tennessee St Univ
Dovie.Kimmins@mtsu.edu

Jeremy Winters
Middle Tennessee St Univ
Jeremy.Winters@mtsu.edu

Keywords: Standards, Instructional Activities and Practices,

Recent shifts in curriculum standards have led to a greater emphasis on students' conceptual understanding of mathematical topics (National Council of Teachers of Mathematics, 2000; National Governors Association, 2010). The discussion in mathematics education about the benefits and limitations of both teaching for conceptual understanding and teaching for procedural understanding is not a new one (Brownell, 1947; National Research Council & Mathematics Learning Study Committee, 2001; Skemp, 1976). Using lesson study as a tool to study the implementation of reform-oriented instruction (R Huang, Kimmins, Winters, Douglas, & Tessema, 2017; Rongjin Huang, Barlow, & Prince, 2016; Rongjin Huang, Zhang, Chang, & Kimmins, 2018), the researchers have identified a unique perspective to enrich this discussion by Comparing and contrasting two research lessons from two lesson study groups.

Using data collected from each separate lesson study group (one in China and one in the United States), key differences in pedagogical choices between the two research lessons are salient -. The data for this comparison consisted of lesson plans and verbatim transcripts of instruction. As the Chinese lesson was taught in Chinese, it was transcribed verbatim in Chinese, and then translated to English for analysis.

The Chinese lesson introduced the topic additive comparison, using a game to develop a best strategy for comparing sets. The students then worked several variations of practice problems to emphasize the mathematical structure of additive comparison. The lesson was intended to guide students to understand the underlying invariant structure within various contexts and representations. Instead, it led to an emphasis on rote student procedures relating to the comparison of sets.

The United States lesson introduced the topic by developing an illustrative definition of additive comparison. The students engaged with different representations of comparison using familiar objects before working independent practice problems. This afforded students the opportunity to find strategies to solve problems and explain their reasoning to the class. While the students produced many strategies for attaining solutions, many were not representative of the additive comparison model.

These lessons offer a distinct perspective on instructional balance (Sfard, 2003). The Chinese lesson offers an emphasis on the use of structure; however, students are given few opportunities to make sense of the structure. The United States lesson offers students a chance to reason about a problem solving method; however, it fails to provide connection to the mathematical structure. Baroody, Fiel, and Johnson (2007) proposed that conceptual and procedural knowledge develop mutually. However, both lessons presented material from a procedural aspect, leaving conceptual development minimally attended within the lessons. Structure and strategy must both be present to attain the goal of conceptual understanding.

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TEACHERS' STORIED PERCEPTIONS OF STUDENTS AS LEARNERS OF MATH

Mari Altshuler
Northwestern University, School of Education and Social Policy
marialtshuler@u.northwestern.edu

Keywords: Affect, Emotion, Beliefs, and Attitudes; Teacher Knowledge

Math is a highly polarizing subject, and there are great inequities in the field within and outside of schools (Boaler & Greeno, 2000). In order to expand access to positive mathematical trajectories for young students, we need to understand how students view themselves and how others view them both as people at large and in relationship to mathematics specifically (Aguirre, Mayfield-Ingram, & Martin, 2013). Students' math identities emerge through experiences and through stories that students, peers, parents, and teachers tell about who students are as math learners, and those identities impact students' interest in, commitment to, and success in mathematics (Langer-Osuna & Esmonde, 2016). In particular, teachers' perceptions of students and their responses to students in math class play critical roles in how students begin to identify themselves in relation to math (Erickson, 2011; Russ, 2018; Horn, 2007; Bertrand & Marsh, 2015). This study explores teachers' perceptions of their students by gathering teachers' stories. The study asks: How do K-3 teachers describe their students as learners of math?

Data for this study consisted of 11 semi-structured, one-on-one interviews with K-3 classroom teachers in a diverse, public, midwestern school district. During the 45 min–1hr interviews, teachers moved through their rosters in an order of their choosing, describing each student in relation to math. Teachers described between 8 and 15 students each, and approximately half of the teachers referred to student work samples during the interviews. I used in vivo coding (Miles, Huberman, & Saldaña, 2014) of the interview transcripts to identity themes across stories about students and highlight characteristics that emerged as salient to teachers. Clustering descriptors and comparing students who were described similarly revealed emergent patterns in the ways teachers understand their students as math learners.

Analysis revealed that when K-3 teachers described their students as math learners, they included distinct judgements about students' performance in math class and their aptitude for math in addition to descriptions of students' personalities, effort and engagement, emotional and behavioral development, and family context. Teachers described these characteristics in repeated combinations, suggesting that there are archetypal ways in which teachers categorize math students. These archetypes are predominantly not described by teachers in wholly positive or negative lights, but rather paint nuanced pictures of students' engagement with math class and the discipline of mathematics.

What that nuance means for the complex relationship between how teachers understand their young students as math learners and their instructional decisions and interactions with students remains an open question. Continuing to unpack these relationships is critical for understanding how young children begin to develop mathematical identities and for supporting students on their mathematical journeys through and beyond their time in school.

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USING PROFESSIONAL DEVELOPMENT TO CREATE A COLLABORATIVE KNOWLEDGE BASE FOR LAUNCHING

Rob Wieman
Rowan University
gomathman@yahoo.com

Kim Johnson
West Chester University
kjohnson2@wcupa.edu

Keywords: Instructional Activities and Practices, Teacher Knowledge, Research Methods

Background: The Problem

The work described by this poster lies at the intersection of two problems in mathematics education, the need to develop a knowledge base for teaching that bridges the gap between research and practice and the need to develop a shared set of methods for effective launching.

There is widespread agreement that how teachers launch demanding tasks determines the learning opportunities that students may have (Jackson, Garrison, Wilson, Gibbons & Shahan, 2013). Experts agree that students need support in making sense of problems, accessing prior knowledge, and identifying important mathematical relationships, but too much support can lower the task's demand. The field lacks research that describes, in detail, the purpose of a launch, typical challenges in launching, and effective moves for launches (Wieman, et al., 2018).

There is a profound gap between educational research and practice, contributing to the lack of a widely shared knowledge base for teaching. In response, researchers have called for collaboration with teachers to identify important research questions and share, disseminate, and preserve shared professional knowledge for teaching (Hiebert, Gallimore, & Stigler, 2002).

Types of Launches: Types of Problems

In 2018 the PME-NA working group on launching engaged participants in sorting tasks according to how they might launch them (Wieman et al., 2018), leading participants to discuss which types of problems might go with certain types of launches. In this poster, we describe professional development involving participants using similar activities to answer two questions:

- What types of launches do teachers say they use, and with what types of tasks?
- How does this activity help us build a collaborative knowledge base for teaching?

Connection to Conference Theme

Educational research can disempower teachers, and poorly launched contextual can create barriers for students. This work is an effort to move beyond the present horizon, accessing teacher knowledge to inform research and open up learning opportunities for all students.

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THE BALANCING ACT: MEETING THE SOCIAL-EMOTIONAL NEEDS OF STUDENTS AND ACCOMPLISHING LEARNING GOALS

Sara Gartland
University of Delaware
sgart@udel.edu

Keywords: Equity and Diversity, Culturally Relevant Teaching, Instructional Activities and Practices

As Rochelle Gutierrez notes "giving voice to the contextual factors that enable or constrain learning in a given situation is equally important" as identifying best practices and effective interventions (2012, p. 18). Kozol's (2005) indictment of the conditions under which urban elementary schools must operate highlights an important and understudied contextual factor in mathematics education – that "the issues are big; children are small" (p. 285).

Specifically, the stress associated with living in poverty may manifest in emotional outbursts that can derail classroom activities (Brantlinger, 2018). However, it has been shown that a sense of belonging can influence students' connection to school, academic motivation, and wellbeing (Maestas, Vequera, & Zehr, 2007; Tyler, Uqdah, Dillihunt, Beatty-Hazelbaker, Conner, Gadson, Henchy, Hughes, Mulder, Owens, Roan-Belle, Smith & Stevens, 2008). Furthermore, a sense of belonging fostered by teachers described as "warm demanders" (Delpit, 2012) helps students productively engage with mathematics (Allexsaht-Snider & Hart, 2001). Thus, I hypothesize that addressing students' social-emotional needs by attending to their feelings of belongingness may lead to better learning outcomes for students.

To begin exploring this hypothesis, I observed a third-grade mathematics teacher at a high-needs public elementary school located in a mid-size US city. The school serves approximately 450 students enrolled in Grades 3-5 who identify as: 76% African American, 18% Hispanic/Latinx, 2% Multi-Racial, 2% White, 1% Asian, and 1% American Indian. To best serve the needs of a low-income population, the school employs two sets of curricula: The Common Core State Standards and a social-emotional learning curriculum. Additionally, teachers have participated in training on trauma-informed teaching.

The research question for this study is: in what ways does a third-grade teacher address the social-emotional needs of students and work to meet mathematics learning objectives? Observation data were analyzed using a grounded theory approach (Strauss & Corbin, 1994) and coded using an open coding process (Hatch, 2002). Conversations with the teacher throughout the analysis stage occurred to refine codes and verify interpretations.

Analysis revealed three salient findings. First, the coupling of the teacher's high expectations for behavior and learning with her use of repetition as a support strategy addressed students' social-emotional and mathematical needs. Second, incorporation of student voice to drive the lesson content and structure emerged as a way in which the teacher fostered belongingness. Third, the teacher consistently used student names to personalize the learning process and give students ownership of the ideas they generated.

These findings are unique in the field, providing both actionable best practices and a foundation for continued exploration of the topic. They answer a question not frequently asked specifically within a mathematics context. Also, they serve as an initial step in unpacking what it means to address student's social-emotional needs *and* mathematical needs. Moving forward, this study will inform the framework and methodology for an expanded and more in-depth study.

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THE POWER OF WORDS SPOKEN: HOW AUTHORITY INFLUENCES DISCOURSE

Elizabeth Lambert
Texas State University
EHL9@txstate.edu

Michael Duane Hicks
Texas State University
mh1653@txstate.edu

Christina Koehne
Texas State University
crz7@txstate.edu

Jessica Bishop
Texas State University
jbishop01@txstate.edu

Keywords: Classroom Discourse, Instructional Activities and Practices, Communication

Engaging in productive mathematical discourse supports students to improve their mathematics learning (NCTM, 2014). The authority structure within a class can determine the efficacy of this discourse. Authority has been examined from a variety of lenses. Wilson & Lloyd (2000) distinguish between *pedagogical* and *mathematical* authority, which focus on orchestrating classroom activity versus making mathematical contributions. Herbel-Eisenmann & Wagner (2010) explored authority in mathematics classrooms by relating stance bundles to interpersonal positioning. This proposal investigates how authority distribution might be operationalized within a classroom.

Inspired by Amit & Fried (2005), we define *authority* as a dynamic and negotiated relationship between people where one party defers to another within a mathematical situation. Drawing from Goffman (1981), we consider three key activities to operationalize authority for our framework: (1) Authorship, (2) Animation, and (3) Assessment of the mathematical ideas. We used our framework (referred to as the AAA Framework hereafter) to analyze authority relations during whole-class (WC) discussions in 1 lesson from 4 middle-grade classrooms. Every lesson was broken into *segments*, each a series of turns of talk with a common focus and a consistent form of participation, that were then coded using our framework. From this, we were able to determine (1) whose idea the segment was grounded upon, (2) which individuals were contributing to the conversation and by which mode, speaking and/or scribing, and (3) who was evaluating mathematical aspects of the conversation. (See Table 1 for a coded segment.)

Table 1: Example of AAA Framework

5th-graders discuss strategies and symbolic notation for sharing 3 sandwiches amongst 4 kids
Teacher: Okay. Tell us about person two’s strategy. *Lexi*: I think that person was giving a ½ to 2 people [at board gesturing], like 2 of the 4 kids, and then she had 2 [sandwiches] left. So she did it again. And then she realized that right here [gestures to student work on board] she only had, uh, 1 [sandwich] left. So she couldn't do a ½, because then it wouldn't be equal to everyone. So she did a ¼ 4 times to get that last whole. And then her answer was a h-- not a ½. Was ¾. [Lexi writes ½ + ¼ and sits down.]
Author – Student, **Animate Speak** - Student, **Animate Scribe** – Student, **Assessment** – None

We found that although all students participated in Speaking in whole-class discussion, there was large variation in students’ opportunities to Author mathematical ideas (i.e. students as sole or shared authors ranged from 15 to 100% of WC discussions). Additionally, half of the teachers maintained control of the board; whereas the other half allowed students sole authority to Scribe at the board in over half of the WC discussion. Further analysis will be presented on the poster.

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UNDERMINING A DISCOURSE

Christopher Leatherwood
Northwestern University
chrisleatherwood@gmail.com

Keywords: Classroom Discourse, Instructional Activities and Practices, Communication

Discourse is central to understanding many classroom interactions, and in regard to math classrooms this importance has been seen in shifts within the curriculum standards (Common Core, 2010; NCTM, 1991; 2000), researcher community (Knuth & Peressini, 2001; and for extensive review see Walshaw & Anthony, 2008), as well as preservice instruction and professional development focused on implementing and/or improving discourse in the classroom (McDonald et al., 2013; Kazemi et al., 2016; Borko et al., 2008;). However, discourse is not monolithic; different contexts or forms invite different opportunities for participation and learning. The context of this research was error episodes in middle school math classrooms. Error episodes were defined as bounded events (Bloome et al., 2004) that included the introduction of an error, the interaction that error produced, and the completion of the interaction in which the error was the focus. I focused on these moments because even though there has been a shift away from an identify and eliminate model (in regard to errors) in both researcher and practitioner communities (Kramarski, B., & Zoldan, S. 2008; Lannin et al., 2007; Common Core, 2010), this model continues to influence day to day classroom interactions (Bray, 2013; Cohen, 1990; Santagata, 2004). This study seeks to understand how discourse is shaped within these moments and how participants come to understand and interact within this discourse.

Data for this study consisted of video recordings of one 8th grade classroom at School Insight International Charter School. School Insight is a K-8 charter school located in a Midwestern city. 98% of the student body identified as African-American, and the school is self-described as African-centered. Using conversation analysis notation (Sacks, Schegloff and Jefferson, 1978), I present one portion of an extended error episode. This interaction was chosen because it was one of the longest error interactions in the data corpus.

Analysis revealed that there was a recognizable discourse around errors with particular rules for entry and rules governing the interaction itself. These rules involved not confronting the error itself but giving the correct procedure to solve the problem at hand. The student in the interaction first adhered to these rules by presenting her misunderstanding as a computational issue. She then attempted to reframe this discourse before outright rejecting it by stating that her misunderstanding was not housed within the problem itself, but that her misunderstanding was in how she conceptualized parentheses in general.

There are implications to teaching and learning with this error discourse. For students who understood that their misconceptions couldn't be addressed with the correct procedure, they had to navigate this discourse to force it to resolve their mistakes by reframing or rejecting the discourse. Students who simply followed this discourse without challenge rarely had their underlying misconceptions addressed at all. Moving forward it is necessary to see if this type of discourse is specific to mathematics, or if it also exists in error episodes in other fields. Also, because students internalized this error episode discourse, it is imperative that we analyze how this interpretation impacts individual student identities and participation around errors.

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BENEFITS AND CHALLENGES TO MATHEAMTICS TEACHER EDUCATORS' INSTRUCTION USING ELECTRONIC JOURNALS WITH PRESERVICE TEACHERS

Heather Gallivan
University of Northern Iowa
heather.gallivan@uni.edu

Chepina Rumsey
University of Northern Iowa
chepina.rumsey@uni.edu

Keywords: Technology; Instructional Activities and Practices

The benefits of utilizing technology in the mathematics classroom are numerous and include improved students' learning of mathematical content (Lim, Son, Gregson, & Kim, 2018) and supporting teachers' learning (e.g., Seago, Koellner, & Jacobs, 2018). However, few studies have looked at how technology can be used by mathematics teacher educators (MTEs) to support their planning, instruction, and formative assessment practices. Studying MTEs' practices over time can provide the field with insights into MTE development (Tzur, 2001) as well as practical knowledge for the field (Arbaugh & Taylor, 2008). New electronic journal technology has the potential to support MTEs with their planning, instruction, and formative assessment practices while teaching mathematics content courses with preservice teachers (PSTs). The electronic journals function like a paper journal, but utilize cloud technology such that PSTs' written work can be uploaded to a Google Drive folder as a PDF. As with any new technologies, it can be beneficial to document both the benefits and the challenges faced by MTEs when using them.

Purpose and Methods

The goal of this narrative study is to determine the impact of using electronic journals with elementary PSTs on two MTEs' abilities to manage the challenges of using these journals and to adjust their instruction to meet the needs of their students. The two MTEs were the instructors of a mathematical problem solving course for elementary PSTs across two consecutive semesters where they utilized the electronic journals. Narrative inquiry methodology was employed for this study to document the experiences of the MTEs across two semesters as they used the journals for the first time and made adjustments for the second semester in their planning, instruction, and formative assessment practices in response to challenges they faced. Both MTEs kept a journal to document their experiences using the electronic journals. Additionally, the electronic journals the PSTs' kept throughout the semester were downloaded and they were asked to respond to an open-ended survey regarding their use of the journals. Data from these sources will be used to provide support to the narratives of the MTEs.

Results and Discussion

There were several benefits to using the electronic journals for both MTEs. Because the MTEs had daily access to PST writing, (1) they were able to use the PSTs' writing to plan for future lessons and (2) the MTEs had more flexibility when providing formative feedback because there was no need to collect and carry the physical journals around. However, both instructors experienced similar challenges. For instance, both MTEs struggled with how to hold PSTs accountable for writing during class as well as for reading and responding to feedback. Thus, for the second semester, the MTE's made adjustments to their formative assessment practices to address these issues. With the advent of electronic journal technology, there are going to be challenges to overcome in implementation so that MTEs can use them effectively.

The narratives from both MTEs can provide practical knowledge for the field and the process MTEs go through to learn to implement new technologies.

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MATHEMATICAL MODELING AND CLASSROOM DISCOURSE: A CASE FOR MODELING-SPECIFIC DISCUSSION STRATEGIES

Sarah Brand
Marquette University
sarah.brand@marquette.edu

Hyunyi Jung
Marquette University
hyunyi.jung@marquette.edu

Ashley Dorlack
Marquette University
ashley.dorlack@marquette.edu

Samuel Gailliot
Marquette University
samuel.gailliot@marquette.edu

Keywords: Mathematical Modeling, Classroom Discourse, Instructional Activities and Practices

Mathematical modeling has become increasingly pertinent to mathematics education due to its benefits on student learning and engagement (e.g., Blum & Niss, 1991; Lesh, Hoover, Hole, Kelly, & Post, 2000; Zbiek & Conner, 2006). One integral aspect of mathematical modeling is mathematical discourse (Lesh et al., 2000), which activates and builds students' background knowledge (Henningsen & Stein, 1997). Productive mathematical discourse promotes the sharing of ideas, expands upon student thinking, clarifies understandings, and fosters credible argumentation (Smith, Steele, & Raith, 2017). While research has been sufficiently conducted on classroom discourse (Boaler & Brodie, 2004; Chapin & O'Connor, 2007; Chapin, O'Connor, & Anderson, 2003; Reinhart, 2000), little is known about discourse and teacher discussion strategies unique to mathematical modeling. This study aims to answer the question, "What discussion strategies specific to mathematical modeling promote students' diverse thinking and facilitate productive mathematical discourse associated with mathematical modeling problems?"

The study examined the implementations of three mathematical modeling activities with 21 middle-grade students. We analyzed the video transcripts of classroom interactions and students' written work using analytical-inductive methods, which included a-priori codes from literature on discourse (e.g., Chapin & O'Connor, 2007; Chapin et al., 2003; National Council of Teachers of Mathematics, 2014) and new codes for mathematical discourse developed using the constant comparative method (Glaser, 1965).

Preliminary Findings

Analyses and coding of the lesson transcripts demonstrated a toolkit of discussion strategies which harnessed the unique aspects of mathematical modeling. Teachers facilitated discourse by relevantly recontextualizing the problem, probing students' responses for assumptions, encouraging multiple perspectives, stimulating students' decision-making, emphasizing students' reasoning, promoting common language, and prompting revision of students' thinking. Students presented diverse thought processes within the mathematical discourse, as well. Specifically, analysis of students' written solutions and the transcript showed diversity in students' mathematical approaches, which resulted in manifold, valid solutions. Further, the use of these strategies generated productive mathematical discourse; that is, students shared their ideas, expanded on each other's thinking, formed convincing arguments, and clarified understandings. In our session, we will present how productive discourse can be used in conjunction with authentic, open-ended tasks and report the positive student outcomes associated with discussion strategies specific to mathematical modeling.

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HOW DO TEACHERS' ARTICULATED VIEWS RELATE TO THEIR IMPLEMENTATION OF EQUITABLE TEACHING PRACTICES?

Jonee Wilson
North Carolina State University
jwilson9@ncsu.edu

Keywords: Inclusive Education; Culturally Relevant Teaching; Instructional Activities and Practices; Classroom Discourse

Mathematical *Standards* (NCTM, 2014; CCSSM, 2010) lay out rigorous goals for "all" students' learning (Franke et al., 2007). However, *Standards*-based mathematics instruction remains uncommon and opportunities for students to develop the understandings outlined in the *Standards* are not equally distributed. In fact, it has been well documented that the quality of mathematics instruction and learning opportunities in US schools, particularly in classrooms serving majority students of color and students for whom English is not their first language, is typically of inadequate quality and at best supports the development of procedural competence (e.g. Boston & Wilhelm, 2017; Hill & Lubienski, 2007; Nasir & Cobb, 2002).

One proposed reason for this phenomenon is that most teachers do not view _all_ students as capable of participating in *Standards*-based mathematics activity and many teachers attribute the sources of students' mathematics difficulty to inherent traits of the students. In fact, research offers evidence of a relationship between teachers' views and the *quality of the activities in which students have an opportunity* to participate (Jackson et al., 2017; Wilhelm et al., 2017). However, research connecting teachers' views of students to the extent to which the *practices implemented by the teachers are equitable* is absent from the literature (i.e. research already connects teachers' views to *what* the students have a chance to experience, but not *who* is participating or *how* students are being supported).

My analysis addresses this gap by examining data (surveys, interviews, and video-recorded lessons) collected from 16 in-service teachers at multiple points in their first-year of teaching. Munter (2014) provides a framework for classifying teachers' articulated visions of high-quality mathematics instruction and Jackson et al. (2017) provide a framework for classifying teachers' views of students' mathematics capability. I used these frameworks to guide my analysis and coding of the interview and survey data. Boston (2012) provides a framework for classifying the quality of instruction and Shah and Reinholz (Reinholz & Shah, 2018; Shah et al., 2016) provide an approach for conceptualizing equity in terms of participation patterns, which I used to analyze and code the video-recorded lessons. I also used a set of rubrics (the Equity and Access Rubrics for Mathematics Instruction, EAR-MI) developed to capture practices aimed to support marginalized students in gaining access and more equitably participating in rigorous mathematics activity (Wilson et al., Accepted) to analyze and code the extent to which the practices implemented by the teachers were equitable. I report on differences and patterns seen across these different codes within the group of 16 teachers and share qualitative findings from "deep dive" comparative case studies based on scores from the rubrics.

Preliminary findings indicate that despite articulating strong commitments to equity and productive views of students' capability, the practices implemented by some teachers still reify inequalities found in many mathematics classrooms. Findings also suggest that in order to support teachers in developing more equitable practices, it is not only important to support shifts in their views, it is also important to support a reconceptualization of their roles as math teachers.

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THE INSTRUCTIONAL PRACTICES OF SECOND-YEAR ELEMENTARY TEACHERS: DOES THE TEACHER PREPARATION PROGRAM MATTER?

Temple A. Walkowiak
NC State University
tawalkow@ncsu.edu

Heather M. West
NC State University
hmwest@ncsu.edu

Elizabeth L. Adams
Southern Methodist University
eladams@smu.edu

Keywords: Elementary education, Instructional practices, Teacher education-preservice

Teacher education has been subject to much scrutiny in recent years about whether preparation programs matter (e.g., Peck et al., 2014) with calls to conduct investigations after graduates enter their career (Cochran-Smith & Zeichner, 2005). Although this type of work is resource-intensive, there are examples of researchers investigating impact of teacher preparation (e.g., Henry et al., 2014). However, there are very few examples of researchers systematically following graduates (e.g., Morris & Hiebert, 2017), particularly in the area of elementary mathematics. The current study is unique in its aim to understand if a mathematics-intensive elementary teacher preparation program (TPP) matters for the instructional practices adopted by teachers early in their careers. Using a quasi-experimental design, the overarching research question was: Are there differences between the *frequency* and *quality* of learning opportunities provided by teachers from the focal TPP and comparison TPPs?

This study took place in the context of a STEM-focused elementary teacher preparation program where the candidates take more mathematics-related courses than is typical. Specifically, they take 12 credits (4 courses) of mathematics content (two of which were specifically designed for them), 2 mathematics methods courses that anchor pedagogy around K-5 number/operations mathematical content, and 1 instructional design seminar focused on developing, implementing, and analyzing a series of mathematical tasks.

The participants in this study were recruited purposefully using propensity score matching. We matched graduates of the focal TPP (n=49) to graduates of other elementary TPPs (n=96) in the same state using pre-program covariates (e.g., high school GPA, SAT score). During their second year of teaching, the teachers completed a teaching log (Walkowiak et al., 2018) for 45 days at three 15-day time points. The log includes questions about the *frequency* of opportunities for students to engage in particular behaviors (e.g., using pictorial representations). Also, three video-recorded mathematics lessons were coded using a quantitative observational protocol that attends to the *quality* of the students' learning opportunities (Walkowiak et al., 2014).

We fit a series of doubly-robust hierarchical linear models with inverse propensity weights to test the effect of the focal TPP on second-year teachers' use of instructional practices. Results indicated that graduates of the TPP reported using math talk more frequently, as measured by the log. Also, graduates of the focal TPP: facilitated higher quality discourse in their classrooms; utilized and translated between representations in more meaningful ways; and facilitated lessons that were more coherent in structure. These results suggest that mathematics-intensive preparation matters for elementary teachers, but more research should investigate why there were no effects in relation to the mathematical tasks the teachers implemented.

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Chapter 14:
Theory, Research Methods, and Miscellaneous Topics

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

MEANINGFUL MATHEMATICS: NETWORKING THEORIES ON MULTIPLE REPRESENTATIONS AND QUANTITATIVE REASONING

Nicole L. Fonger
Syracuse University
nfonger@syr.edu

Nigar Altindis
Syracuse University
naltindi@syr.edu

This paper articulates a stance on the study of students' meaningful mathematics understanding with multiple representations. We integrate Thompson's theory of quantitative reasoning and Dreyfus' theory of multiple representations in our approach to frame and conduct empirical investigations of the study of meaningful understanding of function. We provide empirical data to support our approach to examining representational fluency and functional thinking from this networked stance. Our research articulates how the coordination of theories can be productive in informing the design, conduct, and analysis of contexts aimed to understand students' meaningful math learning with a focus on functional thinking.

Keywords: Functional thinking, Representational Fluency, meaning, Quantitative Reasoning, Multiple representations, Networking theories

Motivation and Aim

The complexity of studying and bolstering students' conceptual understanding of mathematical ideas remains a key challenge in mathematics education (Drijvers, 2019). In studying such a complex phenomena, Sfard (1998) articulates a tension of relying on one metaphor or theory of learning to the detriment of another. Consistent with this view on a need for multiple theories, in this work we aim to coordinate two theoretical orientations on understanding the mathematical idea of function—multiple representations and quantitative reasoning. Networking theories is one approach to theory development to advance knowledge in the field of mathematics education that may help to address such key challenges (BiknerAshbahs & Prediger, 2010; Cobb, 2007; Johnson & McClintock, 2018; Johnson, McClintock, Gardner, 2019). As such, this theoretical report is driven by an aim to coordinate theories for the purpose of developing new theory (cf. Cobb, 2007), not to supplant or replace existing theories.

We aim to network theories to better articulate a local theory for the purpose of addressing an enduring challenge of characterizing meaningful learning of mathematics with multiple representations. This theoretical report seeks to extend prior studies by offering both (a) a step toward the articulation of theory development focused on meaningful learning vis-à-vis the networking of theories, and (b) additional empirical investigation of the relationship between students' conceptions of functions and representational fluency. This aim emerged from both a review of empirical studies in the domain of scholarship on functions, a functions approach to mathematics, and functional thinking (e.g., Cai et al., 2010; Stephens et al., 2017a; Stephens et al., 2017b), and a discernment for the utility of networking of theories to expand possibilities for framing problems and understandings (e.g., Cobb, 2007).

In this report we also enact some of the recommendations and lessons learned from participating in a recent international conference working group focused on theoretical perspectives, networking theories, and methodological implications (Bikner-Ashbahs, Bakker, Johnson, Chan, 2019). We focus on the following recommendations of the working group as a structure for this report: (a) elaborate how the networking of theories helps to address a research problem, (b) take care in elaborating not only theoretical constructs being coordinated but background theories and underlying assumptions, and (c) communicate how the networking of theories and methodologies are in symbiotic exchange in all phases of research. We draw on empirical findings to bring these issues to life.

Theoretical Orientation on Meaningful Learning

Meaning is a creative act. Our current framing of meaningful understanding involves a critical focus on students conceptual or relational understanding (Skemp, 1976). We follow Lobato's (2013) articulation of concept to include meanings to include: (a) "*meanings*, which refer to one's interpretation of situations, conversations, symbols, and operations (Thompson & Saldanha, 2003; Voigt, 1994); and (b) *connections*, which include the specific links or ways of integrating representations, ideas, objects, and/or situations (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986)" (emphasis in original, p. 2-3). From this stance, a person (e.g., student) is viewed as active in their meaning-making processes through the activities of creating and interpreting invariant features among representations.

Multiple Representations

We conceptualize representations as different "faces" of the same mathematical object; looking at an object through only one representation cannot reveal all features of the object (Kutzler, 2010). Representational fluency (RF) is "the ability to create, interpret, translate between, and connect multiple representations" in doing and communicating about mathematics (Fonger, 2019, p. 1). The notion that "using multiple representations" supports students' conceptual understanding of mathematics is prevalent in mathematics education research, practice, and policy in the United States (e.g., NGO & CCSSO, 2010; NCTM, 2000, 2014). Some argue that creating and interpreting representations are "important cognitive processes that lead students to develop robust mathematical understandings" (Huntley, Marcus, Kahan, & Miller, 2007, p. 117). This work draws predominantly on a theory of multiple representations (e.g., Dreyfus, 1991; Lesh, Post, Behr, 1987).

From this theoretical stance, scholarship contributes to an understanding of how the practices of representing and the processes of learning are complexly intertwined and emerge over time in symbiotic relation (Fonger, 2019; Selling, 2016). However, a lens on characterizing sophistication in representational fluency alone is often insufficient for garnering evidence of the nature of students' meanings of mathematical objects. Thus, to attend to the goal of studying meaningful learning with multiple representations, a coordination of lenses is needed (as we have argued in earlier work Fonger, 2019; Fonger, Ellis, Dogan, 2016). In our work, we aim to grow our theoretical orientation on meaningful learning to include not only students' representational activity in creating and connecting representations, but also hypotheses of the *meanings* students hold of the mathematical objects being represented.

Quantitative Reasoning and Functional Thinking

We narrow our focus in this report on the domain of scholarship on supporting and characterizing students' understanding of students' functional thinking (FT)—the ability to generalize and represent functional relationships (Stephens et al., 2017a; Kaput, Blanton, & Moreno, 2008). There is a growing body of work in this domain that draws on Thompson's theory of quantitative reasoning (Thompson, 1994, 2011). Much of this work is grounded in constructivism a background theory of learning (Glasersfeld, 1995; Piaget, 2001). From this perspective, an individual constructs knowledge through processes of assimilation and accommodation, creating mental models that are viable according to their interactions. A researcher then aims to build models of the mathematics of students (Steffe & Olive, 2010).

From a quantitative reasoning orientation, attention to students' conceptions of attributes as measurable and that vary are key (cf. Johnson, McClintock, & Gardiner, 2019). We see an overlap in literature that focuses on characterizations of students' functional thinking, and students' quantitative reasoning, that tends to fall along three types of reasoning: covariation, correspondence, and recursive.

As Johnson et al. (2019) have reported, investigations of students' covariational reasoning are intertwined with students' creation and interpretations of representations such as Cartesian graphs. Along a related line of investigation, Moore et al. (2013) found that pre-service secondary math teachers' meanings of graphs and functions were in some cases constrained by their attachment to canonical forms (e.g., horizontal axis is x, vertical axis is y), limiting their ability to draw meaning about a graphical representation of x as a function of y. In another study, Fonger, Ellis, & Dogan (2016) found students engaged in covariational and correspondence reasoning to support their sense-making of symbolic function rules as generalizations of relationships among quantities. Consistent with this study, students' reasoning about relationships among quantities can support students' meaningful use of multiple representations (Moore, 2014). Moreover, the meanings students hold of representations and representational conventions may constrain their meanings of mathematical ideas such as angles or linear functions (Moore, 2012; Moore et al., 2013). In the domain of research on functional thinking, we see a need to more explicitly integrate attention to how students are creating and connecting representations to shed light on the creative aspects of doing mathematics with representations as important tools that may support or constrain students' activity and meanings of and for mathematics (cf. Brownell, 1947).

Methodology

In this section we focus on how the networking of theories and methodologies are in symbiotic exchange in all phases of research.

Data Sources and Mode of Inquiry

We employed a case study methodology (Stake, 1995) in our focus on meaningful understanding of quadratic function. The participant of this case study was a secondary preservice teacher in her second semester of a two-semester methods sequence. The authors conducted a 60-minute audio-recorded semi-structured task-based interview with the PT. We prompted the PTs to think aloud and clarify the meanings of her thoughts, such as: Can you tell me what connections do you see?; and Did you see the acceleration in your table/graph? In this task, we intended to observe the PT's functional thinking and representational fluency while they interpreted a diagram and table to make sense of a quadratic relationship. We created enhanced transcripts be embedding participant's written artifacts into verbatim transcripts. This method improved our ability to code students' activity.

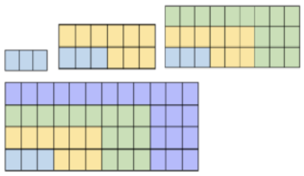
Task. Consider the following series of diagrams illustrating how a rectangle grows in several iterations.  In general, what is the relationship between height, h, and area, A, in the growing rectangle context? Why?	**Structured Interview Probes**. • How did the diagram help you to notice the pattern you observed? • Can you describe any connections across the methods you used? Does it make sense? • What is the connection between the symbolic generalizations you wrote? What about the rules and the diagram, or tables? • How would you sequence representations to teach this task? Why?

Figure 1: Task Design (adapted from Ellis, 2011) and Structured Interview Probes

Data Analysis Techniques

We used cyclic methods of coding (Miles, Huberman, & Saldana, 2013). In the first cycle of coding, we analyzed participants' representational fluency and functional thinking. In the second cycle of coding we employed axial coding (Strauss, 1987) by analyzing instances of coding-cooccurrences (i.e., where in a code for FT and RF was applied in the same data segment). In this round of coding we aimed to identify relationships between participants' functional thinking and representational fluency. The cycles of coding were an iterative process of questioning and reflecting. We compared and contrasted the data related to PTs' functional thinking with preexisting data and situated it in existing literature (Baxter, & Jack, 2008).

Coding for Representational Fluency

In this study, we employed Dreyfus' (1991) theory of learning with multiple representations, to inform decisions about students' creation, interpretation, and connection of representations. We also employed a variant of Lesh, Behr, and Post's "rule of five" or "web" of representation types to inform distinctions of when the participant was drawing on one or more of symbolic, numeric, graphic, or diagram representations. From this conceptualization or theoretical stance on the use and connection of one or more representations, we employed Fonger's (2019) analytic framework for representational fluency (Table 1) to characterize the nature of the PTs discursive activity in creating, interpreting, and connecting *across* multiple representations in solving a task. The framework developed by Fonger (2019) builds on a structure of observed learning outcome taxonomy (Biggs & Collis, 1982) and employs an actor-oriented approach (Lobato, 2012) to problem solving. Each method or approach to a problem was analyzed as a unit of analysis to discern meaningfulness in representational fluency according to eleven levels (8 are below).

Table 1: An Excerpt of Fonger's (2019) Analytic Framework for Representational Fluency

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

Lesser Meaningfulness	*Greater Meaningfulness*
Student expresses a canonically inappropriate or incomplete creation, interpretation, or connection.	*Student expresses canonically appropriate and complete creations, interpretations, and connections.*
Move Between. Moves between more than one representation type, does not correctly create *and* interpret the meaning of the representations to solve the problem.	*Uni-Directional Translation*. Perform a translation from one representation type to another representation type. *Bi-Directional Translation*. Perform translation and complementary translation processes. *Multi-Directional Translation*. More than two representation types are related by translation processes.
Multi-Representationally Grounded. Moves between more than one representation type, giving an incorrect creation *or* interpretation of their meaning.	*Uni-Directional Connection*. Create and give a correct interpretation of an invariant feature across multiple representations or types. *Bi-Directional Connection*. Create and interpret a source representation with respect to a target representation and vise-versa, and recognize invariant features across the two representation types. *Multi-Directional Connection*. More than two representation types are related by translation processes and involve a correct interpretation of an invariant feature.

Lower levels include: *Pre-structural* students create and interpret a representation with incomplete understanding and *Multi-structural* students interpret, create, or connect more than one representation type without making sense of a mathematical object. Higher levels include: *Unistructural* students create and interpret a representation without making a connection to more than one representation, and *Relational* students create, interpret, and connect multiple representations with sophisticated understanding of a mathematical object.

Coding for Functional Thinking

We employed Thompson's (2011) theory of quantitative reasoning to inform our conceptualization of participants' engagement with one or more quantities and relationships between them. From this orientation toward students' quantitative reasoning and ways of thinking about function, we employed an analytic framework on functional thinking as articulated in the literature (e.g., Confrey & Smith, 1991; Stephens et al., 2017a; Thompson & Carlson, 2017; Fonger, Ellis, Dogan, 2016) as an a priori lens. We attended to three types of functional thinking: recursive, correspondence, and coordinated change.

Table 2: A Quantitative Lens on Modes of Functional Thinking

Functional Thinking	includes students' abilities to generalize and represent functional relationships
Coordinated change	students' reason with change in more than one quantity together by comparing or linking change across these different quantities (e.g., change in *x* and change in *y*) with explicit quantification of the magnitude of both changes (e.g., change of 1 in *x* and 3 in *y*) (Fonger, Ellis, Dogan, 2016).
Correspondence	students determine output values (range) related to input values (domain) as a direct mapping or dependency relation
Recursive	"Recursive patterns describe variation in a single sequence of values, indicating how to obtain a number in a sequence given the previous number or numbers" (Stephens et al., 2017, p. 145)

Coding Co-occurrences

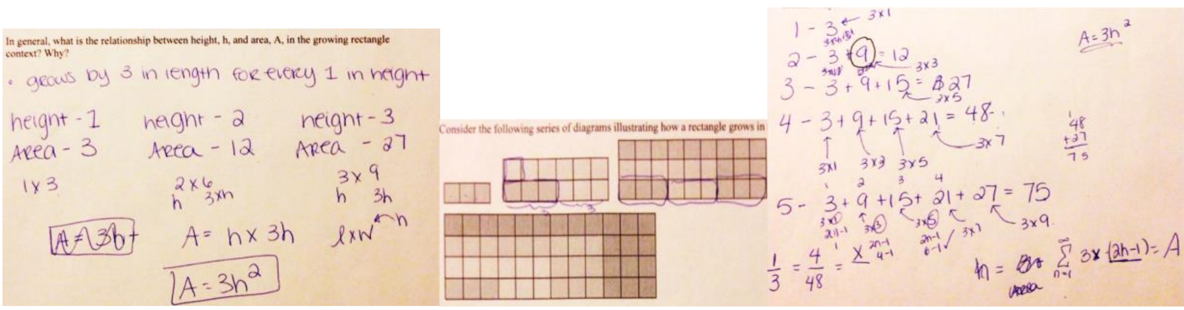
In the second round of analysis we re-analyzed instances of code co-occurrences for themes. This allowed us to identify relationships between the PTs' representational activity and their ways of thinking about function. In essence, this thematic analysis was concerned with answering the question of "what counts" as meaningful understanding when a coordination of analytic lenses it taken together?

Characterizing Meaningful Understanding in Problem Solving

In this section we highlight Vignettes to elaborate how the coordination of lenses enriches researcher sensitivity to characterizing meaning in students' problem solving activity.

Meaning May Emerge from Representational Disfluencies

In Vignette 1 (Figure 2a, b, lines 1-5) we show how meaning may emerge in problem solving.



(a) **(b)** **(c)**

Figure 2: Emy's Problem Solving Activity in Words, Numbers, Symbols, Diagrams

1 Emy: The question is like what is the relationship between height and area and the growing rectangle? So the relationship between the height and length is that it grows a 1 box in height for every 3 boxes in length, which makes it grow. So if the height is 1, the area is 3, if the height is 2 to the area is 12, then 1, 2, 3; 1, 2, 3; 4, 5, 6 times. Height is 3, an area is 27. So it's relation between height and area and the growing rectangle context. So it's 3, 3 times 1. Yeah, 2 times 6, 3 times 9, 3 times 9. So, I would say that the area equals 3, 3 times. Yeah. Three times the height (see left side of Figure 2a)

2 Nicole: How did you get a equals three h [A=3h]?

3 Emy: how we got the area, which would be the height was type 1 times 3 and then I looked in the box and it was 2 for the second one. … No, hold on. … This may be wrong. Yeah. Area equals the height times 3 times of height. So, this is what we are equals 3 height squared because if you look at the height in each one of these … So each of these is just starting out and then this is the high times 3, height times 3 times 3.

4 Nicole: So what was this, like 3h, like how did you get, how did you know to change your rule?

5 Emy: Well, because if you try to apply this to this one, it doesn't work. So area equals 3h would work for a 3 times 1, but it doesn't work for the height of two because 2 times 3 to 6. … if you try to look at that pattern and the next one, … you'll have to get 3h squared.

In interpreting this vignette from our networked lenses, we notice that Emy initially attended to coordinated change in height and length of the growing rectangle diagram by stating "the relationship between the height and length is that it grows a 1 box in height for every 3 boxes in length." Emy created a table of values and interpreted a correspondence between height and area stating "So if the height is 1, the area is 3, if the height is 2 to the area is 12" and concluding "So I would say that the area equals … 3 times the height", yet came to an incorrect generalization of A=3h (lines 1-2, *multi-representationally grounded*, *coordinated change*, *correspondence*).

Prompted to elaborate (line 2), Emy initially expressed uncertainty about her generalization A=3h, "No, hold on. … This may be wrong" drawing on a correspondence perspective of this function rule, Emy articulated independent and dependent quantities that did and did not match her rule, leading her to express a correct generalization and symbolic rule "Area equals the height times three times of height" (lines 3, *bidirectional translation*, *correspondence*). Asked to explain (line 4), Emy interpreted the symbolic rules A=3h and $A=3h^2$, and discerned that the rules give the same (height, area) only when h=1, while $A=3h^2$ accurately generalizes heights 1, 2, and 3 (line 5, *unidirectional connection*, *correspondence*).

From this vignette we learn that Emy's representational fluency changed from lesser to greater sophistication in her problem solving approach (*multistructural* to *relational*), and her engagement in functional thinking (*coordinated change* and *correspondence* reasoning) may have supported that shift as she interpreted the diagram, and created and interpreted a table of values to lead to her generalization in words and symbols. In this case, evidence of Emy's expressions of functional thinking give insight into her meaningful understanding of the growing rectangle situation as represented in diagrams, values, and a symbolic rule.

Flexibility in Meanings of Functions Engenders Representational Fluency

In Vignette 2 (see Figure 2c above, and lines 6-8 below), Emy was asked to solve the same problem in a different solution approach. Emy engaged in recursive thinking with greater sophistication in representational fluency across a diagram, table, and symbols.

6 Emy: Because you're adding a 3 on the bottom to the original 3 and then 2, 3 is on the top. So you're always going to be adding an odd number of 3s … to the original one from the past. … It's the length that's being added. So, you can group them by 3s is what I'm trying to make sense of … So you're adding 3 times. One, 2, 3 times 3 instead of 3 times 2. Um, and then you're adding 3 times 5, but we're skipping 3 times 4 and you're adding 3 times 7 but no 3 times 6 if that's like a pattern. But, um, I saw something like that. I don't know how to write that.

7 Nicole: Does it make sense why that's occurring?

8 Emy: Yes. Because you're adding a 3 on the bottom to the original 3 and then 2, 3 is on the top. So you're always going to be adding an odd number of 3s because I'm here. You're adding two 3s to bottom and then an even number 1, 2, 3, 4, 5, 6, 7, 8. This is an odd number 3 is from before and the third one? … You're adding an odd number of 3s to the original one from the past. … It's a sequence of adding groups of 3s. I don't know how to write the odd groups of 3s. I guess this could be like 3 times 2 N minus 1. Would that work?... This would be the formula [$A=\sum 3*(2h-1)$] for the area and I guess n would actually be h, it would be the height that would be a formula that relates height to area.

From a networking of lenses, we learn about Emy's meaningful understanding of this quadratic growth situation from a recursive perspective. Emy engaged in recursive reasoning, moving fluently between the diagram and the table she created to explain her generalization of adding an odd number of groups of three to the previous area in both the diagram and the table (line 6, *bidirectional connection*, *recursive*). Prompted for sense-making (line 7), Emy elaborated her recursive thinking as a "sequence of adding groups of 3s," as strongly connected to her interpretation of the diagram, again concluding with a correct symbolization of a generalization (*multidirectional translation*, *recursive*).

In this vignette we learn that in her solution approach Emy demonstrated greater sophistication in representational fluency at a *relational* level, and in moving among representations her ability to generalize the functional relationship was interpreted as meaningful from a recursive perspective. Taken together with the first vignette, we see evidence of Emy's representational fluency as going hand in hand with her functional thinking. For instance, if we illustrate Emy's activity from a "web" diagram perspective, with each node taken as a representation type, and Emy's creation, interpretation, and connection across representations as a form of functional thinking, we observe important connections. For instance, Emy began with coordinated change on a diagram (Figure 3), and side bar chart (Figure 2c). Then she built correspondence reasoning on coordinated change by shifting into the summation formula. Emy used a correspondence approach to check if the summation formula she created is meaningful for her by plugging in values of height and area for each step through correspondence reasoning.

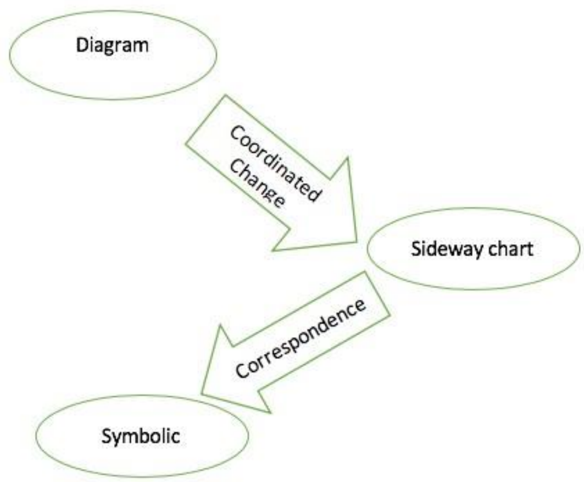


Figure 3: Emy's Solution Web with Representations and Functional Thinking

Discussion and Conclusion Characterize Meaningful Understanding

The networking of theories in this study was borne out of a need to address an enduring challenge in the field of mathematics education: characterizing students' meaningful mathematics learning. In this paper we elaborated how a networking of theories of multiple representations and quantitative reasoning is one productive approach for characterizing meaningful learning of mathematics. The theoretical groundings and assumptions guiding this study span constructivist and semiotic lenses on cognition. In particular, we drew on theories of learning that posit that students learn by creating, interpreting, and connecting multiple representations in doing and communicating about mathematics (Dreyfus, 1991), with covariation, correspondence, and recursive reasoning about relationships among linked quantities as supports for students' meaningful understandings of functions (e.g., Confrey & Smith,1994, 1995; Ellis, 2011).

From this networked stance, we examined how students' representational fluency and quantitative reasoning about relationships between varying and dependent quantities seemed to support and constrain one another. Results of this study highlight (a) how lesser meaningfulness in representational fluency may serve as a productive starting point for more sophisticated thinking and mathematical meaning to follow, and (b) how flexibility in thinking across multiple meanings of function (i.e., correspondence, covariation, and recursive modes) go hand in hand with representational fluency. We see these results as contributing evidence for the productiveness of networking theories to advance tools for characterizing meaningful learning.

Lessons Learned on Networking Theories

We have also learned that the activity of networking theories has contributed to our sensitivities as researchers to the importance of communicating explicit assumptions in background theories, theoretical constructs, and analytic tools. As we reflect on how we've learned new nuances in characterizing meaningful learning, we are left asking how other theoretical backgrounds and assumptions about learning might be brought to bear to paint a different picture of students' meaningful learning. For example, tracing the bodies of work that researchers draw on in defining "meaning" and "meaningful learning" brings us to other theories such as Voigt's (1994) articulation that personal meanings are negotiated through social interaction. Such social interaction and proactive role of the teacher-researcher (e.g., lines 2, 4, and 7) needs to be accounted for in analyses of situations, even interviews that would otherwise be thought of as "purely" psychological in orientation (see also Cobb & Yackel, 1996).

Next Steps

Next steps of this research program are to further investigate how the networking of theories can inform not only our ability to characterize more nuanced models of students' meaningful learning, but also to elaborate design principles as the basis for instructional supports aimed at engendering meaning-making. Future studies can build on this networking across the design, enactment, analysis, and communication of research.

We found the recommendations of the international working group (Bikner et al., 2019) to be a helpful grounding for continuing the work of theory development vis-à-vis networking theories in mathematics education. We encourage others to engage with and extend these recommendations for theory networking to advance theory building in mathematics education toward aims of addressing enduring challenges related to understanding and supporting students' meaningful mathematics learning.

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ELEMENTOS TEÓRICOS Y METODOLÓGICOS PARA ANALIZAR LA PRÁCTICA DE LOS PROFESORES: EL MODELO DE ENSEÑANZA

Ulises Salinas-Hernández
UNAM, CCH-Sur
ulisessh@cch.unam.mx

Luis Moreno-Armella
Cinvestav-IPN
lmorenoarmella@gmail.com

Isaias Miranda
IPN, CICATA-Legaria
imirandav@ipn.mx

En este artículo se presentan y discuten algunos elementos teóricos y metodológicos de una investigación más amplia que analiza la práctica de dos profesores -novato y experto- desde una aproximación teórica de orientación semiótica. Se presenta un encuadre teórico que nos permite atender las dimensiones epistemológica y didáctica de la práctica docente. Se resaltan tres conceptos: signo, artefacto y mediación. Asimismo, se propone, a través de la red de teorías (Kidron & Bikner-Ahsbahs, 2015), el modelo de enseñanza del profesor como constructo teórico-metodológico para analizar la práctica docente. Se reporta el modelo de enseñanza para el fenómeno de la caída libre de objetos; en donde se considera, a su vez, la estrecha relación entre matemáticas y física. Se presentan algunos resultados sobre la aplicación del modelo de enseñanza para el análisis de un profesor novato.

Keywords: Modelos, Artefactos, Recursos, Teoría de la Actividad

El presente trabajo se ubica tanto en el marco de investigaciones enfocadas en analizar la práctica del profesor (e.g., Adler, 2000; Pepin, Xu, Trouche, & Wang, 2017; Sfard, 2005; Stockero et al., 2018), así como en el interés de situar la enseñanza de las matemáticas y de las ciencias como una actividad mediada (e.g., Dobber & van Oers, 2015; Mariotti, 2009). Moreno-Armella y Sriraman (2010) consideran que el acceso a los objetos [matemáticos] ocurre a través de los mediadores (como el lenguaje). De manera que la relación objeto-sujeto no tiene lugar de manera directa.

Dos elementos [*mediadores*] que permiten al sujeto interactuar con los objetos matemáticos tanto en su carácter histórico como en su contexto cultural son el *signo* -en el sentido de Vygotsky (2009)- como instrumento psicológico; y los *artefactos* -en el sentido de Wartofsky (1979)- como representaciones de los modos de la *actividad*. Ambos elementos aparecen, a su vez, como *mediadores* entre el hombre y su entorno. Los *signos* se caracterizan no por su naturaleza representacional sino por su papel funcional: como medios de regulación externa o material de autocontrol. Son utilizados [los *signos*] por los individuos para controlar su propia actividad e influir en la de los demás (Kozulin, 2000). Por lo tanto, al colocarse entre el sujeto y el entorno, el lenguaje y otros medios culturales de significación [*signos*] hacen que el entorno sea percibido por el sujeto ya no en su estado ‘puro’ sino como un entorno transformado por la acción que ejercen inevitablemente los lentes que ofrece la cultura. La noción de *artefactos* se desarrolla en la siguiente sección.

Es así como se resalta el aspecto de la *mediación* como elemento fundamental para el análisis de los procesos de enseñanza-aprendizaje. Es importante señalar que en la presente investigación se considera que la noción de la *mediación* no es suficiente para llevar a cabo el análisis de las prácticas de enseñanza. Para situar la *mediación* en el plano pedagógico se incorpora una aproximación que resalta el *uso* de *recursos* en la práctica docente: la *Aproximación Documental de lo Didáctico* (ADD) (Gueudet & Trouche, 2009; 2012). Con el objetivo de analizar la manera en que los profesores estructuran y representan sus conocimientos en el salón de clases, en este trabajo se aborda el fenómeno de la caída libre de objetos y sus representaciones a través de gráficas cartesianas. En su investigación, Salinas-Hernández y Miranda (2018) analizan cómo las gráficas cartesianas están incluidas en una forma cultural sobre el entendimiento [de formas de hacer y de pensar] del movimiento de objetos. Se plantea así la siguiente pregunta de investigación: ¿cuáles son las características del *modelo* de enseñanza de un profesor novato en torno a la enseñanza de la caída libre de objetos?

Marco conceptual

El marco conceptual de la investigación considera elementos de la *Teoría de la Actividad* (TA) en el trabajo de Engeström (2001), el uso de *artefactos* y la noción de *modelos* (Wartofsky, 1979) y el *uso* de *recursos* (Gueudet & Trouche, 2009; 2012). Estas aproximaciones teóricas se integran alrededor de un objetivo común: interpretar la práctica docente como una *actividad* concreta orientada a objetos (motivos), a través de la *mediación* de *signos*, *herramientas/artefactos* y *recursos*. A su vez, este marco proporciona un encuadre teórico que permite dar atención a las dimensiones epistemológica y didáctica de la pregunta de investigación. La dimensión epistemológica se asocia, por un lado, con el análisis sociocultural del concepto de *sistema de referencia* –dentro de los procesos de producción de significados que ocurren en el salón de clases– para el análisis del fenómeno de caída libre; y por el otro lado, también se asocia al análisis del conocimiento de los profesores sobre el mismo concepto. A este respecto, se sigue una perspectiva que sitúa al conocimiento bajo prácticas sociales que se constituyen social y culturalmente. La dimensión didáctica de la pregunta de investigación está relacionada con los criterios y la manera (a partir de las experiencias docentes y conocimientos propios) que tiene el profesor novato para representar ante sus alumnos el concepto de *sistema de referencia*.

La Teoría de la Actividad a través del trabajo de Engeström

Lo que distingue a las diferentes aproximaciones de la TA, entre sí, es: qué constituye la esencia de la *actividad*. Lo más importante es que los tres elementos: el porqué un individuo [maestro o estudiante, en el contexto educativo] participa en la *actividad* de “hacer matemáticas”, con quién realiza la *actividad* y con qué *herramientas*; estos tres elementos no pueden ser separados y analizados en sí mismos, sino de manera integral. En particular, Engeström (2001) extiende el planteamiento de Vygotsky sobre la *mediación* a través de los *signos* y *herramientas* y el de Leont’ev sobre la labor, a múltiples formas de *mediación* en lo que denominó: *sistema de actividad*. Ahora la unidad de análisis es, al menos, dos sistemas de actividad interactuando entre sí (Figura 1).

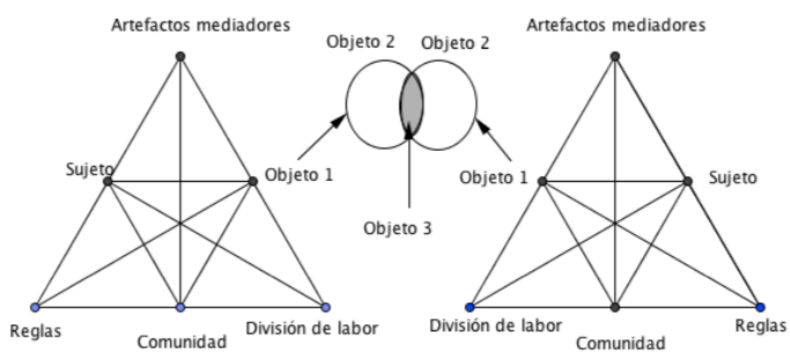

Figura 1: Representación de Engeström sobre los *sistemas de actividad*, basado en la mediación y en el trabajo de Leont’ev

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Una aproximación sobre el uso de artefactos: la noción de modelos

Para Wartofsky (1979) la característica fundamental de la práctica cognitiva humana es la habilidad para crear representaciones. Señala que, a diferencia de los animales no humanos, los seres humanos crean los medios de su propia cognición: los *artefactos*. Además de considerar al lenguaje, Wartofsky (1979) introduce las formas de organización e interacción social, las técnicas de producción, y la adquisición de habilidades, como *artefactos* . De manera que, al producir *artefactos* [medios cognitivos] para su uso, se producen *representaciones*. Porque los *artefactos*, además de su uso, representan el modo de actividad en el que se usan, o el modo de su propia producción [modos de representación]. Es decir, el *artefacto* es tanto un *medio cognitivo* como un *modo de representación*. Por lo tanto, al producir *artefactos* se producen *representaciones*, a través de las cuales el ser humano logra el conocimiento. En el ámbito científico, la producción de *artefactos* [cognitivos] produce a su vez *modelos*. Wartofsky (1979) define los *modelos* como *representaciones* para nosotros mismos de lo que hacemos, de lo que queremos y de lo que esperamos. Son modos de acción adquiridos. En palabras de Wartofsky (1979):

> El modelo se considera una construcción en la cual organizamos símbolos de nuestra experiencia o de nuestro pensamiento, de tal manera que realizamos una representación sistemática de esta experiencia o pensamiento, como medio de entenderlo o de explicarlo a los demás. (p. xv; traducción libre)

De manera que: (1) no hay conocimiento sin *representación* y (2) las *representaciones* y *artefactos* son los elementos que evidencian el conocimiento humano (en nuestra investigación, el conocimiento del profesor).

Aproximación documental de lo didáctico

Gueudet y Trouche (2009; 2012) proponen un enfoque teórico cercano, en la conceptualización de *recursos*, de la propuesta de Adler (2000). Pero ampliando la noción de *recursos* a todos aquellos que intervienen en la comprensión y resolución de problemas, no restringiéndose a los provenientes de objetos materiales. Estructuran su propuesta teórica bajo el nombre de *Aproximación Documental de lo Didáctico* (ADD). En esta propuesta, el *trabajo documental* (TD) es el núcleo de la *actividad* de los profesores y de su desarrollo profesional. Comprende todas las interacciones de los profesores con los *recursos*: su selección y el trabajo con ellos.

Los autores hacen la diferencia entre *recursos* y *documentos*. Así, los *documentos* son desarrollados a través de lo que denominan *génesis documental*. En la *génesis documental*, los *documentos* son creados y desarrollados, en el tiempo, a partir de un proceso en el cual los profesores construyen *esquemas de utilización* de los *recursos* para situaciones dentro de una variedad de contextos, proceso que se ejemplifica por la ecuación: documento = recursos + esquemas de utilización (Gueudet & Trouche, 2009, p. 205). Para dar cuenta de la variedad y del diseño de *documentos* y la manera en que los profesores los articulan en una variedad de situaciones, los *documentos* se estructuran en *un sistema de documentación*, en donde el *sistema de recursos* del profesor constituye la parte del "*recurso*" del *sistema de documentación.* Es importante señalar que en la *génesis documental* uno de los objetivos es concebir la *actividad* del profesor orientada por objetivos, y conceptualizarla como actividad social. Esta consideración de la *actividad* conlleva a poner atención en los contextos sociales de los diferentes grupos en la que ésta esté presente (Gueudet & Trouche, 2012). De manera particular, los autores hacen referencia a la TA, como la teoría sobre la cual desarrollaron su propia aproximación teórica (Gueudet & Trouche, 2012); y señalan que: "[L]a referencia a la teoría de la actividad también está directamente relacionada con nuestro interés en la mediación y los artefactos mediadores" (Gueudet & Trouche, 2012, p. 24; traducción libre). Es así como se integra la ADD en la presente investigación, para abordar la dimensión didáctica de la pregunta de investigación.

Metodología

Se plantea una investigación cualitativa, mediante estudios de caso (Cohen, Manion & Morrison, 2004). Con la finalidad de establecer las categorías y elementos de análisis se propone el modelo de enseñanza del profesor. Para los objetivos de este artículo se expone el modelo de enseñanza sobre el movimiento de objetos. Este constructo teórico-metodológico se produce a partir de la integración de las teorías expuestas en el marco conceptual (*networking of theories* (Kidron & Bikner-Ahsbahs, 2015)).

El modelo de enseñanza del profesor sobre el movimiento de objetos

Modelo C & Modelo E. Se parte de dos suposiciones. (1) La práctica del profesor considera modos (científicos-institucionales) de acción cognitiva humana y modos de representación: *modelos* [científicos-institucionales] (Wartofsky, 1979). En particular, respecto al movimiento de objetos, se define el *modelo C* como el *modelo* científico –newtoniano– actual que considera cada profesor para producir sus propios *modelos de enseñanza E*: sus prácticas y conocimientos pedagógicos. Es en el *modelo C* que se ubica el contenido físico-matemático de la investigación (e.g., los significados científico-institucionales sobre el concepto de sistema de referencia (SR)). (2) Cada profesor produce, a lo largo de su formación académica y de sus años de práctica docente, un *modelo E* que representa, por un lado, el modo de acción adquirido del *modelo C*, esto es, la manera en que cada profesor ha organizado los símbolos de su experiencia y de su pensamiento en torno al *modelo C* para explicarlo a los estudiantes; y por otro lado, los modos de representación de ese conocimiento adquirido (representaciones y artefactos desarrollados por los profesores). El *modelo E* incorpora el *trabajo documental* de cada profesor, en particular sobre los criterios de selección y *uso* de *recursos*, así como planeación e implementación de clase (planeación e implementación de *sistemas de actividad*).

Sistemas de actividad. Se propone que los *sistemas de actividad principales* y los *sistemas de actividad particulares* (*sub-sistemas de actividad*) son producidos a partir tanto del *modelo C* como del *trabajo documental* (TD) de cada profesor.

Los sistemas de actividad principales y *particulares* se producirán a partir de la *actividad* (interacciones en el salón de clase, *uso* de *artefactos mediadores* y *recursos*) que se desarrolla en el salón de clases a partir de una tarea (e.g. resolución de un ejercicio o discusión de un tema; ambos relacionados con el movimiento de objetos) propuesta por cada profesor durante sus clases. Se da lugar así –a partir de los objetivos de la tarea– a los objetivos principales de la *actividad*. A partir del *sistema de actividad principal* (e.g. encontrar la velocidad final de un objeto) y del *modelo C*, se derivan, también, *sub-sistemas de actividad* (e.g. *sistema de actividad* sobre el SR *y sistema de actividad* sobre el sistema de coordenadas cartesianas); que a su vez están en estrecha relación con el *sistema de actividad principal*, pero que tienen, sin embargo, objetivos particulares diferentes que ayudan a que el *sistema de actividad principal* se lleve a cabo (e.g. determinar un SR y/o utilizar un sistema de ejes coordenados).

Para establecer la manera en que se producen y desarrollan los *sistemas de actividad* se toma la noción de Gal'perin sobre la *base orientadora de la actividad*. La epistemología de Gal'perin "se basa en la noción de que la apropiación de nuevo conocimiento y habilidades es la consecuencia de la acción humana.” (Haenen, 1993, p. 97, traducción libre). Por lo que los procesos de enseñanza-aprendizaje tienen como objetivo la mejora cualitativa del repertorio de acciones presentes y reales (Haenen, 1993). “La optimización de este repertorio depende de las representaciones del alumno, del objetivo, la estructura y los medios de mediación para ejecutar una determinada acción.” (Haenen, 1993, p. 97, traducción libre). El término *base orientadora* se usa para referirse a estas representaciones. De esta manera, los *sistemas de actividad principales* y los *sub-sistemas de actividad* se desarrollan a partir de las decisiones de cada uno de los profesores para representar los significados sobre los conceptos que se abordan.

Existen, por una parte, relaciones didácticas/semióticas (R-α) entre el *modelo C* y cada *sistema de actividad principal*, y relaciones didácticas/semióticas (R-α’) entre el *modelo C* y los *sub-sistemas de actividad*; y, por otra parte, relaciones semióticas (R-β) entre el *sistema de actividad principal* y los *sub-sistemas de actividad*; y relaciones semióticas (R-β') entre los *sub-sistemas de actividad*. A continuación, la representación del esquema del modelo de enseñanza del profesor para el movimiento de objetos (Figura 2).

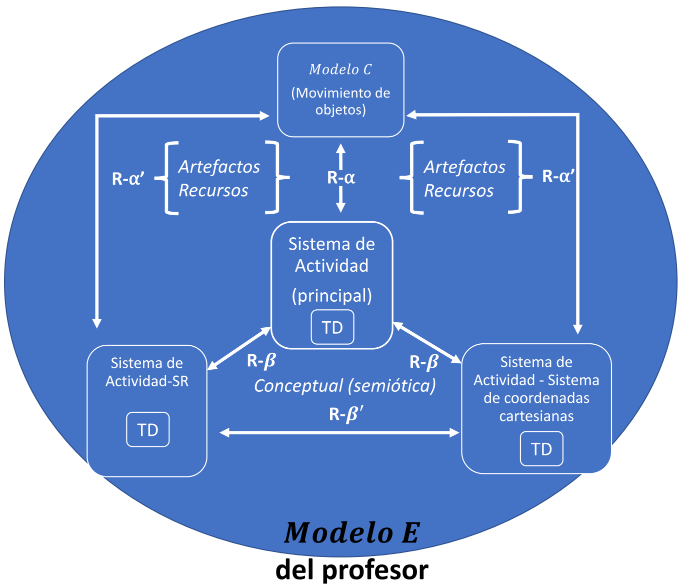

Figura 2: Esquema de las relaciones entre el modelo C y los sistemas de actividad producidos a partir del trabajo documental (TD) de un profesor

Para analizar las características del *modelo de enseñanza* de un profesor novato se trabajó con un profesor de física durante dos etapas, mediante la observación no participativa. El profesor, a quien se denomina Carlos (pseudónimo), contaba con 2 años de experiencia docente al momento de la primera toma de datos, egresado de la carrera de Física. En la primera etapa se videograbaron las clases de Carlos durante el tiempo que enseñó temas de cinemática y dinámica; con el objetivo de identificar los momentos en que abordaba los temas de caída libre y el concepto de SR. Se le observó un total de 20 horas de clase. Se hicieron transcripciones de extractos seleccionados. En la segunda etapa se volvió a realizar videograbaciones de las clases de Carlos, pero ahora solamente cuando enseñó temas particulares seleccionados de la primera etapa, con el objetivo de analizar cambios en su manera de enseñar después de dos años de experiencia docente. En esta segunda etapa se le observó durante 4 horas de clase. Se realizó, al final de la segunda etapa, una entrevista semiestructurada para tener más elementos de su *trabajo documental*.

Resultados y discusión

Para caracterizar el *modelo E* de Carlos se analizan las relaciones entre el *modelo C* y los *sistemas de actividad* (Figura 2). El análisis toma en cuenta tanto los *artefactos* como los *recursos* usados por Carlos. De manera que en lugar de analizar cada elemento (artefactos y recursos) de manera aislada, se realiza un análisis multimodal en donde se considera la estrecha relación entre ambos para establecer las relaciones antes mencionadas. Así, dos elementos importantes para analizar estas relaciones son, por un lado, los artefactos –en particular los gestos– que usa Carlos y que dan cuenta de su pensamiento (estructura cognitiva); y por otro lado el *trabajo documental* (uso de recursos) de Carlos que nos proporciona elementos de la componente pedagógica. Si se determina que en el *modelo E* de Carlos las diferentes relaciones (R-α, R-α’, R-β y R-β') se llevan a cabo de manera satisfactoria, entonces eso se representará en el esquema con una flecha continua. De lo contrario, en caso de haber dificultades para llevar a cabo las relaciones se representará con una línea punteada.

Se presentan algunos resultados de las características del *modelo de enseñanza* (modelo E) de Carlos. En la primera etapa, cuyos datos no se muestran en este artículo, Carlos tuvo dificultad al trabajar con el concepto de SR. En la segunda etapa, continuaron esas dificultades. A continuación, se muestran dos extractos que tuvieron lugar durante una clase de Carlos en la segunda etapa. Su análisis permite caracterizar su modelo de enseñanza. En la clase, Carlos abordó el tema del signo de una cantidad vectorial, en particular la aceleración de la gravedad [g]. El *sistema de actividad principal* de Carlos corresponde a su explicación sobre el significado del signo de la aceleración (se incluye a la aceleración de la gravedad). Así, después de que en la clase Carlos escribió en el pizarrón diferentes ecuaciones para analizar el movimiento de objetos, él dirigió la atención de los estudiantes hacia el significado del signo de la aceleración.

> L1 Carlos: Puede que ustedes [*dirigiéndose a los estudiantes*] tengan que determinar esa aceleración. Pero esa aceleración al momento de determinarla, luego puede que salga un menos, ¿ok? O sea que puede que aquí [*señala las ecuaciones en el pizarrón en las cuales la aceleración puede ser negativa; Figura 3*] tengan un más o un menos en el resultado. Pero ¿que significa eso de un más o un menos? (…) puede haber una aceleración o una desaceleración. Eso es lo que significa cuando les de un menos o un más, ¿ok? (…) por ejemplo, si les da más [*el resultado positivo de la aceleración*], quiere decir que de nueve metros sobre segundo [9 m/s] aumentó a treinta metros sobre segundo [30 m/s]. Y si les da menos [*el resultado negativo de la aceleración*], quiere decir que va frenando. Hay una desaceleración. Entonces quiere decir que de 30 [m/s] baja hasta 9 [m/s], ¿ok? No se les tiene que olvidar ese menos.

Figura 3: Carlos señala los signos de la aceleración

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Desde un inicio, Carlos dirige la atención de los estudiantes hacia el significado del signo de la aceleración (L1). Sin embargo, en su explicación confunde el significado del signo negativo de la cantidad "aceleración" con el significado de "desaceleración". Es importante señalar que en el *sistema de recursos* de Carlos se encuentra un libro de texto, cuyo autor es Giancoli (2006), que Carlos usa como apoyo [*recurso*] durante el desarrollo de sus cursos. En este libro de texto se señala el cuidado que se debe tener de no concebir el signo negativo de una aceleración como una desaceleración. Más adelante en la clase, Carlos aborda el tema del signo en relación con la caída libre de objetos:

> L2 Carlos: Ahora, en el eje vertical, la diferencia es que esta aceleración que ustedes están viendo aquí [*se refiere a la segunda ecuación de arriba para abajo; Figura 3*] que es una constante siempre (…) pero en el eje vertical, esa aceleración ya no la van a tener que buscar. ¿Por qué?, porque esa aceleración va a ser la de la gravedad. (…) Entonces lo voy a poner así [*escribe en el pizarrón:* $a=g=-9.8\ m/s^2$]. (…) Quiero que siempre le pongan el signo menos.

En L2 se observa cómo Carlos se auxilia del sistema de coordenadas cartesianas para situar el fenómeno físico de la aceleración de la gravedad. Así, se puede ver que se introduce otro *sistema de actividad*: *sistema de actividad–sistema de coordenadas cartesianas* (Figura 4). Este *sistema de actividad* debiera estar relacionado en el contexto del SR, es decir con el *sistema de actividad principal-SR*. Esto daría significado a los signos que se usan en el contexto del movimiento de objetos (*Modelo C*). Sin embargo, Carlos nuevamente, tal como lo hizo hace dos años, dice que "g" debe ser siempre utilizada con el signo negativo (L2).

