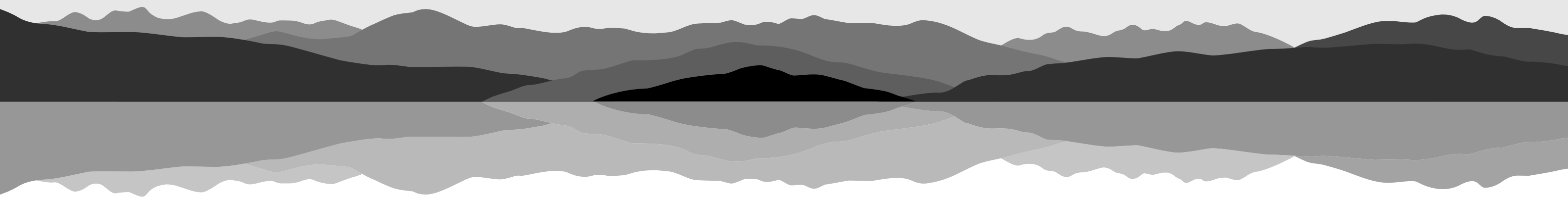
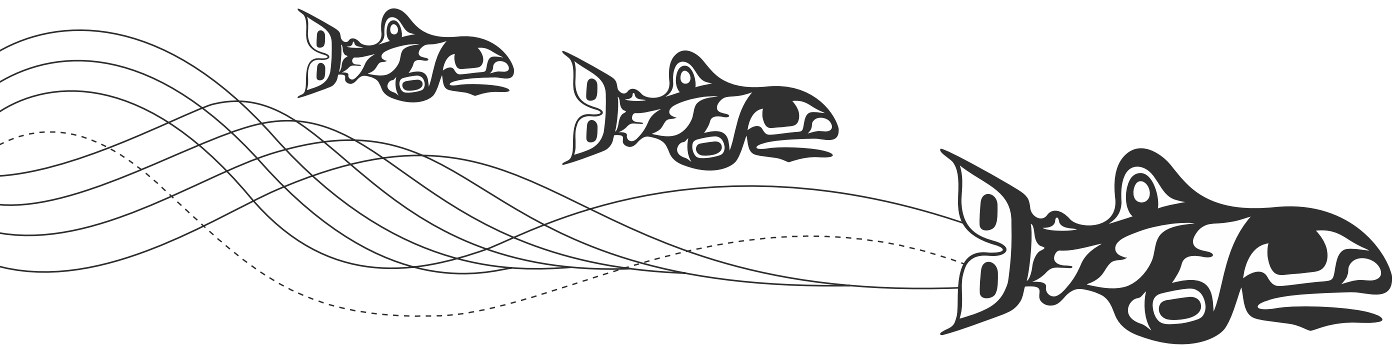


PME 38 / PME-NA 36

Proceedings

Vancouver, Canada | July 15-20, 2014

Of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education



Volume 5

Editors | Cynthia Nicol, Susan Oesterle, Peter Liljedahl, Darien Allan

PME 38 / PME-NA 36

Vancouver, Canada | July 15-20, 2014

Cite as:

Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.). (2014). Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education (Vol. 5). Vancouver, Canada: PME.

or

Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.). (2014). Proceedings of the Joint Meeting of PME 38 and PME-NA 36 (Vol. 5). Vancouver, Canada: PME.

Website: http://www.pme38.com/

The proceedings are also available via http://www.igpme.org

ISSN 0771-100X

Vol. 5 ISBN 978-0-86491-365-4

Set ISBN: 978-0-86491-360-9

Cover Design: Kirsty Robbins, University of British Columbia

Salmon Design: Haida artist, William (Billy) NC Yovanovich Jr. Ts’aahl

Formatting Assistance: Robert Browne, Jacky McGuire

Production: Lebonfon, Québec

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DEMOCRATIZING MATHEMATICAL CREATIVITY THROUGH KOESTLER'S BISOCIATION THEORY

Vrunda Prabhu[1*], Bronislaw Czarnocha[2]

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The presentation challenges a frequently-expressed assertion: "There is no single, authoritative perspective or definition of creativity [in mathematics]" Kattou et al. (2011). It points to difficulties resulting from using accepted definitions in educational research (Wallas, 1926; Thorance, 1975). In this paper, the authors express concern about joining research on creativity with the research into giftedness and suggest the need for democratizing that approach. To that end, they introduce an alternative definition of creativity - bisociation, that is "a creative leap of insight" or an Aha moment (Koestler, 1964). Prabhu and Czarnocha argue for adopting Koestler's bisociation as "the authoritative perspective or definition of creativity."

THE STATE OF THE FIELD

Mathematical creativity may be the only gate through which to reactivate the interest and the value of mathematics among contemporary youth whose engagement in the field is hampered by disempowering habits expressed as "I can't do it," "I am not good in math," "thinking tires me" (Czarnocha et al., 2011). This teaching-research observation is in agreement with the research community: Lamon (2003) emphasizes the need for creative critical thinking and Mann (2005) asks for the explicit introduction of creativity as the component of learning in general. However, the conceptualization of creative learning varies due to the diversity of the proposed definitions of creativity. (Kattou et al., 2011) There is no single, authoritative perspective or definition of creativity (Mann, 2006; Sriraman, 2005; Leikin, 2011, Kattou et al., 2011) leaving practitioners without a clear and supportive viewpoint. However, a clear understanding of the cognitive and affective conditions for the creative act is important at present to be useful as the jumpstart for bridging the Achievement Gap in the US or start the numerical literacy campaign among the Tamilian Dalits of India (Prabhu, Czarnocha, 2008). There are two recently published excellent collections of papers, dealing with creativity in mathematics education, (Sriraman and Lee, 2011; Leikin et al., 2009). Both collections join the issue of creativity with the education of gifted students, indicating that the interest in creativity of all learners of mathematics *is not the central focus of the field*. There can be several reasons for so restrictive a focus on creativity: it could be due to the efforts of globalization so that "the winds are changing" (Sriraman and Lee, p. 2) or it could be that our understanding of the creative process is not sufficiently sharp to allow for the

[*] Vrunda Prabhu passed recently away while working on creativity in mathematics education.

effective focus of research on the mathematical creativity by all students including, of course, the gifted. This observation raises the issue of democratization of creativity in mathematics research and teaching.

There are two definitions of the creative process on which many of the investigations are based. Wallas (1926) puts forth Gestalt-based definition of the process as consisting of preparation, incubation, illumination and verification. More behavioural in approach is Torrance's 1975 definition. It involves fluency, flexibility, novelty and elaboration. Leikin (2007) and Silver (1996) contracted it to fluency, flexibility and originality making the definition one of the bases for understanding creativity in mathematics education. Neither approach, however, addresses itself directly to the act of creativity nor to the structure of the "Aha moment" as the commonly recognized site of creativity itself (Sriraman, 2005). Fortunately, the theory developed by Arthur Koestler in his 1964 work, *Act of Creation*, does exactly that. It builds our understanding of creativity on the basis of a thorough inquiry into the Aha moment, which Koestler calls a bisociative leap of insight. The development of a comprehensive Theory of an Aha Moment is particularly urgent at present from the theoretical research viewpoint given the empirical work of Campbell *et al.*, (2012), who are investigating the Anatomy of an Aha Moment and the work our colleagues from computer creativity, a subdomain of Informatics, who are already employing bisociation for their data mining processes (Dubitsky et al., 2012).

KOESTLER'S PRINCIPLES OF CREATIVITY.

Arthur Koestler (1964) defines "bisociation" as "the spontaneous flash of insight, which …connects the previously unconnected frames of reference and makes us experience reality at several planes at once… " (p. 45) – an Aha moment. Koestler clarifies the meaning of "insight", by invoking Thorpe's 1956 definition of insight: "an immediate perception of relations". Koestler also refers to Koffka's 1935 understanding of insight as the "interconnection based on properties of these things in themselves." In the words of Koestler:

> The pattern… is, *the perceiving into situation or Idea, L, in two self-consistent but habitually incompatible frames of reference, M_1 and M_2*. The event L, in which the two intersect, is made to vibrate simultaneously on two different wavelengths, as it were. While this unusual situation lasts, L is not merely linked to one associative context, but *bisociated* with two. (p. 35)

Consequently, the creative leap or "an immediate perception of relations" can take place only if we are participating in at least two different frames, matrices of discourse. Examples of such simultaneous two frames of thought abound. One of them, present during the instruction of elementary algebra, is the theory of the number line based on (1) the framework of the number theory and (2) the framework of the geometrical line, memorialized through the creativity of the Dedekind axiom of one-to-one correspondence between real numbers and points on the line. Another one is the teaching-research methodology, which is the integration of the teaching framework with the framework of research, a highly creative and effective method of teaching and doing research on teaching and learning at the same time (TR/NYCity in B.Czarnocha et al. 2014). Koestler offers examples of bisociation in the discovery of electromagnetism out of two separate investigations, that of electricity and that of magnetism; he mentions wave-particle duality, of course, as well, and many others.

The depth of Koestler's approach to creativity doesn't rest here. Within his conceptual framework, "creativity is the defeat of habit by originality". That means that bisociation not only is the cognitive reorganization of the concept by "an immediate perception of relations", but also it can be an affective catalyzer of the transformation of habit into originality (Figure1).

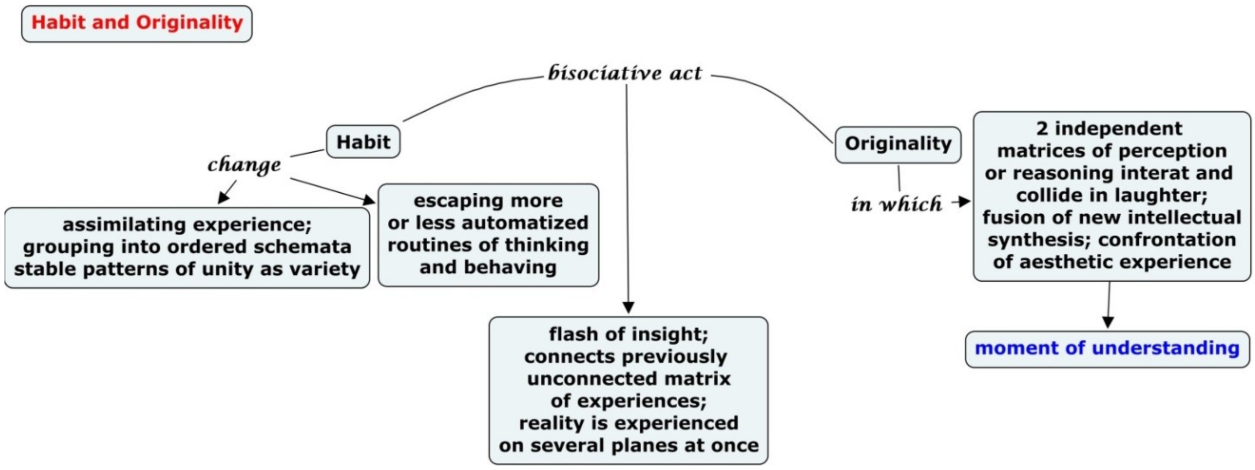

Figure 1: Habit to originality through the "flash of insight" (Prabhu, 2014)

The presence of this cognitive/affective duality of creativity, of the Aha moment, can provide the intrinsic motivation to bridge the Achievement Gap in US and in other centres of educational inequality, according to Prabhu, (2014). In fact, the first teaching experiments introducing the principles of the Act of Creation into classrooms were conducted "to address the emotional climate of learners" in the remedial/developmental classes mathematics classroom. The transformative relationship between habit and originality formulated at the very basis of the bisociation theory confirms Liljedahl's 2004 meta-findings that the "…Aha experience has a helpful and strongly transformative effect on a student's beliefs and attitudes towards mathematics…" (p. 213). However, in stressing that "the aha experience is primarily an affective experience", he is neglecting its equally significant cognitive component. (Liljedahl, 2009). Quoting Poincare, Koestler brings out explicitly the cognitive element of the Aha moment: "Ideas rose in crowds; I felt them collide until pairs interlocked, so to speak, making a stable combination" (Poincare qtd in Koestler,p. 115). Note the process of grasping stable relations of pairs of concepts in accordance with Koestler's definition of bisociation.

THE IMPACT OF BISOCIATION UPON UNDERSTANDING OF CREATIVITY

Koestler's 700+ page *Act of Creation* argues convincingly that bisociation is the common structure across the domains of Humor, Scientific Discovery and Art Sublimation making it the principle underlying any creative act of invention. supporting Hadamard's view that

> Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention, one can say that there is only the difference of degree, the difference of a level, both works being of <u>similar nature</u> (1945, p. 104).

Thus the standard division of creativity into absolute and relative is misleading because it seems to suggest an essential difference between the two. Similarly, in each intellectual domain the tools and the language through which creativity is expressed vary, but the process of insight through bisociation is exactly the same. Hence, the conventional distinction between general creativity and domain specific creativity doesn't hold water.

Situating the definition of creativity in the illumination stage of the Wallas definition itself provides a new perspective upon questions raised in recent discussions on the subject. In particular, Sriraman et al., (2011) assertion can be qualified:

> …when a person decides or thinks about reforming a network of concepts to improve it even for pedagogical reasons though new mathematics is not produced the person is engaged in a creative mathematical activity. (p. 121)

Whether the process described above is or is not a creative mathematical activity can be decided on the basis of Koestler's distinction between *progress of understanding* – the acquisition of new insights, and *exercise of understanding* – the explanation of particular events (p.619). If for example, I decide to design a developmental course of arithmetic/algebra based on my knowledge of the relationship between arithmetic and algebra (generalization and particularization), which involves the redesign of the curriculum, that is its "the network of concepts", I am engaged in *the exercise of understanding of mathematics*, distinctly different from creative *progress of understanding in mathematics.* It may however, depending on the initial knowledge of the teacher, be a creative activity in pedagogical meta-mathematics, that is understanding mathematics from the teaching point of view – the content of professional craft knowledge.

The bisociation theory, in which on the one hand creativity is "an immediate perception of relation(s)", and on the other it is the affective catalyzer of the transformation of habit into originality, interacts well with MST methodology. (Leikin, 2009). It predicts the absence of the difference between absolute and relative creativity observed by authors of the experiment. Moreover, the observed fall in the expression of originality reported by Leikin, (2009) as well as the correlation between creativity and originality is natural in the context of the relationship between habit, creativity and originality – a point made explicit in the often quoted Koestler's assertion "*Creativity* *is the Defeat of the Habit by Originality*". The authors point correctly to the fluency and flexibility as the carriers of the habit which diminished the originality of student subjects: "…when students become more fluent they have less chance to be original". This apparently complementary relationship between fluency and creativity *dictates an utmost care* in conducting the research into creativity with the help of the definition which includes fluency, because it may result in undesired lowering of creativity. And that we don't want, especially in the "underserved communities". This observation brings in the old question to the fore: What is the optimal composition of fluency and creativity in the preparation of teachers of mathematics, as well as in classroom teaching?

CLASSROOM IMPLEMENTATION OF THE THEORY – V.PRABHU (2014)

Design of Triptych based Assignments

The Act of Creation defines bisociation that is "the creative leap of insight, which connects previously unconnected frames of reference and makes us experience reality at several planes at once." How to facilitate this process? Koestler offers a suggestion in the form of a triptych, which consists of three panels…indicating three domains of creativity which shade into each other without sharp boundaries: Humor, Discovery and Art.

Each triptych stands for a pattern of creative activity which is represented on them; for instance

Cosmic comparison ⟷ objective analogy ⟷ poetic analogy

The first is intended to make us laugh, the second to make us understand, the third – to marvel. The creative process to be initiated in our classes of developmental and introductory mathematics needs to address the emotional climate of learners, and here is where the first panel of the triptych comes into play, Humor.

Having found humour and the bearings of the concept in question, the connection within it have to be explored further to "discover" the concept in detail, and finally to take the discovery to a form of sublimation by Art.

Triptych assignments facilitate student awareness of connections between relevant concepts and thus they facilitate understanding. However, what maybe even more important, the accompanying discussions help to break the "cannot do" habit and transform it into original creativity. There was a significant improvement (measured by the instructor's intuitive assessment and tests results) in the experimental statistics class.

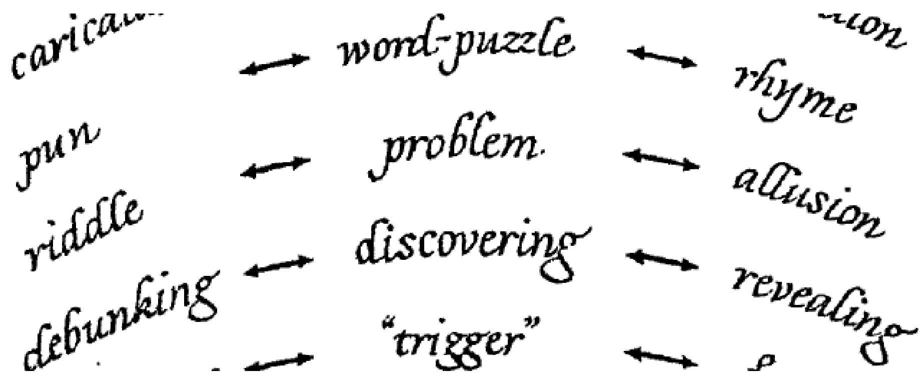

Figure: 2 Koestler’s triptychs

Examples of triptych assignment used in the class of introductory statistics

Trailblazer ⟷ outlier ⟷ originality

⟷ Sampling ⟷

⟷ probability ⟷

⟷ confidence interval ⟷

⟷ Law of Large Numbers ⟷

Lurker ⟷ correlation ⟷ causation
lurking variable.

Figure 3: Triptych assignment

Trailblazer ⟷ OUTLIER ⟷ Original

Random ⟷ SAMPLING ⟷ Gambling

Chance ⟷ PROBABILITY ⟷ Lottery

Lurking Variable ⟷ CORRELATION ⟷ Causation

Testing ⟷ CONFIDENCE INTERVALS ⟷ Results

Sample Mean ⟷ LAW OF LARGE NUMBERS ⟷ Probability

Figure 4: Triptych student response

Use of triptychs in the mathematics class brings back the puzzle inherent in mathematics.

What is the connection between stated concepts? What could be the concepts connected to the given concepts? – A forum for meaning making is created in connecting the prior knowledge, with synthesized, reasoned exploration. The question “how” is answered by the question “why” through the use of mathematical triptychs.”

CONCLUSION: A PROPOSAL

This short review of our efforts to understand creativity indicates serious weaknesses in the field, which undermine the educational effectiveness of creativity.

In light of widely spread conviction that there is no single, authoritative perspective or definition of creativity as expressed by Mann, Sriraman, Leikin, and Kattout et al., we are *proposing bisociation as the authoritative definition of creativity in the field of mathematics*. Its relationship to two basic definitions (1) coming from Gestalt approach as well as (2) from a more behavioristic school depending on fluency, is clear. In the first case it focuses on the stage of illumination, the actual stage of creativity; in the second case, it suggest that fluency, which can correlate well with creativity, can undermine it at the same time. Clearly fluency does not measure nor defines creativity but instead some composition of creativity with a habit. Bisociation, on the other hand, is the "pure" act of creation in the making. Its disassociation from fluency is very important for the facilitation of mathematical creativity in the remedial and elementary mathematics classrooms of community colleges, where it is exactly fluency that's missing. It is the definition of creativity for everyone, because "everyone" knows Aha moment. Koestler flatly asserts that "minor subjective bisociation processes…are the vehicle of untutored learning" (p. 658). Taking bisociation as the definition of creativity ensures democratization in mathematics education. It's interesting to note that our colleagues in computational creativity have discovered recently Koestler's bisociation for the creative information exploration (Dubitsky et al., 2012). The simplicity of bisociative facilitation through the discovery & creative problem solving in the context of a triptych approaches provides us with ready pedagogical techniques of teaching and researching it. It would be very useful to understand better the process of scaffolding the bisociation; this understanding can come only if bisociation is observed en vivo, that is in the classroom, more in the context of qualitative research approach at present than quantitative.

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ASSESSING TEACHERS' PROFOUND UNDERSTANDING OF EMERGENT MATHEMATICS IN A MASTERS COURSE

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Profound Understanding of Emergent Mathematics has been recently proposed as a means of conceptualizing the mathematics knowing of teachers as an open disposition that extends well-defined and fixed categories of knowledge proposed in the literature. The purpose of this paper is to explore how this disposition may be assessed through the assignments submitted by the eleven teachers enrolled in a master's-level course. The assignments included concept study and collaborative lesson design. An analysis of the assignments of one team of teachers is presented suggesting that while previous categories for mathematics knowledge were reflected, evidence of an open disposition was limited.

INTRODUCTION

Davis and Renert (2014) have recently proposed a conceptualization of mathematics knowledge for teachers, *Profound Understanding of Emergent Mathematics* (PUEM), based upon complexity sciences and extending the previous conceptualizations widely reported in the literature. Tracing historical approaches to mathematics knowledge for teachers, Davis and Renert claimed that PUEM includes elements of these conceptualization such as: a) formal mathematical knowledge in terms of postsecondary courses and formal mathematics (Begle, 1972); b) specialized mathematics for teachers, such as pedagogical content knowledge (Shulman, 1986) and didactics in mathematics (Freudenthal, 1983); and c) mathematical knowledge entailed for teaching mathematics, such as *unpacking* as a key process of teachers' practice (Ma, 1999)—this last approach includes a shift from knowing more to knowing different. Davis and Renert saw these prior conceptualizations as important but limited in two key elements. First they proposed that teachers' knowledge of mathematics is "vast, evolving and distributed" (p. 48)—similar to Mathematics as a body of knowledge. Second, teachers should embody an open disposition to emergent mathematics in the classroom, including the capacity to participate in a knowledge building community in which students' ideas, misconceptions and questionings play a major role in extending learning beyond formal mathematics. Davis and Renert provided an example in the form of a classroom episode. Grade eight students were asked to approach the following problem: "*Suppose that the earth is a perfect sphere and that we tie a rope tightly around the equator. How long will the rope be?*" (p. 104). Students calculated an approximation of the length of the rope. In class discussion they agreed that adding ten more meters to the rope would create a gap between the earth and the rope, which they calculated as 1.6 meters long, approximately. Teachers explained that the result surprised them as the slackening effect of adding 10 meters to the 40 000 kilometers long rope must be negligible. Without having an explanation for this result, teachers asked: "*How can it be that the gap is large enough to allow a child through? How would you explain this result to a person who cannot calculate it?*" (p. 105). One student provided the following explanation:

> For us, a gap of 1.6 meters looks big. But this gap 1.6 meters is added to the radius of the earth. If you compare 1.6 meters to the radius of the earth, which is 6391 kilometers, you can see that this it is not large at all. In fact, it's tiny (p. 105).

Teachers indicated that this problem puzzled them for a long time and considered the previous student's explanation as clear and sensible. The open disposition to this collective generation of knowledge was reflected in the planning of the class as teachers asked a question they had not yet answered.

As a means for nurturing teachers' PUEM, Davis and Renert (2014) proposed *concept* study, a mix of lesson study (Stigler & Hiebert, 1999) and concept analysis (Usiskin, Peressini, Marchisotto, & Stanley, 2003). A main focus in concept study is that: "Learning of mathematics should be more structured around *meanings* than *definitions*" (Davis & Renert, p. 38). Rather than providing a prescribed list of steps for concept study, Davis and Renert described four emphases for the collective study of mathematical concepts. The first emphasis is on *realizations* (Sfard, 2008), that is, the learners' possible ways of association used to make sense of a mathematical construct, including: formal definitions, algorithms, metaphors, images, applications and gestures. The second emphasis is on *landscapes*, which are visual ways for representing relations among realizations—usually in form of tables and maps. Grade level has been a very useful criterion for organizing landscapes. The third emphasis is on *entailments* of the different realizations of a concept, which refer to the logical implications of each realization. The fourth emphasis of concept study is on *blends,* which correspond to grander interpretations connecting the realization of a mathematical concept in a more formal fashion. The emphasis on conceptual blends is a deliberate move into a formal, axiomatic world—as described by Tall (2004).

Concept study has been enacted in several courses for mathematics teachers at both the master's and undergraduate levels in western Canadian universities for more than ten years. However, Davis and Renert (2014) still raised the question of "How might PUEM as an open disposition be assessed?" (p. 121). The purpose of this paper is to explore the potential evidence of PUEM, including this open disposition, in the assignments teachers submitted as part of a master's course in mathematics education, which included concept study and collaborative lesson design. Teachers' decisions within the lesson plans may serve to assess the open disposition toward emergent mathematics as enacted in the classroom.

THE COURSE

The master's program in mathematics education was designed exclusively for teachers at a particular school and consisted of four courses delivered on-site during one year. The course described here, Designing Tasks for the Math Classroom, was the second course of the program. In the first course teachers surveyed a variety of theories of learning in mathematics and questioned their current teaching practices. The school, ranging from grades 2 to 12, served the education of students with learning disabilities. Students were streamed in either the *collegiate program or the academy program*. The latter program focused on students coded with learning disabilities. The collegiate program was designated for students who reached academic skills at age and grade appropriate levels, allowing them to stay at the school instead of going to a regular school.

The course for this study included several goals for participants. First, they were expected to consult relevant literature on the social and historical context of selected topics and concepts in mathematics, as well as their related cognitive obstacles and alternative teaching approaches. Second, participants revisited literature regarding different forms of collaborative design—such as lesson study and learning study—in order to be able to design and enact it in their own context. Third, they were expected to develop capacities for the design, in collaboration with other teachers, of mathematical tasks aimed at student engagement in deep mathematical thinking. And finally, they engaged in 'doing' mathematics by solving diverse mathematical problems throughout the whole course—particularly, identifying stages of the problem solving process such as entry, conjecture, verification, specialization, and generalization, as per Mason, Burton and Stacey (2010). The assignments for the course are described in the following paragraphs.

Concept study. This assignment was a deep study of a major mathematical concept or topic from the curriculum comprising its: (a) historical development; (b) cognitive obstacles and students' common mistakes and misunderstandings; (c) images, analogies, metaphors and exemplars used for mathematics and mathematics education; (d) contemporary role/place outside school; and (e) development through the whole curriculum.

Lesson planning. This assignment consisted of planning/creating/selecting learning tasks and activities aimed at engaging students in mathematical thinking. This assignment elaborated from the concept study and extended it to anticipate students' possible approaches and misunderstandings and appropriate teacher's responses. This task was based on lesson study and teachers observed the enactment of the lessons.

Individual enactment report. This was an individual report of the enactment of the lessons, including: (a) a general description of the enacted lessons highlighting relevant moments; (b) proof of students' mathematical thinking; and (c) conclusions.

Debriefing and refinement. A revisited version of the lesson planning including improvements and comments was submitted as a final assignment.

METHODOLOGY

This study took place in the context of a broader research aimed at studying changes in school culture when deliberate support is provided for the professional development of teachers, including the master's program. In order to explore the potential evidence of PUEM in the assignments of the course, I took a qualitative approach. I read these assignments repeatedly to make a general sense of the data as a whole, conducting an open coding and looking for emerging themes. Then, I decided to code for evidence of teachers' knowledge in terms of: *formal mathematics* (Begle, 1972), including knowledge about mathematics such as history and current applications; *pedagogical content knowledge* (Shulman, 1986); *knowledge of content and curriculum* (Ball, et al. 2008); and an *open disposition* (Davis & Renert, 2014).

There were two teams in the group (eleven participants in total). One team decided to focus on the concept of surface area and designed a sequence of lessons for grades four, five, seven and eight. The other group focused on Pythagorean theorem and designed a lesson for grade ten. Due to the length limit in this report, only data from the latter group is presented. Results were, however, similar in the group focused on surface area. In particular, there was a strong interest in promoting relational understanding (Skemp, 1978), as opposed to instructional understanding, in both groups. This was probably a result of the previous course of the master's program.

FINDINGS

The group of teachers focusing on Pythagorean theorem designed a sequence of three lessons for grade ten in both the academic and the collegiate programs. The first lesson focused on visual representations of how the sum of the areas to two squares could yield the area of a third square. In the second lesson students explored triplets of square, most of them Pythagorean triplets. Examples of when the sum of two squares did not yield the area of the third were presented as 'non-examples.' The Pythagorean relationship would be expected to emerge by the end of this class. In the last session students were expected to use the theorem to address a challenging problem in a three dimensional context consisting of finding the length of the diagonal in a closed box. Examples of the evidence found in the assignments that teachers in this team submitted are summarized in Table 1.

Evidence of disciplinary knowledge was clear from the concept study. This knowledge consisted of: references to literature reporting the use of the theorem across time and cultures, connections of related mathematical concepts, extensions such as Fermat's last theorem, and applications beyond school.

Knowledge	Examples
Knowledge of and about mathematics	Historical development including several cultures at different times Connections with other mathematical concepts such as: area, symmetry, square root and algebra Connections to Fermat last theorem Applications out of school mathematics: statistics in baseball, medicine, and microchip technology
Pedagogical Content knowledge	Examples and non examples of right triangles Representation of doubling the area of a square using wooden hinged toys Different images for proofs based on areas Images of the theorem using shapes other than squares Common students mistakes and learning obstacles such as: learning formula without understanding; identify right triangles before applying the theorem, and proper identification of legs and hypotenuse
Knowledge of content and curriculum	Landscapes based on grades as per Alberta's program of studies: K to 3, 4 to 6, and 7 to 9, 10, and 11 to 12 (Related topics for grade 10 included: linear measurement; trigonometric ratios; right triangles, perpendicular lines, right triangles; metric and imperial units; and area.)
Open disposition	Selection of a visual representation to ensure students understand that the sum of the areas of two squares can be the area of a third square. Design of an activity in which students have to figure out a relation between the areas of three squares.

Table 1: Examples of evidence for each type of knowledge

Pedagogical content knowledge could also be identified in the concept study and the decisions for the lesson plans. Common student obstacles, identified from both the literature and teachers' experience, were used in the design of the lessons. For instance, teachers identified that students have difficulties relating the square of a number with the area of a square having the length of its side equal to this number. The team of teachers decided to design an activity in which students would cut two squares into pieces and put them together into a third square: showing that the sum of the area of the two squares equaled the are of the third square. The rationale was that this kinesthetic activity would help students by providing recurring visual representations of squares. Teachers stressed the need for developing relational understanding, as opposed to only instrumental understanding. In particular teachers made the pedagogical decision of modifying the lesson plan format required by the school, which consisted of: 1) quick questions (5 min); 2) interesting idea intended to engage students in the topic, usually presented in a video, new story or other type of media (2 min); 3) a review covering concepts from the previous class (5 to 10 min); 4) introduction of a new concept in which students are led through guided examples and asked to work on independent examples (20 min); 5) seatwork in which students work independently in tasks that may be allocated as homework (10 min); and 6) a summary of the main topics covered in the lesson (2 min). In the concept study, teachers explained their decision for the modification in terms of instructional and relational understanding, as indicated in the following quotation.

> The above lesson plan [required by the school] allows for success with many of our students; however it is based on an instrumental approach to learning. … [This] lesson plan is a guided process, where students are led through the steps they need to successfully perform a required task. In the past, to teach Pythagorean Theorem, we would have introduced the students to the Pythagorean Theorem in the front end of the lesson. Students would have worked through several teacher–led examples on the board, and then completed independent examples. This format of lesson plan has its merits, such as students leave the classroom with a consistent level of understanding of the material. As well, the independent seatwork allows the teacher to discover misconceptions in understanding and immediately correct these errors.
>
> For our lesson study we will be using the general format of a [school's] lesson plan. However, we are excluding the introduction of a New Concept and instead jumping straight into Seatwork. We have decided to take a relational approach to teaching Pythagorean Theorem by creating an inquiry based experience. As a result, we will not be directly teaching the Pythagorean Theorem. In fact, we will not mention the Pythagorean Theorem until the end of the second lesson. Our goal is to have students conceive the Pythagorean Theorem through their own findings, without channelled teaching of the concept. (Concept Study on Pythagorean theorem)

The interest in promoting a relational understanding was explicitly addressed in every assignment, including all the individual enactment reports.

Knowledge of content and curriculum was evident in the landscapes of the concept study based on Alberta's program of studies. Figure 1 shows the landscape for grades 11 and 12. The connection to other topics helped to pay attention to mathematical concepts and skills required to understand Pythagorean theorem, such as: concept of area, square root; square numbers; and algebra.

While there was clear evidence of formal mathematical knowledge, pedagogical content knowledge, and knowledge of content and curriculum, the evidence for an open disposition to emergent mathematics was less obvious. The selection of images for the lessons and the emphasis on understanding that the theorem relates to the sum of areas of squares can be interpreted as attempts to create meaning with the theorem, instead of imposing the formula. Designing an activity, in the second day, in which students would discover the relationship between the areas of squares can be also interpreted as evidence of emerging mathematics in which students work collaboratively in class in order to find this connection. However, evidence of explorations beyond formal mathematics knowledge, as part of the open disposition proposed by Davis and Renert (2014), was lacking in the lesson plans and teachers' reflections. This result was similar for the other group of teachers that focused on surface area.

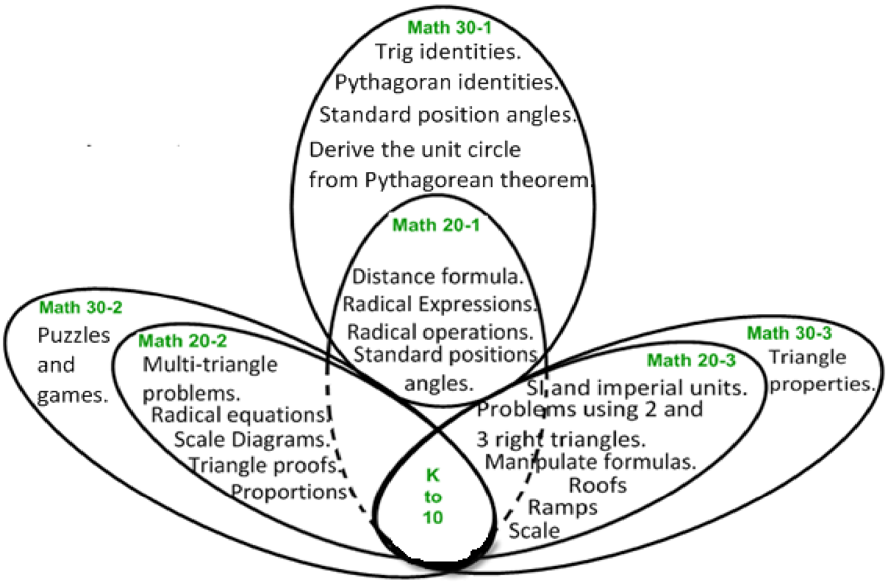

Figure 1: Grade 11-12 Landscape aligned with the Alberta Program of Studies in Mathematics

CONCLUSION

The assignments submitted by the teachers in the course described in this paper served to assess, at least partially, mathematics knowledge for teaching as per PUEM, including prior categories of knowledge as well as the open disposition toward mathematics. Concept study was useful for teachers to explore Pythagorean theorem including historical development and contemporary applications. This may help to extend teachers' understandings about the theorem, including different visual representations for the proofs. Pedagogical content knowledge and curricular content knowledge were also evident in both the concept study and the lesson plans. Teachers were informed by both the literature and their collective experience with this topic in anticipating students learning obstacles. In particular, teachers made a special effort to address relational understanding as they identified an exclusive teaching instrumental approach in the school's lesson format.

The open disposition to emergent mathematics was less obvious in the lesson plans. While common images were selected and students engaged in tasks aimed at the discovery of some relevant properties or relationships, these activities seemed to be more oriented for students to develop a better understanding of pre-existing knowledge. This approach is in contrast with the exploratory approach in the example of the rope and the earth provided by Davis and Renert (2014) and presented in the introduction of this paper. I believe that the combination of concept study and collaborative design is a sound means for teachers to develop an open disposition toward mathematics. However, this disposition may not be immediately reflected in teachers' lesson plans. A more deliberate effort to promote this disposition may enhance the effect of concept study in teachers' professional development.

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MENTAL MATHEMATICS AND OPERATIONS ON FUNCTIONS

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This study is part of a larger research program aimed at studying mental mathematics with objects other than numbers. It concerns operations on functions in a graphical environment with Grade-11 students. Grounded in the enactivist theory of cognition, particularly in problem-posing, the study aims to characterize students' mathematical activity in this mental mathematics environment. The data analysis offers understandings of strategies that students brought forth: algebraic/parametric, graphical/geometric, numerical/graphical. These are discussed in relation to implications for research on solving processes and potential for studying functions.

To highlight the relevance and importance of teaching mental calculations, Thompson (1999) raises the following points: (1) most calculations in adult life are done mentally; (2) mental work develops insights into number system/number sense; (3) mental work develops problem-solving skills; (4) mental work promotes success in later written calculations. These aspects stress the non-local character of doing mental mathematics with numbers where the skills being developed extend to wider mathematical abilities and understandings. Indeed, diverse studies show the significant effect of mental mathematics practices with numbers on students' problem solving skills (Butlen & Peizard, 1992; Schoen & Zweng, 1986), on the development of their number sense (Murphy, 2004; Heirdsfield & Cooper, 2004), on their paper-and-pencil skills (Butlen & Peizard, ibid.) and on their estimation strategies (Schoen & Zweng, ibid.). For Butlen and Peizard (ibid.), the practice of mental calculations can enable students to develop new ways of doing mathematics and solving arithmetic problems that the traditional paper-and-pencil context rarely affords, because it is often focused on techniques that are in themselves efficient and do not require other actions. Overall and across contexts, it is thus generally agreed that practicing mental mathematics *with numbers* enriches students' learning and mathematical written work about calculations and numbers. This being so, as Rezat (2011) explains, most if not all studies on mental mathematics focus exclusively on numbers/arithmetic. However, mathematics taught in schools involves more than numbers, which rouses interest in knowing what mental mathematics with objects other than numbers might contribute to students' mathematical activity. In this study, issues of functions, mainly operations on functions in graphs, are investigated. This paper reports on the strategies brought forth by Grade-11 students.

THEORETICAL GROUNDING OF THE STUDY: AN ENACTIVIST FRAME

Recent work on mental mathematics points to the need for better understanding and conceptualizing of how students develop mental strategies. Researchers have begun to

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 17-24. Vancouver, Canada: PME.

critique the notion that students choose from a toolbox of predetermined strategies to solve mental mathematics problems. E.g. Threlfall (2002) insists on the organic emergence and contingency of strategies in relation to the tasks and the solver (what he or she understands, prefers, knows, has experienced with these tasks, is confident with; see also Butlen & Peizard, 1992). This view on emergence is also discussed by Murphy (2004), who outlines perspectives that conceptualize mental strategies as flexible emergent responses adapted and linked to specific contexts and situations. Because the enactivist theory of cognition (c.f. Maturana & Varela, 1992; Varela, Thompson & Rosch, 1991) has been concerned in mathematics education with issues of emergence, adaptation, and contingency of learners' mathematical activity, it offers a way to contribute to conceptualizations about students' meaning-making and mathematical strategies. In particular, the distinction made between problem-posing and problem-solving offers ways to address questions about the emergence and characterization of strategies.

For Varela (1996), problem-solving implies that problems are already in the world, independent of us, waiting to be solved. Varela explains, on the contrary, that we specify the problems that we encounter through the meanings we make of the world in which we live, leading us to recognize things in specific ways. We do not choose problems that are out there in the world independent of our actions. Rather, we bring problems forth: "The most important ability of all living cognition is precisely, to a large extent, to pose the relevant questions that emerge at each moment of our life. They are not predefined but enacted, we bring them forth against a background." (p. 91). The problems that we encounter, the questions that we ask, are as much a part of us as they are a part of our environment: they emerge from our interaction with/in it. The problems we solve are relevant for us as we allow them to be problems.

If one adheres to this perspective, one cannot assume, as René de Cotret (1999) explains, that instructional properties are present in the tasks presented and that these *causally determine* solvers' reactions. As Simmt (2000) explains, it is not tasks that are given to students, but mainly prompts that are taken up by students who themselves create tasks with. Prompts become tasks when students engage with them, when, as Varela would say, they pose problems. Students *make* the "wording" or the "prompt" a multiplication task, a ratio task, a function task, an algebra task, and so forth. Nonetheless, each prompt is designed following specific intentions in specific ways, which can play a role in how solvers pose problems (e.g. one does not react to two square-root functions in the same way as one does with two linear functions). In sum, each prompt can be seen to have what Gibson (1979) refers to as affordances:

> The affordances of the environment are what it offers the animal, what it provides or furnishes […] I mean by it something that refers to both the environment and the animal in a way that no existing term does. It implies the complementarity of the animal and the environment […]. If a terrestrial surface is nearly horizontal (instead of slanted), nearly flat (instead of convex or concave), and sufficiently extended (relative to the size of the animal) and if its substance is rigid (relative to the weight of the animal), then the surface

> affords support […]. Note that the four properties listed – horizontal, flat, extended, and rigid – would be physical properties of a surface if they were measured with the scales and standard units used in physics. *As an affordance of support for a species of animal, however, they have to be measured relative to the animal. They are unique for that animal. They are not just abstract physical properties*. (p. 127, emphasis added)

These affordances for Maturana and Varela (1992) play the role of triggers in relation to the solver's posing. Hence reactions to a prompt do not reside in either the solver or the prompt: they emerge from the solver's interaction with the prompt, through posing the task. Strategies are thus triggered by the prompt's affordances, but determined by the solver's experiences, where issues explored in a prompt are those that resonate with and emerge from the student, as Threlfall (2002) explains:

> As a result of this interaction between noticing and knowledge each solution 'method' is in a sense unique to that case, and is invented in the context of the particular calculation – although clearly influenced by experience. It is not learned as a general approach and then applied to particular cases. […] The 'strategy' […] is not decided, it emerges. (p. 42)

This emergent/adapted perspective offers a specific way of talking about solving problems, avoiding ideas of possession (acquisition of, choice of, of *having* things, etc.) in favor of issues about emergence, flux, movement, interactions, relations, actions, and so forth. It is this perspective that orients this research.

METHODOLOGICAL ISSUES, DATA COLLECTION AND ANALYSIS

One intention of the research program is to study the nature of the mathematical activity that students brought forth when working on mental mathematics. This is probed through (multiple) case studies conducted in educational contexts designed for the study (classroom settings/activities). This reported study is one of these case studies, taking place in two Grade-11 classrooms. Classroom activities/tasks were designed with the teacher (covering two 75-minutes sessions for each group), in which students had to operate mentally on functions in a graphical environment, that is, they had to solve without paper-and-pencil or any other computational/material aids. For example, using a whiteboard, a typical prompt consisted of showing two functions in the same graph and ask students to add or subtract them (see Figure 1).

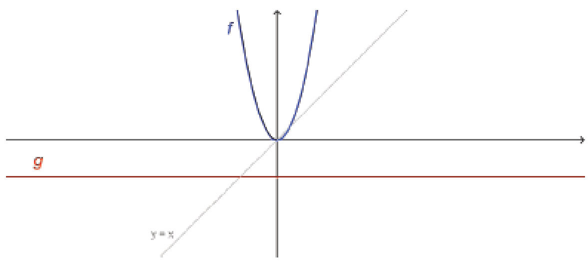

Figure 1: Example of a graphical prompt on operations on functions [$f(x) \pm g(x)$].

The activities were conducted by the regular teacher and had the following structure: (1) a graph is shown on the board and instructions are given orally; (2) students have 20 seconds to think about their solutions; (3) at the teacher's signal, students have 10 seconds to write their answer (on a sheet of paper showing a blank Cartesian graph) and then hold it up to show the teacher; (4) the teacher asks various students to show/explain their answers to others. Six thematic blocks, each composed of 6-10 prompts, were organized. The 1st block introduced students to the ideas, where both the graphs and the algebraic expressions of the functions were offered (prompts consisted of a combination of linear and constant functions). For the 2nd block, graphs of two functions (sometimes three) were given without their algebraic representation, and students had to add them mentally (functions varied from a combination of constant with linear, quadratic, square root, constant, rational, and step functions). In the 3rd block, still on the same graph, students were given the representation of one function and the result of an operation and were instructed to find the function that had been added to or subtracted from the first to obtain the resulting function (functions varied from a combination of two linear, two square-root, or a combination of a constant with a linear or square root functions, see Figure 3). The 4th block was similar to the second, but focused on subtractions. The 5th block differed in that only algebraic expressions of functions were given. These algebraic expressions could not be "directly" computed, like $f(x)=|x|$ or $f(x)=[x]$ with $g(x)=x$ or $g(x)=x^2$. The 6th block focused on symmetry, where students had to add two (linear, quadratic, by parts) functions that looked symmetrical in the graph (see e.g. Figure 2 and 4).

Data collection focused on students' strategies recorded in note form by the PI and a research-assistant, for each of the four sessions. To analyze the data, repeated interpretative readings of the field notes about the various strategies that emerged were conducted, and combined with the existing literature on functions to enrich the analysis. These repeated interpretative readings underlined three strategies, which are reported below: algebraic/parametric, graphical/geometric, graphical/numerical.

FINDINGS – ON STRATEGIES BROUGHT FORTH

Strategy 1. Algebraic/Parametric

Even when prompts were proposed in a graphical context without algebraic expressions, many students engaged in algebraic-related solving. Students referred to what Duval (1988) calls significant units for "reading" the graphical representation of a linear function and offered an interpretation in relation to the algebraic expression. That is, students brought forth parameters from the algebraic expression (the *a* and *b* of the linear function $f(x)=ax+b$) to make sense of the graphs and add them. However, because the resulting function had to be expressed graphically, they explained their answer and strategy algebraically by blending aspects of graphical information. For example, in the following addition prompt (see Figure 2), where neither function had an algebraic expression attached, many students explained that "BOTH FUNCTIONS LOOKED SYMMETRICAL, SO THE 'a' PARAMETER OF EACH LINE WOULD CANCEL OUT, AS WELL AS THE 'b' AND THUS GIVE x=0" (quotations in are taken from students' words and translated from French to English).

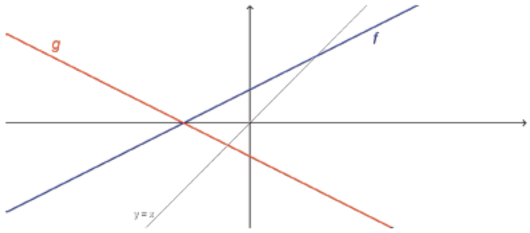

Figure 2: Addition of function graphical prompt.

In prompts where e.g. a linear function *f* would be added to a constant function, even if no algebraic expression was attached to the functions, students would say that the “a” parameter of the function *f* does not change when added with a constant function that “DOES NOT HAVE AN ‘a’ PARAMETER, SO THE FUNCTION’S STEEPNESS STAYS THE SAME AND ONLY THE ‘b’ CHANGES” giving a function parallel to *f* with a *y*-intercept at “b” instead of at 0. Thus students generated algebraic information from the graphs of the functions in order to operate and develop their solutions. They were able to draw out an algebraic context, to pose it as an algebraic task, and to solve with/in that context. Even if no algebraic expression was attached to the functions, students illustrated affordances of the prompt for them, showing that there were potential algebraic pathways in them and for them (of course, students’ algebraic prominence or preference when working with functions is not new, see e.g. Vinner’s, 1989, “algebraic bias”). They thus posed the prompt as an algebraic problem, solving it in relation to algebraic aspects generated for the functions.

Strategy 2. Graphical/Geometric

When facing a function that was not linear (e.g. quadratic, square root, rational, hyperbolic), students generated particular ways of working with slope and parallelism. They assigned a constantly changing rate of change/slope to some nonlinear functions with which they were dealing (students used the expressions *slope* and *rate of change* interchangeably, hence the “/”). E.g. with the addition of a quadratic and a constant function (see Figure 1), students explained that the rate of change of the quadratic function was not affected by the addition of a constant function, because a constant function “DID NOT HAVE A VARIATION” and thus the slope of the quadratic function: “WILL CONTINUE TO VARY IN A CONSTANT WAY”. When students said *constant*, they meant that its appearance was not affected. Thus the resulting function of their addition would have the “SAME RATE OF CHANGE AS THE QUADRATIC FUNCTION” but would simply be “TRANSLATED DOWN” in the graph because the constant function was “NEGATIVE”. Although it is not clear what exactly students meant by this “CONSTANTLY CHANGING” rate of change/slope for nonlinear functions (especially e.g. when they were dealing with $f(\mathrm{x})=1/x$), many of them brought forth a language that enabled them to solve their problem (and talk about it) and not worry about the variation in the function. As one student said about the square-root function, “ITS RATE OF CHANGE IS LEFT UNTOUCHED WHEN I ADD THE CONSTANT FUNCTION, SINCE IT HAS NO VARIATION”.

In cases where students faced more than one nonlinear function, the above constantly changing rate of change strategy appeared insufficient, as they began analyzing functions in terms of "parallelism". For example, in Figure 3 where the function g is to be found, some students expressed that "EACH FUNCTION WAS PARALLEL TO THE OTHER" and that *g* had to be a constant function "FOR THE CURVE TO BE TRANSLATED DOWN" and that it was "NEGATIVE FOR BRINGING THE CURVE LOWER".

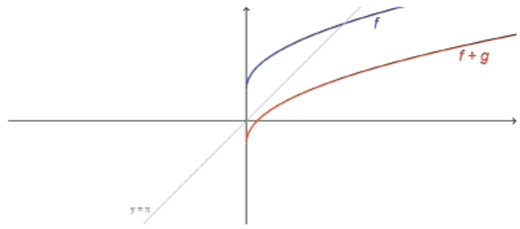

Figure 3: A prompt for which the parallelism strategy was used.

Again, this vocabulary and idea of parallelism (which can be mathematically questioned) emerged as a way of making sense without going into details about the fluctuation in image for each function. Somehow students defined these meanings through their use, in their emergent use for solving their problems. Theirs was a strategy well tailored/generated for their problem, which in turn made their problem about that strategy. To some extent, students offered a geometrical interpretation of rate of change/slope as a property not of the function, but of the curve present on the graph. They were talking about a geometric rate of change/slope, something reminiscent of Zaslavsky, Hagit and Leron's (2002) concept of slope seen as a geometric concept rather than slope seen through the lens of analytical geometry. Through their geometrical rate of change, students brought forth the nonlinearity of nonlinear functions and developed ways of engaging with/in it. By posing the prompt in geometrical terms, they generated a graphical/geometric strategy to solve it.

Strategy 3. Graphical/Numerical

Students brought forth specific points in the graphs of functions (related to Even's (1998) pointwise approach). In sum, the prompts were posed as numerical or pointwise tasks by students. Through those points, they generated exact and approximate answers (Kahane, 2003), which they combined to find the resulting function. In Figure 4 e.g. students had to find the function resulting from the addition of *f* and *g*. In this case, they would bring forth specific points: (1) where *f* cross the *x*-axis (*x*-intercept); (2) where both *f* and *g* intersect; (3) where *f* and *g* cross the *y*-axis (*y*-intercept); (4) where *g* cross the *x*-axis (*x*-intercept). For case (1), the operation is an exact calculation as the addition of the image for *f* (which is of length 0) with the one for *g* results in an image for *f*+*g* that is the same as that for *g* (it has the same image for *g* to which 0 was added). For case (2), the operation is an approximate calculation, as both images at *f* and *g* are the same, so the resulting image is double the value of the intersection point; but a precise location is impossible without knowing the exact location of the intersecting point in terms of precise length. For case (3), the same approximate calculation applies, as both images are added. For case (4), an exact answer is obtained, as in case (1). In doing this, students mingle both exact and approximate calculations to find points for the resulting function.

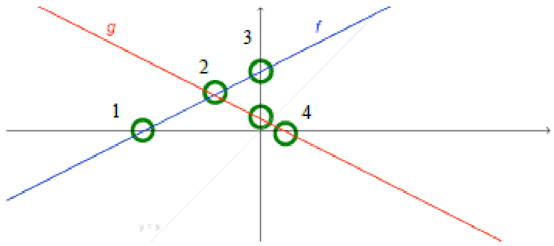

Figure 4: An addition of function prompt for which points were outlined

Students generated precise and approximate points to determine the resulting function. In so doing, they were no longer in an algebraic context, but in a blend of numerical and graphical contexts, generating numbers/coordinates that had meaning for them in the graph. E.g. when they referred to the *x*-intercept, they did not attempt to find its meaning in the algebraic expression (see Moschkovich, 1999), but worked in the graphical context to gain information for computing the resulting function. The same is true for the *y*-intercept, not treated as parameter *b*, but as a point in the graph. Their posing was numerical or pointwise, making the task about points.

DISCUSSION OF FINDINGS AND FINAL REMARKS

These strategies enacted on the spot as emergent reactions tailored to their problems offer illustrations of students' mathematical activity in this mental mathematics environment. Through their entry into the prompts, students posed their problems, making emerge affordances of the problems, that is algebraic, geometric, procedural, and so forth. Thus an algebraic posing of the functions produced an algebraic strategy; a graphical posing produced a graphical strategy; a numerical/pointwise posing produced a numerical pointwise strategy. These affordances are to be seen relative to students and the prompts, as affordances for *those* students interacting with these prompts: they do not exist in themselves, but are brought forth in the interaction with the prompt when posing the task and making them emerge.

Three main lessons can be learned from this analysis. First, it shows how students illustrated significant meaning-making capacities, as they were fluent in linking algebraic (symbolic expression), numerical (coordinate values in *x* or *y*) and graphical aspects of functions. This seems to contrast with what we know from other studies, as students are frequently reported as experiencing difficulties of many kinds when linking graphs of functions with other representations (see e.g. Even, 1998; Hitt, 1998; Moschkovich, 1999). Second, even if more research is obviously needed, this fluency underlines the potential of these mental mathematics activities for studying functions, as it occasioned numerous (and even alternative) ways of conceiving and operating on functions, e.g. algebraic, graphic, and numerical. Third, and possibly most important, the creative and adapted nature of these approaches, seen through problem-posing, underlines the importance of being attentive to students' mathematical activity when they are solving (their) problems. It shows how sensitive we ought to be, following

Threlfall (e.g. 2002), not to constrain students' mathematical doings into specific frames of expected solutions or reducing them to already known categories of solving: it offers a window onto students' mathematical activity that allows us to embrace its creative character and adaptive nature when students are solving (their) problems.

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USING PRACTICAL WORKSHEET TO RECORD AND EXAMINE METACOGNITVE STRATEGIES IN PROBLEM SOLVING

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We adopted Brown's (1987) conceptualisation of metacognition to examine how student teachers can be taught metacognitive control while solving mathematical problems. In addition, tasks given to these student teachers were in-built with opportunities for them to be aware of the need for metacognition. We describe the use of the Practical Worksheet as a way to make visible their metacognition, within the context of solving mathematics problems. Findings suggest that the greater awareness of control due to the use of the Practical Worksheet contributed to the greater employment of control in subsequent problem solving.

INTRODUCTION AND LITERATURE REVIEW

Metacognition is an important idea in the teaching of problem solving. However, its definition has remained elusive. Some have interpreted it as "thinking about thinking" (Lai, 2011; Larson, 2007). Others have attributed this difficulty to the fact that several terms such as self-regulated learning, reflective learning, executive control, meta-memory and monitoring are used interchangeably with metacognition; for example, Holton and Thomas (2001) view students' ability to carry out self-interrogating and using self-scaffolding in problem-solving as metacognition.

Brown (1987) conceptualised metacognition as having two components: the knowledge (what one knows about one's cognition) and the control (what one does to regulate one's cognition). Metacognitive knowledge refers to three different types of knowing: declarative knowledge (about one's skills and intellectual resources); procedural knowledge (about how to execute procedural skills and apply strategies) and conditional knowledge (about when and why to use declarative and procedural knowledge). The control component refers to the actual metacognitive control actions applied to cognitive processes – they are planning (such as goal setting and allocation of resources), monitoring (assessing one's strategy used) and evaluating (appraising the products and efficiency of learning). Metacognitive knowledge and control do not function separately or independently; rather, they complement each other for a person to achieve optimal performance.

According to Desolete (2007), we cannot assume that metacognitive skills will develop in the mathematics classrooms. Veenman, Van Hout-Wolters and Afflerbach (2006) argued that in order for students to develop metacognitive skills, it is crucial that teachers model metacognitive skills since learners acquire metacognitive skills through implicit socialization with experts.

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 25-32. Vancouver, Canada: PME.

In the study reported here, we adopted Brown's (1987) conceptualisation of metacognition to examine how student teachers can be taught metacognitive control while solving mathematical problems. In addition, tasks given to these student teachers were in-built with opportunities for them to be aware of the need for metacognition. In the next section, we describe the use of the *Practical Worksheet* as a way to make visible their thinking, including metacognition, within the context of solving mathematics problems.

THE PRACTICAL WORKSHEET

In Schoenfeld's (1985) framework of mathematical problem solving, control is listed as one of four components essential for success. This idea of the importance of control, together with the well-known Pólya's (1954) model of problem solving, formed the theoretical basis for our design of the *Practical Worksheet* (PW).

The PW consists of four pages exactly corresponding to the four stages of Pólya: (1) Understand the Problem; (2) Devise a Plan; (3) Carry out the Plan; and (4) Look Back. Through the Look Back (which we renamed "Check and Expand") stage, the problem solver may revisit the solution, check the reasonableness of the answer/solution, look for alternative solutions to the problem, and make changes/extensions to the solution; in the process, this looping back is a location where metacognition can be identified. In addition, a "Control" column was added in the Stage 3 so that any conscious metacognitive acts that are utilised can be recorded.

THE PARTICIPANT AND METHOD

One of the authors (hereafter referred to as the "tutor") taught the mathematics methods module in the Postgraduate Diploma in Education (PGDE) programme. The PGDE is a pre-service teacher education programme. This module is taken by university graduates in Mathematics or in a Mathematics-related discipline such as Engineering who are seeking certification to become a secondary school mathematics teacher. Six hours of this 24-hour module are devoted to the idea of teaching of problem-solving and the teaching of mathematics *through* problem solving.

Twenty-two pre-service teachers (PT) participated in this study. The tutor started by explaining what a mathematical problem is, emphasizing that it is different from a routine exercise and that it requires time and effort to solve. The tutor modelled the processes of problem solving before he discussed in detail Pólya's model and Schoenfeld's framework for problem solving. He then demonstrated how the PW should be used by working it through with a specific problem. During the first problem solving lesson, Example 1 was used to explain how the practical worksheet could be used to guide one towards problem solving.

> *Example 1 : ABC is an equilateral triangle. P is a point inside the triangle such that the distances from its three sides are 4, 5 and 6 cm. Find the length of one side of the triangle.*

Before concluding the problem solving part of the course module (which formed the first three lessons), the tutor presented two other problems to be solved in the next two tutorial sessions.

> *Problem 1 : The coordinates of a given point A are (6, 2). Find a point B on the line y = x and another point C on the x-axis such that the perimeter of the triangle ABC is minimum. Find the coordinates of point B and C.*

At this point, it was found that almost all students had not paid much attention to noticing the thought processes involved in solving the problem. The tutor discussed Problem 1 again in class and attempted to get the student teachers to be more aware of their own thinking and to focus more on the metacognitive control action for the next problem. This was followed by another tutor demonstration on how the PW could be used to guide their thinking. However, one of the PTs mentioned that he needed time to examine and write down what was actually happening in his mind while solving a problem. In response to this request, the tutor gave 10 more minutes (making it a total of 30 minutes) for them to solve Problem 2 in their last problem solving lesson.

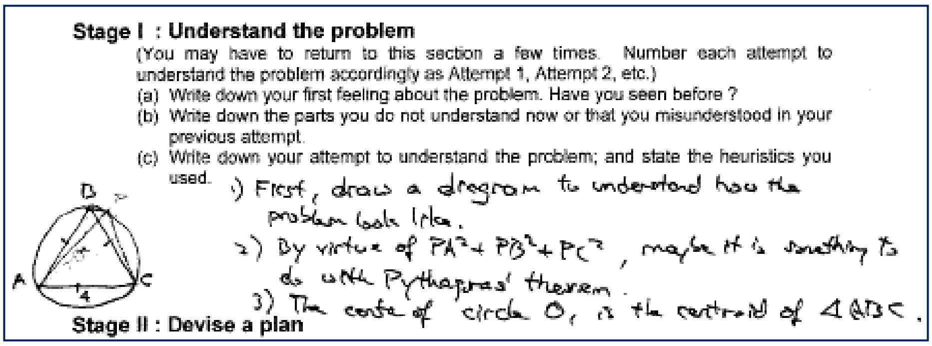
> *Problem 2 : An equilateral triangle ABC with sides 4 cm is inscribed in a circle. If a point P lies on the minor arc BC, find the value of* $PA^2 + PB^2 + PC^2$.

For both problems, the PTs had to solve the problem using the PW in class within the stipulated time. They knew that their performance in the two problem-solving sessions would not be graded.

METACOGNITIVE STRATEGIES DEMONSTRATED IN THE PROSPECTIVE TEACHERS' WORK

In this section, we shall present using some PT's solution of Problem 2 and examine them for evidence of the use of metacognitive strategies in their solution.