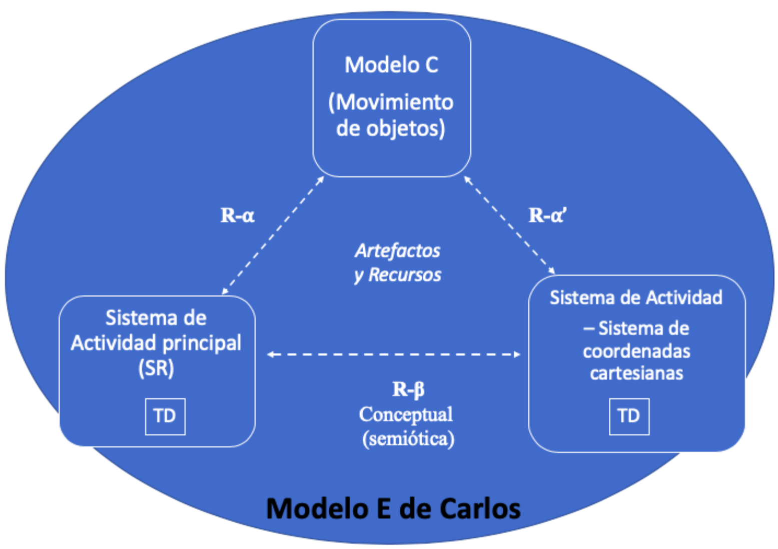

Figura 4: Esquema del modelo E del profesor Carlos

Respecto al *sistema de actividad principal* de Carlos, se observa que éste incorpora al *sistema de actividad-SR* (Figura 4). Sin embargo, se observa la dificultad, por parte de Carlos, de llevar a cabo una relación R-α, entre el *sistema de actividad principal* y el *Modelo C* (institucional). En esta dificultad, el uso del libro de texto –como *recurso* y como *artefacto*– adquiere especial importancia.

Conclusiones

En este artículo se expusieron elementos teóricos y metodológicos para analizar la práctica de los profesores. Se presentó el *modelo de enseñanza* como herramienta teórica y metodológica para analizar las practicas de enseñanza. En particular, se presentaron características del *modelo de enseñanza* de un profesor novato. En el caso de Carlos –un profesor con 4 años de experiencia [antigüedad docente] al momento de que finalizó la segunda parte de la toma de datos– se concluyó, para los resultados presentados, que él tiene un *modelo E* representado por el esquema de la Figura 4. En el nivel epistemológico, este *modelo* se caracteriza por no estar presentes de manera conjunta las relaciones entre el contenido y los objetivos de los *sistemas de actividad principales* –producidos a partir de un *trabajo documental* y derivados a su vez de un *modelo científico-institucional* (*Modelo C*), sobre el movimiento de objetos (relación R-α; Figura 4)– tanto con un *sistema de actividad* sobre el SR (perteneciente al contexto físico), como con un *sistema de actividad* sobre el sistema de coordenadas cartesianas (perteneciente al contexto matemático). Estas relaciones (R-β; Figura 4) permiten dar cuenta de los significados conceptuales que están presentes entre un *sistema de actividad principal* y otros *sistemas de actividad* (*subsistemas de actividad*) que surgen durante la actividad.

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THEORETICAL AND METHODOLOGICAL ELEMENTS TO ANALYZE TEACHER PRACTICE: THE TEACHING MODEL

In this article we present and discuss some theoretical and methodological elements of a wider research that analyzes the teaching practice of two teachers—novice and expert—from a semiotic theoretical approach. The conceptual framework of this work allows for addressing the epistemological and didactic dimensions of the teaching practice. Three concepts are particularly emphasized: sign, artefact, and mediation. Furthermore, we propose, through the networking of theories (Kidron & Bikner-Ahsbahs, 2015), the teacher's teaching model as a theoretical-methodological construct to analyze the teaching practice. This work reports the teaching model for the phenomenon of a free-falling object, in which the close relation between mathematics and physics is addressed. Some results on the application of the teaching model are presented for the analysis of a novice teacher.

Keywords: Models, Artefacts, Resources, Activity Theory.

This work is supported by research analyzing the teacher's practice (e.g., Adler, 2000; Pepin, Xu, Trouche & Wang, 2017; Sfard, 2005; Stockero et al, 2018) and seeks to position mathematics and science teaching as mediated activity (e.g., Dobber & van Oers, 2015; Mariotti, 2009). Moreno-Armella and Sriraman (2010) state that access to [mathematical] objects occurs through mediators as language. Then, the relation object-subject does not take place directly.

Two elements [*mediators*] that allow the subject to interact with mathematical objects, both historically and within the cultural context, are the *sign* —as defined by Vygotsky (2009)— as a psychological instrument and *artefacts*— according to Wartofsky (1979)— as representations of *activity* modes. Both elements appear in turn as *mediators* between the subject and the environment. *Signs* are characterized not by their representational nature but by their functional role as means of external regulation or self-control material. They [*signs*] are used by individuals to control their own activity and influence that of others (Kozulin, 2000). Therefore, when placed between the subject and the environment, language and other means of cultural significance [*signs*] make the subject perceive the environment not in its 'pure' state but as one transformed by the action inevitably exerted by the lens of culture. The notion of *artefacts* is presented in the next section.

This is how the role of *mediation* as a key element to the analysis of teaching-learning processes is emphasized. It must be noted that this research considers the notion of *mediation* is not enough to analyze teaching practices. To locate *mediation* in the pedagogical level, we have added an approximation that highlights the *use of resources* in teaching practice: *Documentational Approach to Didactics* (DAD, Gueudet & Trouche, 2009; 2012). The present work addresses the phenomenon of free-falling objects and their representations through

Cartesian planes to analyze how teachers structure and represent their knowledge in the classroom. In their research, Salinas-Hernández and Miranda (2018) analyze how Cartesian planes are culturally included on the understanding [of ways of doing and thinking] of object displacement. Then, we propose the following research question: Which are the characteristics of a novice teacher's teaching *model* regarding the teaching of free-falling objects?

Conceptual Framework

The conceptual framework of this research includes elements of the *Activity Theory* (AT) in the work by Engeström (2001), the use of *artefacts* and the notion of *models* (Wartofsky, 1979), and the *use* of *resources* (Gueudet & Trouche, 2009; 2012). These theoretical approximations are integrated around a common objective: To interpret teaching practice as a concrete *activity* focused on objects (motifs), through the *mediation* of *signs*, *tools/artefacts*, and *resources*. Furthermore, this framework provides a theoretical approach that allows for focusing on the epistemological and didactic dimensions of the research question. The epistemological dimension is associated, on the one hand, to the sociocultural analysis of the concept of *reference system* —within the processes of production of meanings that occur in the classroom— to analyze the free-fall phenomenon. On the other hand, epistemology is associated to the analysis of the teachers' knowledge on the same concept. Additionally, we follow an approach that places knowledge under social practices that are socially and culturally constituted. The didactic dimension of the research question is related to the criteria and the way (from teachers' experiences and own knowledge) the novice teacher represents the concept of *reference system* in front of the class.

Activity Theory through Engeström's Work

What distinguishes the approximations of AT between them is: What constitutes the essence of *activity*. The most import is that the three elements —why an individual [teacher or student in the education context] takes part in the "doing mathematics" *activity*; with whom the individual carries out the *activity;* and which *tools* the individual uses to do so— cannot be separated and analyzed in themselves but integrally. Particularly, Engeström (2001) expanded Vygotsky's approach on *mediation* through *signs* and *tools* and Leont'ev's on *labor* to multiple ways of *mediation* in what he called *activity system*. Now, the units of analysis are, at least, two activity systems interacting with each other (Figure 1).

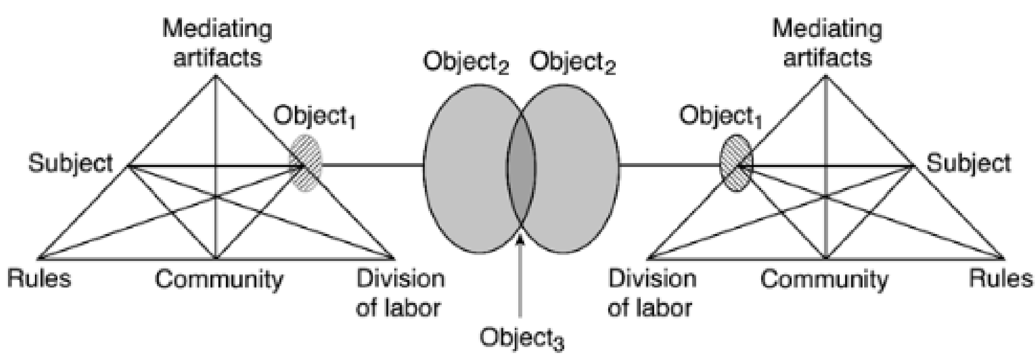

Figure 1: Engeström's Representation on *Activity Systems,* Based on Mediation and Leont'ev's work

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* St Louis, MO: University of Missouri.

An Approximation on the Use of Artefacts: The Notion of Models

Wartofsky (1979) stated that the basic characteristic of human cognitive practice is the ability to create representations. He considers that, unlike non-human animals, human beings create the means of their own cognition: *artefacts*. Besides taking into account language, Wartofsky (1979) introduces ways of social organization and interaction, production techniques, and the acquisition of abilities as *artefacts*. The latter represent not only the use but the way of activity in which they are used or the way their own production [representation modes] occurs. That is, the *artifact* is both a *cognitive medium* and a *representation mode*. Therefore, the production of *artefacts* creates *representations* through which the human being achieves knowledge. In the scientific field, the production of [cognitive] *artefacts* produces *models*. Wartofsky (1979) defines *models* as *representations* for us regarding what we do, want, and expect. They are acquired modes of action. In Wartofsky's (1979) words:

> [T]he model is taken to be a construction in which we organize symbols of our experience or of our thought in such a way that we effect a systematic representation of this experience or thought, as a means of understanding it, or of explaining it to others. (p. xv)

Therefore: (1) there is no knowledge without *representation* and (2) *representations* and *artefacts* are elements that evidence human knowledge (in our research, the teacher's knowledge).

Documentational Approach to Didactics

Gueudet and Trouche (2009; 2012) propose a close theoretical approach in the conceptualization of *resources*, from Adler's proposal (2000). However, they expand the notion of *resources* to all those participating in understanding and solving problems, not restricted to those from material objects. They structure their theoretical proposal under the name of *Documentational Approach to Didactics* (DAD). In this proposal, the *documentational work* (DW) is the core of the teachers' *activity* and professional development. It comprises all the teachers' interactions with their choice of *resources* and the work with them.

The authors distinguish between *resources* and *documents*. Then, *documents* are developed through what they call *documentational genesis*. In the *documentational genesis*, *documents* are created and developed in time from a process in which teachers build *schemes of utilization* of *resources* for situations within a number of contexts. This process is represented by the equation: document = resources + schemes of utilization (Gueudet & Trouche, 2009, p. 205). To provide an account of the variety and design of *documents* and the way teachers articulate them in different situations, *documents* are structures in a *documentational system*, in which the teacher's *resource system* constitutes the "*resource*" part of the *documentational system*. It must be noted that one of the objectives of *documentational genesis* is to consider the teacher's *activity* as goal-oriented and conceptualize it as a social activity. This consideration leads to paying close attention to the social contexts of the different groups in which the *activity* is present (Gueudet & Trouche, 2012). Particularly, the authors refer to AT as the theory from which they developed their own theoretical approach (Gueudet & Trouche, 2012) and state that: "The reference to activity theory is also directly connected with our interest in mediation and mediating artefacts" (Gueudet & Trouche, 2012, p. 24). This is how DAD is integrated into this research to address the didactic dimension of the research question.

Methodology

We propose a qualitative research through case studies (Cohen, Manion & Morrison, 2004). To establish the categories and elements of analysis, we also propose the teacher's teaching model. To achieve the objectives of this article, we present the teaching model on object displacement. This theoretical-methodological construct is produced from the integration of theories presented in the conceptual framework (*networking of theories*, Kidron & Bikner-Ahsbahs, 2015).

The Teacher's Teaching Model on Object Displacement

Model S & Model T. Two suppositions are assumed. (1) The teacher's practice considers (scientific-institutional) human cognitive action modes and representation modes: [scientific-institutional] *models* (Wartofsky, 1979). Particularly, regarding object displacement, *Model S* is defined as the current scientific —Newtonian—*model* that all teachers consider to produce their own *teaching models T*: their practices and pedagogical knowledge. Within m*odel S* is located the physico-mathematical content of the research (e.g., the scientific-institutional meanings on the concept of reference system, RS). (2) Throughout their academic education and their teaching practice years, teachers produce a *model T* representing, on the one hand, the action mode acquired from *model S*; that is, the way in which the teachers have organized the symbols of their experience and thought regarding *model S* to explain it to students. On the other hand, the representation modes of that knowledge acquired (representations and artefacts developed by the teachers). *Model T* adds the *documentational work* of each teacher; especially that on the selection criteria and *utilization* of *resources*, as well as planning and class implementation (planning and implementation of *activity systems*).

Activity systems. We propose that *main activity systems* and *particular activity systems* (*activity subsystems*) are produced from both *model S* and the teacher's *documentational work* (DW).

Main and *particular activity systems* will be produced from the *activity* (interactions in the classroom, *utilization* of *mediating artefacts* and *resources*) taking place in the classroom from a task (e.g., solving an exercise or discussing a topic, both related to object displacement) proposed by the teacher during class. This is how —from the objectives of the task— the main objectives of the *activity* arise. From the *main activity system* (e.g. finding the final velocity of an object) and *model S*, *activity subsystems* are derived (e.g. *activity system* on RS and *activity system* on the Cartesian coordinates system), which are, in turn, closely related to the *main activity system* but that have different particular objectives, allowing for the *main activity system* to be carried out (e.g., determine a RS and/or utilizing a coordinate axes system).

To establish how *activity systems* are produced and developed, we consider Gal'perin's notion on the *orienting basis of the activity*. Gal'perin's "epistemology is based on the notion that the appropriation of new knowledge and skills is the outcome of human action" (Haenen, 1993, p. 97). Then, teaching-learning processes aim to qualitatively improve the repertoire of present and actual actions (Haenen, 1993). "The optimization of this repertoire depends on the pupil's representations of the goal, the structure and the mediational means to execute a certain action" (Haenen, 1993, p. 97). The term *orienting base* is used to refer to such representations. In consequence, *main activity systems* and *activity subsystems* are developed from the decisions made by the teachers to represent the meanings of the concepts addressed.

On the one hand, there are didactic/semiotic relations (R-α) between *model S* and every *main activity system* and didactic/semiotic relations (R-α') between *model S* and *activity subsystems*. On the other hand, there are semiotic relations (R-β) between the *main activity system* and *activity subsystems* and semiotic relations (R-β') between *activity subsystems*. Below is the representation of the scheme of the teacher's teaching model for free-falling objects (Figure 2).

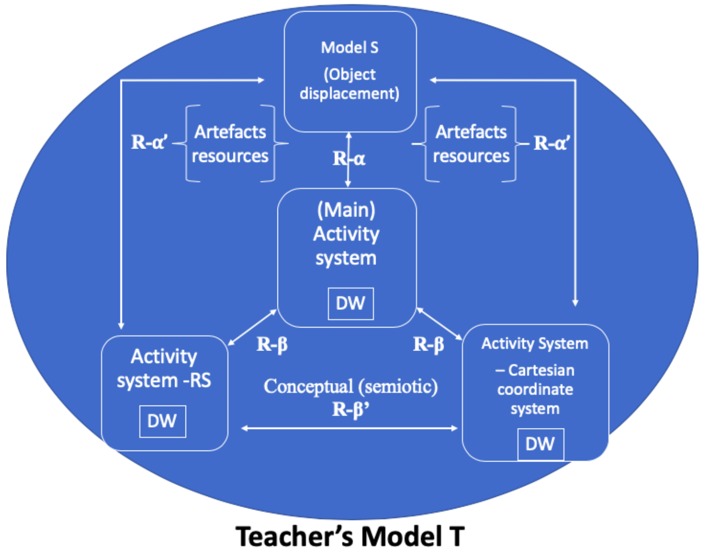

Figure 2: Scheme of Relations between Model S and Activity Systems Produced From a Teacher's Documentational Work (DW)

To analyze the characteristics of a novice teacher's *teaching model*, we worked with a physics teacher in two stages through non-participative observation. The teacher, Carlos (pseudonym), had majored in Physics and worked two years as a teacher when the first round of data was collected. In the first stage, Carlos' classes were video-recorded as he taught kinematics and dynamics topics to identify when he addressed free-fall topics and the concept of RS. The observation comprised 20 hours of class and transcriptions of selected excerpts were done. In the second stage, the teacher's classes were video-recorded as well, but only when Carlos taught specific topics selected from the first stage to analyze changes in the way he taught after two years of teaching experience. The observation in this stage lasted four hours of class. At the end of this stage, a semi-structured interview was conducted to acquire more elements of the teacher's *documentational work.*

Results and Discussion

To characterize the teacher's *model T*, we analyze the relations between *model S* and *activity systems* (Figure 2). The analysis considers both *artefacts* and *resources* used by Carlos. Then, instead of analyzing each isolated element (artefacts and resources), a multimodal analysis is carried out, considering the close relationship between both elements to establish the relations previously described. When analyzing these relations, there are two key elements to consider: the artefacts (particularly gestures) used by Carlos that provide account of his thought (cognitive structure) and his *documentational work* (utilization of resources) that provides elements of the pedagogical component. If the different relations (R-α, R-α', R-β y R-β') are carried out satisfactorily in Carlos' *model T*, then this is represented by a continuous arrow in the scheme. In contrast, it is represented with a dotted line if there are difficulties to carry out the relations.

Below, we present some results of the characteristics found in Carlos' *teaching model* (model T). In the first stage, whose data are not shown in this article, Carlos had difficulties while working with the concept of RS. Those difficulties continued along the second stage. Below are two excerpts from one of Carlos' classes from the second stage. The analysis of these excerpts allows for the characterization of Carlos' teaching model. In the class, Carlos addressed the topic "sign of a vector quantity", especially acceleration of gravity [g]. Carlos' *main activity system* corresponds to his explanation on the meaning of the sign of acceleration (acceleration of gravity is included). Then, after writing different equations to analyze object displacement, Carlos directed the students' attention to the meaning of the sign of acceleration.

> L1 Carlos: [*addressing the students*] You might have to determine this acceleration. But when determining it, we might get a minus sign, ok? That means that here [*points at the equations on the board in which acceleration might be negative; Figure 3*] you might get plus or minus in the result. But, what does plus or minus mean? (…) for example, if you get plus [*positive result of acceleration*], that means that it increased from nine meters per second [9 m/s] to thirty meters per second [30 m/s]. And, if you get minus [*negative result of acceleration*], that means it is slowing down. There is a deceleration. Then, this means that it lowers from thirty [30 m/s] to nine [9 m/s], ok? Don't forget that minus.

Figure 3: Carlos Points at Signs of Acceleration

From the beginning, Carlos directs the students' attention towards the meaning of the sign of acceleration (L1). However, his explanation mistakes the meaning of the negative sign of the quantity "acceleration" for the meaning of "deceleration". It must be considered that within Carlos' *resource system* is a textbook by Giancoli (2006), which Carlos uses as support [*resource*] during his courses. This textbook emphasizes one must be careful not to conceive the negative sign of an acceleration as a deceleration. Later in class, Carlos deals with the topic of the sign and its relation with free-falling objects:

> L2 Carlos: Now, in the vertical axis, the difference is that this deceleration that you see here [*he refers to the second equation from top to bottom; Figure 3*], which is always a constant (…) but in the vertical axis, you will not have to look for this deceleration. Why? Because this deceleration is going to be the deceleration of gravity (…) So, I will put it this way [*writes $a = g = -9.8\ m/s^2$ on the board*]. (…) I want you to always write the minus sign.

In L2, we observe how Carlos is helped by the Cartesian coordinate system to place the physical phenomenon of acceleration of gravity. Then, he evidently introduces another *activity system: activity system-Cartesian coordinate system* (Figure 4). This *activity system* should be related to the context of the RS; that is, to the *main activity system-RS.* This would provide meaning to the signs used in the context of object displacement (*Model S*). However, Carlos says again, just like he did two years before, that “g” must always be used with a negative sign (L2).

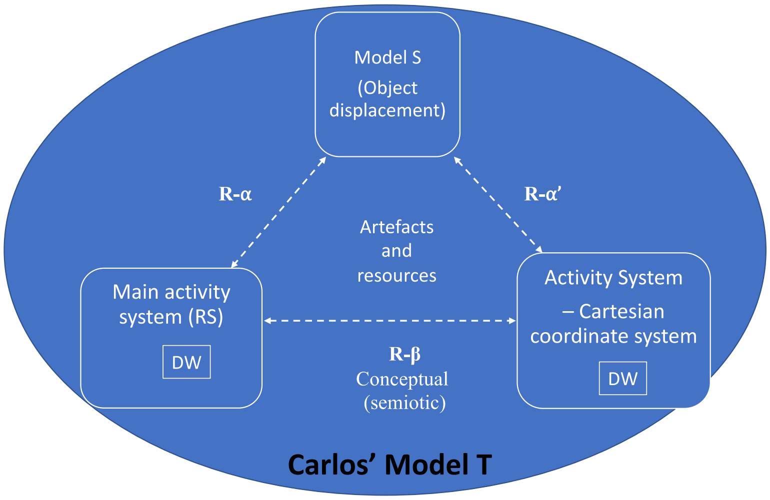

Figure 4: Teacher Carlos’ Scheme of Model T

Regarding Carlos’ *main activity system*, we observed it includes the *activity-RS system* (Figure 4). However, Carlos’ difficulty to carry out a relation R-α, between the *main activity system* and *Model S* (institutional) is evident. Because of this difficulty, the use of the textbook as a *resource* and *artefact* becomes especially important.

Conclusions

In this article, we have presented theoretical and methodological elements to analyze teachers’ practices. We presented the *teaching model* as a theoretical and methodological tool to analyze teaching practices. Particularly, we presented characteristics of a novice teacher’s *teaching model*. In the case of Carlos —a teacher with four years of experience [teaching seniority] at the time the second part of the data collection was completed— we concluded that, according to the results presented, he shows a *model T* represented by the scheme in Figure 4. At the epistemological level, this *model* is characterized by the fact that the relations between content and objectives of the *main activity systems* —produced from a *documentational work* and derived from a *scientific-institutional model* (*Model S*) regarding object displacement (relation R-α; Figure 4)— with an *activity system* on the RS or the Cartesian coordinate system (belonging to the mathematical context) are not present. These relations (R-β; Figure 4) allow for reporting the conceptual meanings present in a *main activity system* and other *activity systems* (*activity subsystems*) that arise during the activity.

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ARTICULATING LINKS BETWEEN STUDENT CONCEPTIONS AND INSTRUCTIONAL ACTIONS IN LEARNING TRAJECTORIES RESEARCH

Amy Ellis
University of Georgia
amyellis@uga.edu

Nicole L. Fonger
Syracuse University
nfonger@syr.edu

Muhammed F. Dogan
Adiyaman University
mfatihdogan@adiyaman.edu.tr

The establishment of learning trajectories has become a key area of research in mathematics education, but there remains a need to characterize the role of instruction on students' developing concepts. This paper offers a way to establish a connection between student concepts, tasks, and pedagogical supports. We draw on two constructs, the epistemic student and DNR-based instruction, and articulate how these constructs can work in concert to inform the three phases of design, enactment, and analysis of learning trajectories. We close with an example from a learning trajectory of middle-school students' understanding of quadratic growth.

Keywords: Algebra and Algebraic Thinking, Learning Trajectories (or Progressions)

The Need to Situate Learning Trajectories Research in Instructional Settings

Learning trajectories have become an important area of research in mathematics education (e.g., Clements & Sarama, 2004; Fuhrman, Resnick, & Shepard, 2009; Sarama, 2018). The influence of this strand of research is evident in the development of the Common Core State Standards in Mathematics (National Governor's Association, 2010), in curricular materials (Confrey et al., 2018), and in special journal issues and reports (Daro, Mosher, & Cocoran, 2011; Duncan & Hmelo-Silver, 2009). There is a need, however, to address the role of instruction in the development of learning trajectories.

The notion of a learning trajectory has different meanings for different researchers. Simon's (1995) original definition described a hypothetical learning trajectory as consisting of "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Clements and Sarama (2004) depicted a learning trajectory as a description of children's thinking and learning in a particular mathematical domain, along with a "conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression" (p. 83). Connections to teaching are typically constrained to a consideration of learning activities or instructional tasks.

We offer a way to establish a connection between students' changing conceptions and instructional supports. Students' conceptual development does not occur in response to tasks alone, but also occurs in a specific setting, in concert with peers, a teacher, and particular material and conceptual artifacts (Voigt, 1995). We articulate two constructs for bridging the gap between student conceptions and the instructional environment: the epistemic student (Steffe & Norton, 2014), and DNR-based instruction (Harel, 2008a; 2008b). Below we describe how these constructs can together inform the design, enactment, and analysis of learning trajectories.

Two Theoretical Constructs: The Epistemic Student and DNR-Based Instruction

The Epistemic Student and the Mathematics of Students

Students who participate in learning trajectory research efforts differ in their mathematical experiences, dispositions, and positioning. A learning trajectory presents an account of students' concepts and the transitions from one concept to the next – but which students? We have found it

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useful to frame a learning trajectory as offering a plausible sequence of concepts for the *epistemic student*. Grounded in Piaget's (1966) notion of the epistemic subject, an epistemic student is a hypothesis of the mathematics of students, consisting of a set of mathematical concepts and operations that can explain students' characteristic activity (Steffe & Norton, 2014). Epistemic students are models to describe, explain, and predict the mathematical actions of similar students who may be operating at a similar level (Steffe & Norton, 2014). A critical goal of learning trajectory research is to engender and explain students' productive thinking. By distinguishing our personal mathematics from students' mathematics, we recognize that students bring significant knowledge and competencies to bear. The job of establishing a learning trajectory then becomes one of explaining students' thinking in a manner that portrays it as meaningful, logical, and internally consistent.

DNR-Based Instruction

DNR is an acronym for the three instructional principles *duality*, *necessity*, and *repeated reasoning* (Harel, 2008a; 2008b). The duality principle addresses two forms of knowledge, Ways of Understanding (WoU) and Ways of Thinking (WoT). WoU can be understood as specific subject-matter knowledge, consisting of definitions, theorems, proofs, and mathematical relationships. WoT, in contrast, are more general conceptual tools, such as deductive reasoning, heuristics, and beliefs about mathematics (Harel, 2013). The duality principle states that students develop WoT through the production of WoU, and, conversely, the WoU they construct are afforded and constrained by their WoT (Harel, 2008a).

The necessity principle states that in order for students to learn the mathematics we intend to teach them, they must experience an intellectual need for it (Harel, 2008a; 2013). Teachers can engender intellectual need through problems that necessitate the creation of the target concept in order to be resolved. Relatedly, the repeated reasoning principle addresses the need for teachers to ensure that their students internalize, retain, and organize knowledge (Harel, 2008a). Repeated experience is a critical factor in achieving this goal, but Harel is careful to emphasize the importance of multiple opportunities to practice mathematical reasoning, rather than the drill and practice of routine problems.

Below we offer an example articulating how the constructs of the epistemic student and DNR-based instruction can inform the design, enactment, and analysis of a learning trajectory. This example draws on a teaching experiment with a group of 6 middle-school students who investigated quadratic growth phenomena, and it highlights the manner in which these constructs can establish a connection between students' changing conceptions and the instructional supports that occur within a particular teaching-and-learning context.

Methodological Approach: Design, Enactment, and Analysis

We conducted a 15-day videoed teaching experiment (Cobb & Steffe, 1983) with 6 8$^{\text{th}}$-grade students. The teacher-researcher was the first author. Three students were in general 8$^{\text{th}}$-grade mathematics, 2 students were in pre-algebra, and 1 student was in algebra. Each session lasted 1 hour. Each student also participated in one pre-interview and one post interview. Data sources included video and transcripts from the teaching sessions and the students' written work.

Learning Trajectory Design

The design phase includes creating a hypothetical learning trajectory (Simon, 1995) informed by DNR-based instructional principles and existing research. We designed a tentative series of tasks to necessitate the WoU that quadratic functions represent a constantly-changing rate of change between two covarying quantities (Saldanha & Thompson, 1998). A broader goal was to engender a WoT that functions can be representations of covariation (Thompson & Carlson, 2017). With those goals in mind we developed a context in which students could compare the areas, heights, and lengths of proportionally-growing rectangles. Students could manipulate the rectangles in order to determine how the quantities grew in relation to one another.

We developed the following three WoU that we wanted to foster during the teaching experiment: (a) the rate of change of a rectangle's area grows at a constantly-changing rate for each same-unit increase in height (or length); (b) given a height, h, the rectangle's area can be determined by $A = \mathrm{a}h^2$ where a is the ratio of length to height; and (c) the constantly-changing rate of change of the area (A) is dependent on the change in height (Δh). In order to foster these WoU, we devised tasks to encourage repeatedly reasoning through predictions of growth, determining areas for specific height values and vice versa, and deciding whether data tables represented proportionally-growing rectangles.

Learning Trajectory Enactment

One purpose of a teaching experiment is to gain direct experience with students' mathematical reasoning in relation to both tasks and teaching actions. Consequently, our set of tasks was not wholly predetermined, but instead we regularly created and revised new tasks in response to our evolving models of the students' thinking. During and between each session, we engaged in an iterative cycle of (a) teaching actions, (b) assessment and model building, and (c) invention and revision of tasks and other instructional supports. In this manner we continually revised our model of the epistemic student throughout the enactment of the teaching experiment (Steffe & Thompson, 2000).

DNR-based instruction provided guidance for introducing new instructional supports beyond tasks, such as teacher moves, question types, and prompts. For instance, we noticed that the students displayed a WoT of identifying number patterns that were divorced from quantities. This then encouraged an emphasis on changes in y without coordinating with corresponding changes in x. For instance, when discussing the increasing area of a rectangle that grew in length only, Ally said, "It would grow by 7, then 10, then 18." Her language emphasized growth in area, but in a manner that did not connect to the amount of growth in length. Further, units were absent from Ally's description. This WoT constrained the students' abilities to develop a WoU that the growth in area is dependent on the growth in height. We responded to this WoT by a) developing tasks to necessitate the creation of tables with different height increments; b) introducing prompts for students to explain their thinking through drawings, necessitating an attention to height increments; and c) encouraging the students to identify and name quantities.

Learning Trajectory Analysis

In addition to ongoing analysis, we also engaged in retrospective analysis (Simon et al., 2010) to support the revision and finalization of the learning trajectory. We applied the constant comparative method (Strauss & Corbin, 1990) to develop codes identifying (a) students' existing and emerging WoT and WoU, and (b) the instructional supports that preceded and appeared to foster shifts in students' WoT and WoU. All three authors independently coded the first eight sessions, met to reconcile discrepancies, and developed an initial framework, which we then applied to the last seven sessions. After these two rounds, we reached a stable framework, which the first and second authors then used to re-code all sessions.

An Example: The Shift from Implicit to Explicit Coordination

We identified three key shifts in the students' WoU: (1) From single-quantity and uncoordinated variation to implicit coordination of changes in quantities; (2) From implicit coordination to explicit quantified coordination; and (3) From explicit quantified coordination to identifying dependency relations of change. We highlight Shift 2, from *implicit coordination of changes in quantities* to *explicit quantified coordination of changes in quantities*. The students first began to explicitly coordinate changes in co-varying quantities when one quantity changed by 1. One of the instructional supports encouraging this coordination was the prompt to create their own ways of recording data. This activity encouraged explicit attention to how each of the relevant quantities grew in order to systematically track the growth. However, the students overwhelmingly created drawings and tables that incremented the height by 1 cm, which could lead to rendering unit changes implicit. We therefore devised a task in which the students had to first rates of change for the growth in area of a 2-cm by 5-cm rectangle, and then devise their own way to record the growth in order to test their predictions. By not specifying the height's magnitude of increase, we anticipated that the students would identify different rates of change for the area, which would engender a disagreement necessitating more explicit attention to changes in height.

Some students predicted that the changing rate of change for the area would be 5 cm^2, while others predicted 20 cm^2. As anticipated, the students did not initially specify for what change in height their prediction referred to. When testing their predictions, some students created tables with a height increase of 1 cm, while others used a height increase of 2 cm. This difference led to a disagreement. For instance, Bianca insisted that Jim's result of 5 cm^2 was incorrect, and that it should be 20 cm^2, which was Daeshim's result. This caused Jim to realize, "*He's* [Daeshim] going by 2's. But *I'm* going by 1's." Jim introduced the need to attend to the change in height, which caused the group to realize that the magnitude of height increase was relevant. The instructional support of encouraging students to create their own ways of recording data, combined with the ensuing conflict that emerged, fostered the students' attention to coordinated changes in quantities for magnitude increases other than 1 cm. Over multiple instances of repeated reasoning, the students eventually became accustomed to identifying different changes in height as coordinated with changes in length and area.

Discussion

Beyond those in the above example, we identified additional instructional supports that fostered Shift 2, including (a) explicitly drawing attention to and identifying quantities, (b) pressing for quantitatively-based justifications, and (c) introducing tables with increments less than 1 for the independent variable. This latter support fostered a WoU that one can coordinate changes in area with height increments less than 1 cm. However, the introduction of tables with $\Delta x < 1$ would not have been effective on its own. Rather, it needed to occur after the establishment of a socio-mathematical norm that one must explicitly identify the magnitude of height increases. This emphasizes the point that a learning trajectory should take seriously the notion that student learning occurs within a highly-contextualized system. This system includes not only carefully-devised tasks, but also students interacting with one another and with their teacher-researcher, the teacher-researcher engaging in particular instructional moves and encouraging particular forms of discourse, and the students and the teacher-researcher together developing a set of norms. One cannot simply lift a task sequence out of a teaching-and-learning system and expect that it will result in similar student outcomes in the hands of a different instructor who may have different mathematical goals, beliefs, and knowledge. Consequently, part of the challenge of translating learning trajectories research is to emphasize and exemplify the ways in which particular instructional moves can be powerful in relation to task sequences, and are dependent on our understanding of the mathematics of students.

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THE APOS-SLOPE FRAMEWORK AND ITS IMPLICATIONS

Courtney Nagle	Deborah Moore-Russo	Rafael Martínez-Planell
Penn State Behrend	University of Oklahoma	University of Puerto Rico at Mayagüez
crt12@psu.edu	dmr@ou.edu	rmplanell@gmail.com

This paper describes the APOS-Slope framework, a tool for studying students' understanding of slope. The paper outlines how this framework merges the theoretical lens of APOS with past work describing multiple conceptualizations of slope. Descriptions of student understanding of slope at each stage of the framework are provided, and implications of the framework for future research are detailed

Keywords: Algebra and Algebraic Thinking, Cognition, Learning Theory, Learning Trajectories

The APOS-Slope framework for studying students' understanding of slope unites the APOS (Action- Process-Object-Schema) theoretical lens with descriptions of multiple conceptions of what slope is and how it is used (Nagle, Martínez-Planell & Moore-Russo, *in press*).

Theoretical and Conceptual Lenses

A brief description of APOS Theory and past slope research are provided.

APOS Theory

APOS provides one possible lens through which data on mathematical understanding may be analyzed (Dubinsky & McDonald, 2001). In APOS, an *Action* is a transformation of a mathematical notion that the student perceives as external. It may be a rigid application of an explicitly available algorithm or the application of a memorized fact or procedure. An *Action* is relatively unconnected from the student's other mathematical knowledge. The student is unable to justify the *Action* and may be limited to using a specific representation. When an *Action* is repeated and the student reflects on the *Action*, it may be interiorized into a *Process* that has meaningful connections to the student's other mathematical knowledge. These connections enable the student to imagine the mathematical transformation, omit some of its steps, and anticipate its results without having to explicitly perform the transformation. The meaningful connections give the student the capability to convert between different representations of the transformation and to justify the *Process*. When encountering novel applications and contexts, the student may feel the need to apply *Actions* on a *Process* in order to deal with new problem situations. When the student is able to view the *Process* as an entity in itself and to apply, or to imagine applying, *Actions* on a *Process*, then it can be said that the *Process* has been encapsulated into an *Object*. A *Schema* is a coherent collection of *Actions*, *Processes*, *Objects*, and other *Schemas* that deal with a specific mathematical notion. The *Schema* is coherent in the sense that its different *Processes*, *Objects*, and sub-*Schemas* are interconnected.

Slope Conceptualizations and Research

Slope serves as a foundational concept in mathematics in part because there are numerous ways to represent and consider the topic across the curriculum (Nagle & Moore-Russo, 2014). Past research has reported 11 ways that students and teachers conceptualize slope (Moore-Russo, Connor & Rugg, 2011): *algebraic ratio* (A), *geometric ratio* (G), *physical property* (P), *parametric coefficient* (PC), *functional property* (F), *trigonometric conception* (T), *calculus*

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conception (C), *real world situation* (R), *determining property* (D), *behavior indicator* (B) and *linear constant* (L). In the past, researchers have used these slope conceptualizations to describe the most prevalent conceptualizations among students and teachers (Nagle, Moore-Russo, Viglietti & Martin, 2013; Stump, 1999, 2001b) and to document deficits in slope reasoning (Stump, 1999, 2001a, 2001b; Teuscher & Reys, 2010). Carlson, Oehrtman and Engelke (2010) gave examples of how *Action* and *Process* views of slope can play a role in students' understanding of functions. Nagle and colleagues (2016) used APOS theory to describe how the 11 slope conceptualizations could be restructured by considering how slope is viewed (as a Number, Ratio, or Invariant) and how slope is used (to Describe Behavior, Measure Steepness, or Determine Relationships). Deniz and Kabael (2017) applied the APOS framework to analyze clinical interviews with eighth grade students constructing meaning for slope. Their findings provide illustrative examples of student thinking at the *Action* and *Process* stages, focusing on *algebraic ratio* and *geometric ratio*. The APOS-Slope framework presented here extends what is proposed by Deniz and Kabael (2017) to a population of more advanced students.

APOS-Slope Framework

The APOS-Slope framework draws on accounts of students' and teachers' interactions with slope in past research and suggests that the concept of slope may be constructed by interrelating the *geometric ratio* (G), *algebraic ratio* (A) and *functional property* (F) conceptualizations until they merge into the *linear constant* (L) conceptualization. These four conceptualizations (i.e., G, A, F, L) are described as ways of thinking about slope. The remaining seven conceptualizations (see Table 1) describe uses of slope. The following paragraphs detail each of the four ways of thinking about slope at Action, Process and Object stages, while drawing on examples of possible uses of slope at each stage. For additional examples and descriptions of possible transition levels between stages, see Nagle, Martínez-Planell & Moore-Russo (*in press*).

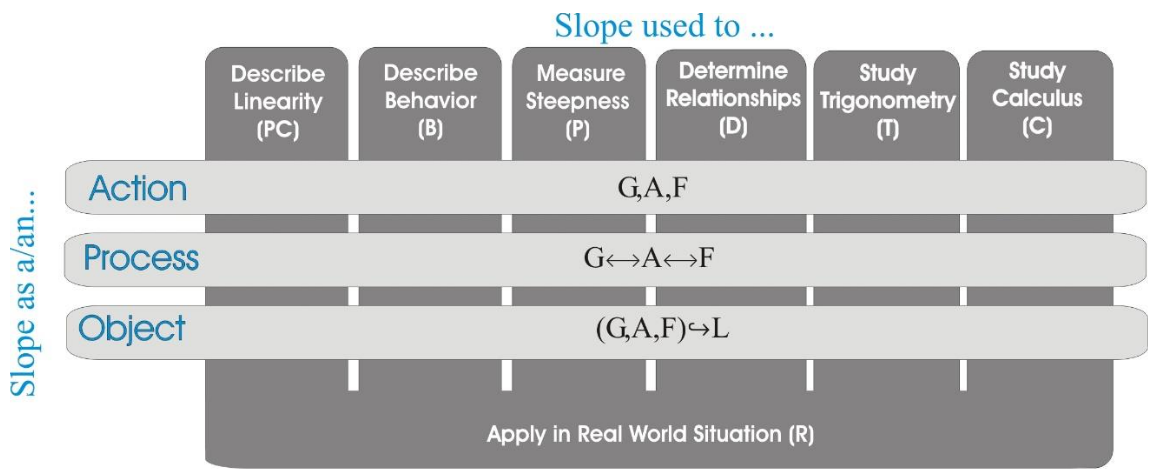

Figure 1: Interrelation between APOS Stages and the 11 Conceptualizations of Slope (from Nagle, Martínez-Planell & Moore-Russo, in press)

Slope as an Action

The *Action* stage can entail thinking of slope as an *algebraic ratio* (A_A), *geometric ratio* (A_G), or *functional property* (A_F). Consistent with Reiken's (2008) "number from a formula", an *Action* stage for the *algebraic ratio* (A_A) would consist of a student being limited to knowing (as a memorized fact) or making use of the formula $\boldsymbol{m} = \frac{y_2 - y_1}{x_2 - x_1}$, without justification or connection to other representations. Like Reiken's (2008) "number form counting", an *Action* stage for the *geometric ratio* (A_G) would consist of a student being limited to thinking about slope as vertical change (rise) over horizontal change (run) with a static mental image and may lead to the incorrect slope for a line graphed on a non-homogenous coordinate system (Zaslavsky, Sela & Leron, 2002). The *functional property* conceptualization involves thinking of slope as the rate of change between two varying quantities. An *Action* stage of *functional property* (A_F) would mean an empty and isolated verbal identification of the word "slope" with the phrase "rate of change" without any ability to further elaborate on what the rate of change represents contextually.

Even if restricted to an *Action* stage understanding of slope, a student might use slope for any of the purposes described in the APOS-Slope framework. For instance, a student who recalls and uses the memorized fact that an equation of the form $y = mx + b$ represents a line will be able to use slope to describe linearity when he or she states that "the graph of the equation $y = 3x + 4$ will be a line because it has a constant slope of 3 and lines have a constant slope." This same student may use slope to describe behavior (e.g., "the line will be increasing because the slope is positive") and measure steepness (e.g., "the line $y = 3x + 4$ is steeper than the line $y = 2x + 4$ because it has a larger slope"). The criteria that place these examples at the *Action* stage involve carrying out of procedures or parroting phrases without understanding of what one is doing.

Slope as a Process

The student showing a *Process* stage understanding of slope would have an understanding that is independent of representation, uniting *algebraic ratio*, *geometric ratio* and *functional property* conceptualizations. The *Process* conception also includes the ability to generate dynamic imagery without having to do explicit computations. Such a student would be able to imagine putting together several slope triangles with unit base to conclude that the vertical change is the slope times the horizontal change, $\Delta V = m\Delta H$. The student would also have the simultaneous awareness that the equation $y - y_1 = m(x - x_1)$ is an instance of this observation.

At the *Process* stage, a student would be able to use dynamic imagery of a line and its equation (see Figure 2) to justify with a geometric argument that any non-vertical line will have an equation of the form $y = mx + b$ (describe linearity) and to relate the sign of *m* in this equation to the increasing or decreasing behavior or the line (describe behavior). This student could also use slope to measure steepness by justifying how the value of slope impacts slope triangles and therefore the steepness of a line.

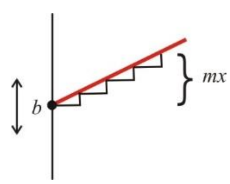

Figure 2: Dynamic Imagery for a *Process* Understanding (from Nagle, Martínez-Planell & Moore-Russo, in press)

Slope as an Object

An *Object* stage understanding of slope results from doing or imagining doing *Actions* on a *Process* of slope as the range of applications extends beyond the context in which it was initially developed. A student who is able to use a *Process* understanding of slope to construct a notion of slope in three-dimensions to think of slopes of tangent planes would be exhibiting behavior consistent with an *Object* stage understanding of slope. Students with an *Object* conception would be able to do *Actions* on their *Process* conceptions of slope to think of slope in units of miles per hour or ounces of juice concentrate per ounces of water or any other combination of units needed for a *real world* application conceptualization. The *linear constant* conceptualization of slope involves thinking of slope as a constant property unique to “straight” figures, independent of representation and thus requires at least a *Process* stage of understanding. Thinking of slope as “a … property” already implies being able to think of the slope *Process* as an entity in itself. In APOS it is conjectured that there might be a stage called *Totality* between the *Process* and *Object* stages (Arnon et al., 2014). *Totality* is proposed to be a stage where the individuals recognize the *Process* as an entity in itself, but they may not yet be able to apply *Actions* on the *Process* (Arnon et al., 2014). We propose that the *linear constant* conceptualization of slope captures the essence of the *Totality* stage of development. That is, the *linear constant* conceptualization implies thinking of slope as a *Process* and as an entity in itself; however the student may not be able to do *Actions* on this *Process*. The ability to do *Actions* on the *Process* is what distinguishes the *Object* from the *Totality* (Arnon et al., 2014).

At an *Object* stage, slope may be used for any of the previously described purposes in a new problem context or setting. While an algebra student may leverage their *Object* stage understanding of slope to a new real-world context such as finding juice concentrate, an advanced calculus student may use it to make sense of slope in three-dimensions.

Implications and Future Research Directions

The APOS-Slope framework provides a lens for interpreting past research while providing direction for future research. Descriptions of possible areas of future investigation follow.

Qualitative Synthesis

A complete qualitative synthesis of past research on slope in light of the APOS-Slope framework would highlight connections between past work describing slope understanding and coverage in the curriculum using a variety of conceptual frameworks. Revisiting this research with the APOS-Slope framework might allow for comparisons that are difficult when authors use different coding schemes and terminology.

Genetic Decomposition

In APOS, a genetic decomposition (GD) is a conjecture of mental constructions students may use in order to understand a specific mathematical concept (Arnon et al., 2014). The APOS-slope framework provides a basis from which to build and refine a GD since it provides detailed descriptions of possible thinking at the *Action*, *Process* and *Object* stages. In addition, because slope is a multi-faceted concept, the APOS-Slope framework describes *Action* stage reasoning where slope can be viewed as an *algebraic ratio*, *geometric ratio*, or *functional property*. Moreover, Nagle, Martínez-Planell and Moore-Russo (*in press*) describe several proposed transitional levels between each of these *Action* stages and a *Process* stage. An important next step is to engage students and teachers in task-based interviews based on the GD, to test and refine these descriptions and the GD itself.

A GD for slope informed by the APOS-Slope framework will provide a new means to analyze the coverage of slope in existing standards documents and curricular materials. One may investigate whether the constructions conjectured in the GD are adequately represented. A GD can also inform the design and implementation of instructional activities intended to help students progress through the GD, resulting in further analysis and refinement. These results can then be used to develop, test and refine an ACE (Activities, Classroom Discussion, Exercise) cycle (Arnon et al., 2014), resulting in pedagogical implications that can inform the teaching and learning of slope throughout the curriculum.

The APOS-Slope framework provides interpretive power to the body of past research on slope and has the potential to inform the direction of future research in the field. Whether used as a research lens or as a guide in curricular development and pedagogical decisions, the APOS-Slope framework has the potential to inform the teaching and learning of slope across the curriculum, from introductory algebra to differential calculus.

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A RADICAL CONSTRUCTIVIST MODEL OF TEACHERS' MATHEMATICAL LEARNING THROUGH STUDENT-TEACHER INTERACTION

Biyao Liang
University of Georgia
leungbiyao@uga.edu

I propose a model of student-teacher interaction with attention to teachers' construction and enactment of knowledge of students' mathematics. I adopt the theoretical perspective of radical constructivism to discuss how it enables us to simultaneously account for individual teacher cognition and the dynamics of student-teacher interaction. Specifically, I characterize a mechanism by which student-teacher interaction is generative to teachers' mathematical learning at a cognitive level. I close the paper by discussing the implications for research on teacher decentering and teacher knowledge, and the potential methodological challenges.

Keywords: Cognition, Social Interaction, Teacher Knowledge, Second-order Models

In the past three decades, many mathematics education researchers have built upon Shulman (1986) and proposed a variety of theories of teacher knowledge specific to mathematics teaching. Some researchers in this pursuit have adopted constructivist epistemology (Harel, 2008; Silverman & Thompson, 2008; Tallman, 2016), believing that teacher knowledge is individually-held, while others have followed the sociocultural perspectives, believing that teacher knowledge is collectively-held and distributed (Adler, 1998; Barwell, 2013; Hodgen, 2011; Putnam & Borko, 2000). There are also perspectives that are not based in an explicit epistemology. In such cases, researchers view teacher knowledge as being based in the work of mathematics teaching (i.e., practice-based theory; Ball, Thames, & Phelps, 2008; Fennema & Franke, 1992; Shulman, 1986). A common critique of the constructivist perspectives (or cognitive perspectives in general) on teacher knowledge is that they fail to account for teachers' knowledge sensitive to specific teaching situations (Barwell, 2013; Depaepe, Verschaffel, & Kelchtermans, 2013). Many radical constructivists have argued against these interpretations, claiming that social interaction is at the core of radical constructivism (e.g., Confrey, 1995; Steffe & Thompson, 2000a; Thompson, 2000, 2013, 2014; von Glasersfeld, 1995). The first goal of this paper is to extend these scholars' discussions to elaborate on how radical constructivism can simultaneously explain mathematics teacher cognition and student-teacher interaction.

The second goal of this paper is to unpack cognitive processes involved in teachers' learning of the mathematics of students through social interaction with them. Mathematics teacher educators have emphasized the importance of teachers' *learning through teaching* (Fennema & Franke, 1992; Leikin & Zazkis, 2010; Sherin, 2002; A. G. Thompson & Thompson, 1996). Underlying this proposal is the assumption that teachers' construction of knowledge is a constructive, dynamic, and adaptive process. While the sociocultural and practice-based theories emphasize the situative aspect of teacher knowledge, they do not explain the mechanism by which student-teacher interaction is generative to teachers' mathematical learning. Here, I discuss how radical constructivism can provide a theoretical tool for addressing this limitation.

First-, Second-, and Third-Order Modeling

A central tenet of radical constructivism (von Glasersfeld, 1995) is that *knowledge*, as the product of knowing, is not a representation of objective truth—rather, it functions and organizes viably within a knower's experience and is idiosyncratic to the knower. Under these assumptions, we have no access to anyone else's knowledge; at best, we can construct hypothetical *models* of others' knowledge that are viable with our observation of their behaviors (Steffe & Thompson, 2000b). Radical constructivists clearly distinguish the living experience of the knower herself from the *observer* who is trying to understand the knower's knowledge. This distinction leads to the notions of *first*- and *second-order models*. *First-order models* are "models the observed subject constructs to order, comprehend, and control his or her experience" (Steffe, von Glasersfeld, Richards, & Cobb, 1983, p. xvi)—i.e., the subject's knowledge that no one else can have access to. The first-order mathematics of teachers (or students) are the teachers' (or the students') mathematical *schemes* (i.e., cognitive entities that consist of a collection of mental actions that govern an individual's perception and activity; see Piaget & Inhelder (1968)) that exist in the teachers' (or the students') minds. Because teachers do not have access to their students' first-order mathematics, what we call teachers' knowledge of students' mathematics is at best the teachers' *second-order models* of the students' mathematics. *Second-order models* are, "[the models] observers may construct of the subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (Steffe et al., 1983, p. xvi). Teachers' second-order models are the teachers' inferences about the students' mathematical thinking that explain the students' observable actions. To be clear, second-order models are also schemes comprising the teachers' knowledge. There is no way for the teachers, the observers, to confirm if their interpretations match what the students, the subjects, are "actually" thinking. Similarly, the researchers who are attempting to infer a teacher's knowledge of the student's mathematics is engaging in constructing second-order models of the teacher's second-order models of the student' mathematics—i.e., *third-order models* of the student's mathematics. These models consist of the researchers' inferences about how the teacher may interpret her student's mathematics. Figure 1 illustrates these different types of models in the context of student-teacher interaction.

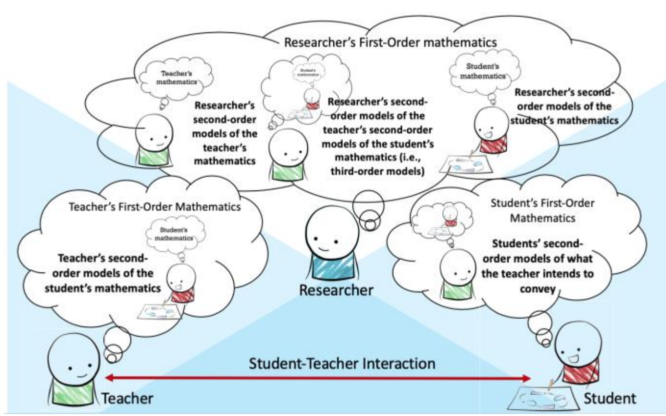

Figure 1: First-, Second-, and Third-Order Models

A Model of Mathematical Learning Through Student-Teacher Interaction

The aforementioned constructs provide the theoretical tools for unpacking the mechanism of student-teacher interaction and teachers' mathematical learning (Figure 2). This proposed model includes components of "teacher activity" (denoted in yellow boxes) and "student activity" (denoted in orange boxes).

I define *teacher actions* as a teacher's observable moves enacted for communicating with her student (e.g., talks, gestures). I limit my definition of *teacher knowledge* to a teacher's mathematical knowledge relevant to her social interaction with students, which includes, among others, the teacher's first-order mathematics and her second-order models of her students' mathematics constructed through *assimilation* and *accommodation* (von Glasersfeld, 1995) (see the bottom blue arrow in Figure 2, left). As the teacher negotiates between these different domains of knowledge, she decides what actions she needs to take to respond to the student (see the left "enact" arrow). Then, the student constructs second-order models of the teacher's meanings by assimilating or accommodating the teacher's actions (see the top blue arrow). The student further enacts her knowledge to respond to the teacher in actions (see the right "enact" arrow), which triggers assimilations and accommodations of the teacher (see the bottom blue arrow). The teacher may also reflect on her previous teaching actions to become aware of how her ways of interacting are consequential to the student's learning and how the student has interpreted her actions (see the left "reflect" arrow).

This interaction cycle continues as the student and the teacher reciprocally and continually assimilate the language and observable actions of the other and construct second-order models of the other's knowledge until (ideally) no accommodations are necessary for successful assimilation (i.e., intersubjective construction of knowledge in social interaction; see Steffe and Thompson (2000a)). In this process, both of the teacher and the student are engaged in interpreting each other's meanings, monitoring and anticipating each other's responses, and correspondingly adjusting second-order models of each other (P. W. Thompson, 2013).

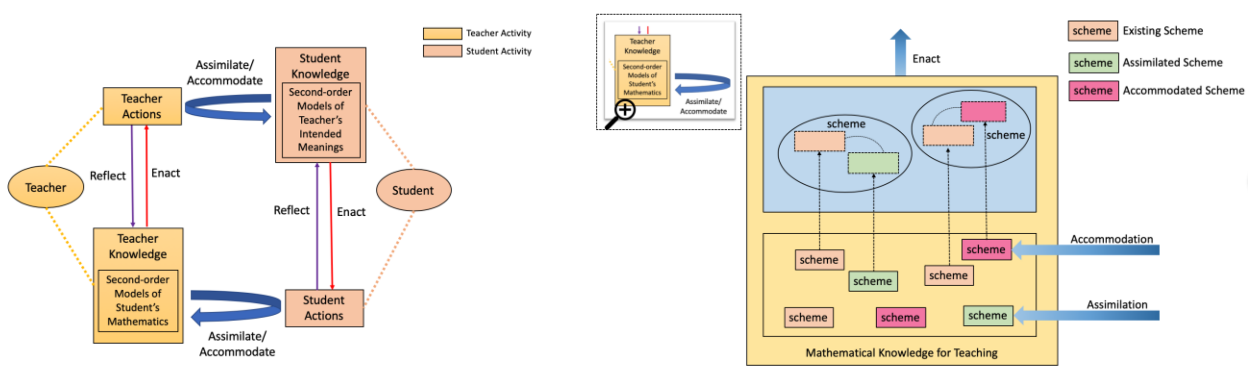

Figure 2: A Model of (left) Student-Teacher Interaction and (right) Teachers' Construction and Enactment of Knowledge of Students' Mathematics

Now I further unpack the mental processes involved in a teacher's construction and enactment of second-order models of students' mathematics during student-teacher interaction (Figure 2, right). I distinguish between three types of schemes of teachers. The first type of scheme (colored in orange) is the teacher's existing scheme prior to her interaction with the student. Constituting the teachers' first-order knowledge, these schemes consist of the teacher's personal mathematical knowledge, her intended mathematical meanings specific to the interaction, and her previously constructed, internalized second-order models of students' mathematics. These schemes are the source knowledge that the teacher leverages to interact with students. The second type of scheme (colored in green) is the scheme to which the teacher *assimilates* the student's actions, hence a subtype of the existing scheme. When the student's actions are similar to the actions that have been abstracted in the teacher's prior learning experiences or prior interaction with students, the teacher recognizes the student's actions by applying her existing schemes. The teacher does not make any significant modifications or reorganizations to her schemes because she does not abstract any novel mental actions from her observation of the student's activity. I note that because no two experiences can be exactly identical at different times and different contexts, assimilation is always accompanied by modest accommodation. The third type of scheme (colored in pink) is the scheme constructed through *accommodation*. When the student's actions generate some perturbations in the teacher, the teacher actively revises her existing schemes to account for the novel features of the student's actions. As a result, the teacher either includes new compositions into her schemes or restructures her scheme system.

As the teacher constructs either type of schemes, she may mentally operate on the different schemes to compare them, identify their relationships, or coordinate them (see the four upward arrows in Figure 2, right). These mental processes have two important outcomes. First, the teacher experiences reorganizations in her schemes and constructs more coherent schemes that consist of a network of related mental actions and operations. The teacher's updated system of schemes may influence her future assimilations and accommodations of the student's actions. Second, by being aware of how the student's ways of operating differ from her own or from the goal understandings, the teacher can determine how to act in ways that are hypothetically beneficial for engendering accommodations in the student.

I note that what types of mental processes come into play is wholly determined by the nature of the teacher's schemes and the teacher's personal ways of organizing and refining her mathematical scheme system in response to her experience of interacting with students—and thus can differ across individuals in face of similar student actions.

Implications for Teacher Education Research

This model characterizes a mechanism that teachers' construction of second-order models of students' mathematics is a process of teachers applying their current schemes to discern students' mathematics and, meanwhile, modifying and reorganizing their own mathematical schemes. This conceptualization can serve as an analytical tool for empirical research aiming at providing a fine-grained analysis of teachers' second-order models or the mental processes involved in their constructions of these models. Relevant findings can (1) help refine the proposed theory of teachers' mathematical learning in relation to student-teacher interaction (i.e., how student-teacher interaction is generative to teachers' mathematical knowledge), (2) provide explanatory power for teachers' decision making during interaction with students, and (3) inform mathematics teacher educators in terms of advancing teachers' sensitivity to students' thinking.

As I have described, a teacher's construction of mathematics of students is an ongoing process that involves continuing scheme assimilation and accommodation. A teacher's second-order models of her students' mathematics can continue to evolve throughout her instruction. I argue that considering the extent to which teachers accommodate their mathematical schemes is crucial to understanding their knowledge growth. I encourage researchers to focus their attention on revealing the constructive and dynamic nature of teachers' knowledge of students' mathematical thinking as it relates to student-teacher interaction. I acknowledge that documenting the evolution of teachers' knowledge of students' mathematics sensitive to their ongoing interaction with students creates methodological challenges. To generate fine-grained third-order models of students' mathematics, researchers may need to triangulate different data sources (e.g., teaching videos, interview videos, written reflections) to test and confirm their hypotheses of teachers' knowledge of students. We may also need to develop methods to document the in-the-moment mental processes involved in teachers' construction of mathematical knowledge of students during their teaching.

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TRANSLANGUAGING TO PERSEVERE: BRIDGING METHODOLOGICAL LENSES TO EXAMINE LATINX BILINGUAL STUDENTS' PROBLEM-SOLVING

Hector Morales, Jr.
Northeastern Illinois University
h-morales3@neiu.edu

Joseph DiNapoli
Montclair State University
dinapolij@montclair.edu

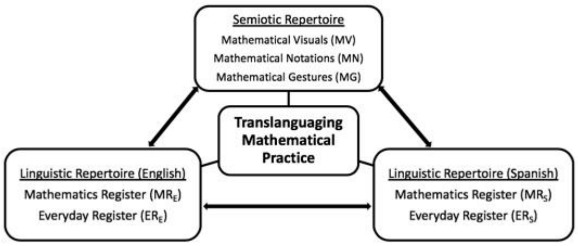
Translanguaging practice has been shown to be an effective resource to support collective perseverance and learning mathematics with understanding. We explore the methodological approach of analyzing the discursive translanguaging and perseverance practices of Latinx bilingual students simultaneously. We found that mapping dialogically tracked components of translanguaging practice to phases of perseverance revealed valuable learning insights.

Keywords: Translanguaging, Equity and Diversity, Classroom Discourse, Latinx Bilinguals

Dehumanizing school practices continue to marginalize the linguistic, social, and cultural capital for Latinx bilingual students in the United States. Research shows Latinx bilingual students can leverage their cultural and linguistic knowledge to construct, negotiate, and communicate about mathematics (Chval & Khisty, 2009; Garcia, Johnson, & Seltzer, 2017; Morales, Jr., 2004), but these resources remain widely ignored in schools that privilege the dominant school language (Langer-Osuna et al., 2016). Reform efforts have made strides toward access for all students to have supports necessary to persevere with challenging mathematics (NCTM, 2014); it is through perseverance in problem-solving that students can learn mathematics with understanding (DiNapoli, 2018). Yet, deficit perspectives on language and culture act as a barrier to knowledge and perpetuate ideas that bilingualism is an obstruction to learning rather than a facilitator (Garcia, 2017; Morales, Jr. & DiNapoli, 2018a). The field of mathematics education must find innovative ways to study how Latinx bilingual students use their bilingualism to learn mathematics with understanding. In this paper, we merge the perspectives of translanguaging and perseverance to posit an analytic method by which this process can be examined.

Translanguaing and Persevering to Learn Mathematics

We draw on translanguaging to reconceptualize bilingualism as a liberating and empowering communicative practice. Translanguaging can be a resource capable of transforming learning that goes beyond the transition to the dominant school language. Translanguaging is more than just a simple shift between two languages. It is a complex and interrelated communicative practice that make up bilinguals' linguistic repertoire (Cenoz, 2017). Garcia (2017) posits, "…speakers use their languaging, bodies, multimodal resources, tools and artifacts in dynamically entangled, interconnected and coordinated ways to make meaning" (p. 258). We extend Garcia's (2017) translanguaging practice framework to accommodate how bilingual students use the mathematics discourse including other linguistic repertoire such as the everyday register, mathematics register, mathematical artifacts (visuals and mathematical notations), and multimodal repertoire to make meanings as they persevere with challenging mathematical ideas (Moschkovich, 2000, 2015; O'Halloran, 2015). Within this Translanguaging Mathematical Practice (TMP) (Figure 1a), participants engaging within the mathematics discourse deploy fluid movement between mathematical and everyday speaking across Spanish and English by using everyday linguistic features and mathematics register resources in dialogically entangled ways with the intention to make meaning (Garcia-Mateus & Palmer, 2017; Morales, Jr. & DiNapoli, 2018b).



(a)

Three-Phase Perseverance Framework (3PP)

Entrance Phase	
Clarity (C-0)	Objectives were understood
Initial Obstacle (IO-0)	Solution pathway not immediately apparent
Initial Attempt Phase	
Initiated Effort (IE-1)	Engaged with task
Sustained Effort (SE-1)	Used problem-solving heuristics to explore task
Outcome of Effort (OE-1)	Made mathematical progress
Additional Attempt Phase (after perceived setback(s))	
Re-initiated Effort (IE-2)	Engaged with task
Re-sustained Effort (SE-2)	Used problem-solving heuristics to explore task
Outcome of Effort (OE-2)	Made mathematical progress

(b)

Figure 1: Frameworks for Translanguaging and Perseverance in a Mathematics Context

In the context of working on a challenging mathematical task, perseverance is initiating and sustaining in-the-moment productive struggle in the face of one or more obstacles, setbacks, or discouragements (DiNapoli, 2018). The Three-Phase Perseverance Framework (3PP) (Figure 1b) is an analytical perspective by which perseverance can be qualitatively described and measured (see DiNapoli, 2018 for detailed framework). The 3PP reflects perspectives of concept, problem-solving actions, self-regulation, and making and recognizing mathematical progress. It considers first if the task at hand warrants perseverance for students (Entrance Phase), considers next the ways in which students initiate and sustain productive struggle (Initial Attempt Phase), and considers last the ways in which students re-initiate and re-sustain productive struggle, if they encounter a perceived setback as a result of their initial attempt (Additional Attempt Phase).

Perseverance is vital for learning mathematics with understanding, so it is important for educators to encourage perseverance via their classroom practice. As such, NCTM (2014) advises for teaching practices that "consistently provide students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships" (p. 48). However, it is still unclear how perseverance might manifest in students working across two languages. An essential next step in this line of research is to consider the ways in which we recognize how Latinx bilingual students leverage their communicative and linguistic repertoires to build understandings (Martinez, Morales, & Aldana, 2017). Next, we describe an analytic process to recognize the ways in which Latinx bilingual students use translanguaging practice to persevere with challenging mathematics.

Recognizing Translanguaging to Persevere: A Methodology

The question of this study involves the nature of mathematical meaning constructions while Latinx bilinguals work collaboratively around a challenging task. Our analytic method consolidated, reduced, and interpreted collective dialogic interchanges to make explicit the interplay of translanguaging and perseverance experiences.

Tier 1: Coding for Translanguaging Mathematical Practice

The goal of this coding is to demonstrate the dialogic nature of the meaning making process. Bilingual students are afforded the possibility of engaging in a translanguaging practice. Students draw on their home and school languages increasing the potential for students to make connections between everyday and mathematical registers across both languages (Bose &

Clarkson, 2016). The main objective was to understand the meaning-making process and to identify the translanguaging resources that students made connections to as a result of this process. Video transcripts were coded so that we could identify which linguistic resources students used across both English and Spanish and non-verbal resources, including the everyday register (ER) and mathematics register (MR) in both English ($_E$) and Spanish ($_S$). Given the nature of communicating mathematically to negotiate meanings, we also identified moments when students transitioned to non-verbal mathematical representations that include visual-graphic (MV) and symbolic representations or notations (MN).

Tier 2: Coding for Perseverance in Problem-Solving

After coding the small group interaction data from the perspective of translanguaging practice, the same video transcripts were considered through a perseverance lens. The Three-Phase Perseverance Framework was used to capture the ways in students collectively persevered, or did not, during their engagement with a challenging mathematical task. The Entrance Phase captured whether the group of students understood the entirety of what a task was asking (C-0) and if the group immediately knew how to solve the problem (IO-0). The Initial Attempt Phase examined whether and how a group of students initiated (IE-1) and sustained (SE-1) their effort, and the outcome of such effort (OE-1) as they worked toward solving all parts of the problem. In the event the group did not solve the problem after making a first attempt, the Additional Attempt Phase aimed to capture if and how the students amended their original problem-solving plan (IE-2, SE-2), and the outcome of such efforts (OE-2) as they worked to overcome setbacks.

Tier 3: Overlay Analysis of Translanguaging to Persevere

We used a multi-tier code mapping process (Anfara, Jr., Brown, & Mangione, 2002) to identity and tag evidence of translanguaging and perseverance practice in the dialogic interchanges. Then, we conducted a constant comparative overlay analysis of both Tier 1 and Tier 2 coding (Glaser & Strauss, 1967), which facilitated the simultaneous analysis of Latinx bilingual students' translanguaging practice during key moments of their problem-solving when perseverance was necessary. Inductively, we analyzed the ways in which evidence of translanguaging practice occupied different phases and components of perseverance. Such analysis results in a theoretical sampling of patterns and themes that help describe the meaning-making actions in those moments of collective problem-solving.

Sample Analysis of Bilingual Students' Perseverance and Discussion

We present a brief example of our method to illustrate how a group of Latinx bilingual students leveraged their translanguaging practice to persevere with a mathematics task (Table 1). The task referenced an exponential relationship within *Alice in Wonderland*. The group had already demonstrated evidence of understanding the goal of the task (C-0) (to describe the mathematical relationship), but did not immediately know a solution pathway (IO-0).

Table 1: Sample Group Vignette with TMP and 3PP Coding

Transcript *(English Translation)*	**Translanguaging**	**Perseverance**
JESSICA: ¿Qué era la primera, se hace así? *(What was the first one, do you do it like this?)* If [*Alice*] eats one ounce, that means that she grows twice, dos ¿qué? *(two, what?)* Double, no double, two…See, so when two is four, and then three is six, and four is eight, y así, y así	**MR$_E$**: Describing and representing how height grows: "grows twice" **MR$_E$ & MR$_S$**: Dos-Double-Two, "Two is four, three is six, four is eight" **MG**: Graphical gesturing **MR$_S$**: Describes the shape verbally:	**IE-1**: Making sense of context **SE-1**: Exploring doubling

vamos hacer la graph *(like this, and this is how we are going to make the graph)*. Going like that (gesturing), para arriba *(up)*. You get it? **ELENA: Um hmm. Pero *(But)*, how to times it?**	"para arriba"	
JESSICA: Porque mira *(look)*, two, times two. Well no…Double it by, nomas *(just)* double the number of ounces, so if she takes… **ELENA: Two times two, y luego *(and then)* four times two, y luego *(and then)* six times two, is that what you are saying?**	**ER_S & MR_E**: Combines everyday and mathematics register: "Porque mira", "nomas double the number of ounces." **MR_E & ER_S**: Everyday and mathematics registers: "Two times two, y luego, four times times two…"	**IE-1**: Making sense of context **SE-1**: Exploring doubling
JESSICA: Más o menos como sumando el mismo número. *(More or less like adding the same number.)* **CARINA: Pero es lo mismo de sumando si lo multiplicas por dos. *(But it is the same as adding if you multiply by two.)***	**MR_S**: Representing doubling with operations: "como sumando el mismo númer"	**SE-1**: Exploring doubling **OE-1**: Better understanding doubling
INES: Lo que parece es como hicimos un in/out table y ya lo sacamos. *(It looks like we just did an in/out table and that's it).* In x 2 = Out In | x | 1 2 3 4 5 6 7 8 9 10 Out | y | 2 4 6 8 10 12 14 16 18 20 **CARINA: Yeah. In times two equal out… ¿Ya no tenemos que hacer su altura? *(We don't have to use her height?)***	**MV**: Tabular representation of the doubling relationship **MN**: Expressing the equation symbolically	**SE-1**: Revisiting representations of function **OE-1**: Recognizing interpretation mistake

This group initiated their effort (IE-1) toward differently expressing what doubling meant to them. Jessica used her linguistic and multimodal repertoire to express her understanding of doubling in a variety of ways: "grows twice," "dos," "double," "two," a sequence of number relationships, and graphical gesturing. Yet, the group did not immediately realize they were multiplying the number of ounces of cake by two instead of by *Alice's* height. Jessica and Elena dialogically moved between everyday and mathematics register (ER & MR) to explore the relation algebraically. Carina and Jessica leveraged the mathematics register in Spanish (MR_S) to agree that doubling is the same as adding the same number or multiplying it by two. They also represented the relationship symbolically as an equation (MN). Exploration of these points of view is evidence of sustaining their effort (SE-1) to make sense of the function. Their equation correctly spanned the table of values, yet did not model an exponential function. Ines helped her peers realize that they were doubling the number of ounces of cake instead of *Alice's* height (OE-1). In Spanish, Ines reminded the group that they must start with *Alice's* initial height. Ines used her mathematics register (MR_S) to describe how she doubled *Alice's* height as each ounce was eaten. Ines' revelation shows how perseverance can emerge from a group dynamic. This change in strategy marked the exit of the Initial Attempt Phase and entrance into the Additional Attempt Phase. We will depict the remainder of this group's efforts toward understanding in our session.

There is an ongoing demand to learn from the ways Latinx bilingual students communicate mathematically, yet it remains largely underserved in practice as teachers continue to have difficulty authentically engaging bilingual students within the mathematics discourse. We aim to apply our analytic method across contexts in which Latinx bilingual students are productively struggling to learn mathematics. If future research can map the specific ways bilingual students are using translanguaging to collectively persevere, we can help practitioners more progressively design learning environments more supportive of productive struggle across multilingual contexts. By prioritizing the TMP by which meanings can be formed, providing freedom to explore challenges, and lessening the focus on just learning vocabulary, teachers can better support Latinx bilinguals' perseverance toward understanding mathematics.

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SUPPOSE A SITUATION: WHAT FOUCAULT HELPS UNCOVER

Travis Weiland
Appalachian State University
weilandtj@appstate.edu

In this paper, I discuss an Archeological analysis of two high school mathematics textbook series in an effort to investigate how actions for the doing of statistics are formed by statements from the statistics lessons the texts. The objective of this paper is not to present all of the findings related to the posed question, but to focus on discussing the methodology employed (i.e. Foucauldian Discourse Analysis), which is relatively novel for analyzing textbooks in mathematics education. One finding will be presented to highlight how the methodology employed can contribute to the field helping to uncover patterns not found using well established deductive frameworks. This methodological discussion is meant to provide an example of how using different methodologies can help to investigate issues in the field and problematize taken for granted norms against a new horizon to embrace the sociopolitical turn.

Keywords: Research Methods, Curriculum Analysis, Data Analysis and Statistics

Objective

In spite of statistics' growing role in the mathematics curriculum and growing importance for citizenship there has been little investigation of the state of the teaching of statistics in K-12 mathematics classrooms (Eichler & Zapata-Cardona, 2016; Shaughnessy, 2007). In an effort to begin to investigate what experiences are made available to students with statistics in the school setting I investigated the research question: How are the actions for the doing of statistics formed by statements from the statistics lessons of two major high school mathematics textbook series? The objective of this paper is not to present all of findings related to the posed question. Instead I am going to focus on discussing the methodology employed (i.e. Foucauldian Discourse Analysis), which is relatively novel for analyzing textbooks in mathematics education and present one finding to highlight how the methodology employed can contribute to the field helping to uncover patterns not found using well established deductive frameworks.