For Metacognitive *planning* activity, we looked for evidence from the PW on statements about possible cognitive resources and heuristics that may be involved in solving the problem. For example, drawing a diagram is a common problem solving heuristic. While it is usually considered a cognitive rather than metacognitive behavior, we think that a diagram can also become a vehicle for stimulating metacognitive strategies, leading to a useful insight of the problem. In the case of Problem 2, a diagram can be seen as a tool for metacognitive planning in this way: when point P is moved along the arc BC to coincide with either points B or C, this special case reveals a way forward in the solution strategy. Using the PW, all except one PT indicated in writing what they planned to do for Stage 1 when they attempted to solve Problem 2. Figure 1 shows one of the PT's metacognitive planning strategies.



Stage I : Understand the problem

(You may have to return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.)

(a) Write down your first feeling about the problem. Have you seen before ?

(b) Write down the parts you do not understand now or that you misunderstood in your previous attempt.

(c) Write down your attempt to understand the problem; and state the heuristics you used.

1) First, draw a diagram to understand how the problem looks like.

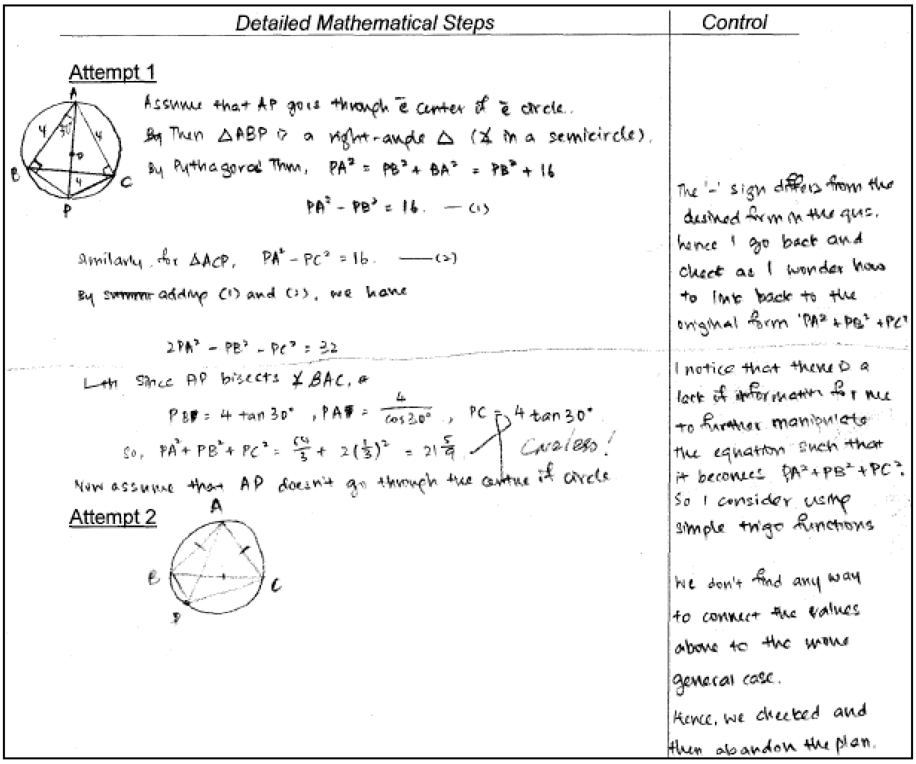
2) By virtue of $PA^2+PB^2+PC^2$, maybe it is something to do with Pythagoras' theorem.

3) The centre of circle O, is the centroid of ΔABC.

Stage II : Devise a plan

Figure 1: PT4's metacognitive planning strategies.

There is also evidence for metacognitive *monitoring* activity. This is illustrated in PT11's work as shown in Figure 2. She used Pythagoras' Theorem and did not manage to obtain the form required by the question. She switched her solution path to using trigonometric ratios to find *PA*, *PB* and *PC* correctly (though she made a slight computational mistake). This is a demonstration of the exercise of metacognitive monitoring strategy, in which the solver constantly monitors his or her solution plan and is ready for error-detection and correction, and to self-question if the current approach is on the correct path leading to the correct solution.



Detailed Mathematical Steps | *Control*

Attempt 1

Assume that AP goes through the center of the circle.

Then ΔABP is a right-angle Δ ($\angle$ in a semicircle).

By Pythagoras Thm, $PA^2 = PB^2 + BA^2 = PB^2 + 16$

$PA^2 - PB^2 = 16$. — (1)

Similarly, for ΔACP, $PA^2 - PC^2 = 16$. —— (2)

By adding (1) and (2), we have

$2PA^2 - PB^2 - PC^2 = 32$

Since AP bisects $\angle BAC$,

$PB = 4\tan 30°$, $PA = \frac{4}{\cos 30°}$, $PC = 4\tan 30°$

So, $PA^2 + PB^2 + PC^2 = \frac{64}{3} + 2(\frac{1}{3})^2 = 21\frac{5}{9}$. Careless!

Now assume that AP doesn't go through the centre of circle

Attempt 2

Control:

The '–' sign differs from the desired form in the qns. hence I go back and check as I wonder how to link back to the original form '$PA^2+PB^2+PC^2$'

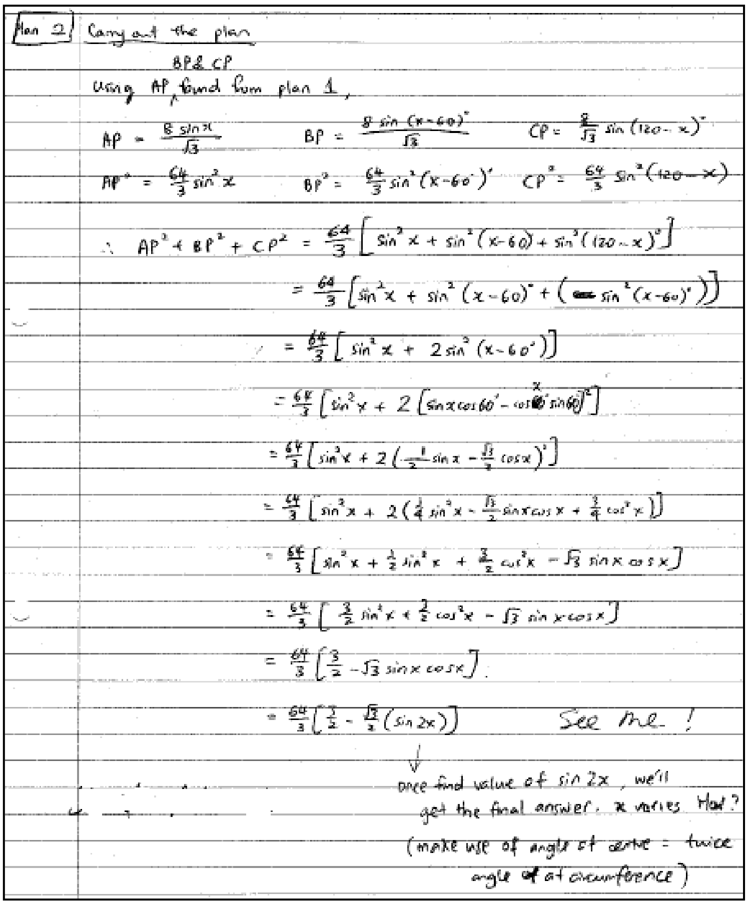
I notice that there is a lack of information for me to further manipulate the equation such that it becomes $PA^2+PB^2+PC^2$. So I consider using simple trigo functions

We don't find any way to connect the values above to the more general case.

Hence, we checked and then abandon the plan.

Figure 2: PT11's metacognitive strategies in the "control" column.

Figure 3 shows PT8's careless mistake as part of the solution to Problem 2. As he did not proceed further on the PW, the tutor spoke with him after the activity about how he would have proceeded if he were given the additional time. He mentioned that he would have checked his working again before abandoning his solution. We think that this is an example of the use of metacognitive *evaluating*.



Plan 2 Carry out the plan

Using AP, find BP & CP from plan 1,

$AP = \frac{8 \sin x}{\sqrt{3}}$ $BP = \frac{8 \sin (x-60)^\circ}{\sqrt{3}}$ $CP = \frac{8}{\sqrt{3}} \sin (120 - x)^\circ$

$AP^2 = \frac{64}{3} \sin^2 x$ $BP^2 = \frac{64}{3} \sin^2 (x-60^\circ)^\circ$ $CP^2 = \frac{64}{3} \sin^2 (120 - x)$

$\therefore AP^2 + BP^2 + CP^2 = \frac{64}{3}\left[\sin^2 x + \sin^2 (x-60) + \sin^2 (120-x)^\circ\right]$

$= \frac{64}{3}\left[\sin^2 x + \sin^2 (x-60)^\circ + (\sin^2 (x-60)^\circ)\right]$

$= \frac{64}{3}\left[\sin^2 x + 2\sin^2 (x-60^\circ)\right]$

$= \frac{64}{3}\left[\sin^2 x + 2\left[\sin x \cos 60^\circ - \cos x \sin 60^\circ\right]^2\right]$

$= \frac{64}{3}\left[\sin^2 x + 2\left(\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right)^2\right]$

$= \frac{64}{3}\left[\sin^2 x + 2\left(\frac{1}{4}\sin^2 x - \frac{\sqrt{3}}{2}\sin x \cos x + \frac{3}{4}\cos^2 x\right)\right]$

$= \frac{64}{3}\left[\sin^2 x + \frac{1}{2}\sin^2 x + \frac{3}{2}\cos^2 x - \sqrt{3}\sin x \cos x\right]$

$= \frac{64}{3}\left[\frac{3}{2}\sin^2 x + \frac{3}{2}\cos^2 x - \sqrt{3}\sin x \cos x\right]$

$= \frac{64}{3}\left[\frac{3}{2} - \sqrt{3}\sin x \cos x\right]$

$= \frac{64}{3}\left[\frac{3}{2} - \frac{\sqrt{3}}{2}(\sin 2x)\right]$ See me !

once find value of sin 2x, we'll get the final answer. x varies How?

(make use of angle at centre = twice angle at circumference)

Figure 3: Metacognitive action by PT8. Note that $x = \angle ACP$

The PWs were examined and marked. Metacognitive strategies written on the PWs were coded and classified into metacognitive planning, metacognitive monitoring and metacognitive evaluating. Table 1 shows the rules for classification.

Metacognitive activity or strategy	Prescriptions	Sample written responses on the Practical Worksheet for Problem 2
Planning	Using heuristics to make sense of the problem.	PT1: trying to find the distance using sine or cosine rules.
	Stating goal and sub-goal of Indicating of possible cognitive resources that may solve the problem	PT5: Draw and label the diagram to translate the problem into a pictorial form.
Monitoring	Indicating the need for answering to the question Indicating of the solution steps make sense	PT11 : … I wonder how to link back to the original form $PA^2 + PB^2 + PC^2$
	Indicating of the 2nd approach	PT19: Stuck at this point, go back to Stage II.

Evaluating	Indication of a problem or error encountered Indication of possible short-coming of the method	PT7. I do not know how to extend – my method does not seem to be able to work for other regular polygons. PT8: Once (I) find the value of sin2x, we'll get the answer. x varies. How ?

Table 1: Summary of the coding scheme for metacognitive activity during mathematics problem solving.

Summary of Data

Table 2 shows the percentages of the three different metacognitive strategies used in solving Problems 1 and 2.

	Metacognitive planning	Metacognitive monitoring	Metacognitive evaluating	Correct answer (out of 22)
% of PT (Problem 1)	90.1%	27.3%	13.6%	9.1%
% of PT (Problem 2)	95.5%	72.8%	50.0%	22.7%

Table 2: Percentage of PTs demonstrating different metacognitive strategies outlined in Table 1.

Stage 1 of the PW provided the PTs an avenue to express in writing how they understand the problem and the plan to solve it—this explains the very high percentage of them (90% for Problem 1 and about 96% for Problem 2) demonstrating the metacognitive planning strategy. Although the percentage of PTs demonstrating metacognitive monitoring and metacognitive evaluating strategies were not high for both problems, especially for Problem 1, we are glad to see that there was an increase in the number of PTs displaying their metacognitive monitoring and evaluating strategies in solving Problem 2 (see Table 2). While this may be attributable in part to the PTs being more familiar with the use of PW to record their metacognitive strategies during problem solving, we think that the additional 10 minutes that were given to the PTs (for Problem 2) also allowed them to record their metacognitive processes on the PW. When we examined the PWs of those who obtained the correct answer for Problem 2, we noticed how PT18—who didn't write about her metacognition at all for Problem 1 (she could not solve Problem 1)—were able to solve Problem 2 correctly. In addition, in the process of solving it, she revealed the metacognitive traces that could have helped her: realizing that her initial conjecture of "$PB + PC - PA = 0$" in her working may be correct but was unable to prove it, she wrote that there "may be another way of calculating area of triangle ABC to develop the relationship" (the relationship she meant is $PB + PC - PA = 0$). As it turned out, that metacognitive evaluating act directed her to more productive ways of exploring the relationship among the three sides and solve the problem successfully.

DISCUSSION AND CONCLUSION

Research has shown that if students have gone through some metacognitive training, they could improve their ability in mathematics problem solving (Jacobse and Harskamp 2009). But the issue of measuring thinking-related variables such as metacognition accurately and effectively is still an issue of concern and it remains an ongoing area of research. In addition, there are debates and discussions about the suitability of instruments. Think-aloud protocols have been commonly used in measuring metacognition (Ku & Ho, 2010; Veenam et al, 2006) but the actual implementation of the measuring process is time-consuming and complex and thus less practical. The search for alternative instruments that are more classroom-friendly and that helps students more directly continues.

From the results of this exploratory study, we think that the PW has the potential to aid learners keep track of their ongoing metacognitive behaviour and strategies used during the whole problem solving process. The PW ‘forces’ them to ‘think-aloud’ in the written form and teases out quite a good variation of metacognitive activities that were going on in problem solving process. Thus the PW can make metacognitive behaviour more visible, allowing the learners as well as the teachers to have information about the thinking processes. This information can, in turn, feedback to the problem solver and the teacher of problem solving.

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FEATURES OF SUCCESSFUL CALCULUS PROGRAMS AT FIVE DOCTORAL DEGREE GRANTING INSTITUTIONS

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We present findings from case study analyses at five exemplary calculus programs at US institutions that offer a doctoral degree in mathematics. Understanding the features that characterize exemplary calculus programs at doctoral degree granting institutions is particularly important because the vast majority of STEM graduates come from such institutions. Analysis of over 95 hours of interviews with faculty, administrators and students reveals seven different programmatic and structural features that are common across the five institutions. A community of practice and a social-academic integrations perspective are used to illuminate why and how these seven features contribute to successful calculus programs.

INTRODUCTION

Calculus is typically the first mathematics course for science, technology, engineering, and mathematics (STEM) majors in the United States. Indeed, each fall approximately 300,000 college or university students, most of them in their first post-secondary year, take a course in differential calculus (Blair, Kirkman, & Maxwell, 2012). In the US, Calculus I is a university-level course that typically covers limits, rules and applications of the derivative, the definite integral, and the fundamental theorem of calculus. Typically, over half of Calculus I students also took a calculus course in secondary school, which usually focuses on techniques of differentiation and integration. In comparison, university-level Calculus I is usually more rigorous in its treatment of concepts (including limits, graphical interpretations, definitions, etc.) and applications. Proofs are typically not part of Calculus I at either the secondary or post-secondary levels.

Internationally, first year university mathematics courses are consistently credited with preventing large numbers of students from pursuing a career in a STEM area (Steen 1988; Wake 2011). In the United States, STEM intending students typically enroll in calculus (though not necessarily Calculus I). In many European countries, STEM intending students instead typically enroll in abstract algebra or proof-based calculus (Wake 2011), as calculus is covered in secondary school.

Recent studies show that in the US and elsewhere students show less interested in a STEM majors paired with an increased need for STEM professionals in the workforce (Carnevale, Smith, & Melton 2011; Hurtado, Eagan, & Chang 2010; van Langen & Dekkers, 2005). Thus for those students that do choose a STEM major, there is a pressing need for them to be successful in first year mathematics courses so that they can continue in their chosen STEM major and ultimately meet the growing demand of the workplace for STEM graduates (PCAST, 2012; Wake 2011). However, student retention in STEM majors and the role of first year mathematics in student persistence is a major problem (Hutcheson, Pampaka, and Williams 2011; Pampaka, Williams, Hutcheson, Davis, and Wake 2012; Rasmussen and Ellis, 2013; Seymour and Hewitt 1997).

In order to better understand the terrain of calculus teaching and learning in the US, we are near completion of a five-year, large empirical study funded by the National Science Foundation and run under the auspices of the Mathematical Association of America. The goals of this project include: to improve our understanding of the demographics of students who enrol in calculus, to measure the impact of the various characteristics of calculus classes that are believed to influence student success, and to conduct explanatory case study analyses of exemplary programs to identify why and how these programs succeed. In this report, we present findings from our case study analyses at five exemplary calculus programs at institutions that offer a doctoral degree in mathematics. Understanding the features that characterize exemplary calculus programs at doctoral degree granting institutions is particularly important because these institutions produce the majority of STEM graduates.

The overall five-year project was conducted in two phases. In Phase 1 surveys were sent to a stratified random sample of students and their instructors at the beginning and the end of Calculus I. The surveys were restricted to “mainstream” calculus, meaning the calculus course designed to prepare students for the study of engineering or the mathematical or physical sciences. Surveys were designed to gain an overview of the various mainstream calculus programs nationwide, and to determine which institutions had more successful calculus programs. Success was defined by a combination of student variables: persistence in calculus as marked by stated intention to take Calculus II; affective changes, including enjoyment of math, confidence in mathematical ability, interest to continue studying math; and passing rates. In Phase 2 of the project, we conducted explanatory case studies at 18 different post secondary institutions, where the type of institution was determined by the highest degree offered in mathematics. In this report, we present findings from analyses of the five case studies at doctoral degree granting institutions.

THEORETICAL BACKGROUND

Analysis of our case study data is grounded in two complementary perspectives, the first of which draws on the *community of practice* perspective put forth by Wenger and colleagues (Lave &Wenger; 1991; Wenger 1998). A community of practice is a collective construct in which the joint enterprise of achieving particular goals evolves and is sustained within the social connections of that particular group. In achieving a particular joint enterprise, such as the teaching and learning of calculus, a community of practice point of view highlights the role of brokers and boundary objects. A broker is someone who has membership status in more than one community and is in a position to infuse some element of one practice into another. The act of doing so is referred to as brokering (Wenger, 1998). Boundary objects are material things that allow people to cross between different communities and facilitate progress on their joint enterprise.

The second set of ideas that we employ to make sense of our case study data draws on research in Higher Education that has extensively studied factors related to student retention at the post-secondary level, with a focus on the effects of student engagement and integration on persistence (e.g., Kuh et al., 2008; Tinto, 1975, 2004). According to Tinto's integration framework (1975), persistence occurs when students are socially and academically integrated in the institution. This integration occurs through a negotiation between the students' incoming social and academic norms and the norms of the department and broader institution. From this perspective, student persistence (a measure of success in calculus) is viewed as a function of the dynamic relationship between the student and other actors within the institutional environment, including the classroom environment.

METHOD

The survey results from Phase 1 provided information on which institutions are enabling students to be more successful in Calculus I (as compared to other institutions of the same type) per our measures of success. From this information, we were able to determine 18 institutions across all institution types that were more successful than others. Success was defined as a combination of increased student interest, enjoyment, and confidence in mathematics, persistence onto Calculus II, pass rates in Calculus I, and previously identified success on national measures of student understanding of calculus. Table 1 provides a brief description of the five selected doctoral granting institutions and why each was selected.

Institution		**Why Selected**
D1	• Large • Public	• Increased confidence, interest in math, and intention to take Calc II; Higher than expected Calc I pass rate
D2	• Small • Private • Technical	• Increased confidence, enjoyment of math, interest in math, and intention to take Calc II; Higher than expected Calc I pass rate
D3	• Small • Public • Technical	• Increased confidence, enjoyment of math, interest in math, and intention to take Calc II; Higher than expected Calc I pass rate
D4	• Large • Public	• Prev. identified implementation of best practices; high scores on national assessment of conceptual understanding of Calc
D5	• Large • Private	• Increased confidence, enjoyment of math, interest in math, and intention to take Calc II

Table 1: Description of case study sites.

Survey results, however well crafted and implemented, are limited in their ability to shed light on essential contextual aspects related to *why* and *how* institutions are producing students who are successful in calculus. The case studies were therefore designed to address this shortcoming by identifying and contextualizing the teaching practices, training practices, and institutional support practices that contribute to student success in Calculus I. As argued by Stake (1995) and Yin (2003), explanatory case studies are an appropriate methodology to study events (such as current practices in Calculus I) in situations in which the goal is to explain why or how, and for which there is little or no ability to control or manipulate relevant behaviors.

Four different case study teams (one per each type of institution—community college, bachelor, masters, and doctoral) conducted three-day site visits at the selected institutions. During the site visit each team, which consisted of 2-4 project team members, interviewed students, instructors, and administrators; observed classes; and collected exams, course materials, and homework. Common interview protocols for all 18 case studies were developed, piloted, and refined in order to facilitate comparison of calculus programs within and across institution type.

At the completion of each site visit the case study teams developed a reflective summary that captured much of what was learned about the calculus program, including key facts and features that were identified by both the case study team and the people interviewed as contributing to the success of the institution's calculus program. A more formal 3-4 page summary report was then developed by reviewing the reflective summary and transcripts and sent to the respective department of each institution as part of the member checking process (Stake, 1995).

At the five doctoral degree granting institutions, we conducted 92 interviews with instructors, administrators, and students for a total of more than 95 hours of audiorecordings. All interviews were fully transcribed and checked by a second person for accuracy and completeness. In order to manage this vast amount of qualitative data, a tagging scheme was developed to facilitate the location of relevant interview excerpts related to one of more of 30 different areas of interest. These areas of interest include such things as placement, technology, assignments and assessments, instructor characteristics, etc. Each interview was first chunked in terms of what we refer to as a "codeable unit." A codeable unit consists, more or less, of an interviewer question followed by a response. If a follow up question resulted in a new topic being discussed by the interviewee, then a new codeable unit was marked. Each codeable unit was then tagged with one or more of the 30+ codes. This data organization strategy then enable us to systematically identify all instances in which any interviewee addressed a particular topic area. Once these instances were located, then a more fine-grained grounded analysis proceeded. We used the facts and features documents to conduct initial cross case analysis to identify common features across the five doctoral degree granting institutions.

The set of 30+ codes was developed by representatives from each of the four different case study teams and consists of both a priori codes from the literature and codes for themes that emerged from the reflective summaries. The final set of 30+ codes underwent an extensive cyclical process in which representatives from each case study team coded the same transcripts, vetted their respective coding, which then led to refining, deleting, and adding new codes and operational definitions. Two different team members coded each transcript and the two coders resolved any discrepancies.

DISCUSSION

Cross case analysis of the five doctoral degree granting institutions led to the identification of seven features that contribute to the success of their calculus program. We first highlight what these seven features are followed by a discussion of the seven features in light of the communities of practice perspective and Tinto's academic and social integration perspective.

- *Coordination*. Calculus I (as well as PreCalculus and Calculus II) has a permanent course Coordinator. The Coordinator holds regular meetings where calculus instructors talk about course pacing and coverage, develop midterm and final exams, discuss teaching and student difficulties, etc. Exams and finals are common and in some cases the homework assignments are coordinated.
- *Attending to Local Data*. There was someone in the department who routinely collected and analyzed data in order to inform and assess program changes. Departments did this work themselves and did not rely on the university to do so. Data collected and analyzed included pass rates, grade distributions, persistence, placement accuracy, and success in Calculus II.
- *Graduate Teaching Assistant (GTA) Training*. The more successful calculus program had substantive and well thought out GTA training programs. These ranged from a weeklong training prior to the semester together with follow up work during the semester to a semester course taken prior to teaching. The course included a significant amount of mentoring, practice teaching, and observing classes. GTA's were mentored in the use of active learning strategies in their recitation sections. The standard model of GTA's solving homework problems at the board was not the norm. The more successful calculus programs were moving toward more interactive and student centered recitation sections.
- *Active Learning*. Calculus instructors were encouraged to use and experiment with active learning strategies. In some cases the department Chair sent out regular emails with links to articles or other information about teaching. One institution even had biweekly teaching seminars led by the math faculty or invited experts. Particular instructional approaches, however, were not prescribed or required for faculty at any of the institutions.
- *Rigorous Courses*. The more successful calculus programs tended to challenge students mathematically. They used textbooks and selected problems that required students to delve into concepts, work on

modeling-type problems, or even proof-type problems. Techniques and skills were still highly valued. In some cases these were assessed separately and a satisfactory score on this assessment was a requirement for passing the course.

- *Learning Centers*. Students were provided with out of class resources. Almost every institution had a well-run and well-utilized tutoring center. In some cases this was a calculus only tutoring center and in other cases the tutoring center served linear algebra and differential equations. Tutoring labs had a director and tutors received training.
- *Placement*. Programs tended to have more than one way to determine student readiness for calculus. This included: placement exams (which were monitored to see if they were doing the job intended), gateway tests two weeks into the semester and different calculus format (e.g., more time) for students with lower algebra skills.

The fact that all five of the more successful calculus programs at doctoral degree granting institutions had someone whose official job included coordinating the different calculus sections is noteworthy. This role of coordinator was not something that rotated among faculty, such as committee assignments do, but rather was a designated and valued permanent position. The existence of this position is, however, only part of the story. An equally important part of the story is the role that calculus coordinator, among others, played in creating and sustaining a community of practice around the joint enterprise of teaching and learning of calculus. In other words, calculus was not seen as being under the purview of one person, such as the coordinator, but rather calculus was viewed as community property.

Nonetheless, the calculus coordinator played a unique role within their community of practice. In particular, the calculus coordinator functioned as a broker between the more central members in the department that typically teach calculus and the many newcomers. At doctoral institutions, these newcomers to the calculus joint enterprise include visiting research or teaching faculty, post docs, lecturers, and graduate teaching assistants (GTAs). The regular meetings that the calculus coordinator convened provided occasions for newcomers to be enculturated into the norms and practices related to calculus. Long-term members of the community also used these meetings to reflect on their own and other’s practices. This reflection contributed to the sense of calculus as community property, as well as to the negotiation of communal practices.

We identified a number of boundary objects that helped to facilitate this enculturation, including historical records of passing rates, current grade and persistence data, student evaluations, various training manuals (especially for GTAs and visiting faculty) and the development of common assignments and assessments. Other brokers in the joint enterprise of teaching and learning calculus included, for some of the five doctoral institutions, the graduate teaching assistant trainers and leaders, department chair and the person whose responsibility it was to collect and disseminate to the department local data concern student pass rates and persistence and/or the correlation between these measures of success and the placement process. We conjecture that their attention to local data and continual improvement efforts contributed to a climate in which those involved with calculus teaching were always striving for improvement. Indeed, it was striking to us that none of the five case study institutions considered themselves to be particularly successful in calculus. That is, none of the five institutions in our case studies felt that they had everything just right.

A community of practice perspective helps to illuminate the how and why particular calculus programs are successful from a point of view that highlights faculty and administration. In our view, Tinto's academic and social integration perspective sheds equally important insight into how and why calculus programs are successful from a student point of view. In particular, almost without exception the students we talked with at the five doctoral institutions noted that they felt their calculus course was academically engaging and challenging (despite the fact that the vast majority had taken calculus in high school) but that there were a number of resources available to them to help them be successful. These resources included well-developed math help centers where students felt they received the help they needed and availability of instructor's and GTAs office hours. Other factors that contributed to students' academic and social integration included student centered instruction, common space in the math department where students could gather to work on homework, dorms that provided them with opportunities to interact with like minded fellow students, and in some places a cohort system or strong student culture that provided cohesion between students.

In summary, our ongoing analysis of the five successful calculus programs at doctoral institutions is highlighting a number of structural and programmatic features that other institutions would likely be interested in adapting. The ongoing theoretical analysis points to the importance of how these structural and programmatic features come together for faculty so that calculus is seen as community property and for the academic and social integration so critical for students' continued interest, enjoyment, and persistence in calculus. Our analysis that combines a community of practice perspective with the seminal work of Tinto on academic and social integration also sets the stage for the development of a more comprehensive model of successful college calculus programs.

Acknowledgments

Support for this work was funded by the National Science Foundation under grant No. 0910240. The opinions expressed do not necessarily reflect the views of the foundation.

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QUALITATIVE FACETS OF PROSPECTIVE ELEMENTARY TEACHERS' DIAGNOSTIC COMPETENCE: MICRO-PROCESSES IN ONE-ON-ONE DIAGNOSTIC INTERVIEWS

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Going beyond measuring accuracy of teachers' judgments of students' achievements, this paper focuses on how prospective elementary teachers proceed in one-on-one diagnostic mathematics interviews. As part of the project ***diagnose:pro****, prospective elementary teachers (PTs) conduct diagnostic interviews with children in grade one and reflect on diagnostic strategies afterwards. Findings of the study lead to a model of strategic elements in diagnostic proceeding and suggest types of diagnostic strategies. It is also discussed how awareness of diagnostic strategies can be developed to foster sensitive every-day qualitative diagnostic attitudes in PTs.*

INTRODUCTION

Based on the domains suggested by Shulman (1986) or Ball et al. (2008), *pedagogical content knowledge* (PCK) includes knowledge about *common* mathematical conceptions or misconceptions that are frequently encountered in the classroom. Besides theoretical instructions in teacher education or through a longer period of teaching experience, acquisition of this knowledge can also be enhanced as teachers examine *individual* cases: Analyzing a student's error to find out more about the underlying misconception refers to *knowledge of content and students* (KCS), which is regarded as subdomain of PCK by Ball et al. (2008). Identifying unique facets of such individual cases may contribute to the understanding of widespread (mis)conceptions (e.g. Peter-Koop & Wollring, 2001; Hunting, 1997), thereby serving the elaboration of KCS and fostering the development of a teacher's diagnostic attitude.

Diagnostic competence is an important element of adaptive teaching competence since detailed information on a student's individual conception can support the design of appropriate learning opportunities (Wang, 1992). Recent studies concerning teachers' diagnostic competence mainly focus on measuring accuracy of teachers' judgments (e.g. regarding a rank order within classes; cf. Südkamp et al., 2012). In these studies, diagnostic competence is "operationalized as the correlation between a teacher's predicted scores for his or her students and those students' actual scores" (Helmke & Schrader, 1987, p. 94). Contrary to this paradigm, there is a wide field in mathematics education research which deals with qualitative aspects of children's wide-ranging learning developments. However, little is known about the *processes of diagnosing* which lead teachers to the evaluation of an individual student's learning development in these process-oriented observations: Focusing on approaches of informal formative assessment (cf. Ginsburg, 2009), *how* do teachers arrive at a diagnosis of a student's conception via oral questioning or observation? As differences in accuracy might be due to teachers' different ways of diagnostic proceeding and analyzing, *how* do they get to an appropriate interpretation of a child`s utterances or can be helped to do so?

Setting the frame for this report, the project ***diagnose:pro*** emphasizes the need to sensitize prospective elementary mathematics teachers (PTs) to varieties, ranges and depth of young children's mathematical thinking. Therefore, graduate students (Master of Education) prepare, conduct and analyze one-on-one interviews about arithmetic problems with children in grade one. One part of the project focuses on the cognitive diagnostic strategies PTs use in the reflection of those interviews. Thereby, it responds to the detected lack of knowledge regarding qualitative facets of interpretation in diagnostic situations. Findings in this scarcely explored domain are likely to strengthen the "power of task-based one-on-one interviews" (Clarke, 2013) in daily practice.

THEORETICAL FRAMEWORK

Diagnostic mathematics interviews in teacher education

As teachers have to cope with an increasingly complex and demanding professional landscape, beginners and experienced teachers need to develop a sensitive, every-day constructivist view on their students' individual mathematical thinking and progress. High-quality professional development engages teachers in concrete tasks (e.g. tasks of assessment or observation) and focuses on students' learning processes (Borko et al., 2010). Preparing, conducting and analyzing one-on-one interviews provide novices with substantial learning opportunities as they study students' mathematical conceptions (cf. Prediger, 2010; Sleep & Boerst, 2012). Developing a sensitive diagnostic attitude is also supported by involving PTs in research projects that include interview assessments (cf. Jungwirth et al., 2001; Peter-Koop & Wollring, 2001).

Diagnostic interviews not only serve as a method in mathematics research and teacher education, but have also reached the classroom. Research-based frameworks (e.g. concerning learning trajectories) resulted in the design of standardized task-based interviews to assess children's thinking in the context of mathematics learning in school – in short, to provide teachers with weighty arguments for sound diagnoses and for the preparation of adaptive learning arrangements. Here, interview tools and the prepared analysis (via empirically based growth points) serve to improve teachers' professional development as they are encouraged to actively explore qualitative facets of children's approaches to mathematics tasks (e.g. ENRP task-based assessment interview/CMIT/EMBI; cf. Clarke, 2013; Bobis et al., 2005; Peter-Koop et al., 2007).

A process-oriented approach to diagnostic competence

Ensuing a comprehensive understanding of diagnostic competence, expertise in this area reaches beyond teachers' accuracy in measuring students' achievements. Besides relating diagnostic competence to KCS as part of PCK, it additionally includes rather vague aspects like diagnostic sensitivity, curiosity, an interest in students' emerging understanding and learning or the aptitude to gather and interpret relevant data in non-standardized settings (e.g. Prediger, 2010). Acting within a diagnostic situation in a one-on-one interview which intends to enlighten students' (mathematical) thinking can be regarded as an integral element of a multidimensional spiral process (Klug, 2011; Klug et al., 2013). According to this model, a *pre-actional phase* (e.g. considerations of preparing diagnostic activities; choice of tasks/methods) prepares an *actional phase* (including data collection and data interpretation) that is followed by a *post-actional phase*. The latter implies taking the necessary action from data collection and interpretation which feeds to the design or the evaluation of a concept for an individual support in a repeated run through phases of the diagnostic macro-process.

Cognitive elements in the micro-processes of the actional phase of diagnosing

Researchers in mathematics education have partially specified the challenges that teachers face within such diagnostic macro-processes. Focusing on *micro-processes within the actional phase*; **collecting** data, **interpreting** and drawing **conclusions** have deep impact on the diagnosis from an interview and are likely based on different kinds of knowledge (e. g. KCS, see fig. 1). In this sense, proceeding in a one-on-one diagnostic interview is vitally influenced by cognitive processes. A person`s (verbal) articulation (e.g. ways of questioning, confirming) and intentional decisions (e.g. switching between tasks) may reveal facets of these ongoing internal considerations.

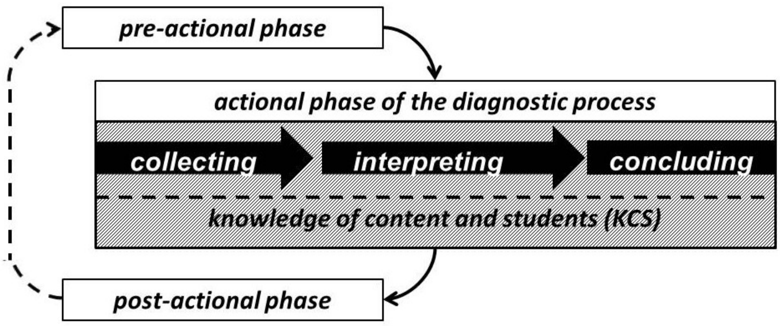

Figure 1: Differentiating the micro-process in the actional phase of diagnosing.

Moyer & Milewicz (2002) identified general questioning categories (check-listing/ instructing/probing/follow-up questions) used by PTs while collecting data in diagnostic interviews. As there is no direct access to students' conceptions in these interviews, they "must be reconstructed by interpreting their utterances" (Prediger, 2010, p. 76) as "the interviewer attempts to construct a model of the student's mathematical knowledge" (Hunting, 1997, p. 149). Thus, it is also important to reach a substantial perception of the diagnostic situation while interpreting. According to Barth & Henninger (2012), this "includes the ability to structure the situation cognitively, the ability to change the focus of attention and the willingness and ability to adopt other perspectives" (p. 51) which leads to the generation/testing of hypothesis. Moreover, there is a demand "to know which information or knowledge sources play the most important role during the process of diagnosing students' learning prerequisites" (Barth & Henninger, 2012, p. 50). But the implications of "gathering information, acting systematically" (Klug et al., 2013, p. 39) within the *actional phase* are not yet entirely clear for one-on-one interviews in mathematics education.

RESEARCH QUESTIONS

Aiming at an empirically grounded theoretical framework for a qualitative view on PTs' cognitive activities in one-on-one interviews with children, the main purpose of the partial study presented in this paper is to detect traits of diagnostic strategies:

- What cognitive elements characterize the PTs' diagnostic strategies when diagnosing individual arithmetic approaches in one-on-one mathematics interviews with children at the beginning of grade one?
- Which types of (flexibly used) diagnostic strategies can be reconstructed from interviews they or others have been conducting?
- What kind of knowledge (e.g. KCS) is used during the diagnostic proceeding?

METHODS

Data collection since 2011 included studies via video-vignettes (which led to written comments of 31 PTs on diagnostic scenes) and switched to video/audiotaped peer-talks among 28 graduate students about video-scenes of diagnostic interviews in 2012. Until fall 2013, retrospective interviews with seven PTs who had conducted a diagnostic mathematics interview with a first-grade child (cf. Moyer & Mielewicz, 2002) complemented data collection (cf. Reinhold, 2013). All PTs attended a mathematics methods course in the last year of their university studies (Master of Education). This course provided the opportunity to conduct individual diagnostic interviews with up to six first-graders per PT in cooperation with an elementary school. First drafts of these interviews were prepared at the beginning of the course where the PTs could refer to previous theoretical work on concepts of arithmetic learning trajectories and the method of task-based mathematics interviews (e.g. EMBI; Peter-Koop et al., 2007).

With only general advice at the beginning of the retrospective interviews, the PTs were asked to "analyze the interview" while watching the video-recording of an interview they had conducted. The PT was requested to stop the video at any point in order to comment on the diagnosis he or she would derive from this specific situation or related observations. If comments were rather short or pure in detail, the PT was asked to explain what knowledge, information or evidence warranted his or her hypothesis. In addition to this concrete task (diagnosis of the child's conception or knowledge), the PT reflected on his or her proceeding in a more general way: Referring to the preliminary design of the interview, the PTs were asked to comment on the choice of some selected tasks, on the wording of questions, on their own gestures or on deviations from the sketch. What prompted them to react to a child's response? What was taken into account to confirm a diagnosis?

The analyses of all re-interviews are based on Grounded Theory methodology and methods which include open, axial and selective coding (cf. Corbin & Strauss, 1990). The interpretation, coding and contrasting comparison of the data are supported by the software ATLAS.ti which enabled the research team to directly code video-data.

FINDINGS

Analyses of the study's data support the notion that cognitive elements of PTs' ways of diagnostic proceeding in one-on-one interviews often resemble basic processes in qualitative data analysis. This includes acts like *collecting, interpreting* and *concluding* within diagnostic micro-processes (see fig. 1). Furthermore, the findings contribute to the identification of sub-categories of collecting, interpreting or concluding and to interrelations among these sub-categories (see fig. 3). Excerpts from re-interviews with Ann and Sue, master's students in their last year of studies, display exemplary *facets of **interpreting** within the diagnostic micro-process of the actional phase*.

Facets of interpreting in a diagnostic micro-process: Comparing and contrasting

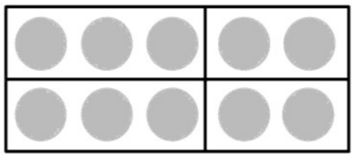

Fig. 2: Structured box

In her interview with six-year old Tom, Ann offers empty boxes for ten eggs and some chestnuts. The boxes of ten are partitioned in four fields (see fig. 2) since Ann intends to find out how children use these structures for counting or for abbreviated enumeration (i.e. counting strategies including subitizing parts of an amount, cf. Besuden, 2003). Ann stops the video and comments on a scene where she has just put five chestnuts into the box (forming a row). Tom is asked to add further chestnuts in order to get a result of eight and fills two, then one more into the box. Answering Ann, he remarks "Because I left two free, one more'd be nine, then ten."

> Ann (07:08): And there I noticed that he, eh, always took ten as a starting point for the higher numbers, well, for eight and a moment ago for nine. He remembers, okay there are ten in the package, and then he always counts backwards.[1]

In her comment, Ann compares and refers to Tom's previous work ("a moment ago"). *Comparing* details to a child's previous utterances or actions, to that of others or to the PTs own concept may also occur in terms of *contrasting* different scenarios:

> Ann (08:30): Here, he saw, okay, there are four in one box and there are another four in the second box, well, four plus four equals eight, but he didn't do it that way in the next task. There he'd count single ones, it was done quite differently.

Facets of interpreting in a diagnostic micro-process: Coding

Sue uses the same kind of tasks in her interview with six-year old Ben. She wants him to find out how many chestnuts have to be added to four chestnuts (which are presented in the "square" on the right side of the box) to get a result of seven. Ben replies by first adding two (forming a "rectangle"), then one more to reach seven (Ben: "These are six, then seven."). Sue codes these actions by creating the new term "auxiliary calculation":

> Sue (05:40): Responding to my enquiry, how he'd done this, now, how many he'd add, actually, I only wanted to hear "three", well, he would seize on his, let's say "auxiliary calculation", six plus one equals seven.

[1] All interview excerpts are translated into English by the author.

PTs are similarly coding observed phenomena as they try to grasp unfamiliar, but obviously central aspects of a child's conception. Codes are often referred to later in the interviews (e.g. Sue's reference to the code "auxiliary calculation", 22:30) and may also substitute established terms (e.g. "shortcut" instead of "subitizing").

Facets of interpreting: References to knowledge of content and students (KCS)

To describe the children's performances in the re-interview, PTs also try to make use of standardized terms. These refer to previously acquired KCS and seize on theoretical concepts that were studied in the methods course before conducting the interviews:

Sue (04:50): At the beginning, Ben definitely used counting strategies. He saw those four and went on counting from that summand. He noticed, if I add two I'll get six, thus, he didn't go like "five…six", but he said, okay, two, that's six.

Although details of the counting strategy "counting on by steps of two" are not reflected here, referring to KCS tends to be an important element of PTs' diagnostic strategies: PTs do use information from their teacher preparation courses. They partially retain general knowledge of children's development of mathematical conceptions, but often remain unfocused in supporting their interpretation with this knowledge as we see in Ann's explanations of the term "understanding of quantities":

Ann (15:17): But, Tom doesn't have, eh, a complete understanding of quantities at his disposal, partly he did, partly he didn't. It's when a child notices that a number is now, eh, bigger than the number before, or that one can draw conclusions from one equation to the next, that is connected to the first one.

Types of diagnostic strategies

Following Grounded Theory methodology, distinct types of diagnostic strategies with a stress on different elements of diagnostic proceeding (i.e. on the exemplified (sub-) categories) are detected. As indicated by the arrows in fig. 3, PTs' diagnostic strategies are far from being a linear process and may be driven by general dimensions of diagnostic strategies (e.g. topographic or symptomatic search; Cegara & Hoc, 2006).

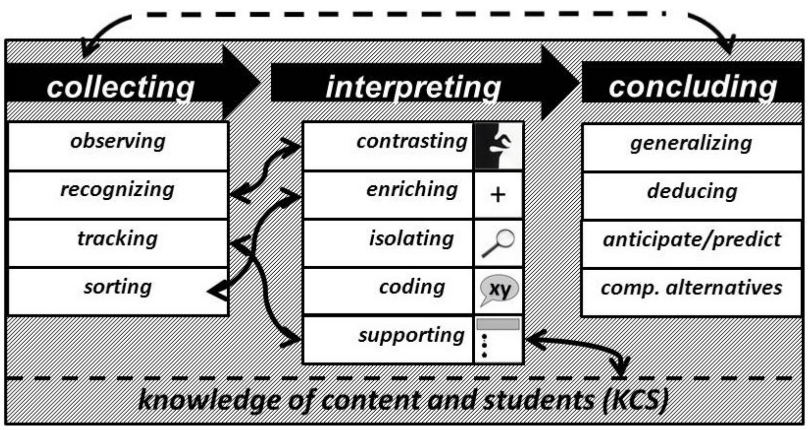

Figure 3: Sub-categories of collecting, interpreting and concluding.

Following the strategy *descriptive collector*, the PT searches rather typographically, focuses on collecting and describing the child's actions and neglects both interpreting and concluding. A *concluding collector* strategy is characterized by skipping elements of interpretation as collecting directly leads to conclusions. Symptomatic searches occur when elements of interpreting prevail in a *branched interpretation*. Here, interpreting, collecting and concluding are intertwined and frequently linked to KCS.

DISCUSSION

The findings of the study provide evidence of sub-categories of collecting, interpreting and concluding within micro-processes of the actional phase of diagnosing, point at KCS within these processes and hint at a variety of strategy types. Thus, results enrich the idea of "interpreting" in the actional phase of diagnosing suggested by Prediger (2010) or Barth & Henninger (2012). Bearing in mind that the findings are restricted to a particular type of tasks (arithmetic) and that they refer to a rather small number of participants (n=28 in peer-talks; n=7 individual interviews), the study outlines new topics in the field of teachers' professional development: It raises the hypothesis that reflecting on facets of proceeding in one-on-one interviews enhances PTs diagnostic sensitivity and increases their knowledge of assessing children's mathematical abilities. As an integral element of PCK, this might include awareness of "strategic diagnostic tools" which help to master diagnostic challenges in the classroom. Thus, further activities of the project ***diagnose:pro*** will explore how the findings (elements of diagnostic strategies/types of strategies) can be taken up in university courses and contribute to appropriate diagnoses of children's concepts in one-on-one interviews.

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SOFYA KOVALEVSKAYA: MATHEMATICS AS FANTASY

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What is entailed in doing mathematics and becoming a mathematician? Working from an autobiographical sketch and biographies of Sofya Kovalevskaya (1850-1891), the first woman generally thought to have gained a doctorate in mathematics, and using the Lacanian notion of desire, I examine the forces that shape and influence engagement with mathematics. This work has consequences for mathematics education in examining the construction of identity/subjectivity in teachers and students alike.

INTRODUCTION

Over the years, mathematics education research has renewed itself in its quest for understanding the nature of learning and teaching mathematics by studying mathematicians: from finding out what mathematicians know, to describing what they do (the practices of mathematicians, e.g. Burton, 2004), and then to exploring who mathematicians are. More recently, researchers, including Tony Brown (2008; 2011), Baldino and Cabral (2006), and Walshaw (2004), have focused on the subjective aspects of being a mathematician, using a postmodern and, more particularly, a psychoanalytic approach. After waves of study on beliefs, emotions, and psychodynamics, the present focus on subjectivity and identity is an inevitable consequence in the search for the factors relating to engagement with mathematics, apart from the usual considerations of students, teachers, classrooms, and tasks.

In this paper, I contend that any effort in understanding how and why students (and, in general, people) take up mathematics must begin with the mathematical subject, the individual in the encounter with the discipline, and that, contrary to expectations, the driving force in the endeavour is desire, on the part of both the subject and the discipline as a cultural phenomenon. I present, as an example of a trajectory in mathematics, the life of Sofya Kovalevskaya (1850-1891), the first woman in the world to have a professional university career in mathematics (Audin, 2013). Kovalevskaya did original work in three areas of mathematics, any of which would have been enough for a doctorate in mathematics. The first area of her work was in solving partial differential equations (one of her results, called the Cauchy-Kovalevsky theorem, is the basic theorem on partial differential equations over complex numbers). In the second, she extended some results of Euler, Lagrange and Poisson involving the reduction of Abelian integrals to elliptic integrals, and in the third, she made some additions to the work of Laplace on Saturn's rings.

Accounts by and about mathematicians regarding their journeys in mathematics have long been neglected as a source of knowledge about the discipline. These accounts have generally been concerned with historiography, showing the unique lives and accomplishments of mathematicians. Using data from an autobiography and biographies, I argue that her journey in mathematics was shaped primarily by the face/t/s of desire. I then discuss the implications for school mathematics.

THEORETICAL INFLUENCES

In keeping with the psychoanalytic approach of the researchers above, I am guided by Lacan's theory of the subject and the construct of desire in the subject's psychic economy played out in three registers or orders which obtain at every stage of the subject's experience: the Imaginary, the Symbolic, and the Real. The Imaginary is the realm of "images, conscious or unconscious, perceived or imagined" (Lacan, 1973/1981, p. 279) of the people and objects in the world present to us. The Symbolic is derived from the "laws" of the wider world in its structure and organization, and shapes or interpellates (Latin: *inter*/between, within, *pellere*/push) the subject, becoming the Other for the subject. The Symbolic is enabled by language as it is language that gives us the structures for the signifiers for the "I" and the Other, and for articulating the lack and misidentification of the self with the specular image. The Real is the unmarked backdrop against which the Imaginary (image-based) and the Symbolic (word-based) come into play, the screen on which images and words unfold and move. In the confluence of these three registers, the subject comes into play or is played with the forces of separation and alienation that lead to desire. Thus, the Lacanian subject experiences desire as a manifestation of lack as the subject seeks to both identify and separate itself from the Other.

In applying these concepts to the interplay of the cultural phenomenon of mathematics and the subjects who encounter and engage with it, I tease out more carefully the notion of the Other, which, despite being a well-known concept in the social sciences, has a meaning in each of the three Lacanian registers. In the Symbolic, the Other is the code or discipline of mathematics, its knowledge and traditions with its concomitant demands and costs. In the Imaginary, the Other is the Imaginary others, the people who engage with mathematics in some form including the mathematicians who embody and make incarnate the code of mathematics. In the Real, the Other is the Lacanian *objet a*, which is the object-cause of desire, and which in some instances of desire will again turn out to be mathematics.

Mathematics as a cultural phenomenon is far-reaching in its forces and effects. With respect to the discourse of literary and cultural criticism, the noted Lacanian theorist, Mark Bracher, writes: "Insofar as a cultural phenomenon succeeds in interpellating subjects—that is, in summoning them to assume a certain subjective (dis)position—it does so by evoking some form of desire or by promising satisfaction of some desire" (1993, p. 19). I demonstrate in the life of Kovalevskaya the desire that mathematics evokes as it holds out the promise of satisfaction.

Bracher (1993, pp. 20-21) elaborates four forms of desire by exploring the ambiguities in Lacan's dictum that desire is desire of the Other. These four forms come from the oppositions of two types of desire, the desire to be (narcissistic) and the desire to have (anaclitic), and the two roles, active and passive, that the subject and the Other take (when the one is active, the other is passive). I present these forms in the following table and give examples of each desire with respect to mathematics:

	Narcissistic (to *be*)	**Anaclitic** (to *have*)
Passive	I can desire to be the object of the Other's love (or the Other's admiration, idealization, or recognition). **"I want to be recognized by mathematics and its community as a mathematician."**	I can desire to be desired or possessed by the Other as the object of the Other's *jouissance*. **"I want to be desired by mathematics as, e.g., a means of adding to its glory."**
Active	I can desire to become the Other – a desire of which identification is one form and love or devotion is another. **"I want to be a mathematician."**	I can desire to possess the Other as a means of *jouissance*. **"I want to possess mathematics as, e.g., enjoyment, as a personal treasure."**

Table 1: Forms of desire (Bracher, 1993)

In describing desire, Lacan used the notion of the asymptote, always approaching but never attaining an object because there is no object of desire, only an object-cause of desire. The origin, nature, and path of desire are elaborated by Grosz (1990), in her feminist introduction to Lacan:

> Lurking beneath the demands for recognition uttered by the subject (to the other) is a disavowed, repressed or unspoken desire. Desire is a movement, a trajectory that asymptotically approaches its object but never attains it. Desire, as unconscious, belies and subverts the subject's conscious demands; it attests to the irruptive power of the 'other scene, the archaic unconscious discourse within all rational discourses. (p. 188)

Further, in order to stage its unconscious, unknown, and unacknowledged desire, the subject resorts to fantasy; "a fantasy constitutes our desire, provides its coordinates, that is, it literally 'teaches us how to desire'" (Žižek, 1997/2008, p. 7)

METHODOLOGY

The data for this study were accounts of Kovalevskaya' life published in English. I did not know of Kovalevskaya as a mathematician despite my many years of learning and teaching mathematics. I came upon her serendipitously from a collection of stories by Alice Munro (2009), *Too much happiness*. Munro had been looking for something else when she came upon Kovalevskaya, and was struck by the unusual combination of mathematician and novelist. I soon found that there was much more material on Kovalevskaya including a memoir, *A Russian childhood*; indeed, there is a small industry on her life and work among historians of mathematics and science, a few mathematicians, and a few who are interested in gender studies. So far these give different perspectives on her life and the myths that have been created around her; none of these takes a psychoanalytical approach. These studies also posed the challenge of translation into English and a related matter, citation. Much of the primary source material about Kovalevskaya is in other languages (Russian, French, and German) and the English translations are not uniform (for example, the word, "imagination", in place of "fantasy").

Using biographies and Kovalevskaya's memoir as data, I searched for evidence of desire related to mathematics and being/becoming a mathematician. These included quotations (oral and written), events, and observations related to mathematics and mathematical activity. I then carried out a thematic analysis, informed by the lens of the above forms of desire.

ANALYSIS AND DISCUSSION

Of the four forms of desire, the one that stands out most clearly in Kovalevskaya's life is **active anaclitic desire**: mathematics was the Other that she wanted to possess. She writes of hiding an algebra textbook under her pillow and reading it through the night as well as of her protracted fascination with the notes of a calculus course that was used as wallpaper of her nursery. Mathematics became a part of her from the long hours spent staring at those hieroglyphics and symbols (in particular, the symbol for the limit) and sleeping with those algebraic equations and expressions. Indeed, when she is introduced to these symbols later, her professor remarks that she understands them as if she had known them in advance. She also writes of the trigonometry she devised in order to understand a physics textbook written by her neighbor. Also her desire for higher education in mathematics was inflamed by the political movement of the times (the serfs had been recently emancipated and there was much hope for reforms such as independence and higher education for women). As a woman, she was barred from classes in mathematics and science and was smuggled into these classes at the university by men sympathetic to the cause, but it was evident that true possibilities for higher education lay outside Russia, further afield in Europe.

Kovalevskaya continued her pursuit of mathematics by journeying though the major university cities in Europe seeking to be admitted or to attend classes (all barred to women). On the recommendation of her professor, Königsberger at Heidelberg, Kovalevskaya finally made her way to Weierstrass in Berlin in the desire and hope of being tutored by the best in the field of mathematics. Weierstrass (some thirty years older and a bachelor who lived with his spinster sisters) was so confused by her presence as a woman wanting to study mathematics that he gave her a list of problems to do in the hope that she would find them too hard and not return. She did return and amazed him with novel solutions that demonstrated unusually great depth of understanding. Her desire to possess higher mathematics led her to a relationship with Weierstrass as a colleague, no longer that of teacher and student.

Closely intertwined with active anaclitic desire is **active narcissistic desire** where the subject seeks to become the Other, to identify with or to be devoted to the Other. The Other of mathematics was embodied for Kovalevskaya in Weierstrass. He was her primary model of a mathematician as she strove to adopt his style: "These studies [with Weierstrass] had the deepest possible influence on my entire career in mathematics. They determined finally and irrevocably the direction I was to follow in my later scientific work: all my work has been done precisely in the spirit of Weierstrass" (Kovalevskaya, 1889/1978, p. 218). This meticulous attention and devotion to the requirements of doing mathematics as exemplified by Weierstrass worked to her detriment in that some mathematicians, in particular Felix Klein, charged that it was Weierstrass, and not her, who did the work for which she was given credit. Klein wrote: "Her works are done in the style of Weierstrass and so one doesn't know how much of her own ideas are in them" (cited in Rappaport, 1981, p. 564), pointing to a loss of boundary in the fusion of her mathematical self in Weierstrass's style of doing mathematics. Weierstrass came to occupy a special place for her as a father-figure, his tone in his letters to her deepening from the formal to one of encouragement and support. In one letter, he refers to himself as her Spiritual father. This was in keeping with the substitutes in her life; she was parented by her nurse (her mother to her nurse when she was brought into company: take your savage away, she is not wanted here), she engaged in a platonic marriage, and she sought a substitute father in Weierstrass.

In seeking to identify with being a mathematician, Kovalevskaya carried out the work of a mathematician in writing and publishing. It was vital to her that her work was published in recognized arenas of mathematics at the time, namely *Crelle's Journal* and *Acta Mathematica*: "At this writing *Acta Mathematica* is regarded as one of the foremost mathematics journals in scholarly importance. Its contributors include the most distinguished scholars of all countries and deal with the most 'burning' questions – those which above all others attract the attention of contemporary mathematicians" (Kovalevskaya, 1889/1978, p. 221). In these way she worked towards belonging to the community of mathematicians (the Imaginary others of mathematics).

Beside the two active forms of desire, there are two passive forms of desire. I begin with **passive narcissistic desire**, the desire to be object of the Other's love, admiration, idealization, or recognition), which Lacan calls the strongest form of desire. Kovalevskaya desired strongly to be recognized as a mathematician; "At that time my name was fairly well-known in the mathematics world, through my work and also through my acquaintance with almost all the eminent mathematicians of Europe" (Kovalevskaya,1889/1978, p. 222). What is in a name and in wanting one's name to be known? For Kovalevskaya, it was her very subjectivity and her desire for posterity in the "mathematics world". Her desire was for a teaching position in a Russian or European university; the times dictated that her only opportunity would be at a school for girls. Despite an enthusiastic endorsement from Weierstrass, this desire went unfulfilled until with the help of Mittag-Leffler (also a former student of Weierstrass) she secured a position teaching mathematics at Stockholm University in Sweden.

Kovalevskaya sought to be desired by mathematics in assuming the position of the first female editor, and only the second editor, of a mathematical journal, *Acta Mathematica* (its first editor and founder was Mittag-Leffler); such a position being occupied by a woman was again unheard of at the time. A further example of seeking the favours of mathematics was in her competing for and winning the *Prix Bordin*, then of the level of a Fields prize in mathematics, the high quality of her submission, *On the rotation of a solid body about a fixed point*, being noted and rewarded with greater prize money.

Her desire for recognition met with mixed results. As a woman studying mathematics, she was often considered a freak of nature which contributed to her gaining the status of a celebrity; people stopped in the street, pointing her out as the young woman who took her studies seriously. Though she could not be a woman lecturer in Russia nor a member of the Russian Academy of Sciences, the local newspaper carried this item: “Today we do not herald the arrival of some vulgar insignificant prince of noble blood. No, the Princess of Science, Madam Kovalevskaya, has honoured our city with her arrival. She is to be the first woman lecturer in all Sweden” (Flood and Wilson, 2011, p.167). But she met with opposition in some quarters, for example, from the playwright, August Strindberg, who abhorred the idea of a female academic.

The remaining form of passive desire is **passive anaclitic desire** or the desire to be desired or possessed by the Other as the object of the Other’s enjoyment. In Kovalevskaya’s life, the desire to be possessed by mathematics did not prove to be so strong as she had an equal passion for literature writing theatre reviews, poems (for herself), a novel, a memoir and play. For her, mathematics was to be revered: “a very lofty and mysterious science, which opened out to those who consecrated themselves to it a new and wonderful world not attained by simple mortals” (Kennedy, 1983, p. 17) but she continues:

> As far as I am concerned, during my life I could never decide whether I had a greater inclination toward mathematics or literature. Just as my mind would tire from purely abstract speculations, I would immediately be drawn to observations about life, about stories; at another time, contrarily when life would begin to seem uninteresting and insignificant then the incontrovertible laws of science would draw me to them. It may well be that in either of these spheres, I would have done much more, had I devoted myself to one exclusively, but I nevertheless could never give up either one completely. (p. 17)

Hence she did not devote herself entirely to mathematics, implying that mathematics was not “everything” for her, and perhaps that she could have been better had she been more faithful to it. Could she have served two masters or two gods? Does mathematics brook no other interests? She pushed away the one (mathematics) as she reached for the other (literature) but then later returned to her first love and passion. For Kovalevskaya, literature provided counterpoint to the “abstract speculations” of mathematics, both literature and mathematics being variations on the theme of the creative. It is interesting that she realizes that she could have accomplished more had she focused exclusively on one of the two but that she was willing to sacrifice that achievement in order to keep a foot in both worlds.

The leitmotif of her life: Asymptotic desire

The four forms above underpin the central theme of asymptotic desire in Kovalevskaya’s life. For Lacan, every desire is born out of lack, out of alienation and separation. Kovalevskaya’s desire arose out of various lacks: of not being allowed to take her place as a mathematician, of not being complete as a mathematician, and of not meeting the requirement of being male (the identities demanded by the "masculinities" of mathematics are explored in Mendick (2006)). Desire, constant, repetitive, and forever circling, in the two spheres of literature and mathematics can be seen as an attempt to address and reconcile the various aspects of herself with respect to the registers of the Imaginary (she was interested in life, its characters, its appearances, and its illusions) and the Symbolic (the words of literature and the symbols of mathematics that she could marshal to give life to her thoughts and ideas). Her desire was fed by both avenues, the one coming to the fore as interest in the other faded or was blocked in some way.