Textbooks in Mathematics Education

In a recent survey of teachers 45% (±2.7) reported using instructional materials for 75% or more of the class time and 74% (±1.5) reported using their text to guide the overall structure and content emphasis of their instructional units (Banilower et al., 2013). This means that classroom texts may largely influence the experiences students might have with statistics in a high school mathematics classroom, particularly in the case of teaching statistics because mathematics teachers often have had little to no past experiences with statistics to draw upon to filter or change the tasks presented in meaningful ways (Shaughnessy, 2007). Up to this point there has been very little work focused on investigating the statistics content of school mathematics texts in the context of the U.S. (e.g. Bargagliotti, 2012; Jones et al., 2015; Jones & Jacobbe, 2014; Pickle, 2012; Tran, 2013). The dearth of research around the statistics lessons of mathematics texts is concerning given that statistics and mathematics are distinct disciplines in spite of statistic's location within the mathematics curriculum meaning there are differences in how it should be taught as well (Cobb & Moore, 1997; Franklin et al., 2007; Groth, 2007).

Theoretical Framework: Discourse

In education the term discourse is frequently linked to talk, language, or other methods of communication (Ryve, 2011). In this study, I draw from Foucault's perspective of discourse (Foucault, 1972); specifically his use of the term "as a regulated practice that accounts for a certain number of statements (Foucault, 1972, p. 80).' The regulated practice is often considered similar to a set of rules, which are generally taken for granted, and also constrain and "specify what is possible to speak, do, and even think, at a particular time" (Walshaw, 2007, p. 19). This makes discourses powerful, producing what is considered knowledge or truth regimes in particular historical, social, and political spaces (Walshaw, 2007). Discourses are important to study because they influence and shape how we see the world around us. They create structures that shape and frame who we are, who we can be, and how the world is perceived. These structures do not represent objective reality, but are socially created and subjective.

Related to the focus of this paper, I am investigating how the discourse of two high school mathematics textbook series, which are from a particular historical, social, and political space produce what is considered the actions for the doing of statistics for readers to take up. Related to textbook analysis in mathematics education, after extensive review I found only two other examples that employed Foucault's notion of discourse and methods; Hottinger (2016) and McBride (1989) both of which focused on how mathematics texts subjectify woman and construct mathematics as a masculine discipline.

Methodology

In this section, I discuss how the methodological considerations taken up in this study. In working towards this end let me first state explicitly the ontological and epistemological assumptions I drew from while conducting this work consistent with the theoretical perspective I discussed earlier drawing from Foucault. For the ontological perspective I reject the existence of knowable, objective, independent reality that individuals can access (Arribas-Ayllon & Walkerdine, 2008; Walshaw, 2007). Objective reality can never truly be known as it is filtered through discourse. Consistent with this ontological perspective I also situate this work in the epistemological where, knowledge or "truth" is viewed as socially constituted by discourses. Various "regimes of truth" are created through rules and regularities in statements in discourses that are historical and situated in context. Rooted in ontological and epistemological stances described I relied upon Foucault's (1972) methodology of Archeology in an effort to answer the research question I posed.

In archaeology, Foucault takes a historical approach to interrogate the "regimes of truth" formed by the rules or regularities of statements in discourse (Walshaw, 2007). In Archeology the unit of analysis is statements from the discourses being investigated and what constitutes a statement changes based on discourse being analyzed (Foucault, 1972). I chose to operationalize statements as sentences as every proper sentence contains at least one verb, which holds the potential to be an action formed for the doing of statistics by the discourse of the text. There is more to discourse than just statements, just as there is more to text than sentences. They may be the building blocks, but there are also rules that function to regulate how statements form and function, similar to grammar in language. These rules are referred to as discursive practices, which operate systematically to form and order objects. Foucault (1972) refers to the 'things' formed by discursive practices as discursive formations. It is regularities in the statements of textbooks that form the actions appropriate for the doing of statistics that are the focus of this study. For example, a statement like "calculate the mean and standard deviation of each data set," functions to position the actions of calculating mean and standard deviation of data sets as actions for the doing of statistics that students can then take up.

For this analysis, regularities in the statements were looked for both within and across lessons in each textbook series. The analysis was iterative through a series of readings of the data with different aspects of the data as the focus of each reading. I used a low bar of three occurrences of an action in the text to constitute a formation to give the benefit of the doubt to the text in terms of the actions formed.

Data Sources

I relied on findings from a survey of a nationally representative sample of science and mathematics teachers in U.S. schools (Banilower et al., 2013) to select texts for analysis. Houghton Mifflin Harcourt and Pearson made of the market share accounting for 35% (SE=1.6) and 30% (SE=2.0) respectively with their traditional Algebra I, Geometry, Algebra II sequence series. I extrapolated from these findings and selected textbook series that explicitly conformed to the CCSSM to make the findings more relevant, selecting the *Houghton Mifflin Harcourt Algebra 1, Geometry, Algebra 2* curriculum (Kanold, Burger, Dixon, Larson, & Lienwand, 2015) and the *Pearson Algebra 1, Geometry, Algebra 2 Common Core* curriculum (Randall et al., 2015). I analyzed all lessons that discussed topics from statistics.

Data Analysis

The analysis was iterative through a series of readings of the data with different aspects of the data as the focus of each reading. The first reading of the data consisted of identifying and recording all the sentences that indicated actions students are intended to perform. Selected statements were then placed in an excel spreadsheet for analysis. The second iteration consisted of taking the sentences identified as relevant statements during the first reading and summarizing the actions constituted in each statement. This process was carried out in the excel spreadsheet where the data from the first reading was recorded. The third iteration consisted of taking all the summarized actions for each statement from the previous iteration and putting them into a word document organized by lesson and then grouping the actions that were similar by lesson. I treated any group that had three or more statements in it as a regularity or action constituted by the discourse of the text. I considered such regularities as a discursive formation consistent with what I described earlier in terms of the methodology of Archaeology (Foucault, 1972). For the forth iteration, I took all the actions formed that met the threshold for being considered a regularity within a lesson into a document and then placed all the statements that did not make the cutoff for being considered regularities within a lesson into a separate document for each textbook. I then looked for regularities that occurred across the lessons that did not come out from the analysis within the lessons. Any regularity found across lessons was then added to those found within lessons in a document.

Findings: Uncovering What Might Be

From the overall analysis I found that the predominant actions formed were associated with analyzing data, which is a narrow view of statistics and how it should be taught. This is a result that has been found by others (i.e. Bargagliotti, 2012; Jones et al., 2015) using deductive frameworks based on the GAISE framework (Franklin et al., 2007). Therefore, the inductive analysis performed supports findings from others using deductive lenses, which helps to strengthen the warrant of those lenses. However, rather than going into detail on the findings that could have been found using deductive frames I will focus on a specific finding that was uncovered that would not have been uncovered by commonly used deductive frames to highlight the use of the methodological approach I took.

The action formation of *supposing a situation* appears frequently in statements throughout lessons in all of the textbooks analyzed. Statements indicative of this action formation typically begin with "suppose …" and go on to describe some sort of fictional situation that students were being given to consider. For example, consider the following example of a series of statements from the text:

> Suppose you asked, "Do you work out every day like a healthy person, or are you a lazy couch potato who only works out once in a while?" Do you think the results of your survey would change? Explain your reasoning. (Randall et al., 2015, p. 752).

Supposing a situation was the most prevalent action formed by the discourse of the textbooks. This action is not specific to dealing with data as this action can be used in any case where students are asked to consider a fictional situation or to take information for granted that is given. One of the most common settings it was formed in was probability settings. For example, consider the following excerpt of statements from the texts:

> 1. A bag contains 12 red and 8 blue chips. Two chips are separately drawn at random from the bag. Suppose that a single chip is drawn at random from the bag. Find the probability that the chip is red and the probability that the chip is blue. Suppose that two chips are separately drawn at random from the bag and that the first chip is returned to the bag before the second chip is drawn. Find the probability that the second chip drawn is blue given the first chip drawn was red. (Kanold et al., 2015, p. 743)

It is not inherently problematic that the discourse of the texts position students to suppose they are in a specific situation or performing a particular task. The problem lies in that students are positioned this way so often. If students are always supposing situations when are they ever investigating actual situations from their own world and reality? In other words how does having students predominant supposing situations when are they being positioned to read and write the word and the world with mathematics (Gutstein, 2006) with statistics (Weiland, 2017)?

Scholarly Significance

The main goal of this paper is to elaborate on how drawing methodologically from Foucault can help uncover realities that may not be visible from using well developed deductive frameworks. Though it is good to have deductive frameworks to help structure analyses and potentially frame what the field values, it is also good to employ inductive analyses to potentially uncover realities that may be hidden to more traditional and structured lenses. Foucault's perspectives are also powerful for interrogating and problematizing the taken for granted values and practices of the field of mathematics education that deductive analysis can be blind to (Kollosche, 2016). Furthermore, using Foucault and the epistemological stance that comes with it helps to make visible the power that textbooks hold in shaping the reality students experience and what is considered knowledge. The prevalence of the discursive formation of "suppose a situation" would not have been laid bare by the frameworks used in previous analyses of statistics lesson in mathematics texts, nor have I found this discussed more broadly in the mathematics education literature. At the same time I hypothesis that this formation is not surprising to anyone reading this who is familiar with mathematics texts. That is the power of this type of analysis is to interrogate taken for granted regimes of truth. As the field begins to realize the sociopolitical turn in mathematics education (Gutiérrez, 2013) it is important more work acknowledges the power that exists in discourse to construct our ways of knowing and being in the world and how they shape how and what people learn as mathematics.

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MODELING RESEARCHER DISCOURSE WITH ANALYTIC PRAGMATISM

Theodore M. Savich
Indiana University
tmsavich@indiana.edu

Erik Jacobson
Indiana University
erdajaco@indiana.edu

Pavneet Bharaj
Indiana University
pkbharaj@iu.edu

Ayfer Eker
Giresun University
ayfer.eker@giresun.edu.tr

In this theoretical paper, we introduce analytic pragmatism (Brandom, 2008) as a paradigm that allows for the simultaneous discussion of findings from different theoretical traditions. We illustrate the use of this paradigm by examining literature about teaching and learning fractions. This novel approach is particularly well suited for the body of fraction research because in this domain the same terms are used differently across traditions, historical periods, and discursive levels. Moreover, different terms are sometimes used in the same way. We assume that the mathematical activity of learners is stable and, using analytic pragmatism (Brandom, 2008), we illustrate how analytic pragmatism can be used synthesize research across time, tradition, and discursive level. We discuss the utility of our approach for the field of mathematics education.

Keywords: Literature review, Analytic pragmatism, Metatheory

Objectives

While conducting a comprehensive literature review of articles focused on fractions and decimals from forty years of research, we discerned ontologically incompatible frameworks that referred to the same phenomena (disagreements) and ontologically compatible claims that argued a particular phenomenon was more nuanced than was previously understood (refinements). In order to bring disagreeing theorists into conversation with one another, we adopted the ontologically agnostic perspective of analytic pragmatism (Brandom, 2008), which splits discourse into two necessarily related components: meaning and use. We argue that Meaning Use Analysis (MUA) can precisely locate disagreements and refinements.

Because the MUA metatheoretic construct is not in common use in mathematics education, we begin by introducing some terminology used in the subsequent analyses. Then we briefly review literature that provides different accounts of students' initial use of (improper) fractions. We follow by applying the MUA framework to this case to locate disagreements and refinements in the literature. We conclude by discussing implications of the framework and method of analysis for the field of mathematics education research.

Theoretical Framework: Analytic Pragmatism, Meaning, and Use

The broad project of analysis, not only in the tradition of analytic philosophy (Brandom, 2008) but also in scientific efforts in general (Habermas, 1971), can be conceptualized as relationships between *vocabularies*. We often talk about understanding a phenomenon as the ability to explain the phenomenon using a different *vocabulary* taken to be more fundamental. For us, vocabularies are subsets of all the words in a language (i.e., English) over which semantic relationships can be defined, including the syntactic components of any associated non-verbal representations (e.g., reading number lines). Brandom's focus is on reducing exceptionally broad types of vocabularies to a single *normative* vocabulary of *commitments* and *entitlements*. We adopt Brandom's approach to study the acquisition of much narrower, math-specific vocabularies, e.g. "fractions," and place no emphasis on category reduction. However, the language of *commitments* and *entitlements* is broadly applicable to the project of mathematics education. Commitments are assertions like "$\frac{1}{2} < \frac{3}{4}$", but we can program computers to spit out that string or train parrots to vocalize that assertion. What makes those assertions become more than marks on a page or vibrations in the air is that we can ask "what must one be able to do to count as saying '$\frac{1}{2} < \frac{3}{4}$'", that is, "what entitles someone to the commitment that $\frac{1}{2} < \frac{3}{4}$". These entitlements are also assertions, like "$\frac{1}{2} < \frac{3}{4}$ because $\frac{2}{4} < \frac{3}{4}$," but they are grounded in *practices-or-abilities* (defined below). As math educators, we find particular value in the notion that one must be able to do one thing to count as saying something else. Adopting the language of commitments and entitlements could make the injunction to "show your work" less opaque in K-12 situations and could possibly lead to more satisfying responses in teacher education where the form of the question is to "explain your answer". That is, teachers and teacher educators are already in the practice of asking learners to express both their commitments and entitlements. In the present paper, we take it as given that some kind of doing is necessary for a mathematical utterance to count as a commitment (i.e., to have meaning).

What makes analytic pragmatism unique and powerful is that vocabularies are treated as separable from the *practices-or-abilities* that give them meaning. It is common to think of word meanings as the kinds of mappings from one vocabulary onto another vocabulary, like what one finds in a dictionary, where a word is associated with a set of words in direct Vocabulary-Vocabulary (VV) relationships. The point of analytic pragmatism is to break out of direct VV relationships and instead consider meaning as mediated by use. Consequently, we are concerned with VV relationships as mediated by sets of *practices-or-abilities*, and we define meaning as coordinated VPV relationships.

Practices-or-abilities (P) are intended not as observable behavior but rather in a discursive sense as the repeatable rules (algorithms) that specify how to use a particular vocabulary. Brandom uses "practices-or-abilities" because analytic pragmatism is ontologically agnostic and seeks to be able to describe practices like "creating common denominators" alongside abilities like "mentally rotating an isosceles triangle", and thus we argue that both positivist and constructivist traditions (as well as others) can be modeled within this paradigm. For us, *algorithms* include mathematical processes such as a standard computational algorithm but can also refer to mental schemes (e.g., Steffe, 1983). In the MUA paradigm, algorithms need not be standardized across individuals or even consistent for an individual; algorithms merely refer to repeatable discursive rules that, by their discursive nature, are specified by a higher-order vocabulary. The vocabularies that describe the algorithms, which constitute practices-or-abilities mediating VV relationships, are called *pragmatic metavocabularies* (henceforth *metavocabularies*). Linking back to our objectives, we can now say that disagreeing theorists use competing metavocabularies in their descriptions of the same practices-or-abilities, whereas refinements occur as researchers critique whether or not a metavocabulary is sufficient to precisely describe the practices-or-abilities associated with a specific domain.

Literature Review

We focus on frequently-cited literature in the well-researched domain of fractions so readers are more likely to be familiar with the issues, and thus might better understand the relevance of MUA. It is important to clearly state that this is but one case, and a similar analysis would be as

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

fruitfully applied in any area of mathematics education that has been addressed in more than one time and from more than one perspective.

Children learn about fractions over a long period of time and with widely differing outcomes in terms of the mathematical knowledge they develop. The issue of how knowledge of fractions is related to knowledge of whole number is a central theme of research in this area, and we summarize three accounts of students' initial use of fractions. The *interference hypothesis* (e.g. Behr, Wachsmuth, Post, & Lesh, 1984) posits that whole number knowledge is thought to interfere with developing knowledge of fractions, and for example, students inappropriately treat the digits of fractions as if they were whole numbers. In disagreement, Steffe (2002) argues with the reorganization hypothesis that whole number knowledge is a resource that is recruited and reorganized to serve fraction knowledge. Whereas Steffe (2002) argued the splitting operation is sufficient for making sense of improper fractions from whole number knowledge, Hackenberg (2007) provided a refinement by identifying the role of coordinating three levels of units.

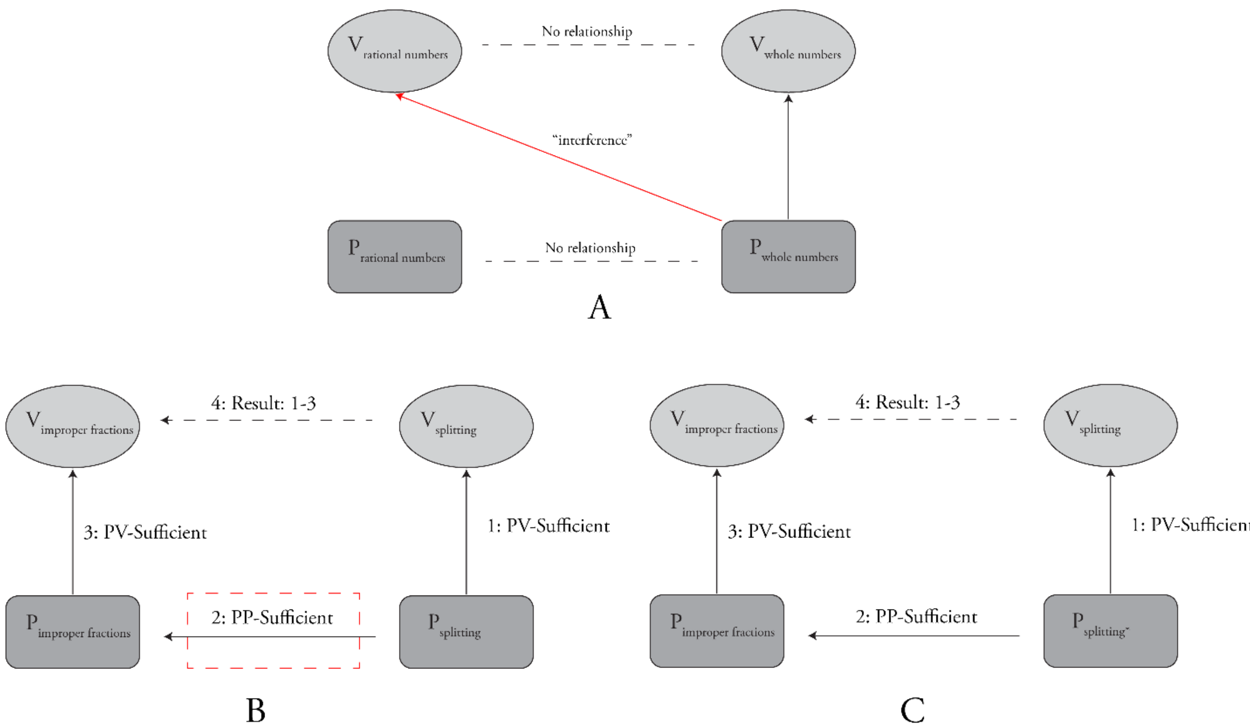

Figure 1: (A) The *Interference Hypothesis* Takes Different Mathematical Vocabularies to be Independent of Each Other; (B) Steffe (2002)Argues that Splitting is Sufficient to Deploy Improper Fractions; (C) Hackenberg (2007) Argues that Splitting Together with the Ability to Doordinate Three Levels of Units (denoted *splitting), is Sufficient to Deploy Improper Fractions**

Results

The results of this project are communicated in Figure 1. Analytic pragmatism allows for a rearticulation of these core perspectives in mathematics education. The interference hypothesis essentially amounts to a claim that children have to develop a new rational number vocabulary, that is independent of their whole number vocabulary. When children deploy a rational number vocabulary under the rules (i.e., practices-or-abilities) associated with their whole number vocabulary, Behr et al. claim that the whole number vocabulary is interfering with the rational number vocabulary. The constructivist alternative to this story is that the practices-or-abilities associated with earlier stages can be elaborated into new practices-or-abilities. In particular, Steffe argues that once learners have established the ability to split, they can deploy a fraction vocabulary to answer questions like "Here's a picture of my stick, which is five times the length of yours. Can you make your stick?" then they have essentially constructed improper fractions, which is to say that they can meaningfully deploy an improper fraction vocabulary. Hackenberg refines Steffe's claim by arguing that the splitting operation that Steffe describes is not sufficient to deploy an improper fraction vocabulary. In terms of MUA, this is taking issue with the second relationship in Figure 1B. Hackenberg argues that a new splitting operation that includes the ability to coordinate three levels of units (denoted *splitting** in Figure 1C) is sufficient to deploy an improper fraction vocabulary. Thus, we demonstrate that MUA allows for the simultaneous discussion of results from traditions that have different ontological commitments, allowing for disagreements and refinements to be located relative to each other with precision.

Discussion

There are two potential contributions of this work. First, reading math education research literature is a high-inference activity because each researcher can approach related phenomena from very different theoretical perspectives. MUA explicates those inferences, which allows readers to lay the arguments of each author side by side. Second, because analytic pragmatism is ontologically agnostic, it is an appropriate tool for synthesizing research across traditions. By looking at the discursive structure of the research reports, disagreements and refinements are readily apparent. Because MUA operates on the most basic elements—words and meaning—no unifying theory is required to bring disparate perspectives into conversation. In a field as rich in theory as is mathematics education, MUA has a great deal of promise.

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METAPHORS BEHIND THE CONCEPT OF TANGENCY

Francisco Sepúlveda
CINVESTAV, IPN, México
francisco.sepulveda@cinvestav.mx

Although a group of students already has advanced knowledge of differential calculus, the first definition they give of a tangent line to a curve remains influenced by the tangent line to a circle. A diachronic analysis on tangency through the conceptual metaphor framework shows how metaphors determine mathematical procedures and definitions and help to understand the possible cognitive obstacles presents on the concept of a tangent line.

Keywords: Precalculus, Advanced Mathematical Thinking, Communication, Geometry and Geometrical and Spatial Thinking

While the word “tangent” comes from the Latin tangere and means ‘to touch’, the word “secant” comes from the Latin secare and means ‘to cut’. As many words, both have a metaphorical etymology that shows, from a modern perspective (e.g., Lackoff & Núñez, 2000), how those terms are cognitively conceived. Many authors (e.g., Font, Bolite & Acevedo, 2010; Talmy, 2000) assert that there is an object metaphor and a fictive-motion phenomenon that pervade conceptions of the tangent and of the secant line. Even though, taking a synchronic approach, those words are not really metaphorical, fictive-motion remains one of the indispensables approaches to make sense of those terms. ¿How metaphors are intrinsically related to modern conceptions of the tangent line?

Methodology

From a reputed Mathematics Education Master in Mexico, we choose seven students and future high school professors. They answered a brief inquiry, from which we expose only two questions. On the first question teachers are asked to write on their own words the definition of a tangent line to a curve on a point P. On the second question they are asked to plot a cubic function on Geogebra and approximate the tangent line on a point P to the curve through a secant line PP’. The abscissa of P’ depends on a slider that goes through an interval between 0.001 and 0.1, with a 0.001 step. This slider permits to simulate the limit concept transforming a hand movement on a displacement of the point P’ to P approximating the tangent line by secants lines. Then students are asked to put the slider on its minimum, and, because the 0.001 value of the slider gives the line the appearance of a tangent to the curve on P, declare if the obtained line is a tangent or not, and justify the given answer.

Results

Five of the seven students mention geometrical properties of a tangent line to a circle in defining a tangent line on a point to a curve, as, for example, “a line that touches only in one point to the curve”, “a line that does not traverse the curve” or “a perpendicular line to the curve on a single point”. Only three future teachers use in their definition the calculus paradigm mentioning the limit of secant lines or the derivative number. On the second question, when they are asked if the line that seems to be the limit of secant lines is a tangent to the curve on point P, they all answer correctly that this is not a tangent. However, of the seven future teachers, four justify their answer by arguing that the line obtained touches more than at one point of the curve, one responds with ambiguity writing that the line obtained "tends to be the tangent line" and only one teacher uses the word "converges" avoiding in this way the argument of the double intersection.

Discussion

A chronological approach to the tangent line shows how the Euclidean Elements influenced Descartes to obtain the algebraic equation of a tangent to a polynomial curve. If the Descartes method is analyzed from a conceptual metaphor frame, then some interesting facts come to the front. Figure 1 represents the simplest method to trace a tangent to a circle, given that the tangent in P to the circle of center A is perpendicular to the radius AP.

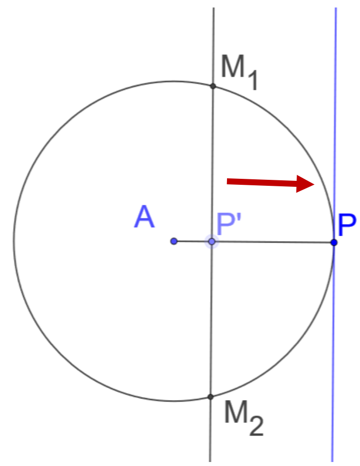


Figure 1: The Trace of a Tangent Line to a Circle by a Perpendicular Displacement of Line M_1M_2

The process in Figure 1 starts with the secant line perpendicular to the radius in P'. This line cuts the circle on two points M_1 and M_2. Then this line is displaced until reach the border of the circle on P. As in this figure, Euclides defined the tangent line through its perpendicularity to the radius, so when Descartes looked for an algebraic equation of the tangent to a curve his first thought was to search for a circle with a radius perpendicular to that curve. Part of Descartes's process is described below where C and E are the two intersections of the supposed circle with the curve.

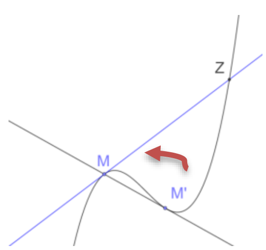
> The more those two points C and E are close to each other, the less difference there is between its two roots; and finally, they are entirely equal, if they are both joined, that is to say if the circle passing through C touches at C the curve CE without cutting it. (own translation from Descartes, 1637, p. 37)

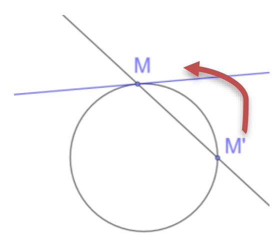
His method is not more used, but the circle that Descartes proposed is called today an osculating circle, or, looking for the initial metaphorical romantic meaning of the verb osculate, a "kissing circle". While this metaphor is not transcendent on the process used by Descartes, three other metaphors are very important on his strategy that come from trying to replicate the method described in Figure 1 on a curve. The first metaphor is the perpendicularity to the radius of a circle replicated on a curve, the second is the fictive motion of the points M_1 and M_2 that approximate to each other, and the third is the coexistence of these two points at the tangency point reinforced algebraically by the double root. Those three metaphors are intrinsically related on the Descartes method, and combined show his cognitive underlying process.

This historical chronology echoes on the mathematics curriculum by introducing first the tangent line to a circle. But, the tangent line to a curve results later from another modern and effective definition of a tangent line to a curve based on the derivative number. In fact, in some curriculums, the derivative of a function at a given point M is introduced as the limit of the slopes of lines passing through M and another point M’ of the curve when M’ tends to M. This process is illustrated in Table 1.

The relationship between tangent and derivative can generate a logical problem representing an obstacle to students (Vivier, 2010). From a linguistic perspective both terms, tangent and secant, are opposed, but in calculus the way to define the tangent line at a given point is as the limit of secants passing through the point. Conceptual metaphors of the calculus paradigm that help to understand the limit concept, and so the tangent concept, are those of collapse and approximation (Oehrtman, 2009) that are related to the Basic Metaphor of Infinity proposed by Lakoff & Núñez (2000). However, as our enquiry and some other researchers supports (e.g., Canul, Dolores, & Martínez-Sierra, 2011; Vinner, 1991; Vivier, 2010), the students’ mental image of the tangent line still influenced by the circle. Let’s analyze on Table 1 which metaphors are behind the calculus paradigm in defining a tangent line.

Table 1: Fictive Motion and the Limit of Secants on a Curve and a Circle

<table>
<tr><th>Limit of secants on a curve</th><th>Limit of secants on a circle</th></tr>
<tr><td>


On the calculus paradigm, to introduce the tangent line to a curve we use the fictive motion metaphor of a point M’ that approximates a fixed-point M. The process could also be understood as a sequence of secant lines that in some point (the limit) becomes the tangent.</td>
<td>


Analogous, this situation can be considered as a fixed-point M and a displacing point M’ that approximates M creating a sequence of secant lines; or as a fixed secant line that rotates around M until it becomes a tangent line.</td></tr>
</table>

This table, in contraposition with Figure 1, helps to understand the cognitive problems behind the conceptualization of the tangent line with a limit paradigm. First, the perpendicular property that was used by Descartes to obtain the tangent line is completely lost. Second, the coexistence of two points on the same place is one impossible phenomenon from the particle’s perspective adopted by Newton and Leibnitz. Moreover, in some cases, the tangent line obtained in one point could remain a secant to the curve (as seen in the left column of Table 1) creating a cognitive paradox and detonating the necessity of a local vision that was not present in the case of the circle.

Finally, a very different fictive-motion metaphor can help to conceptualize a tangent line, not present in the Descartes exposed method neither on the calculus paradigm. That is to interpret the tangent line as the trace of a particle that describe a straight trajectory and “graze” the curve on one point. This interpretation is closer to the original etymology of the word ‘tangent’ and has an algebraic correspondent methodology proposed later by Descartes that consist of searching the equation of a line that intersects twice on the same point with a curve. This approach is closer to the definition of a tangent line as the best linear approximation of a curve on the proximity of a point. Some researchers argue that this definition is the best of a tangent line on mathematics education (De la Rosa, 2009).

Conclusions

A conceptual metaphor approach to the tangent line has served to expose how some of the cognitive process involved on its original conceptualization are lost on the traditional teaching. On the one hand, this historical perspective confirms the power of conceptual metaphors lying in big areas of mathematics to influence procedures and definitions. On the other hand, all the long this article we described different types of motion that can be used to explain properties of the tangent line, accordingly to the sensible approach proposed by Tall (2011, p. 53). As said in the introduction, motion is inherent to the conceptualization of the tangent, however, that does not imply that there is only one cognitive way in which this concept could be introduced. In fact, it is no argued that teachers must use one or other metaphors in classes, but that they may consider a conceptual metaphor approach to some classical delicate concepts before teaching, in order to go deeper and better understand the cognitive processes that were necessary on their conception.

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METAFORAS DETRAS DEL CONCEPTO DE TANGENCIA

A pesar de que un grupo de estudiantes goza de conocimientos avanzados de cálculo diferencial, la primera definición que dan de una recta tangente a una curva permanece influenciada por la recta tangente a un círculo. Un análisis diacrónico de la tangencia a través del marco de la metáfora conceptual muestra cómo las metáforas determinan los procedimiento matemáticos y las definiciones y ayuda a comprender los posibles obstáculos cognitivos presentes en el concepto de recta tangente.

EXPLORING QUALITIES OF A COMMUNITY OF INQUIRY IN A SYNCHRONOUS ONLINE COURSE

Jeffrey Choppin
University of Rochester
jchoppin@warner.rochester.edu

Julie Amador
University of Idaho
jamador@uidaho.edu

Cynthia Callard
University of Rochester
ccallard@warner.rochester.edu

Cynthia Carson
University of Rochester
ccarson@warner.rochester.edu

We studied an online course designed to provide rural mathematics teachers access to high quality professional development. The online course emphasized productive discussions related to high-leverage discourse practices. We applied a community of inquiry framework, which emphasizes deep intellectual work, and its three tenets: cognitive presence, social presence, and teaching presence. In this paper we focus on cognitive presence. A major contribution of our study is the use of mediating processes to characterize cognitive presence; this allowed us to conceptualize the interaction between the three presences (social, teaching, and cognitive) and their connections to our design conjectures.

Online platforms and learning environments are emerging as important contexts for providing opportunities for teachers to engage in professional development (Johnson et al., 2018; Keengwe & Kang, 2012; Means et al., 2009), and thus as sites of research. We argue that *synchronous* learning environments have the potential to provide impactful professional learning experiences for teachers and to incorporate many important features of face-to-face models, but there needs to be an increased focus on the methods to do so. In this study, we explore one component of a three-part online professional development model for middle grades mathematics teachers in rural contexts. We apply a community of inquiry framework (Garrison, Anderson, & Archer, 2000) because of its assumptions about engaging participants in demanding intellectual work, and connect it to *design conjectures* (Sandoval, 2014). Specifically, the mediating process component of design conjectures provides a way to characterize and analyze the cognitive presence in a community of inquiry, which has been an outstanding methodological problem in studying online contexts (Akyol & Garrison, 2011).

Online Professional Development Modules

We designed a three-part online professional development model with the goal of providing rural mathematics teachers access to high quality professional development. Our project utilized a series of synchronous online experiences, including online course modules, teaching labs (akin to demonstration or fishbowl lessons), and online video coaching. The online course modules emphasized discourse practices that facilitate mathematically productive classroom discussions (Smith & Stein, 2011), including anticipating, monitoring, selecting, sequencing, and connecting. The modules also emphasize key aspects of lesson planning, such as goal-setting, in addition to having teachers solve and discuss high-cognitive demand tasks. The modules involved a combination of synchronous and asynchronous work, in order to minimize the amount of time teachers virtually met together (Robinson, Kilgore, & Warren, 2017). We conducted the OMD course in Zoom, which allowed for synchronous whole class and small group interactions. In

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

addition, we simultaneously used Google docs and Google draw, which allowed participants to collectively develop and share artifacts, including approaches to mathematical problems. The instructor presented challenging tasks to the participants, who then worked in virtual breakout rooms to create a document in a common workspace that they shared with the other groups.

Framework

We draw primarily from the community of inquiry framework (Garrison, Anderson, & Archer, 2000), that identifies three components of online learning environments: cognitive presence, social presence, and teaching presence. Garrison and Cleveland-Innes (2005) define a community of inquiry in terms of deep learning that extends beyond simple interactions, stating that a community of inquiry is a place where "ideas can be explored and critiqued; and where the process of critical inquiry can be scaffolded and modeled" (p. 134).

The Three Presences

While not a part of this current analysis, the *social presence* component focuses on the ability of participants to project themselves to establish personal and purposeful relationships (Garrison, Anderson, & Archer, 2000). Rourke, Anderson, Garrison, and Archer (2001) state that the three main components of social presence are affective responses, interactive responses, and cohesive responses. *Teaching presence* defines multiple roles for the teacher, including designer, facilitator, and subject matter expert who provides direct instruction (Anderson, Rourke, Garrison, & Archer, 2001).

The focus of this analysis is on the *cognitive presence*, described as "the extent to which learners are able to construct and confirm meaning through sustained reflection and discourse in a critical community of inquiry" (Garrison, Anderson, & Archer, 2001, p. 11). Garrison (2007) adds that cognitive presence involves the processes of "the exploration, construction, resolution and confirmation of understanding" (p. 65). Akyol and Garrison (2011) elaborate these processes by listing four phases of cognitive presence in online environments: a *triggering event* involving a well-designed activity; *exploration* in which participants seek out relevant information and explanations; *integration*, in which there is a focus on meaning making and integration of ideas; and *resolution*, in which ideas are evaluated. We focus on these aspects to explore the extent to which our participants engaged in meaning making and reflection in an online context.

Using mediating processes to characterize cognitive presence. Because cognitive presence has been harder to define than the other two presences (Akyol & Garrison, 2011), we turn to an aspect of design research, *conjecture maps*, to articulate how cognitive presence can be operationalized analytically. According to Sandoval (2014), "*Conjecture mapping* is a means of specifying theoretically salient features of a learning environment design and mapping out how they are predicted to work together to produce desired outcomes" (p. 19) and is intended to reify the conjectures regarding the learning environment and how they interact to promote learning. There are four main elements to a conjecture map. The first element involves *high-level conjectures* about the learning context and how it supports learning. Those conjectures are then operationalized in the *embodiment* of the learning design, which is the second element. In the third element, this embodiment in turn is intended to generate *mediating processes* that produce desired *outcomes.* To analyze cognitive presence, we turn to the third element, the mediating processes.

Mediating processes focus on the interactions between learners and the learning environment. These interactions can be in the moment and observable or are evident in artifacts that reify learners' engagement in the learning environment. Mediating processes represent the intellectual practices in which we want participants to engage in the online learning context. For example, one mediating process is for participants to explore and discuss the cognitive demand of tasks (i.e. Smith & Stein, 2011); the extent to which participants are able to explain and justify their reasoning is an indicator of cognitive presence. Because mediating processes reflect the immediate aims of the design of the learning environment, they provide a strong articulation of the cognitive presence designers hope to engender.

Our research question was: What were the qualities of participants' cognitive presence, as signaled by an analysis of the mediating processes in which they engaged?

Methods

We video recorded each of the OMD sessions, six in total, using screen capture technology. For each session, we video recorded the host computer as well as each of the breakout rooms, creating up to five video files for each session. We used Panopto so that we could record the Zoom window and simultaneously any Google docs the groups were creating.

We parsed each video into clips, labeling each clip according to whether it was an introduction (e.g., launch, agenda-setting, directions), exploration phase (break-out room in which participants conducted an activity in small groups), or summary discussion (post-breakout room whole-class discussion). We delineated clips by changes in activity structure (e.g., from introduction to breakout room, breakout room to whole group summary, and so on). We then transcribed the clips for the breakout rooms (one for each breakout room) and summary discussions using Transana video analysis software.

Analyzing cognitive presence. For each clip, we created a list of mediating process claims. Subsequently, we sorted the clips into categories we developed according to our design conjectures and emergent categories after an initial pass of the data. Based on our work developing conjecture maps for the OMD course, the categories that we attended to in this analysis include *pedagogical insights*, *instructional practices*, *accounts of instructional episodes*, *analysis of tasks,* and *problem solving strategies*. These categories involved participants engaged in meaningful learning through exploration, joint construction of explanations, and resolution of ambiguities and approaches with the assistance of the instructor. We then tabulated the instances of mediating claims in those categories to understand the content involved in the cognitive presence. However, this did not indicate the extent to which cognitive presence went beyond basic participation in the activity (e.g., rating the cognitive demand of a task) to include more complex forms of participation (e.g., justifying cognitive demand of tasks by explaining features that relate to given levels of demand). To capture the quality of cognitive presence, we sought instances in which participants went beyond low-level types of participation, such as reporting a fact or procedure, providing an opinion, or evaluating a peer's response. These types of participation can be accomplished easily and were not the goal of the project, though we recognize that much of the time, these are how people contribute to conversations. Thus, we coded a participant's contribution as *high cognitive presence* if there was an explanation in which the participant made connections across topics and provided evidence for those connections.

Results

Participants engaged in the mediating processes identified in our conjecture mapping process in a large number of instances, at least at a superficial level. To capture the most robust instances of cognitive presence, we place more weight on the third column in Table 3, which represents the strongest evidence of participants' meaning making, sustained reflection, and inquiry-oriented discourse, key attributes of cognitive presence (Akyol & Garrison, 2011). These instances occurred across the exploration, integration, and resolution phases identified by Akyol and Garrison. We noted that some activities were more likely to facilitate instances rated as involving high cognitive presence. One explanation is that these activities were more explicitly linked to the conceptualization of cognitive presence. For example, explanations of problem solving strategies provided opportunities for participants to make connections between strategies and between mathematical topics within a strategy, which involves exploration and integration. These opportunities were available due to the selection of mathematical problems for the activities and directions by the instructor to develop multiple strategies, indicators of teaching presence. Many of the instances not rated as high cognitive presence involved simpler explanations, evaluations, or reporting of steps. See Table 1.

Table 1: Cognitive Presence as Characterized by Mediating Processes

Mediating Process Types	Number of instances	Number rated as high
Accounts of instructional episodes	63	14
Analysis of tasks	133	27
Instructional practices	32	0
Pedagogical insights	41	5
Problem solving strategies	90	35

Discussion and Implications

Using the mediating processes from our conjecture mapping process to characterize cognitive presence was critical in conceptualizing a rigorous way to explore cognitive presence in online contexts. The specific articulation of these processes derive from our goals related to the design of the learning environment, as evident in the design conjectures from the conjecture mapping process (Sandoval, 2014). In our case, we felt that the mediating processes represented a continuum of ways of showing cognitive presence, with some participants' contributions categorized as a participating in a mediating process even though the contributions did not signal deep reflection or inquiry, and others represented a higher degree of reflection and inquiry, and thus a stronger indicator of cognitive presence. Simple engagement in a mediating process did not always rise to the level of cognitive presence we desired, as was evident in the large number of instances not coded as high. A question that derives from this finding is whether the frequency of instances of high level of cognitive presence was sufficient to engender learning. Our findings indicate, nevertheless, that our instructional design and the instructors' facilitation and direct instruction (the three components of teaching presence) were associated with a considerable number of instances in which participants engaged in reflection and inquiry. Thus, our framing and use of mediating processes provides evidence that we developed in a synchronous online context a community of inquiry that adhered to the assumption of intellectual rigor. A major contribution of our study is the use of mediating processes to characterize cognitive presence. This allowed us to examine the nature of our design conjectures (i.e., to see if our designed learning environment resulted in the desired intellectual activity) in a way that allowed us to conceptualize the interaction between the three presences (social, teaching, and cognitive) and our design conjectures, which helped us to assess more meaningfully our design conjectures.

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CONCEPTUALIZING A FEMINIST POSTSTRUCTUAL DISCOURSE ANALYSIS STUDY: AN APPROACH TO GENDER RESEARCH

Julia Przybyla-Kuchek
University of Georgia
juliapk@uga.edu

The increased use of poststructural theories in mathematics education provides a pathway for novel methodological approaches. In this paper, I introduce feminist poststructural discourse analysis (FPDA) as a methodology to study the intersection of gender and racial inequities in mathematics education. Specifically, I explore one example of a conceptualization of such a study to understand gendered, racialized, mathematical discourses in a middle school classroom.

Keywords: Gender, Research Methods, Discourse Analysis

Poststructural and postmodern theories have been suggested as possible theoretical frontiers in identity and equity research in mathematics education (Darragh, 2016; Langer-Osuna & Esmonde, 2017; Stinson & Bullock, 2015; Stinson & Walshaw, 2017). Some researchers have begun to take up these theories to study gender and race inequities (e.g., Stinson, 2013; Walshaw, 2005). Yet, theories of gender in mathematics education predominately remain unchanged in recent years (Damarin, 2008), with feminism not always being seen as relevant to mathematics education. More specifically, few researchers are taking up feminist *and* poststructural theories in their research (e.g., Parks 2009; Mendick, 2006; Walkerdine, 1998; Walshaw, 2001, 2005). In this paper, I introduce feminist poststructrual discourse analysis (FPDA) as a methodology from both feminism and poststructrual theory that has potential to build upon prior feminist and poststructural work and open spaces for new equity research. Below, I present the theoretical underpinnings of feminist poststructuralism and describe their methodological implications. I then present a conceptualization of a FPDA study.

Feminist Poststructural Theory

Feminist poststructural theory emerged out of and in response to liberal and radical feminisms with/through poststructural theory (Gavey, 1989; St. Pierre, 2000). Some feminists were critical of essentializing tendencies of radical feminism, where "women's" *experience* tended to be generalized from White, middle and upper-class women's experiences, and side-lined women's experiences across race, ethnicity, sexual orientation, age etc. (Davies & Gannon, 2005; St. Pierre & Pillow, 2000). When we put limits on, confine and define what it means to be "woman," we are always excluding someone's experiences and crushing the potential for different work to be done (St. Pierre & Pillow, 2000). Thus, projects of poststructuralism feminism show the complex, contradictory, multiple, and shifting meanings of language, how discourses come into being, become "normal" while others are excluded, and how power operates (St. Pierre, 2000). Feminist poststructualists break from humanist ways of knowing and producing research (i.e., theories assuming a human is a rational, conscious being with agency) in their framing of fundamental constructs, including language, discourse, self, and identity.

Language

Humanist perspectives posit that language is innocent, functioning to *identify* and name natural, present objects/subjects; language *mirrors* the "real" world. (MacLure, 2003; St. Pierre,

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* St Louis, MO: University of Missouri.

2000). Poststructural theory posits that language *produces* the "real" world. By organizing language, we create something more than meaning; language produces real world effects (Walkerdine, 1989). For example, *girl* has no inherent meaning. Instead, *girl* is created from its "position and function within the discourse itself" (Walkerdine, 1989, p. 271). *Girl* becomes meaningful in difference from *boy,* where what is means to be *girl* is a product of a particular organization of practices, such as how one performs *girl* and identifiably not *boy*. Language creates binaries and hierarchies, of which *girls* end up at the bottom (St. Pierre, 2000). Feminist poststructuralists seek to expose what discourses produce and open them up to new discourses. For example, Walkerdine (1989, 1998) demonstrates how language structures are harmful and gendered. She shows that *girl,* through practices in a mathematics classroom, is in a relation to *passivity, rote-learning, rule following,* and *real understanding*. These practices work together in a system of similarity and difference. Walkerdine (1998) argues that, *passivity, rote-learning, and rule-following* exist in similarity, while *real understanding* exists in difference from the previous practices that are often not assigned to *girl* but assigned to *boy*. The ways people talk about subjects in classrooms *produce* real effects as to what learning participants can access.

Discourse

Discourses are traceable and historically constructed "forms of knowledge or powerful sets of assumptions, expectations and explanations, governing mainstream social and cultural practices. They are systematic ways of making sense of the world by inscribing and shaping power relations within all text" (Baxter, 2003, p. 7). Discourses regulate how we interact, how we be "ourselves" (i.e., *subjectivity*), what we know (i.e., truth), and power relations (MacLure, 2003). They allow certain things to be said/done and simultaneously not others (St. Pierre, 2000); they are both constraining/limiting and enabling/generative. Walkerdine argues that the organization of the language into *discursive practices* allows *girls* to do mathematics in particular ways (e.g., rote-memorization) and not others (e.g., real understanding). Furthermore, discourses become so pervasive they become "natural" (St. Pierre, 2000).

Feminist poststructural theory posits that *gender differentiation,* a discourse distinguishing people by gender and positioning them into binary, hierarchical categories, is pervasive in many societies (Baxter, 2003, 2008; Walshaw, 2001). It is discursively produced through modern sciences and maintained through modern social apparatuses, such as school, family, church, etc. (Walshaw, 2001; Walkerdine, 1990). Feminist poststructuralists are interested in how gender differentiation functions, is produced and regulated, and its social effects (St. Pierre, 2000).

The (Human) Subject

A *subject* (i.e., person) in feminist poststructural theory decenters humanist assumptions about the existence of "self." Feminist poststructural theorists do not believe that there is an 'essential self' residing in every human. Walkerdine (1989, 1990) presents the subject as a product of a complex and contradictory system of *discursive practices*—a body of anonymous rules and practices that are constantly changing. Discourses operate simultaneously, not interchangeably, to produce a subject, allowing subjects to be fragmented and even contradictory. A subject is simultaneously "girl" and "student" and "child" and these *subject positions* can be contradictory within the same moment (MacLure, 2003). Discourses impact the ways a subject can be read (i.e., perceived/seen) as "student" and as "girl" differently than how boys are read as "student" and "boy." In being read differently as "boy" and "girl," discourses themselves become gendered. Similarly, being read differently as "black" or "white" simultaneously with "boy" or "girl" discourses are gendered and racialized. Furthermore, a subject does considerable work (i.e., agency) to take up these discourses (Davies, 2003). But, her "agency" is not occurring because of free will. Her fantasies, fears, desires, emotions, and relations (i.e., *subjectivities*) are constructed and regulated through discourse (St. Pierre, 2000).

Power is produced through discourse and is inherent in the relations *between* subject positions (St. Pierre, 2000). Power is not located in or possessed by an individual nor is it held by dominant groups of people, such as men and Whites, in a static and ever-present sense. Instead, power exists within the discursive relations of different subject positionings (Walkerdine, 1990).

Feminist Poststructural Discourse Analysis Methodology

Because of these dramatic shifts in perspective from humanist to feminist poststructural theories, research methods also shift. Feminist poststructural discourse analysis (FPDA) is one methodological approach to poststructural inquiry. Baxter (2008) defines FPDA as "a feminist approach to analysing the ways in which speakers negotiate their identities, relationships and positions in their world according to the ways in which they are located by competing yet interwoven discourses" (p. 1). FPDA is feminist in its focus on gender and goals to be *transformative*: "to open up spaces for those female voices which have been systematically marginalised or silenced" (Baxter, 2008, p. 10). FPDA is a discourse analysis methodology interested in how people become themselves in relation to others by examining spoken words. Based on MacLure's (2003) and Baxter's (2008) work, I describe three interrelated aims of FPDA below, where gender differentiation is a constant lens avoids essentializing girls and women and thus assumes the intersectionality of gender and other identifiers such as race.

Challenge the Reduction of the (Human) Subject and Subjectivities

Researchers who challenge the reduction of subject and subjectivities seek to demonstrate how humanist perspectives of "self" is reductionist by means of presenting subjects and subjectivities as complex, multiple, contradictory, and fragmented, rather than as unified, singular, and coherent (Baxter, 2008; Maclure, 2003). This aim of FDPA stems from the feminist poststructural conceptualizations of subjects and subjectivities. Furthermore, FDPA emphasizes how human subjects and their subjectivities are produced through discourse (MacLure, 2003).

Disrupting Epistemologies and Research Traditions

FPDA aims to disrupt epistemological and ontological assumptions regarding researchers' relationship to research, the use of written texts, and to insert ambiguity and uncertainty into taken-for-granted frames of research (Davies & Gannon, 2005; MacLure, 2003). The poststructural conceptualization of truth impacts how one conducts FPDA research, specifically related to how to treat and re-present data. Data is not proof or evidence of a certain kind of "reality." Rather, "accounts or descriptions or performances of gendered ways of being reveal *the ways in which sense is being made of gender*, or *the way gender is being performed* in that particular text" (Davies & Gannon, 2005, p. 319). That is, the data we collect/produce illustrate how discourses work through/on us, our participants and research, and demonstrates the discursive construction of reality. We cannot position ourselves outside the production of knowledge as objective knowers of truth (Baxter, 2003; Davies & Gannon, 2005). Throughout the research and writing process FPDA self-reflexivity is necessary (MacLure, 2003).

Social Transformation

Rather than the perpetually oppressed position of women in a critical perspective, the feminist poststructural perspective gives rise to a more complex and optimistic perspective of subjects. Feminist poststructural theory seeks to disrupt gendered binaries altogether (Baxter, 2002, 2008; MacLure, 2003). Thus, FPDA aims to be *transformative*, rather than emancipatory.

Deconstructing gendered discourses opens possibilities for new discourses within/between contradictory subject positions making social transformation possible (Walkerdine, 1998).

One Conceptualization of a FPDA Study

Adapting the work of Judith Baxter, I designed a FPDA study around the aims of FPDA to study how discourses operate in a middle school mathematics classroom. Although feminist poststructual theory appears to focus on gender, I consider the intersectionality of gender and race. The study sought to answer the following research questions: (a) What mathematical, gendered, and racialized discourses operate in a middle school mathematics classroom? (b) How do discourses constitute and regulate students' gendered and racialized subject positions and subjectivities in mathematics? (c) How do students resist gendered, racialized, mathematical subject positions and create sites for production of new and generative subject positions?

For 12 weeks I collected daily fieldnotes and video and audio data in a public, gender and racially diverse 7th grade mathematics classroom to track changes in students' subject positions across time, mathematical content, group dynamics, and instructional formats. Classroom observations address two aims of FPDA. First, I *challenged the reduction of subjects* by observing the multiple and contradictory subject positions in which participants are foisted into and take up across classroom interactions over time. Second, daily observations allowed me to note ways in which students resisted restrictive subject positions and discursive practices in local, unpredictable, and constant ways (i.e., *social transformation*). Furthermore, the extended observation time coupled with audio and video recordings allowed for both a broad level analysis of discursive practices and subject positions, as well as detailed moment-by-moment analysis of utterances and non-verbal routines. This broad level and fine-grained analysis differs from other discourse analysis methods which are often restricted to more turn-by-turn speech analyses.

Additionally, this FPDA design makes use of unique interviewing methods to seek out the various contradictory and complex subject positions participants performed. Through interviews with the classroom teacher and small groups of students (usually self-grouped by gender and race), I gained access to students' subjectivities related to mathematics by focusing on students' view of and feelings towards mathematics, experiences and routines in the classroom, and perspectives of narratives told about who does mathematics and what it means to do mathematics. In addition to interview responses, their interactions in the interview—how they positioned each other in relation to mathematics—provided a second level of material to analyze.

A second round of interviews was conducted near the end of the semester that made use of the video data obtained from the classroom. I selected a lesson I believed students resisted gendered, racialized, mathematical discourses. I created a series of clips to represent lesson events, which the teacher and groups of students then watched. As they watched, I encouraged them to stop the video at any time to share their reactions to and interpretations of the events. I sought to uncover why students performed in particular ways of doing mathematics. In doing so, I opened the analysis to multiple possible readings so there was no one authoritative voice determining the "truth" about the interactions in the classroom. That is, the researcher, teacher, and several groups of students provided analyses of the interactions with no one reading claiming the truth. This unique interviewing technique *challenged the reduction of subjects* and *disrupted epistemological assumptions* about the subject positions of researchers and participants. This interview approach challenges the discursive practices by which research produces truth.

Discussion

I have presented but one conceptualization of a FPDA study. This is by no means “the” way to conduct feminist poststructural research or even make use of FPDA—this work is always left undefined. This one example contributes to the ongoing discussion of theoretical and methodological innovation in mathematics education, particularly related to equity research and presents a starting point for generative ways of conducting feminist postructural research.

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PSYCHOMETRIC MODELING AND REASONING ABOUT FRACTION ARITHMETIC

İbrahim Burak Ölmez
University of Georgia
i.burakolmez@hotmail.com

The purpose of this study was to coordinate psychometric modeling with reasoning about fraction arithmetic. Recent research has demonstrated the existence of three distinct latent classes in a nationwide sample of in-service middle grades teachers based on psychometric analysis of responses to survey data. The survey focused on four components of reasoning about fraction arithmetic: Referent units, Partitioning and iterating, Appropriateness, and Reversibility. The present study sought to better understand the three classes by interviewing 8 future teachers and comparing to what extent characteristics based on the three latent classes were associated with reasoning evidenced during interviews about fraction arithmetic. The psychometric analysis and qualitative analysis provided consistent results about the reasoning of these future teachers' facility with fraction arithmetic.

Keywords: Rational Numbers, Reasoning and Proof, Research Methods

Having highly qualified teachers is a main goal of many Western, institutionalized systems of education because teachers are responsible for creating classroom environments in which students share observations, make and justify arguments, connect ideas, and use appropriate representations (National Council of Teachers of Mathematics [NCTM], 2000). Because one cannot teach what one does not know, having sufficient mathematical knowledge is one obvious qualification that teachers need to possess. One content area with which teachers and students experience considerable difficulty is fractions, which have been known to be critical for success in mathematics at later grades and further study in mathematics (e.g., Hackenberg & Lee, 2015).

Although recent curriculum standards such as the Common Core State Standards for Mathematics (e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) have emphasized the necessity of developing reasoning about fraction arithmetic when solving problems embedded in situations, the present study lies at the intersection of two challenges. One challenge is to support teachers' reasoning about fractions in terms of quantities. Despite most teachers' computational fluency with multiplying or dividing two fractions, numerous studies have pointed out the difficulties that teachers experience in reasoning about products or quotients of fractions in problem situations (e.g., Jansen & Hohensee, 2016). A second challenge is to study reasoning in large samples because most research on reasoning is based on detailed case studies with small samples.

In response to these challenges, several recent studies have used traditional item response theory (IRT) models (e.g., Lord, 1980), which use survey item responses to locate teachers on unidimensional scales to measure teachers' mathematical knowledge. These studies include the Learning Mathematics for Teaching (LMT) project (e.g., Hill, 2007) and the Diagnostic Mathematics Assessments for Middle School Teachers (DTAMS) project (Saderholm, Ronau, Brown, & Collins, 2010). Traditional IRT models assume that all examinees in a given sample belong to a single population. Some recent studies (e.g., Izsák, Orrill, Cohen, & Brown, 2010; Izsák, Jacobson, de Araujo, & Orrill, 2012), however, have reported the existence of distinct latent classes of middle grades teachers when it comes to reasoning about fraction arithmetic in terms of quantities. Existence of distinct latent classes violates local independence, a key assumption of traditional IRT models. In this case, different psychometric models need to be used such as the mixture Rasch model (Rost, 1990), which is an appropriate model to use when distinct latent classes are present. For the present study, distinct latent classes correspond to distinct patterns of strengths and weaknesses in middle grades teachers' reasoning about fraction arithmetic as evidenced by different patterns of correct and incorrect responses to survey items.

To measure teachers' capacities for reasoning about fraction arithmetic, Bradshaw, Izsák, Templin and Jacobson (2014) developed the Diagnosing Teachers' Multiplicative Reasoning (DTMR) Fractions survey and reported on content and item-level validity of the survey. Recent research (Olmez & Izsák, Under Review) analyzed responses from a nationwide sample of 990 in-service middle grades teachers to the survey using the mixture Rasch model. Olmez and Izsák reported the existence of three distinct latent classes, and identified strengths and weaknesses of each class with respect to reasoning about fraction arithmetic. The results from the nationwide sample led to the present study that focused on interview data of 8 future middle grades teachers recruited from each of the three classes. I asked to what extent were strengths and weaknesses associated with each class detected by the mixture Rasch analysis consistent with the 8 future teachers' reasoning about fraction arithmetic across multiple survey items and related tasks?

Theoretical Framework

My theoretical framework considers middle grades teachers' reasoning about quantities and focuses on using drawings (e.g., number lines and area models) to learn and teach fraction arithmetic. I think that reasoning, which goes beyond computational fluency, requires a teacher to make sense of quantities in fraction arithmetic problems using drawings. Thus, a teacher's capacity to reason about quantities with drawings depends in turn on paying particular attention to distinct components such as *Referent units*, *Partitioning and iterating*, *Appropriateness*, and *Reversibility* — each of which is grounded in past research. Moreover, I take the stance that reasoning depends on context. That is, a teacher might use any one of these components appropriately in one situation, but not another, depending on the wording of the situation, the arithmetic operations required, or the representations provided. Hence, it is important to examine teachers' capacities to employ particular components across a range of problem situations.

The DTMR survey measures four components of reasoning about fraction arithmetic: *Referent units*, *Partitioning and iterating*, *Appropriateness*, and *Reversibility*. *Referent units* deals with reasoning about units when numbers are embedded in problem situations and includes *Norming*, *Referent units for multiplication*, and *Referent units for division*. *Norming* refers to the formation of standard units for measurement and occurs either in case of selecting a standard unit from alternate choices or making at least two different choices for a measurement unit in a given situation (i.e., renorming). *Referent units for multiplication* and *Referent units for division* concern the problem situations that can be modeled by the equation $M \bullet N = P$ where M and N refer to different units. *Partitioning and iterating* refers to separating a quantity into equal-sized pieces and concatenating unit fractions, and consists of three sub-components: *Partitioning in stages*, *Partitioning using common denominators*, and *Partitioning using common numerators*. *Partitioning in stages* refers to repartitioning to obtain a desired partition. *Partitioning using common denominators* and *Partitioning using common numerators* refer to using common denominators or numerators to obtain common partitions. *Appropriateness* concerns identifying situations that can be modeled by multiplication and division and includes three sub-components:

Identifying multiplication, *Identifying partitive division*, and *Identifying quotitive division*. *Reversibility* deals with returning to an original state.

Latent Classes Based on Mixture Rasch Analysis

The mixture Rasch analysis (Olmez & Izsák, Under Review) found three latent classes, each with strengths and weaknesses. Specifically, Class-C teachers performed well only on *Appropriateness*, but they had persistent difficulties in the remaining three components of fraction arithmetic. Class-B teachers performed well on *Appropriateness* like Class-C teachers and also on *Reversibility*, but they struggled with items that measure certain aspects of *Referent units* such as *Referent units for multiplication*, and *Partitioning and iterating* such as *Partitioning using common denominators* and *Partitioning using common numerators*. In addition to having the strengths of Class-B teachers, Class-A teachers performed well on *Partitioning using common denominators* and *Partitioning using common numerators* (*Partitioning and iterating*). At the same time, teachers in all three latent classes were not proficient on renorming in the presence of improper fractions (*Referent units*).

Methods

I administered the survey to a cohort of 18 future middle grades teachers at the beginning of a number and operations course offered in Fall 2017. The course focused on reasoning about fraction arithmetic in terms of quantities. The future teachers completed the survey as the first homework assignment during the first week of the course. I used the item parameters generated by the mixture Rasch analysis to analyze the item responses and estimate latent class membership for all 18 future teachers. Of the 18 teachers, 10 were in Class-C; 4 in Class-B; and 4 in Class-A. I then selected 8 future teachers (4 in Class-C; 3 in Class-B; and 1 in Class-A) to be focal participants for this study. To insure strong warrants for class membership, I recruited the participants who had a high probability (over .90) of being a member in a particular class. The survey consists of 27 items (19 multiple choice and 8 constructed response) that measure *Referent units*, *Partitioning and iterating*, *Appropriateness*, and *Reversibility*.

One week after completing the survey, I conducted one individual video-recorded interview with each of the 8 focal participants (8 interviews in total). Each interview consisted of two parts. In the first part (about 45 minutes), I went through items from the survey and asked follow-up questions such as "Why did you select the choice you did?" The focal participants did not experience instruction in fractions between completing the survey for homework and the interview. During the second part (about 45 minutes), I presented additional tasks that were not part of the survey to obtain further information about future teachers' reasoning.

For each participant, I analyzed the interview data by listing item-by-item evidence for and against the proficiency of each component of reasoning. Among the three categories of coding ("for"; "against"; and "other comments"), I included evidence for and against proficiency wherever I found it in the interviews. I used "1" to indicate evidence consistent with proficiency, "0" to indicate absence of evidence for proficiency, and "X" to indicate cases where there was too little evidence or there was a mix of contrary evidence for and against the proficiency of each component. I then focused on 5 of the sub-components that characterize and distinguish the three classes, and used the results of the mixture Rasch analysis to predict each of the 8 teachers' strengths and weaknesses on these sub-components using a "1" or a "0".

Results

My research question asked to what extent strengths and weaknesses associated with each of the three latent classes detected by the mixture Rasch analysis were consistent with the 8 future middle grades teachers' reasoning during the interviews. I compared latent class membership for each participant based on analysis of interview data with latent class membership as diagnosed by analyzing survey item responses using the mixture Rasch model. In other words, while interview data were based on a subset of item responses from the survey and additional tasks, the mixture Rasch analysis was based on responses to the complete set of survey items.

Table 1 presents latent class memberships for the 8 focal participants as diagnosed based on the interview data and as diagnosed with the mixture Rasch analysis. Each column indicates a diagnosis of proficiency (1), non-proficiency (0), or uncertain proficiency status (X) for each component of reasoning. Given 8 focal participants and 5 (sub)-components of reasoning that each interview has addressed, there existed 40 cases. I assigned a 1 in 15 cases, a 0 in 19 cases, and an X in the remaining 6 cases. Of the 15 cases where I assigned a 1, my diagnoses and the class memberships generated by the mixture Rasch analysis agreed in 11. Of the 19 cases where I assigned a 0, my diagnoses and the class memberships agreed in 13. Of the 6 cases where I assigned an X, indicating uncertain proficiency status, the mixture Rasch analysis diagnosed proficiency in 1 case and non-proficiency in 5 cases. The results indicate reasonable consistency between the analysis of interview data and the mixture Rasch diagnoses. This, in turn, suggests that latent classes based on the mixture Rasch model do reflect relative strengths and weaknesses in reasoning about fraction arithmetic in terms of quantities. This result provides significant additional evidence for the validity of our interpretation of the three distinct latent classes.

Table 1: Diagnoses Based on Interview Data and the Mixture Rasch Analysis

Component	Sub-component	Stu-1		Stu-2		Stu-3		Stu-4		Stu-5		Stu-6		Stu-7		Stu-8	
		Int	MR	Int	MR	Int	MR	Int	MR	Int	MR	Int	MR	Int	MR	Int	MR
RU	Norming	0	0	1	1	1	1	1	0	1	0	0	0	X	1	1	1
RU	RU for Multiplication	0	0	X	0	X	0	X	0	1	0	0	0	0	0	1	1
PI	Partitioning Using Common Numerators	0	0	0	0	X	0	0	0	X	0	0	0	0	0	1	1
APP	Identifying Multiplication	0	1	0	1	1	1	1	1	1	1	0	1	0	1	0	1
REV	Reversibility	0	0	1	1	1	1	0	0	1	0	0	0	0	1	1	1

Note. RU=Referent Units; PI=Partitioning & Iterating; APP=Appropriateness; REV=Reversibility.
Int=Diagnosis based on interview data; MR=Diagnosis based on the mixture Rasch analysis.

Discussion

I found reasonable consistency between the class memberships generated by the mixture Rasch analysis and the qualitative analysis based on the interview data of the focal participants. These results demonstrate that psychometric models applied to data gathered with appropriately designed surveys can capture information about strengths and weaknesses in reasoning, particularly in the critical domain of fraction arithmetic. Regarding the high number of qualitative research using small samples in mathematics education, coordination of contemporary psychometric models with qualitative studies is promising to illuminate fine-grained components of reasoning on core issues in the teaching and learning of mathematics. The approaches in the present study can provide helpful diagnostic information to mathematics educators in teacher preparation programs about strengths and weaknesses of each future teacher, and educators can plan instruction accordingly by focusing on those weaknesses. Future research should focus on generalizing these results to other cohorts of future teachers.

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LET'S GET META: USING META-AGGREGATION TO SYNTHESIZE QUALITATIVE RESEARCH IN MATHEMATICS EDUCATION

Rachael Kenney
Purdue University
rhkenney@purdue.edu

Yukiko Maeda
Purdue University
ymaeda@purdue.edu

Secil Caskurlu
Purdue University
scaskurl@purdue.edu

Qualitative research studies are used in our field to explore the complexity of teaching and learning. There are, however, persistent barriers to using results from qualitative research to inform policy in schools. There is a growing interest in education for integrating findings across qualitative empirical studies to unpack common threads that can serve as actionable recommendations for practice. We describe qualitative synthesis and its relationship to other research synthesis methods, identify a specific method, meta-aggregation, used to integrate research into practice, and share a critical appraisal protocol for in this method.

Keywords: Research Methods

Qualitative research studies are frequently used in our field to explore the complexity of teaching and learning in mathematics and can generate deeper understandings of teacher and student knowledge and needs. With a wide range of qualitative methods and results, it can be difficult for educators to look across a set of studies to integrate their findings into teaching practices. There are also persistent barriers to using results from qualitative research to inform policy related to teaching and learning in schools (Tierney & Clemmons, 2011). Thus, there is a growing interest in mathematics education, and education in general, in integrating findings across qualitative empirical studies to deepen our understanding of evidence-based practices and to unpack common threads across a related topic (Thunder & Berry, 2016; Yore & Lerman, 2008) that can serve as actionable recommendations for practice in learning environments.

Qualitative synthesis is a methodology for pooling qualitative research data and drawing conclusions on the collective meaning of the research (Bearman & Dawson, 2013). It is a methodology that is still evolving (McCormick, Rodney, & Varcoe, 2003; Noyes & Lewin, 2011). Thunder and Berry (2016) suggest that fields such as mathematics education can benefit greatly from qualitative synthesis, but we need to formalize this process and recognize it as providing important findings that cannot be generated from quantitative synthesis (i.e., meta-analysis) or a single qualitative study. In this paper we identify a specific qualitative synthesis method, meta-aggregation synthesis, used to integrate research into practice, and share a tool we have developed for use in this method.

What is Qualitative Synthesis

Qualitative synthesis is not merely a qualitative version of meta-analysis (Mohammed, Moles, & Chen, 2016). It provides: (a) an accumulative understanding of how qualitative research contributes to our understanding of a studied phenomenon; (b) new insights into the existing theories and knowledge; (c) research-based evidence for the development, implementation and evaluation of intervention; and (d) directions for policy, public perception, and future research (Campbell et al., 2011; Mohammed, et al., 2016; Nye, Melendez-Torres, & Bonell, 2016; Suri & Clarke, 2009; Tong, Flemming, McInnes, Oliver, & Craig, 2012). A qualitative synthesis is a specific type of systematic review that differs from other types of narrative review because of its explicit intent to treat reported findings in individual studies as *data* for analysis to derive a new theory or interpretation from original findings. It requires a systematic and transparent process for analysis and interpretation of the findings from selected studies. While a literature review can take weeks to months to complete, a qualitative synthesis can take six months to two years (Khangura, Konnyu, Cushman, Grimshaw, & Moher, 2012).

Meta-aggregation as a Tool for Informing Practice

Meta-aggregation has gained attention as an effective qualitative synthesis methodology for bridging research findings to practice (Aromataris & Munn, 2017). It involves analyzing empirical qualitative findings across multiple studies to understand evidence-based practices and unpack common threads (Thunder & Berry, 2016; Yore & Lerman, 2008). Meta-aggregation informs practical-level theory or lines of action that are directly applicable to decision-making practices in, for example, a classroom (Lockwood, Munn, & Porritt, 2015). It serves an *integrative* purpose in the synthesis, focused on developing recommendations in response to a defined area of interest (Heyvaert, Hannes, & Onghena, 2017; Lockwood et al., 2015).

Why Focus on Meta-aggregation?

Meta-aggregation is a rigorous process of synthesizing qualitative data that moves beyond the production of theory and attends to the practicality and applicability of results (e.g., specific recommendations for teachers) (Hannes & Lockwood, 2011). Developed by the Joanna Briggs Institute (JBI), it has been used in the fields of nursing and medical science, but a limited number of methodologically sound qualitative syntheses are currently available in the field of education (e.g, in our review of 25 studies identified as qualitative syntheses in education since 1988, only two have used a meta-aggregation approach). However, according to Hannes, Petry, and Heyvaert (2018), meta-aggregation is important to educational research because, “the conceptual clarity in many educational research subdomains invites review authors…to accurately summarize the existing empirical evidence in that domain in order to inform and advise policy and practice. (p. 292). We see meta-aggregation as a useful and needed tool in mathematics education for building bridges between research and practice.

Meta-aggregation Synthesis Procedures

We outline below a detailed procedure for using the meta-aggregative approach, drawing on work from the JBI (Aromataris & Munn, 2017) and a recent example from Hannes et al. (2018) to explain the steps involved in this research method.

Finding potential data for the synthesis. A meta-aggregation synthesis begins with identifying a research question or purpose for the study. The question drives the search for qualitative studies to include to in the synthesis. Before searching for existing studies, we must set criteria for consideration in the search. The reviewer sets this criteria and systematically tracks what databases and keywords used to search for and find potential studies for inclusion.

Appraising the data for inclusion. A quality appraisal, also referred to as quality assessment, validity assessment, and assessment of risk of bias (Higgins & Green, 2008; Hong & Pluye, 2018), is the systematic process used to judge the trustworthiness of reported findings in studies and their relevance to the synthesis goal. After collecting potential qualitative studies for inclusion, researchers then should use a critical appraisal protocol (Aromataris & Munn, 2017; Lockwood et al., 2015) and evaluate how findings are supported with data (e.g., direct interview quotes, observation notes). Studies identified as “unsupported” are excluded from the synthesis pool. Selected studies become the collection of primary studies that now serve as data for the meta-aggregation (we expand on our development of a critical appraisal tool useful for meta-aggregation studies in a later section). Following recommendations for sufficiency and manageability of data (Major & Savin-Baden, 2010) and a review of 12 recent qualitative syntheses, it is reasonable to include 10-20 studies as data.

Analyzing the data. Using the pool of existing research studies chosen for inclusion, the researcher then analyzes the findings from these papers as "data" for the meta-aggregation study. Findings are extracted from individual qualitative studies and coded for literal descriptions of themes, categories, or metaphors reported by the original authors using pre-determined coding protocols and memo (Aromataris & Munn, 2017; Major & Savin-Baden, 2010). Key components in the coding protocols are *general information* about the context and methodological characteristics of a primary study and *interpretive findings*, which serve as the main data for the synthesis. Researchers also use *coding memos* to record reactions to the reported findings, reflect subjectivity, and identify potential impacts on synthesis validity.

It is important to seek supporting evidence, such as interview quotes, work samples, or rich descriptions of observed practices to understand the derivation and trustworthiness of findings for synthesis. Findings are classified into three categories: (a) *unequivocal (not open to challenge)*, (b) *credible (open to challenge)*, or (c) *unsupported by data*. Unequivocal or credible findings are grouped inductively for conceptual and descriptive similarities. This is referred to as *first stage clustering*.

Synthesizing the findings. The fourth step in the meta-aggregation is to take the clustered categories and move them to "lines of action" or particular recommendations (for educators, administrators, policy makers, etc.). These are called the *synthesis findings*. Following Hannes et al. (2018), we see the core purpose of the recommendations as to try and fix problematic situations, and their value lies in the practical usefulness of accepting them or and interrogating them. As a final determination of the robustness of the recommendations, the reviewers should return to the critical appraisal results. Previously excluded unsupported studies are returned to the analysis to evaluate if the developed recommendations for lines of action remain the same with their inclusion. We next share a specific critical appraisal protocol that our team has developed (through modification of existing tools) as one that is useful for meta-aggregation.