CONCLUSION

The analysis above shows that the dimensions of subjectivity and desire in the mathematical endeavour are significant in probing human relationships with mathematics. Kovalevskaya's journey is an effect of desire orchestrated by the cultural phenomenon of mathematics and by her lack as a subject. Kovalevskaya was circumscribed by the signifiers of 'woman', 'Russian' and 'mathematician', none of which would have come up as an issue of struggle in a given society or community. Only when her desire was hemmed in by these did they become forces by which she was buffeted. Looking back on Kovalevskaya's life, it seems to me that the distance from the place of mathematics as fantasy that she accessed through her mathematical work to the reality of her life in the circles in which she moved was too great. The metric needed to conceptualize that distance would take a century and more of social upheaval. The costs were too inordinate to bear and the cold realization is that mathematics is indeed, even with the gifts of genius and charm, not for the faint of heart. In the end, she was unable to realize her dreams to the extent of her desire. She had started with quadratures and asymptotes. Her life was an ode to her attempts of squaring the circle amid her trajectory of asymptotic desire in search of her old friend and lost object, the limit.

The implications for the kind of knowledge that has been excavated above in mathematics and mathematics education are far-reaching. What is truly entailed in doing mathematics and being a mathematician? If what is true for Sofya Kovalevskaya is also true of others in becoming mathematicians, then desire is a paramount consideration. To this day, her life and work remains an inspiration for mathematicians and those who would be mathematicians. The most recent work on Kovalevskaya is a passionate subjective account by Michèle Audin (2008/2011), a renowned present-day mathematician who found Kovalevskaya from her own work on integrable systems.

Further, official documents that describe what is entailed in learning and/or doing mathematics (such The *Adding it up* document, Kilpatrick *et al*, 2001) as a means of helping children develop as mathematicians has five strands, none of which addresses the subjective aspect of the endeavour on both the parts of the student and the teacher. In the encounter between the rational discourse of mathematics and the bundles of subjectivity that are students and teachers, how, as teachers of mathematics and researchers in mathematics education, how must we conduct ourselves? I contend that we consider that mathematics may not be for all, and that it is near transgressive that we do not address the personal costs and defences (Nimier, 1993). What is at stake is more than the curriculum, the delivery, and tasks; it is a recognition of the very nature of subjectivity, of who we are when we do mathematics, of the subject positions we are called to be in order to do mathematics. It is a matter of what drives the engagement with mathematics, namely, desire.

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GRAPHING CALCULATOR SUPPORTED INSTRUMENTATION SCHEMES FOR THE CONCEPT OF DERIVATIVE: A CASE STUDY

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This paper reports on the role of the graphing calculator (GC) in the learning of derivatives and instantaneous rate of change. In a longitudinal study, we administered task based interviews before and after the introduction of calculus. We analyzed students' use of the GC in these interviews. This paper reports on the case of one student, Andy, who is a resilient user of the GC while he develops into a flexible solver of problems on instantaneous rate of change. His case demonstrates that, although the GC is meant to promote the integration of symbolical, graphical and numerical techniques, it can facilitate a learning process in which symbolical techniques develop separately from other techniques.

INTRODUCTION

Graphing calculators (GC) are widely used in mathematics education because they support a multiple-representational approach to the learning of mathematics. The GC gives opportunities to interactively discover relations between functions and graphs. Burrill et al. (2002) report on evidence that the use of the GC improves the ability to link symbolical, graphical, and numerical representations, in particular for the understanding of functions and algebraic expressions. Also for learning the concept of derivative, the GC can make a possible contribution, as Delos Santos (2006) notes. So, on the one hand there is evidence that using the GC promotes students to develop strong relationships between symbolical and graphical forms of functions and derivatives. On the other hand the question remains: what are effects of handheld technology on students' mathematical thinking (Burrill et al., 2002)?

This study will contribute to this question by zooming in on one particular student during the period, in which he is learning about derivatives at pre-university level. His learning context is Dutch mathematics education, in which the GC is used as a tool during the introduction of derivatives. The GC offers, for example, options to draw the graph of the derivative, such as NDeriv, or to find dy/dx in a point of the graph. Depending on the textbook series and the teacher, different GC-options are used in mathematics lessons.

We will report on student Andy. He was part of a group of ten students in a longitudinal study (Roorda, Vos & Goedhart, in press). In that study general patterns of students' thinking were reported. Andy showed an a-typical pattern, which we left largely unreported as he was an outlier. Unlike the other students, in Andy's thinking the GC

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 57-64. Vancouver, Canada: PME.

played an important role. The goal of this paper is to present evidence of how a student's understanding of the concept of derivative can be affected by the use of a GC.

THEORETICAL FRAMEWORK

To study the relationship between the use of digital technology and students' mathematical thinking we use the theoretical framework of *instrumental genesis* (e.g., see Drijvers, Godino, Font & Trouche, 2013; Guin & Trouche, 1999). In this theory, artefacts are distinguished from instruments. The latter refers to a psychological construct, actively constructed by an individual, which consists of the user's mental scheme for using the artefact for a type of tasks. As such, the instrumental scheme integrates technical knowledge of the use of the artefact and the (in our case mathematical) knowledge involved. Instrumental genesis is the process of an (in our case digital, handheld) artefact becoming an instrument; it is a process in which techniques for using the digital tool and mathematical insights co-emerge. The resulting *instrumentation scheme* is the more or less stable way to deal with specific situations or tasks, guided by the opportunities and constraints of the artefact, as well as by the available knowledge.

The theory of instrumental genesis provides a widely applicable framework for investigation of the use of ICT-tools in mathematics education, and avoids an oversimplified separation of mathematical thinking and outsourcing calculations to the artefact. By explicitly describing instrumentation schemes, the instrumental genesis lens may help to identify the relationships between the use of the digital tools and the mathematical knowledge a student develops. This is exactly the way in which we will exploit this theory.

Guin and Trouche (1999) conclude that there is a great diversity in instrumental geneses. However, schemes related to using a GC for studying the derivative so far have hardly been described. In our study, therefore, we will identify such schemes and investigate how these develop over time. In terms of the instrumentation framework, the research question is: how do students' instrumentation schemes develop while studying the concept of derivative with the use of a GC?

METHODS

To gain insight into the development of students with regards to derivatives, we opted for a detailed description and analysis one student's work over a time period of a year. The case study of Andy is part of a longitudinal, multiple case study, in which ten students were followed (Roorda, Vos & Goedhart, in press). The students were in a pre-university science track, which means that they take science and mathematics courses at an advanced level. When we discovered that Andy's development contrasted with the other nine students, we decided to gather additional data on Andy's development.

The data were gathered at four different moments in time, together spanning the period before and after the introduction of calculus at school. In April and November task-based interviews (Goldin, 2000) were administered. The first interview (TBI-1) was held while Andy was still in grade 10 and the concept of derivative had not yet been introduced in his mathematics classes. The second interview (TBI-2) was held a few weeks after the introduction of differential calculus (difference quotient, differential quotient, derivatives of polynomials) with Andy being in grade 11. Also, we collected his work on two calculus tests, which were set by his teacher (CT-1 and CT-2). CT-1 was immediately after the lesson series, while CT-2 was about two and a half months later for those students with a low mark on the first calculus test. Andy was one of the low performers on the first test.

According to the Dutch curriculum for the pre-university science stream, at the beginning of grade 11 the derivative is introduced in mathematics classes. The introduction starts with the transition from graphs to functions and with the transition from a difference quotient to a differential quotient. Textbooks start with exercises on distance-time graphs to illustrate the meaning of average and instantaneous rate of change. The distance-time situation serves as an example to introduce the mathematical concept of derivative. After this physics-based introduction the slope of the tangent at a graph in the *xy*-plane is approximated by the slope of a line through two points on successively smaller intervals. The rate of change is directly linked to the tangent of the graph. Thereafter, the basic rules of symbolical differentiation are introduced and practiced.

Instruments and analysis

The two task-based interviews were designed to provide in-depth information about students' mathematical thinking while studying the concept of derivative. The tasks offer different representations (graphs, symbols, tables, etc.). Special about the tasks is, that the mathematical terms derivative, slope or differentiation and the symbols f' and dy/dx are explicitly avoided. In the tasks, the concept of derivative is asked for within situated contexts whereby variables have a physical meaning, such as time, volume or distance. The interview protocol prescribed, that a student, after completing a task, was repeatedly asked to check the obtained answer through other techniques. In this way, we were assured to observe a range of Andy's techniques.

In this paper we will focus on two tasks, *Barrel* and *Monopoly* (see Figure 1). These two tasks were selected because they offer students opportunities to use different techniques to solve the tasks, including numerical, graphical and symbolical approaches. The tasks were used in both task-based interviews, and therefore we can compare between the two interviews that were six months apart. We analyzed the interview transcripts and Andy's written answers to the problems, focussing on his techniques and the GC-options used.

The two calculus tests were designed by Andy's mathematics teacher. The tests contained similar tasks, and for this paper we will focus on two tasks: (1) a velocity-task, in which a distance-time formula is given and an instantaneous velocity has to be calculated, and (2) a tangent-task, in which the formula of a function is given and the tangent has to be calculated for a certain point of the graph. Based on Andy's writings we analyzed his techniques and the GC-options used.

Barrel: A barrel contains a liquid, which runs out through a hole in the bottom. The volume of the liquid in the barrel (V in m^3) decreases over time (t in minutes). The volume of the liquid is expressed with a formula
$V = 10\,(2 - {}^{1}/_{60} \cdot t)^2$. Also its graph is presented.
a. Calculate the outflow velocity at $t = 40$.
b. When a pump is used, the out-flow velocity can be expressed with the formula
$V = 40 - 1/3 \cdot t$. When will the out-flow velocity by pumping be equal to the velocity of out-flow through a hole in the bottom?

Monopoly: For a company the revenue function is $R(q) = -0.5q^2 + 12q$ and the cost function is $TK(q) = 0.03q^3 - 0.5q^2 + 4q + 15$.
a. For which amount of sold products do the costs increase at the slowest rate?
b. At what production level will the costs and the revenue increase at the same rate?

Figure 1: Short descriptions, without figures, of the Barrel and Monopoly tasks

RESULTS

We present the results in chronological order. Due to space limitations, Andy's work in TBI-1 and 2 is strongly summarized.

Task-based interview-1 (April, grade 10)

Andy solved the *Barrel*-a task by calculating the volume at $t = 40$ and $t = 41$ and subtract these from each other. For the *Barrel-b* task he plots the linear graph of V into the diagram of the worksheet, and by drawing a parallel tangent to the curved graph (see Figure 2), he estimates that at $t = 60$ the out-flow velocity of both barrels is equal. He checks this estimation by using the trace-option of the GC to move to the volume at $t = 60$ and $t = 61$ and calculate their differences. So, in the *Barrel*-task Andy calculates rates of change on a unit-interval by using his GC as a graph-plotter and value-calculator.

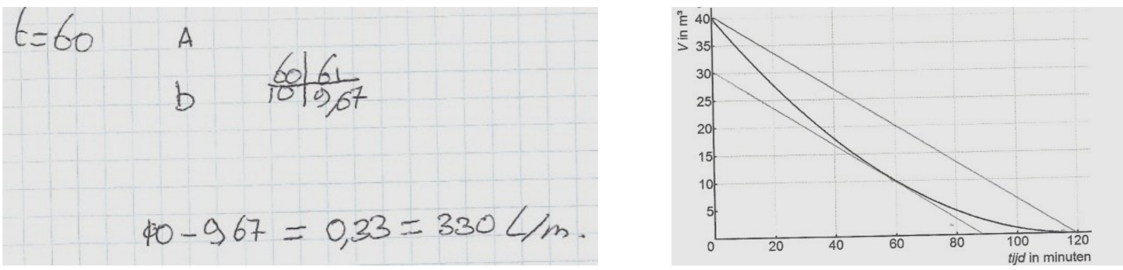

Figure 2: Drawing and calculation of Andy in the *Barrel-b* task

In the task *Monopoly-a* he uses the trace-option of his GC again to move the cursor over the graph (see Figure 3) and to look where the costs increase least. In the task *Monopoly-b* Andy plots the graphs of TK and TO. He uses the option Intersect and calculates the two points of intersection. But then he remarks that this is not correct, because "*the task is about increase*".

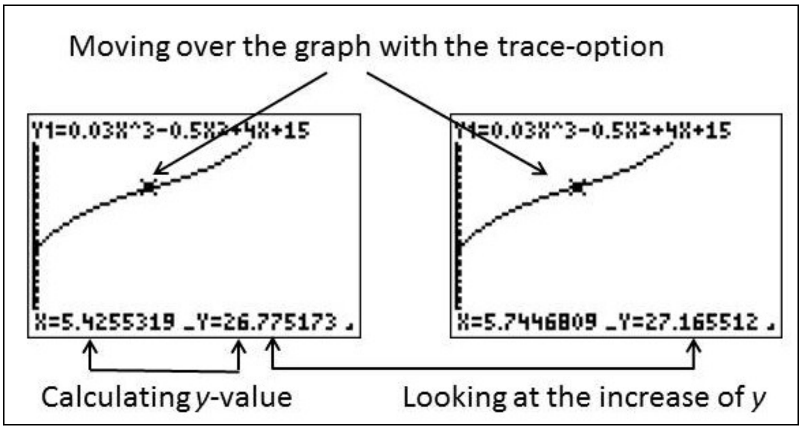

Figure 3: Example of Andy's plot-trace-scheme

Compared to the other nine students, Andy stands out by using his GC for the plot and trace options to explore the given functions. Andy is among the three students (out of ten) who solve the *Barrel*-task correctly. So, although derivatives and instantaneous rate of change have not yet been introduced, Andy is able to give meaning to rate of change in a volume-time situation and in a product-cost situation in terms of steepness of a curved graph. He does this by skilfully using plot, window and trace options of his GC. We refer to this as Andy's *plot-trace-calculate*-scheme (see Figure 3). This scheme reflects a graphical view on instantaneous change as the increase of the function at a small interval on the graph.

CT-1 Test on calculus (October, grade 11)

Andy solves the *velocity*-task about a falling object at $t = 6$ (given a formula for the height) in a remarkable way. Although it is a mathematics test on derivatives, he uses his GC and knowledge of physics to correctly calculate the velocity. Other students use symbolical differentiation for this task. In the *tangent-task*, Andy calculates the derivative function, but then he 'gets stuck' in an incorrect calculation. The test shows that Andy is able to calculate derivatives, but he does not use derivatives, neither to calculate the slope of a tangent, nor to calculate velocity.

TBI-2: Task-based interview (November, grade 11)

In the second task-based interview, six months after the first interview, the tasks *Barrel* and *Monopoly* are used again. To calculate the out-flow velocity in the task Barrel-a, Andy mentions three different procedures. He starts by plotting the graph on his GC and uses the option d*y*/d*x* in the CALC-menu to reach a correct answer. When asked to check his answer he mentions two additional techniques: (1) drawing on paper a tangent and calculating its slope, and (2) calculating the difference quotient on a small interval (he puts $t = 40$ and $t = 40.0001$ into his GC to find the corresponding values of V). He remarks about this small-interval technique: "*It is somewhat the same as dx-dy, dy-dx* (option of GC), *but then calculated by hand*." We notice at this point that Andy does not mention the derivative.

In the *Barrel-b* task Andy estimates the answer t = 60 by looking at the graph. He checks with the CALC-option dy/dx whether the slope at t = 60 is exactly -.333333. The interviewer asks if he is able to calculate the point. Andy says: "*To find this value in a direct way?[...] The line is always 1/3, so you have to find a point on the other graph where it is the same.*"

Andy also uses the dy/dx-option on his GC in the Monopoly-task. By looking at the plotted graphs he estimates the x-value, for which the steepness of both graphs is equal. He makes his cursor jump up-and-down between the two graphs using the dy/dx-option for calculating the steepness (see Figure 4). It is time-consuming and he says: "*I have no idea how to do this in another way*."

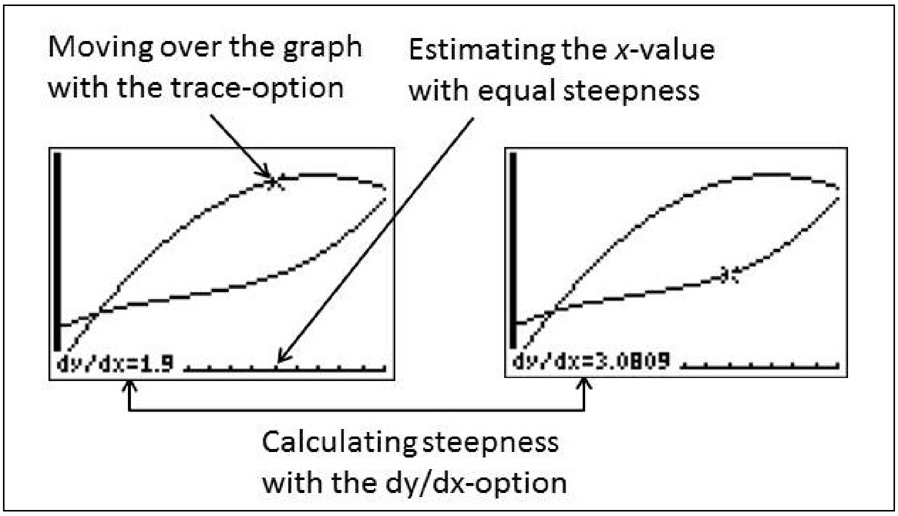

Figure 4: Example of the plot-trace-dy/dx-scheme.

Compared to the other nine students, Andy is the only one who uses the dy/dx-option of the GC. Other students work symbolically with the derivative combined with drawing a tangent.

So, in situated tasks about instantaneous rate of change Andy first explores the situation by plotting and tracing, he proceeds by using the dy/dx-option of his GC. In his explanations he relates the GC-option dy/dx to the tangent and also to the increase at a small interval. We call this the *plot-trace-dy/dx*-scheme. For Andy, this scheme is related to *tangent* and a *difference quotient on a minimal interval*. When asked for other techniques for these tasks, Andy never mentions the derivative. To him, symbolical differentiation apparently is not related to the *plot-trace-dy/dx*-scheme.

CT-2 (15 January, grade 11)

On the second test on calculus Andy solves the velocity-task correctly using the dy-dx-option of his GC. He solves the tangent-task by using derivatives. Thus, in velocity-tasks Andy's *plot-trace-dy/dx*-scheme becomes active, but apparently this scheme is not activated in tangent-tasks in the xy-plane.

CONCLUSIONS AND DISCUSSION

Before the introduction of calculus Andy's preferred instrumentation scheme is characterized as *a plot-trace*-scheme: he uses the plot and trace-options of his GC to calculate a rate of change. After the introduction of calculus we observe an uptake of another GC-option, d*y*/d*x*. His instrumentation scheme can be characterized as a *plot-trace-dy/dx*-scheme with links to *tangent* and *small interval* procedures. His skill in working with derivatives, which is observed in CT-1 and CT-2, is not used or mentioned by Andy in several situated tasks about velocity and increase. So, options of the GC become part of his instrumentation scheme for situated tasks on rate of change, but this scheme seems to develop separately from the symbolical procedure to calculate derivatives. Compared with nine other students, Andy is unique in his use of the GC. For solving the same tasks, the other students prefer symbolical differentiation combined with the use of a tangent.

The idea that the use of the GC encourages students to create links between graphical and symbolical representations as reported by Burrill et al. (2002) and Delos Santos (2006) does not hold for Andy. Andy's initial, resilient use of plot-options in his GC assimilates the d*y*/d*x*-option in situated rate-of-change tasks. Andy does not once mention or use symbolical differentiation in the task-based interviews, despite repeatedly being asked for alternative procedures. Nevertheless, Andy has learnt to use derivatives, as demonstrated in both calculus tests.

It is not clear why Andy does not relate symbolical and GC techniques. Our hypothesis is that Andy's instrumentation scheme is affected by the structure of the textbook. The textbook makes a clear distinction between tasks on the steepness of distance-time graphs, and tasks on tangents in the *xy*-plane. Solutions to the first type of tasks can often be approximations, solutions to the latter type of tasks always have to be exact.

One can wonder if it is a problem that Andy does not relate symbolical techniques and GC-options. An advantage of Andy's approach is his early uptake of graphical and numerical techniques with his *plot-trace*-scheme. A disadvantage is that he has few reasons to replace or supplement his GC-techniques with symbolical differentiation. We surmise that if Andy succeeds in linking symbolical differentiation to his *plot-trace-dy/dx*- scheme, he will have an excellent conceptual understanding of the concept of derivative in all representational aspects.

The theory of instrumental genesis is helpful to identify relationships between the use of the GC and Andy's knowledge about steepness, instantaneous rate of change and velocity in situations. Just as Trouche and Drijvers (2010) point out, the case of Andy shows that the use of technology in education can have complex and subtle effects: instead of being a tool that promotes links between representations, it can facilitate a learning process in which symbolical techniques develop separately from other techniques.

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A FRAMEWORK FOR THE ANALYSIS OF VALUES THROUGH A MATHEMATICAL LITERACY LENS

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This paper aims to offer a framework to interrogate how learners make sense of values in Mathematical Literacy lessons. Through an exploration of the curriculum materials, a framework that uses, the Bloom's taxonomy to analyse the cognitive levels of the tasks in the materials against mathematical competency and knowledge areas was developed. Findings from this study show that there is merit in the use of the framework to analyse values in Mathematical Literacy

INTRODUCTION

Pundits concur that education systems play a pivotal role in fostering and developing values in learners. In South Africa the values and rights enshrined in the Constitution and the Bill of Rights resonate in the Schools Act. Society is making greater demands on its citizens to be numerate and demands that learners become more engaged with school mathematics (Bishop, 2007). The introduction of mathematical literacy into South African classrooms further reinforces this engagement by students. The OECD/PISA (2003) defines mathematical literacy as:

> … an individual's capacity to identify and understand the role that mathematics plays in world, to make well- founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (p. 10)

The National Curriculum Statement (Grades 10-12) for Mathematical Literacy (Department of Education, 2001), states the following about education and values:

> Values and morality give meaning to our individual and social relationships. They are the common currencies that help make life more meaningful….An education system does not exist to simply serve as a market…..It's primary purpose must be to enrich the individual. (pp. 4-5)

Values may be imparted by example (Nieuwenhuis, 2007) or clarified through discussion, debate and negotiation (2004) suggest that values are implied in the mathematics curriculum, mathematics teaching and mathematics itself. This also applies to the mathematical literacy curriculum and classroom.

Mathematical literacy may be taught through contexts giving rise to difficulties such as issues of language and the Nieuwenhuis (2007). Values in mathematical literacy are inculcated through the nature of the content, context and an individual's experience in the mathematics classroom. These values provide the cognitive and affective lenses which modify and shape one's perception and interpretation of the world (Seah &

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 65-72. Vancouver, Canada: PME.

Bishop, 1999). Since explicitly alerting students to values in a mathematical literacy lesson or articulating them anywhere is not a norm in mathematical literacy, values education may appear to have a hidden agenda. This paper forms a part of a larger study which aims to understand how learners make sense of values in Mathematical Literacy lessons. This paper, therefore reports on the development of tools necessary to carry out this analysis.

In an attempt to understand the socio-moral and cognitive development of the learner in the classroom, I chose to refer to the work of Jean Piaget, Lawrence Kohlberg and Lev Vygotsky. Piaget (1932), whose focus was the moral development of children, was of the opinion that individuals use their interactions with the environment to construct and reconstruct their knowledge of the world and considered morality to be a developmental process His theory, if applied to values education, suggests that the teacher whose task is to provide students with opportunities for personal discovery through problem solving rather than indoctrination through societal norms is instrumental in the moral development of a learner (1932). In attempting to define moral development in terms of cognitive growth, Piaget identified the following four moral judgment dimensions which demonstrate a distinct correlation to his concept of cognitive development: (i) absolutism of moral perspective, (ii) concept of rule as unchangeable, (iii) belief in immanent justice, and (iv) evaluation of responsibility in terms of consequences (Lickona, 1976). Criticism levelled at Piaget is that he paid very little attention to the impact of social interactions and differing cultures on development (Sigelman & Rider, 2009). He paved the way for the theory of moral development advanced by Lawrence Kohlberg.

Kohlberg (1984) developed a system for categorizing the moral reasoning in human beings into six stages. Central to this theory is the notion that the moral growth of human beings progresses through an invariant sequence – a fixed and universal order of stages, each of which represents a consistent way of thinking about moral issues that differs from the preceding or following stage (Sigelman & Rider, 2009;). The age of the individual, regardless of cross-cultural moral norms and beliefs, plays a vital role in this development. The six stages, as identified by Kohlberg, relate to moral thinking. He further suggested that associated with moral judgement is the concept of sociomoral perspective; a reference that is made to the point of view an individual takes in defining both social facts and sociomoral values (Kohlberg, 1976). There are three broad levels of social perspective that correspond to the three levels of moral judgement. His non-subscription to the view that values education comprises of a moral agenda that prescribes a list of values to be learnt (cited in Nucci, 2001), resonates with my view on values education. Simpson (1974) suggests that Kohlberg’s stages are not culturally universal as they are based on western philosophical tradition. Her proposal for the transformation of his cognitive-developmental theory into a cognitive-affective-conative developmental theory is based on the claim that it will give equal regard to three facets of the human personality: thought, emotion and motivation (Simpson, 1974). Carol Gilligan (1982) points to a gender bias in

Kohlberg's theory Simpson (1974) suggests that Kohlberg's stages are not culturally universal while Gilligan (1982) points to a gender bias in the theory.

The socio-cultural theory of learning suggests that characteristic to human evolvement is the development of higher order functions through social interactions. Lev Vygotsky (1978) was of the view that in order to understand the human development of an individual, a study of the individual and the external social world associated with him needs to be undertaken. Vygotskian theory (Vygotsky, 1978) suggests that each developmental stage is determined by genetic, maturational and socio-cultural factors. Socio-cultural theory (Vygotsky, 1978) differentiates between experiences produced by the individual's contact with the environmental stimuli and those shaped by interactions with symbolic mediators. Central to Vygotskian theory is the Zone of Proximal Development (ZPD). Vygotsky (1978) defined the ZPD as the:

> … distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers. (p. 86)

Within the ZPD, through social and cultural interactions, learners receive instructional support from experienced peers and teachers in a particular mathematical literacy context.

The Values Education Study Report of the Australian Government (2003) states that cognitive-developmental theorists' argue in favour of values education being "promoted through the development of reasoning" while critics of this approach focus the on the neglect of the behavioural and emotional components of character and the absence of any attempt to determine whether the stated values resulted in behavioural change" (p. 35). I do not claim that any one approach to be most effective. The adoption of an approach is context dependent. At times a combination of approaches could prove to be more effective than the adoption of any single theory. While the literature does offer suggestions about possible strengths and weaknesses of approaches, I did not find any strong claims in the literature to warrant my not using the cognitive-developmental approach.

FRAMEWORK FOR THE ANALYSIS OF VALUES

This section outlines how the framework for analysing the values in a mathematical literacy lesson was developed. The framework is divided into five domains, namely: content categories, expected mathematical literacy competencies, Bloom's taxonomy, evidence of values and the value itself. In order to appreciate how the learners understand, identify and implement the values inherent in the Mathematical Literacy lessons, I carried out an analysis of the classroom materials looking at the mathematical content in the learner materials. A content review of the textbooks, learner worksheets, assignments, tests and examination papers was conducted and classified into the following five content areas: compound growth and finance; measurement; numbers and calculations; patterns, relationships and representations;

and representation of statistical data. Table 1 provides details with respect to the content areas and the related sections.

Content	Related learning objectives
Compound growth & Finance	Banking loans; break-even analysis; budgets; cost & selling price; exchange rates; income & expenditure; inflation; interest; profit & loss and tariff systems.
Measurement	Calculation of area, perimeter & volume; conversions; measuring length, weight & volume; temperature and time.
Numbers & calculations	Fractions; number formats & conventions; operations using numbers; percentages; proportions; rates; ratios and rounding
Patterns, relationships & representations	Patterns & relationships; representation of relationships in tables, charts & equations.
Representation of statistical data	Non-violence

Table 1: Content categories in ML & related learning objectives.

The content in Mathematical Literacy may be situated in contexts requiring learners to apply their mathematical and critical thinking abilities. Table 2 below provides a more detailed description of each competency category. The six competency categories suggested by Jaftha, Mhakure and Rughubar-Reddy (2012) in their study on Quantitative Literacy and social justice was adapted and used to analyse the competencies required in the Mathematical Literacy classroom.

Bloom's Taxonomy consisting of the knowledge dimension and cognitive process was also used in the analysis the cognitive levels of activities from the learner materials. For the purpose of analysis the six cognitive process dimensions, namely: remembering, understanding, applying, analysing, evaluating, and creating were used.

Finally I examined the mathematical content of the activities in the learner materials for evidence of values. Discussions with colleagues and observations of their lessons in the Sathya Sai schools in South Africa and abroad, together with my own study of values in education, have convinced me that values are also embedded in the mathematical content. Sathya Sai schools in my analysis of the Mathematical Literacy lessons. I do not claim that that the methods used by the Sathya Sai schools nor the set of values to be the best. I found the list of values very comprehensive and suitable for my analysis. The values embedded in the South African constitution are integrated in this list.

Competencies	Core descriptors of competencies
Comparing numbers	Conversions of numbers from one form to another
Critical thinking ability	Interrogates the content & contexts; checks the validity of the solutions.
Data representation	Familiar with data representation (tables & graphs); analysis & interpretation of data from varying formats
Reading data from texts, charts & tables	Making sense of numbers in charts, tables & texts; comparing data in graphs, tables and texts
Procedural competencies	Routine calculations; relationships between quantities; substitution and manipulation of formulae
Writing proficiency	Effective communication of information; explains understanding of concepts; applying knowledge to novel situations

Table 2: Expected Mathematical Literacy Competencies.

Content	Competencies	Bloom's Taxonomy	Evidence of Values	Value
Compound growth & Finance	Comparing numbers	Remembering	Explicit in either context or content	Love
Measurement	Critical thinking ability	Understanding	Implicit in either context or content	Truth
Numbers & calculations	Data representation	Applying	Explicit in both context or content	Right conduct
Patterns, relationships & representations	Reading data from texts, charts & tables	Analysing	Implicit in both context or content	Peace
Representation of statistical data	Procedural competencies	Evaluating	Value absent from context and content	Non-violence
	Writing proficiency	Creating		

Table 3: Summary of the framework.

The one hundred and eight sub-values are pooled into five major groups, namely; truth, right action, peace, love and non-violence. I also sought to establish whether the values are explicit or not in the contents and contexts of the Mathematical Literacy lesson. Table 3 provides a summary of the framework that resulted from the analysis of learner material. Table 3 was then used to analyse a mathematical literacy lesson.