The Development of Critical Appraisal Protocol in Meta-aggregation

Explicit use of the results from an appraisal form during data selection and synthesis is unique for meta-aggregation compared to other methods of synthesis, and is used to enhance the credibility of synthesis results. While a variety of appraisal tools are available, few have documented evidence of their validity and reliability (Katrak, Bialocerkowski, Massy-Westropp, Kumar, & Grimmer, 2004). Hong & Pluye (2018) indicated in their conceptual framework that, when doing an appraisal, reviewers may look for three dimensions of a piece of qualitative research: (a) methodological quality (trustworthiness), (b) conceptual contribution (insightfulness), and (c) reporting practices (accuracy, transparency and completeness). Assessments of methodological quality and conceptual contribution, however, are more complex than the assessment of reporting practice because they involve reviewers' thorough understanding and operationalization of how to evaluate and what they look for in the trustworthiness and insightfulness in each primary study. The evaluation of trustworthiness and insightfulness, by nature, is more likely to be affected by the reviewers' own ideological and theoretical perspectives, preferences and experiences (Sandelowski, 2015).

In our own meta-aggregation work using existing *quality appraisal* forms adapted from existing sources (e.g., Blexter, 1996; Hannes & Lockwood, 2011; MacEachen, Clarke, Franche

& Irvin, 2006; Thunder & Berry, 2016), we found our coding was inconsistent because we provided multiple interpretations when making a judgment of quality of a particular component of the research. Quality checklists have, in the past, been found to generate low inter-rater reliability scores even among experienced qualitative systematic reviewer (Dixon-Woods et al., 2006). We also found that dissertation studies, which have little space limitations, were meeting all measures of quality simply because they had the room to detail their work, while journal articles often had to skip reporting certain quality elements due to page limits. Thus, our measures of methodological quality were based on our assumptions of what they "probably" did. We began to wonder how we might limit assumptions and inferences needed by the reviewers.

Through four rounds of pilot testing and discussions and revisions, we made the following decisions on the revision of our appraisal form: (a) the form should avoid adjectives in the appraisal questions to minimize a room for inferences and subjective judgment by reviewers, (b) not all criteria in the form are equally important for appraising a study; some serve as useful information about the study, and others are critical for deciding on the inclusion in the synthesis, and (c) we did not need to measure the methodological quality and conceptual contribution of the study, but rather focus on the transparency and completeness of the information provided to understand their findings. We therefore followed other's examples of labeling the resultant appraisal form as a *critical appraisal form* to remove a sense of subjectively judging quality of the primary studies. Our new critical appraisal tool has questions that can be used as key questions for pre-screening that address the "What?" and "How?" regarding the study (shown in Table 1); and optional additional questions that can be used for data gathering that address *why* the researchers chose the ideas, tools and processes for their study. These additional questions allow the form to produce general information findings for the meta-aggregation.

Table 1: Critical Appraisal Form Key Questions

Criteria	**Question**
Purpose of Study	1. Did author(s) provide clear research questions? If not, is there a clear purpose statement to guide the investigation?
Research Design	2. Is a qualitative research design appropriate for addressing the research purpose? (This may include considering the qualitative component of a mixed-methods study)
Sampling	3. Did author(s) describe the setting/context for data collection?
	4. Did author(s) describe the specific sample of the population being studied? (e.g., Administrators; Secondary science teachers; 4-6th grade math students; History textbooks before 2010; etc.)
Data Collection	5. Did author(s) provide any description of the data collection procedures? (*e.g., If interviews were used, are there details on how they were conducted*)
Data Analysis	6. Did author(s) provide description of data analysis procedure? (*e.g., If thematic analysis is used, is it clear how the categories/themes were derived from the data*?)
	7. Were the findings explicit and clear?
Findings	8. In the concluding sections, did the author(s) describe implications for teaching/learning/practice AND/OR implications for future research?

Summary

We encourage researchers interested in qualitative synthesis review in mathematics education to consider the method of meta-aggregation as a useful and productive approach. Through our own work, we have developed a new research tool that is supported by reliability and validity to enhance the quality of meta-aggregation studies and thus help take us a step closer to move beyond knowledge generation to knowledge application in our *field*.

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ANALYSIS OF THE E-TEN CALIBRATION DATA

Ying Liu
University of Southern California
liu385@usc.edu

Arthur J. Baroody
University of Illinois at Urbana-Champaign
baroody@illinois.edu

Elizabeth M. McCarthy
WestEd
bmccart@wested.org

Michael D. Eiland
University of Illinois at Urbana-Champaign
meiland@illinois.com

The present report summarizes the analyses of the calibration data for the electronic Test of Early Numeracy (e-TEN), an adaptive, iPad-based test of early numeracy achievement. A total of 794 children age 3 years to 8 years-11 months from two states were tested. Items were designed to map onto seven theoretical domains: verbal counting, numbering, numerical relations, numeral literacy, single-digit calculation, multi-digit calculation and base-ten place-value. Analysis indicated a strong single factor—a unidimensional structure across all domains. A Wright map of calibrated item difficulty indicated that items were fairly evenly distributed along the entire continuum of the early numeracy.

Keywords: Assessment and Evaluation, Early Childhood Education, Number and Operations

Purpose and Background

The primary aim of the calibration analysis is to develop the adaptive and scoring algorithm that is psychometrically sound yet feasible to implement through standard programming (e.g., for the iOS platform). Secondary aims include (a) examination of the dimensionality, (b) refinement of the item or sublevel scoring rule, (c) item calibration, and (d) identification of "problematic" items. This report addresses these four secondary aims, which are prerequisites for achieving the primary aim.

Methods

Sample

For each 6-month interval between 3.0 to 8.99 years (12 age groups), data were collected with a total sample size of 794 (55 young 3s, 77 old 3s, 66 young 4s, 80 old 4s, 88 young 5s, 72 old 5s, 56 young 6s, 56 old 6s, 57 young 7s, 61 old 7s, 60 young 8s, and 66 old 8s). This sample size was chosen to allow appropriate use of the item response theory (IRT) model, our primary analytic framework.

Procedure

Participants were administered all items deemed appropriate for their 6-month age group regardless of their individual performance to determine the success rate for each item for each age group. The number of items scored per age group ranges from 17 (young 3s) to 28 (older 6s).

Results

Dimensionality

Dimensionality & issue of age. Among the seven proposed dimensions, all three "calculation-2" items loaded on the calculation-1 dimension. Collapsing calculation-1 & -2 resulted in a 6-dimensional model. All items loaded on a single dimension, except two (writes 2-

and 3-digit numbers), which loaded on both the “grouping-and-place-value” and the “numeral literacy” dimensions.

Multi- versus unidimensionality. Table 1 compares the goodness of fit information across models. By comparing model 1 and 3 (unidimensional vs. 6-dimensional Rasch), the 6-dimensional structure is slightly more favorable (smaller AIC and BIC). All dimensions are highly correlated, with pairwise correlation between grouping and place value and the other five dimensions being the lowest (ranging from 0.78 to 0.90). The remaining pair-wise correlations arranged from 0.90 to 0.99. Exploratory factor analysis on the estimated correlation matrix suggests a strong single factor (eigenvalue = 5.48, with the remaining eigenvalues all less than 1), which is consistent with what the correlation matrix indicates. Therefore, **a unidimensional structure across all domains is most appropriate** (see Purpura, 2010; Ryoo et al., 2015)**.**

Table 1: Summary of Model Fit Statistics.

Model #	Item Score	Dimension	Age regressor	Deviance	AIC	AICc	BIC	# Parameters
1	Full credit or not	1	No	16778.8	16886.8	16880.1	17139.4	54
2	Full credit or not	1	Yes	18042.3	18152.3	18145.3	18409.5	55
3	Full credit or not	6[a]	No	16160.9	16308.9	16296.5	16655.0	74

[a] Calculation-1 and Calculation-2 were collapsed into one dimension, because Calculation-2 was not well identified. A Monte Carlo method was used to estimate the 6-dimensional model.

Refinement of Partial Credit

We started with the finest partial credit scoring based on content area expertise, and refined the scoring rules using item analysis. In particular, we relied on mean-square fit statistics (MNSQ), an item misfit index frequently used in Rasch analysis that signals the amount of distortion in the measurement system due to mis-specification (i.e., misfit between data and model). Values greater than 1.0 suggest unmodeled noise in the data (i.e., something in the data is not yet explained by the model), whereas values less than 1.0 imply that the item is not very informative (i.e., the item may not provide much unique information above and beyond other items). Items with greater MNSQ values are more of a concern in our case, and we revised such items’ scoring to improve the fit. In particular, we collapsed sublevels that were not well differentiated with adjacent sublevels. Two examples: (a) Item “composing & decomposing ten & base-10 equivalents of ten” contained two trials: “10 ones=ten” and “ten=10 ones.” As the vast majority of students were incorrect or correct on both, we removed the original partial credit of being correct on one trial, so a student either receives a full credit if he is able to do both, or no credit. (b) The original scoring for subtraction items (e.g., “subtract to 18”) entailed correctness and fluency (i.e., completion within 3 seconds). However, even the oldest group (old 8) in our calibration sample showed little evidence of fluency. In the revision, we removed the reference to fluency, and considered correctness solely for scoring.

Item Calibration

Figure 1 summarizes the calibrated overall difficulty levels of the items. As expected, the sample is fairly evenly distributed along the entire continuum. Items are located across the entire continuum, implying good coverage of the full spectrum of the developmental levels. Item “blocks” by domain are located, in general, consistent with the developmental orders predicted by theory. For instance, “numbering” appears in the low end of the scale, signaling that it is one of the first few things that a young child learns about numeracy, whereas "grouping and place-value knowledge" and "calculation-2" appear in the high end, implying that they are most

```
        XXXXXX|50 53     | KEY:
           XXX|          |
          XXXX|          |
           XXX|47        |
           XXX|52        |
 4        XXXX|37        |
          XXXX|          |
          XXXX|48        |
           XXX|44        |
         XXXXX|26 45     |
           XXX|40        |
 3       XXXXX|36        |
          XXXX|35 49     |
          XXXX|33 51     |
          XXXX|34 42 46  |
          XXXX|          |
           XXX|41        |
 2        XXXX|          |
          XXXX|          |
          XXXX|39        |
          XXXX|25 43     |
          XXXX|5         |
 1        XXXX|          |
          XXXX|          |
          XXXX|          |
          XXXX|31        |
          XXXX|32        |
          XXXX|          |
 0         XXX|          |
          XXXX|          |
         XXXXX|4 18      |
          XXXX|38        |
          XXXX|          |
          XXXX|24 30     |
-1        XXXX|          |
          XXXX|          |
          XXXX|23        |
          XXXX|          |
         XXXXX|12        |
         XXXXX|15        |
-2      XXXXXX|14 17     |
         XXXXX|10 16 29  |
       XXXXXXX|          |
         XXXXX|1 3 22    |
        XXXXXX|          |
        XXXXXX|20        |
-3       XXXXX|13        |
          XXXX|28        |
          XXXX|11        |
         XXXXX|8 19      |
          XXXX|          |
          XXXX|21        |
-4        XXXX|27        |
          XXXX|2 7       |
          XXXX|9         |
           XXX|          |
           XXX|          |
            XX|          |
-5          XX|          |
    XXXXXXXXXX|6         |
```

Domain 1: Verbal Counting
- Item 1: Verbally counts to "52" (Set 1A)
- Item 2: Number after to 9 with a running start (Set 1B)
- Item 3: Number-after to 9 without running start (Set 1C)
- Item 4: Number-after: 2-digit numbers to 100 (Set 1D)
- Item 5: Number-after: 100 terms (Set 1E)

Domain 2: Numbering
- Item 6: Subitizes 1 & 2 (Set 2A)
- Item 7: Subitizes 3 (Set 2A)
- Item 8: Subitizes 4 (Set 2A)
- Item 9: Counts sets to 5 (Set 2B)
- Item 10: Cardinality principle to 5 (Set 2B)
- Item 11: Counts sets to 10 (Set 2C)
- Item 12: Cardinality principle to 10 (Set 2C)

Domain 3: Numerical Relations
- Item 13: Identifies "more" sets 2:1 (2.0 x) apart (Set 3A)
- Item 14: Identifies "more" sets 3:2 (1.5x) apart (Set 3A)
- Item 15: Identifies "more" sets 4:3 (1.33 x) apart (3A)
- Item 16: Identifies "more" verbal 1-10 far apart (Set 3B)
- Item 17: Identifies "more" verbal neighbors 1-5 (Set 3C)
- Item 18: Identifies "more" verbal neighbors 6-10 (Set 3D)

Domain 4: Numeral Literacy
- Item 19: Relates numerals to sets of 1 to 3 (Set 4A)
- Item 20: Relates numerals to sets of 4 to 9 (Set 4A)
- Item 21: Reads numerals 1 to 5 (Set 4B)
- Item 22: Reads numerals: 6 to 9 (Set 4B)
- Item 23: Reads teen numerals (Set 4C)
- Item 24: Reads numerals 20-99 (Set 4D)
- Item 25: Reads 3-digit numerals (Set 4E)
- Item 26: Reads 4-digit numerals (Set 4F)

Domain 5: Calculation-1 (single digit)
- Item 27: Concrete add/subtract – basic (2+1, 3–2) (Set 5A)
- Item 28: Concrete add/subtract – intermediate (3–1 & 3+1) (Set 5A)
- Item 29: Concrete adds/subtracts – difficult (4–3 & 2+2) (Set 5A)
- Item 30: Accurately solve addition word problems to 10 (Set 5B)
- Item 31: Fluency with add-1 facts to 9 (Set 5C)
- Item 32: Fluency with doubles facts to 9 (Set 5C)
- Item 33: Fluency with near doubles to 18 (Set 5D)
- Item 34: Fluency with add-9 to 18 (Set 5D)
- Item 35: Accurate with subtraction doubles to 18 (Set 5E)
- Item 36: Accurate with other differences (Set 5E)
- Item 37: Accurate with 2-digit differences (Set 5F)

Domain 6: Grouping and Place-Value Knowledge
- Item 38: Writes 2-digit numerals (Set 6A)
- Item 39: Composing/decomposing 10 & equivalents of 10 (Set 6A)
- Item 40: Equates 2-digit numbers with groups(s) of 10 & 1s (Set 6A)
- Item 41: Strategic counting by tens and ones (Set 6A)
- Item 42: Strategic counting by 10s and 1s with renaming (Set 6A)
- Item 43: Writes 3-digit numerals (Set 6B)
- Item 44: Composing/decomposing 100 & equivalents of 100 (Set 6B)
- Item 45: Equates 3-digit numbers to groups(s) of 100, 10, 1 (Set 6B)
- Item 46: Strategic counting by hundreds, tens, and ones (Set 6B)
- Item 47: Strategic counting by 100s, 10s, and 1s with renaming (Set 6B)
- Item 48: Composing and decomposing 1000/equivalents of 1000 (Set 6C)
- Item 49: Strategic counting by 1000s, 100s, 10s, and 1s (Set 6C)
- Item 50: Strategic counting by 1000s, 100s, 10s, and 1s with renaming (Set 6C)

Domain 7: Calculation-2 (multiple digits)
- Item 51: Accurate 2-digit + & - /no renaming (Set 7A)
- Item 52: Accurate with 2-/3-digit addition/renaming (Set 7B)
- Item 53: Accurate 2- & 3-digit subtraction with renaming (7C)

-5 to 4+ indicates item difficulty along the same continuum of early numeracy.
X = Estimated Person Abilities; Each X represents 3.2 cases
Colored numbers = Calibrated item difficulty

Figure 1: Wright Map of Calibrated Item Difficulty

challenging and most challenging and typically achieved by older and more capable children. Within each domain block, items spread across a range, suggesting varying degrees of abilities that such items try to capture. This also gives us leverage to further develop the adaptive algorithm, so that given a child's performance on earlier items, we will be able to choose the next best item that is closest to the child's true ability along the continuum.

Conclusions

All six dimensions were highly inter-correlated, and a unidimensional structure was strongly supported by the data. These results are not surprising given the interdependency of numeracy knowledge.

Item scoring originally provided partial credit for reasonable answers (e.g., incorrect answers that at least honored the meaning of an operation and were in the right direction), indicated a child basically understood how to determine an answer but slipped up and was off by 1 or 2, or could at least make a reasonable estimate of an answer (i.e., off or within 25% of a correct answer). However, such reflections of "operation sense" occurred so infrequently that they had to be eliminated as a candidate for partial credit.

The item calibration provides reassurance that the e-TEN assesses the whole range of numeracy ability. Some similar items, such as Items 25 (reads 3-digit numerals) and 43 (writes 3-digit numerals) and Item 33 (fluency with large near doubles such as 8 + 7) and Item 34 (fluency with combinations involving 9) had the highly similar or even the same level of difficulty. In such cases, one item may be eliminated to shorten testing times.

The item calibration largely corroborated the developmental trajectories on which items were ordered. For example, consistent with previous research (Huttenlocher, Jordan, & Levine, 1994; Levine, Jordan, & Huttenlocher, 1992), success on a nonverbal addition/subtraction task emerges in a step-like manner (with collections of 1 and 2 first, then those involving 3, and finally with 4 items; Items 27, 28, and 29, respectively) and does so before children are successful with verbally presented word problems (Item 30). The results confirmed that the small doubles such as 3 + 3 and 4 + 4 (Item 32) are among the easiest of combinations and, in something of a surprise, were even slightly easier than the highly salient add-1 combinations (Item 31; cf. Baroody, Purpura, Eiland, & Reid, 2015). Moreover, the large doubles such as 8+8 (Item 33) were as difficult as combinations involving adding 9 (Item 34). Another unexpected result was that Item 42 (strategic counting by 10s and 1s with renaming) proved as easy as Item 41 (strategic counting by 10s and 1s). Theoretically, flexibly counting representations of tens and ones by tens and then by ones should be easier than doing so when there are more than 10 representations of ones, which requires regrouping them into a ten (Chan, Au, & Tang, 2014). Moreover, in something of a surprise, Item 42 (strategic counting with 1s and 10s with renaming) was easier than Item 40 (equating 2-digit numbers with groups of 10s & 1s). (It is less surprising that Item 41 (strategic counting 1s and 10s without renaming) was easier than Item 40 or that Item 46 (strategic counting with 1s, 10s and 100s without renaming) was somewhat easier than Item 45 (equating 3-digit numbers with groups of 100s, 10s & 1s), because flexible enumeration may or may not involve understanding grouping and place-value ideas). Additional research is needed to see if these curious result holds and, if so, why.

For a full report, see: https://www.researchgate.net/project/NSF-funded-Development-of-the-Electronic-Test-of-Early-Numeracy-and-IES-funded-Evaluating-the-Efficacy-of-Learning-Trajectories-in-Early-Mathematics.

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GTAS' LIVED EXPERIENCE IN THEIR DUAL ROLE AS STUDENTS AND TEACHERS OF MATHEMATICS

Nate Jewkes
North Carolina State University
njjewkes@ncsu.edu

Karen A. Keene
North Carolina State University
kakeene@ncsu.edu

As both students and teachers of mathematics, graduate teaching assistants' (GTAs') identities are shaped by their experience in both learning and teaching. Through interviews with four GTAs in a large mathematics department, the purpose of this pilot study was to gain insight into their identities and lived experience as they fulfill these two roles simultaneously. Preliminary findings highlighted GTAs' perceptions of other GTA peers' prioritization of teaching and schoolwork, navigation of relationships with and expectations from faculty, and the maintenance of a week-by-week balance between teaching, research, and coursework.

Keywords: Graduate teaching assistants, Doctoral education, Undergraduate-level mathematics

Purpose

The challenges that departments of mathematics face in preparing graduate teaching assistants (GTAs) to teach undergraduate mathematics served as the motivation behind this pilot study. The need for better support for this unique group of novice teachers first requires a better understanding of "who [G]TAs are… and how their communities help define themselves and their profession" (Speer, Murphy, & Gutmann, 2009, p. 7). This study aimed to gain insight into the identities and lived experience of mathematics GTAs as they fulfill both roles of *student* (i.e., their own coursework and research) and *teacher* (i.e., teaching and/or recitation responsibilities for undergraduate mathematics courses). The guiding question of this research was: What is the interplay between GTA student and teacher roles, and to what extent does each role play in defining their lived experience and identities?

Literature Review

The literature surrounding mathematics GTAs' identities and lived experience is still developing. Much of the research on mathematics GTAs addresses their challenges (e.g., Hauk et al., 2009; Meel, 2000; Park, 2004), their preparation for teaching (e.g., Belnap & Allred, 2009; Deshler, Hauk, & Speer, 2015; Kung & Speer, 2009; Speer, Gutmann, & Murphy, 2005), and their knowledge and beliefs related to teaching (e.g., Musgrave & Carlson, 2017; Speer, 2008; Speer & Hald, 2008; Speer, King, & Howell, 2015; Speer, Strickland, Johnson, & Gucler, 2006; Speer & Wagner, 2009; Wagner, Speer, & Rossa, 2007). Alternatively, more research is found about GTAs' identities and experiences when searching for work about GTAs in other disciplines (e.g., science, business, humanities, etc.). The following sections draw on both bodies of research.

GTA-Faculty Relationship

GTAs' identities are partially shaped by their relationship with faculty that teach and mentor them. GTAs often experience role ambiguity, or difficulty knowing faculty's expectations for their teaching, student, and research responsibilities (Duba-Biederman, 1991; Park, 2004). Furthermore, because GTAs have dual roles, so do the faculty that advise and supervise them. While many faculty members are adept advisors (i.e., advising GTAs as students about courses, research, etc.), some may lack supervisory skills (i.e., supervising and providing feedback to GTAs in their teaching roles) (Duba-Biederman, 1991; Nyquist & Wulff, 1996). GTAs often report positive relationships with faculty, but they also express the need for more feedback and support (Duba-Biederman, 1991; Latulippe, 2009; Meel, 2000). GTAs may rely on their GTA peers more than faculty for information and support about their multiple roles (Duba-Biederman, 1991; Park, 2004), or when planning to teach (Winter, DeLong, & Wesley, 2009).

GTA-Student Relationship

In pre-semester trainings, GTAs are often advised on how to maintain professionalism with the students they teach (Deshler et al., 2015). Because GTAs are often novice teachers (Speer et al., 2005), and relatively close in age to the undergraduates they teach, establishing professional relationships with students can be a challenge. Nyquist and Wulff (1996) describe how beginning GTAs may feel a desire to become close friends with their students, sometimes to the point of "becoming advocates for undergraduates who are experiencing what they feel to be unjust practices by faculty members" (p. 24). On the other hand, GTAs might resent their emotional investment in their students' welfare after students miss class, fail to hand in homework, or perform lower than expected on exams (Nyquist & Wulff, 1996).

GTAs' Experiences and Challenges

Research on the experiences of mathematics GTAs reveals that they struggle with balancing home, instructor, student, and junior researcher roles, searching for jobs, learning about potential responsibilities as a future faculty member, classroom management issues, and creating an active learning environment (Hauk et al., 2009; Meel, 2000). They often feel lack of autonomy in their teaching and perceive their teaching duties are need-based and not planned out (Austin, 2002; Park & Ramos, 2002).

Methods

The purpose of this phenomenological study (Moustakas, 1994), is to begin to capture the "essence" (Creswell & Poth, 2018, p. 80) of GTAs' lived experience as both students and teachers of mathematics, as well as how the challenges associated with these roles shape their identities. We use Gee's (2001) *institutional* and *affinity* identity sources. This conception is useful since GTAs have designations of "graduate student" and "teaching assistant" bestowed upon them by an institution, which establishes their *institutional* identity. GTAs also engage in common practices and share common interests in mathematics and teaching, which helps define their *affinity* identity.

Participants and Data Collection

Two of the GTAs identified as female, two as male, and all GTAs described themselves as White, with one identifying as Latina (Sarah). Sarah and Greg were recitation leaders and graders for Calculus I, David was a lecture assistant and grader for Calculus III, and Heather was the instructor of record for a large-lecture section of Precalculus. All were doctoral students in mathematics at a large, public research institution in the eastern United States. Consistent with the primary data source in phenomenological studies (Moustakas, 1994), semi-structured interviews with the four mathematics GTAs were conducted near the end of the Fall 2018 semester. Two of the interview questions were given to the participants beforehand to allow interviewees time to think about specific examples of student- and teacher-related challenges they had experienced as GTAs. During the interview, GTAs were asked to respond to questions such as: What do you think is a struggle or challenge that most mathematics GTAs deal with in balancing their roles as both students and teachers? The interview questions were meant to gather data leading to a "textual and structural description" (Creswell & Poth, 2018, p. 79) of the GTAs' common experiences of being a teacher and student of mathematics and were modeled after Moustakas' (1994) broad questions that solicit participants' experience with a phenomena.

Data Analysis

Each interview was audio-recorded, and subsequently transcribed verbatim by the primary author. The transcripts were analyzed (also by the primary author) through the process of *horizontalization* (Moustakas, 1994), where 18 preliminary themes were generated from identifying and grouping significant statements that captured the essence of balancing roles as both students and teachers, and how GTAs' described their experiences related to their institutional and affinity identities (Gee, 2001). The second author then examined the preliminary themes and corresponding significant statements, and together we consolidated them into eight emergent themes.

Researcher Positionality and Bracketing

As a former mathematics graduate student with teaching assistant experience, and as a current mathematics education student with continued GTA duties, the primary author can identify with many of the experiences that the GTA participants shared during the interviews. As part of the phenomenological lens, the primary author thus attempted to *bracket* (Moustakas, 1994) himself from the study by identifying and setting aside his own experiences with the phenomena in order to "take a fresh perspective toward the phenomena under examination" (Creswell & Poth, 2018, p. 78).

Preliminary Findings

While the interviewed GTAs said many things that related to the student or teacher roles separately, we constructed themes focusing on those statements that specifically addressed the phenomena of being a student and teacher of mathematics simultaneously, or the balance between these two roles. All eight themes are listed in Table 1 but due to space constraints, we highlight just three of the themes.

Table 1: Emergent Themes

Theme	**GTAs whose statements were coded for the theme**
Week-by-Week Balance	Sarah, Heather, Greg, David
Mixed Messages from Faculty on Role Prioritization	Heather, Greg
Other GTAs Prioritize Research and Coursework Over Teaching	Sarah, Heather, Greg, David
Maintaining Professionalism with Undergraduate Students	Greg, David
Relying on GTA peers	Sarah, Heather, Greg
Feeling Out of Place	Sarah, Heather, Greg, David
GTA-Faculty Relationship	Sarah, Heather
Sitting on Both Sides of the Table	Sarah, Heather, Greg, David

Mixed Messages from Faculty on Role Prioritization

Heather perceived some incongruence in how teaching duties are assigned and how her advisor and other faculty members tell her that her teaching role is not as important as her own studies:

> Heather: So like when someone says 'teach a 200-person class, but teaching isn't your first priority', that doesn't compute in my brain, like I'm going to have to pour myself into this to really be successful.

Feeling Out of Place

In discussing challenging coursework and teaching responsibilities in her first semester, Sarah felt displaced.

> Sarah: I felt uh, what does everybody call it? Imposter syndrome?... It was hard to get used to, living in a new place not knowing anyone, feeling like you don't belong.

Other GTAs Prioritize Research and Coursework Over Teaching

Even though only two of the participants said that they plan to teach for a career (Sarah and David), all four GTAs expressed a love for teaching and a desire to be helpful to students, and all four also described how they feel that not all GTAs share that same positive attitude towards teaching.

> David: Maybe it's just me who kind of feels concerned about that; about maybe wanting to improve like attendance in office hours, you know? Other [GTAs] might see it as more of a blessing or something. They get their hour or two back.

Greg, a recitation leader for Calculus I, shared his perception of his peers' prioritization of the two roles.

> Greg: The majority of students are here for careers in research and then teaching is what pays the bills getting to that point.

Limitations

One of the limitations of this pilot study is the small, voluntary sample that may not be representative of all mathematics GTAs. For example, the GTAs who agreed to participate may have also been those more likely to enjoy their teaching roles. It would be enlightening to interview GTAs who express dislike for teaching and how that affects their experience. All four GTAs identified as White with one identifying as Latina, and there were no international GTAs. A larger and more diverse group of GTAs would yield a more authentic description of GTAs' lived experience and identity.

Discussion and Conclusion

The lived experience of being a mathematics GTA with both student and teacher roles is complex. The GTAs in the study expressed a love for teaching and a desire to positively impact students but acknowledged that time and energy dedicated to their teaching and student roles can sometimes be out of balance. Like the findings of Winter et al. (2009), the GTAs relied on peers for assistance in both roles, especially in teaching. This suggests that professional development programs for GTAs should leverage the natural environment of peer-centered support and collaboration. The identities of mathematics GTAs that fulfill multiple roles is equally complex. In GTAs' minds, their institutional designations of 'student' and 'teacher' may be partly defined by the emphasis that faculty place on these roles. The GTAs in this study felt like they received mixed messages regarding the proper balance of these institutional identities. This suggests that faculty members that supervise and advise graduate students should be mindful in the way that the prioritization of these roles is discussed, and that neither role should be devalued even if the student role is the primary focus of a GTA's experience. In conclusion, this pilot study yields new information that contributes to all GTA professional development work by hearing from the GTAs themselves. As mentioned before, those involved in improving GTA support could benefit by building on this pilot and future work in this area.

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INNOVATING SCAFFOLDED PROTOTYPING FOR DESIGN EDUCATION: TOWARD A CONCEPTUAL FRAMEWORK DERIVED FROM MATHEMATICS PEDAGOGY

Hadi Ali
Arizona State University
hwali@asu.edu

Barbara Kinach, Ed.D.
Arizona State University
barbara.kinach@asu.edu

Micah Lande, Ph.D.
South Dakota School of Mines & Technology
micah.lande@gmail.com

The practice of prototyping is challenging to novice designers as they underutilize insights that prototyping offers to solving design problem. Central to this challenge is the abstract nature of design concepts like idea representation, iteration, and problem solution-space exploration. A unique opportunity from mathematics education presents itself for design educators and facilitators; that is, teaching with manipulatives. We seek to transfer such practices in mathematics education to design education and practice. Challenges exist for design researchers to carefully craft activities in design education mainly because of the open-endedness of problems, the decision-making that takes place while designing, and the inherent uncertainties in the design problems. Ultimately, the goal is to develop students' ability to flexibly transfer expertise to other contexts and new design challenges.

Keywords: Prototyping, Manipulatives, Design education, Mathematics education

Objective: Navigating the Prototyping Challenge in Design Education

Engineers build and create with materials ("stuff") to fashion new designs. The practice of prototyping in the design process can be challenging for novice designers however. Students learning to become engineers often underutilize prototyping as a step for their design process (Ali & Lande, 2018). The abstract and nebulous nature of what and how to prototype is central to this challenge. We propose to navigate the prototyping challenge within design education with insights from mathematics education.

Theoretical Perspective: Prototyping in Design Education Against a New Horizon

Drawing parallels for design education from the manipulative "prototyping" literature within mathematics education we glean insight into the learning and teaching of abstract notions and processes. Borrowing teaching with manipulatives as a pedagogical approach from mathematics education presents a unique opportunity for design educators and facilitators. In mathematics education, modeling mathematical ideas with manipulatives has been demonstrated to be effective in teaching students abstract concepts and processes of mathematics, like addition and subtraction (Carpenter, Fennema, Franke, Levi & Empson, 2015), fractions (Cramer, 2002, 2003) and bases of ten (Dienes, 1960), and function (Arcavi, Tirosh & Nachmias, 1989). Established theories of mathematics learning (Dienes, 1960; Bruner, 1963) support the practical development of activities to introduce abstract mathematical ideas to students by means of manipulating physical objects while reasoning about the represented mathematical abstraction (Thompson & Thompson, 1990; Arcavi, 2003). Students are able to complete sophisticated mathematical processes with understanding through this visual externalization of their "thinking with" tangible items (Carpenter et al., 2015). We map such practices in mathematics education to transfer to design education and practice. Specifically, we aim to develop examples of scaffolded prototyping in manipulatives as a way to move from concrete representations of prototypes to abstract understanding of design.

There are parallels between the conceptualizations of prototyping within the “design” and “mathematics” education literature. Within each domain, there is a specific subject-matter content to be applied: mathematical ideas applied through algebraic symbols, built and tangible creations applying scientific and engineering principles. The phenomena of thinking moving from concrete to abstract though is similar. Both design artifacts and manipulative representations provide conceptual affordances otherwise difficult to access for learners.

Within engineering design, a prototype provides a means for feedback (Jørgensen, 1984), between the designer and the material (Lim, Stolterman & Tenenberg, 2008), amongst the design team (Hartmann, 2009; (Hollan, Hutchins & Kirsh, 2000; Hutchins, 1995; Star & Griesemer, 1989), or between the team and the user (Buchenau & Suri, 2000; Kelley & Littman, 2006; Schrage, 2013). Similarly, the function of the “prototype” physical embodiment (manipulative representation) of an abstract mathematical idea is to feature certain aspects of the mathematical notion, ideally in a one-to-one correspondence between mathematical objects, processes, and relations. Representational theories of mathematics learning (Dienes, 1960; Bruner, 1963) posit that humans learn abstract ideas (no matter the field) not from the abstraction or symbolization itself but through generalizing the abstract idea from multiple concrete representations of it. No single “prototype” (aka manipulative representation or concrete embodiment) will ever fully depict a mathematical abstraction. But from each prototypical manipulative representation, the learner gleans an aspect of the abstraction not captured in the same way by the other prototypical manipulative representations. From this collection of concrete (prototypical) representations, the learner generalizes the abstract mathematical concept. There are clear connections to a design artifact and mathematical education research provides language to more precisely develop mindful design learning experiences and expectations for physical prototyping.

Like aspects of prototypical engineering thinking, algebraic thinking involves the practices of representation; pattern finding and generalizing functions from the relationship between quantities; modification; and abstraction (Kieran, 2006). These are practices that are transferrable to the abstract representation of ideas in design prototyping and defining the relationships between shapes and functions in prototypes. One key misconception in the teaching and learning of design is that students approach prototyping as a final product, as opposed to intermediate steps to learn about the design problem (Cross, 2018), gain insight (Lawson & Dorst, 2009), and reflectively learn design (Beckman & Barry, 2007). Moreover, the focus, in teaching design currently, is not on simplifying the approach to the solution, nor attempting to find the most efficient one; rather to breakdown the process and make it visible, with the ultimate goal for students to take this learning to other contexts. Thus, just as the literature on mathematical knowledge for teaching recognizes that being able to “do” mathematics is different from being able to “teach” mathematics (Ball & Bass, 2002), design instructors would do well to recognize the same for engineering and engineering design. Similar to the expectations for a mathematics instructor, the design instructor should be able to carefully plan the lesson, creating a range of possible, yet expected, paths that the students may take.

Modes of Inquiry: Results on the Horizon

Design educators do well on encouraging students to be creative, but they do not take this to the next step of actually teaching design, harnessing this creativity, and guiding the process. There is a gap, beyond students sharing their ideas and methods in the class, to actually learn design. The questioning approach to teaching is very much needed (Boaler & Humphreys, 2008). A design educator should build on these teaching practices, as intermediate milestones, asking questions, and using the artifact to elicit understanding (Cramer, 2003). There is an opportunity to use case studies to capture such moments in design education, which is often beyond simply characterizing students' behavior or observing current practices in teaching.

The overarching research question for this research design is as follows: *How can the utilization of object representation of mathematical concepts in mathematics education be transferred to object representation in scaffolded prototyping in design education?*

Preliminary Research Design

Overview and Participant Characteristics

In order to conduct the research study, our strategy involves creating a case for building prototypes in a controlled observation environment. The research design emphasizes producing relevant and meaningful results through substantiated rationale and supported by evidence from the collected data. In the preliminary stage of the research, participants of four students from four different disciplines (design, engineering, business and sustainability) will be recruited to complete a structured design task. Because different disciplines emphasize different epistemologies and practices, the setting will evoke the unique utilization of manipulatives as boundary spanning objects between the different participants. The research design will aim for case creation and comparisons across multiple groups of participants. Ideally, design projects span multiple months; however, the controlled task setting in a laboratory setting will allow a focused laboratory-based study.

Characterizing the Design Task

The participants will be provided an open prompt accompanied by a carefully selected set of manipulatives. Observations of group interactions will be video- and audio-recorded to allow for careful analysis. Control for the objects that participants will have access to should allow for augmentation of the underlying cognition and the corresponding exhibited behavior by participants. In order to select the manipulatives that participants will have access to, key considerations should be provided for the following issues surrounding prototyping (Lewrick, Link & Leifer, 2018):

- What are the basic functions for the user?
- What hasn't the user taken into consideration at all yet?
- How has nobody ever done it before?

The manipulatives should allow physical modeling to represent products, spaces and environment, in a way that provides an executable version of the design but with only the most necessary functions. In addition, the observatory lab setting should allow the reproduction of specific situations with team members doing physical acting.

Example. In an attempt to pilot the study to understand students' approaches to prototyping, we asked students to create three different prototypes of "an exercise machine that saves time and space." In this project, the idea was to push the students beyond the machine itself, thinking about larger contexts of exercising and healthy living—a readily available machine in a dorm room, for example, can save time for the students not needing to walk for the gym if it is designed in a way not to take much space as well.

Contextualizing the Design Task

Generally, the prompt that will be provided to the students should be planned to cross the boundaries between their different areas of expertise. More specifically, it should be framed to push the students to construct a collective, shared understanding in a very direct and visual manner. The *directness* and *visibility* of the idea representations should be a center in the design activity, so that playful insights are attained across multiple interactions with the objects. The goal is to provide a deeper, shared understanding of the solution vision while prompting collaboration and discussions. The design task tentatively has the following structure, which moves participants in a series of divergent and convergent modes of thinking:

- Given a set of provided initial solution scenarios, identify which one you would like to test. In the process, you should ponder which functions are absolutely critical
- Think about which variant should be built
- At this point, an intervention on how to allow the "objects to talk" should be perceived
- The built prototypes should allow feedback, moving to the next phase of essential features and solutions in a scaffolded manner. Breaking the groups of four into pairs should allow some variety for testing ideas
- Finally, revised new variants of prototypes should be conceived.

Example. In our pilot study of the "exercise machine," students were expected to combine analytical thinking with creative synthesis. Looking ahead at the implementation of the study, a promising opportunity exists because the learning objectives of the design course are aligned with the mathematical and physical knowledge that the students have acquired at this stage. For example, students are expected to utilize springs as means to achieve the desired functionality in their proposed design. This presents an opportunity to develop course modules, using manipulatives, to present the different behavior of springs to the students. Once they understand and are able to mathematically analyze the behavior of the spring, they can start using it in their prototype that is directed to fulfill the need of a user. Figure 1 below provides an example of the students' work in the pilot study.

Figure 1: Sample of a Students' Prototype in the Pilot Study: "Resistive Core Trainer"

Conclusions and Future Work

Ultimately, the goal is to allow students to flexibly transfer the learned expertise in the proposed activities to other contexts and new design challenges. Building on the understandings for learning from mathematics can have useful implications for the teaching and learning of engineering design. Furthermore, the outcome of this study could guide future bidirectional studies by benefiting not only design education, but also mathematics education, especially at the college-level courses.

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ABSTRACTED QUANTITATIVE STRUCTURES

Kevin C. Moore
University of Georgia
kvcmoore@uga.edu

Biyao Liang
University of Georgia
leungbiyao@uga.edu

Halil I. Tasova
University of Georgia
halil.tasova25@uga.edu

Irma E. Stevens
University of Georgia
istevens@uga.edu

Research on quantitative and covariational reasoning has emerged as a critical area of study in recent decades. We extend this body of research by introducing the construct of an abstracted quantitative structure. In addition to introducing the construct, we illustrate it by presenting empirical examples of student actions. We close with implications for research and teaching.

Keywords: Cognition, Algebra and Algebraic Thinking, Calculus, and Learning Theory

Steffe and Thompson enacted and sustained research programs that have characterized students' (and teachers') mathematical development in terms of their conceiving and reasoning about measurable or countable attributes (see Steffe & Olive, 2010; Thompson & Carlson, 2017). Thompson (1990, 2011) formalized such reasoning about measurable attributes into a system of mental actions and operations he termed *quantitative reasoning.* In this paper, we extend this work by introducing the construct *abstracted quantitative structure*: a system of quantitative relationships a person has interiorized to the extent they can operate as if it is independent of specific figurative material (i.e., representation free).

Background

Thompson (2011) defined quantitative reasoning as the mental operations involved in conceiving a situation in terms of measurable attributes, called *quantities*, and relationships between those attributes, called quantitative relationships. One form of quantitative reasoning involves constructing relationships between two quantities that vary in tandem, or covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Carlson et al. (2002), Confrey and Smith (1995), Ellis (2011), Johnson (2015a, 2015b), and Thompson and Carlson (2017) are a few researchers who have detailed frameworks and particular mental actions associated with students' covariational reasoning. We narrow our focus to that of Carlson et al. (2002), who identified five mental actions associated with covariational reasoning. A critical mental action of their framework, especially for differentiating between various function classes, is to compare *amounts of change* to draw inferences about quantities' covariation (**Figure 20**).

Piagetian notions of *figurative* and *operative* thought (Piaget, 2001; Steffe, 1991; Thompson, 1985) differentiate between thought based in and constrained to figurative material (e.g., perceptual objects and sensorimotor actions)—termed *figurative thought*—and thought in which figurative material is subordinate to logico-mathematical operations and possibly their transformations—termed *operative thought*. Quantitative and covariational reasoning are examples of operative thought due to their basis in logico-mathematical operations (Steffe & Olive, 2010). Moore, Stevens, Paoletti, Hobson, and Liang (online) illustrated figurative graphing meanings in which prospective secondary teachers were constrained to constructing graphs with particular perceptual features (e.g., drawing a graph solely left-to-right) even when they acknowledged those features did not enable them to viably represent a conceived relationship. In contrast, Moore et al. (online) described that a prospective secondary teacher's graphing meaning is operative when perceptual and sensorimotor features of their graphing actions are persistently dominated by the mental operations associated with quantitative and covariational reasoning.

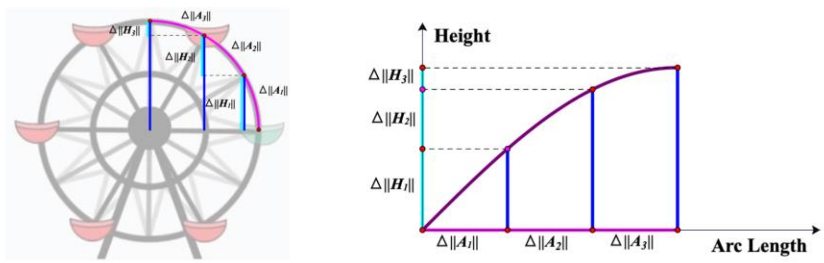

Figure 20: For Equal Increases in Arc Length (counterclockwise direction from the 3 o'clock position), Height Increases by Decreasing Amounts

Abstracted Quantitative Structure

We define an abstracted quantitative structure as a system of quantitative relationships a person has interiorized to the extent he or she can operate as if it is independent of specific figurative material. An abstracted quantitative structure can also be re-presented to accommodate novel contexts or situations permitting the associated quantitative operations. Our notion of an abstracted quantitative structure is an extension of von Glasersfeld's (1982) definition of concept to the context of quantitative and covariational reasoning. von Glaserfeld defined a *concept* as, "any structure that has been abstracted from the process of experiential construction as recurrently usable…must be stable enough to be re-presented in the absence of perceptual 'input'" (p. 194). An abstracted quantitative structure entails both of these features; an abstracted quantitative structure is recurrently usable beyond the initial experiential construction and it can be re-presented in the absence of available perceptual (or figurative) material.

An abstracted quantitative structure can accommodate novel situations through another process of experiential construction within the context of figurative material not previously experienced in the context of using that structure. This action is a hallmark of operative thought due to the action entailing an individual using the operations of their quantitative structure to accommodate novel quantities and associated figurative material. It is in this way that the quantitative structure of an abstracted quantitative structure is abstract; mathematical properties are understood as not tied to any particular two quantities and associated figurative material.

Indications and Contraindications

The way we have defined abstracted quantitative structure presents an inherent problem in characterizing a student as having abstracted a quantitative structure: it is impossible to investigate a student's reasoning in every situation in which an abstracted quantitative structure could be relevant. For this reason, we find it necessary to discuss a student's actions in terms of indications and contraindications of her or him having constructed an abstracted quantitative structure. What follows are examples that we use to illustrate contraindications or indications of students having constructed abstracted quantitative structures.

Lydia and re-presenting. Lydia was a prospective secondary teacher in a teaching experiment focused on trigonometric relationships and re-presentation (Liang & Moore, 2018). Lydia initially engaged in a task in which she constructed incremental changes compatible with those presented in Figure 20 (left). We took her actions to indicate her reasoning quantitatively and presented her the *Which One?* task. The task (Figure 21, left) presented Lydia with numerous red segments that varied in tandem as the user varied a horizontal (blue) segment, which represented the rider's arc length traveled along the circle. We designed only one red segment to covary with the blue segment in a way compatible with the vertical height and arc length of the rider. Lydia's task was to choose which, if any, of the red segments represented that relationship. Lydia chose the correct red segment by re-orienting it vertically and checking whether the heights matched pointwise within the displayed circle (Figure 21, middle). We then asked her if the chosen segment and blue segment entailed the same covariational relationship constructed in her previous activity involving the Ferris wheel (see Figure 20, left) (from Liang & Moore, 2018):

> Lydia: Not really…Um, I don't know. [*laughs*] Because that was just like something that I had seen for the first time, so I don't know if that will like show in every other case…Well, for a theory to hold true, it like – it needs to be true in other occasions, um, unless defined to one occasion.
> TR: So is what we're looking at right now different than what we were looking at with the Ferris wheel?
> Lydia: No. It's – No…Because I saw what I saw, and I saw that difference in the Ferris wheel, but I don't see it here, and so –
> TR: And by you "don't see it here," you mean you don't see it in that red segment?
> Lydia: Yes.

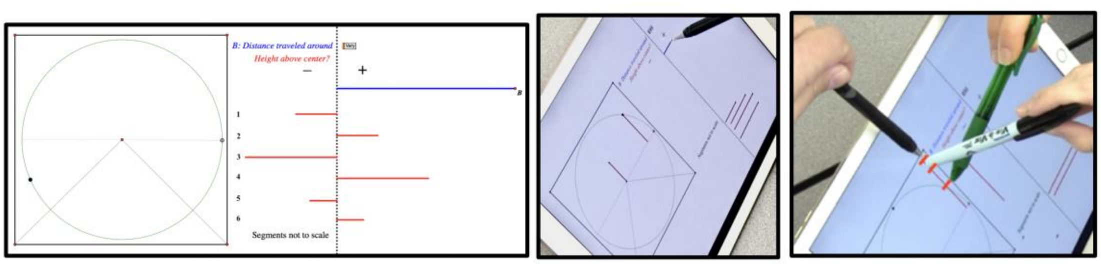

Figure 21: (left) Which One?; (middle) Lydia Checking Segment Pointwise; and (right) Lydia Attempting to Re-present a Quantitative Structure

Lydia's actions are a contraindication of her having constructed an abstracted quantitative structure during her previous activity. Specifically, she could not re-present the activity she engaged in with respect to the Ferris wheel situation (Figure 20, left) when provided with what was to her novel figurative material in the *Which One?* task. As a further contraindication of her having constructed an abstracted quantitative structure, it was only after much subsequent teacher-researcher guiding and their introducing perceptual material using their pens (**Figure 21**, right) that she was able to conceive the red and blue segments' covariation as compatible with the relationship she had constructed in the Ferris wheel situation.

Noli and the inverse sine relationship. We draw on a prospective teacher's response to the graphs in Figure 22 (left and middle). We presented a graph consistent with Figure 22 (left) to

Noli, a prospective teacher, as hypothetical student work and asked whether the graph represents the inverse sine (or arcsine) function. Noli identified that Figure 22 (left) can be thought of as representing the inverse sine function by considering the vertical axis the input of the function (see Figure 22, right). In response, we presented Figure 22 (middle) and explained that a hypothetical student claimed Figure 22 (middle) represents the inverse sine function rather than Figure 22 (left). Noli claimed that both graphs could represent the inverse sine function (completed work in Figure 22, right):

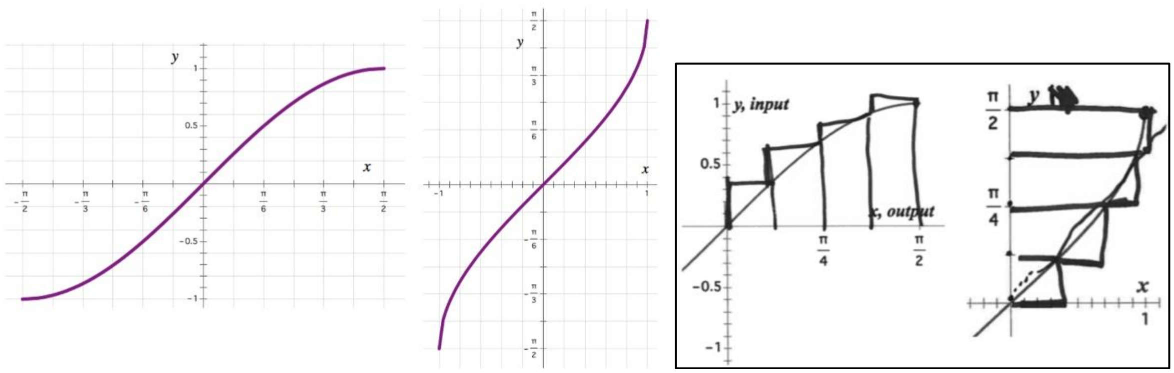

Figure 22: Graphs of (left) $x = \sin^{-1}(y)$, (middle) $y = \sin^{-1}(x)$, and (right) Noli's work.

> Noli: [*Noli has identified that Figure 22, left and middle, represent $x = \sin^{-1}(y)$ and $y = \sin^{-1}(x)$, respectively*] They're both representing the same thing just considering their outputs and inputs differently [*referring to axes*]…So it's like here [*referring to Figure 22, middle, $y > 0$*], with equal changes of angle measures [*denoting equal changes along the vertical axis*] my vertical distance is increasing at a decreasing rate [*tracing graph*]…here [*referring to Figure 22, left, $x > 0$*] it's doing the exact same thing. With equal changes of angle measures [*denoting equal changes along the horizontal axis*] my vertical distance is increasing at a decreasing rate [*tracing graph*]…this one looks like it's concave up [*referring to Figure 22, middle from $0 < x < 1$*] and this one concave down [*referring to Figure 22, left from $0 < x < \pi/2$*], it's still showing the same thing.

We interpret Noli's actions as indicating her having constructed an abstracted quantitative structure that she associates with the "inverse sine function…sine function." Noli understood each graph as representing equivalent covariational properties despite their differences in shape; she understood that both "concave up" and "concave down" graphs represent one quantity increasing by decreasing amounts for equal successive variations in the other quantity. Furthermore, she flexibly moved between re-presentations of this covariational relationship, adjusting for the alternative coordinate orientations, which is a contraindication of her reasoning being dominated by figurative aspects of thought.

Implications

The construct of an abstracted quantitative structure provides a specified criterion for claims about students' and teachers' quantitative and covariational reasoning. In our previous work (e.g., Moore, 2014), we made claims about a student constructing a particular function or relationship based on her or his activity in one, or at most two, contexts. If we frame the construction of a function or relationship in terms of an abstracted quantitative structure, then our evidence within those previous studies is insufficient to make such claims. Making such claims requires studying a student's actions in a variety of contexts in which her or his actions can provide indications or contraindications of such a structure. Likewise, and transitioning our focus to teaching, it is speculative at best to claim one has taught a function or relationship concept if one has not explored their students' reasoning in more than one context and relationship.

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UNIT TRANSFORMATION CAPACITY: ASSESSING COGNITIVE DEMAND WITH UNITS COORDINATION AND WORKING MEMORY

Sarah Kerrigan
Virginia Tech
stk123@vt.edu

Anderson Norton
Virginia Tech
norton3@vt.edu

Catherine Ulrich
Virginia Tech
culrich@vt.edu

Understanding student cognition has been a crucial part of mathematics educational research for decades. The construct of units coordination has proven to be central in students' mathematical development. The psychological construct of working memory has also been shown to have an important role in cognitive development. This report combines units coordination and working memory to create a new theoretical construct called unit transformation capacity, aimed to capture nuances in modeling students' mathematics.

Keywords: Cognition, Research Methods, Learning Progressions

Motivation

Understanding students' struggles and cognitive development in mathematics is important in developing better curricula that foster mathematical foundations. Building models of students' mathematics provides an avenue for understanding what cognitive structures are needed for students to progress in mathematics. Scheme theory has been a primary mode for modeling students' mathematics, through describing assimilatory processes, mental actions, and expected results. The construct of schemes has been applied to various mathematical domains in K-12 (Steffe, 1992; Olive & Çağlayan, 2008; Olive & Steffe, 2010) and post-secondary schooling (Akkoç & Tall, 2002; Arnon et al., 2014; Cottrill, Dubinksy, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996).

In research on primary school education, units coordination (UC) provides a particularly valuable construct for modeling mathematical cognition. UC has been shown to be connected across several mathematical domains including students' construction of number sequences (Ulrich, 2015, 2016), fractions knowledge (Hackenberg & Tillema, 2009; Izsák, Jacobson, de Araujo, & Orrill, 2012; Norton & Boyce, 2013; Steffe, 2010), and algebraic reasoning (Hackenberg & Lee, 2015; Olive & Çağlayan, 2008). Since UC is an integral component of students' mathematics, it is beneficial to continue to investigate how to engage students in tasks that facilitate mathematical development.

This report characterizes a new theoretical framework that combines UC and working memory as a research tool for building models of students' mathematics, and as possible assessment tools for designing tasks and curricula.

Theoretical Framework

Our proposed theoretical framework utilizes the mathematics-specific construct of UC (Hackenberg & Tillema, 2009; Steffe, 1992; Ulrich, 2015) and the psychological construct of working memory to better capture students' mathematics. UC describes students' engagement with various levels of units in mathematical tasks (Steffe, 1992). Pascual-Leone (1970) developed the psychological construct of M-capacity to model working memory. This report presents a new construct that combines UC and working memory to capture and explain nuances in students' engagement in mathematical tasks.

Units Coordination

UC describes students' ability to construct and transform various levels of units in mathematical tasks across mathematical domains. This construct emerged out of Steffe's (1992) research on children's construction of whole number knowledge: "I think of units coordinating as the mental operation of distributing a composite unit across the elements of another composite unit" (Steffe, 2001, p. 279). For example, for a task to be classified as multiplicative, there needs to be at least two levels of units that are coordinated so that one composite unit is distributed over the second composite unit. In the task of 7 times 5, the composite units are the 5 composed of five 1s and the 7 composed of seven 1s. A student could solve this task by distributing seven units of 1 across each single unit of 5 to produce a new unit of 35. In this case, the student sees 35 as composed of five units of seven units of 1 (Figure 1).

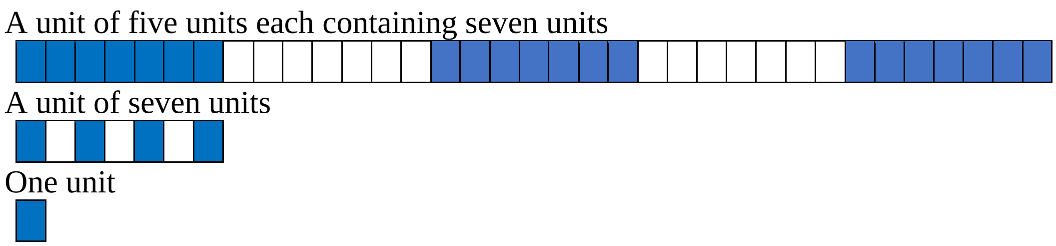


Figure 1: An Example of Coordinating Composite Units in the Problem of 7 Times 5

UC is classified in three stages depending on the number of levels of units the student is able to take as given and operate on. For example, a Stage 2 student is able to take a composite unit, or two-level structure (e.g., a unit of seven units of 1) as given without needing to construct the composite unit in activity. The stages are representative of the psychological structures and schemes with which the students operate (Piaget, 1970). Students build new structures by acting on different units in a task either imaginatively or through sensori-motor activity. A students' success on a particular task will depend on the cognitive demand of the task and, in part, on the student's UC stage. The cognitive demand of a task not only depends on the number of units in the problem but also the mental actions and relationships between the units. Thus, to know whether the cognitive demand of a task exceeds the cognitive capacity of a student, researchers and teachers must identify the units and mental actions needed to transform them successfully, in order complete the task. The psychological construct of working memory can account for the additional cognitive strain of accounting for units and actions, or relations between units, when modeling students' mathematics.

Working Memory

Working memory is involved in retrieving, maintaining, and sequencing the information a person must hold in mind and work on at any moment in time. It has been shown to be crucial in recalling numerical facts, updating and maintaining numerical representations, and implementing mathematical problem solving strategies (Bull & Lee, 2014). Working memory has also been shown to be a predictor of children's mathematical achievement levels (Blankenship, Keith, Calkins, & Bell, 2018). This paper uses the following definition of working memory: "Working memory involves the process of holding information in an active state and manipulating it until a goal is reached" (Agostino, Johnson, & Pascual-Leone, 2010, p. 62). There are several models for working memory; however, Pascual-Leone's (1970) M-capacity model aligns with the Piagetian scheme theory associated with UC.

In the context of UC, a student's working memory holds the units and transformations between units in an active state in order to perform the mental operations required to solve a task. M-capacity refers to the number of separate schemes a person can actively hold in their mind at a time (Pascual-Leone, 1970). However, Pascual-Leone's schemes are simpler than Piaget's, referring to a single action, such as the construction or transformation of a unit. If a person is able to mentally keep track of five units or actions between them, that person would have an Mcapacity of five. Pascual-Leone (1970) empirically found that children ranging in age from 5-11 years-old have M-capacity from 1.723 to 4.149; therefore, elementary school students typically can hold between 2 to 4 units/actions at a time when working on a problem. With a constraint on the number of units a students can hold in mind, having two and three-level units structures helps explain why Stage 2 and 3 students can handle more advanced mathematical tasks. The integration of M-capacity and UC may help describe students' mathematics and explain the power of higher-level mental structures.

Unit Transformation Capacity

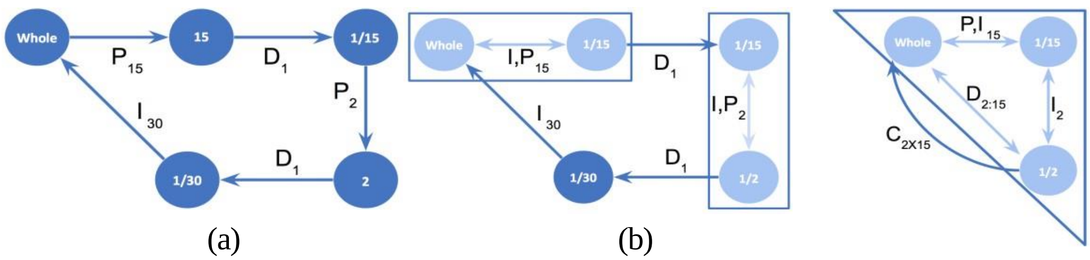
The theoretical framework proposed here combines UC and M-capacity to create an analytical tool for modeling students' mathematical cognition. We believe that students' ability to solve cognitively demanding tasks depends on their stage of UC and M-capacity. This codependence motivates the need for the Unit Transformation Capacity (UTC) construct, which we define as the number of units and transformations of units a student can mentally sequence while solving a task; units may be chunked into two and three-level structures. Thus, the UTC framework accounts for two kinds of mental resources available to a student when predicting whether that student will be successful on a task or not.

To model students' UTC, we began by ranking the cognitive demand of mathematical tasks found in the literature in the domains of whole number arithmetic, fractions, and algebraic reasoning. Rankings were based on hypothetical solutions, developed independent of student work, and assuming the student would be operating at UC Stage 1. Analyzing the tasks from the perspective of a Stage 1 student allowed for unpacking all the different units and transformations required to solve a particular task. The total number of units and transformations involved in a task became the rank of that task, which we call unit demand. We developed a diagraming system to model such solution paths (see Figure 2 below).

The diagrams incorporate units (circles), transformations (arrows), and chunking higher-level structures (boxes, triangles). Each transformation is explicitly identified with a corresponding letter: P=partition, I=iterate, D=disembed. The subscripts on the transformations indicate how many of each transformation occurred. For example, in Figure 2, P_{15} represents the student partitioning a unit into 15 equal pieces. The two models in Figure 2 are theoretical models built from the following task based off Hackenberg and Tillema's (2009) cake problem: "Imagine a rectangular cake that is cut into 15 equal pieces. You decide to share your piece of cake fairly with one other person. How much of the whole cake would that person get?" (p. 7). This task was analyzed to be a task of rank 10 because there are five units and five transformations is the diagrammed solution (Figure 2a).

A student would need to have a UTC of at least 10 to solve this task. For a Stage 1 student to solve this task without relying on figurative material they would need an M-capacity of 10.

Because Stage 1 students aren't able to chunk units and transformations into two- or three-level structures, each unit and transformation taxes their M-capacity. However, Stage 2 students are able to take some units and transformations together as given within two-level structures, represented by boxes, which allows for off-loading of working memory. Thus, a Stage 2 student would only need an M-capacity of 6 to solve the task (see Figure 2b). For a Stage 3 student, a single three-level structure contains many of the units and transformations so that only an M-capacity of 2 is needed to solve this task (see Figure 2c). Note that, for a Stage 3 student, demand is further reduced because there is no need to disembed units within the three-level structure. In fact, the three-level structure captures all units and transformations except for the need to compute the final result.



(a) (b) (c)

Figure 2: Student Diagrams for (a) Stage 1; (b) Stage 2; (c) Stage 3

Conclusion

The UTC framework explains why it is advantageous for student to develop higher-level structures in mathematical development. This added power of working with two or three-level structures has been demonstrated across several domains in mathematics (Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009; Norton & Boyce, 2013) and is captured in our diagrams. These findings fit with Hackenberg and Tillema's (2009) results showing a Stage 1 struggling to solve the cake task without figurative material while a Stage 2 student was able to solve the problem with little figurative material or teaching support.

The diagraming also helps capture nuances in student's mathematics found in the literature. For example, Hackenberg and Tillema (2009) documented a Stage 2 student who was able to solve the task of finding 2/5 of 1/3 but was unable to solve the mathematically equivalent task of finding 1/3 of 2/5. Our analysis finds that the 2/5 of 1/3 task has a rank of 12 while the 1/3 of 2/5 task has a rank 14. The added unit demand is accounted for in having the extra unit of 2 at the beginning of the task instead of at the end. The Stage 2 student solved the 1/3 of 2/5 with an answer of 1/15 indicating that the factor of 2 was lost during her solution path. This fits with what our model would predict for a Stage 2 student with an M-capacity of 5 or less, which is consistent with the student's age.

Our framework and models aim to capture nuances in students' mathematics by integrating UC and working memory. The models offer new psychological perspective on why higher-level structures allow for more advanced mathematics, which is a common theme throughout the literature (Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009; Norton & Boyce, 2013). The diagraming has also provided researchers with a tool for investigating different nuances about how students are thinking about mathematics. We hope to continue to capture nuances in students' mathematics that are consistent with the literature and help explain where students struggle with particular types of tasks. We are currently analyzing behavioral data from a pilot study with pre-service teachers to test the framework's utility in predicting student outcomes on fraction tasks. From this study, we hope to refine our framework and elucidate its use as a tool for researchers and teachers in selecting appropriately cognitively demanding tasks for students.

Overall, the goal for this new framework is to continue to build models of students' mathematics and broaden the scope to encompass multiple age groups and mathematical domains beyond fractions.

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AN EMERGING METHODOLOGY FOR THE STUDY OF PRESERVICE TEACHERS' LEARNING ABOUT EQUITY IN STEM EDUCATION

José F. Gutiérrez
University of Utah
jose.gutierrez@utah.edu

Lauren A. Barth-Cohen
University of Utah
lauren.barthcohen@utah.edu

Rachel Francom
University of Utah
rachel.francom@utah.edu

Kevin Greenberg
University of Utah
kevin.greenberg@utah.edu

Kelly MacArthur
University of Utah
macarthur@math.utah.edu

Tracy E. Dobie
University of Utah
tracy.dobie@utah.edu

A major challenge for elementary STEM teacher educators is incorporating social justice considerations across the span of professional program coursework. Recognizing that standards and policy documents are pressing for diversity and inclusion in STEM education, there is a growing need to support preservice teachers' learning about critical theories and how to develop an equitable vision of teaching. This paper describes ongoing research on our University's elementary STEM teacher education program. We focus our discussion on instrument development and the methods we used for eliciting preservice teachers' understandings of equity and diversity issues related to teaching STEM content. We designed a number of math, science, and technology scenarios in tandem, as means of building coherence across disciplinary boundaries; this report focuses on math teaching and learning.

Keywords: Qualitative Research Methods, Data Collection Instruments, Preservice Teacher Education, Equity and Diversity

Background & Goals: Building Coherence in STEM Learning Opportunities for PSTs

Commonly in teacher preparation programs, preservice teachers (PSTs) learn math and science content, technology in education, and teaching methods separately. Similarly, multi-cultural issues and the historical, sociopolitical foundations of schooling are also typically discussed separate from other topics and subject matter domains, if covered at all. There are known challenges with this lack of coherence and fragmentation in teacher education programs (Sirotnik, Goodlad, & Soder, 1990; Howey & Zimpher, 1989; Zeichner, Gore, & Houston, 1990). Broadly speaking, this fragmentation can lead PSTs to encounter contradictions or a lack of common language, and is overall an inefficient use of their time that can hinder PSTs in developing a clear vision of equity and the work of teaching.

Building on prior calls for coherent teacher education programs (Darling-Hammond & Bransford, 2005; Ritchie, An, Cone & Bullock, 2013), we are currently engaged in a long-term teacher education project aimed at building coherence in PST learning opportunities across disciplines and departments in undergraduate elementary teacher education STEM coursework. Our research team is comprised of math education experts, STEM content experts, and critical-theory scholars. Together, we have been working on a larger project and developing a vision of coherence based on theories of learning, cross-cutting concepts in math and science, as well as principles for fostering equity and social justice. In this paper, we focus on how we created, theorized, and piloted math tasks and in-class activities for the larger project.

Approach and Purpose of Study

Our research team has developed a class activity using a set of hypothetical teaching scenarios. These scenarios feature disciplinary content, student thinking, instructional design, and principles and dispositions around equity and social justice. These activities combine written tasks and prompts for group dialogue, through which PSTs engage in critical discussions that encompasses different aspects of classroom teaching and learning.

Using these tasks and in-class activity, we are collecting data in the form of PSTs' written work and transcripts of group discussions involving these teaching scenarios. With these data, we hope to survey teachers' prior knowledge, experiences, and assumptions pertaining to math content, student thinking, and, equity and social justice in math education. We seek to answer the following questions: What do PSTs already know? What are their productive resources and intuitions related to equitable teaching specifically in math? How do PSTs make sense of complex relational situations that they will likely encounter in future practice?

The Design and Rationale for Multi-Discourse Problematized Teaching Scenarios

During the Fall 2018, the research team designed and piloted instruments for eliciting and analyzing preservice teachers' knowledge related to equity and justice in math education. These instruments involve a number of specially designed tasks, each reflecting a *multi-discourse problematized teaching scenario* (MDPTS). Each MDPTS is a group activity for pre-service teachers and designed to elicit their perceptions and attitudes concerning a hypothetical, yet realistic, classroom scenario. MDPTS design was informed by previous research based on critical and sociocultural theories of math education (see below) and combines different elements highlighting various dimensions of classroom teaching such as content learning, social context, and power dynamics (cf. R. Gutiérrez, 2009).

The MDPTS were loosely inspired by existing work on scenario or case-based assessments in teacher education (e.g., see Shaughnessy & Boerst, 2018; Selling et al., 2015). Each MDPTS was designed to elicit different types of discourses, from about content, to classroom practice, to equity-based dialogue. Where applicable, MDPTS were designed to align with relevant math and science standards for elementary grades. Furthermore, some MDPTS were inspired by scholarship focused on equity, gender, and race in math education (e.g., R. Gutiérrez, 2009; Leyva, 2017; Martin, 2009).

MDPTS Example: "Mathematical Equivalence"

The mathematical equivalence problem presented in this scenario (Figure 1) is intended to build on student prior knowledge, and it creatively combines (and goes beyond) two core standards from earlier grades, namely, CCSS-M Standards 1.OA.6 and 1.OA.7 (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). This scenario aims to highlight the intersection of mathematical content, cognition, and gendered dynamics as a means of examining PSTs' knowledge within and across these areas.

Specific elements of "Mathematical Equivalence" were designed by adapting transcripts of utterances and emulating the tone of social interaction and mathematical behavior of students appearing in existing empirical work (Gutiérrez, Brown, Alibali, 2018; Heyd-Metzuyanim & Sfard, 2012). Specifically, it was designed to foreground mathematical content and cognition on the one hand, and gendered dynamics on the other. The psychology literature involving equivalence problems indicates the importance of noticing the location of the equal sign on individual strategy use and learning outcomes (e.g., Alibali, Crooks & McNeil, 2018). The hypothetical student (“Pat”) wants to point out the (accurate) location of the equal sign, which is the crux of the reasoning exhibited by the students in the scenario. The prompting strategy we chose for this scenario is intentionally open-ended for the PSTs to interpret the mathematical as well as the social factors that shape the interaction. Pat is not gender-identified, thus we hypothesize that the PSTs’ interpretations of this scenario will vary according to whether/how they assign a gender to Pat.

Mathematical Equivalence

A class of 3rd graders is in the middle of an explorative unit on the equal sign that incorporates the following goals:

- Accurately notice the location of the equal sign in equations involving addition and subtraction
- Generate and compare different strategies for determining the unknown number that makes an equation true

A group of three students are working on solving the following equivalence problem:

$$\mathbf{8 + 5 + 4 = 4 + \_\_\_}$$

A teacher overhears the following conversation:

Rebecca:	I’m not sure what to do. I’m confused. Do I fill in the blank?
Pat:	This is so easy guys! The answer is just 13.
Rebecca:	I don’t think it’s easy. That was rude Pat.
Gabe:	Plus, I don’t think you did it right Pat. I think the answer is 17. 'Cause 8 plus 5 plus 4 is 17.
Rebecca:	Yeah. That seems smart.
Pat:	You guys are so dumb. You have to pay attention to the equal sign...
Gabe:	Don’t act like the boss of us. You always act bossy.
Rebecca:	I think it’s 21. Cause I added it all up.
Pat:	If you guys would just listen I could teach you how to do it.
Gabe:	We can figure it out ourselves. Thanks anyway.

Imagine you are an observer in the classroom. Please discuss the following questions:

1) What is the dynamic between the three students?
2) How do you think the dynamic came about?
3) If you were the teacher, when would you intervene? How? What would you do?

Figure 1: MDPTS– “Mathematical Equivalence”

There are subtle elements that were designed into this scenario which might allude to either stereotypical female or male gender roles in mathematics, which adds further complexity to our analysis of the PSTs' perceptions of gender and math. At first read, one might assume Pat is a boy given the ways "he" asserts himself in the beginning and finds the solution procedure "easy." Yet, whereas these aspects of Pat's behaviors are aligned with dominant narratives of a masculine mathematician (Hottinger, 2016), Pat's behaviors also exhibit a feminine quality that resist this stereotype. Through it all, Pat wants to teach Rebecca and Gabe, and *teaching* mathematics is not typically associated with the historically gendered role of the masculine mathematician (cf. Hottinger, 2016). Another element in the scenario that could elicit PSTs' perceptions of not only gender and math, but also race, is the fact that it refers to Pat as "bossy." The term bossy can be perceived as a gendered as well as racialized term in math contexts (Langer-Osuna, 2011; McGee & Bentley, 2017), and these troubling narratives should be better understood in teacher education.

Preliminary Analytic Approach for Examining PSTs' Responses to MDPTS

We recently collected audio recordings and written work of 15 small groups (2-4 undergraduates in each group) of PSTs engaging with a set of four MDPTS, including the one presented in this paper. We started the process of open coding, reducing data, and articulating a preliminary analytic approach (Saldaña, 2013).

In Figure 2, we provide a few data excerpts of statements made from one group of PSTs during the activity and our tentative codes for those utterances. These data examples are not meant to be "final products," instead they represent our initial step toward a systematic and rigorous analysis and interpretation of our data. We include them to illustrate the range of codes that might be possible that highlight the multiple intersecting discourses elicited from one group of PSTs in response to "Mathematical Equivalence."

: [*responding to prompt no. 2 in Math Eq.*] It came about because Pat told them it was super easy. : [1]They were all like Pat you're a jerk! Stupid idiot! : [2]Because saying it's easy is ob--like ob-ject-ive, right? : Yeah, objective.	[1]STUDENT BEHAVIOR - JUDGEMENT [2]OBJECTIVITY/ MASCULINITY
: 'Cause I mean they're like none of them are *wrong* because [3]eight plus five plus four is= : Is seventeen. : =seventeen.	[3]STUDENT MATH THINKING
: I don't know if Pat is a boy or a girl but I feel like if they're both boys [*lower volume*] it could have been [4]toxic masculinity. 2 & 3: [5][Laughing] : Just throwing that out there. PST 2: [5]Yeah!	[4]GENDERED DISCOURSE [5]JEST

Figure 2: Data with Tentative Codes for Multiple Discourses for a Group of PSTs

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CAPTURING INSCRIPTIONAL PRACTICES IN DIGITALLY COLLABORATIVE CLASSROOM SETTINGS: A FOCUS ON CONSTRUCTING

David M. Bowers
Michigan State University
bowersd2@msu.edu

Alden J. Edson
Michigan State University
edsona@msu.edu

Amit Sharma
Michigan State University
sharma79@msu.edu

Keywords: Research Methods, Middle School Education, Technology

Representations often refer to the information in material form on paper or computer screen such as written text, graphical displays, tables, equations, diagrams, maps, and charts. The term, representation, is ambiguous because it also refers to a learner's internal or mental thought; while internal representations can serve as a resource for learning, they are not publicly accessible or directly available to others. From a situative learning perspective, knowing and learning mathematics are "situated in social and intellectual communities of practice, and for their knowing of mathematical knowing to be active and useful, individuals must learn to act and reason mathematically in the settings of their practice" (Greeno, 1988, p. 482). Therefore, the term inscription builds on the situative perspective and refers to external representations of thinking that exist in material form where meanings are developed in social settings.

As a part a four-year design research study that explores student learning and engagement with digital technologies, we advance an analytic framework to capture middle grades students' inscriptional practices in collaborative settings. As called for in the PME-NA 2019 Conference Theme, "...against a new horizon," the examination of inscriptions in mathematics classrooms invites new opportunities to consider the expansion and growth of mathematics education research through shifting attention from focusing on representations of the individual mind towards viewing representations as a social practice (Roth & McGinn, 1998).

The purpose of this presentation is to advance an analytic framework to capture inscriptional practices in middle grades mathematics classrooms. In this poster, we focus on the practice of constructing because it (a) foregrounds the relationship among people and their use of digital resources being developed in the larger project, (b) builds on empirical research on the teaching and learning of mathematics, and (c) provides the project with insights on how students use inscriptions to develop, communicate, and retrieve their understandings.

The framework was developed using a two-phase iterative process. The first phase involved identifying, compiling, and incorporating inscriptional practices that existed when reviewing empirical studies in the educational literature. The second phase involved engaging in iterative cycles of axial coding (Strauss & Corbin, 1998) and discussion to reach consensus.

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MODELING MATHEMATICS STUDENTS' TASK SOLUTION PROMULGATION

Thomas E. Ricks
Louisiana State University
tomricks@lsu.edu

Keywords: Classroom Discourse, Research Methods, Technology

Challenging tasks focused on the big mathematical concepts of the mathematics curriculum provide a core set of experiences from which students can develop powerful mathematical meaning (NCTM, 2000). As social beings, students notice and adopt the more efficient strategies of their peers, leading to natural mathematics learning with minimal teacher-talk. This paper continues previous research (Ricks, 2017) that models student engagement in challenging tasks and tracks the spread of student solutions across the classroom landscape by simulating classroom mathematics learning using discreet numerical analyses (various computer models simulate how student solutions to a central, challenging task promulgated across a classroom of twenty students arranged in five rows, each four seats deep). Over 14 million task completions were modeled, across varied parameters, such as an idea-spread ratio to adjacent desk locations and overall task difficulty. Results corroborate previous findings, and suggest that even in the most simple modeling, promulgation behavior is chaotic—hinting that current neo-liberal reform attempts to corporate-ize education, strip students of their humanity by transforming education into industrialized production of human capital, and reduce schooling to quality-control programs is problematic. When properly contextualized, this data contributes to furthering the goals of mathematics education in a variety of ways, including differentiating instruction for specific student needs, making mathematics accessible to all students, encouraging positive mathematics dispositions, learning to talk about personal mathematics understanding and make sense of others' mathematical thinking, and increasing the rigor of mathematical tasks. Educators, educational researchers, and educational policy-makers are increasingly being asked to improve the schooling process with progressively limited resources. We can do more with less—through better design. Proper reform of schooling to align with the various interests of standards-pressures can be recast as engineering problems—the role of the computer modeler then becomes a student of maximization and optimization. Modeling students' mathematical task solution promulgation across the classroom terrain can contribute to reform attempts to optimize pedagogical approaches. Computer modeling holds promise to augment the possibilities of reform by testing at accelerated rates various pedagogical practices for recent reform proposals. As these models become more sophisticated and simulate actual classroom behavior with greater fidelity, various reform scenarios can be tested virtually to see which ones educators might wish to adopt for their specific classrooms to differentiate instruction for their own students. As this study is limited in scope, more research is needed to understand how these models can be improved to better simulate real class dynamics.

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EFFECT OF NUMBER CHARACTERISTICS ON PERFORMANCE IN COMPUTATIONAL ESTIMATION

Coskun Erden
Iowa State University
cerden@iastate.edu

Keywords: Cognition

Computational estimation is a form of mental arithmetic which requires the temporary storage of initial information and partial results, retrieval of information from long term memory and the use of a variety of calculation strategies that determine solution sequences. Research revealed that computational estimation skill positively correlates with a variety of measures of mathematical ability including SAT score (Paull, 1972 as cited in Siegler & Booth, 2004) and arithmetic fluency (Dowker, 2003). However, previous research mainly focused on understanding the strategies participants use and the development of computational estimation skills. Only a few studies have examined the effects of problem characteristics on the performance of participants. Ganor-Stern (2015) showed that the certain characteristics of the numbers used in computational estimation have substantial effects on the performance. The purpose of the present study is to contribute to the knowledge base on the effects of number characteristics on the performance in computational estimation. This study examines the effects of values in the ones' digits of numbers on university students' performances in computational estimation. To this end, two-digit by two-digit multiplication problems are characterized by two main aspects: the values in ones' digits of the numbers (homogeneous-lower, homogeneous-upper, or heterogeneous) and the sum of distances of ones digits to nearest tens values (adjustment size of 3,4,5,6, or 7). For example, 64 x 23 is a homogeneous-lower problem with adjustment size 7 as both ones' digits are less than 5 and the sum of the differences between each operand and the nearest tens value is 7. On the other hand, 37 x 82 is a heterogeneous problem with adjustment size 5. The main question in the present research was "How do homogeneity type and the adjustment size of a computational estimation problem affect participants' performances in terms of accuracy and reaction time?"

The author developed a computational estimation test (CET) consists of 30 two-digit by two-digit multiplication problems as a tool of data collection. Thirty undergraduate students (20 female and 10 male) at a Midwestern university in the United States participated in the study. To address the research question, a two-way (3x5) repeated measures ANOVA and two Chi-square goodness of fit tests were conducted. The results revealed a significant main effect for adjustment sizes on reaction time, $F(4, 116) = 4.090$, $p = .004$; no main effect for homogeneity on the reaction time $F(2, 58) = 0.221$, $p = .802$; no main effect of homogeneity on accuracy, $\chi 2(2, N=30) = 0.69$, $p = .966$; and no main effect of adjustment size on accuracy, $\chi 2(4, N=30) = 0.031$, $p = .994$.

The results show that reaction times differ significantly in problems with different adjustment sizes. The results can be used to gain an insight into the amount of cognitive loads different types of problems exert on the participants. Research indicates that reducing the cognitive load of mathematics problems improves students' performance (Gillmor, S.C., Poggio, J. & Embertson, S., 2015). Thus, findings of this study may have implications for mathematics education.

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A THEORETICAL MODEL FOR THE DESIGN OF EQUAL SHARING PROBLEMS

Kiera Vaccaro
Concordia University
kiera.vaccaro@concordia.ca

Helena P. Osana
Concordia University
helena.osana@concordia.ca

Anne Lafay
Concordia University
anne.lafay@concordia.ca

Katherine Foster
Lester B. Pearson School Board
kfoster02@lbpearson.ca

Keywords: Problem solving, Equal sharing, Fractions, Measurement

Our research examines the effects of different types of equal sharing problems on children's solution strategies. In a recent study (Foster & Osana, 2018), we assessed whether the familiarity level of the item being partitioned in the problem had an effect on the strategies used by fourth graders. Children had more difficulty partitioning objects that were semi-familiar (e.g., thread) than familiar (e.g., brownies). The results were difficult to interpret, however, because of two confounding variables. First, all the semi-familiar problems involved partitioning length models, whereas all the familiar problems involved partitioning area models. The second confound was the type of unit – that is, the familiar items used arbitrary units of measure (e.g., one brownie) and the semi-familiar problems involved standard units of measure (e.g., one meter).

In this poster, we present a more specified theoretical model for investigating problem-related factors that impact children's solutions to equal sharing problems. The first dimension of the model involves the attribute of the object being partitioned (e.g., length, area). Children's partitioning of brownies, for example, may involve different challenges than partitioning lengths, such as ribbons. Evidence suggests that children's understanding of measurement and their partitioning strategies vary across attributes (e.g., Curry et al., 2006; Hiebert & Tonnessen, 1976). The second factor in the model involves distinguishing between standard and arbitrary units. In our own research (Foster & Osana, 2018), we found that children's intuitive partitioning strategies were compromised as soon as the term "meter" was introduced. Indeed, children's measurement strategies depend on the type of unit used, and this relationship looks different depending on the attribute (Bouton-Lewis et al., 1996; Nunes et al., 1993).

Finally, while familiarity remains a factor in the model, it is not sufficient because it overlooks the potential moderating effects of "groundedness," or the extent to which the to-be-partitioned quantity refers to a physical object in the real world (Belenky & Schalk, 2014; Fyfe & Nathan, 2018). When children are familiar with the items being partitioned, the extent to which the items are grounded may not matter: Children can reason appropriately about mathematical relationships even when representations are abstract (Mix et al., 2017). In contrast, groundedness may have an impact when items are unfamiliar (Koedinger et al., 2008).

In sum, our model can serve as a framework for future research on children's strategies for solving equal-sharing problems. Identifying the types of equal sharing problems that optimize the development of students' problem-solving strategies and fractions knowledge will be informative in the design and implementation of classroom instruction.

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PROBLEMS AND INSTRUCTIONAL APPROACHES FOR MATHEMATICS ACROSS METROPOLITAN AND NON-METROPOLITAN SCHOOL DISTRICTS

Phi Nguyen
University of Missouri
phinguyen@mail.missouri.edu

Cassandra Quinn
University of Missouri
cdquinn@mail.missouri.edu

Cara Haines
University of Missouri
carahaines@mail.missouri.edu

Keywords: Instructional leadership, Policy, Frames, Institutional setting

Teacher quality, educational funding, and curricular resources differ across urban, suburban, and rural settings (Bouck, 2004; Harmon, Gordanier, Henry, & George, 2007). Recognizing that context matters, we acknowledge that practitioners' perspectives on key issues in mathematics education may vary with respect to communities' size and proximity to metropolitan centers. For instance, Bouck (2004) described two districts, one metropolitan and one non-metropolitan, that faced a similar teacher quality challenge, but framed it differently—the former as a problem of teacher retention and the latter as a problem of teachers' educational preparation. In an effort to learn about and ultimately address school districts' challenges with respect to mathematics education, we talked with district leaders, administrators, and teachers about their mathematics-related challenges and identified commonalities and differences across metropolitan and non-metropolitan districts.

Our sample included 50 Missouri school districts, 23 metropolitan and 27 non-metropolitan. We interviewed one representative in each district, 46 of whom were district leaders (e.g., superintendent, assistant superintendent of curriculum and instruction, mathematics curriculum coordinator) and 4 of whom were middle and/or high school mathematics teachers. All interviews were audio recorded and summarized. For this poster, we focus only on the responses for questions regarding districts' problems and instructional approaches for mathematics. In our analysis of interview summaries, we identified a primary problem and a primary instructional approach, if any, for each school district. We then revisited the results of our initial pass to identify whether different framings were related to metropolitan or non-metropolitan classifications, and to refine our characterizations of the categories.

Our findings revealed that non-metropolitan districts, as compared to metropolitan districts, were more likely to articulate student outcomes problems and less likely to promote a particular instructional approach. The most common problem for mathematics was student outcomes, though participants framed outcomes (e.g., standardized test scores; course-taking patterns) and ways to improve outcomes (e.g., through teaching; through curriculum adoption) in different ways. For 48 school districts, 22 promoted a particular approach to mathematics instruction (e.g., direct instruction, inquiry-based instruction, etc.); 26 did not. Both metropolitan and non-metropolitan districts that did not promote a particular instructional approach were also more likely to articulate a student outcomes problem for mathematics. We will also discuss implications for research and policy.

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A HYPOTHETICAL LEARNING TRAJECTORY FOR PRIME DECOMPOSITON

Courtney Vahle
Southern Illinois University Edwardsville
courtneyvahle@gmail.com

Keywords: Learning Trajectories, Number Concepts and Operations

In my research, I created a hypothetical learning trajectory (LT) for the concept of decomposing numbers into products of primes. In the context of an after-school math league aimed at providing enrichment for students who perform in the bottom quartile, I gained insight into the complex way that students consider prime and composite numbers, divisibility, and decomposition of numbers. Through the league, I examined and tracked students' understanding related to these big ideas.

Theoretical Perspective and Research Questions

This research employs an LT (Clements & Sarama, 2004) as a lens through which to view children's concepts and strategies for prime and composite numbers, divisibility, and decomposition of numbers. An LT is defined in terms of three constituent parts: a mathematical instructional goal, a likely path for learning, and instructional tasks that help move students along that path. Thus, this study seeks to address the questions: How can students' responses to questions regarding prime number decomposition be sorted into an LT? How do students' concepts and strategies surrounding prime number decomposition grow in response to instructional interventions?

Data Collection and Analysis

Using a review of related literature, a hypothetical LT was created prior to the selection of students for the math league. Thirty-eight students in the league were given a pre-test before practices began in three leagues around the country. For comparison, students from a school in the Midwest were also given a pre-test before they started a prime factorization unit in their classroom. I then used thematic analysis (Jin & Anderson, 2012) to refine and revise the LT using pre-test results. Post-test results were analyzed by sorting student responses into the revised LT to trace students' growth along the LT over the course of the program.

Results and Conclusions

Preliminary results suggest that most students' responses to questions regarding prime decomposition can be sorted into the hypothetical LT. Findings suggested that level two required two sub-levels. The revised LT consists of the following three levels:

1. Students understand whole number divisibility and factors.
2. (a) Students apply concepts of divisibility to break down known factors and create more factors of the number; they can recognize prime numbers as an outcome of factoring (its only factors are one and itself) or by exclusion (the number is not composite, so it is prime) (Zazkis & Liljedahl, 2004).
   (b) Students recognize that these prime numbers, multiplied together, create a separate representation of the number.
3. Students understand the implications of the Fundamental Theorem of Arithmetic and manipulate alternate representations of a natural number such as its prime factorization.

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Chapter 15:
Working Groups

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

DESIGNING AND RESEARCHING ONLINE PROFESSIONAL DEVELOPMENT

Julie M. Amador
University of Idaho
jamador@uidaho.edu

Cynthia Callard
University of Rochester
ccallard@warner.rochester.edu

Jeffrey Choppin
University of Rochester
jchoppin@warner.rochester.edu

Cynthia Carson
University of Rochester
ccarson3@u.rochester.edu

Ryan Gillespie
University of Idaho
gill6762@vandals.uidaho.edu

In this working group, we continue with previous efforts to consider design and research methodologies related to teacher learning in online professional development contexts. We then describe an innovative project designed to support the development of middle school mathematics teachers, with a focus on three distinct forms of online learning: digitally communicated teaching lab lessons, an online course, and online video coaching. Given recent technological advances and demands to support teachers in various contexts, we contend that researching and understanding these online models, as well as other online models is important for the broader field of mathematics education. As a result, Year Three of this proposed discussion group will combine whole-group and subgroup time to converse about: (a) the challenges of online professional learning experiences, (b) research tools, methods, and analyses, (c) the connections among different projects and studies, (d) scaling up online models, and (e) future collaborations and research.

Keywords: Teacher Education-Inservice/Professional, Research Methods, Learning Theory

All teachers need access to high quality professional development in order to meet the needs of students and to teach rigorous mathematics as outlined in college and career-ready standards (Marrongelle, Sztajn, & Smith, 2013). Online professional development has the potential to provide access to a wider range of teachers than what is possible face to face. Furthermore, given the propensity of millennials to seek online learning experiences, we feel that more attention needs to be given to the design, dissemination, and research of online professional development. Given the emerging importance and availability of online professional development, we propose the continuation of a working group that met at PMENA 2017 and PMENA 2018. We will continue focus on the design, dissemination, and research on online professional development. The working group participants will analyze current practices in online professional development, including the technology affordances and limitations. Major themes that will be addressed are:

- affordances of online platforms,
- affordances and constraints of synchronous versus asynchronous experiences,
- challenges related to scaling up high-quality online professional development,
- methodologies used to research professional learning in online contexts.

This year the working group will move beyond prior conversations that centered on the specifics of our model to a broader conversation about online opportunities in the greater mathematics education field. Focus will be given to connections among projects, scalability of projects, and future directions for both research on online professional learning and implementation of professional learning. We plan to focus part of the session on the challenges and opportunities of implementing online models in rural contexts. A goal of the working group is to organize a conference to be held in summer, 2020, for which we already have funding.

Importance of Online Opportunities

As schools turn to digital learning contexts, it is inevitable that professional development will follow a similar trend. It is imperative to have research-based models that demonstrate how the features of high quality face-to-face professional development can be matched or augmented in online contexts. As an example of necessity, teachers in rural areas face constraints in terms of accessing the expertise and resources required for high-quality professional learning experiences, often because of a lack of proximity to such resources as institutions of higher education and critical masses of teachers required to collectively reflect on problems of practice (Howley & Howley, 2005). Rural contexts are thus ideal sites for online professional development, which can be offered at a distance and can involve geographically dispersed participants (Francis & Jacobsen, 2013). At the same time, teachers in urban and suburban areas may have more regular access to professional development, but online formats afford conveniences and customized learning opportunities that may not be available in face-to-face settings. Digital learning contexts provide opportunities for connections and visual supports that may otherwise not be accessible in face-to-face professional development. As a result, we consider it necessary to research online professional development and to engage with other mathematics educators and researchers about online professional learning. This working group is intended to advance the practices of designing and researching online professional learning experiences by investigating the challenges of balancing high-quality learning experiences and accessibility for teachers. The focus is also on reconnecting with those in attendance during the 2017 and 2018 conferences for updates on discussions about current happenings and experiences with online learning.

Below we provide an overview of the literature related to professional learning in online contexts. Then we revisit the NSF-funded model of online professional development discussed over the last two years, describing what we have learned in terms of the learning environment and our efforts to research its impact. We will devote part of the first session to providing updates on the project as a means of introducing possible models and methodologies to study online professional development, leaving opportunities over the next working sessions to incorporate discussion of other models and methodologies. We then conclude with aims for the 2020 working group.

Literature Related to Online Professional Learning

Digital Technologies

Online professional learning experiences combine longstanding and emerging digital technologies to provide high-quality, interactive, content-focused professional development. Longstanding digital technologies (e.g., electronic learning management systems) have been used to implement online courses to design and implement professional development for the past couple of decades. Emerging digital technologies involve an internet-based platform to implement online video coaching, or other online communications, in ways that augment the interactivity of face-to-face coaching. Online video coaching emerges from the content-focused face-to-face coaching that the project personnel have engaged in over the last ten years.

Research shows that while online communication lacks some of the modalities (e.g., gestures, facial expressions) and spontaneity of face-to-face communication (Tiene, 2000), there are also affordances unique to its asynchronous and text-based nature. In online asynchronous discussions, communication tends to be more exact and organized (Garrison, Anderson, & Archer, 2001; McCreary, 1990), involve more formal and complex sentences (Sotillo, 2000) and incorporate critical thinking, reflection, and complex ideas (Davidson-Shivers, Muilenburg, & Tanner, 2001; Marra, Moore, & Klimczak, 2004). Research on synchronous online communication – which can include text chat windows and shared space in learning management systems – shows that it is experienced as more social than asynchronous spaces (Chou, 2002). Synchronous sessions induce *personal participation*, which Hrastinski (2008) compared to *cognitive participation* in that personal communication in synchronous spaces “involves more intense interaction … while cognitive participation is a more reflective type of participation supported by asynchronous communication” (p. 499). Furthermore, synchronous communication fosters multiple communication channels based on emerging networks within the larger group, including the use of chat boxes and personal email during synchronous sessions (Haythornthwaite, 2000, 2001). Researchers have reported positive outcomes from professional development involving synchronous exchanges via typing (e.g. Chen, Chen, & Tsai, 2009). However, synchronous verbal online discussions and group activities have not been a focus of research.

Online Professional Development in Education.

Despite the growing popularity of online professional development, there is a continued need for empirical research regarding its quality and effectiveness (Dede, Ketelhut, Whitehouse, Breit, & McCloskey, 2009). Prior research has not demonstrated advantages for online professional development in terms of teacher outcomes (cf. Fishman et al., 2013), in part due to the lack of online professional development contexts that involve teachers in sustained, intensive reflection on their practices. Furthermore, teacher learning in online spaces can be challenging, especially related to complex forms of learning. Sing and Khine (2006) found that a number of factors make it difficult for teachers to engage in complex or difficult forms of learning in an online context, such as teachers’ roles as implementers rather than producers, cultural norms where disagreement is seen as confrontational, and the cognitive demands relative to the available teacher time. Teacher learning in online contexts is discussed in more detail below.

In order to illustrate professional learning in an online context, we present a model that the authors are currently using in a project situated in rural contexts. We present the model to continue the discussion of this model and other potential models, as well as the learning platforms and other features, such as the synchronous or asynchronous nature of learning in online environments.

Outcomes from Implementing a Model of Online Professional Development

During the working group sessions, we will report on outcomes from an innovative professional development model we have implemented for three years. We will provide an overview of findings from each component of the project, and describe dilemmas and challenges related to the implementation of the model. The innovative online professional learning experiences in this project focus on the development of teacher capacity to enact ambitious, responsive instruction aligned with the rigorous content and practice elements of the Common Core State Standards for Mathematics (CCSSM). We use the term *professional learning experiences* to denote that the professional development we employ differs from traditional workshop or other models that are too short or fragmented to be effective (Garet, Porter, Desimone, Birman, & Yoon, 2001).

In the project, we identified three primary research goals, which were to study and understand: (a) the ways online-based professional development can help teachers improve their instructional practices and their ability to notice and respond to student thinking; (b) the characteristics of the cycles in the online coaching, the role of video, and the asynchronous components; and (c) the features of the professional development model that would inform efforts to scale up the model, including the resource commitments, the requisite capacity of the course instructors and coaches, and the logistical requirements of the courses and coaching. We are currently in year three of four years of the project. The following describes the three online components of our project. In the working group, we envision these and other components used by other researchers serving as the catalysts for dialogue around online professional learning and will engage participants with conversation around our learning in the past year while encouraging them to share their recent experiences.

Online Course - Orchestrating Mathematical Discussions

The first component of our project is two online course modules, *Orchestrating Mathematical Discussions Parts One and Two*, aimed at orienting the participants toward high-leverage discourse practices that facilitate mathematically productive classroom discussions (Smith & Stein, 2011). In this course, the participants solve and discuss a series of high cognitive demand tasks, activities that will be accompanied by synchronous and asynchronous discussions around the *5 Practices for Orchestrating Productive Mathematics Discussions* (i.e. anticipating, monitoring, selecting, sequencing, connecting; Smith & Stein, 2011). The courses are designed to develop awareness of specific teacher and student discourse moves that facilitate productive mathematical discussions, to understand the role of high cognitive demand tasks in eliciting a variety of approaches worthy of group discussions, and to further develop participants' mathematical knowledge, particularly the rich connections around big mathematical ideas that are helpful to teach with understanding (Author, 2007a, 2007b; Ball, 1991; Boaler & Staples, 2008; Chapin, O'Connor, & Anderson, 2003; 2014; Herbel-Eisenmann, Steele, & Cirillo, 2013; Ma, 1999; O'Connor & Michaels, 1993).

In order to take advantage of the affordances of both asynchronous and synchronous characteristics of online communication, the course is embedded in a learning management system (LMS) that: allows for synchronous whole class and small group interaction; the sharing of artifacts, including those collectively developed in the LMS; and asynchronous discussion threads. In the online course modules in the LMS, the facilitator verbally presents a challenging task to the participants, which is viewed in a shared work space. The course instructor then assigns participants to virtual breakout rooms, in which the participants work synchronously in a common workspace, creating virtual white boards to share with the other groups. They can talk to each other, work simultaneously in the virtual space, and use the chat window to communicate. The course instructor can listen to and participate in these group discussions to determine when the groups are ready to present their solutions. The course instructor then closes the virtual breakout rooms, which automatically returns all participants to the main room to conduct a summary discussion of the different strategies, in effect modeling the practices in the *5 Practices* book. Asynchronously, the group can continue to go back and reflect and comment on the task and related solutions, as well as on the readings from the *5 Practices* book using discussions threads in the LMS. Participants are also encouraged to share resources, lesson plans, and student work as appropriate. The working group will discuss this format for online professional learning as well as other formats and tools that have proven beneficial for users.

Teaching Labs

In order to address the challenges of engaging teachers in learning complex practices in an online context, we include a component aimed at initiating and reinforcing relationships between participants and project personnel and at helping participants to understand the types of learning experiences and design and feedback cycles that will be the core of the project. Research on lesson study (e.g., Author, 2016; Stigler & Hiebert, 1999) has led to an emphasis on demonstration lessons where teams of teachers collectively plan, enact, and reflect on lessons in ways that make public the features of the lessons and teachers' instructional practices (Saphier & West, 2009). Consequently, one component of our project is a collaborative classroom activity, a Teaching Lab, that builds from the *studio classroom* model developed by the Teachers Development Group (2010), with features consistent with content-focused coaching (West & Staub, 2003). To do this in an online space, we first teach a lesson in a participant's classroom and video record and edit the footage. Then, for each lesson, a group led by project personnel meet to discuss the task of the lesson, which is typically of high cognitive demand. The group explores the task, the mathematical learning goals embedded in the task and anticipated student approaches to solving the task, and the related CCSSM practice and content standards. The group then immediately watches the edited video of the lesson with a focus on productive teaching moves and evidence of student thinking and learning in relation to the lesson goals. In the same meeting, the group collectively reflects on the experience, with a focus on describing evidence for student understanding using the data gathered by the teachers and observers. This process is repeated regularly with participants.

In the beginning of the project all demonstration lesson activities were face-to-face. However, at this point in our project, this component has moved to an online format of synchronous activities. The entire process is held via a video conferencing platform, Zoom, allowing for synchronous engagement in both whole group and small group discussions of the lessons. Discussion in the working group will center on the affordances and constraints of the online teaching lab model and possible modifications to ensure the intended professional development goals are met.

Online Video Coaching

The third – and most innovative – component of our project's professional development program is the online video coaching that builds from models of content-focused coaching (West & Staub, 2003). More recently, thanks to the advent of improved internet-based software aimed at increasing collaboration around video data, the project personnel have begun conducting online video coaching cycles with teachers. The coaching cycles are focused on identifying and unpacking the mathematics with the teacher, while anticipating likely student strategies, conceptions, and misconceptions. The coach helps the teacher identify evidence for demonstrating how students are thinking (from the video as well as from student artifacts) and make connections between different student approaches in order to help the teacher structure the summary discussion of the lesson.

The online coaching experiences involve synchronous and asynchronous components, with the goal of engaging participants in reflective or deliberative practice. The online coaching has features similar to face-to-face coaching, such as video conferencing conversations via Zoom, in which the coach and participant collaborate to plan lessons and reflect on the qualities of lessons. However, the online coaching includes an innovative component that involves asynchronous collaboration and feedback that structures the post-lesson collaborative reflection, features that augment or surpass the kind of feedback that can be given face-to-face. Teachers video-record themselves using Swivl, which allows them to place a camera (iPhone or other device) on a robot that tracks them around the room, allowing for teacher-focused video without the necessity of someone operating the camera. The video is automatically uploaded into a password-protected site and processed, and is immediately accessible to view and annotate. The annotation feature in Swivl allows the coach and the teacher to separately view and annotate the video. For example, a teacher can stop the video by hitting the pause button and type in a comment or question that is synced with the video, so that when the coach watches the video, she can read the comment during the point in the video referenced by the comment. The coach can do the same. The video can be viewed repeatedly, which allows for more thorough reflection and analysis. The notation provides for more in-depth and substantive feedback, pointing to specific instances of practice and student thinking. The discussion group will focus on this model for professional coaching as well as other models or avenues for supporting individual teachers in online professional learning.

Researching Online Professional Learning Experiences

There is a dearth of research on online professional development, especially online professional development that is sustained and intensive. Similarly, while there have been years of intensive efforts to implement coaching in schools, much of the research has revolved around the role and impact of coaches (Coburn & Russell, 2008; Penuel, Riel, Krause, & Frank, 2009), and less around the impact on reflective or deliberative practice. Although coaching has now been around for over ten years, there is limited research on the effectiveness of coaching in terms of improving teacher quality (Matsumura, Garnier & Spybrook, 2012). The greatest dearth of research involves online video coaching in education, as opposed to face-to-face coaching, which has no peer-reviewed research yet associated with it.

Structure of the Working Group Sessions

Within this working group we propose to explore the following questions related to researching online professional learning experiences:

1. What are various platforms and models for online professional development?
2. What theoretical framework and methodologies are salient for researching online digital technologies and online professional learning experiences?
3. What data analysis methods are suited to the data captured in online environments?
4. In what ways can online professional learning experiences help teachers improve their instructional practices and their ability to notice and respond to student thinking?
5. In what ways does the use of video in an online professional learning experience project maximize teacher learning?
6. What features of the professional development model would inform efforts to scale up the model, including the resource commitments, the requisite capacity of the course instructors and coaches, and the logistical requirements of the courses and coaching?

Plan for Working Group

In Session 1, the organizers will present brief update reports on the Author's project, research design, and evolution of the model, as well as a recap of the 2018 working group discussions. Subgroups will be formed to continue conversations around design and implementation efforts with online professional learning experiences from their own research and current efforts in the field; attendees from this working group at PMENA 2018 will provide updates on their respective projects in the small-group setting.

During Sessions 2 and 3 we will provide the subgroups time to continue collaborating on themes identified in the PMENA 2018 working group: a) identifying the challenges of online professional learning experiences that are the most challenging and why—this will include a specific look across the projects presented and with a focus on activities on the last year, b) refining research tools, methods, and analyses, c) exploring connections among different projects and studies for former and new attendees, d) discussing scaling of online project, and e) discussing future collaborations and research. We will close Session 3 with time to review group progress and discuss next steps for our work as shown in Table 1. Meeting notes, work, and documents will continue to be shared and distributed via our Google Folder (set up for this Working Group). The use of Google documents allows members to create an institutional memory of activities during the working group that we will continue to use and add to following the 2019 working group. This shared folder will also provide a shared space for future collaborations and writing projects related to online professional learning experiences within the working group members.

Table 1: Overview of Proposed Working Subgroup Sessions

	Activities:	**Guiding Questions:**
Session 1	1. Introductions and Agenda 2. Brief Presentations of Authors' Project and Research Questions 3. Brief Presentations of former Attendees' and New Attendees' Projects and Research Questions 4. Subgroup formation and initial work time - designing Online PD experiences	1. What are the different forms of online professional development? 2. What research is being done related to online professional development? 3. Which aspects of online professional learning experiences are the most challenging to implement or research? 4. What are new questions that have arisen within the last year?
Session 2	1. Overview of subgroup's work from previous day 2. Subgroup work time - engagement in online professional learning experiences 3. Brief sharing of work in subgroups	1. How can online learning support teacher learning? 2. What are the affordances and constraints of various platforms? 3. What are the affordances and constraints of synchronous and asynchronous experiences?

Session 3	1. Overview of subgroup's work from previous day 2. Subgroup work time - researching online professional learning experiences 3. Brief sharing of work in subgroups 4. Final reflections – future collaborations and research	1. What theories and theoretical frameworks have informed the design of your research project(s)? 2. How might your work inform theory in researching online professional learning experiences? 3. What issues and challenges have you faced in designing studies in this area? 4. What challenges may exist for scaling up high-quality online professional development?

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EMBODIED MATHEMATICAL IMAGINATION AND COGNITION (EMIC) WORKING GROUP

Erin R. Ottmar
Worcester Polytechnic Institute
erottmar@wpi.edu

Candace Walkington
Southern Methodist University
email

Dor Abrahamson
Univ. of California, Berkeley
dor@berkeley.edu

Mitchell J. Nathan
Univ. of Wisconsin-Madison
mnathan@wisc.edu

Avery Harrison
Worcester Polytechnic Institute
aeharrison@wpi.edu

Carmen Smith
University of Vermont
carmen.smith@uvm.edu

Embodied cognition is growing in theoretical importance and as driving a set of design principles for curriculum activities and technology innovations for mathematics education. The central aim of the EMIC (Embodied Mathematical Imagination and Cognition) Working Group is to connect with inspired colleagues in this growing community of discourse around theoretical, technological, and methodological developments to advance the study of embodied cognition for mathematics education. Our thriving, informed, and interconnected community of scholars organized around embodied mathematical cognition will continue to broaden the range of activities, practices, and emerging technologies that contribute to mathematics teaching and learning as well as to research on these phenomena. This year's proposed EMIC working group builds upon our prior working groups with a specific focus on collaboratively creating embodied activities for mathematics learning that utilize different types of physicality, from full-body to gestural movements. In particular, we aim to develop and evaluate novel activities that apply principles of embodied cognition to foster mathematics learning through engaging in the enactment of carefully crafted movement. Our ongoing goal is to connect researchers and educators as we all create activities which can be implemented in mathematics classrooms.

Keywords: Learning theory, Cognition, Technology, Instructional activities and practices

Motivations for This Working Group

Empirical, theoretical, and methodological developments in embodied cognition and gesture studies provide a solid and generative foundation for the continuation of the established, **regularly held Embodied Mathematical Imagination and Cognition (EMIC) Working Group** for PME-NA. The central aim of EMIC is to attract engaged and inspired colleagues into a growing community of discourse around theoretical, technological, and methodological developments for advancing the study of embodied cognition for mathematics education, including, but not limited to, studies of mathematical reasoning, instruction, the design and use of technological innovations, and learning in and outside of formal educational settings.

The interplay of multiple perspectives and intellectual trajectories is vital for the study of embodied mathematical cognition to flourish. While there is significant convergence of theoretical, technological, and methodological developments in embodied cognition, there is also a trove of questions that must be addressed through formulating and implementing experimental design principles. As a group, we aim to (1) synthesize the work of leading scholars into a coherent theory of EMIC, (2) identify the most promising ideas for opportunities for methodological and technological integration, (3) curate and disseminate a set of evidence-based design principles for enhancing mathematics education and broadening participation in STEM fields, and (4) articulate a future research agenda in the growing area of embodied design.

We aim to address basic theoretical questions such as, *What is grounding?* And practical ones such as, *How can we reliably engineer the grounding of specific mathematical ideas?* We want to understand how variations in actions and perceptions influence mathematical reasoning, including self-initiated vs. prescribed actions, and actions that take place in intrapersonal versus interpersonal interactions; how gestural point-of-view when enacting phenomena from a first-versus-third-person perspective, including how gestures move through space, influences reasoning and communication; how actions enacted by oneself, observed in others, or imagined influence cognition; how gestures connect with external visual representations, and how gestures are used to forge collaborative thinking (Abrahamson, 2018; Abrahamson & Bakker, 2016; Alibali & Nathan, 2012; in press; Walkington et al., in press).

From an applied level, we are also witnessing the emergence of a new genre of educational technologies and interventions for promoting STEM, rooted in theories of embodied cognition. These new uses of technology, in turn, offer novel opportunities for students and scientists to engage in math visualization, symbolization, intuition, and reasoning. In order for these designs to successfully scale up, they must be informed by research that demonstrates both ecological and internal validity. As technology becomes more affordable, more integrated in mathematics education spaces, and more common in classrooms, we need the proposed intellectually rigorous synthesis of theory and design principles to help shape approaches and activities that help make mathematics education accessible to a wider range of students.

Focal Issues in the Psychology of Mathematics Education

Emerging, yet influential, views of thinking and learning as embodied experiences have grown from several major intellectual developments in philosophy, psychology, anthropology, education, and the learning sciences that frame human communication as multimodal interaction, and human thinking as multi-modal simulation of sensory-motor activity (Clark, 2008; Hostetter & Alibali, 2008; Hutto, Kirchhoff, & Abrahamson, 2015; Lave, 1988; Nathan, 2014; Newen, Bruin, & Gallagher, 2018; Varela et al., 1992; Wilson, 2002). As Stevens (2012, p. 346) argues in his introduction to the *JLS* special issue on embodiment of mathematical reasoning, “it will be hard to consign the body to the sidelines of mathematical cognition ever again if our goal is to make sense of how people make sense and take action with mathematical ideas, tools, and forms.”

Four major ideas exemplify the plurality of ways that embodied cognition perspectives are relevant for the study of mathematical understanding: (1) *Grounding of abstraction in perceptuo-motor activity as one alternative to representing concepts as purely amodal, abstract, arbitrary, and self-referential symbol systems.* This conception shifts the locus of “thinking” from a central processor to a distributed web of perceptuo-motor activity situated within a physical and social setting. (2) *Cognition emerges from perceptually guided action* (Varela, Thompson, & Rosch, 1991). This tenet implies that things, including mathematical symbols and representations, are understood by the actions and practices we can perform with them, and by mentally simulating and imagining the actions and practices that underlie or constitute them. (3) *Mathematics learning is always affective*: There are no purely procedural or “neutral” forms of reasoning detached from the circulation of bodily-based feelings and interpretations surrounding our encounters with them. (4) *Mathematical ideas are conveyed using rich, multimodal forms of communication, including gestures and tangible objects in the world.*

In addition to theoretical and empirical advances, new technical advances in multi-modal and spatial analysis have allowed scholars to collect new sources of evidence and subject them to powerful analytic procedures, from which they may propose new theories of embodied mathematical cognition and learning. Growth of interest in multi-modal aspects of communication have been enabled by high quality video recording of human activity (e.g., Alibali et al., 2014; Levine & Scollon, 2004), motion capture technology (Hall, Ma, & Nemirovsky, 2014; Sinclair, 2014), eye-tracking instruments (e.g., Abrahamson, Shayan, Bakker, & van der Schaaf, 2016), developments in brain imaging (e.g., Barsalou, 2008; Gallese & Lakoff, 2005), multimodal learning analytics (Worsley & Blikstein, 2014), and data logs generated from embodied math learning technologies that interacts with touch and mouse-based interfaces (Manzo, Ottmar, & Landy, 2016).

Past Meetings and Achievements of the EMIC Working Group

The first PME-NA meeting of the EMIC working group, "Mathematics Learning and Embodied Cognition," took place in East Lansing, MI in 2015. Our group has been growing ever since. In addition to the PME-NA meeting each year, there are a number of ongoing activities that our members engage in. We have built an active website which connects members, provides updates on projects, and hosts resources. We have also created a space for members to share information about their research activities – particularly for videos of the complex gesture and action-based interactions that are difficult to express in text format. In addition, we have a common publications repository to share files or links (including to ResearchGate or Academia.edu publication profiles, so members don't have to upload their files in multiple places). Our members collaborate on ongoing projects and have presented at other conferences and workshop events annually. Several research programs have formed to investigate the embodied nature of mathematics (e.g., Abrahamson 2014; Alibali & Nathan, 2012; Arzarello et al., 2009; De Freitas & Sinclair, 2014; Edwards, Ferrara, & Moore-Russo, 2014; Lakoff & Núñez, 2000; Melcer & Isbister, 2016; Ottmar & Landy, 2016; Radford 2009; Nathan, Walkington, Boncoddo, Pier, Williams, & Alibali, 2014; Soto-Johnson & Troup, 2014; Soto-Johnson, Hancock, & Oehrtman, 2016; Walkington et al., in press), demonstrating a "critical mass" of projects, findings, senior and junior investigators, and conceptual frameworks to support an ongoing community of like-minded scholars within the mathematics education research community.

In order to sustain collaboration and connect emerging and established scholars, eight of our members will also host an NSF funded Synthesis and Design workshop at the University of Wisconsin–Madison (UW–Madison) in May 2019. This workshop will bring together leading scholars in mathematical reasoning, teaching, and learning who work on embodied design with the goal to form a ten-year research agenda that will provide a coherent set of evidence-based design principles for enhancing mathematics education and broadening participation in all STEM fields. The organizers will seek to attract an interdisciplinary set of 30 scholars from education research, cognitive science, the learning sciences, developmental psychology, movement science, computer science, and mathematics, as well as 6 teachers. The research presented will span K–16 topics in content areas such as arithmetic and algebra, proportional reasoning and fractions, geometry, complex numbers and functions, statistics, and calculus. It will focus on design of systems for classroom learning settings, with attention to equity and access for underrepresented groups, while examining evidence of learning both in *and* outside of school. The reach of such an endeavor can extend to studies of mathematical intuition and reasoning, learning in and outside of formal educational settings, professional development, classroom instruction and assessment, and STEM integration. The hope is that the 2019 PME workshop can serve as a time to follow up, disseminate, and extend what was learned from this workshop.

Current Working Group Organizers

As the Working Group has matured and expanded, we have a broadening set of organizers that represent a range of institutions and theoretical perspectives (and is beyond the limit of six authors in the submission system). This, we believe, enriches the Working Group experience and the long-term viability of the scholarly community. The current organizers for 2019 are (alphabetical by first name):

- Candace Walkington, Southern Methodist University
- Carmen J. Petrick Smith, University of Vermont
- Caro Williams-Pierce, University at Albany, SUNY
- David Landy, Indiana University
- Dor Abrahamson, University of California, Berkeley
- Erin Ottmar, Worcester Polytechnic Institute
- Hortensia Soto–Johnson, University of Northern Colorado
- Ivon Arroyo, Worcester Polytechnic Institute
- Martha W. Alibali, University of Wisconsin-Madison
- Mitchell J. Nathan, University of Wisconsin-Madison

Some of **our collaborative accomplishments** since last year's PME-NA working group include:

1. A successful proposal to host a three-day NSF Synthesis and Design workshop for researchers and teachers on "The Future of Embodied Design for Mathematical Imagination and Cognition" with 8 members serving as organizers https://www.embodiedmathematics.com/
2. Invitation by Springer to write a book on our collective work on Embodied Cognition in Mathematics for the "Research in Mathematics Education" Series
3. Submission of an additional NSF Workshop Proposal to host a 3 day Workshop in 2020 on Embodied Cognition for K-16 math educators)
4. Several members creating and teaching Embodied Cognition and Gesture seminars at their institutions.
5. Several grants awarded by IES CASL program to study embodied cognition, including the role of action in pre-college proof performance in geometry (Funded 2016-2020 for Nathan & Walkington) as well as the use of perceptual learning technology to study algebra learning (Ottmar & Landy, 2018)
6. Expanding a group website using the Google Sites platform to connect scholars, support ongoing interactions throughout the year, and regularly adding additional resources/activities https://sites.google.com/site/emicpmena/home
7. Extending the embodied-design agenda into special education in dialogue with Universal Design for Learning (Abrahamson, Flood, Miele, & Siu, in press)
8. Some senior members joining junior members' grant proposals as Co-PIs and advisors

EMIC 2019: Creating Embodied Instructional Activities for Mathematics

Last year, the EMIC working group focused on extending theoretical frameworks of embodied cognition (Melcer & Isbister, 2016) as well as exploring the role of technology in assessing and assisting mathematics learning. This year, we will continue this discourse as we effectively transition from theory-driven discussion and research to application in mathematics education. Specifically, we will continue to focus on theoretical frameworks which tie various perspectives on embodiment to different forms of physicality in educational technology (Melcer & Isbister, 2016; see Figure 1 below) as foundations to collaboratively design novel activities for mathematics education which utilize action, objects, and the surrounding environment in distinct ways. We will follow up on several of the discussions from the workshop held at UW-M, plan the proposed 2020 workshop, and explore the ten-year research agenda that aims to provide a coherent set of evidence-based design principles for enhancing mathematics education and broadening participation in all STEM fields.

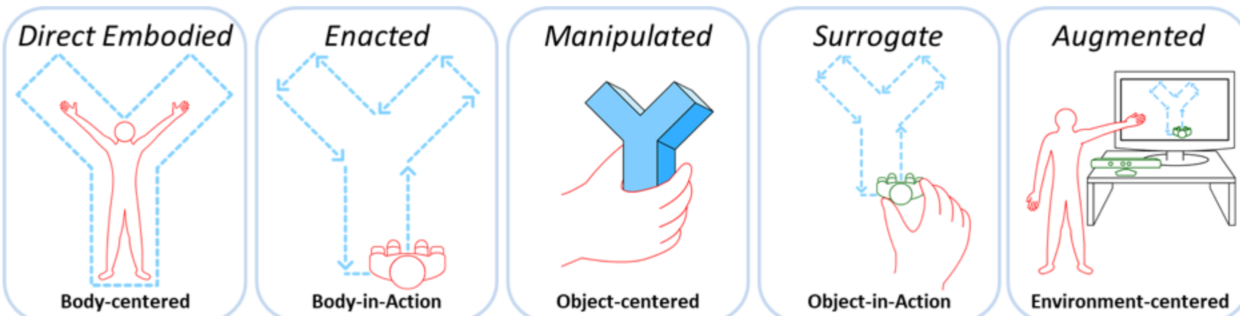

Figure 1: Five Distinct Approaches to Facilitating Embodiment through Bodily Action, Objects, and the Surrounding Environment in Educational Technology

In previous years, participants experienced theory-driven embodied instructional activities for mathematics learning. Two years ago, the EMIC working group focused on embodied instructional activities in geometry and most recently, the group focused on instructional activities which emphasize gestures through novel technologies. Examples include exploring mathematical transformations while using a dynamic technology tool (Ottmar & Landy, 2016), playing and creating embodied technology games to teach mathematics and computational thinking (Arroyo et al., 2017; Melcer & Isbister, 2018; Nathan & Walkington, 2017); using dual eye tracking (Shvarts & Abrahamson, 2018), and a teacher guiding the movements of a learner exploring ratios (Abrahamson & Sánchez-García, 2016). From experiencing these embodied activities, we explored questions such as: what role does technology play on supporting connections between the brain, body, and action? Even as we continue to bear these formative questions in mind, this year, participants in our EMIC workshop will shift from experiencing such activities and reflecting on the roles of physicality, technology, and collaboration to applying their perspective of embodied cognition to the creation of future novel activities for research and/or learning contexts.

Figure 2: A Small Selection of Embodied Activities Created by EMIC Organizers and Experienced by EMIC Participants

Plan for Active Engagement of Participants

In the past years of PME-NA working groups, we successfully engaged participants in open-ended math activities at the beginning of each session that steered our discussions towards elements of mathematics that the group found most provocative. This year, we intend to engage participants by facilitating one central open-ended activity that places participants in the driver seat for their working group experience—collaborating in small groups to design novel embodied instructional activities for mathematics (Figure 3).

On **Day 1,** we will focus on introductions and goals for the three sessions. After introductions to one another as well as an overview of the EMIC working group, we will discuss the goals for PME-NA 2019. The overarching goal will be to design and demonstrate a small collection of embodied activities for mathematics learning that utilize the enactment of goal-oriented movement in unique ways. This will give way to a discussion of the theoretical framework of physicality in embodied activities (Melcer & Isbister, 2016), which will drive the structures of our instructional activities. We will divide into small groups which will work together over the course of three days to create an instructional activity. The groups will mix educators, researchers, and students to create synergistic groups with different interests and knowledge of mathematics education. After a brainstorming session in small groups, the EMIC working group will close with a general discussion of ideas for instructional activities. On **Day 2**, we will primarily work in our small groups to continue designing embodied instructional activities. Towards the end of the session, groups will share their progress on activity design and participate in a guided discussion about the role of physicality, technology, and collaboration in the designed activities to reflect on how the structure of the activity contributes to connections between the mind, body, and action. **Day 3** will be focused on finalizing the created activities among small groups as well as planning for continued engagement and the dissemination of these activities.

Figure 3: Agenda Outline for the 2019 Embodied Mathematical Imagination and Cognition Working Group

Building on the diverse work in embodied cognition and among our group members, possible topics for these activities may include:

1. Grounding Abstractions
    1. Conceptual blending (Tunner & Fauconnier, 1995) and metaphor (Lakoff & Núñez, 2000)
    2. Perceptuo-motor grounding of abstractions (Barsalou, 2008; Glenberg, 1997; Ottmar & Landy, 2016; Landy, Allen, & Zednik, 2014)
    3. Progressive formalization (Nathan, 2012; Romberg, 2001) and concreteness fading (Fyfe, McNeil, Son, & Goldstone, 2014)
    4. Use of manipulatives (Martin & Schwartz, 2005)
2. Cognition emerges from perceptually guided action: Designing interactive learning environments for EMIC
    1. Development of spatial reasoning (Uttal et al., 2009)
    2. Mathematical cognition through action (Abrahamson, 2014; Nathan et al., 2014)
    3. Perceptual boundedness (Bieda & Nathan, 2009)
    4. Perceptuomotor integration (Ottmar, Landy, Goldstone, & Weitnauer, 2015; Nemirovsky, Kelton, & Rhodehamel, 2013)
    5. Attentional anchors and the emergence of mathematical objects (Abrahamson & Bakker, 2016; Abrahamson & Sánchez–García, 2016; Abrahamson et al., 2016; Duijzer et al., 2017)
    6. Mathematical imagination (Nemirovsky, Kelton, & Rhodehamel, 2012)
    7. Students' integer arithmetic learning depends on their actions (Nurnberger-Haag, 2015).
3. Affective Mathematics
    1. Modal engagements (Hall & Nemirovsky, 2012; Nathan et al., 2013)
    2. Sensuous cognition (Radford, 2009)

4. Gesture and Multimodality
    1. Gesture & multimodal instruction (Alibali & Nathan 2012; Cook et al., 2008; Edwards, 2009)
    2. Bodily activity of professional mathematicians (Nemirovsky & Smith, 2013; Soto-Johnson, Hancock, & Oehrtman, 2016)
    3. Simulation of sensory-motor activity (Hostetter & Alibali, 2008; Nemirovsky & Ferrara, 2009)
5. Universal Design for Learning in Special Education (Abrahamson et al., in press)

Follow-up Activities

Unique to this year, the EMIC working group intends to focus on collaboratively creating instructional activities for mathematics education. By the end of the third session, we aim to have a collection of novel activities for mathematics classrooms that utilize action and movement in various ways. After the conclusion of the working group, participants will be invited to assist the organizers with the dissemination of these activities on the EMIC website, in journal articles, and other math education forums.

Beyond dissemination of our instructional activities, we envision an emergent process for the specific follow-up activities based on participant input and our multi-day discussions. As in previous years, we will continue to develop a list of interested participants and grant them all access to our common discussion forum and literature compilation. Additionally, the EMIC organizers will continue to plan workshops that continue this line of work. All EMIC participants will be invited to attend these events.

In the past several years, we have seen a great deal of progress. This is perhaps best exemplified by the development and continuation of the EMIC community, the NSF workshops, the website, the ongoing collaborations between members, and the annual PME-NA workshops that draw participants from across the country. We will strive to explore ways to continue to reach farther outside of our young group to continually make our work relevant, while also seeking to bolster and refine the theoretical underpinnings of an embodied view of mathematical thinking and teaching.

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DEVELOPING THEORY, RESEARCH, AND TOOLS FOR EFFECTIVE LAUNCHING: DEVELOPING A LAUNCH FRAMEWORK

Andrew M. Tyminski
Clemson University
amt23@clemson.edu

Rob Wieman
Rowan University
gomathman@yahoo.com

Gloriana González
University of Illinois
ggonzlz@illinois.edu

Aaron Trocki
Elon University
atrocki@elon.edu

Kim Johnson
West Chester University
kjohnson2@wcupa.edu

Jill Perry
Rowan University
perry@rowan.edu

Jennifer A. Eli
University of Arizona
jeli@math.arizona.edu

This working group continues to develop a research program on the practice of launching mathematical tasks and its resulting impact on learners. Our research agenda contains two strands of inquiry exploring (1) theory and framework building concerning effective launches and (2) empirical examination of the link between launching and opportunities for students to engage in "productive struggle" while solving worthwhile mathematical tasks. Participants will (a) examine empirical data collected from K-12 mathematics teachers in order to further develop our emerging Launch Framework and (b) discuss the next steps to be taken in the development of an empirical research agenda examining launching and high demand tasks. Participants will collaborate to shape the development of the emerging research agenda and plan future research and dissemination.

Keywords: Classroom Discourse, Instructional Activities and Practices, Curriculum, Problem Solving

The new *Standards for Preparing Teachers of Mathematics* calls for supporting new teachers in building a foundation of "effective and equitable mathematics teaching practices" including introducing, or "launching" demanding tasks (AMTE, 2017; p. 6; 14). When doing so, teachers need to activate prior knowledge, ensure understanding, establish expectations, and remove barriers to productive engagement (Van de Walle, Lovin, Karp, & Bay-Williams, 2014). However, there is little research that describes, in detail, what effective teachers do when launching, and how what they do changes in response to changes in their teaching context.

Research has given some guidance to teachers seeking to launch tasks effectively. Initial descriptions of effective launches stress the importance of clear expectations (Stein, Smith, Henningsen, & Silver, 2009) and creating a shared understanding of the problem and the problem's context (Ball, Goffney, & Bass, 2005; Lubienski, 2000). Jackson and colleagues (2013) found that launches which supported students in developing a taken-as-shared understanding of the key ideas and quantities as well as a shared vocabulary to describe those ideas produced greater opportunities to learn in subsequent mathematical discussion. Despite this guidance, teachers and teacher educators continue to struggle with what constitutes an effective launch. Teachers wrestle with how to give exactly the right amount of support that provides access to students without taking over mathematical thinking (González & Eli, 2017). In addition, different goals for the launch may conflict with each other (González & Eli, 2017; Jackson & Shahan, 2013). Furthermore, the features and goals of a specific launch may depend on the task, the learning goal, and the needs and strengths of a specific group of students. In the face of this complexity, continued examination and discussion about the purpose and features of effective launches are necessary in order to help guide teachers and teacher educators as they work to develop launching expertise.

History of Launching Working Group

While the launching working group first officially met at PME-NA 2018 in Greenville, SC., this meeting was an important culmination of sorts for the group and its interests. The original impetus for the formation of this working group is based upon the work of mathematics methods instructors who designed a module to support prospective teachers' ability to launch cognitively demanding tasks. As these researchers sought to examine the theoretical underpinnings behind launching tasks it became evident there was a dearth of research or theory on the subject. "There were few common images of effective launches in the research literature, nor were there descriptions of the kinds of problems that students and teachers experienced during launches" (Wieman, Perry, et al., 2018, p. 1501). As a result, the content and design of the module was not a representation of "professional knowledge" which is established by the research community or as a result of empirical studies, but rather "practical knowledge" which is built by teacher educators as a result of doing the work of teaching and reflecting upon that work (Arbaugh & Taylor, 2008, p. 2).

At PME-NA 2016, the designer-researchers presented some early results of their empirical examination of their methods course module (Wieman & Jansen, 2016). The session was well attended and included a spirited discussion concerning the salient features and non-features of an effective launch. Several attendees shared examples of launches from their own methods course work, examples that contrasted starkly with each other, and the launch depicted in the module. Both presenters and several attendees continued the conversation after the conclusion of the session. After the conference the session organizers collaborated with a group of mathematics teacher educators to propose a symposium session at AMTE 2018 on launching (Wieman, Jansen, et al., 2018). This symposium brought together researchers who had studied launches, teacher educators who were teaching teachers to launch effectively, and professional developers with extensive experience in schools. In this session, we began a discussion that moves the field towards an explicit, shared understanding of how to effectively launch demanding tasks, including (1) What is the purpose of a launch? (2) What are the elements of an effective launch? and (3) What are common challenges in launching? Again, this symposium was well attended, and generated extensive discussion, as well as a striking diversity of thought and experience. Clearly, there was a need among mathematics teacher educators to examine the practice of launching more closely.

PME-NA 2018 Working Group

The launching working group was formed to give educational researchers and practitioners and opportunity and space to work toward the following long-term goal:

> Create shared, empirically-based knowledge about launching cognitively demanding tasks, that would support mathematics teachers in launching tasks effectively, mathematics teacher educators in supporting teachers learning to launch, mathematics education researchers in

generating knowledge about launching, and curriculum writers in supporting teachers' launching. (Wieman, Perry, et al., 2018)

The three working group sessions at PME-NA 2018 were consistently attended by a group of 10-12 mathematics teacher educators. Across the three sessions, participants engaged in a series of activities and discussions designed to answer the following questions:

- What is the purpose of an effective launch?
- What challenges do teachers face when planning, enacting and evaluating launches?
- What are typical experiences for students in launches?
- How do we support teachers and pre-service teachers in developing skill in planning, enacting and reflecting on launching (and how might we improve these efforts)?
- How do we support teacher educators and professional developers in helping others get better at launching (and how might we improve these efforts)? (Wieman, Perry, et al., 2018)

Day 1. After orienting participants to the prior work of the group, organizers and participants shared interests and questions concerning launches. After identifying some common questions and themes, participants were introduced to the following mathematical task:

Last year the national weather service recorded _____ tornadoes in the United States. They recorded some tornadoes in other parts of the world. They recorded a total of ____ tornadoes. How many tornadoes were in other parts of the world?

(18, 28) (26, 48) (22, 75) (83, 150) (95, 194) (101, 183)

We asked participants to consider and discuss how a teacher might launch this task and what other questions they might have about launching this task. We then viewed a video of a teacher launching this task to a class of second graders. While they watched, we asked participants to consider the following:

- What do you notice about the launch that you found especially interesting or surprising?
- What does the teacher think the purpose of a launch is?
- What supports does the teacher use to help students, and the teacher during the launch?
- What is the impact of this launch on students? How do you know?
- What mathematical activity do you predict the students will engage in?
- How might you evaluate the effectiveness of this launch?

After discussing these questions in relation to the launch of this task, we introduced participants to a second task, "*Write an equation that you can use to find the number of one foot by one foot square tiles you would need to make a one-foot fringe around an "n x n" square pool".* Participants then discussed how a teacher might launch this task, and viewed a video of a teacher launching this task in an 8th grade classroom, considering the above set of questions. This was followed by a discussion in which attendees compared and contrasted the two launches, and how they informed our original interests and questions.

This final discussion resulted in five salient questions moving forward: (1) How do teachers balance support with maintaining the cognitive demand while launching? (2) What might be the role of a “Precursor Task” and when should one occur? (3) Can students “internalize launch routines” or develop the capacity to make sense of problems? (4) Are there routines, steps, or structures for launches? (5) How do features of a task relate to features of a launch?

Day 2. In Day 2 we took up two versions of the final question listed above, “What is the relationship between key features of tasks and key features of launches?” and “Are there certain launch structures that go with certain tasks?” In order to begin examining these questions participants worked in small groups to sort a collection of 10 mathematical tasks according to how they would launch them – they were group together tasks that they would launch in a similar way and be ready to explain how that type of launch would support students doing that task. Once participants had come to a consensus within groups, we engaged in a gallery walk, in which groups struggled to make sense of other group’s categories. The ensuing discussion illuminated a great diversity of ideas about task features, launch features, and the lack of a shared language or understanding about the details of launching.

Day 3. Our final session began with a review of the prior day’s sorts in which we created a list of possible problem features and a list of launch types or structures which might be appropriate for the given tasks (e.g. three reads, act out the story/process, make a prediction, etc.)

Participants then were given this set of launch types and worked in groups to sort a second set of tasks according to which launch type they would use to launch it. From the discussions generated by the two task sorts we were able to begin work on a launching moves framework to support teachers’ practice of effective launching for demanding tasks. We present these in our next subsection.

Progress Since 2018 Working Group

We extend our work from 2018 by presenting preliminary ideas related to the question, “What is the reciprocal relationship between central features of demanding mathematical tasks and central features of launches? or “What types of launches are effective for what type of problems?” One result of the discussions in the working group was a *Launching Framework*, which serves as an initial attempt to describe key features of different types of launches, as well as key features of demanding tasks that may inform choices about launching. We first unpack and define the key features of important launching types (Table 1). We then identify specific problem features and give illustrative examples (Table 2).

Table 1: Launch Types

Launch Types	Example/Description
Act out the process	Eric the sheep: Act out Eric skipping two places whenever a sheep gets sheared.
Provide a visual representation of the context	Create a representation of the task - i.e. showing a picture of two punch bowls with different amounts of Orange Juice and Ginger Ale next to them for a comparing rates problem (Which tastes more like Orange Juice?)
Noticing	Three-act lesson: Show a video, ask what they notice, what they wonder

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Three Reads	First Read: What is this about? Second Read: What is the question? Third Read: What is important information/quantities?
Make a prediction or conjecture	Ask students to make a guess about the answer
Evaluate an incorrect answer	Johnny had some money, he went to the store and worked until he had twice as much as he started with. Then he went to the diner and had dinner, for $12. He now has $34. How much did Johnny start with? I say he started with $50. Am I right or wrong? How do you know? Then how much DID he start with?
Engage students in Precursor Tasks: (i.e. a number talk or routine for reasoning)	Contemplate then calculate: What do you notice about this figure? How many tiles in this figure without counting? Then, can you come up with a rule for the number of tiles in the "nth" figure?
Set Expectations	Remind or create expectations for how to work together in a group, how to address logistical concerns, and the nature of the final product

Table 2: Problem Features and Sample Tasks/Descriptions

Problem Features	**Description and Sample Task**
Process with end-point given	Golden Apples (Start with some apples; Meet three trolls, each one takes half your apples and two more. You end up with 3 apples. How many did you start with?
Process with beginning point given	Eric the sheep is 50^{th} in line to get sheared. Each time a sheep gets sheared, Eric cuts two sheep in line. When will Eric get sheared? (How many sheep will get sheared before Eric is at the front of the line?)
Graph/Diagram Focused	Match the picture of people at a bus stop with a graph of age versus height.
Familiar context	Context that draws on student experiences
Unfamiliar context	Context that may be foreign and confusing to students

Context free problem	Explain why cos(x) = sin (90-x)

An Illustrative Example

As an example, suppose a teacher wanted to pose the Eric the Sheep problem:

It's a hot summer day, and Eric the Sheep is at the end of a line waiting to be shorn. Each time the shearer takes one sheep from the front of the line, Eric sneaks past two sheep to get closer to the front.

1. Suppose there are 10 sheep in front of Eric. How many sheep will be shorn before Eric gets to the front of the line.
2. Suppose there are 25 sheep in front of Eric. How many sheep will be shorn before Eric gets to the front of the line?
3. Suppose there are 50 sheep in front of Eric. How many sheep will be shorn before Eric gets to the front of the line?
4. How could you predict the answer for any number of sheep in the line?
5. What happens if Eric sneaks past 3 sheep? 4 sheep?
6. What happens if there are 2 shearers? 3 shearers? (Driscoll, 2001).

The teacher may anticipate that students may have trouble making sense of this particular situation, so she may decide to act out this specific situation, having students take the place of sheep, and physically acting out the process of getting shorn, moving up in line, and, for Eric, cutting in line each time a comrade is shorn.

This task would be categorized as a *Process with a beginning point given*. A process is described and students are asked to explain what will happen as this process unfolds. The teacher employs an "Acting it out" launch, by having her students act out the shearing. One could also argue that the task itself employs a "precursor task" structure, asking students to solve specific cases of the process before engaging in the task of generalizing. It is key however to consider what number(s) might make for good examples without reducing the cognitive demand of the task. For example, if Eric is in a line with 6 sheep, 2 will be shorn before him. If Eric is in a line with 7 sheep, there are still 2 sheep shorn before him. Revealing the idea that multiple starting numbers have the same answer in the launch of the problem would not be beneficial as it can be a somewhat surprising outcome and one students have to consider when answering the generalizable questions in the task. If the teacher wanted to show an odd number example and an even number example, 7 and 8 would be better number choices than 6 and 7.

Once we have begun to identify important launching types and important problem features, we can begin to empirically determine if curriculum writers and teachers tend to launch problems with specific features using specific types of launches. For instance, we can ask teachers to engage in the same kind of sorting exercise as we described above, and then keep track of their responses on a grid.

Figure 1: Launch Framework - Matching Problem Features and Launch Types

Launching Framework		Launch Types							
		Act out process	Provide a visual representation of the context	Noticing (I notice, I wonder)	Three Reads	Prediction/ Conjecture	Evaluate an incorrect answer	Precursor Tasks	Setting Expectations
Problem Feature	Process with Starting Point Given								
	Process with Ending Point Given								
	Graph/Diagram focused								
	Familiar Context								
	Unfamiliar Context								
	Context Free								
	Other								

Plan For PME-NA 2019 Sessions

Day 1

Prior to the meeting we will send potential attendees a file of mathematical tasks along with instructions how to complete the launching task sort described above. The goal of this is to help those unfamiliar with the sort to understand the task sort activity prior to our first session. We will begin session 1 with a quick introduction of the participants and their interests in examining the practice of launching demanding tasks, followed by an orientation to the previous work and progress of the group. Our main activity for the first session will be to engage participants in examining data and or/results from an enactment of the launching task sort activity with inservice teachers. Our session organizers are planning a professional development activity in the spring of 2019 in which they will engage a group of K-12 in-service mathematics teachers in the launching tasks sort activity, and collecting data on the choices they make and their rationales. This will enable leaders and participants to both make sense of the launching framework and examine teacher perspectives on launching. We also hope to discuss revisions and additions to the framework based on this data analysis.

Day 2

On the second day of our working group we intend to engage participants in two different launch related activities. In the first activity participants will utilize the *Launch Framework* to engage in small group examination of a curriculum series to identify relationships between high demand problem features and prescribed launches. Each group will examine the same curriculum materials so a common base of discussion can be had. Through this activity we hope to generate a protocol for utilizing the framework for curriculum analysis that could be used to continue this work in the months following the conference. The second activity will afford participants an opportunity to examine videos of teachers launching the same task utilizing different launch types and to consider the initial discussion of students following/during the launch. Discussion of this activity will help conceptualize a follow up study examining the relationship between categories of selected launch move, teacher enactment of launch move, and student reaction to the launch. At the end of the session participants will be asked to consider the two different lines of inquiry and to select one to discuss in small group the following day.

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Day 3

The third day will begin by forming special interest groups (SIGs) (i.e. curriculum analysis, effect of launch types on student learning, etc.). Each group will be provided time to begin planning a research project to be carried out over the upcoming months for their topic. At the end of the session groups will have selected a SIG leader and will submit a brief outline of their plans moving forward including possible timeline. Each group will share these plans with the group as a final activity. We will encourage SIGs to plan for proposals to present at AMTE 2021 (proposals usually due in mid-May 2020).

Focus on Specific Framework: Rationale

Given the wide range of initial questions about launching that this working group was interested in, we have chosen to focus on examining and developing a specific framework related to launching structures and features of tasks for several reasons. First of all, we hope that a more focused discussion will provide access to new members of the working group while pushing the work of the group forward. Second, we think that this focus on launch types, task features, and the connections between them will provide numerous opportunities to talk about other important and interesting questions. For instance, we believe that discussing features of problems and launches will also involve discussing student thinking and creating hypotheses for the kinds of struggles students and teachers experience during the launch phase of a lesson. We think that anchoring these discussions in specific problems and launches will help participants more rigorously articulate questions and hypotheses that can drive research.

Connection to Conference Theme

The conference theme for the PME-NA 2019 Annual Conference, "Against a New Horizon," is an explicit acknowledgement of how a quest for progress can also reinforce systematic exclusion. The extant research has demonstrated that a teacher's launch has the ability to provide or deny opportunity and access for students to engage in worthwhile mathematics and as such it is paramount we understand what constitutes effective launch practices. The promise of problem-based learning remains unrealized for many mathematics students, especially those who have traditionally been under-served by larger educational institutions and systems. While we have come to understand there exists an unexplored variety and complexity to launching cognitively demanding tasks, we have been able to agree that the underlying goal of any launch is to provide "access and opportunity for ALL students to grapple with challenging mathematics" (Wieman, Perry, et al., 2018, p. 1502). In addition to access and opportunity, launching is also, fundamentally, about agency and empowerment. Effective launches support all students in drawing on their own funds of knowledge (Moll, Amanti, Neff, & Gonzalez, 1992) to make sense of problems, provides them with opportunities to engage in the productive struggle that results in deep conceptual understanding, and empowers them to decide for themselves what solution strategies to pursue. Learning more about launches will help us empower all of our students so that they can all build on the knowledge and experience they have to construct new mathematical understandings, and engage powerfully in a diverse mathematical community.

Anticipated Follow Up Activities

Following the completion of our working group sessions we foresee several follow up activities. Our first priority will be the completion and submission of a National Science Foundation DRK-12 Conference Grant Proposal for a 2020 conference focused on launching for teachers and teacher educators. Second is the empirical examination of our hypothesized framework. We have three lines of inquiry for this: (1) Through collection and analysis of empirical data examining tasks, launches and student reactions to them we will attempt to verify and/or refine the framework in terms of student effectiveness; (2) We intend to utilize a survey approach to gather more Launching Task Sort data from MTEs in order to further refine the *Launch Framework*; and (3) We will continue the empirical examination of curriculum materials begun at this working group. As mentioned above we will be encouraging members of the SIGs forms at this working group to plan proposals for AMTE 2021.

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SIMULATIONS OF PRACTICE FOR THE EDUCATION OF MATHEMATICS TEACHERS

Heather Howell
Educational Testing Service
hhowell@ets.org

Yvonne Lai
University of Nebraska - Lincoln
yvonnexlai@unl.edu

Carrie Lee
East Carolina University
leecarr16@ecu.edu

This working group addresses the challenge and opportunity presented by the use of simulations of teaching practice as an educative tool for preservice and practicing teachers. We focus particularly on the development of teachers' skills in enacting the content-intensive work of teaching, including how simulations may be able to cultivate or reveal teacher MKT, either as propositional or applied knowledge or as an essential component of a larger teaching practice in which MKT is activated.

Keywords: Mathematical Knowledge for Teaching, Teacher Knowledge, Teacher Education-Preservice, Teacher Education-Inservice/Professional development

History of Topic and Elaboration of the Problem Space

We situate this working group in two complementary strands of work, one focusing on teacher knowledge and the other on teaching practice. Since Shulman (1986) introduced the construct of pedagogical content knowledge, scholars have identified specialized mathematical knowledge in and for teaching (e.g., Ball & Bass, 2003; Krauss et al., 2008; Thompson & Thompson, 1996). With this scholarship have come efforts to focus teachers' education on learning this specialized knowledge (e.g., Ball, Sleep, Boerst, & Bass, 2009; Ghousseini, 2017; Silverman & Thompson, 2008). This knowledge, sometimes identified as mathematical knowledge for teaching (MKT), coordinates purposes and reasoning from both mathematics and teaching. Developing teacher MKT has increasingly become a core component of teacher preparation for mathematics teachers (e.g. Association of Mathematics Teacher Educators, 2017; Conference Board of the Mathematical Sciences, 2012) with assessment of MKT generating a body of scholarship (e.g., Gitomer, Phelps, Weren, Howell & Croft, 2014; Herbst & Kosko, 2012; Hill, Schilling & Ball, 2004; Krauss et al., 2008) and recently crossing over into mainstream licensure tests (see https://www.ets.org/praxis/about/ckt/).

MKT is, by definition, knowledge that is demanded in response to the content-intensive work of teaching (Ball, Thames & Phelps, 2008) and has been described by scholars as *practice-based*, or more closely grounded in teaching practice than other forms of mathematical knowledge (Ball & Bass, 2003). Other scholars have noted that the examination of MKT as it is learned by teachers or used in teaching benefits from more clearly articulated conceptualizations of how that knowledge is held, enacted, activated, or drawn on. (Heid, Wilson, & Blume, 2015; Rowland, 2013) The Knowledge Quartet work, for example, provides a framework designed "to guide attention to, and analysis of, mathematical knowledge-in-use within teaching", the purpose of which is to help teachers reflect and learn from their teaching (Rowland & Ruthven, 2011, p.85). Ghousseini (2017) similarly distinguished between propositional knowledge and knowing as a form of action, and Heid et al. (2015) described mathematical knowledge in terms of proficiency and activity. These lines of work arguably push toward understandings of MKT as it is used in teaching in ways that are even more closely aligned to the practice of teaching.

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A complementary line of practice-based research has emerged that seeks to conceptualize critical teaching practices and to understand how teachers learn to engage in them effectively. Recent years have seen widespread effort to identify and prioritize teaching practices (e.g., http://www.teachingworks.org/work-of-teaching/high-leverage-practices, https://www.corepracticeconsortium.com/), shifting the field toward the systematic description of the most critical competencies for teachers to develop during preparation. At the same time, theories such as the Grossman, Hammerness, and McDonald (2009) pedagogies of enactment help to describe learning opportunities likely to support teachers' development of those competencies. A number of studies utilizing simulations of teaching practice are grounded in this theory, which describes the key pedagogy of *approximation of practice* as "opportunities to rehearse and enact discrete components of complex practice in settings of reduced complexity" (p. 283).