ANALYSIS OF A LESSON USING THE FRAMEWORK

This paper reports on the exploration of the views of five grade 10 learners from a public secondary school. The school is situated in a low income, residential area in the Cape Flats region of Cape Town, South Africa. The learners were from the first cohort of students taking mathematical literacy at the senior secondary level. A discussion on what the study was about and an illustration of how to identify the values was presented. Classroom lessons were observed and videotaped to capture the learners' participation and attitude during the lessons. This paper focuses only on a lesson on simple and compound interest. The following were the questions under review:

1. "Zandile takes out a bank loan of R13 500 to pay for an urgent medical operation. The bank terms are 12% p.a. compounded over two years, compounded annually. How much money must Zandile repay the bank?"
2. "Bongani invests his first Christmas bonus of R750.00 in a bank that offers interest rates of 9% p.a. compounded yearly. How much interest will he have earned after 12 years?"
3. A bank charges 11% interest p.a. on loans over 4 years. Oluwethu borrows R12 000. Calculate: (a) the amount of interest due, (b) The total amount to be repaid, and (c) the monthly repayments needed.

A focus group meeting was held with these five learners after the completion of the classroom visits. This was a two part meeting. Firstly the learners had to comment on the values that they attributed to their general experiences in the mathematical literacy classroom. Thereafter they were shown the video footage of the lesson and asked to identify values in the lesson. A list of values to assist with the identification of values was given to the learners. The learners had to support their claims with evidence of where or how these values manifested themselves in the lessons.

The findings based on the viewing of the video are divided into three sections as follow: general values associated with the classroom experience, values in the context and values in the mathematical content. Table 4 gives a summary of the values identified by learners together with the evidence. During the focus meeting, consideration was first given to the values learners attributed to their experiences in the mathematical literacy classroom. Learners were of the opinion that the subject mathematical literacy was looked down upon by both the educators at their school and the learners who did mathematics. One learner commented that:"… the other learners think that they are better because they do maths … They think that we are stupid … they make us feel stupid …"

Value	Evidence of value in the lesson
General values associated with the classroom experience	
Values about classmates:	
Consideration	Learner that was finding difficulty understanding a concept and required repeated explanation.
Focus	Learners concentrating on lessons
Spirit of inquiry	Learners asking questions to learn more about investments
Values about educator:	
Dedication and sacrifice	Teacher was at school despite being sick & had lost her voice.
Helpfulness	Walks around the classroom assisting learners
(Lack of) Good manners	Screams at learners & stamped her foot on the floor to get learners' attention
Values identified in the context of investments and loans	
Right action	By investing his money Bongani was able to increase his bank balance.
Discrimination (against Zandile)	Zandile had to pay more than was required for her medical operation (as compared to Bongani)
Values identified in the mathematical content	
Equality (sub-value of truth)	The use of the equal to sign:

Table 4: Values identified by focus group from video footage

CONCLUSION

The study has developed and validated a framework to interrogate how learners make sense of values in Mathematical Literacy lessons. Findings show that the learners found it easy to identify and talk about the values associated with the classroom environment. The fact that they were able to link an action to the value identified demonstrated their understanding. Although it did take the learners in the focus group a longer time to be able to identify values they thought were related to the content and context of the mathematics literacy lesson, they were able to do so quite effectively when they understood what was required of them. Using the framework will allow educators to ascertain whether their course materials provide the opportunity to sensitise students to the extensive social issues in their communities.

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WRITTEN REASONING IN PRIMARY SCHOOL

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Currently, language competences in mathematics lessons gain more attention in Germany. The paper reports an interdisciplinary study of linguistics and mathematics education on reasoning. A model to rate the competences in arithmetic reasoning at primary level will be presented for discussion: mathematical reasoning is coded separately from its linguistic realization. In a pilot study, 243 students of 3rd, 4th, and 6th grade solved different arithmetic reasoning tasks. The results show a one-dimensional scale for the model of reasoning. Its specific components provide differentiated requirements, which are formulated concretely in the coding guidelines. They may unfold didactical potential for language support in mathematical reasoning as well as in mathematics lessons itself at primary level.

THEORETICAL BACKGROUND

Reasoning in mathematics and language learning

Mathematical argumentation can be divided into four steps: detecting mathematical regularities, describing them, asking questions about them and giving reasons for their validity (Meyer, 2010; Bezold, 2009). The content base of an argumentation is achieved by description of the detected structures or by reference to common knowledge (Ehlich & Rehbein,1986; Krummheuer, 2000); reasoning then is necessary to acknowledge the described regularities as true (Toulmin, 2003/1958; Schwarzkopf, 1999).

The didactical value of reasoning in mathematics learning is seen in gaining deeper insights into mathematical structures and thereby as a development of one's mathematical knowledge. In this sense, reasoning leads to ask questions about mathematical statements, to make sure they are right and to develop new mathematical connections (Steinbring, 2005). Two intertwined processes may be distinguished: one's own understanding and the process of sharing this understanding with others. Therefore, in its epistemic function mathematical reasoning may be monologic and lead to deeper individual understanding, in its communicative function it is dialogic and depend¬ent on other people if mathematical structures are explained and justified (Neumann, Beier, & Ruwisch, 2014).

Mathematical reasoning in this sense has to be distinguished from reasoning in language classes, especially at primary level. While both are seen as concepts which develop out of situated everyday ("vernacular") speech (Elbow, 2012), reasoning in language learning focusses much more on self-evident facts and personal meanings instead of provable structures in special content areas. So, argumentation in language learning leads to a more addressee-oriented cognitivization (Krelle, 2007); reasoning

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 73-80. Vancouver, Canada: PME.

in this kind is much more persuasion than proving. Nevertheless, typical linguistic formats of reasoning are learned in these everyday situations and students have to learn how to use them in different content areas. So, in combining the mathematical and the linguistic view on early reasoning, we try to get a broader and deeper understanding of early reasoning, like it can be found in written argumentation of primary students.

Modelling written mathematical reasoning

Although mathematical reasoning is seen as a key issue for students already at the primary level, which for example can be seen in the National Mathematics Standards, there is only few reasoning requested. A textbook analysis showed that not more than 5-10% of all textbook tasks ask for reasoning (Ruwisch, 2012). As well, models which try to describe mathematical competences of this age regard reasoning as important but very specific and classify these competences only to the highest mathematical level (Roppelt & Reiss, 2012). This gap between importance for all and performance of only few was one reason for us to develop a model which may represent different stages of reasoning in early years.

DATA AND METHOD

Sample

The data include 477 written justifications of 243 students. 41 third-graders (♀21; ♂20), 96 fourth-graders (♀43; ♂53) and106 sixth-graders (♀52; ♂54) worked out two out of four designed arithmetic reasoning tasks (s. below).

Arithmetic reasoning tasks

All working sheets are divided into three sections (s. figure 1): In the first section given arithmetic tasks have to be solved and regularities have to be recognized and transferred to more tasks. Following this part of detection, the children are asked to describe their observations, before giving reasons for them.

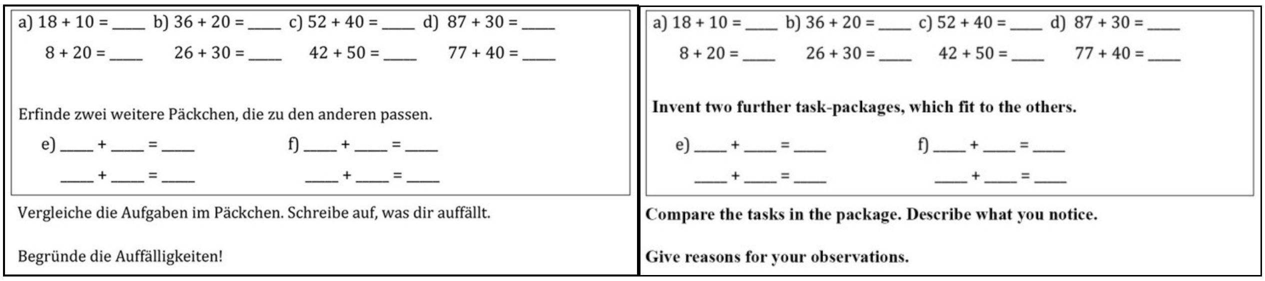

Figure 1: Complex addition tasks (CA) as a sample item.
(on the left: original version; on the right: English translation)

Four different arithmetic tasks were designed for this study. Although the tasks differ in the complexity of regularities, all of them are easy to compute and focus on detection and reasoning. In format ZF three number sequences need to be continued: +9, +7, and +2n. The format EA asks to continue a given additive structure in increasing all three summands by one, so the sum increases by three. In solving formats CA and CM the children need to recognize two structures at the same time. To answer the complex addition task which is given in figure 1 children need to find two tasks with the same sum. At the same time they had to take into account that the summands have to be changed by 10 in opposite directions. The multiplication tasks CM show a constant difference in the product, caused by the difference between the multipliers while the multiplicands remain constant.

Data analysis

Rating scales

Fundamental for our data analysis is the separate evaluation of detecting the mathematical structure and giving reasons for its validity. The argumentation itself is distinguished as well: we separate mathematical from linguistic aspects of reasoning. So, students' writings are rated by one detection-scale and two reasoning-scales (see table 1, explanations below). This separation allows a differentiated grasping for sub-skills of reasoning.

Mathematical detections	Mathematical aspects of reasoning	Linguistic aspects of reasoning
irrelevant aspects as regularities	regularities (partially) described	indicators without reason-effect-structure
	rudimentary reasoning	reason-effect structure
regularities partly transferred	reasoning through examples	explicit linguistic reference to the task
regularities totally transferred	partially generalized reasoning	completeness and consistency
	generalization / formal reasoning	use of math. terminology / decontextualization

Table 1: Rating-scales to evaluate written mathematical reasoning.

Mathematical detections: Children have to compute the arithmetic tasks given on the sheet to find out the underlying structure and transfer it to two more packages with tasks. This process may be realised fully or only partly; sometimes only irrelevant aspects are used to create new tasks. If the structure is transferred fully, the results of the tasks given are also correct, so three stages of this rating scale seem sufficient.

Mathematical aspects of reasoning: Reasoning needs a description of mathematical aspects as basis. If only some regularities are described without giving reasons this leads to stage 1. If a rudimentary reasoning is given despite a description, the work is coded by stage 2. To be rated by stage 3 to 5 all relevant aspects have to gain attention in the argumentation. If this is done by examples, the work is rated by stage 3, if it is already partly generalized, it is rated by 4, and if it is totally general or a formal proof, by 5.

Linguistic aspects of reasoning: The realisation of a mathematical argumentation by written language is also rated by 5 stages which were gained theoretically, especially in focussing on linguistic categories like the use of connectors and identifiable coherence of the text. If explicit linguistic indicators are already used without any structure of reasoning, the text is classified in stage 1. If the text shows a reason-effect-structure it is coded at least as stage 2. If also an explicit linguistic reference to the tasks is visible, the text is classified in stage 3. A text of stage 4 shows a consistent and complete argumentation. To be assigned to stage 5, the use of mathematical terminology must be given in addition, so a decontextualization is identifiable.

Process of coding

14 raters which concentrate either on the mathematical or the linguistic scales were included in the coding process. This process ensured an independent coding by the two professions.

The raters found it easy to code the texts with respect to the detection scale. More difficulties were reported concerning the aspects of reasoning. So the decision between description and rudimentary reasoning was difficult for the mathematical raters. The trade-off between stage 2 and 3 (use of connectors without/with explicit reference to the tasks) as well as between 4 and 5 (use of mathematical terminology) was reported by the linguistic raters as difficult.

Despite the many rater-combinations high absolute agreement in judgments can be reported (62% across all tasks and scales). Deviations of more than one stage occurred in 8% of the cases and showed three important results:

- The multiplication task cannot be compared to the others, because up to now only 35 encodings made by only one pair of raters exist in the data.
- The linguistic scale is the most difficult. Throughout all tasks and raters deviations of more than one stage are observable.
- During the project an increase of coding quality can already be determined. Although acceptable internal consistencies exist across all tasks (Cronbach's α=.80), these values increase, if only ZF (α=.82) and EA (α=.84) which were used later in the project are considered. Nevertheless, large individual deviations can still be observed.

With respect to these results the multiplication task was excluded for the following overall scaling. Thereby, an acceptable average internal consistency of the individual scales over the remaining tasks was achieved: α=.86 for the mathematical detections, α=.81 for the mathematical aspects of reasoning and α=.71 for the linguistic aspects of reasoning.

RESULTS

Due to the great number of rating persons and on the basis of an acceptable inter-rater-consistency (α >.70) we worked on with the means of the ratings for reporting first results.

Overall scale

The IRT-scale of the three tasks and all texts shows a common scale over all components (see table 2). The items are conform to the model as well (WMNSQ .85-1.09). Therefore, early mathematical reasoning in arithmetic like it is measured by the three tasks and the ratings with our scales can be described as a one-dimensional construct.

	Mathematical detections		Mathematical aspects of reasoning		Linguistic aspects of reasoning	
Item	Estimate	WMNSQ	Estimate	WMNSQ	Estimate	WMNSQ
(ZF) number sequences	-1.556	1.02	-0.459	1.06	0.124	0.85
(EA) simple addition	-1.628	1.09	1.057	1.09	1.570	0.93
(CA) compl. addition	-0.845	0.98	0.506	0.92	1.230	0.97

Table 2: Item parameters (Estimate) in IRT scaling.

Looking at the three scales, it becomes obvious that – as expected – it is easier to detect and transfer mathematical structures than to give reasons for their validity (negative deviation from zero). Comparing the two scales of reasoning it seems to be easier to realise mathematical aspects of reasoning than to do this in an appropriate linguistic structure. At the same time, *mathematical detections* is the most stable dimension with a maximum difference of .783 compared to 1.446 for the linguistic and 1.516 for the *mathematical aspects of reasoning*.

Comparing the three tasks it seems as if the complex addition is the most difficult to be transferred whereas the simple addition and the number sequences show nearly no difference. The justifications show that it was most easy to realise mathematical as well as linguistic aspects of reasoning in the format number sequences, followed by the complex addition and then by the simple addition task. Despite these differences, all tasks can be characterized as well suited to capture mathematical reasoning in arithmetic.

Students' performances

The performance of the total sample is distributed normally to slightly right-shifted: On the raw scores level 21.2% are one standard deviation above, 9.6% one standard deviation below the mean; 6.2% are two standard deviations above, 4.2% two standard deviations below the mean.

All scores were transformed onto a scale with the mean of 100 and a standard deviation of 20 to make comparison between the three groups of students easier (see figure 2): 3rd graders (M=102/SD=29), 4th graders (M=98/SD=19) and 6th graders (M=101/SD=17) showed nearly the same mean performance.

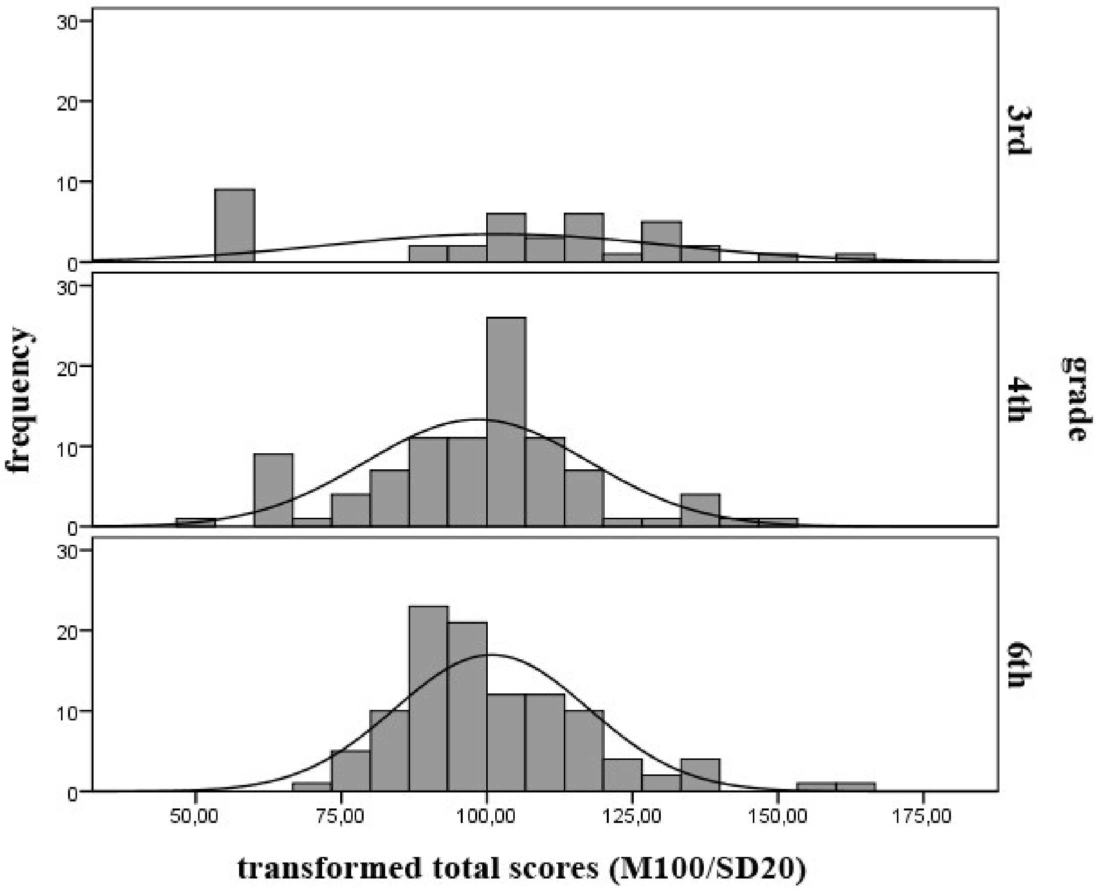

Figure 2: Students' performances by different grades.

Unexpectedly, reasoning competences as they were measured by our tasks and ratings do not increase over time. Even though our data were collected cross-sectionally and not longitudinally a significant increase of competences could have been expected. In interpreting the differences of standard deviations over the three groups, it seems as if 3rd graders differ more in their results than 4th graders and both more than 6th graders, so a homogenization seems to take place during schooling. But, due to the fact of missing comparative data and the small number of our data this remains speculative at the moment.

CONCLUSIONS

Our aim was to describe and report the competences of primary students in dealing with written arithmetic reasoning tasks by different aspects. The results show on the one hand one consistent scale as a one-dimensional construct from detecting and transferring mathematical structures to mathematical and linguistic aspects of reasoning. This one-dimensional construct confirms the approach of Roppelt and Reiss (2012) who assume that process-oriented mathematical skills at primary level are more or less interwoven, interdependent, and therefore one global construct, which will differentiate in higher mathematics learning.

On the other hand, the detailed descriptions of the three scales allow an awareness of different components of mathematical reasoning which will be missed by only one global scale (Neumann 2013). So, the described stages may help to understand which aspects have to be taken into account to be successful in written arithmetic reasoning tasks.

The internal relationships between mathematical and linguistic requirements in solving written reasoning tasks need further verifications and investigation. For instance, we cannot exclude that the difficulties during the coding process (see above) will have spilled over into the variance of the difficulty gradations in the students' results. It might also be that linguistic aspects of reasoning are such difficult, because students do not expect them in mathematics classes. This effect may be reinforced by our anticipation of a very explicit use of "reasoning language" as can be seen in the coding table. So maybe the tasks are too demanding concerning the use of appropriate language to reason in mathematics.

Another critical question concerns the multiplicative task, which did not fit into the model. This may be caused by a too small number of students solving this task (N=35) up to now. But we could also see that a more complex task produces more dropouts as well as more difficulties for the raters. Maybe, the multiplicative task is also too complex to gain information about written reasoning. This may lead to a deeper understanding of the critical aspects of a task to be a "good reasoning task" in mathematics classrooms. High complexity may require too much cognitive and motor capacity to assume a successful writing process (Hayes, 2012). As a consequence, we need more items to check which task is suitable to which function in reasoning processes.

An open question is the stagnation of the students' performance at the level of grade 4. This result may be caused by demotivation, because the sixth graders may think the tasks were too easy to give explicit reasons for the structures. Another argument could be that students still are not used to reasoning in mathematics lessons and competences do not increase by themselves without being taught.

The design of the tasks and the scales of rating show already that written reasoning processes in mathematics at the primary level may be challenged as well as described in more detail then by only a global measure. Hopefully, such interdisciplinary projects help to sharpen the construct and lead to criteria for teachers how to focus on the different aspects of reasoning as well as to unfold didactical potential for language support in mathematics lessons.

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LEARNING WITH INTERACTIVE ANIMATED WORKED-OUT EXAMPLES IN GROUPS OF TWO

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This exemplary case study describes the learning process of two sixth-graders that learn from an animated worked-out example and an accompanying self-explanation prompt in the domain of fractions. It is based on a corresponding field study. The analysis focuses on the interaction with the computer, the communication between the students, the metacognitive aspects of the learning process and self-explanations. Supported with quantitative data, the qualitative results show that worked-out examples are proper materials for learning in groups of two. Furthermore, it is shown that self-explanation prompts have positive effects on the learning process and the analysed aspects. With detailed scenes it is elucidated, how the interactive capabilities and the animations are used during the learning process.

WORKED-OUT EXAMPLES

Studying worked-out examples is a well-known method for novices to increase their knowledge (Sweller & Cooper 1985). A huge body of research has shown the positive effects on knowledge acquisition and learning, whereat it often focuses on a single learner, who processes an example silently and alone. Only a few studies examine, how worked-out examples can be processed in groups (e.g. Retnowati, 2010). These studies emphasize quantitative aspects, but neither give detailed insight into the learning process nor consider differences in communication processes or learner behaviour when interpreting the results. There are different research findings about the role of *animated worked-out examples* in abstract domains (e.g. Tversky et al., 2002). A plausible position seems to be that such examples should be used, if a content analysis reveals benefits of a dynamic presentation (Höffler & Leutner, 2007).

SELF-EXPLANATIONS

The use of cognitive learning strategies has a crucial influence on the learning of mathematics (Murayama, 2012). Especially, when students learn from worked-out examples, they often do not apply meaningful strategies, but process the examples in a superficial or passive way (Renkl, 1997). *Self-explanations* form one class of cognitive learning strategies. A self-explanation is defined as

> a constructive activity that engages students in active learning and insures that learners attend to the material in a meaningful way while effectively monitoring their evolving understanding. (Roy & Chi, 2005, p. 272)

Research about worked-out examples and self-explanations shows that self-explanations are a main predictor for the learning outcome (Chi, 1989; Renkl,

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 81-88. Vancouver, Canada: PME.

1997). Several concrete self-explaining processes in the domain of mathematics are integrating textual, iconic and symbolic representations (e.g. equations), goal-operator combinations and the determination of assumptions and special cases (e.g. dividing by zero). To encourage learners to learn actively and meaningfully, several methods of eliciting self-explanations and their implications on learning outcome and transfer have been analysed (e.g. Chi et al., 1994). A successful method is the use of open self-explanation prompts. These prompts are short questions or impulses that focus on key-concepts of the material or common misconceptions, or ask the learners to explain the presented procedure in their own words. The effect of self-explanation prompts on the processing of static examples is well known, whereas little is known about the combination of animated worked-out examples and self-explanation prompts (Betrancourt, 2005; de Koning, 2011).

Self-explanations are activities inside the learner's head – hence, they cannot be observed directly. However, verbal and nonverbal data can provide more or less obvious hints that allow the researcher to characterize the underlying cognitive processes (Chi, 2000). To distinct self-explanations from the observed phenomena, this paper uses the following definition: If a phenomenon (an utterance, gesture, action, etc.) gives rise to the interpretation that an underlying cognitive process is a self-explanation, this phenomenon is called a *projection of a self-explanation*.

METACOGNITIVE PROCESSES

The given definition of self-explanations names the importance of *monitoring* processes – without being aware of the need for an explanation, the learner probably will not give a self-explanation (Chi, 1989). Other important metacognitive strategies are planning and regulating (Pintrich, 1989). *Planning* means to organize the learning process. Possible manifestations are identifying task-requirements or formulating learning strategies. *Regulations* are alterations of the learning process like asking the partner or the teacher for help, or restructuring the learning process or details of it. An important group of regulations when considering learning processes with interactive animated learning material are meaningful interactions with the material such as controlling the pace of an animation or skipping animation-steps (Kettanurak, 2001). While metacognitive processes are often measured in studies concentrating on self-explanations, the concrete learning process and the effects of prompts on the metacognitive behaviour of students is rarely analysed.

CONCEPTUAL FOCUS AND RESEARCH QUESTIONS

From the former mentioned research gaps, we formulate two questions that should lead the analysis of the following case study.

- What characteristic patterns and behaviour can be observed concerning computer-interaction, metacognitive processes and communication between students, when animated worked-out examples and self-explanation prompts are processed in dyads?

- How does an open self-explanation prompt affect the processing of the interactive animated worked-out examples in dyads, the metacognitive processes and the occurrence of self-explanations?

SUBJECTS AND MATERIALS

To answer the research questions, a field study with 85 sixth-graders from three classes of a German middle school was conducted (Salle, in press). The students of one class worked with interactive animated worked-out examples in a self-regulated learning scenario and accompanying open self-explanation prompts (cf. Salle, 2013).

Materials

The used fractions curriculum focuses on the construction of concepts by connecting the mathematical characteristics of fractions to meaningful activities and familiar real world situations to enable the students to operate flexibly in a syntactic and semantic way (English & Halford, 1995). One part of this curriculum deals with reducing of fractions. On an iconic representation level, this transformation is visualised by altering the equal segmentation of a given figure. To connect the symbolic operation of reducing to its dynamic iconic counterpart, an interactive animated example was designed (Figure 1). The accompanying prompt reads: "What is the meaning of 'altering a segmentation'? What changes, what remains?"

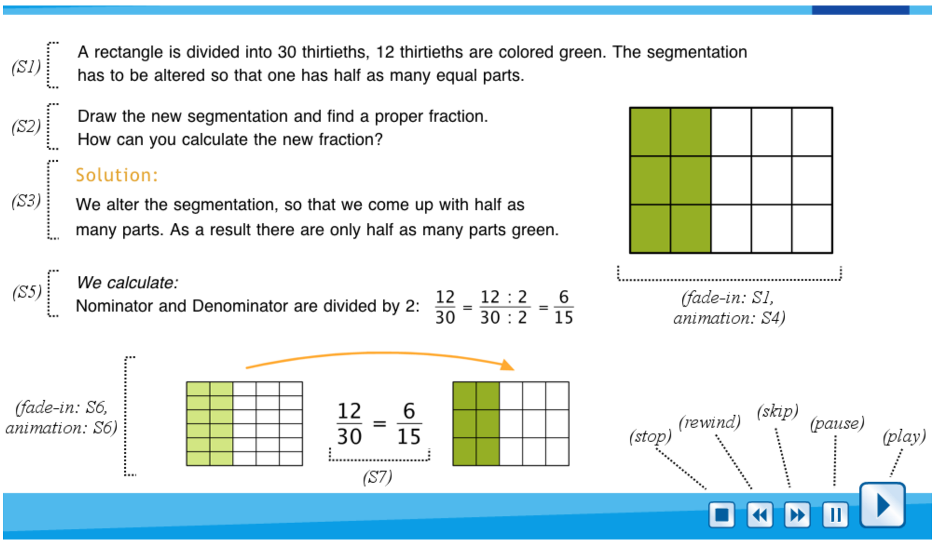

Figure 1: Screenshot after the last step of the interactive animated worked-out example *(dotted lines and italicised text in parentheses added as explanation for the reader)*.

The solution is divided into 7 segments: the context with the rectangle on the right (S1), the task (S2), the fade-in of the first part of the solution-text (S3), an animation of the altering of the rectangle (S4), the fade-in of the equation (S5), an animation of the arrow-scheme (S6) and the last equation (S7). By highlighting certain words and fractions with boxes during the animation, the dynamic processes are connected to the textual and symbolic representations. With a bar of control-buttons the students can control the different steps of the animated worked-out example.

QUANTITATIVE RESULTS

The example processing can be partitioned into a three-phase structure that could be derived from the data of the field study. In the *first phase* the students process the worked-out example without noticing the prompt. In the *second phase* they read the prompt and process it. In the *third phase* the students write down an answer. This sequence occurs in about 95% of all cases from the analysed class (Salle, in press).

Various quantitative results of this field study are published in Salle (2013), especially concerning the observed argumentation processes. The coding of metacognitive processes, self-explanations and argumentation processes shows the influence of the prompt and the differences between the phases (Figure 2). A comparison of phase one with phase two and three reveals obvious increases after the transition to the prompt-centred phases in all diagrams. Furthermore, especially questioning and reasoning statements increase in the latter phases and shape the content-related dialogues between the learners.

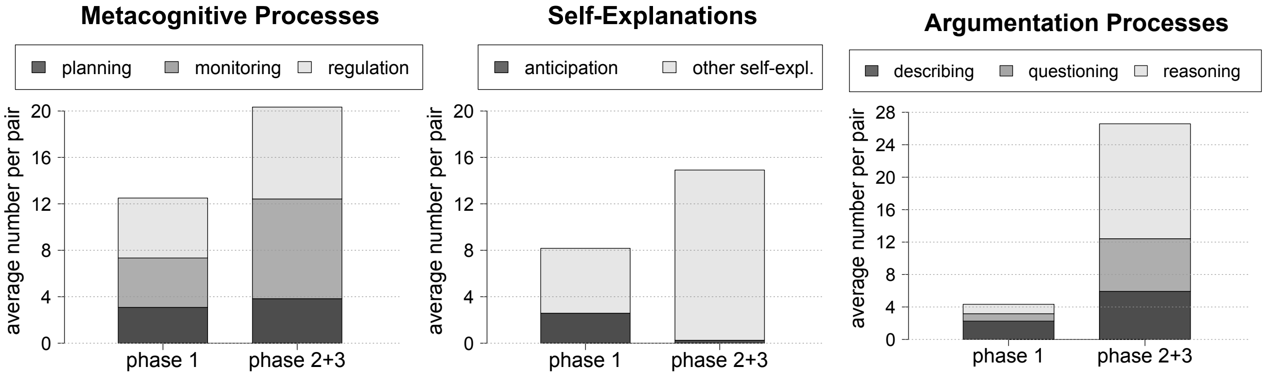

Figure 2: Quantitative results of the average number of coded metacognitive processes, self-explanations and argumentation processes.

The presented case of Ayla and Elli is chosen out of the transcribed learning processes of the analysed class because it contains exemplary aspects with regard to the research questions. Only phases one and two are considered in the following section, because these two phases reveal most of the aspects concerning the research questions. Due to the limited space, parts from the whole transcript are depicted, followed by a short description. Finally, the whole phase is summarized.

THE CASE OF ELLI AND AYLA

Elli and Ayla are two female students of the described class. The duration of their content-related dialogues during the processing of the examples is average. During the processing of the self-explanation prompts, they show the second longest duration in their class. The quantitative data of the girls' metacognitive processes, self-explanations and argumentations show average results related to their class.

Phase 1

Ayla and Elli start to process the described interactive animated worked-out example:

1 **Ayla**: (*with context on screen, Ayla hits the play-button, segment S2 – the task –*

appears. Immediately, she hits the play-button again, segment S3 with the first two lines of the solution appears.)