We situate this working group proposal at the intersection of these two lines of work. While not all simulated teaching is content-intensive, the simulation of content-intensive teaching practices may provide us with opportunities to observe, develop, and measure MKT, as recommended across elementary (e.g., Ball & Cohen, 1999; Hill, Sleep, Lewis, & Ball, 2007; Stylianides & Stylianides, 2013 and secondary levels (e.g., Ticknor, 2012; Wasserman, Weber, Villanueva, & Mejia-Ramos, 2018). And increasing attention to how MKT is activated and used in the practice of teaching can help us to better understand the relationship between knowledge and practice and how a teacher's MKT may relate to the quality of his or her practice or opportunities to learn from practice.

Enacting this approach brings a number of challenges. When working with simulations, teachers' attention can emphasize pedagogical concerns, "eclipsing" the intended specialized knowledge (Creager, Jacobson, & Aydeniz, 2016, p. 3); and at other times teachers may attend to mathematical ideas in a way that sidelines pedagogical reasoning (e.g., Schilling & Hill, 2007; Suzuka et al., 2009). In enactments of these activities, whether teachers' attention slides away from mathematics or away from pedagogy, the opportunity for using mathematics and pedagogy to inform each other is lost. In other words, one challenge facing the simulation designer is that of focus. The proposed working group will provide opportunities for simulation designers to work toward clearly defining objectives and determining which simulation design characteristics are critical in creating circumstances likely to meet those objectives.

A second challenge is that of common vocabulary. As Grossman, Compton et al. (2009) noted, simulations such as approximations of practice "can vary significantly, both in terms of comprehensiveness and authenticity" (p. 2065), ranging from responding to written cases to role-playing in mixed reality settings. In addition, the simulations under consideration vary significantly with respect to factors that matter in simulation design such as the level of interactivity or the types of records in which the practice is made visible. This variability in turn may amplify the potential for confusion across lines of work. Lesson planning, for example, might be considered a teaching practice to be simulated by one project, or as a part of the preparatory cycle for a simulation of interactive teaching by another, and it is likely to be conceptualized, supported, standardized, recorded, related to teacher knowledge, and attended to in very different ways as a result. The proposed working group will provide opportunities to work toward a common vocabulary or way of describing simulations of teaching practice

Rationale for the Working Group

This working group is timely and critical. The field is still at the early stages of developing a robust understanding of the ways in which practice-based professional preparation occurs within teacher education, especially in terms of how to leverage simulated environments to productively develop teachers' competencies (Sykes & Wilson, 2015). Recent years have seen multiple nationally-funded efforts to improve the mathematical preparation of teachers by developing materials with simulations of teaching practice for use in mathematics and methods courses. Most of these efforts build on the success of small-scale pilot studies (e.g., Lischka, Strayer, Watson, & Quinn, 2017; Straub, Dieker, Hynes, and Hughes, 2015; Wasserman, Fukawa-Connelly, Villanueva, Mejia-Ramos, & Weber, 2017).

While approximations of practice such as role playing or peer teaching are common practices in teacher preparation, more systematic approaches such as those using trained actors to provide standardization of opportunity (e.g., Dotger, Masingila, Bearkland, & Dotger, 2014; Shaugnessy & Boerst, 2018) are relatively rare, and investigators have noted the novelty and rarity of the very idea of incorporating technologically-supported simulations into mathematics courses (e.g., Ensley & Fiorini, 1998; Lai & Patterson, 2017; Wasserman et al., 2017). As noted in the 2018 PME-NA call, such technologies create opportunities to "explore how technology can be used in the service of mathematics education and research," (Hodges, Roy, & Tyminski, 2018) both by allowing us to explore best practices around the use of such technologies and because of the windows of opportunity they create to observe teacher learning in more systematic ways.

There is thus an emerging community of mathematician and teacher educators who are interested in attending to teachers' mathematical development through simulations of practice. For example, one of the NSF-funded projects listed previously supported a recent conference on the topic of simulation use in teacher education attended by one of the working session co-leads, who reported that major learning from sessions included (1) a significant variation across models of implementation, structure of cycles of enactment, and purposes and (2) a need expressed among expert panel members for the field to describe the parameters of simulation task design in a way that helps to build mappings between those parameters and task design.

Focus of Work

This working group seeks to explore the following questions:

1) How can we conceptualize the theories of action by which teacher learning is expected to result from engagement in simulation activities in contexts such as mathematics and methods coursework?

Theories of how teacher learning occurs, characteristics of simulation activities, the role of MKT, and how the activities are evaluated differ in substantial ways across projects. For instance the ULTRA project (Wasserman et al., 2018) is based on a learning theory of transfer and has revised its simulations in tandem with revising decompositions of teaching practice, whereas the MODULE(S^2) project (Lai, Strayer, & Lischka, 2018) is based on a learning theory of decentering and has revised its simulations in tandem with revising an adaptation of the Knowledge Quartet framework for use with simulations. This working group will invite comparison of theories of action and explore the variability in how researchers are conceptualizing simulation.

2) How can we articulate design principles rising out of or grounded in the theories of action identified in question 1?

Many simulation projects utilize iterative cycles of design to revise and refine simulation tasks. That is, simulations are designed based on a provisional theory of action, the simulations are then enacted in instruction, the enactments and instruction are studied in relation to intended outcomes, and action is taken to revise the simulations and underlying theory based on studying enactments.

This revision process is the foundation for a growing understanding of how characteristics of simulations provide opportunities for teachers to develop their teaching practice. But we lack a common language for what these characteristics are, and how they differ by context. Without a common language, it is difficult for the field to build systematically on this emerging base and capitalize on interest. So now is the time to contextualize what different educators are doing, how they draw on theory, and the rationale for choosing these theories. As Kennedy (2016) argued, the rationale matters more than the particular actions, because it's the reasoning that can change practice.

3) How can we measure the development of MKT, teaching practice, or other valued outcomes through the use of simulations?

Simulation use is additionally diverse in the varying degree to which it focuses on assessment and with respect to the targets of that assessment. Shaughnessy and Boerst (2018), for example, describe a program designed to measure preservice teachers' skill in eliciting student thinking through the use of a standardized on-demand interactive simulation. While the context of their work is indisputably content-intensive and they note that skill in eliciting may interact with MKT, the primary focus of their study is the teaching moves involved in elicitation. One could easily imagine, however, analyzing such data for evidence of MKT in use, similar to the approach taken by Lai et al. (2018). In contrast, Mikeska, Howell, & Straub (2017) acknowledge and propose to study the potential relationship between MKT and the targeted teaching practice of leading small group discussions in mathematics, but provide preservice teachers with preparatory materials in advance of the interactive simulated teaching to support their ability to make sense of the student work samples they will be discussing. Of note across these examples is that despite similar conceptualizations of MKT and a common understanding that it relates to the teaching practices simulated, MKT holds a very different role in what each project seeks to measure.

Organization and Plan for Active Engagement

The overall goal of this new working group is to create a community in which researchers and practitioners can explore how simulations of practice can be optimized to provide opportunities for teacher learning. Prior to convening in Missouri, we see value in collecting information from participants through a short survey on participants' conceptualizations of simulations of practice and how they use them in their teaching and/or research. This information will help create a canvas in which to begin to co-create our community.

The working group will consist of three sessions during the conference followed by virtual meetings through the following year. Across the three sessions participants will engage with facilitators to examine the learning theories that ground current work in simulations of practice, identify key design principles of the simulation tasks, and analyze our methods of assessing the impact of this work. In each session, participants will engage with simulation tasks of their own or from the facilitators, make explicit the areas they want to dive into deeper, and link with others that want to dive in with them.

Session 1: Exploring our Simulations of Practice

For the first session we will use information from our short survey to ensure that we are anticipating how different researchers and practitioners are conceptualizing simulations of practice. We will introduce our conceptions of simulations of practices within our own work and expand to include those from the survey. Next, small groups will be formed based on common interpretations, and group members will participate in an activity to map features of simulations to the issues of teacher learning that they are attempting to address. These simulations of practice will be provided by attendees' own work or examples from co-leaders' projects.

This mapping activity will result in a working document of simulation characteristics linked to the objectives they are intended to meet. This mapping will also allow outlets for us to problematize the simulation space. These activities are intended to surface commonalities and differentiations in our theories of action and our intended use of simulations.

Session 2: Analysis of Design Elements of Simulation Tasks

For the second session we will continue utilizing the working document from the first session and delineate the characteristics or design elements of the simulations that provide opportunities for teachers to enact MKT and teaching practices. Participants will begin by placing examples of simulations on a continuum of authenticity and completeness in relation to their approximation of teaching (Grossman, Hammerness, et al., 2009). This activity will lead to group efforts to identify key components of simulations that afford or hinder teachers' opportunities to learn. From this we will identify common themes in how simulations are designed for particular contexts. This record of design characteristics will be the beginnings of our efforts to create a collection of design principles for simulations that will inform the field and help link theory to practice. Session participants will apply this preliminary collection of design principles to revise or write simulations of practice to incorporate these characteristics. Through this process we will also identify issues of congruence and contrast within the design process.

Session 3: Assessment of the Impact of Simulation Tasks

In the last session of the working group we will again employ the working document to dive deeper into how learning objectives are measured active projects represented by attendees. In our efforts to identify key components of simulations that afford learning we need to be transparent in what outcomes we are measuring as learning and how they align with the underlying theories guiding the simulation development. Outcomes and metrics used within projects by co-facilitators will be presented and examined as a starting point for this session. Next, participants will identify their outcomes and metrics within their own work and consider any possible revisions. The power of utilizing a consistent working document is that we will have a record of the revision processes and analyses that will further inform future simulations and underlying theory.

In the closing segment of this session we will collect contact information from participants, document subgroups and their future research interests, and share next steps.

Follow-up Activities

This working group is a key step in establishing a network of teacher educators and researchers that are engaged in utilizing simulations of practice to develop teaching knowledge and practice. Subgroups formed during the sessions will be encouraged to continue collaborations on co-constructed areas of inquiry. The facilitators will support the continuation of collaboration by hosting virtual meetings through the year to check-in. We anticipate subgroups pursuing design revisions, creating simulations, exploring learning theories, examining the impact of simulations across varying contexts, and other emerging themes.

One of the future priorities of this working group is to channel the productivity of our sessions into a proposal for a special issue in the *Journal of Mathematics Teacher Educator*. In 2007 there were a series of special issues in *JMTE* that described the field's then-current understanding of mathematics-related tasks for the education of mathematics teachers. Those special issues highlighted the development process around task and as a result the editors saw "how the focus on tasks leads to 'meta-tasks' through which knowledge in and of teaching grows in practice through a research process" (Jaworski, 2007, p. 201). We see simulations as a form of "meta-task" and envision this special issue as a platform to describe our current understanding of simulations of practice for the education of mathematics teachers. Subgroups formed during the working group will be well positioned to contribute and advance this work.

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WORKING GROUP ON GENDER AND SEXUALITY IN MATHEMATICS EDUCATION: EXPERIENCES OF PEOPLE ACROSS CULTURES

Rebecca McGraw
University of Arizona
rmcgraw@math.arizona.edu

Katrina Piatek-Jimenez
Central Michigan University
k.p.j@cmich.edu

Lynda Wiest
University of Nevada, Reno
wiest@unr.edu

Ana Dias
Central Michigan University
dias1al@cmich.edu

Harryson Júnio Lessa Gonçalves
State University of São Paulo
harryson.lessa@unesp.br

Jennifer Hall
Monash University
jenniferhall@monash.edu

Angie Hodge
Northern Arizona University
Angie.Hodge@nau.edu

Betsy Kersey
University of Northern Colorado
elizabeth.kersey@unco.edu

Laurie Rubel
University of Haifa & City University of New York
LRubel@brooklyn.cuny.edu

The Gender and Sexuality in Mathematics Education Working Group convened in 2018 with a focus on (1) language use, multidimensional understandings of gender and sexuality, and influences of these on methods, results, and interpretations; (2) interactions between gender/sexuality and students' self-perceptions; and (3) the roles of curriculum, pedagogy, and teacher education in students' experiences of gender and sexuality. The 2019 Working Group will continue with these foci, but with an added dimension of learning through examination of work in gender and sexuality in mathematics education across the world, including country/culture-specific studies, and cross-cultural studies.

Keywords: Gender and Sexuality; Equity and Justice; Affect, Emotion, Beliefs, and Attitudes

The Gender and Mathematics Working Group (GMWG) began more than 20 years ago at PME-NA 20 in Raleigh, North Carolina. At this conference, mathematics education scholars came together "in order to weave together the findings of various strands in research and understanding of issues in gender and mathematics" (Damarin et al., 1998, p. 78). The GMWG continued to meet from 1998 to 2007, with the exception of the 2003 joint conference between PME and PME-NA in Honolulu, Hawaii**.** The group also reconvened for the 2011 PME-NA conference in Reno, Nevada. The GMWG's past discussions have involved reviewing the scholarship surrounding gender and mathematics, defining research strands, determining gaps in the literature, and establishing directions for future work. Some notable accomplishments of this group have been creating a visual representation of the field of gender and mathematics and the social and psychological complexities of the topic (Erchick, Condron, & Appelbaum, 2000) and publishing a monograph on gender and mathematics research as a joint project with scholars from the International Group for PME (Forgasz, Becker, Lee, & Steinthorsdottir, 2010).

The Gender and Sexuality in Mathematics Education Working Group convened in 2018 at PME-NA 40, after a hiatus of a seven years and with an updated title to reflect current trends in the field. Prior to the conference we identified preliminary themes across our work, including (1) the choice of language that we as scholars use when conducting research on gender and sexuality in order to be inclusive of all individuals and to reflect a multi-dimensional understanding of gender and sexuality, and how our choices of language and methods may affect

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our research results and interpretations; (2) interactions between gender/sexuality and students' perceptions of themselves as mathematical learners, including experiences inside classrooms, in less formal learning spaces (e.g., summer camps, homes); and (3) the role of curriculum, pedagogy, and teacher education in the study of gender and sexuality in mathematics education.

Much of our time during the 2018 conference was spent becoming familiar with one another's recent work, and more broadly exploring the experiences and interests of the working group participants, including both the co-authors and others in attendance. Dr. Ana Dias, who was not a co-author for 2018, gave a presentation to the working group about the current social and political context of education, gender, and sexuality in Brazil. In addition, two of the co-authors (Dr. Elizabeth Kersey and Dr. Jennifer Hall) led the group in a very informative presentation and discussion about language and methodology in gender and sexuality research. We also welcomed and heard from new participants who joined us on the final day due to previous unavailability, and we considered opportunities for sharing resources and collaborating in the future. Based on the work of these three days, and to further develop and contextualize our knowledge and research related to gender, sexuality, and mathematics education, the working group leaders decided to propose for 2019 a focus on how these topics are experienced by students and teachers, and studied by researchers, outside of the U.S. In addition to work conducted within the U.S., among our 2019 proposal co-authors are individuals who have studied gender and/or sexuality as related to mathematics education in Australia, Brazil, and Israel. In addition, for this proposal, we have examined the literature from China, Ghana, Jordan, and Saudi Arabia. Short summaries of this work are provided below.

Rebecca McGraw, University of Arizona

Rebecca McGraw's work related to equity, gender, and mathematics includes co-facilitating a residential summer math camp for middle grades girls (led by Lynda Wiest, a co-author of this paper), studying teacher classroom practice from an equity perspective (e.g., distribution of images by gender in classroom resources, patterns of participation, course-taking, and achievement), and investigating pre-service secondary teacher preparation (Bay-Williams & McGraw, 2008; Eli, McGraw, Anhalt & Civil, forthcoming; McGraw & Lubienski, 2007; McGraw, Romero & Krueger, 2009; Rubinstein-Avila et al., 2014). Currently, Rebecca McGraw is particularly interested in the development of teacher and student beliefs about mathematics and learning, and the development of middle/high school students' mathematical identities. Recently, she has begun working with several colleagues, including one at King Saud University, to plan a year-long mathematics education program for Saudi teachers. This program would be modeled, in part, after the Building Leadership for Change through School Immersion (Khbrat) programs currently happening across the U.S. In the following paragraphs, Dr. McGraw has summarized some of the recent research related to mathematics, gender, and sexuality in Saudi Arabia.

According to Abu-Hilal et al. (2014), public schooling in Saudi Arabia began around 1930 with girls beginning to enroll in the 1960s. Rapid progress was made thereafter in terms of the number of girls enrolled, reaching 50% of the total students enrolled in 2000, according to the Ministry of Education. The school system, policies, and curriculum are strongly centralized, and teaching methods are frequently targeted towards rote learning, and not particularly oriented towards skills needed for a global economy (Hein, Tan, Aljughaiman, & Grigorenko, 2015). Pre-college education is entirely single-sex; however, a co-educational university, King Abdullah Science and Technology University, opened in 2009. As of 2018, enrollment included 35% Saudi students (65% international students) and the student body was 37% female (https://www.kaust.edu.sa/en/about/media-relations#part3).

In many countries around the world, researchers have found that mathematics self-concept is highly related to achievement (Mohammadpour & Ghafar, 2014). In Saudi Arabia, boys exhibit a higher self-concept while girls exhibit higher achievement (Abu-Hilal et al., 2014; Marsh, et al., 2014). Some researchers have argued that differences in the ways that boys and girls are raised in Saudi Arabia and in their expected future roles in society, lead boys to have a higher, or even inflated, self-concept, while girls are motivated to prove themselves academically and secure places at local universities through achievement (Abu-Hilal et al., 2014). To some extent, the Big Fish Little Pond Effect (Marsh & Parker, 1984) may be at play, with boys comparing themselves to only to other boys and girls to other girls (Abu-Hilal et al., 2014; Marsh, et al., 2014). Research on Saudi Arabian teacher practices by gender is sparse; however, in one study of teachers' Mathematical Knowledge for Teaching (MKT), researchers found that female teachers scored significantly higher than male teachers on both number and operation content knowledge and knowledge of content and students scales of a translated version of the Learning Mathematics for Teaching (LMT) (2008) instrument (Haroun, Ng, Abdelfattah, & Alsalouli, 2016). This difference may explain some component of the difference found in students' achievement scores by gender.

With regard to student interest in future careers, it is in the more gender-egalitarian countries (such as the U.S. compared to Saudi Arabia) that boys and girls are *less* interested in careers involving mathematics, and in which researchers find gender differences in interest in such careers, with girls significantly less interested than boys (Goldman & Penner, 2016). The research of Charles and Bradley (2009) suggests that gender-egalitarian contexts can "encourage girls and boys to express societally approved gender ideals as part of their gender performance…. Education in these contexts [serves] . . . an expressive function, so that students' educational choices are thought to reflect important aspects of who they are." (Goldman & Penner, 2016, p. 415). Currently, Saudi Arabian women have access to higher education exclusively through women's colleges (with the exception of King Abdullah Science and Technology University). These colleges are on completely separate campuses that are attached to male-only universities. The one exception is the all-female Princess Nourah Bint Abdulrahman University. Founded in 1970, it currently serves approximately 60,000 women, making it the largest women's university in the world. Altogether, women currently account for over 60% of all Saudi university students (Islam, 2017); however, access to jobs is very limited. For example, in 2015, 57% of science graduates were women, but their share of the total labor force was only 16% (Islam, 2017). As in a number of other places throughout the world, gender discrimination in STEM workplaces, and particularly in computer science and engineering, as well as the masculine gendering of STEM, continue to create barriers to entry and advancement (DeBoer & Kranov, 2017).

Katrina Piatek-Jimenez and Ana Dias, Central Michigan University

Katrina Piatek-Jimenez's research interests focus on what motivates women to study mathematics at the undergraduate level and what influences their decisions whether or not to continue in mathematical careers, including factors such as the development and role of one's mathematics identity (Cribbs, Piatek-Jimenez, & Mantone, 2015; Piatek-Jimenez, 2015), images of mathematicians (Piatek-Jimenez, 2008a), knowledge of mathematical careers (Piatek-Jimenez, 2008b), and equity within mathematics textbooks (Piatek-Jimenez, Madison, & Pzybyla-Kuchek, 2014). Ana Dias's research interests include the politics of mathematics education, adult numeracy, ethnomathematics, and mathematics in vocational education and training. Ana Dias and colleagues have conducted comparative studies of professional education curriculum in the U.S. and Brazil (Gonçalves & Dias, 2017), Freirean mathematics education (Gonçalves & Dias, 2016), and the history of professional education in Brazil (Gonçalves, Pires, Dias, & Monteiro, 2013a; 2013b; 2013c). In the following paragraphs, these scholars summarize some of the recent work on gender and mathematics in Jordan.

Drs. Piatek-Jimenez and Dias chose to study the country of Jordan for two reasons. First, Jordan is a country in which girls repeatedly score higher in mathematics on standardized exams than boys. Given that this is not the case in the United States, these scholars decided that it would be interesting to learn more about Jordan's culture and educational system to better understand what influences girls to be more successful on standardized examinations in mathematics. Second, Central Michigan University (CMU) has a large number of Jordanian students in the doctoral program in mathematics. Since the program's first PhD graduates in the year 2000, CMU has had 21 students from Jordan earn their PhD from the mathematics department. Given that from 1995 to 2015 only 90 students from Jordan earned a PhD in mathematics in the United States (National Science Board, 2018), CMU has awarded a large percentage of those degrees. Further, CMU's mathematics department currently has eight Jordanian students pursuing PhDs in mathematics. Therefore, these scholars wanted to learn more about the country of Jordan and its culture, with the intent to design a qualitative study to better understand gender and mathematical interest and achievement in the country of Jordan.

A substantial amount of research shows that girls in Jordan score higher than boys on standardized exams in mathematics. This is true not only with large-scale international assessments, such as the TIMSS (Innabi & Dodeen, 2006, 2017), but also with a national exam developed by the Jordanian Ministry of Education (Al-Bursan et al., 2018). Although these researchers found that certain nuances exist based on specific variables, such as the mathematical content or context of items, or whether the students attended coeducational or single-sexed schools, a clear pattern that shows girls achieving higher overall than boys on mathematics assessments exists.

This is not the case only for the subject of mathematics. In Jordan, "girls outperform boys at all levels and in all subjects" (Education Reform for Knowledge Economy Project II (ERfKE), 2014). At the collegiate level, a larger percentage of the female population attend college than the male population (UNESCO Institute of Statistics, 2019). According to one report (Ripley, 2017), at the University of Jordan, which is the country's largest university, women outnumber men by a ratio of almost two to one in their undergraduate programs. Amongst the students in the sciences, the division is even greater, with women outnumbering men by a ratio of almost 3.5 to 1 (University of Jordan, 2012). Despite these advances in their education, women consist of just under 20% of the actual workforce.

There may be multiple reasons why girls in Jordan academically outperform boys. One conjecture could be that it is because the girls have fewer freedoms and more restrictions put on them by their parents and therefore spend more time studying than the boys; if true, this may not the whole story (Ripley, 2017). A difference in the quality of schooling may play a role as well (ERfKE II, 2014; Ripley, 2017). In the public school system in Jordan, children attend coeducational schools only until third grade. After that point, boys and girls are filtered into separate single-sex schools, and, where difference exists in the quality of teachers, this may likely play a role in achievement differences. In Jordan, because of societal expectations, women rarely work jobs that require long hours or evening hours, or that involve close involvement with men (Kawar, 2000; Shteiwi, 2015). As such, the teaching profession is common for women because the hours are conducive to societal expectations and it is an all-woman environment in the single-sex schools. On the contrary, teaching is not a preferred profession for many men. Due to cultural norms, men are expected to be primary breadwinners, yet the pay for teachers in Jordan is quite low. Therefore, many male teachers work two or three jobs to help earn enough money to pay their bills. This leads to lower job satisfaction for male teachers (ERfKE II, 2014). Further, the boys' schools tend to be more violent than the girls' schools, which also leads to a less conducive environment for learning (ERfKE II, 2014; Ripley, 2017). Although the government has acknowledged these differences between the boys' and girls' schools and generally place strong women teachers in the lower elementary grades so that boys can develop a stronger foundation during their initial co-educational years (Ripley, 2017), in many ways these differences have not yet been systemically addressed.

Lynda Wiest, University of Nevada, Reno

Lynda Wiest's scholarly interests involve understanding factors that influence gender differences in mathematics, including those that relate to dispositions and beliefs, and strategies and opportunities for supporting and encouraging females in mathematics. Dr. Wiest has particularly focused on the role that out-of-school-time (OST) learning can play in this regard because it is an area of rising scholarly interest in education, especially in STEM education (e.g., McCombs et al., 2012; Slates, Alexander, Entwisle, & Olson, 2012). Dr. Wiest developed and has directed a residential summer math program for middle school girls for 20 years, conducting research in association with the program and recently publishing the first book of its kind on OST STEM programs for females (Wiest, Sanchez, & Crawford-Ferre, 2017). Dr. Wiest has worked for multiple years with graduate students from Ghana at University of Nevada, Reno, which has spurred her interest in mathematics education in that country. In the following paragraphs, Dr. Wiest summarizes some of the recent research related to mathematics, gender, and sexuality in Ghana.

Gender equality in Ghana is promoted not only as a human rights issue but also as one that is vitally linked to the pursuit of sustainable national development, such as one that has the potential to increase the socioeconomic well-being of all people (Zaney, 2014). Some gender issues in education in Ghana include the fact that only two girls for every three boys graduate from senior high school, that girls are more likely than boys to be over-age for their grade level and to drop out of school, and that females have a higher adult illiteracy rate than males (Camfed Ghana, 2012; UNESCO, 2016). Despite barriers to girls' education in Ghana, improvements have been made due to the joint efforts of governmental and nonprofit organizations, which have included eliminating school fees and providing training for school counselors and teachers (Eppenauer, 2018).

Nevertheless, Ghanaian males outperform females in school mathematics, only a little more than half as many females as males pursue elective mathematics in secondary school, and males far outnumber females as college mathematics/statistics students (Asante, 2010; Baah-Korang, Gyan, McCarthy, & McCarthy, 2015; Frempong & Asare-Bediako, 2016). Reasons offered for these differences are sociocultural, such as societal stereotypes that lead to greater familial financial investment in boys' education due to a greater expectation for social mobility for boys, whereas girls are generally expected to preserve current traditions and are more subject to early marriage (Asante, 2010). Such social pressures, as well as a lack of female role models in mathematics, also contribute to weaker mathematics-related dispositions, such as self-esteem and self-confidence, for females in Ghana (Asante, 2010, 2012). Girls who do break with this pattern and pursue elective mathematics seem inclined to do so based on their higher social status, which includes family support and role models (Baah-Korang et al., 2015; Boateng, 2017). The minimal research conducted on gender in mathematics education in Ghana has been conducted at the upper academic levels (secondary, tertiary), with little attention to the younger grades, where the roots of gender inequity might first take hold.

Ana Dias, Central Michigan University, and Harryson Gonçalves, State University of São Paulo

As previously stated, Ana Dias's research interests include the politics of mathematics education, adult numeracy, ethnomathematics, and mathematics invocational education and training. Harryson Gonçalves's background and research are in the area of curriculum studies and mathematics education, in particular diversity and inclusion or curricula as gendered and racialized texts. In the following paragraphs, Ana Dias and Harryson Gonçalves describe some related research from Brazil.

Previous research on Brazilian mathematics education has focused on comparing boys' and girls' scores on standardized international tests (Machado, 2014), examining gender stereotypes and gendered discourse in mathematics education (de Souza & Fonseca, 2010), and analyzing mathematics textbook illustrations and story problems in relation to gender stereotypes (Casagrande, 2006). Our focus is on research that encompasses a non-binary view of gender and sexuality. Diversity in gender expression and sexual identities is a reality in Brazilian schools, and teachers, including mathematics teachers, often find themselves unprepared to fight homophobia, transphobia, bullying, and other problems that directly affect students, often resulting in school dropout, self-mutilation and suicide attempts.

Examining the public policies and legislation in Brazil, three distinct moments are evident; naming, recognizing, and including (Pinto, 1999). In many schools there are efforts to name and classify (e.g., activities clarifying or defining gender and sexuality, different gender and sexual expressions and identities); efforts and strategies to recognize (e.g., who the different groups are, how they organize, what their political and social demands are), and mechanisms of inclusion (accepting, respecting, and valuing diversity, respecting human rights, combating school drop-out and violence against minorities). However, as Seffner (2013) points out, policies of inclusion inevitably generate procedures of labeling, and end up acting as mechanisms of oppression, even when guided by good intentions. Seffner (2013) addresses the challenges faced when schools attempt to promote mechanisms of inclusion and of respect towards sexual and gender diversity.

Even with the best intentions, sometimes these mechanisms have unintended results, and Seffner (2013) argues that this is due to a certain naïveté in relation to the ways in which heteronormativity works. In his school ethnography he describes five cases, in one of which he found that students responded to a campaign to value diversity by mocking the campaign slogan "value diversity" by changing it to "value Arthur" (Arthur is a pseudonym that the author uses for a homosexual student in the school). The slogan was later spread around as "value the literature teacher" in reference to the school's literature teacher who was openly gay. In another case described by Seffner (2013) a student, here called Renato, complained that in two situations, when the teachers invited a gay leader for a collective interview and showed some videos with anti-homophobia campaigns, the classmates – both boys and girls – claimed that the event was happening “just because of Renato.” A friend of Renato´s said that she would “rather attend a math class than talk about stuff like that,” leaving Renato in an awkward position.

How prepared is the mathematics teacher to deal with such situations? What is the role of mathematics in the curriculum? Although it may seem that these topics are outside the field of mathematics education, they directly affect students and teachers of mathematics. When the student said that she would rather have a math class than hear a speaker talk about inclusion, she was not necessarily referring to how much she enjoys her math classes, but probably also signaled the fact that students will be held accountable for their mathematics performance in exams at school, and especially, in the “vestibular” exams which determine entrance into higher education institutions in the country.

A few authors, such as Peralta (2019), have questioned how school mathematics may reinforce labeling, categorizing, and comparing social groups. Peralta’s focus is in preschool mathematics activities, which consist mainly of seriation, classification, and ordination. We believe that adequate preparation of mathematics teachers in gender and sexuality issues need to move beyond examining stereotypes about women in mathematics textbooks (which is currently the emphasis of the literature used in Brazilian teacher preparation programs). Further efforts need to a) promote educational efforts in partnership with social movements, and not only scholars; b) create curricular materials that focus on the hegemonic mechanisms of heteronormativity, instead of focusing on homosexuality (that is, problematizing the norm itself and not those presently out of the norm); and c) avoid activities that have as sole focus defining different gender and sexual identities, which end up being almost like an effort in taxonomy and often result in labeling.

Jennifer Hall, Monash University

Jennifer Hall has long had an interest in “gender issues” research in mathematics education. Dr. Hall began formally investigating this topic through a Master’s degree research project, in which was an exploration of the experiences of women mathematics majors at a Canadian institution (Hall, 2010). More recently, Dr. Hall has conducted research in Australia and Canada (Hall & Jao, 2018a, 2018b) about the general public’s views of gender and mathematics. Notably, for this study, the data collection instrument from a previous, similar study by Forgasz and Leder was altered (1) to make all the questions non-binary (e.g., “For which gender…”) and (2) to add questions that explicitly queried the participants’ views on gender. In another recent project, Dr. Hall and colleagues (e.g., Hall, Robinson, Flegg, & Wilkinson, in press) investigated the supports and challenges faced by undergraduate mathematics majors in Australian universities. This project has an explicit focus on gendered aspects of the students’ experiences, and, like the previous project, gender-related data are collected in non-binary ways. Data collection is complete for both projects, and analysis and writing are underway.

In Australia, there is a long tradition of “gender issues” research in mathematics education, primarily linked to the work of Helen Forgasz and Gilah Leder, the latter of whom is the winner of the 2009 Felix Klein Award. Forgasz and Leder have widely published in the area of gender and mathematics for decades, with notable publications such as *Mathematics and Gender* (Fennema & Leder, 1990), *International Perspectives on Gender and Mathematics Education* (Forgasz, Becker, Lee, & Steinthorsdottir, 2010), and *Towards Equity in Mathematics Education: Gender, Culture, and Diversity* (Forgasz & Rivera, 2012).

With regard to achievement on international large-scale assessments, such as the Programme for International Student Assessment (PISA), which assesses 15-year-old students, no statistically significant gender differences in mathematical literacy were found between Australian boys and girls from 2003 to 2015 (Thomson, De Bortoli, & Underwood, 2017). In contrast, on the Trends in International Mathematics and Science Study (TIMSS; Thomson, Wernert, O'Grady, & Rodrigues, 2016), gender differences in mathematics achievement in favoring boys were found in 2015 for Grade 4 students; this marks the first statistically significant gender difference since the 1995 iteration of the test. No statistically significant gender differences were found in 2015 for Grade 8 students, and there has only been one such difference since 1995 (in boys' favor).

With respect to participation at non-mandatory levels of study, women remain a minority of students in university programs in the mathematical sciences, and women's proportion of the enrolments has been declining in recent years (Australian Academy of Science, 2016; Johnston, 2015). For instance, in 2014, only one-third of the students who graduated with an honors bachelor's degree in the mathematical sciences were women (Johnston, 2015). Over the past 15 years, there has been a slight downward trend in the number of women, compared to a substantial upward trend in the number of men, graduating with honors degrees in the mathematical sciences (Johnston, 2015).

Angie Hodge, Northern Arizona University, with contributions by Dakotah Wilkey, Northern Arizona University

Angie Hodge's research interests lie at the intersection of active learning (Ernst, Hodge & Yoshinobu, 2017) and gender equity in the STEM disciplines. Most recently, Dr. Hodge conducted research on a four-week summer camp for underrepresented middle school girls (Hodge, Matthews, & Squires, 2017). In addition, Dr. Hodge conducted a research study on why some women choose STEM fields and what has made them successful in such majors (Weber & Hodge, 2012). In the following paragraphs, Angie Hodge, with contributions by Dakotah Wilkey, has summarized some of the recent research related to mathematics, gender, and sexuality in China.

In China, gender equity has been a common topic studied in mathematics education research as mathematics has been seen as a male dominated field of study. The achievement levels of girls in mathematics have not been found to be significantly different than boys, except that the highest scorers on the college entrance examinations are more likely to be boys (Tsui, 2007). Based on findings such as this, some researchers have advocated for analyses of societal factors, including home and school backgrounds, to further investigate male domination of mathematics-related fields (Zhu, Kaiser, & Cai, 2018). For example, researchers conducted a study in poor, rural areas of China in which they aimed to measure the factors of health, nutrition, and background and how these factors might affect children (Zhou et al., 2016). These researchers concluded that health and nutrition did not seem to be related to engagement with mathematics fields later in life; however, girls were found to have lower levels of self-efficacy and self-esteem, and higher levels of anxiety (Zhou et al., 2016), which may contribute to their opting out of certain academic areas and/or making different career choices than boys do.

The lack of a gender difference academically in China may be related to the recently changed policy that formerly allowed families to only have one child. Prior to this law, boys were found to have more value placed on their education than girls. There was a time where the laws were more relaxed, leading to some children having siblings. In one study, researchers found that having a male sibling negatively impacted female students potentially due to parents having a male preference (Kubo & Chaudhuri, 2017). Chinese parents often depend on their children when they grow older, so seeing their children succeed is crucial, regardless of gender. Parents who only have one child have expectations that are more gender-neutral, removing competition and value placed on some children over others (Tsui, 2007). In fact, having a single, female child may lead to more educational spending to ensure that the child is successful in the competitive job market, leading to more success in academic endeavors. This may be related to the ways gender, achievement, and selection of mathematics-related careers is experienced by students in China.

Betsy Kersey, University of Northern Colorado

Betsy Kersey is primarily concerned with ways that we can research gender without reinforcing the gender binary. In her dissertation study, she used a narrative framework to analyze the experiences of transgender students in STEM fields, some of whom were nonbinary, in order to gain insight into how the treatment of students in mathematics and other STEM fields are gendered, both for those within and those who transcend a binary notion of gender. Several interesting findings emerged. Perhaps most surprisingly, those participants who were assigned male at birth and then transitioned to align their presentation with their gender identity as women reported that being treated as male, even though it meant they had more social power, felt uncomfortable and sometimes even insulting. Once they transitioned and others in their programs perceived and treated them as women, they had less social power, but experienced this as affirming of their identity and saw it as an improvement in their circumstance. This suggests that there are subtleties in how we regard gender-based privilege and discrimination to which we should attend.

Betsy Kersey is collaborating with Rowen Thomas, a colleague pursuing their doctorate in Higher Education and Student Affairs, to propose a variety of ways that researchers can ask their participants about gender to provide some structure without reducing gender to a binary. The first model is the gender oppression plane (Kersey, 2018). The second model is a pictorial model based on the color wheel that would allow participants to select both what gender(s) they identify with and to what degree, if any, they identify in that direction. These models would be best suited for qualitative research. The first model focuses on how participants are perceived by others and how their treatment varies based on their perceived gender; the second model focuses on how an individual identifies. The third model is a Likert-scale adaptation of the model for gender used by the Gender Unicorn (Trans Student Educational Resources, 2017) and would be best suited to quantitative research. This model is also readily adaptable to asking about sexual/romantic attraction. Finally, Dr. Kersey is collaborating with Jennifer Hall on an article about the ethics of studying gender. The work of these researchers draws from relevant statements from professional organizations in mathematics and mathematics education around the world, as well as previous work conducted by researchers in other fields, such as science education. This work focuses primarily on the collection and analysis of data, how to avoid forcing participants to choose a category that does not fit them, and some considerations when analyzing data related to gender.

Laurie Rubel, University of Haifa and City University of New York

Laurie Rubel's current research focuses on the educational experiences of women Palestinian citizens of Israel (Arab Israelis, who represent 25% of the population of Israel). Arab Israelis are a minority group with limited access to opportunity and resources and lower socioeconomic standing in Israel (Bar-Tal & Teichmann, 2005; Zuzovsky, 2010). Arab Israeli girls outperform boys in mathematics (Rapp, 2015) and have been found to have beliefs about success in mathematics different from Western or Westernized peers (Forgasz & Mittelberg, 2008). It has been hypothesized that girls appropriate academic success to overcome the lower status that they hold, as a minority in Israel as Palestinians exacerbated by their role as women in a patriarchal society (Rapp, 2015), and as an unintended consequence of less diversity among course offerings in Arab high schools (Ayalon, 2002).

Laurie Rubel's research methodology consists of life-story interviews with Arab Israeli women in undergraduate and graduate programs in mathematics education, from diverse Arab ethnic groups and geographies, in which they are asked how they understand and interpret the effects of local racialized narratives in relation to learning, how they interpret their achievement and others' lack of achievement, and which factors they identify as supporting or inhibiting their success. Dr. Rubel's analysis will pursue the educational experiences of these women, as told in their own stories and counterstories, with particular attention to how these stories converge or contrast with Western narratives. The commentary will include analysis of researcher positionality (Milner, 2007) and Dr. Rubel will discuss methodological tensions inherent to research about equity in mathematics in relation to social groups known to espouse conservative, patriarchal, and homophobic ideologies.

Plans for Active Engagement During the Conference

During sessions 1 and 2 we plan to begin our work with a series of short presentations by the working group authors about research that has been or is being conducted across the various parts of the world described previously. We will focus on situating the research within the respective cultures, and we will explore the ways gender and sexuality are constructed and re-enforced, especially, but not only as they relate to mathematics education. We believe it is important to attend to cultural contexts, and, where possible, we will include short, publicly available video highlighting stories of the experiences of individuals within these cultures and/or artifacts of student/teacher work. We will highlight themes that emerge across cultures and discuss ways in which we see our own cultures anew based on studying the experiences of others. We will also include time for non-co-author participants who have engaged in mathematics-education related work outside of the U.S. to share their work and experiences. It is important that this time is both informative and interactive, and thus will likely take us well into the second session. During sessions 2 and 3, our time will be devoted to break-out group work, followed by a large group-sharing session at the end of Session 3. Break-out groups will be arranged according to individual interest, with the goal of generating questions to be investigated and creating (small) sets of online resources and potential collaborators. One aim is to begin to consider opportunities for cross-cultural collaborations, leveraging our own connections and those of other participants. Each break-out group will generate a plan for moving the work forward, and collectively we will determine a timeline for engaging in this work after the conference ends. These sessions might initiate a process for developing materials that could lead to an edited volume.

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STATISTICS EDUCATION: (RE)FRAMING PAST WORK FOR TAKING A HOLISTIC APPROACH IN THE FUTURE

Travis Weiland
Appalachian State University
weilandtj@appstate.edu

Gemma Mojica
North Carolina State University
gmmojica@ncsu.edu

Chris Engledowl
New Mexico State University
chriseng@nmsu.edu

Ryan Seth Jones
Middle State Tennessee University
ryan.jones@mtsu.edu

The goal of the proposed working group is to create a space for those interested in researching issues around the teaching and learning of statistics to meet, discuss, synthesize past research, and begin to strategize ways of leveraging multiple perspectives and expertise to identify and address current challenges in statistics education. The nature of statistics being a methodological discipline make it such that statistics education is made up of a diverse array of people from various backgrounds, disciplines, fields, interests and expertise. We propose this working group to create a space for dialogue among people with diverse perspectives to tackle important issues in the teaching and learning of statistics. Diverse perspective help to look at problems in new ways and come up with new solutions. However, we also want to pragmatically make progress towards a goal, which requires some common direction as well. To balance these tensions we frame our work in the notion of learning environments as a way of organizing past work as well as ideas for future projects into a meaningful structure. Additionally we are layering on the consideration of teachers and teaching in the design and enactment of learning environments.

Keywords: Data analysis and statistics

Introduction

Creating opportunities to develop statistical literacy in public schooling is crucial for education in the 21st century in data centric societies, and thus a top priority for statistics education (Franklin et al., 2007; Steen, 2001). The focal point of statistics education, an interdisciplinary field, is the teaching and learning of statistics and can trace its beginning to the early eighties (Zieffler, Garfield, & Fry, 2018). While statistics education has emerged as a discipline in its own right, it evolved from statistics and mathematics education drawing on the practice of statisticians, theories of learning and development, as well as methods of inquiry (Ben-Zvi & Grafield, 2008; Garfield & Ben-Zvi, 2008; Zieffler et al., 2018). In chronicling the emergence of statistics education, Ben-Zvi and Garfield (2008) identified core areas of importance in this emerging field: statistical literacy, challenges related to learning and teaching statistics, development of the field of statistics education, collaborations among statisticians and mathematics educators, and the relationship between statistics and mathematics. They argue that statistics can be perceived as a bridge connecting mathematics and science.

While much progress has been made to establish statistics education as a discipline, there are still many challenges that educators and researchers face in order for the field to evolve as a collective community of practice. Because of the interdisciplinary nature of statistics education and its roots in statistics, mathematics and science education, statistics educators come from a diverse array of disciplines, backgrounds, and expertise and, as such, have a wide variety of visions for the direction of statistics education. This is no surprise to anyone who agrees with Cobb and Moore's (1997) characterization of statistics:

> Statistics is a methodological discipline. It exists not for itself but rather to offer to other fields of study a coherent set of ideas and tools for dealing with data. The need for such a discipline arises from the omnipresence of variability (p. 801).

While having a diversity of perspectives is a strength, it is also a challenge. Another challenge for statistics educators is that the teaching and learning of statistics is often situated within the mathematics curriculum and in mathematics departments. Perhaps as a result, statistics is often still positioned as a branch of mathematics as opposed to a distinct discipline that draws heavily upon concepts and practices from mathematics (Cobb & Moore, 1997; Franklin et al., 2007; Groth, 2015).

The goal of the proposed working group is to create a space for those interested in researching issues around the teaching and learning of statistics to meet, discuss, synthesize past research, and begin to strategize ways of leveraging multiple perspectives and expertise to identify and address current challenges in statistics education. Organizing within the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA) acknowledges the roots of statistics education in mathematics education, and acknowledges the importance the authors place in drawing from educational theories, research, and practices in studying the teaching and learning of statistics. Additionally, PME-NA is also well known for promoting and stimulating interdisciplinary research; thus, it would be an ideal space for mathematics and statistics educators to work and collaborate, along with mathematicians, statisticians and scientists, on issues relating to the teaching and learning of statistics. Furthermore, in line with the conference theme, in a world where we are saturated with data in our daily lives, we view the new horizon as a place where we have the opportunity to prepare individuals to critically engage with data and make evidence-based decisions. This new horizon points to the critical need for fostering statistical literacy among all members of society to critically make sense of the world around them (e.g., Arnold, 2017). The new horizon also holds great potential for the future, with the explosion of interest in statistics and statistics education from the many fields it offers a coherent set of ideas and tools for dealing with data.

Statistics Education as a Field

In an effort to start a new working group focused on issues of teaching and learning statistics, we begin by briefly discussing the past history and current status of the field of statistics education in conjunction with mathematics education—particularly because of PME-NA's situatedness in mathematics education. We also use this section to make explicit some of the major challenges that statistics education currently faces as a field. This section serves as both a background and as a rationale for the work of the group.

Statistics Education and Mathematics Education

Mathematics education and statistics education have quite a long and, some might argue, complicated history. Mathematics education can trace its origins back to such writings as Thorndike (1922), Moore (1923), and *The First Yearbook* by the NCTM (Smith, 1926). In the US, although the National of Council of Teachers of Mathematics (NCTM) was established in

1920 (NCTM Board of Directors, 2017), the field of mathematics education was not formally recognized in the academe until 1962 when it was included as an option for doctoral graduates to choose as their primary field of study in the Survey of Earned Doctorates—a survey that has been used in the US to track earned doctorates since 1920 (Shih, Reys, & Engledowl, 2017). Historically, statistics has been included as part of the mathematics curriculum—dating back to at least 1923 in the US (National Committee on Mathematics Requirements, 1923).

It wasn't until the 1950s and 60s that statistics began to form its own identity apart from mathematics. For instance, although the American Statistical Association was founded in 1839 (ASA, 2019a)—49 years before the American Mathematical Society (AMS, 2019) and 76 years before the Mathematical Association of America were founded (MAA, 2019)—a separate department for statistics did not exist at Harvard until 1957, when founded by Frederick Mosteller (Powell, 2006). Further recognizing the distinct disciplines of mathematics and statistics, in 1968, the ASA and NCTM formed the *ASA/NCTM Joint Committee on the Curriculum in Statistics and Probability*, chaired by Frederick Mosteller (Scheaffer, 1982)—the outgoing ASA President. This first joint committee set out to produce a book that would "help the layman, and those in charge of secondary school mathematics curricula, understand the uses of statistics and probability" (Mosteller, 1971, p. 341). The committee went on to publish two curricula: *Statistics by Example* and *Statistics: A Guide to the Unknown* (Scheaffer, 1982)—the former of which was well received, with one book review stating that "from the point of view of interest and motivation, these problem sets are far superior to what has been used previously" (Greenhouse, 1975, p. 483). This relationship between NCTM and ASA would result in supporting roles in curriculum standards documents, such as the *Curriculum and Evaluation Standards* (NCTM, 1989), *Principles and Standards* (NCTM, 2000), and the *GAISE* report (Franklin, et al., 2007), and it played a role in the development of probability and statistics content in NSF-funded curricula, such as *Connected Mathematics Project* (Franklin, et al., 2015). Moreover, the *GAISE* report made a lasting impression on the field at large, and played a leading role in the probability and statistics standards included in the *Common Core State Standards for Mathematics* (NGA & CCSSO, 2010; Franklin et al., 2015). The joint committee is still in existence, and its charge still the same:

> To provide national leadership for the inclusion of statistics and probability in the Nation's mathematics curriculum; to promote awareness programs and quantitative literacy among teachers; and to support the development of appropriate curriculum materials. (ASA, 2019b, para. 1)

The current 6 members of the committee (3 from ASA, 3 from NCTM, Chair alternates between NCTM and ASA) are listed in Table 1 (taken from ASA, 2019b).

Table 1: Current ASA/NCTM Joint Committee Members

Name	Position/Affiliation	Term
Perrett, Jamis	Chair/ASA	2019–2019
Bargagliotti, Anna	NCTM	2016–2021

Gould, Robert	ASA	2018–2020
Maddox, Kathleen	NCTM	2014–2019
Miller, Stephen	NCTM	2015–2020
Tyson, Douglas	ASA	2016–2021

Through some of the efforts of ASA and NCTM described above, as well as others, there was a general recognition in the 1980s that statistics should be included in the school curriculum and improved at the tertiary level (Garfield & Ben-Zvi, 2008). In 1982, the first International Conference on Teaching Statistics (ICOTS) was held in order to improve statistics education at all levels, from elementary school to the training of professionals. Zieffler et al. (2018) suggest that statistics education can trace its roots as a discipline to the first ICOTS. ICOTS has continued to be held every four years ever since.

Past History of Working Group

While PME-NA 41 will be the inaugural meeting of this working group, past working groups have focused on a closely related topic, the teaching and learning of probability. Although probability plays an important role in statistics education, past working groups have placed more emphasis on probability rather than statistics. The first group's work centered on learning to reason probabilistically (Maher, Speiser, Friel, & Konold, 1998) and has convened at ten annual conferences up to 2006 (Lee, Tarr, & Powell, 2005; Maher & Speiser, 1999; 2001; 2002; Powell & Wilkins, 2006; Speiser, 2000; Stohl & Tarr, 2003; Tarr & Stohl, 2004). In 2009, Lee, Lee, Wilkins, and Angotti extended the focus of the working group to include statistics by exploring ideas related to learning to reason probabilistically and statistically through experiments and simulations. The following year, the focal point of the working group returned to using technology to teach and learn probability (Radakovic, Karadag, & McDougall, 2010).

Challenges of the Field

Mathematics and statistics. Though both mathematics and statistics are part of the mathematical sciences, statistics is its own distinct discipline—not a sub-discipline or branch of mathematics (Cobb & Moore, 1997; Franklin et al., 2007; Gattuso & Ottaviani, 2011; Groth, 2013). As Steen (2001) points out, "Although each of these subjects shares with mathematics many foundational tools, each has its own distinctive character, methodologies, standards, and accomplishments" (p. 4). Statistics relies heavily on mathematics, but there are distinct practices and habits of mind in statistics which are non-mathematical as well (Groth, 2007, 2013). Part of this reliance is through probability, which is necessary for statistical inference and firmly a part of mathematics (Fienberg, 1992).

There is a strong literature base that discusses important differences that should be considered between the discipline of mathematics and statistics in undergraduate and school settings (Cobb & Moore, 1997; Franklin et al., 2007, 2015; Gattuso & Ottaviani, 2011; Groth, 2007, 2015; Scheaffer, 2006; Usiskin, 2014). The most prominently discussed differences include the treatment of context, variability, inductive versus deductive reasoning, and uncertainty. Scholars generally support the location of the teaching of statistics in the mathematics curriculum (Gattuso & Ottaviani, 2011; Scheaffer, 2006), and some scholars also point out it should also be distributed cross the teaching of all disciplines (Usiskin, 2014). This situatedness can also be viewed in the earlier discussion of the relationship between mathematics and statistics education.

While there are interrelations between mathematics and statistics, differences in the disciplines mean there are some important differences in the thinking and reasoning encouraged in both disciplines, which relates to teaching and learning. For example, mathematics primarily relies on deductive reasoning using definitions, axioms, and theorems, in a logical chain of reasoning, to come to a conclusion. A student could use Euclid’s definition of a circle and his first and third postulates to construct an equilateral triangle. At the same time, Euclidean geometry is based on certain unprovable assumptions such as the parallel postulate, which if changed creates an entirely new type of geometry and way of viewing the world (Katz, 2009).

Statistics, on the other hand, begins with a question for which data is collected to answer (Franklin et al., 2007; Wild & Pfannkuch, 1999). It is from the data that information is empirically derived. This can lead to issues in teaching statistics, as teachers who have had few experiences with statistics may attempt to deduce solutions from rules and assumptions rather than inducing them from the data. Such differences in teaching result in the need for the investigation of issues relevant to the teaching and learning of statistics, which may differ in some regards to the teaching and learning of mathematics. It is these differences that have fueled the need for the field of statistics education. However, As Groth (2015) cautions,

Although the evolution of a statistics education community of practice can be viewed as a positive development for research on the teaching and learning of statistics, there is also a danger that the disciplines of statistics education and mathematics education may become increasingly insular and non-communicative with each other. Sustained boundary interactions are vital to preventing insularity from contributing to the stagnation of interrelated communities of practice. (p.5)

This caution is why a working group such as the one being proposed is so important—to create a community of practice that focuses on sustained boundary interactions between mathematics and statistics education. The need for such a space is discussed in the section that follows.

Collaborating and communicating across fields and disciplines. Statistics education as a discipline has established its own organizations, conferences and journals. At the international level, the International Association of Statistics Education (IASE) serves as an organizing body, which also houses the Statistics Education Research Journal (SERJ) to help disseminate the work of its members. The IASE also helps to organize a number of events and research conferences, both on a yearly basis in conjunction with the International Statistics Institute, and every four years for the main research conference (i.e., ICOTS). In the US, the ASA disseminates scholarly work through the *Journal of Statistics Education* (JSE).

While these organizations, conferences and journals make important contributions to the knowledge base of scholarly work and best teaching practices, we conjecture that work may not be as widely accessed by mathematics educators and researchers, or others, who do not identify as statistics educators and scholars. In turn, statistics educators and scholars, especially those who identify more broadly as mathematics educators, science educators, learning scientists, mathematicians, or statisticians likely publish their work outside of journals such as SERJ and JSE. Thus, the interdisciplinary nature of statistics education contributes to challenges in disseminating work through organizational efforts, conferences and scholarly publications. We argue that while it is a challenge to access research across different fields and disciplines, it is necessary to build on all contributions to move statistics education forward.

An additional challenge to collaborating and communicating about important issues in statistics education is the multitude of stakeholders with vested interests in improving the teaching and learning of statistics: K-12 mathematics teachers, university and community college mathematics and statistics instructors, statisticians, mathematics education researchers, statistics education researchers, mathematics teacher educators, learning scientists, as well as others. When statistics educators and scholars disseminate their work, they are faced with the challenge of appealing to such a wide audience.

Lastly, another challenge in working across disciplines is the relationship between science education and statistics education. Statistics has very strong roots in science, and as a methodological discipline is drawn upon heavily by all the sciences, which has implications for standards. For example, the new Next Generation Science Standards (NGSS Lead States, 2013) includes science and engineering practices that are deeply connected with the practice of statistics, such as analyzing and interpreting data and engaging in argument from evidence. While there are strong disciplinary boundaries between mathematics and science at the K-12 level in the US, we argue that there should be stronger ties between these communities. The work of Lehrer and Schauble (e.g., Lehrer, Kim, & Schauble, 2007; Lehrer & Schauble, 2000, 2004; Petrosino, Lehrer, & Schauble, 2003) around data modeling is an exemplar of how drawing on multiple fields can contribute to improving the teaching and learning of statistics. Work along these boundaries holds much promise for the future, particularly given Ben-Zvi and Garfield's (2008) call for statistics to serve as a bridge between mathematics and science. It is our hope that the proposed working group will foster a space for such boundary work to occur, and to consider how best to disseminate that work to the various stakeholders and audiences of interest.

Researching the Teaching and Learning of Statistics

The challenges of the field we describe in the previous section are part of the reason why it is so important to form spaces to bring together people from various backgrounds, disciplines, fields, interests and expertise to tackle important topics in the teaching and learning of statistics. We propose this working group for that very purpose, to create a space for dialogue among people with diverse perspectives. Diverse perspective help to look at problems in new ways and come up with new solutions. However, we also want to pragmatically make progress towards a goal, which requires some common direction as well. To balance between acknowledging and drawing from diverse perspectives and working towards a common goal we have decided to take the approach of framing our work as a way of organizing past work as well as ideas for future projects into a meaningful structure.

To frame the activities of the working group, we draw on Ben-Zvi, Gravemeijer, and Ainley's (2018) recommendations in taking a holistic approach to investigating the teaching and learning of statistics by focusing on learning environments and drawing from a multitude of theories from various levels of generality. They call for more researchers to take up a holistic perspective of investigating many or all of the factors of a learning environment, and furthermore ask researchers to consider which supports for teachers and teacher education are necessary for creating and facilitating such learning environments. Their work highlights the importance of understanding how various dimensions of learning environments are interrelated.

Responding to the call of Ben-Zvi et al. (2018), the initial goal of the working group is to review what the field knows about each of the components of a learning environment described, and to also conceptualize the role of teaching and teacher education in relation to learning environments and synthesize the literature related to that as well. The rationale for this approach is to synthesize the literature through this new lens to identify areas of need and have a basis for identifying specific problems for the working group to brainstorm ways of investigating and form collaborations for future work around. To frame the work of the working group, we briefly describe the factors in learning environments described by Ben-Zvi et al. (2018) below and then begin to conceptualize the role of teaching and teacher education in relation to learning environments.

Learning Environments

Taking a learning environment perspective acknowledges the inherent complexity in a classroom, which is at the intersection of social, cultural, temporal, and spatial dimensions and represents an interactional space that includes the influence of many different stakeholders including students, parents, teachers, administrators, politicians, etc. Such complexity requires design efforts that do not just take a narrow focus on a single element of a learning environment, such as the written curriculum used, but instead focuses on how to take into account many interrelated dimensions of the environment. Ben-Avi et al. (2018) identify key interrelated dimensions that are critical to designing learning environments to support students in developing productive statistical thinking: focus on central ideas, well designed tasks, real or realistic data, technology tools, classroom culture, and assessment to monitor and evaluate. Ben-Zvi et al.'s proposed interrelation of these elements can be seen in Figure 1.

Researchers taking up this perspective often take a design research approach (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Our goal in drawing from this framework is not to subscribe to particular orienting or background theories or specific methodologies. Instead our initial goal is to use this framework as a way of organizing past research in a manner that will serve as a starting point for designing research projects that leverage what is known by the field and target areas of need in a holistic manner attending to the learning system as a whole versus only its parts. In framing our review of the literature, we feel it is important to explicitly incorporate the role of teachers, teaching and teacher education in learning environments. We elaborate on these dimensions in the section that follows.

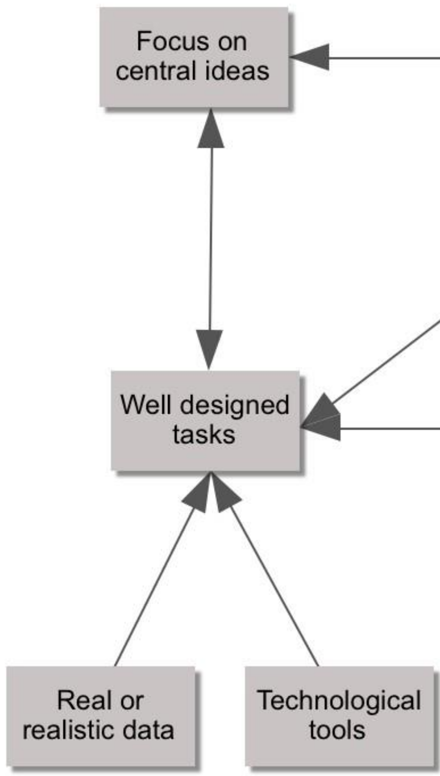

Figure 1: Ben-Zvi et al.'s (2018) Interrelated Dimensions of a Learning Environment

Teaching and Teacher Education

There is a wide variety of models or notions of teaching. In teachers' enactment of their curriculum, they must work as both diagnosticians and river guides (Russ, Sherin, & Sherin, 2011) by both probing and making sense of student learning, and being flexible to navigate the complexities and dynamics of the practice of teaching in the moment. This includes navigating the complex interactions between teacher and student, teacher and content, student and content, and between students situated in learning environments (Cohen, Raudenbush, & Ball, 2003). These interactions are influential in shaping what is taught as mathematics/statistics, how to do mathematics/statistics, and students' identification in relation to mathematics/statistics (Boaler, 2002; Boaler & Greeno, 2000). To be able to navigate these interactions requires that teachers have a strong content knowledge—be familiar with the content they are teaching—as well as have a strong pedagogical content knowledge—be familiar with how to create effective learning experiences for others to learn content knowledge (Ball, Thames, & Phelps, 2008; Baumert et al., 2010; Hill et al., 2008; Shulman, 1986). Furthermore, content knowledge and pedagogical content knowledge are deeply intertwined and are inseparable in the practice of expert teachers (Baumert et al., 2010). As a result it is important for mathematics teachers to not only have a strong understanding of the mathematics they teach, but to also have powerful pedagogical practices for effectively teaching mathematics content to students from diverse backgrounds (Chao, Murray, & Gutiérrez, 2014).

Scholars argue that merely discussing teaching in terms of the interactions Cohen et al. (2003) describe, does not give enough credit to the historical, cultural, societal, and political environments that learning environments are situated in and influenced by (Chazan, Herbst, &

Clark, 2016). Classrooms are situated at the intersections of many different contexts, such as the specific social norms, policies, and physical layout of the classroom and school within which such interactions are occurring, as well as larger community and societal environments within which the school is situated (Davis & Sumara, 1997; de Freitas & Sinclair, 2014).

The importance of statistics education has gained traction in K-12 school settings in the US with the CCSSM (NGA Center & CCSSO, 2010). However, a common issue related to this is the teaching of statistics and the statistical education of teachers (Franklin et al., 2015). As Shaughnessy (2007) pointed out over a decade ago, and Horizon has found is still the case from two separate nationally representative surveys of K-12 mathematics teachers (Banilower et al., 2012; Banilower et al., 2018), many mathematics teachers have had little to no prior experiences with statistics. This is an issue because, as Cobb and Moore (1997) describe, the teaching of statistics has some differences from that of mathematics and this position has been echoed by a number of others since (Franklin et al., 2007; Gattuso & Ottaviani, 2011; Groth, 2007). More recently, the Horizon surveys found teachers across K-12 are not confident in teaching statistics in general (Banilower et al., 2012; Banilower et al., 2018), and, regarding specific statistics content, Lovett and Lee (2017) found that secondary teachers are not confident in their ability to teach a number of the new statistics and probability standards they are expected to teach. The consideration of teachers and teaching is crucial to developing statistics education in school settings. Teaching is however a complex social practice at the intersection of many different communities of practices and influencing factors.

To incorporate the consideration of teachers and teaching the working group will include considering the knowledge and pedagogy necessary to create and enact learning environments as well as how teacher education may need to be modified to help prepare teachers for such roles. These layers will be considered in addition to the learning environment framework from Ben-Zvi et al. (2018).

Plan for Active Engagement of Participants

The working group meetings at the PME-NA 2019 conference will be organized to introduce members to one another, explore frameworks for synthesizing statistics education research, synthesize findings from statistics education research, and make connections with researchers that have a similar focus within statistics education research. To make the most of our time together during the conferences the meetings will be organized as described below.

Session 1

- Participants introduce themselves to the group and share research focus in statistics education
- Working Group leaders introduce the focus of the group and share initial framework from Ben-Zvi and elaborate on components
- Participants break out into small groups to discuss and engage with the framework considering guiding questions provided by the working group leaders
- Participants share summary of their small group discussion and the group as a whole refines the framework and goals of the group

Session 2

- Participants brainstorm in small groups dimensions that are not in the Ben-Zvi framework that would be important to consider
- Groups share out and Working Group leaders facilitate discussion to build shared framework

- Participants identify areas of the framework that align with their expertise and/or interest

Session 3

- Participants work in groups to identify sources for literature within each dimension of the framework and brainstorm a plan for systematically reviewing the literature
- Participants share out sources of literature and thoughts on a plan for reviewing the literature
- Group refines plan for reviewing the literature
- Outline of draft of publication created
- Participants plan meeting times and shared work for after conference
- Possible venues for publication of work are discussed

After conference

- Small groups follow the plan for searching for literature along different dimensions of the framework
- Small groups add literature to google sheet repository
- Working Group leaders curate literature and write synthesis for each area
- Teams work on adding to the outlined draft
- Working group leaders plan 2020 conference time

It is our hope that this working group will serve as a basis for a sustainable working group that can continue on in later PME-NA conferences and serve not only to connect scholars currently in the field, but to also provide an environment that welcomes in early career scholars and graduate students interested in investigating issues around the teaching and learning of statistics.

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EXPLORING THE NATURE OF MATHEMATICAL MODELING IN THE EARLY GRADES

Jennifer M. Suh
George Mason University
jsuh4@gmu.edu

Erin Turner
University of Arizona
eturner@email.arizona.edu

Cynthia Anhalt
The University of Arizona
canhalt@math.arizona.edu

Mary Alice Carlson
Montana State University
mary.carlson5@montana.edu

Megan Wickstrom
Montana State University
Megan.wickstrom@montana.edu

Amy Roth McDuffie,
Washington State University
mcduffie@wsu.edu

Julia Aguirre
University of Washington
jaguirre@uw.edu

Helena P. Osana
Concordia University
helena.osana@concordia.ca

Padmanabhan Seshaiyer
George Mason University
pseshaiy@gmu.edu

Kathleen Matson
George Mason University
kmatson@gmu.edu

Melissa A Gallagher
University of Louisiana
magallagher27@gmail.com

Sara Birkhead
George Mason University
Sbirkhea@gmu.edu

Amy Been Bennett
University of Arizona
amybeen@email.arizona.edu

Emily Yanisko
Urban Teachers at Johns Hopkins
eyanisko@gmail.com

Dong Hwan Lee
Busan National University of Education
dhdhdh@bnue.ac.kr

This working group will engage PMENA members to better understanding the nature of mathematical modeling in the early grades while considering the student perspective and recognizing the importance of teachers knowing their students and the contexts that are meaningful to their students. We will investigate how PK-6 teachers demonstrate the interdisciplinary nature of mathematical modeling, the diversity of mathematical approaches taken by student modelers, and the multiple pathways the teacher can use to elicit students' mathematical thinking. We will explore how mathematical modeling bridges equity and social community in teaching and learning mathematics for all students. Exemplar tasks that emphasized local contexts and tapped into students' funds of knowledge and student artifacts will be shared to illustrate the child's perspective and the developmental progression. These topics will facilitate group discussions exploring the learning progression for mathematical modeling thinking and habits of mind that can develop for emergent mathematical modelers from an early grade. Finally, based on the interests of the participants, we will devote work time to finding synergistic collaborative topics to pursue for future research and practice.

Keywords: Mathematical Modeling, Elementary Education, Teaching Practices, Professional Development, Learning Progressions, Knowledge of Content and Pedagogy

Overview of the Working Group

This working group began at the 2017 PMENA in Indianapolis, IN and continued its meeting in 2018 in Greenville, SC. We proposed that this working group have a special focus on early mathematical modeling and continue to build on PMENA's long tradition of working groups on

Models and Modeling. Through the 2017 and 2018 working group meetings, we found that there are researchers and practitioners with a keen focus on broadening the access of mathematical modeling to diverse learners in the elementary grades and advancing the field's collective understanding of the interrelated processes of mathematical modeling in the elementary grades and beyond. Although there has been a long history of mathematical modeling at PME and PMENA, the focus was primarily middle, high school and university levels. We believe it is critically important to understand the learning progression of mathematical modeling from early elementary to secondary grades to ensure coherence and rigor in the mathematics curriculum.

In our first year, the working group leaders proposed an edited volume and a special issues journal venue for MM where participants interested in submitting manuscripts could work together to provide a comprehensive research trajectory documenting the progression of mathematical modeling from emergent levels to more sophisticated levels of modeling. We are excited to share that this working group was able to secure a contract with Springer to publish an edited volume on this very topic. In our second year of the working group, one of our lead facilitators announced an exciting networking meeting at the upcoming MSRI meeting in 2019 focused on modeling and the connection to community and cultural contexts. It is clear that this PMENA working group is facilitating ways to bring synergy among researchers across North America, and we hope to continue this working group so that we can invite more mathematics educators to take part in the important research of MM in the early grades.

Implementing MM in the elementary grades is not just going "light" with the high school math modeling curriculum. Instead we advocate integrating aspects of mathematical modeling in the early grades effectively to enhance student learning and to help build their competency in real-world problem solving using their current mathematical knowledge. The latter content knowledge is expected to develop and evolve as students progress towards high school and beyond. So what does mathematical modeling look like in the elementary grades? Why focus on early grades? In addition to the direct benefits of modeling, the elementary school environment affords many advantages that complement work in mathematical modeling. Elementary students often rely on concrete referents such as objects, drawings, diagrams, and actions that can support the conceptualization and construction of carefully formulated arguments to solve a problem. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades (CCSSO 2010). Young students have great potential to become fluent – native speakers, thinkers, and dreamers of mathematics. Thinking creatively may come more easily to children first learning and exploring mathematical concepts. Kindergarteners can use manipulatives to independently solve traditional multiplication or division problems they have never seen before, which is evidence that students come with knowledge--it is not necessary to wait to incorporate modeling activities until we have "shown them how" to do everything. Because early grade teachers are generalists, they can address several subjects simultaneously through modeling activities. Mathematical modeling is of interest and relevance to the mathematics education community especially because it connects to the need for professional development focused on MM in the elementary grades.

We are also interested in focusing on different research methodologies used in mathematical modeling research. Some of our researchers use Design-Based Implementation Research methodology, DBIR (Fishman, Penuel, Allen, Cheng, & Sabelli, 2013) to examine the design of the professional development and to study and enhance the design through feedback from iterative implementation cycles. Each year at our working group meetings, we meet researchers who are also using Networked Improvement Communities (NICs) collaborating with school districts working with diverse populations to examine what works, for whom, and under what conditions, which helps us better understand the nature of MM in the elementary grades with diverse learners across geographic regions. For example, in one of our NSF-funded projects, each university site worked with the collaborating district's teacher leaders to co-plan the professional development. Teachers became co-designers of the MM curriculum for the elementary classrooms. In our project, we engaged elementary teachers in MM using real world tasks that contained several of the following attributes: (a) Openness; (b) Problem-posing; (c) Creativity and choices; d) Iteration and revisions.

Through our work, we are gaining a better sense of teaching practices and classroom routines that support modeling. We are contributing to the understanding of what is possible in early elementary grades and how these processes support the development of critical 21st century skills. As we continue in our research to consider what constitutes the practice of Mathematical Modeling (MM) and how it could be implemented in classrooms at different grain size, we invite the larger PMENA community to build on this knowledge. Over the past decades, working group leaders have individually, and in subgroups, been theorizing and collecting, analyzing, and reporting on data relating to mathematics modeling. This Working Group builds on and extends the work of previous Model and Modeling traditions by discussing current work from leading scholars from diverse perspectives.

Relevance to Psychology of Mathematics Education

In the spirit of exploring the theme of 2019 PMENA "Against a New Horizon," we will offer differing views of "expansion" and "growth" in relation to ways mathematical modeling can provide opportunities for learners, families, and their communities to engage in mathematics and supporting *all* students through a concerted focus on equitable teaching practices. In addition, this working group will attend to the interdisciplinary nature of mathematics and how it connects to social justice, STEM, and civic responsibility for our citizen in our country. Finally, we invite researchers from different countries in North America to broaden our understanding of how different national curricula attend to mathematical modeling as mathematical literacy, situational problem solving, and other curricular initiatives that develop critical thinking skills.

The purpose of this working group is to invite individuals across the research community interested in synthesizing the literature and collaborating on research focused on mathematical modeling along the developmental continuum. Our goal of mapping a learning progression of mathematical modeling from K-12 education, particularly starting from elementary to middle grades, is critically important to provide coherence in the mathematics curriculum.

The primary focus for this working group will be centered on the following three goals:

1. Examine current research and discuss the nature of mathematics modeling and detailing the development of teachers' content knowledge, teaching practices, and students' modeling competencies.
2. Map the learning pathways for mathematical modeling and task design for K-6 mathematics education and explore how mathematical modeling can bridge equity and social community in teaching and learning mathematics for all students.
3. Engage in dialogue and collaboration among individuals and groups conducting research on student- and teacher-related outcomes related to implementing mathematical modeling, ways mathematical modeling promotes social justice, 21st century skills, and ways in which early modeling can develop interdisciplinary skills in STEM.

Related Research

Mathematical proficiency, in today's world, moves beyond computational ability. It includes the development of 21st century skills (i.e., critical thinking, creativity, communication, and collaboration), conceptual understanding of mathematics (NCTM, 2014), and mathematics that has practical relevance outside of the classroom (Gravemeijer, Stephan, Julie, Lin, & Ohtani, 2017). Mathematical modeling (MM) is a powerful tool for developing students' 21st century skills (Suh, Matson, & Seshaiyer, 2017), advancing their conceptual understanding of mathematics, and developing their appreciation of mathematics as a tool for analyzing critical issues in the world outside the mathematics classroom (Greer & Mukhopadhyay, 2012). It provides the opportunity for students to solve genuine problems and to construct significant mathematical ideas and processes instead of simply executing previously taught procedures and is important in helping students understand the real world (English, 2010).

There is broad agreement among mathematics educators on the relevance of MM in schools, but the field has yet to come to a consensus on the definition of mathematical modeling or on how it might be taught and learned in schools (Kaiser, 2017). Although mathematical modeling has traditionally been reserved for secondary and college students, its enactment in schools contributes to broad educational goals that are relevant to learners of all ages (Ferri, 2018). In addition, scholars have argued that engaging in mathematical modeling is important for elementary school students (Carlson, Wickstrom, Burroughs, Fulton, 2016).

Mathematical modeling has received increased attention in the United States since the release of the Common Core Standards in Mathematics (the Common Core hereafter) in 2010. Modeling is incorporated as a specific area of expertise that teachers should cultivate in students across Grades K–12. The Common Core's Standards for Mathematical Practice, SMP4 is called *Model with Mathematics.* Although SMP4, as a mathematical practice, cuts across Grades K–12, mathematical-modeling opportunities are not highlighted in connection with the K–8 content standards, presenting an implementation challenge for teachers (Cirillo, Pelesko, Felton-Koestler, & Rubel, 2016). Modeling with mathematics, the topic of SMP4, refers to both modeling mathematics and mathematical modeling. The distinction between modeling mathematics and mathematical modeling is not clear to many teachers (Meyer, 2015), nor is it clear in Common Core documents or in mathematics education literature (Cirillo et al., 2016). The key difference between mathematical modeling and modeling mathematics is where the mathematical activity begins. Modeling mathematics begins in the mathematical world (Van de Walle, Karp, & Bay-Williams, 2016), whereas mathematical modeling begins in the unedited real world (Pollak, 2007). The explicit focus on getting a problem outside of mathematics into a mathematical formulation and explicitly translating the mathematical solution back into the real world is what differentiates mathematical modeling from modeling mathematics. The real-world focus also distinguishes mathematical modeling from problem solving and application problems (Lesh & Caylor, 2007; Schukajlow et al., 2012).

One of the ways the researchers in this working group have approached MM in the elementary grades was to immerse students in a relatable and personally-meaningful real-world situation within their local contexts. In bringing mathematics closer to social community spaces, mathematical modeling became a vehicle that brought teaching and learning mathematics closer to all students. Reforms in mathematics have advocated for mathematics to be more related to students' lives by building on community and cultural knowledge and practices with issues that matter to them, which then helps students view mathematics as a vehicle through which they learn to be active change agents for social justice (Bartell et al., 2017; Civil, 2007).

To keep the initial problem open, students were encouraged to develop the habit of mind of being problem posers by identifying the many questions around the real phenomenon, then defining a mathematical problem that can be solved by way of mathematics. After the identification process of the problem, the modeler makes assumptions, eliminates unnecessary information, and identifies important quantities to develop a solution. The mathematical solution focuses on the usefulness of mathematics to solve a real-world problem. It should be noted that there can be several mathematical solutions for a given real-world situation. After solving the problem, the results are translated back to the real-world and interpreted in the original context. The problem-solver then validates the solution by checking whether it is appropriate or reasonable for the purpose. This process of making assumptions, identifying variables, formulating a solution, interpreting the result, and validating the usefulness of the solution is iterative in nature and modified and repeated until a satisfactory solution is obtained and communicated (Blum, 2002).

It is important to note that teachers play a crucial role in MM and must be able to: (1) find appropriate questions to move students through the modeling cycle, (2) handle discussions in nondirective but supportive ways, (3) allow students time for productive struggle, and (4) provide scaffolding without directing the problem or its solution (Burkhardt, 2006). Teachers also need to develop problem-posing expertise (Suh et al., 2017) and to base their instructional decisions on responses to students’ work (Bleiler-Baxter et al., 2016). Thus, learning to teach MM requires teachers develop multiple knowledge bases. For example, teachers must understand modeling processes and tasks, including the potential mathematical content embedded within tasks; learn about students’ mathematical and personal experiences to predict the strategies they might use when responding to modeling tasks; know what content is on the mathematical horizon to anticipate what mathematical ideas students might construct; and learn to engage individual and groups of students in the modeling process (Blum, 2011; Ferri, 2018).

Previous work with elementary school children demonstrated it is feasible for them to develop a disposition towards realistic mathematical modeling (Verschaffel & De Corte, 1997). One of the issues in implementing MM at the elementary level is that MM can be difficult for both teachers and students to implement (Blum & Ferri, 2009). MM can be difficult for teachers to implement as they must be able to merge mathematical content and real-world applications while teaching in a more open and less predictable way (Blum & Ferri, 2009). Mathematical modeling can be a challenge for students because each step of the modeling process presents a possible cognitive barrier (Blum & Ferri, 2009). As stated in the Common Core Standards for Mathematical Modeling,

“Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. These real-world problems tend to be messy and require multiple math concepts, a creative approach to math, and involves a cyclical process of revising and analyzing the model” (Carter et. al., 2009).

To model, students need support to develop mathematical modeling competency -- i.e., the ability to independently carry out the various phases of the modeling process (Vorhölter & Kaiser, 2016) and its related sub-competencies (Schukajlow et al., 2015). Based on the Standards for Mathematical Practice 4 in the Common Core: Model with Mathematics, Bleiler-Baxter, Barlow, and Stephens (2016) identified the mathematical modeling sub-competencies needed by students as simplification (e.g., making assumptions), relationship mapping (e.g., identifying important quantities and their relationships), and situation analysis (e.g., interpreting results in the context of the situation). In addition, students need to develop a metacognitive modeling competency because it is "indispensable in order to enable students to solve complex modeling problems independently, which is an indispensable part of true modeling activities" (Vorhölter & Kaiser, 2016, p. 279).

Emerging Research on Early Mathematical Modeling

Developing modeling competencies in young mathematicians. English (In press) argues for the importance of incorporating mathematical modelling into the years of early education, where young children's learning potential often remains untapped. She states MM is ideally suited for early learning and should not be reserved for the later school years; Early MM facilitates interdisciplinary learning, in particular, linking the STEM disciplines as well as humanities (e.g., literature). She elaborates on (a) "pre-modelling" experiences (e.g., working with simple data and their representations), (b) mathematical modelling within different disciplinary contexts (e.g., engineering in the production of confectionary for a modelling problem that targeted this topic; using literature as introductory and supporting contexts), (c) the inclusion of supplementary science content, and (d) modelling experiences posed by children.

Anhalt, Cortez, and Aguirre (In press) share a construct for developing competency across grades and propose key modeling competencies that can be targeted for development early on, and thus, classroom activities can be designed with the goal of developing those competencies in students through modeling tasks and through other activities that do not necessarily engage the entire modeling process. They refer to it as *mathematical modeling thinking (MMT)*. In order to define MMT and determine what to include in it, they address the questions: (1) what are key competencies to be successful in mathematical modeling? And (2) what activities are effective at laying the foundation for the development of those key competencies?

Osana and Foster (In press) report on the genesis of modeling in kindergarten and unpack the key ingredients for professional development. The MEA that was orchestrated in the teachers' classrooms invited the children to design their ideal Kindergarten classroom. The activity encouraged the children to answer the questions they themselves generated through modeling (English, 2010), such as "How many tables and chairs do we need for all the kids in the class?" "How big should the tables be and where should we put them?" "How many kids should be at each table and why?" Manipulatives and other tools were made available to the children as they worked through the modeling cycles, and children worked collaboratively on the construction and revision of their models. The results yield valuable insights on the elements that are necessary for professional development in mathematical modeling with young children, particularly in classrooms with at-risk students.

Turner, McDuffie, Aguirre, Foote, Chappelle, Bennett, Granillo, and Ponnuru (In press) provide a description of the Upcycling Jump Rope task and state several important implications for mathematics education researchers to consider. They recommend that given the salience of children's funds of knowledge across all phases of the modelling process, teachers should explicitly elicit students' experiences and perspectives, and position these experiences as resources to support meaningful engagement in mathematical modeling. Second, their findings highlighted pedagogical tensions in mathematical modeling lessons that demand further investigation. For example, while students readily shared experiences related to the broader task context (plastic consumption, recycling, and pollution), teachers had to determine when to encourage this sharing and when to redirect the conversation to key features of the specific modeling task. A third implication is related to more effective engagement with the final phase of the modeling process – generalizing. While generalization of models remained elusive, emerging evidence suggests that teachers can reframe tasks to facilitate shareable and re-usable models for similar situations.

Wickstrom and Yates (In press) analyze students' notions of mathematics as they consider how elementary students define mathematics and view themselves as learners while doing mathematical modeling and also during traditional instruction. Findings suggest that the third-grade students conceptualized math as computations and often compared themselves to peers to determine success in mathematics. In contrast, during mathematical modeling, students discussed that the task was more difficult than traditional mathematics, but also more rewarding and inclusive.

Developing core teaching practices for early mathematical modeling. Suh, Matson, Birkhead, Green, Rossbach, Seshaiyer, and Jamieson (In press) report ways in which researchers are collaborating with teacher designers to develop personally relevant and rigorous MM tasks for elementary students. The essential design skills include : 1) Leveraging problem posing routines to develop questioning skills: When posing an MM problem, teacher-designers adopted instructional routines for problem posing and worked on developing teacher and student questioning competence; 2) Connecting familiar context that engages students: Teachers, as designers, looked for situational features that warranted mathematizing and searched for contexts that were relevant and important to support students' engagement in modeling. In addition, teachers elicited students to think about how their solution was shareable, reuseable, or generalizable in order to evaluate whether a systematic model was created; 3) Connecting context with content: Teachers connected the need for mathematics in a modeling task with the curricular objectives of their grade level; 4) Considering categories of MM tasks: The modeling tasks tended to fall into four general categories (described below) where a mathematical solution or model could be used to describe, predict, optimize, and make decisions about real world situations.

- Descriptive Modeling - Using math to describe, represent, and analyze a situation or a phenomenon.
- Optimization Modeling - Using data to find the "best" by optimizing or in some cases minimizing some variable (i.e., cost, space) in a situation.
- Rating and Ranking - Using a criterion where one assigns weights or mathematical measures as a way to rate and rank options to make decisions.
- Predictive Modeling - Using trends and data analysis to predict an outcome or using patterns (data analysis and algebra) to predict a situation and make decisions. In some tasks, probability and statistical modeling is used to search for patterns in data to explain a phenomenon (i.e., scientific phenomenon used in STEM contexts).

In addition, researchers are examining core practices that are essential in supporting student learning through modeling. Suh and Matson (In press) found four main categories of core teaching practices that emerged as being central to the success of enacting mathematical modeling in the elementary classroom: a) Questioning practices: Developing competence in asking productive questions of students; b) Data Practices: Connecting relevant data by formulating the problem and eliciting student thinking about important variables and assumption in a problem situation; c) Modeling Practices: Building solutions/models that can be communicated and are useable to others through records of student work, concrete tools, written and verbal explanations, number sentences, and pictorial representations; d) Analytic and Interpretive Practices: Facilitating productive analysis of a model for the purpose of refining it.

Carlson (In press) report on teacher knowledge bases for engaging young children in MM. She explores the mathematical and pedagogical knowledge teachers need in order to engage young children in mathematical modeling. Analysis of data collected suggests that teachers developing and implementing modeling tasks draw on three knowledge bases: (1) knowledge of real world contexts around which students might pose and investigate mathematical problems, (2) knowledge of students' mathematical and local knowledge resources, and (3) knowledge of curricular mathematics – including the mathematical tools children may have at their disposal and ideas that are on the "mathematical horizon" and might be constructed during a modeling activity.

Formal and informal learning of modeling across disciplines and settings. Mathematical modelling is central to understanding different disciplinary contexts. Elementary teachers are generalists who teach multiple disciplines. Early caregivers, parents, and preschool teachers also have opportunities to leverage real-world situations to teach mathematics. Gallagher and Jones (In press) describe how elementary teacher candidates were introduced to mathematical modeling (MM) in their math methods course and how one of those candidates implemented a MM task related to economics. Students were asked to make economic decisions on the supplies needed for their classroom based on a list of choices and a budget prepared by their principal. The students chose items they found most necessary within the budget. They discussed how economics, as a field based on MM, provides a natural way to integrate MM into the elementary curriculum.

Yanisko and Minicucci (In press) share design features of a course designed for K-5 prospective teachers and aligned with two curricular goals – that students feel empowered by learning mathematics and that teachers recognize the assets of their students and leverage those assets to improve the effectiveness of mathematics instruction. This acquired knowledge is unpacked through an asset-focused lens and leveraged to build students' capacity for geometric modeling that is aligned with math standards. Additionally, teachers examined how to foster students' personal sense of power by investing them to use acquired math knowledge to positively impact themselves or their community.

Gilbert and Suh (In press) highlight how modeling principles across mathematics, science, and engineering converge toward a construct of integrated STEM modeling. These processes are framed as a disciplined inquiry approach that embrace cross-disciplinary connections to solve problems or better understand real-world phenomena. In particular, this instrumental case study investigated preservice teachers enrolled in a graduate-level integrated STEM course where activities were steeped in modeling tasks surrounding content and pedagogy involved in meaningful integrative processes. Findings suggest that preservice teachers could both prepare and enact integrated STEM approaches, and after teaching children in elementary contexts, recognized the creative freedom and motivation it brought to the children they taught as well as themselves. These researchers propose a model for the convergence of STEM practices and articulate the value for STEM modeling in elementary contexts.

Civil, Bennett, and Salazar (In press) look at ways to encourage MM in informal settings with professional development with families and caregivers. They report on their learning from Modeling with Mothers. They maintain that parents are intellectual resources and have a wealth of knowledge about various topics that interest their children. In alignment with the larger project's overarching goal to bring together parents and teachers, the co-development exercise intertwined the roles of those inside and outside of the classroom, fusing researchers' knowledge of mathematical modeling with parents' funds of knowledge about the school community. Additionally, the mothers' insistence to involve their children signaled to the researchers the importance of collaboration between parents and children in curriculum development.

Plan for Active Engagement of Participants

The working group will meet three times during the conference and virtually during the course of one year. In each session, PMENA members will engage in mathematical modeling while sharing their perspectives in teaching and learning mathematics, considering synergistic areas fruitful for future research and practice, and finding collaborators within our group. At each session, we will have facilitators share a research theme and provide participants time to get into smaller research groups so that participants can join one or visit with three smaller groups to network and find synergistic research interests with others at our working group.

WG-Research Group 1: Student Development Focused: Tapping into Students' Funds of Knowledge and Assessing Student MM Competencies and the Developmental Trajectories

Research Group 1 will focus on better understanding the nature of mathematical modeling in the elementary grades while considering the student perspective and recognizing the importance of teachers knowing their students and the contexts that are meaningful to them. We will investigate how K-6 teachers can assess math modeling in the elementary grades while appreciating the diversity of mathematical approaches taken by student modelers and the multiple pathways the teacher can use to elicit students' mathematical thinking. We will explore how mathematical modeling bridges equity and social community in teaching and learning mathematics for all students. Exemplar tasks that emphasized local contexts and tapped into students' funds of knowledge and student artifacts will be shared to illustrate the child's perspective and developmental progressions. These topics will facilitate group discussions exploring the learning progression for mathematical modeling thinking and habits of mind that can develop for emergent mathematical modelers from an early grade. We will map out productive learning pathways for mathematical modeling and task design for K-6 mathematics education and beyond.

WG-Research Group 2: Teacher Development Focused: Identifying Core Teaching Practices for Early Math Modeling and Unpacking the Knowledge of Content and Pedagogy Needed for Mathematical Modeling in the Elementary Grades

Research Group 2, we will focus on clearly defining modeling teaching practices and competencies needed for mathematical modeling and outlining research goals and objectives to monitor the enactment of these practices. We will detail classroom routines, such as the "organize - monitor - regroup" cycle (Carlson et al., 2017), and the Core Practices for Mathematical Modeling (Suh & Matson, in press) as we share designed activities and lesson vignettes to solicit more ideas around high leverage MM teaching practices. We will explore what mathematical knowledge is needed to "successfully" facilitate mathematical modeling tasks in elementary grades. As we synthesize the current research on early modeling, we will define the nature of mathematics modeling and detail the development of teachers' content knowledge, teaching practices, and students' modeling competencies.

WG-Research Group 3: Professional Development Focused: Detailing PD Research and Development of Tools and Modules for PD & Finding Convergence between PD on Mathematical Modeling and Interdisciplinary Areas

Research Group 3 will focus on detailing components of effective mathematical modeling professional development for educators, examining relevant research methodology and instruments for studying the nature of MM in the early grades, and outlining several 21st century skill frameworks and teaching approaches for mathematics educators, researchers, and practitioners. We will share PD modules designed for elementary teachers that engage learners to use mathematical modeling through problem-based tasks, STEM, and teaching social justice through MM. Connecting interdisciplinary topics across subjects afford modeling opportunities that will help educators value the complementary connections between subjects and common classroom practices that support MM. We will engage in dialogue and collaboration among individuals and groups conducting research on student- and teacher-related outcomes related to implementing mathematical modeling, ways mathematical modeling promotes 21st century skills, and ways in which early modeling can develop interdisciplinary learning.

Anticipated Follow-up Activities and Goals of Working Group

Each session will engage participants to share their research interests related to mathematical modeling and form groups that might pursue research collaboratively based on the interests of the participants. Some of the questions that we will engage in include:

- What defines successful mathematical modeling at different grade levels?
- How does mathematical modeling support each and every learner?
- How does mathematical modeling connect to issues of social justice, STEM, andcivic responsibility of citizens?
- What can we learn from teachers who implement MM regularly in their classrooms?
- How is mathematical modeling ambitious teaching and how can we support teachers enacting MM through lesson plans and other resources?
- How can we map out the learning pathways of MM across grade levels?
- How and what can we learn about models elicited from student artifacts from MM tasks?
- What do “successful” modeling practices look like in our elementary mathematics classrooms? How are they similar or different from practices in secondary classrooms?
- What does it mean to “see the math” in the components of mathematical modeling?
- How do teachers select and/or develop modeling problems? How can Professional Learning Communities or Teacher Study Groups help teachers anticipate how students will answer the MM questions?

Our goal is for the working group leaders to propose an edited handbook or a special issues journal venue for mathematical modeling where participants interested in submitting manuscripts can work together to provide a comprehensive research trajectory documenting the progression of mathematical modeling from emergent levels to more sophisticated levels of modeling.

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MATHEMATICAL PLAY: ACROSS AGES, CONTEXT, AND CONTENT

Caro Williams-Pierce
University at Albany, SUNY
cwilliamspierce@albany.edu

David Plaxco
Clayton State University
davidplaxco@clayton.edu

Paul N. Reimer
AIMS Center for Math & Science
preimer@aimsedu.org

Amber Simpson
Binghamton University
asimpson@binghamton.edu

Chandra Hawley Orrill
UMass Dartmouth
corrill@umassd.edu

James P. Burke
UMass Dartmouth
james.burke@umassd.edu

Nathalie Sinclair
Simon Fraser University
nathsinc@sfu.edu

Victoria Guyevskey
Simon Fraser University
vga@sfu.edu

Amy B. Ellis
University of Georgia
amyellis@uga.edu

Muhammed F. Dogan
Adiyaman University
mfatihdogan@adiyaman.edu.tr

Mathematical play has a fairly short history, with strong roots further back in time (e.g., Papert, Montessori), and understanding the role of mathematical play from early childhood to adulthood is, as yet, unmapped. This working group will build on the success of last year's working group and continue to provide a community space to explore and discuss mathematical play broadly, ranging from informal to formal contexts and from 3rd grade students to teachers in professional development. We will emphasize physical and digital interactions designed specifically to support mathematical play. Each day will focus specifically on a different approach to and definition of mathematical play. Day 1 will focus on mathematical play and making by 3rd to 5th graders across formal and informal environments; Day 2 will focus on intellectual play within mathematical microworlds; and Day 3 will focus on mathematical play in teacher professional development. Throughout the sessions, we will be examining threads of common ground that will assist in developing a more flexible and appropriate model of mathematical play that can inform design of environments and activities across age groups, content, and context.

Keywords: Instructional activities and practices; Design experiments; Technology

Mathematical play has a fairly short history, with strong roots further back in time (e.g., Papert, Montessori). The majority of research on this topic stems from early childhood research on play and researchers have begun to identify the mathematical play children naturally engage in during open-ended play activities and explores how to further mathematize that play and the consequent learning (see Wager & Parks, 2014, for a review). In addition, researchers have explored how mathematicians in the course of their work engage in mathematical play (e.g., Holton, Ahmed, Williams, & Hill, 2001). Given that young children and mathematicians both engage naturally in mathematical play, there is an intriguingly underexplored area of promise between those two populations. A small number of researchers have examined how to support students in approaching mathematics problems with a playful bent (e.g., Steffe & Wiegel, 1994; Holton et al., 2001), but understanding the role of mathematical play from early childhood to adulthood is, as yet, unmapped.

Following the success of our initial working group at PME-NA 2018, we reached out to new collaborators who will be conducting the activities this year. Our goal is to continue providing a community space to explore and discuss mathematical play broadly, which requires that we extend into new mathematical play activities. In 2018, we had a different focus each day: early childhood mathematical play with wooden blocks (Reimer); middle-school mathematical play with touchpad games (Williams-Pierce); and undergraduate mathematical play with Rubik's cubes (Plaxco). This year, our three foci are: mathematical play in informal makerspaces (Simpson); intellectual playgrounds (Sinclair and Guyevskey); and mathematical play professional development with teachers (Burke and Orrill).

Our goal for the second instantiation of this working group is to continue facilitating mathematical play experiences and discussions around open research questions that transcend our individual lines of research, such as:

- What is the nature of mathematical play across the age/grade bands?
- What are the features, characteristics, and affordances of mathematical play?
- How might context (e.g., physical, digital) influence mathematical play?
- How might content (e.g., fractions, group theory) influence mathematical play?
- How might factors such as gender, race and ethnicity, and parental income/education level influence experience of and access to mathematical play?
- How are mathematical play and mathematical learning related?
- How does mathematical play influence problem solving?
- How do mathematicians (experts) engage in mathematical play, and how might that mindset be fostered for learners (novices)?
- When might a didactical introduction to the content support more productive mathematical play?
- How might mathematical play support or influence learning in other disciplines (e.g., a broader STEM perspective)?

Although answering all of these is beyond the scope of possibility for our working group, we will use these questions to facilitate and orient discussions during each of the three days, and as potential topics for future collaborative investigations. In order to ground the discussions, we will facilitate a mathematical play experience each day, then guide the discussion towards the mathematics at play (pun intended) and the specific characteristics of that mathematical play. During these group discussions, we will regularly orient the conversation specifically towards the open research questions listed above. We will take notes during these conversations and conclude each session by collecting names and emails of working group attendees.

Definitions of Mathematical Play

There are numerous definitions of mathematical play, each emerging from different contexts and with students of different ages. Contexts can range widely, such as digital (Steffe & Wiegel, 1994; Williams-Pierce, 2016, 2017; Sinclair & Guyevskey, 2018), physical (Sarama & Clements, 2009; Simpson, this submission), or paper and pencil-based (Holton et al., 2001). One of the crucial open questions that we highlighted in our discussions at the first working group that we plan to continue discussing is how the definitions may vary in features they prioritize due to the differing contexts. For example, might Holton et al. (2001) and Williams-Pierce (2016) find common ground if they examined similar contexts? Or are their approaches too fundamentally different to ever come to agreement? Or how might Reimer's wooden block activities with young children relate to Plaxco's Rubik's cubes work with undergraduates learning about groups? How might all these examples of play compare to what mathematicians do? We strive to ensure that each session of the working group is oriented around a specific definition and operationalization of mathematical play, so that attendees have concrete experiences grounded in different definitions to facilitate discussion across these different frameworks.

In the below sections - Retrospective (about our initial working group in 2018) and Working Group Schedule and Activities (where we outline the plan for this year's working group) - we describe in more detail our definitions of mathematical play along with activities they inspired.

The History of This Working Group

As mentioned above, this proposal is for a second year of the mathematical play working group, following the success of our first at PME-NA 2018 (Williams-Pierce, Plaxo, Reimer, Ellis, & Dogan, 2018), where we had about 30 attendees each day - most returning each day. We followed the same approximate format for each session: Frame - a brief introduction to the theoretical grounding of that day's topic; Play - a mathematical play activity in order to ground the discussion in a common experience; and Discuss - a broad discussion of the activity, the theoretical underpinnings, and implications for designing and understanding mathematical play and learning. Importantly, we had found in previous working groups (e.g., Nathan, Williams-Pierce, et al., 2017), that beginning with a relevant activity quickly develops fruitful discussions between participants - especially given that working groups tend to have a variety of attendees with wide-ranging expertise in the topic - so we considered the Play component of our format to be crucial for grounding the discussion in shared experiences. One of the crucial goals for our working group was - and is - to ensure that every attendee actually *experiences* mathematical play, and we were delighted that during the 2018 sessions, considerable laughter emerged from our assigned room. However, it is important that we also emphasize the richness of the discussions about mathematical play and learning that emerged alongside the laughter.

The first author was bemused by some of her conversations with PME-NA attendees who did not attend the working group, who often asked, "Are you just playing in the sessions?" An implicit attendant to this question is, "Are people *learning* anything in your working group?" In our retrospectives below, we describe the activities and the discussions that took place, but we would like to take this opportunity to answer this question head-on. First, in order to understand and discuss the role of mathematical play in mathematical learning, we must *experience* what mathematical play feels like. Second, we posit that many PME-NA attendees regularly enjoy their experiences with mathematics, and consequently often engage in some type of mathematical play already in their daily professional lives. Then, discussions about mathematical play can help attendees develop language in order to explicate their own experience with mathematics, and consider new ways in which to introduce elements of mathematical play in their own work with learners. Third, discussion and collaborations that emerged from the relationships established in the working group give empirical evidence that attendees were already considering how play might be infused into mathematical learning - we know of at least four different partnerships that have emerged as a direct consequence of this working group.

Here, we give succinct descriptions of the activities and discussions that occurred in each session last year.

Retrospective - Day 1 (Reimer)

Frame. The first session began with a brief background on early childhood play that highlighted the spontaneous and emergent nature of early play. We focused this session around early childhood play frameworks that emphasize child agency through spontaneous interaction, choice, and opportunities for repeated trials (Wager & Parks, 2014). Because attendees would be engaging in exploratory play in solo and collaborative forms, we also drew on conceptualizations of play that suggest players use novel ways to generate norms, players create new rules in contextual ways to continue play, and play continues through the creation of new meaningful constraints (Di Paolo, Rohde, & De Jaegher, 2010).

Play. We distributed colorful nontraditional pattern blocks (e.g., a mix of both concave and convex hexagons) to participants and encouraged them to begin with individual construction or puzzle play to explore the characteristics of the blocks. We asked them to pay attention to any material constraints that contributed to the ways they developed norms in their individual play. Then we asked attendees to orient each other to their play by explaining their norms and sharing their constructions. This led to opportunities for negotiated play in which participants joined each other in coordinated constructions. Finally, we asked participants to form small groups and bring their shapes together into one as an example of an intertwined sense-making activity. Participants circulated the room and shared the ways their groups had coordinated their play, pointing to specific aspects of their constructions that were made possible by constraints, unexpected possibilities, or the breaking of their established rules.

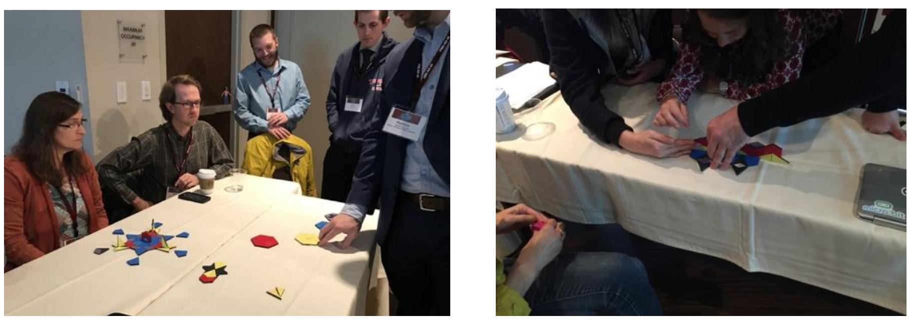

Figure 1: Attendees Orient Peers to Their Play (left) and Negotiate Coordinated Play (right)

Discuss. We offered several questions to guide participants' discussions after their play experiences, including what mathematics learning opportunities emerge in children's play, and how can teachers support children's mathematics learning without disrupting play? Discussions around these questions centered on the role mathematical properties of the blocks played as constraints in construction play. One aspect of these blocks is that they are stable enough to allow construction in an upwards direction (such as in Figure 1), which a small number of participants noticed and then spread to other participants in the room. Participants also noted productive tensions between self-direction in mathematical play and the ways norms and rules of play are emergent and negotiated. For example, one group had a long discussion about their differing views of using symmetry as a design tool - most group members were striving specifically for a fully symmetric shape, while another group member went so far as to (playfully) hide a piece in order to prevent symmetry from being achieved.

Retrospective - Day 2 (Williams-Pierce)

Frame. This session began by defining mathematical play as *voluntary engagement in cycles of mathematical hypotheses with occurrences of failure*, and introducing five features of digital contexts that support mathematical play: (1) consistent and useful feedback; (2) high enough levels of difficulty and ambiguity that players experience frequent failure that is closely paired with the feedback; (3) non-standard mathematical representations and interactions; (4) mathematical notation introduced late or not at all; and (5) the legitimate possibility of alternative conceptual paths for successful progression (Williams-Pierce, 2017; Williams-Pierce & Thevenow-Harrison, in revision). This framework emerges from a blend of scholarship on mathematical learning and videogames research, the latter of which has embraced the conceptualization of *failure* as an important and often enjoyable experience within the realm of gameplay (e.g., Juul, 2009; Litts & Ramirez, 2014), which highlights the importance of a non-threatening environment where mistakes are perceived as natural and appropriate.

Play. We paired attendees and distributed iPads with *Dragonbox 12+* by WeWantToKnow (a commercial learning game that focuses on balancing equations) and a variety of other mathematics learning games. We chose *Dragonbox 12+* for three reasons: it instantiates four of the five features above, and at least partially instantiates the remaining feature, #2; it is highly popular (over a million downloads worldwide, and a *Wired* article that touted the original game's release; Liu, 2012); and because two of the authors (Williams-Pierce and Ellis) had strongly divergent views about how mathematical the play in *Dragonbox 12+* actually is. Figure 2 shows a screenshot of *Dragonbox 12+* and attendees at the working group playing the game together.

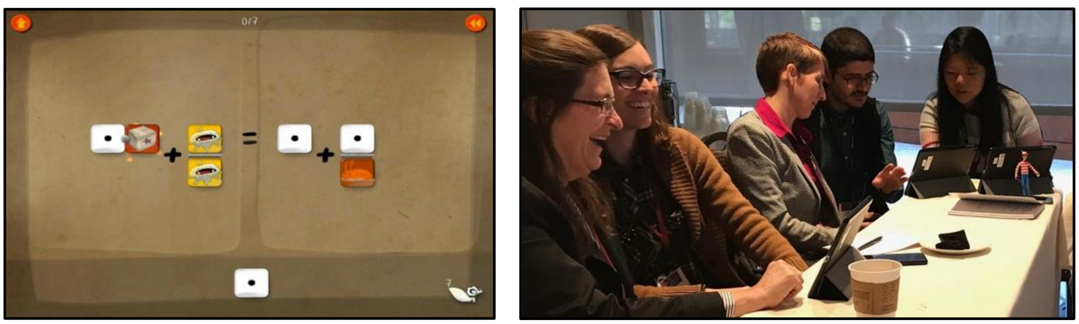

Figure 2: *Dragonbox 12+* (left) and Working Group Attendees Playing Together (right)

Discuss. To support attendees in analyzing their play experiences, we developed broad guiding questions: (1) Where's the math? (2) Where's the learning? (3) Where's the play? Did you experience *voluntary engagement in cycles of mathematical hypotheses with occurrences of failure*? (4) Does the game have all five features for supporting mathematical play? Paired players discussed the game and these questions as they played *Dragonbox 12+* and the discussions grew naturally into larger groups until we were all debating together as a single large group. The primary group discussion focused on the idea of target content - that is, we discussed how the game designers' intended outcomes may differ from the intended content goals of the individual instructor. One insight that developed was that we need to reframe our discussions about mathematics games from "What math games are good?" to "What games are good for *x*?" where *x* represents a specific learning goal or mathematical behavior. For example, while it may be easy to dismiss many mathematics games as digital versions of flash cards, some instructors may have learning goals that involve practicing math facts - and consequently, in that context, digital flashcards can be a good game. We also discussed how the intended learning goals of designers may not be accomplished - for example, while *Dragonbox 12+* was enjoyable for most attendees, there was debate as to whether the game actually teaches the balancing of equations as intended. To that end, we discussed ways in which games that use innovative interactions and representations can be bridged into more formal and rigorous mathematical learning.

Retrospective - Day 3 (Plaxco)

Frame. The Rubik's cube session began with an introduction to the history of the cube, a discussion of some rules governing the cube's movement and arrangements, and an overview of how to communicate moves on the cube (for examples, see Figures 3 and 4). Because research in this area of mathematical play is less developed and we anticipated our participants' unfamiliarity with solving the cube, we focused on first engaging the participants in play intended to support later discussion of the possibilities for educational research.

This play was more structured than the previous days' free play. Given the nature of Rubik's cubes, and how quickly new players can lose their place (mix up the cube), we opted to scaffold working group participants in their interactions by providing simple goals that still supported open-ended playfulness and also provided safety in "failure" to keep the cube solved. The purpose of this variety of structured play was to move participants beyond the typical singular (and often daunting) goal of solving the Rubik's cube in order provide them with alternative goals that would begin and end with the solved cube and other goals that participants could work toward from an unsolved cube. This supported a discussion of how using artifacts of play in alternative ways can open opportunities for creativity and improvisation in play.

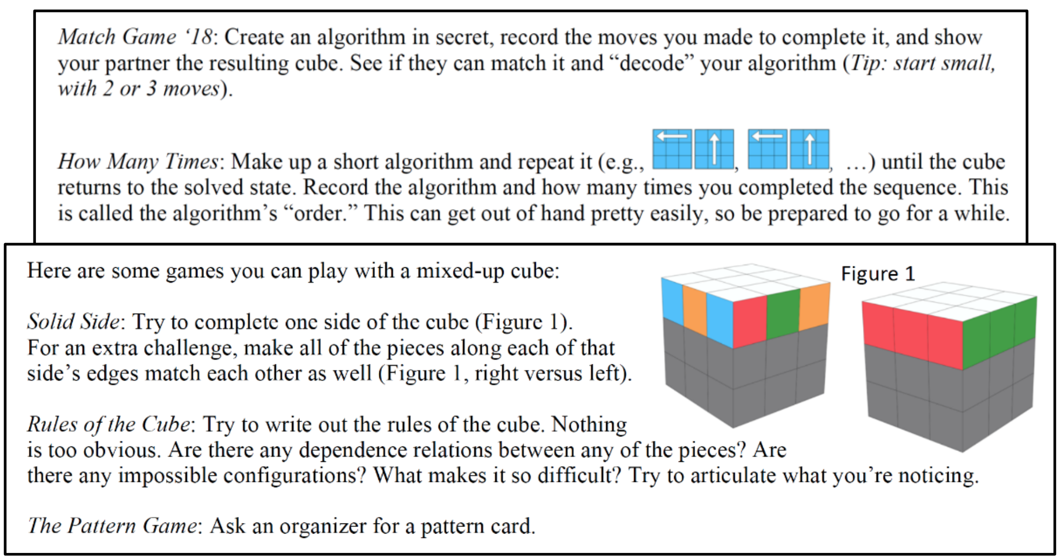

Figure 3: Structured Play Beginning and Ending with a Solved Rubik's Cube

Play. We distributed 50 solved 3x3x3 Rubik's cubes to the working group participants, who engaged in a number of types of structured play. We guided this play with use of a handout that provided a list of games that begin (and hopefully end) with a solved cube (Figure 3, top) as well as some that drew on the likelihood of the participants' eventual mixing up of the cube during the first type of organized play (Figure 3, bottom). Throughout this time, the organizers of the working group walked around to engage with participants as they explored the Rubik's cubes. Very fortunately, a few of the participants happened to have experience solving Rubik's cubes, which allowed for some "safety" in the other participants' *failure* to keep the cube near a solved state. This occurred when participants "lost track" of the moves that they needed to make relative to the moves they had already made. Initially, we had planned to incorporate iPads in the activity so that partners could record each other to help keep track of moves. However, we decided that we could forego this by incorporating tasks and games that could be completed on unresolved cubes. Specifically, we included a "Pattern Game" that tasked participants with matching a face of the cube to an image on a small, numbered card. These cubes were then collected with the cards and used to construct a mosaic (Figure 3, bottom) that read **PME NA 2018**!

Figure 4: The Last Remaining Working Group Participants with a Rubik's Cube Mosaic

Discuss. During the discussion portion of the third day, the organizers brought up a few themes that had continued to emerge throughout the sessions. For instance, the group tried to elaborate on what aspects of the Rubik's cube made play with it mathematical. This included a distinction between an intended content focus of mathematical play and a practice focus of mathematical play. For instance, the Rubik's cube exemplifies the mathematical structure of an algebraic group. However, for many K-12 students who are interested in cubing, the formal construct of a group is unnecessary to engage in and develop their own mathematical practices of problem solving, communicating, generalizing, and reasoning structurally that play with the Rubik's cube can afford. This conversation supported a more general discussion about the values that educators have when focusing on students' mathematical play, specifically the need for an awareness of our focus on content and practices as we develop mathematical play spaces.

Working Group Schedule and Activities

Each day will focus specifically on a different approach to and definition of mathematical play. Day 1 will focus on mathematical play and making by 3rd to 5th graders across formal and informal environments (Simpson); Day 2 will focus on intellectual play within mathematical microworlds (Sinclair and Guyevskey); and Day 3 will focus on mathematical play in teacher professional development (Burke). While each session has distinct differences in their definitions of mathematical play and approaches to fostering such play, there are commonalities across the sessions. For example, the art produced by the Art Bots on Day 1 have distinct similarities to the mathematical microworld focused on in Day 2. Day 2 and Day 3 both highlight the use of open-ended microworlds for supporting mathematical play, although the microworlds focus on different types of content (geometry versus ratio and proportion). By concluding the working group with a session on teacher professional development, we hope to foster conversations about how to support both others and ourselves in fostering mathematical play experiences.

Day 1 Session – Following Williams-Pierce (2016)

The first session will focus on Dr. Simpson's preliminary research on mathematical play of 3-5 graders in making and tinkering contexts across formal and informal learning environments. As such, building upon the scholarship of Williams-Pierce (2016), mathematical play is defined as *voluntary engagement in cycles of mathematical hypotheses, wonders, and curiosities that lead to or stem from occurrences of failure*. Making and tinkering is not a new phenomenon but involves youth in the process of designing, constructing, testing, and revising of physical and/or digital products for play or for purpose. It involves the use of a variety of materials and tools such as low-tech (e.g., conductive tape), high-tech (e.g., 3D printers), household items (e.g., cotton balls), and recyclable material (e. g., yogurt containers). Scholarship on making and tinkering contexts has illustrated youths' engagement in "experimental" play as scientists and engineers (e.g., Simpson, Burris, & Maltese, 2017), but less is known regarding youths' play as mathematicians in such contexts (Pattison, Ruben, & Wright, 2016).

The session will begin by engaging participants in a making activity – participants will make an Art Bot that draws a hands-free art piece (see Figure 5). During and after this activity, we will guide discussions about how (and if) participants felt their experience involves mathematical play. For example, was mathematical play a genuine part of their making experience or did it seem like an add-on or an outlier to their experience? Did mathematizing happen during play, or only afterwards, when we engage in overt discussions about mathematical play? If the latter, can we trace the mathematics brought up afterwards back to the play, to see how the play provoked our own mathematizing and consequent play? We will particularly focus on the role of failure in their experiences of mathematical play, given Simpson's previous work examining failure episodes in makerspaces (e.g., Simpson & Maltese, 2017) and the emphasis on failure in Williams-Pierce's (2016) definition of mathematical play.

Figure 5: Example of an Art Bot (left), and the Resulting Art Piece (right)

Next, Dr. Simpson and Dr. Williams-Pierce will present episodes of how one youth's mathematical play in making and tinkering context span space (e.g., after-school program vs. home) and time. This is a dynamical view of youths' play, as opposed to a "flat" view, as we consider how mathematical play and the physical and abstract objects transcend time (time scales; Lemke, 2000) and environment (boundary crossing; Akkerman & Bakker, 2011). We will conclude this session by seeking feedback on our view of the ways in which mathematical play and learning can transcend contexts, and discussing future research possibilities.

Day 2 Session – Following Featherstone (2000)

The second session, conducted by Dr. Sinclair and Ms. Guyevsky, will focus on Featherstone's (2000) *intellectual play*. Drawing on the work of Huizinga, Helen Featherstone (2000) argues for the central importance of what she calls "intellectual play" in mathematics learning. She finds many parallels between the characteristic features of play and the way mathematicians work. After having observed her students in an elementary classroom engaged in serious mathematical activity in a manner she described as “math in the ludic zone”, Featherstone wondered whether we could help children see mathematics as an arena for play. She is not arguing that play and mathematics are identical, nor that teachers should include play periods as part of their mathematical lessons; instead she is interested in “the moments in which doing mathematics becomes playful and about the ways in which play might expose children to aspects of the discipline that may not ordinarily be visible to them” (p. 16).We conjecture that certain well-designed and open-ended computer-based environments might be especially effective at providing “playgrounds” or mathematical microworlds in which play can occur. As participants engage in playful activities, they will be invited to reflect on how such play might free learners “from the dictatorship of concrete objects,” as Vygotsky (1933/1966) writes, and enable them to “develop the capacity to behave in accordance with meaning” (p.19).

The Web Sketchpad version of *The Geometer’s Sketchpad* (Jackiw, 2012) will provide the digital context for intellectual play. In contrast with traditional definition of a game, the participants will not be offered a set of explicitly prescribed rules that the player has to abide by; rather, the rules of the “game” will be determined by the fixed geometric properties of the figures and communicated to the player via visual feedback as participants engage with the microworld. The player “wins”, if the outcome of the interaction is a robust dynamic construction that behaves according to the criteria of the assigned task. However, being “at play” is more of our focus here than “winning”, as we hypothesize that those participants who “lost” the game have nonetheless come a step closer to understanding a certain geometric concept. We chose a dynamic geometry environment because it offers various modes of feedback that can enable students to experiment, test conjectures and debug without having to appeal to an outside authority, which we think is a crucial aspect of enabling play (Sinclair & Guyevskey, 2018).

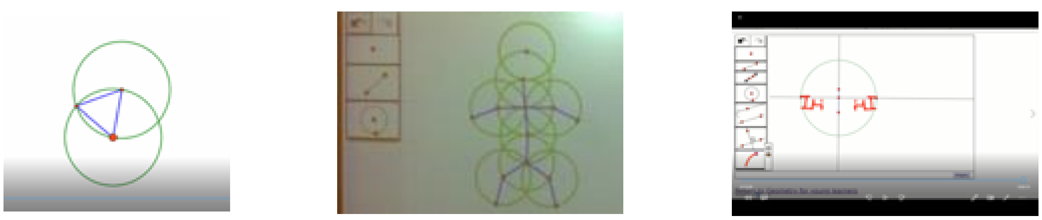

Figure 6: Regular Polygons (left), Stickman (middle), and Mirror Machine (right).

Participants will be asked to engage in computer-based mathematical tasks in which they will explore geometric concepts and relations, and model mathematics in contextualized experiences. Activities will be carried out in pairs. Participants will be offered three different tablet-based activities that are potentially relevant to play. These activities will vary in the level of difficulty, i.e., the geometric relationships involved in a task will range from less to more complex. A progression of tasks will focus on regular polygons, properties of a circle, and the concept of symmetry. In Task 1, Regular Polygons, participants will construct a variety of shapes, beginning with an equilateral triangle, and then gradually tessellating with that triangle to construct a rhombus, regular trapezoid, and regular hexagon (Figure 6, left). In Task 2, Stickman, participants will be asked to construct a stickman in such a way, that the arms are equilateral, and the legs are equilateral. As extension, equilateral fingers could be added, as well as neck, knees and elbows. Participants will need to accommodate for the stickman's ability to grow limbs while maintaining their congruency (Figure 6, middle). In Task 3, Mirror Machine, participants will be shown a picture of Leonardo da Vinci mirror-writing machine that he designed to encode his writings, and then shown a dynamic version of such a machine in web sketchpad. They will be invited to create such a machine themselves (Figure 6, right).

Day 3 Session – Following Burke (2017)

The third and final session, conducted by Dr. Burke and Dr. Orrill, will focus on mathematical play in teacher professional development. Dr. Burke will coordinate an exploration of the environments and tasks of Dr. Orrill's Proportions Playground project. This approach to professional development relies on engagement with digital "toys" designed by Dr. Burke and Dr. Orrill to encourage participants to "play" with mathematics to strengthen their understanding of important aspects of ratio and proportion. The toys we'll be playing with in the workshop were designed to be explored, requiring us to conjecture, justify, and explain as we collaborate. We will focus on using the Bars Toy, which is built around a relatively simple browser-based simulation of two bars of interdependent length that can be edited by dragging. In the PD and in the working group session, we will use three separate scenarios to provide contrasting mathematical relationships that will elicit discussions in which mathematical language and ideas are valuable to differentiate among them.

For our purposes, we define playing with math to mean engaging teachers in problem solving in a way that relies on making and testing conjectures and mathematical arguments that can be reasoned about, tested, illustrated, and explained through the use of dynamic tools. In order for this play to occur, an environment (including the toys and the implementation of the PD) has to allow for playfulness. It has to exist as a space where participation - and consequently play - is a safe activity. We use language and norms to lower the stakes of participation, and provide activities that teachers can immediately explore and use to form conjectures (Burke, 2017).

Central to the goal of our PD were three key ideas of proportionality: quantity, constant, and covariation. When we talk about proportional situations, what quantities do we refer to? What do we identify as remaining constant? How do we, and do we, use covariation to make sense of proportion? As Sutherland and Balacheff (1999) have written, students must learn to use the language of algebraic thinking in order to develop algebraic ways of solving problems. While teachers have shown themselves to be adept at cross-multiplying missing value problems, we want them to be able to see a ratio as a comparison between two quantities (Lamon, 2007). In particular, we want them to be able to talk to students about proportional situations as ones in which "the ratio of one quantity to the other is invariant as the numerical value of both quantities change by the same scale factor" (Lobato & Ellis 2011, p. 11). To this end, we developed a PD in which rich discussions about proportional situations can take place, allowing teachers to use their knowledge of proportion together with the key ideas we emphasize.

In this session, after facilitating a mathematical play experience with the Bars Toy, we will highlight some of our unexpected findings or struggles from our PD research, and foster a discussion that feeds into our re-design plans. In particular, we plan to use this opportunity to both share our work on mathematical play with teachers, and to gain insight from a community of deep thinkers about mathematical play in order to further our ongoing PD design. Our broad discussion questions will revolve around the relationship between mathematical play and learning for teachers, and how can support teachers in bridging from their own experiences of mathematical play in the PD to fostering such play with their students.

After Dr. Burke's group activity, we will conclude the working group by discussing our experiences across each of the three days. In particular, we will focus on the three mathematical play approaches emphasized on different days, and how the differing contexts and content may have influenced the development of these approaches, following the open research questions identified above. We will seek to find threads of common ground that will assist in developing a more flexible and appropriate model of mathematical play that can inform design of such environments and activities across age groups, content, and tools. Finally, we will conclude by identifying next steps for the working group members, as outlined in the following section.

Future Plans

In order to establish clear next steps at the working group, we will investigate potential NSF conference proposals in advance, and share a draft submission action plan with attendees, with the goal of collaboratively conducting a workshop that draws across different areas of mathematical play expertise, in order to begin productively synthesizing the phenomenon.

In addition, this working group will continue establishing a network of support for designing and examining mathematical play at all ages. To that end, we will continue curating the listserv we established after last year's session, so that attendees find it convenient and simple to continue the pattern of fruitful collaborations that the first session spawned.

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THE MATHEMATICS EDUCATION OF ENGLISH LEARNERS

William Zahner
San Diego State University
bzahner@mail.sdsu.edu

Sarah A. Roberts
Univ. of California, Santa Barbara
sroberts@education.ucsb.edu

Craig Willey
Indiana University, IUPUI
cjwilley@iupui.edu

This Working Group builds on the accomplishments of the Working Groups in 2015, 2016, and 2018. We will continue considering multiple aspects of research and practice related to mathematics learning and teaching with English Learners. Our goals for the 2019 Working Group include: (1) sharing an opportunity to publish empirical research related to mathematics and English learners; (2) identifying additional venues for dissemination of mathematics education research on English learners, including novel outlets that connect research to practice; and (3) developing and refining the work that will be shared. In Session 1, the organizers will engage participants in a structured sharing and offer feedback using a protocol. During Session 2, we will engage in a second round of co-working to improve our developing empirical studies. In our final day of the Working Group, we will discuss alternative outlets for sharing our work. We will close with time to review group progress and discuss next steps for our collective and individual work.

Keywords: Equity and Diversity

Introduction/Rationale

English Learners (ELs) are the fastest growing group of U.S. students (Verplaetse & Migliacci, 2008). U.S. schools have seen an increase in the percentage of ELs in all but 10 states between 2000 and 2015 (National Center for Educational Statistics [NCES], 2017). This increase in the population of ELs has created a need for schools and teachers to create inclusive and equitable mathematics classrooms. No longer is supporting ELs a concern only for educators in states like Arizona, Texas and California with traditionally high numbers of EL students. With all but 15 states across the country seeing increases in their EL populations between 2004-05 and 2014-15 (NCES, 2017), there is growing nationwide pressure for support in addressing the needs of these students. (Figure 1).

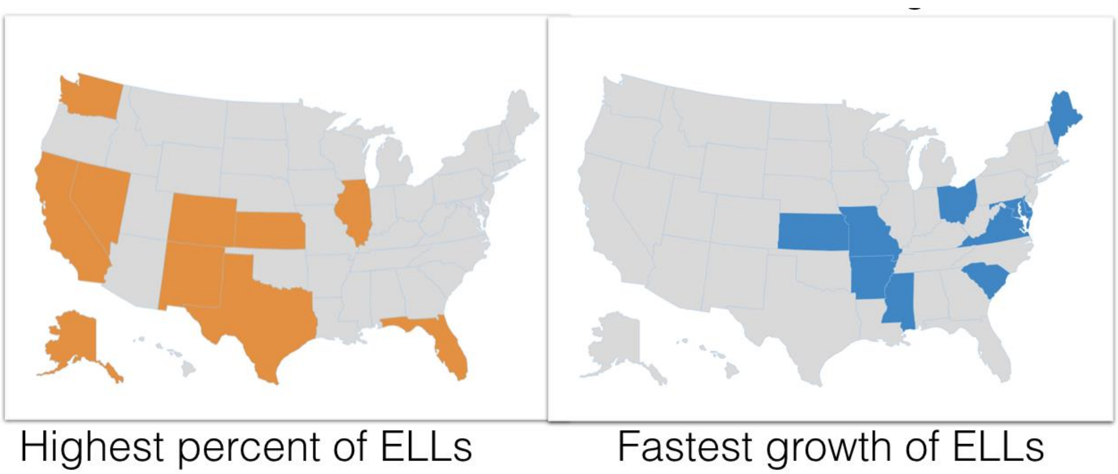

Figure 1: The 10 states with the highest percent of ELs in 2016 (left) and the 10 states with the fastest growth in EL population (right). (Source: National Center for Educational Statistics) https://nces.ed.gov/programs/coe/indicator_cgf.asp

This year, our Working Group aims to look backward to synthesize research related to the mathematics education of ELs, while also looking forward to consider new opportunities to publish research in the field. We will focus on the question: How can we produce empirical research that addresses gaps in the literature related to the mathematics education of English Learners?

In looking back at research that the field has completed, the organizers will share insights from a recently completed a literature review of research on ELs and mathematics education (de Araujo, Roberts, Willey, & Zahner, 2018). We will also discuss the political landscape in relation to this work, as we acknowledge the sociocultural and political dimensions of mathematics, school mathematics, and the complex intersection of language(s) culture and mathematics in multilingual settings (Gutiérrez, 2013). There is still much work to be done; some contexts and critical dimensions of the question of how best to educate English learners in mathematics remain unexamined. During this meeting of the working group, we will look forward as a group to continue to contribute to the field and to consider the contexts, dimensions, and work that still needs to be done related to ELs and mathematics education. The product of this Working Group will be a special issue in a research journal focusing on ELs in mathematics education, which features the empirical research discussed in this Working Group.

Brief History of the Working Group

The facilitators of this Working Group initially came to work together through the NSF-funded Center for the Mathematics Education of Latinas/os (CEMELA). CEMELA brought together researchers from across the country to collaborate on research focused specifically on critical issues related to Latinos/as in mathematics. Prior to CEMELA, researchers interested in such a focus worked mostly in isolation. In considering issues related to Latinos/as in US schools, issues of language and culture were central to CEMELA's work, which often had direct implications for ELs more broadly. While not all Latinos/as are ELs, and not all ELs are Latinos/as, these two groups have significant overlap. For example, about 80% of ELs speak Spanish as a first language, and Spanish-speaking ELs appear to struggle on measures of academic achievement (Goldenberg, 2008).

CEMELA expanded the field's knowledge of ELs in mathematics through conducting interdisciplinary studies that helped researchers and practitioners better understand the reality of Latinas/os and mathematics teaching and learning. CEMELA's research focused on teacher education, research with parents, and research on student learning, resulting in over 50 publications and presentations. Again, several of these studies involved the investigation of questions related to the interplay of language, culture, and mathematics education. CEMELA also had the goal of connecting the network of scholars focused on these issues, as a means to build capacity and sustain the work.

Following the conclusion of CEMELA's funding, Zandra de Araujo, Sarah Roberts, Craig Willey, and William Zahner continued to meet regularly. These meetings focused on examining intersections among these early career scholars' work related to the mathematics education of ELs. To date these meetings have resulted in a number of national presentations at the annual meetings of the National Council of Teachers of Mathematics, the American Educational

Research Association, and PME-NA. Currently, this group is working on several manuscripts and follow-up studies related to the preparation of teachers to work with ELs. Perhaps most notable is an extensive review of the international literature focused on mathematics and ELs (de Araujo et al., 2018).

The Mathematics Education and ELs Working Group met at PME-NA in 2015 (de Araujo et al., 2015), 2016 (de Araujo et al., 2016), and 2018 (de Araujo et al., 2018). At those meetings, we brought together a diverse group of about 30 researchers who started working together on several projects related to the mathematics education of ELs (see descriptions of these projects in the section below titled *Previous Work of the Group*). Our aim for the 2019 Working Group is to provide a space for these scholars to continue their work, to focus on producing and sharing products from our research, and to bring new scholars into the fold.

Previous Work of the Group

The 2015 and 2016 Working Groups (de Araujo et al., 2015, 2016) began with whole group discussions aimed at examining five aspects related to the mathematics education of ELs including: a) Student Learning; b) Family and Community Resources; c) Language Perspectives; d) Teacher Education; and e) Curriculum. In the following sections we briefly summarize our discussions and subsequent work in each of these areas.

Student Learning

Building upon situated and sociocultural perspectives (Moschkovich, 2002), the student learning group started from the premise that ELs, like all students, learn mathematics through a process of appropriating discourse practices, tool use, and perspectives of mathematics. In reviewing the literature on the mathematics learning of ELs, we have identified a need to better understand how research in mathematics education at large is connected with research on the mathematics learning of ELs. Much of the content-focused work in mathematics education is isolated from research on how ELs develop specific mathematical understandings. Previous discussions at this Working Group have supported ongoing research and development efforts focused on bridging the literature on supporting ELs in mathematics classrooms and the literature on students' mathematics learning through focused instruction.

Family and Community Resources

Families and communities can serve as resources for ELs in their mathematics learning in myriad ways. Families can advocate for their children and provide and support learning experiences both in and out of the classroom. Communities can also provide a wealth of support mechanisms and learning possibilities. Moll, Amanti, Neff, and Gonzalez (1992) described how students studied candy making and selling within their neighborhood to explore mathematics within this context, such as discussing and analyzing production and consumption. In doing so, the teachers and students acknowledged the value of these community experiences. Additionally, Civil and Bernier (2006) explored the challenges and possibilities of involving parents in facilitating workshops for other parents around key math topics. These studies and others like them illustrate the promise of family and community resources in fostering ELs' mathematics learning. At our 2015 meeting, Civil shared her recent work on how school language policies impact ELs' engagement and how teacher educators can draw on family and cultural resources in support of ELs. We followed up on this work during our 2016 with small group discussions.

Language Perspectives

Teachers' and researchers' conceptions of language, second language acquisition, and bilingualism affect teaching and learning mathematics for ELs. In 2015, Judit Moschkovich shared her work, highlighting how perspectives of language, second language acquisition, and bilingualism appear in both theory and practice. We also discussed, in particular, how work focused on ELs can draw on current work on language and communication in mathematics classrooms, classroom discourse, and linguistics. Looking for these intersections and connections was crucial, because it ensures that work in mathematics education is both theoretically and empirically grounded in relevant research, preventing researchers from reinventing wheels.

Teacher Education

Much of the prior work on teacher education related to ELs has focused on more general strategies (e.g., sheltered instruction, as in Echevarria, Vogt, & Short, 2007), such as using visuals, modifying texts or assignments, and using slower speech. We argue there is a need for content-specific support for mathematics teachers of ELs. At our previous meetings, we explored ways to support teachers, both pre-service and in-service, in better understanding students' strengths and meeting the needs of ELs in the mathematics classroom.

During the 2015 and 2016 Working Groups, the teacher education subgroup focused on the primary issues that arise in the preparation of teachers to teach ELs at the various institutions. As a group, we recognized that there were few attempts to include the teaching and learning of mathematics to ELs beyond the states where there was a high population of ELs. Given that some of the group members were meeting for the first time, a significant portion of the allotted time was spent sharing the details of the research we did and our interest in being part of this particular subgroup. One of the members shared a survey about examining preservice teachers' conceptions about teaching mathematics to ELs, and the other members agreed to administer the survey at their locations. Together, the responses could provide us with some insight about possible conceptions that need change and the steps we can take to make that happen. The group stayed in touch online and continued the discussions about potential collaborations.

Curriculum

In 2015, the curriculum subgroup focused on the role of textbooks, specifically teachers' guides and student work pages, in demonstrating how one might approach supporting ELs in building mathematical understanding and developing mathematics language. We inquired about the process publishers undergo to incorporate and offer support to teachers. What assumptions do they make? Who do they consult? What motivates them to invest in serving ELs better? What is/are their end goal(s)? The group decided to conduct an analysis of various middle grades curriculum to ascertain what supports and guidance are offered to teachers. It was suggested that we might build on the work of Pitvorec, Willey, and Khisty (2010), who explored the features of Finding Out/Descubrimiento (FO/D) that proved to be successful with bilingual children of migrant families in the 1980's and partially contributed to the development of complex instruction (Cohen, Lotan, Scarloss, & Arellano, 1999).

At the 2016 meeting, this group continued to examine textbooks to understand better the supports they provide for ELs. Participants considered language issues in mathematics texts for ELs, especially as related to word problems and assessment items. We shared a short literature review of relevant research on linguistic complexity and vocabulary for mathematics word problems. Based on that research, we summarized recommendations for addressing language complexity and vocabulary in designing word problems for instruction, curriculum, or assessment. We then used examples of released sample Smarter Balanced Assessment Consortium items to illustrate how to apply those recommendations to design word problems and to design supports for ELs to work with word problems. Several of the participants have continued this work examining curriculum accommodations for ELs and are completing a research study based on the work started at the 2016 Working Group.

2018 Working Group

The 2018 Working Group was an opportunity to share recent empirical work from members of the group and to provide participants with a chance to engage in discussions around data collection and analysis. During the first half of the working group, several members presented current research. For example, Erin Smith presented her work on curriculum supports for ELs in elementary mathematics classrooms. Marta Civil introduced the Working Group to the forthcoming National Academy's (2018) *English Learners in STEM Subjects* report, on which she had contributed. Sarah Roberts shared a framework she and a colleague have developed for supporting ELs that they use in their content methods course. Such presentations represented topics from a variety of researchers who conduct research across the spectrum of grade bands and contexts of research previously mentioned. Zandra de Araujo wrapped up these presentations by sharing an overview of the findings of the literature review of de Araujo et al. (2018). These presentations provided an overview of a variety of research taking place in the field around English learners and mathematics, and with the literature review presentation and the National Academy report, some of the research that is currently lacking.

The second half of our 2018 Working Group was dedicated to a *data dive*. In the first part of the activity, participants learned about Bill Zahner's NSF CAREER study. He then introduced the group to a framework he has developed for linguistic and mathematical demands (Zahner, Milbourne, & Wynn, 2018). The group used this framework to analyze a task and interviews from his study, entering into the *data dive*. The Working Group considered transcripts, the accompanying audio, student work, and the original task.

Sharing this existing work along with participating in the data dive engaged the Working Group in conversations around data. Our discussions and activities during the 2018 Working Group allowed us to model and to discuss analysis and data collection. Now members of the Working Group have works in progress that they are getting ready for dissemination. In 2019, we will focus our efforts on dissemination, providing structured time together for improving the products of our research. Between March 2019 and November 2019, the Working Group organizers will be proposing a special issue to journal editors to lay the foundation for this group publication.

Aims for the 2019 Working Group

Previous meetings of this Working Group have brought together a large and diverse group of attendees. For example, in 2016 there were approximately 30 attendees including teachers, preservice teachers, researchers, graduate students, and teacher educators from a range of institutions. In 2018, the Working Group attendees included experienced leaders in the field, mathematics education faculty with experience in the topic of English learners in mathematics who were still at the early stages of their careers, and graduate students and faculty who were new to the topic of English Learners in mathematics. During the 2018 meeting, we successfully shared some of our existing and in-progress research in the form of presentations. For the 2019 iteration of this Working Group, we propose to narrow our focus and to home in on the goal of producing high-quality scholarship in widely accessible venues. Our goals for the 2019 meeting include: *(1) sharing an opportunity to publish empirical research related to mathematics and English learners; (2) identifying additional venues for dissemination of mathematics education research on ELs, including novel outlets that connect research to practice; and (3) developing* *and refining the work that will be shared.* The goal for 2019 will be realized through specific activities and interaction formats. These activities include: (1) structured time for participants to share their existing work; (2) structured feedback protocols through which participants can offer constructive feedback with the goal of advancing the quality of our research, and (3) concrete discussions of timelines and plans for publication of our work in a refereed journal as well as alternative forms of dissemination.

Plan for the 2019 Working Group Sessions

During the three sessions in 2019, participants will engage in collaborative efforts related to dissemination and the production of research based on the prior research of Working Group participants. Each meeting of this Working Group at PMENA will have a narrow focus. Prior to the Working Group, we will contact past participants and invite them to come to the 2019 Working Group with a paper in progress for discussion and feedback sessions. We will also send out an email to the PMENA list-serv and offer this as an option to new participants, to capture individuals who may be working on papers and would fit in the session.

Participants who wish to receive feedback on their manuscripts in progress will be asked to upload draft work ready to be shared and workshopped in a working group format before the first meeting. Aware that some members will be new in 2019, we will invite new members to join in the shared writing and editing efforts using Google Docs or a similar tool to share files as appropriate.

Session 0 (Pre-PME-NA): Propose Special Issue & Virtual Study Group

The summer before PMENA 2019, we will propose two special issues on ELs in mathematics to peer reviewed journals. One issue will be written for a research audience and the other will address teachers. We will also build momentum for the working group before the PMENA meeting by organizing a virtual study group in which potential Working Group members can participate. Over the course of six sessions, we will meet monthly and organize a study space around translanguaging in mathematics classrooms. The group will be open to anyone who is interested in learning more about translanguaging to inform their teaching and/or scholarship. This will also allow us to provide an introduction to the special issue and possible contributors.

Session 1: Introductions and Brief Reports

The initial session will include some time to allow participants to meet fellow attendees and to share current perspectives on extant research related to mathematics and English learners. The organizers will present a brief report on the outcomes of prior working groups. Then we will share goals for this year's iteration of the Working Group. After this preliminary step, we will devote a substantial amount of time to sharing and workshopping three manuscripts that are close to completion.

Activities

- Brief (re)introduction to the Working Group.
- Sharing of 3 papers by potential contributors to the special issue.
- Structured feedback to authors.

The structured feedback will happen in small groups. Participants in the Working Group will be divided into groups and invited to read an excerpt of each authors' work and then offer constructive feedback that can be used to refine the manuscripts in development. Feedback will be structured to maximize the probability that the feedback is useful to authors. Guiding Questions for the feedback will be:

1. What does this paper add to the literature on mathematics education and English learners?
2. Are the frameworks (theoretical and conceptual), research questions, methods, analysis, and claims aligned? If not, how can the author bring these elements into alignment?
3. Are the discussion points illuminating in relation to the literature review and the existing state of knowledge in this field? Is there anything that should be added to the discussion to make it more powerful?

At the end of session 1, the authors who will be sharing work in session 2 will give a brief (1 minute) precis of their work. Participants will be divided into groups and assigned to one author. The author will give these participants access to a manuscript in progress that they will read before Session 2.

Session 2: Continued Workshopping

Session 2 will have a structure similar to Session 1. In Session 2, three more potential contributors to the special issue will share their work. This will follow a format similar to the format of Session 1: Each potential author will give an overview of their work in the form of a short Powerpoint. Then the Working Group will divide into subgroups in which the members of the Working Group provide structured feedback to authors.

We will conclude Session 2 with an overview of some of the general themes of the articles and the feedback that authors are receiving. This work is twofold. First, it helps to begin to organize the special issue topically. Second, it helps the authors to consider some of the major feedback that authors are receiving and how that might support their writing moving forward. What are some of the missing pieces across the manuscripts? With what are authors conceptually struggling? How can we make our methods stronger to make arguments stronger? These conversations will help us wrap-up our second day.

Session 3: Consider Alternative Modes of Sharing Work

Aware that much of our work winds up in journals that are only read by other researchers, on Day 3 of the Working Group, we will convene a discussion of potential alternative modes of sharing the work we produce. We will consider the following outlets: video exemplars, blog posts, op-eds, apps, curriculum production/development, professional development. We will invite a panel who has completed such work to share their experiences, challenges, successes, and suggestions. This will take the first half of the last day of our Working Group.

During the last half of our final day of Working Group, we will create a timeline for our collective and individual next steps of the special issue and for participants' individual manuscripts.

Follow-up Activities

We anticipate that this Working Group will again attract other researchers interested in issues related to the mathematics education of ELs. Therefore, an important component of this fourth meeting of the Working Group will be to maintain current relationships while also continuing to establish connections with other interested researchers in order to build opportunities for future collaborations. We will provide space for new researchers to contribute to our collective work, to suggest new directions, and to add to the growing body of research on mathematics and ELs. At the first session of our Working Group, we will share our ongoing online Google Community, which uses Google applications (Hangout, Groups, Drive, etc.). Google's applications are freely available and allow for a number of collaborative opportunities, including video conferencing, group messaging, collaborative document development, and shared web and social media space. Through this collaborative Google Community, we have organized follow up meetings both virtually and at conferences such as TODOS and the NCTM Research Conference.

Our long-term goal is to publish a special issue on ELs and mathematics. We hope to take the articles that are workshopped during Working Group 2019 and continue to work with participants to develop a full-fledged special issue. We currently plan to co-edit two special issues in two different journals: *TEEM (Teaching for Equity and Excellence in Mathematics)* and *Qualitative Studies in Education.* This work will be the culmination of four PME-NA working groups and can provide a product of the work from bringing individuals together who are committed to completing work around ELs and mathematics.

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COMPLEX CONNECTIONS: REIMAGINING UNITS CONSTRUCTION AND COORDINATION

Beth L. MacDonald
Utah State University
beth.macdonald@usu.edu

Steve Boyce
Portland State University
sboyce@pdx.edu

Jessica H. Hunt
North Carolina State University
jhunt5@ncsu.edu

Cameron Byerley
University of Georgia
cameron.byerley@gmail.com

Diana L. Moss
Utah State University
diana.moss@usu.edu

Claudia Bertolone-Smith
SUNY Plattsburgh
cbert006@plattsburgh.edu

Students' construction, coordination, and abstraction of units underlie success across multiple mathematics domains. Structures for coordinating units underscore notions of numbers as composite units (e.g., five is a unit of five and five units of one). In this working group, we seek to facilitate collaboration amongst researchers and educators concerned with units construction and units coordination. The aim for this working group is two-fold: (1) to extend our research around units construction and coordination to new grade levels; and (2) to collaborate with researchers who investigate students with learning differences in school settings to determine diverse students' mathematics learning trajectories.

Keywords: Number Concepts and Operations, Learning Trajectories, Learning Theories

In Steffe's 2017 plenary for PME-NA, he substantiated particular needs for investigating how children develop operations when constructing and coordinating units. The Complex Connections: Reimagining Units Construction and Coordination working group began at PME-NA 2018, with the aim of facilitating collaboration amongst researchers and educators sharing Steffe's concerns about units construction and coordination for all learners. The main goals of the working group are to extend research around units construction and coordination to new grade levels and to facilitate collaborations with researchers who investigate students with learning differences in school settings to determine diverse students' mathematics learning trajectories. We frame this proposal to continue this working group by first providing a background about units construction and coordination. We next describe our progress toward meeting our goals stemming from our inaugural meeting. Lastly, we describe our goals and plans for continuing this work at PME-NA 2019.

Background and Theoretical Perspective: Composite Units – Old and New

We provide a theoretical background for units construction and coordination, focusing on the role of units coordination in students' number sequences (Steffe, 1992), multiplicative concepts (Hackenberg & Tillema, 2009), and fractions schemes (Steffe & Olive, 2010).

Units Construction and Coordination

Unitizing, or setting an object (a unit) aside for further action or activity (Steffe, 1992), is the basis for units construction. Students initially rely on concrete, pictorial, fingers, symbolic numerals, and language to evidence internalized (being able to mentally re-imagine contextual actions) or interiorized (being able to draw on de-contextualized actions) actions. Consider the construction of additive reasoning as an example. When considering how to add eight and seven, students might not yet see the cardinality of eight, counting a set of objects to create eight, then another set of objects to create seven, and finally combining the two sets, beginning at one to quantify the total. Should students see the cardinality of eight, they might use it as an input for solving the problem. They will count on from eight, using objects (cubes, fingers) to keep track of the addition (e.g., 8…9 [raises a finger], 10 [raises another finger]…). The double counting involved in this activity (e.g., 8…, 9 (1), 10 (2), 11 (3), 12 (4), 13 (5), 14 (6), 15 (7)) promotes a coordination of the start value and the stop value. That is, eight, seven, and 15 are taking on some meaning as composites (8 and 7) and a unitized whole (15). Evidence of this meaning includes the breaking apart of one or both of the numbers to arrive at the total (e.g., 8 is 5 and 3; 7 is 5 and 2; 8 + 5 is the same as 5 + 5 + 3 + 2, or 15). This type of units coordination can be explained through the type of numerical sequence students produce and rely upon.

Number Sequence Types (INS, TNS, ENS, GNS)

Steffe and Olive (2010) described four different counting sequences that children may develop and evidence when solving mathematics tasks: (1) Initial Number Sequence (INS), (2) Tacitly-Nested Number Sequence (TNS), (3) Explicitly-Nested Number Sequence (ENS), and (4) Generalized Number Sequence (GNS). Each number sequence can illustrate stages of children's development of units coordination.

Initial number sequence. Steffe (1992) explained that children who segment and interiorize number sequences have developed Initial Number Sequence (INS). Children who develop an INS are characterized by their counting of single units and then their segmenting of a numerical sequence (evidenced through "counting on" activity). When children segment numerical sequences, they are interiorizing patterned templates for counting, which allows them the ability to count on from a composite unit (e.g., developing one composite unit to use when counting on, 4…5-6-7-8). Thus, through counting actions, numerical patterns are developed and become interiorized (evidenced through less reliance on sensory-motor experiences; e.g., verbalizing counts, using fingers, or tapping) before being segmented into a composite unit.

Tacitly-nested number sequence. Once children have developed composite units through their INS activity, they can begin to coordinate these composite units, treating the result of counting activity as both a unit to count on from and one to keep track of when counting. These actions evidence children's development of a Tacitly-Nested Number Sequence (TNS). Steffe (1992) explains that when children have a numerical sequence interiorized and segmented they can use their segmented numerical sequence as material for making new composite units within these numerical sequences. The awareness of one number sequence contained inside another, or double-counting, is an indication of TNS, as is a skip count (i.e., 4, 8, 16,…) to solve early multiplicative kinds of problems, such as how many 3's are contained in 12.

Explicitly-nested number sequence. Children who are described as having part-whole number reasoning in place are capable of disembedding parts from wholes and developing iterable units of one. These children are described as reasoning with an Explicitly-Nested Number Sequence (ENS) (Olive, 1999; Steffe, 1992; Ulrich & Wilkins, 2017). Children capable of multiplicatively understanding a single unit and a composite (whole) unit without disrupting either are said to be operating with an ENS (Ulrich & Wilkins, 2017). The two given composite units (e.g., parts and whole) provide children material to coordinate while constructing a third composite unit; e.g., a unit of units of units (Steffe, 1992). This part-whole reasoning with abstract units provides children multiplicative number structures.

Generalized number sequence. Children capable of developing iterable composite units where units of units of units can be coordinated, are described as operating with a GNS. For example, Olive (1999) explained that when children are asked to find common multiples, they are required to keep track of two series of composite units (e.g., 3, 6, 9, 12; 4, 8, 12; 12. The LCM of 3 and 4 is 12). Children evidencing successful completion of tasks like this are described as reasoning about two iterable composite units (e.g., 3 and 4), while keeping track of the common composite unit in each sequence. At the root of much of this number sequence development, units construction and coordination explain how and why children are capable of transitioning from additive/subtractive operative structures towards multiplicative/division operative structures towards rational number understanding.

Multiplicative Levels of Units Coordination

A student is said to assimilate with one level of units when she conceives of multiplication situations, such as seven iterations of four, by counting on from the first or second set of four by ones and double-counting the number of fours to reach a stop value (e.g., 4, 8,…9- 10-11-12; 13-14-15-16; 17-18-19-20; 21-22-23-24; 25-26-27-28). Here, the child has to model or carry out the situation by using internal (e.g., subvocal counting) or external (e.g., fingers or objects) representations. This is referred to coordinating two levels of units in activity. Units coordination in activity is ephemeral: in a follow-up task, such as how many ones are in eight iterations of four, the student would likely need to repeat a similar process rather than count-on four more from 28.