Elli: *(reads the example)* Draw the segmentation...

Ayla: (*moves the cursor above the play-button. Then she hits the play-button, segment S4 with the first animation begins, the part half as many parts is highlighted.)*

Elli: No, hold on a second... *(Meanwhile, the animation starts. Several lines of the iconic representation of the 12 thirtieths disappear successively.)*

In this short part from phase one, the two students follow the interactive animated worked-out example. Ayla controls the mouse and clicks on the play-button to start the segments of the example (line 1). Elli reads a short part of the text, but before she finishes, Ayla moves to the next segment and starts an animation (5-7, see S4 in Figure 1). Elli asks Ayla to hold on, obviously because Elli hasn't finished reading (8).

Summary of the first phase: Either the two students process the example silently or one of them is reading the text aloud. At the beginning of their processing, different processing paces can be observed. But with ongoing time, they coordinate their learning process and read the same segments ("hold on a second"). The girls process the example in linear fashion – the succession of segments is not interrupted. In the whole first phase only the play button is used. No projections of self-explanations can be observed. Their behaviour can be characterised as passive and receptive. After the last segment, the students read the accompanying open self-explanation prompt.

Phase 2

Elli: (*while both girls are looking into their workbook, Elli reads*) Open the computer-example. Try to comprehend every example-step. Then answer the following question … What is the meaning of 'altering a segmentation'? What changes, what remains?

Ayla: (*looks up*) I can't do that.

Elli: (*looks into her workbook*) Well, look. What is the meaning of 'altering a segmentation? What changes, what remains? (*looks up at the computer-screen*) Ok, look, here is something changing. (*grabs the mouse and hits the rewind-button several times*.)

When Ayla hears the questions of the prompt, she states that she "can't do that" (14). Elli tries to find something that helps to answer with the prompt (15-16). Simultaneously, she addresses her words to Ayla to involve her in the conversation (e.g. "Well, look", 17). Elli does not want to surrender too early. She grabs the mouse and rewinds some segments to navigate to a part of the example. There she discovers "something changing" that could help with answering the prompt (17).

After the depicted scene, the two girls continue their approach to the prompt-answer and repeat the animations. Then they stay at a point, at which the lighter green pieces become darker in an animation (can be seen in the smaller left and right rectangle).

Ayla: Hm? That doesn't make sense. This has only changed its color.

Elli: (*hits the play-button, segment S5 – the calculation – appears*) That has become darker .. And then, here. In absolute terms, numerator and denominator are divided by two … throughout by two, by two, by two..

Ayla: Ah, ok, now. That's because herein are such strange lines.

Elli: (*looks up*) Yes. (*5 seconds pause*) Whatever. Divided by two, hm? .. Because, …, because one box (*forms a rectangle with her hands*), there were two boxes in it. Then, they removed the line there (*the back of one flat hand touches the palm of the other, then she separates them vertically*) and then there was a very big box, now again. And, uhm, then they have, .. two, well, divided against [sic!] two. 12 divided by 30, why 30? (*2 seconds pause*) One two three ... Ah yes! (*then she speaks a bit slower*) because previously there were 30 (*moves her fingers in circles*).

After an animation step (S5), Ayla is confused concerning the graphical alteration of the pieces (19). Simultaneously, this is a specific monitoring statement – she expresses which part of the animation does not make sense to her. Elli describes, that on a computational level, "numerator and denominator are divided by two" (21-22). She repeats "by two" three times, obviously to clarify that there are more dividing-processes than one (22). Ayla gets the point and connects Ellis explanation to the "strange lines" in the animation (23). Then Elli explicates her first utterance by an explanation how two rectangles were put together to one (25-29). During this explanation, she uses her hands to form a rectangle and to visualize the removal of the lines. Finally, she successfully connects the rectangle-pieces to the fraction (30).

Summary of the second phase: The second processing-phase is characterised by a much more active behaviour of the two students. They regulate their learning process frequently by using the control-bar to navigate through the example, try to identify relevant information with regard to the prompt and make their partner aware of this information. The reading of the prompt oftentimes causes immediate monitoring utterances ("I can't do that", 14). Various projections of self-explanations can be observed – *verbal projections* (e.g. "by two, by two, by two" (22) suggests that she breaks the division down into a division of pieces) as well as *nonverbal projections* (imitating a rectangle and the altering-process with her hands). In following scenes, Elli continuously uses gestures to imitate depicted processes or to clarify aspects. Elli explains the altering-process in her own words and does not need to use the example to refer to it.

RESULTS

The case study of Ayla and Elli is exemplary in many aspects with respect to the analysed class that worked with open self-explanations in groups of two. In the following section, I refer to characteristic results of the whole class.

Characteristic patterns and behaviour during example processing: The *first* phase is characterised by a passive and receptive processing of the steps of the animated examples. Observed regulations are often adjustments of the processing pace, only in a few cases segments are rewinded or skipped. Mostly, they are followed in a linear way. The students follow the steps silently, or one of them reads out the text aloud. Monitoring statements are often unspecific. Only a few self-explanations can be reconstructed, a focus on special aspects of the content is identified seldom. With respect to the new content of the example, this behaviour could be expected.

The *second* phase is shaped by lots of student activities. Many more of the monitoring statements are specific ones. Meaningful interactions with the animated worked-out examples can be observed frequently – with concrete aims in mind, the students use the buttons to navigate through the examples, heading for various segments or animations. The students talk much longer than in phase one – explanations, argumentative and coordinative statements are verbal features of their learning process. These utterances are often induced by parts of the prompt questions that want the students to explain or to reason. Many more self-explanations can be reconstructed from verbal as well as nonverbal projections than in the first phase. These cognitive activities focus on key-concepts of the depicted solution. During this lasting involvement in the example- and prompt-processing, the students can organise the depicted processes more and more mentally without referring to the animations.

Effects of the open self-explanation prompts: Having watched the entire animated example in phase one, the first contact with the self-explanation prompt constitutes a caesura in the learning process. The prompt-question induces content-related conversations, especially argumentations, explanations and coordinative dialogues (Fig. 2, see also Salle, in press). The whole learning process is more focussed towards key-concepts and -principles. Self-explanation prompts serve as focal points and support students during the engagement with the examples. Only a few seconds after reading, the students often utter monitoring statements and self-explanations. This can be reconstructed in many transcripts. Subsequently, they navigate through the examples and break through the linear, superficial and passive processing of the first phase. Altogether, self-explanation prompts foster meaningful, active and self-regulated learning, content-related talk and argumentations during the learning with animated worked-out examples.

PERSPECTIVES

This paper shows the different positive implications that learning with animated worked-out examples and open self-explanation prompts can have despite their well-known properties. Nevertheless, a lot of questions remain unanswered and further research is needed to shed more light on cooperative learning from worked-out examples and the implications that prompts, trainings or design features of examples have on the learning process.

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IMPACT OF SINGLE STUDENT MATHEMATICAL FIELD EXPERIENCE ON ELEMENTARY TEACHERS OVER TIME

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Mewborn and Stinson (2007) explored three tasks implemented in a preservice teacher education program, which supported awareness of beliefs and reflection on teaching practice. In this study, we investigated one of these tasks, a single student mathematical field experience, to study its impact on learning and to determine its effects over time. We studied elementary teachers starting at their preservice teaching experience into their second year of teaching and after their tenth year teaching. We found this experience fostered development of implementation of multiple questioning strategies during their preservice teaching experience, which grew during their career as educators and is consistent into their tenth year of teaching.

Some literature reported that teacher education programs have minimal impact on preservice teachers' future teaching styles (e.g., Hiebert, Gallimore, & Stigler, 2002). However, we investigated a teacher education program that had a significant impact on four elementary teachers' teaching practice as well as their beliefs (Spangler, Sawyer, Kang, Kim, & Kim, 2012). Mewborn and Stinson (2007) explored three tasks from this education program, which supported belief awareness and change: critiquing a reflective teaching essay, participating in a single student mathematical field experience, and observing a mathematics lesson from an experienced teacher. In this study, we explored the impact of the single student mathematical field experience (SSMFE) to investigate the teachers' learning trajectory through the field experience and determine the staying power of the teaching practices developed from this event. The SSMFE was an activity where one preservice teacher assisted one elementary student over 8 weeks in various tasks. Four teachers were followed from their junior year in their teacher education program into their second year full time teaching and after their tenth year teaching to study the lasting affects of this experience. We found the SSMFE helped to foster teachers' development of a variety of questioning styles as described by Boaler and Brodie (2004).

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Mewborn and Stinson (2007) explored the interplay between students' personal theories, field experiences, and mathematical methods courses to suggest that specific tasks in teacher education program could assist preservice teachers' examination of their beliefs about teaching. The authors identified the single student mathematical field experience (SSMFE) as one of the activities that influenced preservice teachers to become aware of beliefs and reflect on their teaching practice. However, the authors explicitly stated their goal was not to identify a belief change but rather to illustrate the

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 89-96. Vancouver, Canada: PME.

tasks. Mewborn and Stinson (2007) advised that future research was needed, "to develop robust descriptions of their learning trajectories and to ascertain the staying power of the teaching practices they began to develop *with assistance* in their preservice program" (p. 1484). With this field experience, we investigated the teachers' learning trajectories and determined the teaching practice's staying power.

Field experiences provide opportunities for preservice teachers to develop questioning strategies to gain knowledge of students' mathematical thinking (Chamberlin & Chamberlin, 2010; Mewborn & Stinson, 2007). Mewborn and Stinson explained, "Field experiences provide a rich ground for questioning why we do the things we do and how we might do them differently if we are serving the goal of creating opportunities for preservice teachers to engage in assisted performances" (p. 1483). Chamberlin & Chamberlin (2010) found, "Many of the teachers mentioned questioning the students to stimulate their thinking, to refocus them on the problem at hand, to understand the students' thinking, or to challenge the students in their thinking" (p. 402) in the preservice teachers' field experience. However, the articles did not explain the kinds of questioning occurring during field experiences.

One of the most common forms of questioning patterns initiated in schools is teacher Initiation, student Response, and teacher Evaluation (IRE) (Cazden, 2001). This form of mathematical conversations sets a norm where the teacher asks the questions, and the students provide answers. IRE has the teachers dominating the mathematical talk and determining what mathematics is "correct" in the classroom. The United States of America's Common Core Standards (NGA Center and CCSSO, 2010) advocate that students need to be *constructing viable arguments and critiquing the reasoning of others*. The IRE mode of questioning does not support students' construction of thought because the teacher validates the mathematics. Other questioning strategies were advocated, and many researchers sought to determine the actions teachers should make to support students' mathematical thinking (Boaler & Brodie, 2004)

Boaler and Brodie (2004) categorized nine different forms of questions: gathering information, leading students through a method, inserting terminology, exploring mathematical meaning and relationships, probing, generating discussion, linking and applying, extending and thinking, orienting and focusing, and establishing context. The authors determined that the majority of questions given in a traditional classroom were focused on gathering information while reform-oriented teachers implemented a variety of questions in their classroom (Boaler & Brodie, 2004). Therefore, supporting development of a variety of question types in preservice teachers may help them develop into reform-oriented teachers.

METHODS

The data corpus for this study was collected in two parts. The first study included 15 participants across a 4-year period from their first year in a teacher education program through the end of their second year of teaching. The second study included 3 of the 15 previous participants 10 years after the initial 4-year study. In this investigation, we explored 4 of the participants, 3 of whom span both studies.

Participants

We began by analysing existing data on participants who we knew were still teaching and selected two pairs of participants for detailed analysis. These 4 were chosen because they entered the teacher education program with similar beliefs, but their teaching practices differed markedly by the end of their second year of teaching. One of these participants became a reading specialist immediately prior to his tenth year in the field, so he did not participate in the follow up study. The 4 participants were assigned the following pseudonyms: Laura, Jennifer, Jayne, and Alex. The initial study began during their junior year in college where they took one mathematics content course for elementary education majors prior to the study. During the study, they completed 2 mathematics methods courses for elementary education majors, the first of which included the SSMFE. During the second and third semesters they participated in 4-week field experiences in local schools; the fourth semester was a traditional student teaching experience. After graduating, the participants were employed at elementary schools for at least 10 years.

Single Student Mathematical Field Experience (SSMFE)

During the first mathematics methods course, the participants assisted the mathematical learning of one elementary student once a week for 8 weeks focusing on understanding the student's thinking, explanations, and interpretation of mathematical problems the preservice students constructed. This interaction was designed to focus on the mathematics of the students, allowing preservice teachers to build confidence in their abilities to develop problem-solving activities.

During the SSMFE, the instructor of the mathematical methods course and her two teaching assistants assisted the preservice teachers by coaching them in real time with questioning, adjusting instructional pace, and paying attention to the student's mathematical thinking. For each session of the SSMFE preservice teachers prepared comprehensive written plans and wrote follow up reflections. They also constructed a final portfolio documenting their growth and the growth of the child with whom they worked over the course of the semester.

Data Collection

During the initial study, the participants were interviewed once per semester for four years, observed once during an early field experience, twice during student teaching and approximately 4 times during each of the first two years of teaching. They were also asked to complete the Integrating Mathematics and Pedagogy (IMAP) web-based beliefs survey (Ambrose, Philipp, Chauvot, & Clement, 2003). The survey is designed to assess beliefs about mathematics, about learning or knowing mathematics, and about children's learning and doing mathematics.

During the tenth year study, the participants were individually interviewed and observed 3 times over the course of a semester, and they participated in a focus group. They were also asked to complete the IMAP web-based beliefs survey and Known Factors Affecting Belief Change survey created by one of the authors.

Analysis

Data from both studies were analysed using the constant comparison methods. We first identified what the participants stated they learned from the SSMFE in their first methods course's final portfolio and then compared what they said with their teaching practices over the first two years of teaching and then 10 years afterward. We understand that just because a participant stated something does not necessarily mean it will be enacted in their classroom practices. Individuals often are not aware of their beliefs, so we interpreted the participants' understanding using multiple strategies to ensure an accurate representation of their views (Leatham, 2006).

FINDINGS

We followed our participants' progress through this SSMFE in three stages. First, we identified what they said influenced their teaching practice as stated in their final portfolios from their first methods course. Next, we described the four teachers' teaching practices through their first 2 years of teaching to assess the preservice teachers' self-identified impact. Finally, we asked the teachers in their tenth year of teaching about these experiences and observed their teaching practices to see what impact still existed from the SSMFE.

Laura

In Laura's final portfolio of her first methods course, she stated that she learned how to address issues in behaviour, how to assess student understanding from student's work, and how to ask appropriate questions from the SSMFE, and these teaching practices were observed during her first two years of teaching.

When she began full time teaching, Laura demonstrated learning when to press a student mathematically and when to "take a break" in her own teaching. Laura implemented her skills of assessing students' understanding of concepts throughout her first two years of teaching though her use of questioning. Laura learned a lot about questioning from her SSMFE. She showed this by implementing different questioning styles in her classroom. When she was observed even as soon as student teaching, Laura would ask questions like, "Does this make sense?" "How do you know?" Laura explained that there was a difference between her student in the SSMFE and her kindergarten students by saying "I guess with the kindergartners, I had to ask a lot more leading questions, but with the [SSMFE student], I could just assume that she could make a lot of connections." But she still admitted to learning how to construct different forms of questions from the SSMFE.

Jennifer

From the SSMFE, Jennifer identified in her final portfolio learning how to engage gifted students, how to implement appropriate wait time, and how to help students feel successful. However, the skills that had a lasting impact on her teaching were how to use questioning strategies to determine what students learned and how to help students enjoy mathematics.

When she started teaching, Jennifer had a deep desire to make mathematics fun, and each lesson had some element to engage her students. This went along with her belief in helping students feel successful. Jennifer stated that she found a connection between questioning and her SSMFE. She did not originally identify questioning as learned from the SSMFE, but she explained, "I think personally I grew, like as a teacher" from the field experience because it taught her to think about questioning. She said, "I don't know if I was asking as many of the right questions then as I might have. So yeah, I think I got better at it." Although Jennifer did not initially identify questioning as learned from the SSMFE, reflecting on the process helped her to see her own mistakes, thus helping her develop questioning skills. In her observations, she showed improvement in asking a variety of questions over the course of the two years. In the beginning, Jennifer mainly implemented the IRE pattern, but over time she developed probing, exploring, and orienting questioning patterns as was suggested by Boaler and Brodie (2004).

Jayne

In Jayne's final portfolio, she expressed that she had a strong belief in doing what was best for her student, which influenced what she learned from her SSMFE. Jayne identified from her SSMFE learning: how to select appropriate mathematical tasks, how to implement mathematical discourse, and how to follow students' thinking.

Jayne demonstrated selecting appropriate mathematical tasks by having a student-centred stance in her first two years of teaching. Jayne insisted students needed to conceptually understand the mathematics, not just recall the facts. Jayne believed that all teachers should prepare students for what they would need in the future rather than just for standardized examinations. In addition, she emphasized that knowing her students was the critical foundation in teaching, and she assessed her students' understanding individually through asking questions. She explained that appropriate questioning was one of the most efficient means to assess students' thinking. This matches with Boaler and Brodie's (2004) categories of probing or getting students to explain their thinking. Jayne admitted assessing students' performance takes a lot of time, and teachers need to be patience in this process. Finally, she continually implemented her belief in following her students' thinking through her first two years of teaching. Jayne admitted that her school's curriculum did influence what was taught, but she believed she still controlled how that material was taught to her students. Thus, Jayne preferred maintaining her student's pace of learning, rather than a pace dictated by others.

Alex

In working with his student in the SSMFE, Alex learned how to ask effective questions, how to explore and understand a child's mathematical thinking, and how to learn a student's unique problem solving techniques.

In Alex's first two years of teaching, he emphasized the use of questioning strategies to explore and understand children's mathematical thinking. Alex assessed students' understanding through a multitude of questions and also encouraged his students to ask questions in mathematical activities. For example, he had his students asking "how does this work" and "why did this happen" probing questions which is constant with Boaler and Brodie's (2004) fourth questioning type. Although he tried hard to build on his students' mathematical thinking, Alex felt a lot of pressure from the school administration for his students to preform on standardized tests. Thus, many of his teaching practices were defined by the curriculum designated by the school, which hindered him constructing mathematical tasks.

10 Years Later

Ten years after the SSMFE, Laura was the only participant to remember the activity. The student she was paired with was unresponsive and lacked key mathematical knowledge necessary to explore many of the tasks she planned for their sessions. Laura explained, "I should have approached it differently, but at the time I didn't really know. I mean I think it was a growing experience. It was struggle time." Laura still remembered this experience because of her struggles with her student, but what she learned about behavioural management and questioning during that time was apparent during her observations.

Jayne and Jennifer did not remember the SSMFE when asked after their tenth year of teaching. Yet, they both explained experiences like that were beneficial. As Jennifer stated, "because, you know, how else are you going to learn about how kids think without sitting down and working one on one with them and listening to them?" They explained field experiences helped them to develop skills in building relationships with their students. Jayne still showed a desire to do what is best for her students through her pacing based on students' needs and the student centred hands on activities she implement in her classroom. They both still demonstrated using multiple questioning types in their lessons. Altogether, the teachers explained that they mostly did not remember details about what they did in the SSMFE, but they believed it to be valuable and they still demonstrated knowledge of different questioning styles ten years after they were introduced.

DISCUSSION

Each of our participants identified that his/her questioning strategies were initiated from their SSMFE, and over time they improved in their use of different questioning types. We believe the field experience was a beneficial activity that fostered development in this area, yet beliefs developed before entering into the teacher education program did influence the preservice teachers learning trajectory as well.

Beliefs about Teaching Influenced Learning

The four participants demonstrated a focus on students' learning from their preservice teaching experience into their second year of teaching. However, even though they all consistently showed this behaviour into their second year of teacher and some into their tenth year of teaching, this teaching style appears to have been more influenced by their personal beliefs about students' mathematical learning rather than any particular activity they did in their preservice experience. For example, Jennifer held a strong belief in making mathematics fun, and she even expressed that she found mathematics only to be meaningful if the students enjoyed what they were doing. Her actions to make students feel more successful were consistent with her belief in making mathematics fun. Jayne, on the other hand, had a strong belief in doing what is best for students. Her actions were influenced by her beliefs about student learning rather than a single activity from her teaching program.

Questioning as a Learned Skill

The participants demonstrated using multiple questioning types in their classrooms after the SSMFE, and there is evidence that they learned this skill from that activity. Each of our participants identified that his/her previous experiences with mathematics were in traditional classrooms. Questioning was not stressed in the traditional classroom, but it was stressed during their preservice teaching experience (Cadzen, 2001). Laura, Jayne and Alex initially identified questioning as a skill they learned from their field experience, and Jennifer later asserted that she originally began learning questioning from her SSMFE even though she did not feel proficient in the skill at the time. Laura identified learning the different properties of questions, which is similar to what Boaler and Brodie (2004) categorized. Jayne worked on phrasing questions carefully to assess her children. Although Alex struggled with the accountability-driven system, he believed the advantage of questioning benefitted both teachers and students and applied it in assessing students' mathematical understanding. Questioning students can be an unnatural activity for teachers. By giving them the chance to practice this skill, they can become more proficient. Thus, it appears that this SSMFE reinforced questioning strategies for these teachers.

Lack of Memory of the Activity Does Not Influence Past Learning

Ten years after the SSMFE most of the participants could not identify what they learned or what they did in the experiences. However, just because they could not identify that specific activity does not mean that it was not significant in their growth. Many people are not aware of their own beliefs or know why they developed (Green, 1971). Jennifer, Laura, and Jayne were able to show how they focused on student thinking by using questioning during their observations. Because the participants did not remember the activity but still demonstrated skills learned during the SSMFE, we take this as evidence that SSMFE does have a lasting effect on preservice students.

CONCLUSION

The SSMFE fosters the development of teachers' questioning over time. This experience reinforced their ability to construct multiple questioning types and facilitate student thinking in their classroom. To focus on understanding students' mathematical learning, the activity provided the preservice teachers an opportunity to learn about what they should do to become teachers without the pressure of dealing with classroom management. Further research is needed to determine if these findings can be duplicated in other schooling environments, but we can say for the four participants studied ten years after the introduction of the activity it continues to influence their teaching practice.

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MAKING GENERALIZATIONS EXPLICIT: AN INFERENTIAL PERSPECTIVE ON CONCEPT-FORMATION OF VARIABLES

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The transition from preformal and propaedeutic generalization-actions to a symbolically explicit use of the concept of variable has been a matter of significant attention in mathematics education, for example in the context of generalization processes on a preformal level and regarding the specific nature of algebraic concepts. This contribution offers an inferential theoretical perspective to study the relation of geometric and arithmetic notions when dealing with figural growing patterns and arithmetic sequences. By reconstructing "individual commitments" we show results of the impacts of this relation on the individual notion and explicit use of the concept of variable. Finally, the results also show that the concept of "propaedeutics" itself gains an extension in the light of the theoretical framework.

GENERALIZING IN THE CONTEXT OF VARIABLE

The concept of variable in algebraic expressions is one of the most important mathematical objects within mathematics classroom (Kieran, 2007, Usiskin, 1988, Lee, 1996). Its fundamental role is due to its conceptual nature as a tool to make explicit patterns and structures in form of generalizations. This paper presents results of a study (Schacht, 2012, Young Researchers Promotion Award 2012 by the GDM) on the propaedeutic and then symbolically explicit use of the concept of variable in early algebra. The explication of the concept of variable in mathematics classroom (usually in 5th grade) does not mark the beginning of algebraic experiences for the students. These experiences are often deeply rooted within the conceptual dimensions of dealing with patterns, a functional context, equivalence and equations on a preformal level (e.g. Cooper & Warren, 2011). Cooper & Warren especially point out the importance of generalizing for learning algebra: "improving one's ability to generalise lies at the foundation of efforts to enhance participation in and learning of algebra." (Cooper & Warren, 2011, p. 190) There are many productive examples that use this idea within the context of early algebra (cf. Lee, 1996). For example, Lee (1996) points out to start early with generalization-actions: "Generalization is one of the important things we "do" in algebra and therefore something students should be initiated into fairly early on." (Lee, 1996, p. 103) There are two different types of generalization, a recursive and an explicit one (e.g. Lannin, 2005), whereas "young students develop these abilities from recursive to explicit" (Cooper & Warren, 2011, p. 197). With variables in algebraic expressions, students can make these generalizations explicit, for example in dynamic figural growing patterns or in arithmetic sequences.

It is important to note, that the notion of generalizing can both be done on a geometric level (with figural growing patterns) and on an arithmetic level, whereas the geometric level can be seen as essential in order to build a solid foundation for the future understanding of functional contexts (Moss & London McNab, 2011, p. 297). Becker and Rivera (2011, p. 363) even point out that the ability to generalize figural growing patterns supports the conceptualization of the concept of variable significantly.

Within our study, a learning environment was used (Hußmann et al., 2012) that is mainly influenced by these ideas and that follows the following aspects: First, students use numerical expressions to describe static figural patterns (e.g. $2 \cdot 3+6 \cdot 5$), then they use expressions like these to describe the number of dots in figural growing patterns (e.g. for the linear sequence $2+1 \cdot 5$, $2+2 \cdot 5$, $2+3 \cdot 5$, $\cdots$). Then, in order determine the number of dots in for example the 1000th element of the sequence, they use this linear structure to determine $2+1000 \cdot 3$. Finally, they use the algebraic expressions with variables to make the generalization of the pattern explicit.

Within this learning environment, the concept of variable is used to both determine the number of dots for different elements of a given sequence and pattern and to describe the structure of a given (figural or arithmetic) sequence. The potential of dealing with patterns and describing can be seen in the specific nature of patterns:

> Mathematical visualization and growing patterns (…) can mediate between the mathematical structure and the student's thinking because of their special ‚double nature' (they are on the one hand concrete objects, which can be dealt with, which can be pointed at and counted, (…) and at the same time they are symbolic representatives of abstract mathematical ideas). (Böttinger, et al., 2009, p. 151)

Sfard et al. (1994) point out a similar notion the dual nature of algebraic concepts: "the operational outlook in algebra is fundamental and the structural approach does not develop immediately" (p. 209). Regarding the variable, this dichotomy means that it can be used specifically as a tool for example to find out the number of dots in a certain element of a growing pattern. At the same time the variable can be used as structural objects in algebraic expressions with variables for example to describe mathematical growing patterns.

These insights refer to the different epistemological statuses of mathematical concepts and especially to the variable itself. Our results suggest, that these insights for algebraic concepts can be extended to a propaedeutic understanding. Within the context of the study and within the processes of concept-formation toward the variable, the students were engaged in many generalization processes when dealing with geometric and arithmetic sequences without using the concept of variable explicitly. Regarding these processes on a micro-level, there is a need for research concerning the question, how far these geometric and arithmetic generalization processes correspond to the (individual) later notion of the concept of variable as a tool or as a theoretical object.

THEORETICAL FRAMEWORK, RESEARCH QUESTIONS AND DESIGN

The inferentialist perspective

In order to describe both individual processes of concept formation as well as the process of constructing learning environments and in order to structure the subject matter we developed a consistent theoretical framework. Its foundations pick up philosophical ideas of Kant, Frege, Wittgenstein, Heidegger and Frege. Also, the theory of inferentialism (Brandom, 1994) was adopted to develop this framework.

Within this framework, commitments are seen as (reconstructed) assertions in a propositional form, that the individual student acknowledges and holds to be true. Commitments can be made explicit. In this perspective it is one of the central background-theoretical assumptions, that doing mathematics a highly social process: “At the core of discursive practice is the game of giving and asking for reasons.” (Brandom, 1994, p. 159) This basic theoretical assumption is used within a qualitative psychological research design. *Individual commitments* are seen as the smallest units of thinking and acting, that do not necessarily have to be true, but which are being held to be true by the individual. Within the interpretative process, these individual commitments are reconstructed turn-by-turn. Commitments can be acknowledged by the individual and they can be attributed to our discursive partners. As reasons in argumentation, individual commitments are inferentially related. That does not mean an inferential relation in the sense of classical logic though: inferential relations between two commitments do not have to be true or false, but they are held to be true or false by the individual!

In this commitment-based theoretical perspective, the notion of individual concept-formation is being extended: individual concept-formation is modeled as the development of individual commitments, that underlie our use of concepts (c.f. Schacht, 2012). Since our concepts are always inferentially related by the commitments we acknowledge, this implies a holistic perspective on concepts themselves: “One immediate consequence of such an inferential demarcation of the conceptual is that one must have many concepts in order to have any. For grasping a concept involves mastering the proprieties of inferential moves that connect it to many other concepts (...). One cannot have just one concept.” (Brandom, 1994, p. 89) This fundamental insight is one of the foundations of the analysis, since the theoretical framework gives respect to the many concepts that are involved when learning the concept of variable for example.

Finally, this theoretical perspective has an important consequence for the notion of propaedeutics of concepts. Even if a certain concept is not yet symbolically explicit, there may be still identified a variety of individual commitments, that refer to the concept implicitly or explicitly. This means that even before the explication of the concept of variable it might be possible to reconstruct individual commitments that refer to the concept of variable for example in the context of generalizing arithmetic or geometric patterns. This way, the genesis of individual concepts can be reconstructed and described within a fine-grained analysis (cf. Schacht, 2012, Hußmann et al., 2009).

Research Questions and Design

The inferential theoretical framework offers potential especially regarding its conceptual foundations. First the *inferential notion* of the framework faces an important nature of concepts: we never learn only one isolated concept (e.g. the concept of variable), but furthermore many other concepts, that may get a different and new shape within the practice, operations and new situations. The explication of the variable in mathematics classroom, that follows the exploration of geometric and arithmetic patterns, may have also influence on the concepts of *balance, equivalence or recursivity*. Within this perspective, the following research questions are posed:

- How do individual commitments, that refer to geometric and arithmetic concepts, influence the individual notion and use of the concept of variable?
- Within the process of concept-formation, how far do individual commitments relate, depending on i) the preformal and propaedeutic use within generalization processes and ii) the symbolically explicit use of the variable?

These research questions are posed within a qualitative research design. The case study (Schacht, 2012) was planned and conducted within the design-research project KOSIMA (see Barzel et al., 2013). About 60 students (11 years) worked within a learning-environment which introduces to the concept of variable. The theoretical framework was used to understand and describe individual processes of concept-formation by reconstructing individual commitments and their inferential relation. In this paper, the case *Orhan* is discussed in detail.

RESEARCH RESULTS AND DISCUSSION

Geometric and arithmetic commitments within generalization processes

The following scenes show the student Orhan dealing with linear dynamic figural patters with the arithmetic structure 2+6x. Within the learning process, Orhan has already dealt with static figural patterns. In the scenes below (Fig. 1), Orhan is given the figural pattern and he is asked to draw the next element of the sequence. He then counts the number of dots in the first three elements of the sequence, determines the arithmetic rule of the figural pattern (*add 6*, indicated by “2 + 6 = 8” and “8 + 6 = 14”, see Figure 1) and then he calculates the number of dots for the next element: 20 dots. Then he draws the next element with the shape of a triangle.