A student's use of strategic reasoning in such situations may be evidence that she assimilates the situation with two levels of units. For example, a student assimilating with two levels of units might conceive of seven iterations of four as five iterations of four plus two iterations of four (e.g., five 4s is 20; 21-22-23-24; 25-26-27-28). As opposed to modeling the entire coordination, the child can anticipate breaking apart the composite unit of seven into five and two and use each of those parts to solve the problem. For a student assimilating with two levels of units, the result of operating is simultaneously 28 ones and 7 fours; hence a follow-up task of finding the number of 1s in 8 fours would not require building up from 5 fours again.

A student is said to assimilate with three levels of units when she can conceive of a situation such as seven iterations of four as resulting in three distinct yet coordinated units: (a) one unit of 28 that contains (b) seven units of four, each of which contains (c) two units of two. Students assimilating with three levels of units have flexibility to reason strategically with each of the units. For instance, a student assimilating with three levels of units might solve the task, "How many more twos are in 32 than in 28?" by reasoning that 32 is one more 4, which is thus two more 2s. This reasoning involves assimilating three levels of units, multiplicatively.

Norton, Boyce, Ulrich and Phillips (2015) conducted a cross-sectional analysis of 47 sixth-grade students' reasoning in whole number multiplicative settings in clinical interviews. Figure 1 displays descriptors of attributions of students' activities Norton and colleagues identified as corresponding with students' reasoning with one, two, or three levels of units. For instance, they describe students' activity when transitioning from reasoning with one level of units to reasoning with two levels of units with descriptors G-K (Norton et al., 2015, p. 62).

Though there are commonalities with descriptions of students' counting schemes (e.g., descriptors C and J), the focus of the descriptors are more generally about how and whether students are able to flexibly reverse and reflect on their multiplicative reasoning.

Descriptors			
A. Iterates units of 1	E. Works with at least one level of units within a figurative three-level structure	I. Works with at least two levels of units within a figurative three-level structure	M. Assimilates three levels of units within a task
B. Builds composite units	F. Indicates a reciprocal relationship between the size and number of parts in a whole	J. Distributes composite units	N. Reflects on all levels within a three-level structure
C. Iterates composite units	G. Reverses multiplicative relationships	K. Immediacy of response	O. Conflates or confuses units
D. Uses multiplicative relationships	H. Builds a three-level structure	L. Flexibly works with a three-level structure	P. References the structure of units

Figure 1: Descriptors of Sixth-grade Students' Units Coordinating Activity as They Progress from Coordinating 1 Level of Units to Coordinating Three Levels of Units, as Discussed in Norton et al.'s (2015) Findings

These stages of units coordination, as delineated in Norton et al.'s (2015) findings, comprehensively explains transitions children make from additive operations to multiplicative operations. These findings also provide fundamental explanations for fractional unit development, as recursive units coordination in which to act upon and explain how students develop fractions as mental objects.

Fractional Units

Reasoning with fractional units requires unitizing a fractional size, 1/nth. When children first conceptualize fractional units, Steffe (2001) posited that they would reorganize natural number schemes to develop fractional schemes. Olive (1999) and Steffe (2001) argued that they would have to re-interiorize their units coordination operations, to consider a fractional unit as a result of *equi-partitioning* a unit whole into a size that, when iterated *n* times, would result in the size of 1. This re-interiorization of schema requires students to recursively construct and coordinate new composite units relationships (Olive, 1999). Thus, children's production of numerical sequences and their associated units coordination provide children necessary operations for their fractional units coordination.

To conceive of a fraction *m/n* as a number, one must understand m/n as equivalent to *m* 1/*n*ths, n of which are equivalent to 1. In the case of $m > n$, the meaning of 1/*n* must transform from thinking of 1/*n* as one out of n total pieces (*a parts-out-of-wholes scheme*) to thinking of *1/n* as an amount that could be iterated more than *n* times without changing its relationship with the size of 1 (*an iterative fraction scheme*). This measurement conception of fraction (Lamon, 2008) involves coordinating three levels of units of nested units: 8/3 is 8 times (1/3), 1=3/3 is 3 times (1/3), thus an 8/3 unit contains both a unit of 1 and a unit of 1/3 within 1 (Hackenberg, 2010). Students in the intermediate stages of constructing such a measurement conception may reason about the size of proper fractions of form m/n by counting the number of parts of size 1/*n* within a whole of *n/n*. Such students are not yet iterating the amount of 1/*n*, which limits their ability to iterate unit fractions beyond the size of the whole (Tzur, 1999).

Issues Related to the Psychology of Mathematics Education

Steffe's 2017 plenary included both a summary of contributions and important extant problems for mathematics educators pertaining to units coordination. Given these advances in research surrounding K-8 students' units construction and coordination, our mathematics education field is still limited by the context in which students' units construction and coordination develops and how preschool and high school students' units construction and coordination may inform these trajectories. Further, Steffe (2017) proposes that about 40% of first grade students have very different learning trajectories than their peers, suggesting a need to develop alternative means in which units construction and coordination may be developed by children. Finally, Steffe suggests that our field would benefit by investigating children's transitions in scheme development. For instance:

> It is especially crucial to investigate possible changes that indicate *fundamental* transitions between reasoning with two levels of units and three levels of units induced in the construction of quantitative measuring schemes and their use in the construction of multiplicative and additive measuring schemes (Steffe, 2017, p. 46).

One of our intentions for this working group is for researchers from different backgrounds to collaborate to work toward solving such problems. Consider that in their review of research preparing the recent (2016) compendium chapter on quantitative reasoning, Smith and Barrett (2017) note the following:

> [We] found it striking how often the same conceptual principles and associated learning challenges appear in the measurement of different quantities… Despite the clear focus in research on equipartitioning, units and their iteration, units and subunits… curricula (and arguably most classroom teaching) focus students' attention on particular quantities and the correct use of tools, as if each was a new topic and challenge. (p. 377).

Consistent with Arbaugh, Herbel-Eisenmann, Ramirez, Knuth, Kranendonk, and Quander's (2010) call to "develop mathematics proficiency in various school, cultural, and societal contexts" (p. 13), our goal is to connect research programs involving units construction and coordination with research programs that stem from other theoretical perspectives.

Unfortunately, many students with mathematics learning difficulties do not transition from two levels of units to three levels of units at the same pace as their more successful peers. In fact, the construction of ENS is one pervasive mathematical impediment for students with mathematics difficulties (Landerl, Bevan, & Butterworth, 2004). Compared to students without mathematics difficulties, students with mathematics difficulties develop less sophisticated strategies for number computation problems over time, suggesting a lack of a conceptual basis for ENS engagement that actually plays a part in their later disability identification (Butterworth, Varma, & Laurillard, 2011). Therefore, particular research programs with these foci are desperately needed to nurture multiplicative and rational number conceptions (Boyce & Norton, 2017; Grobecker, 1997; Kosko, 2017) and operations (Grobecker, 1997; Grobecker, 2000; Norton & Boyce, 2015) for students who need our support.

Alignment with Conference Theme

By designing interventions with children's mathematics and their units construction and coordination at the center, the field has grown over the years, yet there is much opportunity for improvement. Relations between cognitive factors and test performance may be important, yet these relationships are only one way to conceptualize "cognition." Instead of focusing on aspects of students' working memory or processing, researchers can revolutionize access for students by intervening on the malleable cognitive factors that can be improved upon through students' own development. Research stemming from units coordination and construction is "*against* the new horizon" in the sense that the goal of supporting students' units construction and coordination does not align with goals or initiatives that focus entirely on helping students to meet grade-level, task-based learning objectives. We argue that equitable instruction for all students begins with increased opportunities to adapt their own thinking grounded in a construction within their own mathematical realities. When well-intentioned educators provide children interventions that promote procedures and actions, not only are they not serving their children's mathematics learning needs, they may be preventing them from engaging in learning situations that support the children to adapt their thinking structures and advance their learning.

Research Designs and Methodologies

The primary methodology for investigating units construction and coordination has been the radical constructivist teaching experiment (Steffe & Thompson, 2000). A main role of these teaching experiments is to generate (and refine) epistemic models of students' mathematics – models for how students with common underlying conceptual operations learn within a particular mathematics domain (Steffe & Norton, 2014). Such teaching experiments involve close interactions with a teacher-researcher modeling the dynamics of students' ways of operating longitudinally. Teaching experiment methodology is also used as part of design research (Cobb, Confrey, diSessa, Leher, & Schauble, 2003), to inform instructional approaches or interventions that could be "scaled up" to heterogeneous classroom settings.

Results from analyzing teaching experiments have also informed methods for assessing a child's ways of constructing and coordinating units at a particular moment. In addition to task-based clinical interviews (Clement, 2000), Norton and Wilkins (2009) created written instruments for assessing middle-grades' fractions schemes and operations associated with units coordination. These instruments have been used to validate conjectured learning trajectories for children's construction of schemes for coordinating fractional units (e.g., Norton & Wilkins, 2012). These instruments currently serve as tools for selecting research participants in teaching experiments with middle-grades students (e.g., Hackenberg & Lee, 2015).

Consideration of these methodologies for researching units construction and coordination suggests areas for collaborative work to build our understanding not only of the research programs Steffe (2017) described, but also opportunities and needs regarding related research domains. For instance, Norton and Wilkins' (2009) written instruments have been modified to assess units coordination with fractions with prospective elementary teachers (Lovin, Stevens, Siegfied, Wilkins, & Norton, 2016). Thus, research development with these methodologies are better served in collaborative designs to allow for more perspectives in the design and analyses to more closely determine students' mathematics.

First Aim: Extending Units Construction and Coordination Research

The first aim of this working group proposal is to extend units construction and coordination research to investigations that include both older students and younger students. For instance, questions regarding whether differences in secondary students' units coordination persist beyond eighth grade, and, if so, how these differences manifest in older students' learning, remain underexplored. In interview and teaching experiment settings, Grabhorn, Boyce, and Byerley (2018) have found that students enrolled in university-level calculus do not necessarily coordinate three levels of units. Further expanding understanding of students' units coordination beyond eighth grade would contribute to the development of "coherent frameworks for characterizing the development of student thinking" (Arbaugh et al., 2010, p. 15).

In addition to studies focusing on relationships between units coordination and older students' mathematics, studies of pre-kindergarten children's units construction are also warranted. For instance, Wright (1991) found that students entering kindergarten had a wide variance in their number knowledge, which suggests critical mathematics learning may occur prior to the elementary school experience. With a significant dearth of research studies in the early childhood years (De Smedt, Noel, Gilmore, & Ansari, 2013) we posit that studies of how young children construct their earliest units are a critical area of research. Another potential need regards equity and access in mathematics education.

Researchers might investigate whether analyses of students' units construction and coordination provides insight to the mathematical reasoning of diverse and underserved populations. We envision collaborations around similar conceptual principles and learning challenges, to investigate units construction and coordination of children enrolled in different grade levels and representing different population groups (i.e., special education, early childhood, secondary education, teacher education), which would allow more coherent mathematical learning theory and practical means with which to link research to classrooms.

Steffe (2017) estimated that about 40% of first grade students rely solely on perceptual material when counting all items (Counters of Perceptual Unit Items – CPUI) and that 45% of first grade students are capable of counting figurative unit items (CFUI), a necessary precursor for interiorizing counting actions and developing "counting-on" (e.g., INS) (see Figure 2). Steffe posits that these distinct groups of children require different learning trajectories due to the differences in their construction of units.

With more U.S. students attending preschool programs and an increase of 48% in national funding towards preschool programs, it would be advantageous to develop research that could directly inform early childhood mathematics curricula (Diffey, Parker, & Atchison, 2017; Sarama & Clements, 2009). Also, given the need for these curricula should be coherently aligned with elementary grade mathematics curricula initiatives, it would serve early childhood curriculum designers to bridge research programs around number development in early elementary grades to preschool grade levels. For instance, one author found in a case study that one preschool student may be using subitizing activity to construct prenumerical units (MacDonald & Wilkins, 2019). Investigating how early, perceptual actions may relate to students' actions and operations development around units construction and coordination would serve these foci. Thus, our first aim is to extend research around units construction and coordination to new grade levels.

Grade/N Seq.	CFUI or INS	ENS	GNS
First	≈ 45 Percent	≈ 10 to 15 Percent	≈ 0 to 5 Percent
Second	≈ 30 Percent	≈ 25 to 30 Percent	≈ 0 to 5 Percent
Third	≈ 5 Percent	≈ 45 to 50 Percent	≈ 0 to 10 Percent

Figure 2: Steffe's (2017) Estimated percentage of Students Capable of Counting Figurative Unit Items (CFUI), Engaging with Initial Number Sequences (INS), Engaging with Explicitly-nested Number Sequence (ENS), or Engaging with Generalized Number Sequence (GNS) p.41).

Second Aim: Widening Units Coordination Research

Our second aim is to widen units coordination research by collaborating with researchers who investigate students with learning differences, from diverse cultures, and from low-socioeconomic households to determine diverse students' mathematics learning trajectories. For instance, one author who works closely with students with learning differences in case study research uncovered three important challenges students experience as they work toward more sophisticated coordination of units. First, when engaging in tasks that support the construction of composite units, the use of memorized fact combinations or teacher taught strategies eclipsed the use of one student's natural reasoning (Hunt, MacDonald, & Silva, in press) yet supported a second's (Hunt, Silva, & Lambert, 2017). Both students evidenced initial or tacit reasoning; one reverted to pseudo-empirical abstractions, tricks, or algorithms that she could not explain to solve the tasks. Conversely, the second student leveraged his knowledge of number facts and alternative representations to advance his fractional reasoning and compensate for his perceptual motor differences (2017). For both students, teacher encouragement and support to engage in each student's own ways of reasoning was imperative. Second, Hunt & Silva (in press) found evidence that confirms previous research (Geary, 2010) that one student sometimes lost track of counting during a count on, possibly due to working memory. Hunt & Silva (in press) conjectures that this learning difference interferes with the move from counting on to more sophisticated additive reasoning due to sequential (as opposed coordinated) counting. Yet, the problem was alleviated through opportunities for within-problem reflection through experiences that the child had to construct addends involved in number problems through reprocessed figurative counting (e.g., closing the firsts to recognize ten), sweeping small numbers as lengths, and improving the usability of small composites like 2, 4, 3, 5, and 6. Hunt's research is currently conducting cross-case analysis to discern whether the students' activity is indicative of similar or unique trajectories in number and/or fractional reasoning to students without learning differences.

Specific Goals and Aims

To extend and widen units construction and coordination research this working group intends to accomplish the following: (a) delineate tasks used in various areas of units construction/coordination research/teaching, (b) critically consider student reasoning associated with necessary task features, (c) explore develop means to organize tasks and cross reference with associated study, means for perturbation, materials, etc., (d) discuss webpage revision and creation to house organized tasks, and (e) embark upon collaborations leading to reading groups, research endeavors, and funding opportunities.

Goals and Outcomes from 2018 Working Group

The Complex Connections: Reimagining Units Construction and Coordination Working Group was well attended each day at PME-NA 2018. Including the organizers, there were 16 participants attending all three sessions who also provided their email contact in order to foster continued collaborations.

Session 1: Concept Formation

GOAL: Generation of research questions that are important to the group and/or sub-groups

ACTIVITIES: Introduce focus for the working group by asking "what types of problems would members like to explore?", by viewing/discussing short video clips of students working through various mathematical concepts to better understand the students' thinking, and developing potential research foci (e.g., overall purpose/goals of this working group, ties between composite units, coordination of units, and particular mathematics content). Finally, subgroups will be developed to form research questions that can cross-domains and use questions to form collaborations based on each members' area of interest and expertise.

OUTCOMES: Participants spent a bulk of the session defining terms and discussing students' responses to tasks designed to assess units construction/coordination. Small groups foci included participants' experience in this field with teaching/research.

Session 2: Theoretical Frameworks and Methodologies

GOAL: Explore appropriate research methodologies.

ACTIVITIES: Formulate plans for research and collaboration across group members by examining a variety of methodologies. Means for these examinations would include but not be limited to the following: (a) view videos of work already conducted to highlight possible methodologies for future studies; (b) discuss other potential methodologies not highlighted during the video viewing; (c) discuss how to design robust collaborative studies. Small group would entail: (1) work already done; (2) research agenda development; Large group would entail: (1) sharing of small group discussions; (2) delineate session 3 goals

OUTCOMES: Foci shifted in the session to include task development and website organization.

Session 3: Planning and Writing

GOAL: Embark on collaborations.

ACTIVITIES: Small group will entail: (1) work on written product of research agenda; (2) develop shared conceptual framework and the relationship of our framework to what is currently being done; (3) identify target journals and outlets or grants and funding sources. Large group will entail: (1) share progress and commitments from small group discussion; (2) finalize a plan for individual groups to continue updating progress to the larger group; (3) creation of working group website or blog

OUTCOMES: Collaborative plans around theoretical questions related to particular tasks were developed including task development/organization and webpage development.

Anticipated follow-up activities

Throughout the year, the members of this working group will continue working on research problems of common interest. They will contribute to a common website in which they will update other members of the working group about the progress of the various research collaborations. In the future, this working group will propose a special issue to a leading journal in the field and/or construct a grant proposal to a nationally recognized funder.

Results from 2018 Working Group

Resulting from the 2018 working group were development of three projects (extending to Calculus students, Preservice teachers, early elementary students; widening to special education), five manuscripts, and 11 conference proposals at four (inter)national conferences. Discussions from the 2018 working group further informed nuances in manuscript and project development while also including at least two new members to research initiatives in the units construction/coordination field. The webpage has become further developed to improve organization and include more readings.

Goals and Plan for 2019 Working Group

We primarily plan to build upon the successes of the inaugural Working Group, with two exceptions. We realized during the 2018 working group that the term "unit" needed to be discussed further. This year, we include distinction between research focusing on units construction (to include pre-numerical activity) and research focused on interiorized units coordination. We also plan to spend more time in the first meeting setting up the research paradigm and inviting participants to engage in task-based units coordinating activity themselves (rather than merely watching video excerpts of students' reasoning).

Session 1: Task Organization and Constructivist Groundings

GOAL: Describe main tenets of the Constructivist paradigm; generation/organization of units coordinating tasks set in this paradigm. Introduce focus for the working group by delineating particular assumptions of constructivism:

1. Knowledge is actively created or invented by the child, not passively received from the environment,
2. Children create new mathematical knowledge by reflecting on their physical and mental actions,
3. No one true reality exists, only individual interpretations of the world,
4. Learning is a social process in which children grow into the intellectual life of those around them,
5. When a teacher demands that students use set mathematical methods, the sense-making activity of students is seriously curtailed (Clements & Battista, 2009, p. 6-7).

To draw parallels between these tenets and this working group, members will actively engage with tasks while reflecting on their actions. In particular, they will consider "how did you respond to this task?," "how might students respond to this task?" and "why are these tasks effective for teaching/research purposes?." By breaking into groups to engage with tasks, which exemplify various units construction/coordination, participants can reflect on their own actions to determine how students may develop units and coordinate units through their engagement. Finally, subgroups will come together and discuss possible relationships between student actions and task features when examining students' units construction/coordination.

Session 2: Student Reasoning Relative to Units Construction/Coordination Theories

GOAL: Connect student reasoning to units construction/coordination learning theories.

ACTIVITIES: Participants will discuss students' responses to tasks and explanations relative to appropriate learning theories: (a) documenting anticipated student responses to categorize intended outcomes for tasks; (b) connecting details for student actions to theories in units construction/coordination; (c) documenting learning theories with tasks and references. Small group would entail: (1) task/student actions organization; (2) learning theory discussion to explain importance of particular student actions; large group would entail: (1) sharing of small group discussions; (2) delineate session 3 goals.

Session 3: Webpage Expansion

GOAL: Organize tasks on the webpage. Small group will entail: (1) collaborative webpage expansion by organizing tasks and possible student responses on the webpage; (2) associate each task with intended perturbation/assessment, reference, learning theory, etc. Large group will entail: (1) share progress and commitments from small group discussion; (2) finalize a plan for individual groups to continue updating progress to the larger group; (3) further creation of working group website or blog.

Anticipated Follow-up Activities

Throughout the year, the members of this working group will continue working on research problems of common interest and develop several reading groups. They will contribute to a common website in which they will update other members of the working group about the progress of the various research collaborations and discussion from reading group ideas. In the future, this working group will propose a special issue to a leading journal in the field and/or construct a grant proposal to a nationally recognized funder.

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MATHEMATICS TEACHER EDUCATORS' EXPLORING SELF-BASED METHODOLOGIES

Elizabeth Suazo-Flores
Purdue University
esuazo@purdue.edu

Signe E. Kastberg
Purdue University
skastber@purdue.edu

Dana Cox
Miami University
dana.cox@miamioh.edu

Jennifer Ward
Kennesaw State University
jward105@kennesaw.edu

Olive Chapman
University of Calgary
chapman@ucalgary.ca

Melva Grant
Old Dominion University
mgrant@odu.edu

Historically underused methodologies in mathematics teacher education such as narrative inquiry, self-study, and autoethnography (i.e., self-based methodologies) are becoming a more frequent choice of mathematics teacher educators (MTEs). This has opened new challenges for MTEs as they try to disseminate their findings in mathematics education journals. Building from our working group at PME-NA 2018, we respond to the need for creating spaces (communities) where MTEs can feel supported in their study design, implementation, representation of findings, and publication using self-based methodologies. This year, we shift our focus from discussion to mentoring and scholarship on self-based methodologies. We invite MTEs with research projects in any stage of preparation to join us to in discussions meant to promote growth, sustainability, and continued insight into the use of self-based methodologies.

Keywords: Mathematics Teacher Educators, Research Methods, Narrative Inquiry, Self-study, Autoethnography

Significance and Historical Context of the Working Group

Mathematics teacher educators (MTEs) are responding to calls to employ research approaches less often used in mathematics education (e.g., Bullock, 2012; Stinson & Walshaw, 2017). Spaces to support scholars in the design of studies, implementation, representation of findings, and publication of findings using these approaches is needed. In this proposal, we continue our effort begun in 2018. Influenced by the work of Hamilton, Smith, and Worthington (2008), we have adopted the language of *self-based methodologies* (Chapman, Kastberg, Suazo-Flores, Cox, & Ward, 2019 in press) to refer to narrative inquiry (Clandinin & Connelly, 2000), self-study (LaBoskey, 2004), and autoethnography (Ellis & Bochner, 2000). These research methodologies focus on self-understanding based on personal professional experiences and are promoted in teacher education to enhance practice. These methodologies are often used in teacher education (e.g., Grant & Butler, 2018; Ross, 2003; Sack, 2008; Samaras & Freese, 2009) and are growing in use in mathematics education (e.g., Chapman, 2011; Chapman & Heather, 2010; Grant & Butler, 2018; Kastberg, Lischka, & Hillman, 2018a). However, MTEs interested in these methodologies tend to attend conferences or workshops outside of mathematics education that are more supportive of these ways of conducting research (e.g., Self-study of Teacher Education Practices (S-STEP) International Biennial conference, or Invisible College). Although some MTEs have been successful in conducting and publishing research under self-based methodologies (e.g., Chapman, 2011; Chapman & Heather, 2010; Grant & Butler, 2018; Kastberg et al., 2018a), incorporating "outsider" practices to the mathematics education field is not an easy task. Yet, we cannot dismiss this need as every year PME-NA accepts more studies using self-based methodologies (Kinser-Traut, 2018; Lischka, Kastberg, & Hillman, 2018; McGraw & Neihaus, 2018).

Another challenge to creating access to less used research approaches (e.g., Bullock, 2012; Stinson & Walshaw, 2017) is the dissemination of studies in peer reviewed mathematics education journals. Challenges are usually related to describing methodologies that are new to editors and reviewers. Resulting frustration might motivate MTEs to publish their research in "outsiders" journals. For instance, Ross (2003) and Sack (2008) have backgrounds in mathematics education and their narrative inquiry studies are published in teacher education journals.

We see a need to create a community of practice (Lave & Wenger, 1991) to support conducting and publishing results of studies using self-based methodologies. Some of us have experience conducting and disseminating work using self-based methodologies and being editors of well recognized journals in mathematics education. Our experiences conducting studies using these methodologies have allowed us to inquire into our practices (Chapman et al., 2019 in press), learn about ourselves (Cox, D'Ambrosio, Keiser, & Naresh, 2014; Cox & D'Ambrosio, 2015; Grant & Butler, 2018; Kastberg, Lischka, & Hillman, 2018b), and become more empathetic researchers (D'Ambrosio & Cox, 2015). The development of a community of practice would support colleagues with similar methodological interests.

History of the Working Group

The work of this group began as a desire to build a network of MTEs using self-based methodologies (Suazo-Flores, Kastberg, Ward, Cox, & Chapman, 2018). We reasoned that such a group could support scholars at different points in their professional learning and research by creating a community of practice dedicated to understanding and using self-based methodologies. We share an overview of our PMENA 2018 working group sessions (90 minutes each), that included Melva Grant, to outline the history of our working group.

Session one provided background about each of the methodologies, including terminology, and common tools used across the methodologies. Methodologies were described by outlining (1) communities of practice, (2) focus, (3) characteristics, (4) methods. Olive engaged the group in the role and use of stories in mathematics education research from an analytic and an empathetic approach.

Session two addressed the nuances of each methodology. Melva and Jennifer described their experiences conducting studies using self-study and autoethnography methodologies, respectively, and participants were invited to add their own questions or concerns to the discussion.

Session three addressed participants' lingering questions and discussions of possible physical or virtual spaces to continue our collaboration.

PME-NA 2018 working group participants had a variety of questions and interests related to the self-based methodologies. Some attendees were interested in learning about theoretical underpinnings and differences among self-based methodologies: How do I align these [methodologies] with a theory/positionality? What is the difference between autoethnography and ethnography? How do we use these [methodologies] to help/support teachers (prospective/in-service) capture growth over time? What are the connections between the methodologies? Is there danger in too much blending between these methodologies? What is the meaning of "truthful" within the methodology of autoethnography? Truthful –relative to what and to whom? Others wanted to know about methods that can be used under these methodologies: What are effective prompts, tasks, artifacts, contexts for engaging adolescents in narrative inquiry/autoethnography? How do you find a critical friend? Finally, attendees were also interested in learning about dissemination of studies conducted using these methodologies: Who is publishing? How is this research valued? Can I be promoted and tenured using these methodologies to inform my research? How do I prove "quality/significance/impact"? How are the conclusions/discussions different with these methodologies?

Progress Since 2018 Working Group

When we presented our working group proposal for PME-NA 2018, our intention was to gauge mathematics education researchers' interest in self-based methodologies. This goal informed the decision to structure the 2018 working group sessions to provide information about each methodology and share our experiences conducting studies. For the first two days of the 2018 working group we had approximately 20 attendees. We created an email list and during the year have maintained communication. Many of our working group participants are early career scholars with a passion for learning about self-based methodologies and a desire to gain experience conducting such studies. Therefore, we see a need for building a community of practice for scholars in all stages of career to be academically supported. Over the last year, we have worked to build a network of scholars. Central to our role as organizers and facilitators of such a network is our continued communication about and use of self-based methodologies in the research community. One vehicle for such communication is our co-authored book chapter in the *International Handbook of Mathematics Teacher Education* (Chapman et al., 2019 in press).

Each author has continued present and publish using self-based methodologies. Kastberg, Lischka, and Hillman (2018a) explored MTEs' use of written feedback and MTEs' use of questioning (2018b). In addition, the trio presented findings from their use of self-study methodology to explore MTE questioning at Association of Mathematics Teacher Educators (AMTE) 2019 conference.

Dana has continued to explore the role of empathy in her teaching and scholarship. Empathy for others in that work is supported by the use of narrative inquiry as a methodology. This positions her participants, be they teachers or students, as partners and knowledgeable people from whom she learns, and together construct new understandings of mathematics and its teaching (Cox & D'Ambrosio, 2015). She has used narrative inquiry as a methodology to explore and communicate experiences working with her students on problem solving and posing in technological environments, and mathematical design (Cox & Harper, 2017). After engaging with this working group in 2018, she was inspired to write a personal narrative about her professional trajectory (Cox, 2019) that revisits an earlier call for empathetic methodologies (D'Ambrosio & Cox, 2015), but also conceptualizes *rough draft spaces* (Jansen, Cooper, Vascellaro, & Wandless, 2017) in mathematics teacher education.

Elizabeth continued building from her dissertation study using narrative inquiry and self-study methodologies. A manuscript narrating an eighth-grade mathematics teacher's personal practical knowledge (Elbaz, 1981, 1983) was submitted to *Journal for Mathematics Teacher Education*. Another manuscript describing her experiences working with an eighth-grade mathematics teacher was accepted for presentation at American Educational Research Association 2019 annual meeting. Elizabeth lead a group of graduate students and postdoctoral fellows to explore their experiences conducting narrative inquiry studies in mathematics education (Suazo-Flores, Kersey, Bloome, Richardson, & Burdick, 2018). Signe, Elizabeth and a colleague presented findings from a study of MTEs' motivations for using mathematics autobiographies (Drake, 2006) in mathematics courses at AMTE 2019 and have a manuscript focused on unpacking these motivations (Kastberg, Suazo-Flores, & Richardson, 2019).

Following PME-NA 2018, Jennifer has been using her experiences teaching mathematics for social justice with young children to explore how this informs her work with prospective teachers in mathematics methods focused on grades K-2. A current focus has been reflecting on her scaffolding of prospective teachers to engage in equity based mathematics teaching in their field placements. Jennifer has been disseminating her dissertation research focused on experience and the lessons she implemented including sharing her findings at 2019 National Council for Teachers of Mathematics conference and submitting proposals for 2019 National Association for the Education of Young Children conference. She has built from her 2017 (Ward, 2017) work examining her experience in teaching mathematics for social justice from a critical lens. This involves her looking at the times she missed opportunities to engage in conversations with her students during the study; revisiting these moments to reflect on what could have been done to enhance conversations around social justice.

Olive has used narrative inquiry as a research methodology in studies of her teaching and mathematics teachers, a research tool to obtain phenomenological data in studies with mathematics teachers, and an approach to in-service and prospective mathematics teachers' learning. With the goal of understanding mathematics teachers from a holistic context that considers their perspectives, Olive has been communicating mathematics teachers' personal meanings (Chapman, 1994; Chapman, 1997) and teachers' agency on their practice change (Chapman & Heater, 2010; Chapman, 2011; Chapman, 2013). Olive has also used narratives as a research tool and pedagogical tool. When working with prospective teachers (Chapman, 2008a), Olive has asked them to write and share narratives of teaching mathematics and re-write them later in the course. This pedagogical tool has triggered prospective teachers' reflection of mathematics teaching practices. Olive's research and teaching experience using narrative inquiry and narratives to approach teachers in different stages of their career had brought her to lead one of the PME 2017 plenary session, chapters in handbooks (e.g., Chapman, 2008b; Chapman, 2009), and discussion groups at 12th International Congress on Mathematical Education, PME, and ICME (Beswick & Chapman, 2015; Beswick, Chapman, Goos, & Zaslavsky, 2012; Beswick & Chapman, 2013) that focused directly or indirectly on mathematics teacher educator knowledge and learning that included narratives and self-study. Olive recently gave a presentation on self-based methodologies in mathematics teacher educators' learning at the International Symposium on Mathematics Teacher Education in UK (Chapman, 2019) and has a paper in press on mathematics teacher educators' use of narrative in research, learning and teaching (2019, in press).

We invited Melva to join our working group as a presenter and organizer in 2018. Signe became familiar with Melva's work at the Self-study of Teacher Education Practices (S-STEP) International Biennial 2018 conference. Melva has brought together her Black feminist and post-structural academic formation with self-based methodologies in mathematics education and higher education. Professional experiences in higher education led to an auto-ethnography study where she explored the construct of epistemic oppression (Grant, 2019 in press). As an entry into self-study research, Melva explored teacher educators' motivations to conduct self-study as she learned and practiced the method collaboratively (Grant & Butler, 2018). She is currently engaged in self-study work with Signe and others to explore ways to improve elementary teacher preparation using instructional technology. The technology afforded her online mathematics method students opportunities to teach within a virtual environment and includes a small group of student avatars (i.e., virtual students). Melva is trying to determine ways to leverage the instructional technology to improve prospective teachers' capacity to facilitate productive math talk for developing student's mathematics understanding (Lamberg, 2013). In 2019, she plans to share findings through conferences and journal publications related to: (a) developing the instructional technology for improved teacher preparation; (b) prospective teachers' learning and teaching; and (c) professional learning emerging from her own practice. This line of inquiry started in spring 2018 and has spanned several elementary mathematics methods courses taught in both online and face to face settings. Melva plans to continue this self-study research until she is satisfied that her elementary method students' learning is not hindered because her teaching is maximally effective.

Energizing MTEs has been identified as critical to sustaining the field of mathematics teacher education (e.g., Whitcomb, Liston, & Borko, 2009). Our individual and collective work conducting studies under self-based methodologies have created such feelings of energy. Wilson (2006) described, being a "teacher educator-researcher requires understanding the practice of teacher education and the practice of teacher education research" (p. 316). Yet, defining a teaching and research path involves negotiating internal passions with external factors such as institutional structures, and funding policies. This could constrain MTEs' practices making them feel that academic life is unsustainable. We argue that this working group could be the beginning of a community of practice where MTEs interested in conducting studies using self-based methodologies feel sustained and empowered in their professional practices (Jaworski & Wood, 2008). Below is our plan for the 2019 working group.

Outline of Working Group Session

Our focus on participant discussion in our first year as a working group enabled us to build an emerging community of practice around the foundation of self-based methodologies (Suazo-Flores et al., 2018). Given the personal nature of that work, it was paramount that we create an atmosphere of trust and care. Extensive time was spent on setting norms, getting to know one another and our research foci, but also getting to know the methodologies and setting goals, both individual and collaborative, for our group going forward.

In 2019, we shift our focus from discussion to mentoring and new scholarship. We welcome new and returning members to our working group, thus we will begin again by building norms and (re)establishing an atmosphere of trust and care. The remainder of our work together will be oriented around project and manuscript review teams, with the long-term goal to start outlining a potential book. Working group participants are encouraged to bring works-in-progress to the group for feedback. Each day will focus on different phases of the work: Day One will focus on projects in progress, Day Two will focus on beginning projects, and Day Three will be devoted to discussing the year-long book project.

Beginning with projects in process on Day One is an intentional choice. By starting with projects with a defined methodology and whose research questions or goals are identified, we set the stage for productive mentoring and feedback cycles for beginning projects on Day Two. Sharing the origins of scholarship with others can be daunting and these ideas are the most fragile. Beginning with established projects may 1) assure new authors that their ideas will be handled gently, 2) give new authors a chance to hear the ways these methodologies shape potential studies, and 3) give us a chance, as a community, to review the tenets of each methodology in an educative way.

Day Two will start with a progress update from each of the authors who shared work on Day One. These project updates will include where projects are going next and action(s) scholars hope to take in 2019-2020. The group will take time to consider and adjust the ways we provide (as well as the nature of) feedback to respond to the needs of participants. The remaining time will be devoted to projects that are in initial or introductory stages. These projects could include those that are still being conceptualized, those where data is still being collected, or those that still feel fragile to authors. Encouraging participants to share their work publically within our community of practice may help crystalize or generate ideas. In a whole-group setting, we will support participants as they give an overview of their current work/idea and invite us to give feedback on data sets, research questions, relevant literature, or early writing. Feedback can and should focus on specific areas identified by the author, so as to respect the fragility of these ideas and give the author agency to ask for what they need.

Discussion of a year-long project designed to support scholars in different career stages and to disseminate findings from studies using self-based methodologies will be a focus of Day 3. Building on the ideas shared in the previous two days, we will identify themes that will comprise the table of contents of a book. Before the conference, we solicited manuscripts from those who attended our 2018 working group. We have maintained contact with those participants and have continued to develop our community including scholars who expressed an interest in joining us following PME-NA 2018. Manuscripts will also be shared with the members of our email list in advance of the conference and distributed to mini review groups (2-4 people). 2019 PME-NA working group participants can also propose studies or share manuscripts designed under self-based methodologies to be included in the potential book.

Conclusion

As MTEs use self-based methodologies, spaces (communities) are needed to support their practices including study design, implementation, representation of findings, and publications in mathematics education journals. This working group intends to be such a space (community) where over time MTEs feel sustained in their use of self-based methodologies.

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ADDRESSING EQUITY AND DIVERSITY ISSUES AROUND EXPANSION, DISPLACEMENT, AND GROWTH IN MATHEMATICS EDUCATION

Juanita Silva
Texas State University
jsilva@txstate.edu

Marrielle Myers
Kennesaw State University
mmyers22@kennesaw.edu

Ricardo Martinez
Iowa State University
ricardom@iastate.edu

Jennifer M. Langer-Osuna
Stanford University
jmlo@stanford.edu

Joel Amidon
University of Mississippi
jcamidon@olemiss.edu

Embracing the theme of this year's meeting, we seek as a community to consider the role of expansion, displacement, and growth in mathematics education in the privileging of some and marginalizing of others. Following on the topics discussed at the Working Group between 2009-2018, this year the focus is on balancing the need to reflect and collect around issues of equity and diversity in mathematics education and orienting toward action. Each session is designed to address the needs of (or to create opportunities for) attendees interested in equity, generating and brainstorming new subtopics, potential projects, and/or working to establish standalone working groups dedicated to furthering research on equity. The purpose being to encourage a move away from "big-tent" equity thinking and toward more productive working collectives.

Keywords: Equity and Diversity

Brief History

This Working Group originates from the Diversity in Mathematics Education (DiME) Group, one of the Centers for Learning and Teaching (CLT) funded by the National Science Foundation (NSF). DiME scholars graduated from one of three major universities (University of Wisconsin-Madison, University of California-Berkeley, and UCLA) that comprised the DiME Center. The Center was dedicated to creating a community of scholars poised to address critical problems facing mathematics education, specifically with respect to issues of equity (or, more accurately, issues of inequity).The DiME Group (as well as subsets of that group) has engaged in important scholarly activities, including the publication of a chapter in the *Handbook of Research on Mathematics Teaching and Learning* which examined issues of culture, race, and power in mathematics education (DiME Group, 2007), a one-day AERA Professional Development session examining equity and diversity issues in mathematics education (2008), a book on research of professional development that attends to both equity and mathematics issues with chapters by many DiME members and other scholars (Foote, 2010), and a book on teaching mathematics for social justice (Wager & Stinson, 2012) that also included contributions from several DiME members. In addition, several DiME members have published manuscripts in a myriad of leading mathematics education journals on equity in mathematics education. This working group provides a space for continued collaboration among DiME members and other colleagues interested in addressing the critical problems facing mathematics education.

It is important to acknowledge some of the people whose work in the field of diversity and equity in mathematics education has been important to our work. Over time, the Working Group has encouraged building on and featuring senior scholars' work, including Jo Boaler (Boaler, 2002), Walter Secada (Secada, 1992), Marta Civil (Civil, 2007; Civil & Bernier, 2006; González, Andrade, Civil, & Moll, 2001), Eric Gutstein (Gutstein, 2003, 2006; Gutstein & Peterson, 2013), Jacqueline Leonard (Leonard, 2007; Leonard & Martin, 2013), Danny Martin (Martin, 2000, 2009, 2013), Judit Moschkovitch (Moschkovich, 2002), Rochelle Gutiérrez (2002, 2003, 2008, 2012, 2013) and Na'ilah Nasir (Nasir, 2002, 2011, 2013; Nasir, Hand & Taylor, 2008; Nasir & Shah, 2011). We have as well been building on the work of our advisors, Tom Carpenter (Carpenter, Fennema, & Franke, 1996), Geoff Saxe (Saxe, 2002), Alan Schoenfeld (Schoenfeld, 2002), Megan Franke (Kazemi & Franke, 2004), and Anita Wager, (Wager & Stinson, 2012) as well as many others outside of the field of mathematics education.

Previous iterations of this Working Group at PMENA 2009 – 2013, and 2015-2016 have provided opportunities for participants to continue working together as well as to expand the group to include other interested scholars with similar research interests. Experience has shown that collaboration is a critical component to this work. These efforts to expand participation and collaboration were well received; more than 40 scholars from a wide variety of universities and other educational organizations took part in the Working Group each of the past six years. Starting in 2017, an effort was made to “reset” the group toward providing opportunities for a new generation of scholars whose work intersects with issues of equity/inequity, diversity/inclusion, privilege/oppression, and justice in mathematics education research, practice, and development.

Focal Issues

Under the umbrella of attending to equity and diversity issues in mathematics education, researchers are currently focusing on such issues as teaching and classroom interactions, professional development, prospective teacher education (primarily in mathematics methods classes), factors impacting student learning (including the learning of particular sub-groups of students such as African American students or English learners), and parent/family/community perspectives. Much of the work attempts to contextualize the teaching and learning of mathematics within the local contexts in which it happens, as well as to examine the structures within which this teaching and learning occurs (e.g. large urban, suburban, or rural districts; under-resourced or well-resourced schools; and high-stakes testing environments). How the greater contexts and policies at the national, state, and district level impact the teaching and learning of mathematics at specific local sites is an important issue, as is how issues of culture, race, and power intersect with issues of student achievement and learning in mathematics. There continues to be too great a divide between research on mathematics teaching and learning and concerns for equity.

The Working Group has begun and will continue to focus on analyzing what counts as mathematics learning, in whose eyes (and for whose benefit), and how these culturally bound distinctions afford and constrain opportunities for traditionally marginalized students to have access to mathematical trajectories in school and beyond. Further, asking questions about systemic inequities leads to methodologies that allow the researcher to look at multiple levels simultaneously. This research begins to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.

Some of the research questions the Working Group will continue to consider are:

- What are the characteristics, dispositions, etc. of successful mathematics teachers for *all* students across a variety of local contexts and schools? How do they convey a sense of purpose for learning mathematical content through their instruction?
- How do beginning mathematics teachers perceive and negotiate the multiple challenges of the school context? How do they talk about the challenges and supports for their work in achieving equitable mathematics pedagogy?
- What impediments do teachers face in teaching mathematics for understanding?
- How can mathematics teachers learn to teach mathematics with a culturally sustaining approach?
- What does teaching mathematics for social justice look like in a variety of local contexts?
- What are the complexities inherent in teacher learning about equity pedagogy when students come from a variety of cultural and/or linguistic backgrounds all of which may differ from the teacher's background?
- What are dominant discourses of mathematics teachers?
- What ways do we have (or can we develop) of measuring equitable mathematics instruction?
- How do students' out-of-school experiences influence their learning of school mathematics?
- What is the role of perceived/historical opportunity on student participation in mathematics?
- Whose mathematics is accepted? (displaced?) across a variety of local contexts and schools?
- How are students' out-of-school experiences valorized in their learning of everyday mathematics?
- How do we make space for this work to continue with open opposition to critical approaches to mathematics education?

We believe the continued support of this working group provides a partial response to the last question. Specific to the intent of this year's Working Group, we will organize around questions like the ones above in order to create specific, targeted working groups that are charged to address and act around such questions.

Plan for Working Group

Based on feedback from the previous year and the emergence of new working groups related broadly to "equity," this working group has shifted toward a renewed focus on facilitating "collaboration within the growing community of scholars and practitioners concerned with understanding and addressing the challenges of attending to issues of equity and diversity in mathematics education." We have reconfigured the working group toward being a catalyst for new spaces instead of a "destination" for the inclusion of equity discourse within the PME-NA organization. To put it differently, our vision for the working group is to bring together attendees toward developing their own agendas and specific working groups related to equity-oriented themes--or toward themes that push the field beyond traditional equity discourses yet adhere to the needs and challenges of inequity within mathematics education.

Our plans for PMENA 2019 will proceed as follows. Each session will build on previous sessions, beginning with a facilitated conversation around the previously stated purpose of the working group. The format for the sessions will include:

- DAY 1: Opening up a day of reflection: Welcome, Introductions, Norm Setting. We will then ask each attendee to reflect on their experiences and write out examples of displacement, expansion, and growth in mathematics education that privileged some and marginalized others. Session attendees can generate examples from any time period. To encourage a wide variety of examples, we will ask attendees to think about these experiences as they relate to K-12 classrooms, with preservice teachers (methods or content courses), in student teaching, in undergraduate mathematics courses, in professional development settings, in the media, etc. From there, we will devote time to trying to categorize all of the examples (written on sticky notes) by themes to identify what types of categories emerges from this activity. We will conclude Day 1 by engaging in a whole group discussion where we will operationalize the ways in which mathematics has been used as tool to displace and marginalize. After operationalizing these examples, we will generate guiding questions to be used throughout the conference. As a collective, we will use (and ask) these questions at plenary sessions, individual presentations, etc to ensure that the conference theme remains at the forefront for all attendees.
- DAY 2: Finding common ground: Developing problem trees. Utilizing emerging themes from day 1, we will engage participants in an activity problematizing issues in mathematics education. The activity, a problem tree (Cammarota, Berta-Avila, Ayala, Rivera & Rodriguez, 2016) is used to distinguish between superficial symptoms and underlying root causes of societal issues, towards the development of a research question. Participants will work in small groups to create their own problem trees, generating a research question that will be presented to the whole group further enriching the collective discussion on issues of equity and diversity in mathematics education and resulting in potential collaborations to be built on in Day 3.
- DAY 3: Working working groups: Newly established subgroups will “take flight” and initiate plans to support their chosen topics/research questions through continued collaboration.

Previous Work of the Group

The Working Group met for productive sections since 2009. In 2009, participants identified areas of interest within the broad area of equity and diversity issues in mathematics education. Much fruitful discussion was had as areas were identified and examined. Subgroups met to consider potential collaborative efforts and provide support. Within these sub-groups, rich conversations ensued regarding theoretical and practical considerations of the topics. In addition, graduate students had the opportunity to share research plans and get feedback. The following were topics covered in the subgroups:

- Teacher Education that Frames Mathematics Education as a Social and Political Activity
- Culturally Relevant and Responsive Mathematics Education
- Creating Observation Protocols around Instructional Practices

- Language and Discourse Group: Issues around Supporting Mathematical Discourse in Linguistically Diverse Classrooms
- A Critical Examination of Student Experiences

As part of the work of these subgroups, scholars have been able to develop networks of colleagues with whom they have been able to collaborate on research, manuscripts and conference presentations.

As a result of the growing understanding of the interests of participants (with regard both to the time spent in the working group and to intersections with their research), we began to include focus topics for whole group discussion and consideration and continued to provide space for people to share their own questions, concerns, and struggles. With respect to the latter, participants have continually expressed their need for a space to talk about these issues with others facing similar dilemmas, often because they do not have colleagues at their institutions doing such work or, worse yet, because they are oppressed or marginalized for the work they are doing. These concerns, in part, informed the focus topics for whole group discussion and consideration. For example, in 2009 research protocols (e.g., protocols for classroom observation, video analysis and interviewing) were shared to foster discussions of possible cross-site collaboration. In 2012, the Working Group explicitly took up marginalization in the field of mathematics education with a discussion about the negotiation of equity language often necessary for getting published; this was done in the context of the 'Where's the mathematics in mathematics education' debate (see Heid, 2010; Martin, Gholson, & Leonard, 2010). Dr. Amy Parks was invited to join Working Group organizers to share reflections on their experiences. In 2013 the Working Group hosted its first panel in which scholars (Dr. Beatriz D'Ambrosio, Dr. Corey Drake, Dr. Danny Martin) shared their perspectives on the state of and new directions for mathematics education research with an equity focus. In recent years, we have had collections around topics that have resulted in several proposals for sessions and working groups for PMENA.

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MODELS AND MODELING WORKING GROUP

Corey Brady
Vanderbilt University
corey.brady@vanderbilt.edu

Jeffrey McLean
Univ. North Carolina Chapel Hill
mclean@unc.edu

Hyunyi Jung
Marquette University
hyunyi.jung@marquette.edu

Angeles Dominguez
Tecnológico de Monterrey
angeles.dominguez@itesm.mx

Courtney Baker
George Mason University
cbaker@gmu.edu

Aran Glancy
University of St Thomas
aran.glancy@gmail.com

The Models and Modeling Working Group at PME-NA has provided a forum for discussing and collaborating on projects fundamental to research on mathematical modeling since the first PME-NA conference in 1978. We propose to convene this Working Group at PME-NA 41 with a dual purpose: (1) to build on a theme begun in Greenville, holding a focused workshop to pursue innovations in activity design to connect modeling activities to mathematically rich, socially-engaged inquiry into questions from the world outside of school; and (2) to continue to invite newcomers to the Models and Modeling Perspective (MMP), giving them an introduction to this design research tradition.

Keywords: Modeling, Problem Solving, Design-Based Research, Design Experiments

The Models and Modeling Working Group was initiated in the inaugural year of the PME-NA conference in 1978, and it has met frequently since then. Over the 41 years of its existence, the working-group format has offered a vehicle for coordinating substantial research efforts and for fostering collaborations and mentoring relationships within the “Models and Modeling Perspective” (MMP). The MMP takes a pragmatic approach to fundamental questions in mathematics education, such as “What ... beyond having a mathematical idea ... enables students to *use it* in everyday problem-solving situations?” (Lesh, Landau, & Hamilton, 1983, qtd. in Lesh & Doerr, 2003, p. *i*., emphasis added). Such questions foster interdisciplinary collaborations connecting a broad range of theoretical perspectives.

A function of the Working Group has been to pursue innovations in design-based research (cf, Kelly, Lesh, & Baek, 2008) – to discuss and extend the ways in which a focus on models and modeling can be used both to support learning in STEM, and to study such learning processes in action. Indeed, calls for STEM integration (English, 2016; English & King, 2015) and for attending to Engineering perspectives (Diefes-Dux, Moore, Zawojewski, Imbrie, & Follman, 2008; Roehrig, Moore, Wang, & Park, 2014) have only increased the potential of this long-standing tradition to contribute to efforts to innovate in teaching and research.

This year, we propose to convene the working group to pursue an area of innovation in activity design for STEM integration. We aim to connect a signature *genre* of mathematical modeling activity created by the MMP, Model-Eliciting Activities (MEAs), with real-world modeling contexts that engage with important broader societal questions. In particular, we will workshop a particular MEA, refining it (and/or creating a cluster of thematically related MEAs), and connecting it with (a) related Citizen Science projects; (b) resources and information housed in centers for informal learning (e.g., zoos and museums); and (c) bearing on deep issues of social and ecological justice.

History of the Working Group

Early in its history, the Group focused heavily on the design and analysis of particular, self-contained activities that enabled groups of learners to engage realistic and deep forms of modeling and that produced an auditable trail of thinking, exposing their thought processes to teacher and researcher observers. Thus, this initial line of research pursued modeling activities as research instruments, analogous to group-level versions of Piaget's interview tasks. In this phase of the field's development, a primary effort involved elaborating design principles for these and documenting images of *idea development* that they promoted.

Quickly, these activities were recognized for their potential as powerful learning environments, yet this "turn" raised questions about how different student groups' work on open modeling problems could be "processed" by the whole class, bringing out common themes and connecting them to more conventional mathematical terminology, algorithms, and procedures. Gradually over time, researchers associated with the Group expanded their perspectives to consider implementations and curricular sequences that had longer time-duration, and that integrated models and modeling into the experience of learning mathematics in more extensive ways.

Several broad patterns in this more extensive and disseminated approach to modeling in the curriculum have emerged, and there is no sense that the MMP has yet exhausted the space of possibilities. These broader perspectives open both exciting opportunities and significant challenges. On the one hand, new questions can be researched, opening the way for new forms of contact and interaction with classroom practice and with learning outside of school; on the other, the approach raises new challenges at the level of methodology, data analysis, and forms of evidence that are convincing backings for claims about learner activity.

We propose convening the Group at PME-NA 41 to continue a style of work that has characterized the Group's collaborations over the past several years. In particular, based on our experiences in 2015-2018, we propose a work-session structure that can serve two dual purposes: (a) making substantive progress in work on a particular cluster of Model-Eliciting Activities, exploring how to use activities in the MEA genre to spark student connections with citizen science and ecological conservation; while also (b) integrating newcomers to Models and Modeling as a research area.

For this Working Group, these two goals are ***both*** essential: we propose to gather, not as a closed expert group, but as a broad group of educators and researchers. To welcome newcomers and new perspectives into the group, the structure we are proposing for the Working Group will provide initial introductions to the approaches and characteristics of the MMP and the activity designs the tradition has developed, but we also plan to engage both newcomers and returning participants in the work of continuing the refine the MEA structure in general and the Pelican MEA in particular to address issues of equity, sustainability, and environmental stewardship, which represent urgent opportunities and problems of research and practice in mathematics education.

In the following sections, we provide a very brief overview of the field of research represented by the Models and Modeling Perspective; we outline patterns in research efforts that have extended modeling activities over longer timescales; and we describe our plan of work in detail, illustrating how these goals are addressed as well as how we plan to productively integrate newcomers to the Group over the three working sessions offered in the Conference.

The Models and Modeling Perspective (MMP)

Since the 1970s, MMP researchers and educators have engaged in design research directed at understanding the development of mathematical ideas among groups of learners. A key principle behind this work has been that learners' ideas develop through, and in relation to conceptual entities called *models*. The core construct of a "model" and the activity of "modeling" are both central to the MMP; and they are also multidimensional, playing multiple roles in the MMP theory.

Models & Modeling Working Group founder Dick Lesh and Helen Doerr provide the following working definition of *models*:

> conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)—perhaps so that the other system can be manipulated or predicted intelligently (Lesh & Doerr, 2003, p. 10)

In this spirit, "being a good modeler is in large part a matter of having a number of fruitful models in one's 'hip pocket.'" (Lesh, 1995, personal communication, qtd in Lehrer & Schauble, 2000).

But the term "model" not only applies to features of "target knowledge" that are built through learning: the interpretive systems that people *bring* to problems are also "models." When explicitly expressed through representational media, such models—personal interpretive systems—can provide illumination into how students, teachers, and researchers adapt, formulate, and apply relevant mathematical concepts in particular situations or contexts (Lesh, Doerr, Carmona, & Hjalmarson, 2003). In fact, "models" are the shape and vestment of most all knowledge—whether this knowledge appears as the patterns of perspectives and pre-conceptions that learners bring with them "in the door"; the shared ways of thinking that a group of learners build in solving a problem; or the systematic accounts of phenomena that represent the normative views of a scientific discipline at a given moment in its history.

An early finding of research in the Models and Modeling Perspective (MMP) was that, under appropriate conditions, groups of learners can be supported in producing external representations of the models they bring to a situation, and that when these groups put their initial models into conversation with one another and in contact with the real world, they can be supported to revise, and refine them in rapid and iterative cycles, building toward a more robust model that reflects their achievement of a shared way of thinking. In particular, when individuals and groups encounter problem situations with specifications that demand a model-rich response, their models can be observed to grow through such relatively rapid cycles of development toward solutions that satisfy these specifications.

While the elicitation of initial ways of thinking is valuable, MMP researchers' interest quickly turned to this process of model refinement—that is, to the dynamics of model***ing*** (as opposed to the statics of mod***els***), and to the features of activity environments that foster modeling and make it visible for teachers and researchers. The dynamics of modeling represent, for the MMP, an account of *idea development*, as observed in the discourse and other representations produced by groups of learners as they iteratively work to mathematize and formulate a solution that meets the needs of a concrete *client* in a realistic setting.

Thus, the MMP tradition became focused squarely on local conceptual development (Lesh & Harel, 2003): that is, on investigating the micro-evolution of ideas in groups of students (and teachers). The resources and tools its researchers produced were first and foremost designed to study idea development and the range of possibility for this mode of learning activity. The results of this work include a body of *Model-Eliciting Activities* (MEAs), in which students are presented with authentic, real-world situations where they repeatedly express, test, and refine or revise their current ways of thinking as they endeavor to generate a structurally significant product—that is, again, a *model*, comprising conceptual structures for solving the given problem. These activities differ markedly from some other environments dedicated to applications of specific mathematical concepts and procedures. In contrast, MEAs give students the opportunity to create and adapt mathematical models in order to interpret, explain, and/or predict the behavior of real-world systems (Zawojewski, 2013). An extensive body of MMP research has produced accounts of learning in these MEA environments (Lesh, Hoover, Hole, Kelly, & Post 2000; Lesh & Doerr, 2003), design principles to guide MEA development (Hjalmarson & Lesh, 2007; Doerr & English, 2006; Lesh, et. al., 2000; Lesh, Hoover, & Kelly, 1992) and accounts and reflections on the design process of MEAs (Zawojewski, Hjalmarson, Bowman, & Lesh, 2008).

As an example of the activity-type of Model-Eliciting Activities (MEAs), consider the Pelican Colony problem (Moore et al, 2015; Pompei, 2010) (below, and **Figure 1**). In this problem, the client is Alice Heart, a Wildlife Biologist from the U. S. Fish and Wildlife Service. The problem provokes students to grapple with notions of area, and for many (though by no means all) groups it prompts them to invent a measure analogous to “nest density” in the context of a pelican conservation effort to arrive at a procedure that the client can use.

An excerpt from the client letter, which includes the “call to action” for students, is given below:

> The U.S. Fish and Wildlife Service needs a procedure to estimate the number of nests at each pelican colony. Because pelicans are very sensitive to disturbances while they are incubating their eggs, we are not able to physically walk through every colony and count nests (this would also take too much time and cost too much!). We have hired pilots to fly our biologists over nesting colonies so they can take aerial photographs of the sites. As pelican colonies can be quite large (hundreds or thousands of nests), each photograph shows only a portion of the entire site. We have maps based on satellite images that are taken annually, which show us the shape and size of each colony site. We are enlisting your team’s help to create a procedure that will allow us to estimate the number of nests in a pelican colony, based on the photograph that shows a sample of the colony, and a map that shows the size and shape of the entire site.

Students are provided with aerial photographs and colony outlines (Figure 1) for two sample pelican colonies. The client would like to be able to determine the size of a colony (in terms of the number of nests) from these pieces of data, and students are tasked with developing a procedure to 1) compare the sizes of different nests and 2) estimate the size of a nest based on an aerial photograph and an outline of the colony.

Student groups iteratively develop solutions to this problem in the time allotted—usually 50-60 minutes for this MEA. Afterwards, groups may present their work for a discussion of the modeling possibilities of these presentations (see Brady & Jung, 2019a; b, for suggestions of the value of such presentations); the students solutions can be used as a springboard for a class discussion following the five-practices structure described by Stein, Engle, Smith, and Hughes (2008); or the teacher may organize a “poster session” for the groups to share and learn from each others’ solutions (Lesh, 2010). In one version of this activity structure, one member of each group hosts a presentation of the poster showing their results. The other students circulate to learn about other groups' solutions, using a Quality Assurance Guide to assess the results produced by others in the class. These instruments are submitted to the teacher and contribute to assessment in various ways, providing evidence for the achievements of both individuals and groups.

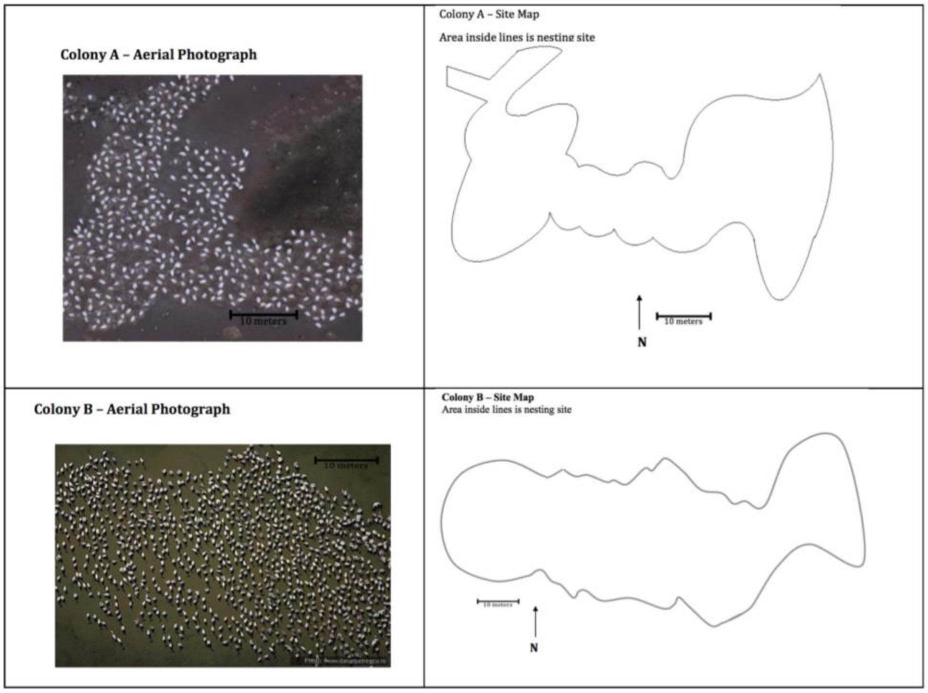

Figure 1: Two Pelican Colonies. Outlines, and Aerial Photographs.

MEAs like the Pelican problem present learners with situations where familiar procedures, ways-of-operating, and constructs may be *applicable*, but where they are also *insufficient*. In this sense, they support a learning environment that has a "low threshold" but a "high ceiling" (cf Papert, 1980). That is, on the one hand they are accessible to learners from a wide range of levels of ability, experiences, or knowledge (from upper elementary school through graduate school). On the other hand, learners encountering these problems find that they have no ready-made solution they can apply to address the client's needs. As a result, groups learners must engage in sense-making and solution-construction processes that position them as mathematical creators and also put them off balance in comparison to typical school-mathematics tasks. Indeed, this uncertainty is part of the design of MEAs, illuminating fundamental conceptual issues associated with the core mathematical structures involved.

MEA Design Principles

As individual MEAs emerged, an intense period of design research ensued to understand them as a genre of learning tasks that could (a) stimulate mathematical thinking representative of that which occurs in contexts outside of artificial school settings (Lesh, Caylor, & Gupta, 2007; Lesh & Caylor, 2007); (b) enable the growth of productive solutions through rapid modeling cycles; and (c) leave behind "auditable trails" - researchable traces of learners' ways of thinking during the process (Kelly & Lesh, 2000; English et al., 2008; Kelly, Lesh & Baek, 2008). The

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success of MEAs as an activity genre and as research tools was both enabled by and illustrated by the MMP's articulation of a set of six design principles (Lesh & Harel, 2003; Lesh et al., 2000; Hjalmarson & Lesh, 2007). These principles indicate essential elements of MEAs and their classroom implementations, enabling them to serve as rich contexts for student problem solving. Table 1, below, also indicates "touchstone" tests for whether each of these six principles has been realized in a given implementation setting.

Table 1: Six Design Principles for MEAs and Touchstones (see also Brady et al, 2017)

Principle	Touchstone Test for its Presence
Reality Principle	Students are able to make sense of the task and perceive it as meaningful, based on their own real-life experiences.
Model Construction Principle	To solve the problem, students must articulate an explicit and definite conceptual system (model).
Self-Evaluation Principle	Students are able to judge the adequacy of their in-process solution on their own, without recourse to the teacher or other "authority figure".
Model Generalizability Principle	Students' solutions are applicable to a whole range of problems, similar to the particular situation faced by the "client" in the MEA.
Model-Documentation Principle	Students generate external representations of their thinking during the problem-solving process.
Simplest Prototype Principle	The problem serves as a memorable representative of a kind of mathematical structure, which can be invoked by groups and by individuals in future problem solving.

Nested Levels of Modeling: Multi-Tiered Design Research

In parallel with learner-focused research using MEAs, researchers also have observed that *teachers'* efforts to understand their students' thinking involve yet another process of modeling: In this case, teachers engage in building models of student understanding. Although these teacher-level models are of a different category from student-level models, students' work while engaged in MEAs does provide a particularly rich context for teachers' modeling processes. Following this line of inquiry, the MMP community has also produced tools and frameworks that can be useful to teachers in making full use of MEAs in classroom settings, while also providing *researchers* with insights into *teachers'* thinking.

Finally, at a third level of inquiry, researchers' own understandings of the actions and interactions in *curricular activity systems* (Roschelle, Knudsen, & Hegedus, 2010) involving students, teachers, and other participants in the educational process can also be studied through the lens of model development. Multi-tier design experiments in the MMP tradition have done precisely this, involving researcher teams in self-reflection and iterative development as well (Lesh, 2002). Therefore, the MMP version of multi-tier design research can involve at least three levels of investigators— students, teachers, and researchers—all of whom are engaged in developing models that can be used to describe, explain, and evaluate their own situations, including real-life contexts, students' modeling activities, and teachers' and students' modeling behaviors, respectively. The situation can be further enriched by considering other educational stakeholders and learning settings, such as interactions between academic coaches and teachers (Baker & Galanti, 2017), and between schools and community organizations, and between students and parents.

Constructing Curricular Materials to Support Modeling at Larger Timescales

Over the past 15 years, MMP researchers have continued this direction of work in their own teaching and in partnerships with K-12 classroom teachers. Within the domain of statistical thinking in particular, this effort has produced resources and tools sufficient to support *entire courses* in several versions and including accompanying materials related to learning and assessment aimed at both student and teacher levels.

In their work on MEAs, students have rich but idiosyncratic mathematical experiences that need to be *unpacked* and placed into relationship with each other and with more canonical concepts, practices, and procedures from the discipline. To investigate such matters, MMP researchers attend to learner activity beyond the scope of single MEAs, formulating tools and designs for Model Development Sequences, or MDSs (Arleback, Doerr, & O'Neil, 2013; Doerr and English, 2003; Hjalmarson, Diefes-Dux, and Moore 2008; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Work here includes approaches for extending the modeling dynamics that MEAs foster and for unpacking and making explicit learners' ways of thinking, so that they are available to be reflected upon by the classroom group as a whole and systematized in relation to big ideas in the discipline (see also Brady, Eames, & Lesh, 2015 for tentative MDS design principles).

Because the courses supported by these materials were designed explicitly to be used as research settings, for investigating the interacting development of students' and teachers' ways of thinking, the materials were modularized so that important components could be easily modified or rearranged for a variety of purposes in different implementations. In particular, by selecting from and adapting *the same core collection of MEAs*, and surrounding them with MDS activities tailored to the learning goals and emergent ideas of different classrooms, parallel versions of the course have been developed for: (a) middle- or high-school students, (b) college-level elementary or secondary education students, and (c) workshops for in-service teachers. Existence-proof versions of these courses have produced impressive gains (see, e.g., Lesh, Carmona, & Moore, 2009).

As a result of the breadth of the MMP, other models for engaging with MEA-style modeling at larger timescales have also emerged. These larger structures reflect other professional and theoretical interests and concerns, such as a commitment to iterative design-based inquiry (Eames, et al, in press); a focus on socio-mathematical norms (Yackel & Cobb, 1996), or the dynamics of coaching (Baker & Galanti, 2017).

Connecting to Themes Supporting Sustained Inquiry

While Model Development Sequences and other MEA-based approaches (cf Eames et al, in press) explore ways to sustain authentic modeling experiences over longer timescales, we have recently become interested in ways to build upon the *thematic* interests that MEAs awake in learners. Entering the world of the MEA's client, learners become deeply engaged in a world-view, and they are positioned as assisting in a cause or concern that they adopt for the duration of the problem, often quite energetically. We are asking ourselves, how might teachers or educators at a larger scale (e.g., schools, districts) build upon student interest in MEA clients' problems to offer sustained inquiry into the issues and concerns raised there? And the reciprocal question, how might important social issues (e.g., questions of social or ecological justice), or movements (e.g., citizen science or citizen activism projects) be a fruitful source for MEA design, yielding not only opportunities for rich mathematical modeling but also entry points for engaging of the student population with these out-of-school themes, contexts, and groups.

Research and Discussion Themes to Guide the Working Group

In the past several, our previous PME-NA working group meetings in East Lansing, Tucson, Indianapolis, and Greenville have brought together over 40 participants from the US, Canada, and Mexico. Participants have described ongoing implementation and research across a wide range of grade levels and educational settings. In their work, attendees reported that they apply a variety of interpretive lenses and frameworks to modeling, and they situate their work in a variety of ways with respect to other current trends in mathematics education research. Moreover, in pursuing their practice, they developed definitions of core MMP constructs that were both broadly compatible (enabling productive discussion) and differently specialized and exemplified (enabling illuminating debate).

We have thus found Working Group meetings to offer unique opportunities to connect research voices and viewpoints, spurring conversations between research groups that have common inspiration and compatible interests, but very diverse local experiences and perspectives. The Working Group meetings have also consistently been a vehicle for connecting "old timers" with "newcomers." Some of the giants in the PME-NA community (e.g., Dick Lesh, Lyn English, Helen Doerr, Margret Hjalmarson, Jim Middleton, Tamara Moore, Eric Hamilton, and others) have also been leaders in the MMP, and in each of our sessions, we have invited participation from one or more of these leaders to offer perspectives on our thinking and on the field as a whole. We see these interactions as an important aspect of newcomers' (and old-timers') experience of the conference as a site for the exchange of wisdom, perspective, and enthusiasm among participants.

In our planning leading up to the Greenville we identified a compelling opportunity to experience and workshop an MEA that was still being designed. Thus we configured our work as a "researcher-level MEA" – where our "client" was the research team preparing the MEA for its first implementation, and where we experienced the problem as learners (working on the problem together during session 1) and as researchers (reflecting on our experiences and work-shopping possible revisions or additions to the activity).

As we learned more about the work of our "client" through workshopping the MEA, we became interested in the subject matter of the problem (which involved Box Turtles and conservation efforts surrounding them), and we identified connections between the way conservation efforts and issues entered into this problem and into work that members of our group had done in their own MEA designs. Our client's connections with a major ecological foundation were distinctive, but we felt that the idea of building an MEA that would awaken students' interest in ecological or social issues was not only compelling but possibly a repeatable and generalizable approach. This year, we aim to prove out that emergent conjecture. Specifically, we aim to explore how conservation and sustainability can serve as a context for authentic mathematical modeling—and how such modeling efforts can stimulate interest and concern for conservation and sustainability.

Session Outline: Advancing the Research Agenda while Building Community and Capacity

The working group will meet in three sessions over the course of the conference. As the organizers and facilitators do the preparatory work for the conference, these plans will be refined, but the broad outlines here reflect our current thinking.

As mentioned above, our experience of the working group over the past three PME-NA conferences has highlighted the value of these meetings for both (a) establishing and "hashing out" plans for innovative collaborative research, and (b) inviting interested newcomers to the

MMP, providing them opportunities to engage with its principles and practices and to interact with some of its founding members. Although it imposes an intense challenge for organizing and facilitating the working group, we aim to continue to support these two strands of activity. In part, we are committed to both because we recognize the importance to the MMP both to advance its agenda and rejuvenate its participant group. But in addition, we recognize that these two threads are in fact inseparable. Some of our most interesting theoretical discussions have come out of the friendly challenges from newcomers/outsiders, and we aim to cultivate and integrate rather than cordon off these voices and perspectives.

Key among the preparatory efforts for the facilitator group will be to select the focal MEA. One proposal is to return to the Box Turtle MEA workshopped last year, and use the session to explore adaptations and variations that could enable learners from diverse geographic settings to connect with local conservation efforts and/or citizen science projects. An alternative would be to pursue another of the design efforts emerging among others in the facilitator group, which also combine conservation, community engagement, and citizen science. In either case, the focus will be on creating MEA-style activities that foreground the mathematical modeling challenges inherent in pursuing such efforts. Below, we describe the session plan assuming the "Box Turtle" choice; changing the focal MEA will change the details but not the high-level purpose of each session or the "story arc" of the Working Group overall.

Session One

Our goal in Session One is to equip a diverse attendee group to participate in workshopping the focal MEA. With Box Turtles, this will involve a slightly time-compressed experience of the MEA as learners, followed by a report of the implementations since last PME-NA of this activity, and patterns in both students' modeling work and in the interests that students and teachers have expressed in conservation topics.

Sessions Two and Three

The goal of these sessions is to make substantive contributions to the effort to identify opportunities to connect the MEA with local conservation efforts, and to provide suggested structures for activating those connections. For instance, one of the emergent themes in our prior work with the Box Turtle MEA was the challenge of structuring observations (of turtle's features) and converting observed phenomena into measurable quantities (and into a *procedure* for measuring them). This is both an important theme in the history and philosophy of science (Daston and Lunbeck, 2011) and a theme recognized as appropriate and valuable for science learning, even at a young age (Eberbach & Crowley, 2009). Moreover, as Trumbull et al (2005) describe how identifying and stabilizing clear and repeatable procedures for observation is critical efforts like those central to organizing successful citizen science efforts.

A possible goal, then would be to refine the design of this MEA so that it (1) uncovers and delves into an important issue and practice involved in participating in citizen science projects, and (2) raises awareness and interest in the social and ecological themes that often *inspire* participation in citizen science projects. Together, these features of a refined MEA could both motivate and prepare students to engage in conservation projects, preparing them to experience citizen science through the lens of modeling, and modeling through the lens of engaged citizenship.

Session Three will be dedicated to making concrete research plans for continuing work after the Conference. This will include plans for multiple implementations of the revised focal MEA and other thematically-connected MEAs. We will foreground implementation opportunities, identify opportunities for cross-institution IRB proposals and additional collaborations to create new MEAs that exhibit similar connections with social issues and movements.

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The end.