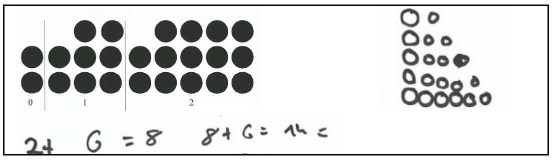

Figure 1: First three elements of a growing pattern. Orhan draws the fourth element.

Analyzing the transcript in detail, the reconstruction of the individual commitments shows that his actions in this scene are prototypical for many other scenes in a way, that they follow a certain scheme, that he repeats in different other scenes: Orhan first counts the total number of dots of each element, he then determines the balance between each two elements and then generates the arithmetic rule that underlies the figural growing pattern. He then uses this arithmetic rule to determine the number of dots of the next element (see table 1 for a reconstruction of his commitments). He then continues the figural growing pattern by drawing the next element with a shape of a triangle. This is also a typical scene, because Orhan uses this shape in different scenes to continue differently structured growing patterns. By using the term "wall", he refers to the triangle-shaped structure that he often uses to continue the sequences geometrically.

1 Int.: Why did you draw the dots like this and not in a different way?

2 Orhan: Because, eh, the wall is easier for me.

The following table shows the reconstruction of Orhan's commitments within this scene. The arrow indicates the inferential relation, that means that a given commitment serves as a reason for the next one.

Commit-ment number	**Orhan's individual commitments (reconstruction)**	**inf. relation**	**geo. or arith.**
1	I determine the number of dots by counting them one-by-one	↙	geo.
2	I can find the arithmetic rule by determining the differences.	↙	arith.
3	The rule of the growing pattern is: you have to add 6 dots to the last element.	↙	arith.
4	I use the (arithmetic) rule of the pattern to determine the number of dots in the next pattern.	↙	arith.
5	The next element has 14+6=20 dots.	↙	arith.
6	I can use a triangle shape to visualize a given number of dots.		geo. / arith.

Table 1: Orhans individual commitments in an inferential structure

This reconstruction of Orhan's individual commitments shows a number of results. First, it shows the strong relation between commitments that refer to arithmetic (commitments number 2, 3, 4, 5 in Table 1) and to geometric concepts (1, 6 in Table 1). Also – typical for situations like these – Orhan initially changes from geometric commitments to arithmetic commitments when continuing a figural growing pattern. Originally, this task was posed in order to observe, if students see the geometric pattern (there is always a block of 6 dots added in 2 rows of 3 dots each) and if they use this pattern to continue the sequence. By doing so, we figured, they would initially use commitments that refer to geometric concepts (rows, geometric pattern etc.). But Orhan activates commitments that refer to arithmetic concepts. For him, it is a viable way of operating in situations like these. Continuing the sequence, Orhan uses a triangle shape to draw the next element. Asked for his commitment (line 2 of the transcript above), he says that it is easier for him to use "the wall". It is one important result to show, that Orhan uses both geometric and arithmetic commitments to find the rule of the pattern and then to continue the sequence. This will play an important role in the light of the symbolically explicit use of the variable (see below). Second, it is an important result that these changes between geometric and arithmetic commitments do not occur without mathematical frictions: Although, for Orhan, it is a proper conclusion to use a triangle shape when having determined the number of dots for the next element. But, mathematically, he does not use the given geometric structure to continue the sequence. This is an interesting result insofar, as it is typical for Orhan's learning process that he is very flexible and shows strong competences when dealing in arithmetic situations but at the same time he often acknowledges geometric commitments that are mathematically not viable. As a third result in this scene, for Orhan, the "wall" is a tool to visualize a given (or determined) number of dots. Even more: The wall, for Orhan, is *the easiest* tool to visualize in situations like these. This scene shows – on a propaedeutic level – how far geometric patterns can be *tools* for students in visualization-situations. In contrast to these results dealing with figural growing patterns, the next section will show, that – also on a preformal level – Orhan acknowledges arithmetic commitments, that refer to the structural notion of the variable.

The concept of variable in algebraic expressions between the implicit and the explicit

In a different scene, Orhan works on a given arithmetic sequence: 2, 10, 18, 26, 34,…. Orhan is first asked to determine the rule and he answers: "You always have to add 8." Using that rule, Orhan then determines the next three elements 42, 50 and 58 of the sequence. Meanwhile, the students have learned in class, that the (general) rule of a given arithmetic or geometric pattern can be made explicit with the help of an algebraic expression with a variable. Orhan writes down: "2+8x = ". His commitment can be reconstructed here: *I can make explicit generalizations of arithmetic patterns in algebraic expressions with variables*. Being asked, what the x stands for, Orhan answers: "The x means a number, let's say you want the 35th element of a sequence."

Here, Orhan uses the concept of variable first to describe the arithmetic structure of the sequence and then, second, as a tool to determine certain elements of the sequence. Orhan uses the concept of *element* of a sequence, that can be determined with the help of the variable. For him, the meaning of the concept in this situation is mainly rooted in the calculation of elements with high numbers. For Orhan, the explicit concept of variable marks a specific character of a tool (c.f. Sfard & Linchevski, 1994) and he acknowledges the following commitment: *Elements of sequences can be calculated with expressions using the variable. The x stands for the element*.

The importance of this result can be seen by analysing the last scene. Orhan is asked, if 90 was an element of the sequence.

1	Orhan:	90 (…) yes
2	Int.:	(…) And why is 90 an element of the sequence?
3	Orhan:	90, eh, because 88 can be divided by 8 (88 ist in der 8er Reihe)
4	Int.:	Yes. And 90?
5	Orhan:	90 cannot be divided by 8, but 90 is part of the sequence. Then I subtracted 2 and that makes 88.

Here Orhan activates commitments, that explicitly refer to structural algebraic concepts. For Orhan, the sequence 2, 10, 18, … in this situation is a structural object with certain properties. One of these properties is that each element of the sequence can be produced by taking a number that can be divided by 8 and then add 2. In this scene, Orhan puts 2 different sequences (0, 8, 16, 24,… and 2, 10, 18, 26,…) in a certain relation and his argumentation uses the properties of both sequences. It is important to note now, that by dealing with arithmetic sequences, Orhan acknowledges commitments, that not only refer to central algebraic concepts (Kieran 2007) but that also constitute a stable concept of function (Healy & Hoyles 1999). It is a central insight here, that Orhan uses the concept of sequence not as an operational but as a structural object with certain properties, that he uses as reasons in his argumentation. Also, this scene shows that his commitments refer to an elaborated concept of equivalence and the transformation of algebraic expressions although these concepts themselves are not symbolically explicit. Still, his commitments refer to them in on a preformal level (lines 1-5 in the transcript above).

Although Orhan's symbolically explicit use of the concept of variable refers to an operational character, the detailed analysis of his commitments shows that they also refer to important structural algebraic concepts on an implicit level. His commitments, that the properties of two sequences may be compared and used within a complex argumentation, reveal the concept of equivalence, variable and equality as structural objects that are not used symbolically explicit but on a propaedeutic level. This result extends the notion of the duality of concepts within the context of generalizing to a propaedeutic dimension of conceptual use.

CONCLUSION

The empirical results presented above allow some insights into processes of concept-formation when learning the concept of variable within a learning environment, that focuses on dealing with growing figural patterns and arithmetic sequences. The inferential theoretical approach unfolds its potential especially regarding the (inferential) relation of arithmetic and geometric commitments when learning the concept of variable. The data especially shows examples of students' geometrical reasoning within generalization processes, whereas the reconstructed commitments refer to arithmetical concepts. Although we only refer to one case in this contribution, it not only shows the interplay between arithmetic and geometric commitments but also the different notions of structural and operational concept-usage on a propaedeutic as well as on a symbolically explicit level. Besides the very fine-grained analysis to study, this theoretical framework offers a tool to describe frictions and to interpret them within the individual argumentation processes that are being reconstructed with individual commitments and inferences. Finally, the data shows that this theoretical framework offers a perspective to extend and differentiate important insights (Sfard et al., 1994) regarding the symbolically explicit use of the variable to its propaedeutic usage and – more generally – it offers insights into the individual formation of concepts itself.

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COGNITIVE PROCESSES UNDERLYING MATHEMATICAL CONCEPT CONSTRUCTION: THE MISSING PROCESS OF STRUCTURAL ABSTRACTION

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The purpose of this paper is twofold: On the one hand, this work frames a variety of considerations on cognitive processes underlying mathematical concept construction in two research strands, namely an actions-first strand and an objects-first strand, that mainly shapes past and current approaches on abstraction in learning mathematics. This classification provides the identification of an often overlooked fundamental cognitive process, namely structural abstraction. On the other hand, this work shows a theory-driven and research-based approach illuminating the hidden architecture of cognitive processes involved in structural abstraction that gives new insights into an integrated framework on abstraction in learning mathematics. Based on our findings in empirical investigations, the paper outlines a theoretical framework on the cognitive processes taking place on mental (rather than physical) objects.

INTRODUCTION

Attributed as a crucial cognitive process in concept construction, abstraction has been the focus of many researchers in diverse research areas. Caused by both a confusion between abstraction and generalization and a characterization of abstraction aimed at *de*contextualization instead of *re*contextualization (see, van Oers, 1998), the term 'abstraction' has been almost "removed from the discourse of learning" (Sfard, 2008, p. 10). Though attention in research on abstraction has steadily declined since its peak in the pre-cognitive science era, some researchers still have advanced our all understanding on this issue by integrating 'modern' perspectives on past and current theories of learning in a broader theoretical frame. *The Nested RBC Model of Abstraction* originally described by Hershkowitz, Schwarz, and Dreyfus (e.g., 2001), for instance, provides an interesting conceptual undertaken in this area. The following pages present a further theoretical approach addressing the issue of abstraction in learning mathematics more broadly. In this work, the purpose is not to compete with other theories but to shed lights on a neglected cognitive process, namely on *structural abstraction*.

The proposed outline of the theoretical framework on structural abstraction results from (a) reconsidering Davydov's (1972/1990) *ascending from the abstract to the concrete* from a *dialectical* point of view as expressed by Ilyenkov (1982), (b) taking fundamental findings in cognitive science and psychology into consideration, (c) embedding the framework into philosophical grounds, and undertaking a reanalysis and presentation of data obtained in a previous study (Pinto, 1998).

THEORETICAL BACKGROUND: TWO FUNDAMENTAL STRANDS IN RESEARCH ON ABSTRACTION IN LEARNING MATHEMATICS

Several approaches, partly distinct and partly overlapping, shape the theoretical landscape in mathematics education research on abstraction. Taking as poles of a wide spectrum, we can distinguish two strands of cognitive processes underlying concept construction, namely (1) an *actions-first* strand and an *objects-first* strand. The former has to do with processes of *focusing on the actions on objects*, in particular, individuals' reflections on actions on known objects, grounded in Piaget's work of 'genetic epistemology' that puts 'actions' in its heart with the underlying philosophy that knowledge is basically 'operative'. The latter has to do with processes of *focussing on the objects themselves*, in particular, paying attention to the properties and structures inherent in those objects. As shown in Fig. 1, in both strands, the focus of attention may take place on *physical* objects (referring to the real world) or *mental* objects (referring to the thought world). Both strands capture the bulk of theoretical and practical work in past and recent years, however, it seems that the mathematics education research literature has nearly limited its focus on actions-first theoretical approaches. Research within the actions-first strand has made considerable progress considering both physical and mental objects as a point of departure in abstraction processes, while the focus of attention within the objects-first strand is limited, with few exceptions, to physical (instead of mental) objects. The current study considers cognitive processes underlying concept construction that take *mental* objects as a point of departure. Based on philosophical grounds and findings in psychology and cognitive science, we argue that *structural abstraction* is the key cognitive process in this issue. Furthermore, the paper outlines how an integrative framework might conceptualize the functional interplay of cognitive processes building the architecture of structural abstraction.

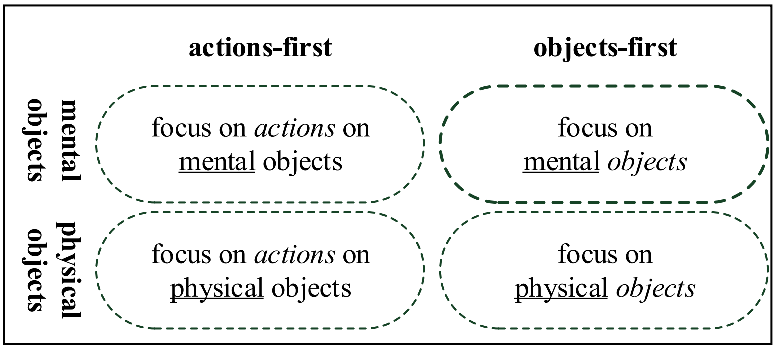

Figure 1: Actions-first and objects-first strand.

Actions-first Strand

Within this strand, two fundamental cognitive processes can be distinguished, namely (1) focussing on *actions* on <u>physical</u> objects and (2) focussing on *actions* on <u>mental</u> objects. The former refers to Piaget's *pseudo-empirical abstraction*, while the latter refers to Piaget's *reflective abstraction*. In his *Recherches sur l' abstraction réfléchissante*, Piaget (1977/2001) describes pseudo-empirical abstraction as a process by which individuals discover in objects the properties that have been introduced into them by their own activity. In other words, the results covered by pseudo-empirical abstraction are read off from material objects but the observed properties are actually introduced into the objects by the subject's activities. Yet, reflective abstraction is abstraction from the subject's actions on objects, mostly from the coordination between these actions. Abstracting properties of an individual's action coordinations is thought as the crucial function of Piaget's reflective abstraction. In mathematics education, its highest impact is considered in its process of *encapsulation* (or *reification*). From Piaget's reflective abstraction, Dubinsky et al (e.g., 1991) and his colleagues developed the APOS theory, describing the construction of concepts through the encapsulation of processes. Similar to the latter is *reification* – the main tenet of Sfard's (e.g., 1991) framework emphasizing the cognitive process of forming a (structural) concept from an (operational) process. In the same line, Gray and Tall (e.g., 1994) describe this issue in terms of an overall progress from procedural thinking to *proceptual* thinking.

Objects-first Strand

Symmetrical to the actions-first strand, two fundamental cognitive processes can be distinguished within this strand, namely (1) focussing on <u>physical</u> *objects* and (2) focussing on <u>mental</u> *objects*. The former refers to empiricist approaches in the sense of *seeing similarities* among objects that fall under a particular concept. *Empirical abstraction*, in the sense of Piaget, describes a process when an individual abstracts sensory-motor properties from experiential situations. In Piaget's (1977/2001) own words, empirical abstraction "draws its information from objects" (p. 317) but "is limited to recording the most obvious and global perceptual characteristics of objects" (p. 319). However, as argued by diSessa and Sherin (1998), though these abstraction processes (abstraction of dimensions that can be perceived) work well for *category-like* concepts, classical approaches (such as *classifying* or *categorizing* that are based on identifying commonalties from a set of specific exemplars) do not provide fertile insights into cognitive processes underlying concept construction in mathematics. An approach that goes beyond Piaget's empirical abstraction has been developed by Mitchelmore and White (e.g., 2007). Drawing on Skemp's (1986) conception on abstraction, their work on *empirical abstraction in learning elementary mathematics* describes abstraction in terms of the *underlying* structure rather than from superficial characteristics. This study of the *underlying* structure (of a mathematical concept) is considered as the heart of the objects-first strand in mathematics education research on abstraction. While Mitchelmore and White consider physical objects, the following subsection describes a cognitive process that takes *mental* objects as a point of departure.

STRUCTURAL ABSTRACTION

The notion of structural abstraction has been already used by Tall (2013) in the sense of a superordinate abstraction for empirical and platonic abstraction. Its "fundamental role […] throughout the full development of mathematical thinking" (ibid., p. 39) has been highlighted in Tall's (2013) work *How humans learn to think mathematically*. As described in earlier work (Scheiner, 2013) and argued in this paper, structural abstraction goes beyond Tall's conception of this particular kind of abstraction. The crucial puzzle lies in the observation that structural abstraction has a dual nature, namely (1) 'complementarizing' the aspects and structure underlying specific objects falling under a particular mathematical concept and (2) facilitating the growth of coherent and complex knowledge structures. From this point of view, structural abstraction takes place both on the objects-structure and on the knowledge-structure (see, Figure 2).

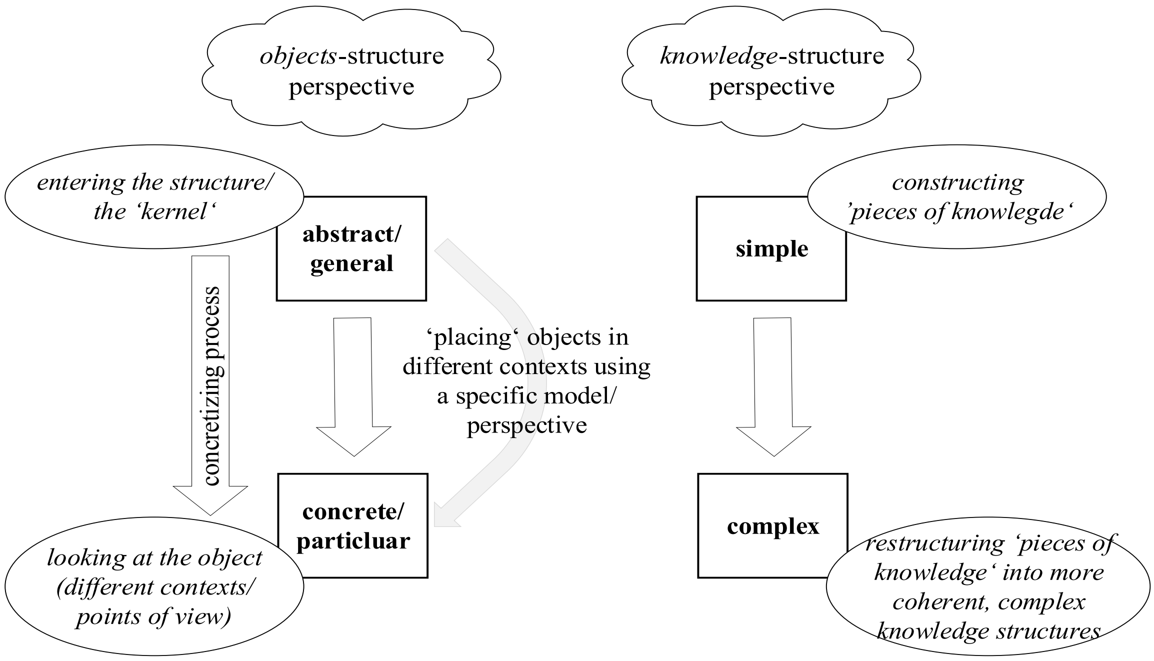

Figure 2: The dual nature of structural abstraction.

From the *objects*-structure perspective, structural abstraction means (mentally) structuring the diverse aspects and the underlying structure of specific objects that have been particularized through placing the objects in a variety of different contexts. However, structuring the diverse aspects and the underlying structure of objects falling under a particular concept requires a concretizing process where the mathematical structure of a specific object is entered by looking at the object in relation with itself or with other objects that fall under the particular concept. Through placing objects into different specific contexts using a realistic model or perspective that provides theoretical structure in constructing a concept the meaningful components of the object may be highlighted. Models are, in this sense, intermediate in abstractness between 'the abstract' and 'the concrete'. This means that at the start of a particular learning process a model is constituted that supports the 'ascending from the abstract to the concrete' as described by Davydov (e.g., 1972/1990). Davydov's strategy of ascending from the abstract to the concrete draws the transition from the general to the particular in the sense that learners initially seek out the primary general 'kernel' and, in further progress, deduce multiple particular features of the object using that 'kernel' as their mainstay. The crucial aspect in this approach is Ilyenkov's (1982) observation that "the concrete is realized in thinking through the abstract" (p. 37). Taking this view, models are embedded in goal structures and used by embodied agents. The key feature within the objects-structure perspective, however, lays in the idea that various specific objects falling under a particular concept mutually *complement* each other, so that the abstractness of each of them, taken separately, is overcome. From this perspective, structural abstraction is a movement towards *complementarity* of diverse aspects creating conceptual unity among objects. This is in line with a dialectical perspective described by Ilyenkov (1982) and differs from empiricist approaches in Skemp (1986).

From the *knowledge*-structure perspective, structural abstraction, on the other hand, implies a process of restructuring the 'pieces of knowledge' constructed through the mentioned processes. Further, it also implies restructuring knowledge structures coming from current concept images, essential for the new concept construction. The cognitive function of structural abstraction is to facilitate the assembly of larger, more complex knowledge structures. The guiding philosophy here is rooted in the assumption that learners initially acquire mathematical concepts on their backgrounds of existing domain-specific conceptual knowledge through progressive integration of previous concept images or by the insertion of a new discourse alongside them. The crucial aspect of structural abstraction, from the knowledge-structure perspective, is that structural abstraction moves *from simple to complex* knowledge structures, a movement with the aim of establishing highly coherent knowledge structures.

RESEARCH QUESTION AND METHOD

Which insights does the above outline on structural abstraction reveal for the analysis of an individual's striving for making sense of a mathematical concept and which aspects may be illuminated that have been hidden? These questions are addressed by returning to an earlier study (Pinto, 1998) that identified mathematics undergraduates' strategies of making sense of formal mathematics, which were not fully captured by "action-first" models of concept construction (e.g., Dubinsky, 1991). The original data collected undertook an inductive approach throughout two academic terms during students' first year at a university in England. It consists of classroom observation field notes and transcriptions of semi-structural individual interviews that took place every two weeks with eleven students. From a cross-sectional analysis of three pairs of students, two prototypical strategies of making sense could be identified, namely 'extracting meaning' and 'giving meaning'. Here the latter is our focus; through new lenses provided by the notion of 'structural abstraction' (Scheiner, 2013). Meanwhile, scrutinizing the old data contributes to the development of the very notion of structural abstraction itself. Due to the limited scope of the paper, we limit our focus on the case study of the learner Chris, who "consistently understood [the formal concepts] by just reconstructing it from the concept image" (Pinto, 1998, p. 301).

SELECTED FINDINGS

The above outline on structural abstraction provides indications to refine the characteristics of the 'giving meaning' strategy expressed by 'reconstructing a formal object from the concept image' (Pinto, 1998). If we return to examine the earlier study (Pinto, 1998), we find that several students take the formal definition of a mathematical concept as just one amongst other related representations built in earlier experiences at school and out of school – a full meaning for considering the concept definition inside the concept image cell. The formal concept definition does not necessarily have primacy over the other representations but has a *complementary power* to give deeper insights into the 'bigger picture' of the concept. Moreover, we could identify some learners who 'give meaning' but simply 'add' the formal definition to their concept image. By merely juxtaposing pieces of knowledge, occasionally conflicting, the structure underlying the different facets of the concept may stay inconsistent, hampering the structural abstraction process. On the other hand, there are modes to succeed. Reasons for our claim rely in part on the analysis of Chris' written formal definition of the limit of a sequence. We interpret that Chris firstly evokes a representation of a constructed object to start with, based upon his visual representation of a convergent sequence (see, Figure 3) and on his explanation of the meaning of the definition which starts as "... and you've got like the function there, and I think that ... ... it's got the limit there..." (Chris, first interview). Yet, his written discourse seems to recall a specific representation of a sequence tending to L, as he starts "if a_n tends to L" instead of "a_n tends to L if", as he was told in the lessons, self correcting and crossing out the first line. Chris' responses show that he developed and is guided by a *generic representation* of the limit concept. By taking a retrospective view, he described that he has developed this representation, looking at other sources than the lectures, through 'complementarization' of a variety of representations.

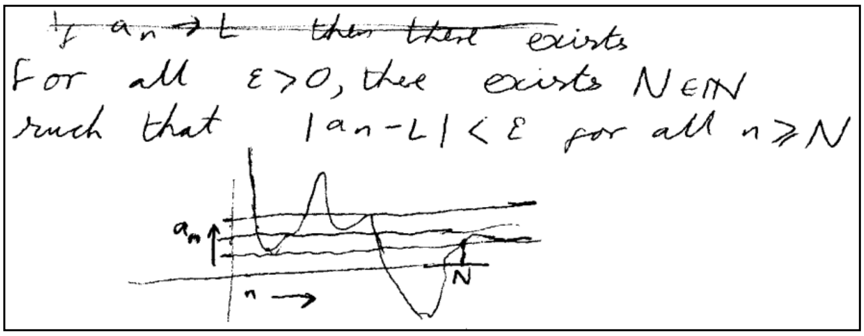

Figure 3: Chris' representation of the limit concept.

Chris expresses his doubts when responding whether the sequence 1,1,1,… has a limit:

> "(Laughter) I don't know really. It definitely it will ... it will always be one ... so I am not really sure (laughter) ... ... umm ... it's strange, because when something tends to a limit, you think of it as never reaching it ... so if it's ... 1 ... then by definition it has a limit but ... you don't really think of it as a limit (laughter) but just as a constant value."
>
> (Chris, first interview)

He evokes a dynamic view of the limit concept and an understanding (limit as unreachable) coexisting with the formal definition. His seriousness expressing his doubts suggest that, even immersed in the classroom culture at university, he will not simple let go 'old images' when faced with the formal definition, acknowledging that he is not making a complete sense of the concept in its overall structure, which at the time is composed by conflicting 'pieces of knowledge'. In a certain sense, there is no primacy of the formal definition in relation to other representations and he goes through a process of restructuring them into a coherent and complex whole proudly announcing in his last interview where he could express the formal definition of limit of a sequence "without making it formal" as follows:

A sequence has a limit if and only if as the sequence progresses, eventually, all values of the sequence gather around a certain value.

(Chris, last interview)

Modes to reconstruct earlier dynamical views of limit into the static version above, which seems unifying the various representations and we interpret as movements across levels of complexity, are only recovered through scrutinizing Chris' descriptions of his attempts to make sense of the formal definition. During the second interview, when Chris comments "[I could] see what the definition meant", may be referring to "... ... when you actually ... think that you can ... you make ε small." (Chris, seventh interview). Notice that "you can" suggests an experiment, which seems to be guided by his generic representation of a convergent sequence. He then self corrects, mentioning an action, "you make", in order to *define* a convergent sequence. Other instances from the first interview suggest that he experimented by giving N and finding a related ε, in a logical inversion of what is stated in the definition:

> ... you decide how far out ... and you can work out an epsilon from that ... or if you choose an epsilon you can work how far out.

However, moving N to the right and determining ε allows a dynamical feeling that the sequence is tending to a limit. Such thought experiments may have guided him to "... thinking about why you are doing it ... ... you find out why you are choosing *N* so they lie all there in, so ... it gradually tends towards the limit" (Chris, seventh interview). Finally, a central aspect in this reanalysis is related to modes of dealing with cognitive conflicts, which appear as a pivot issue during the process of structural abstraction. Since there are learners who are not aware of a cognitive conflict, as further findings indicate, a realistic model/perspective, as described in the outline of the framework, may be a helpful 'guide' in order to construct the right idea of the concept. Further, the impact of cognitive conflicts and learning through conceptual change in our approach on structural abstraction reflects crucial issues in cognitive science.

CONCLUDING REMARK

Structural abstraction, from our point of view, is considered as a movement 'from particular to unity' in terms of 'complementarizing' particularized meaningful components/structure into a whole, and, on the other hand, as a movement 'from simple to complex' in terms of restructuring already constructed 'pieces of knowledge' into coherent and complex knowledge structures. In synthesis, structural abstraction acknowledges abstraction as a movement across levels of complexity rather than levels of abstractions or generality. With this approach, we call to free of the term abstraction from connotations that have been associated with it through decades in many works.

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TYPES OF ARGUMENTS WHEN DEALING WITH CHANCE EXPERIMENTS

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This paper contributes to the discourse in stochastic education of how young students deal with learning settings that allow a data-based approach to probability. By using the micro-structure of arguments by Toulmin (1958), it explores which arguments students use and which role they play in the learning process. The data stems from design experiments with students at the beginning of their stochastic career (aged 11 to 13) and is analysed with an interpretative approach.

THEORETICAL BACKGROUND

Integration of theoretical and experimental approach to probability

There are several perspectives on probability, two of which will be taken into account here (Fig 1): the so-called 'classical' approach focusses on calculating probabilities theoretically, for instance by determining the ratio of outcomes favourable to the number of outcomes unfavourable of an event when all cases are equally likely (Jones et al. 2007, p. 912). The 'experimental' (or 'frequentist') approach is more centred on data: "the probability of an event is defined as the ratio of the number of trials favourable to the event to the total number of trials" (Jones et al. 2007).

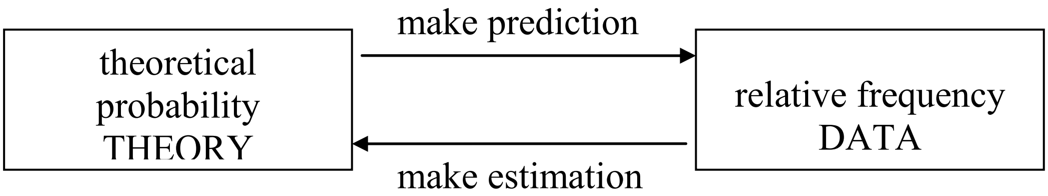

Figure 1: Interplay of theoretical probability and relative frequency.

Coming from theoretical considerations, trends in data collected via experiments with random devices (such as dice) can be predicted. Or, having analysed data first, the relative frequencies can serve as estimation for the probability distribution underlying the experiment. The estimation and prediction will become better the more often the experiment is repeated. This is the definition Moore (1990) uses for the term 'random': "Phenomena having uncertain individual outcomes but a regular pattern of outcomes in many repetitions are called *random*. 'Random' is not a synonym for 'haphazard' but a description of a kind of order different from the deterministic one that is popularly associated with science and mathematics" (p. 98 emphasis in original).

Open to question is how students gain and integrate understanding of these two perspectives (Jones et al., 2007, p. 946).

Reasoning and arguments in stochastics

To investigate this, this paper focusses on the activity of *reasoning* as it gives insights into students' sense-making processes. 'Informal inferential reasoning' was first used in statistics, a term which in probability education refers to exploring and making inferences about trends in data generated by random devices such as dice, without explicitly using probability or statistical terms (Pratt et al., 2008).

In this paper, 'reasoning' is understood as using arguments in interaction. To get further insight, the micro-structure of arguments proposed by Toulmin (1958) is useful: Reasoning is finding arguments which are a series of propositions in which a *Claim* is inferred from *Evidence*. The so-called *Warrant* links the Evidence and Claims, for instance the fictive argument in Fig 2. The triple is called 'argument'.

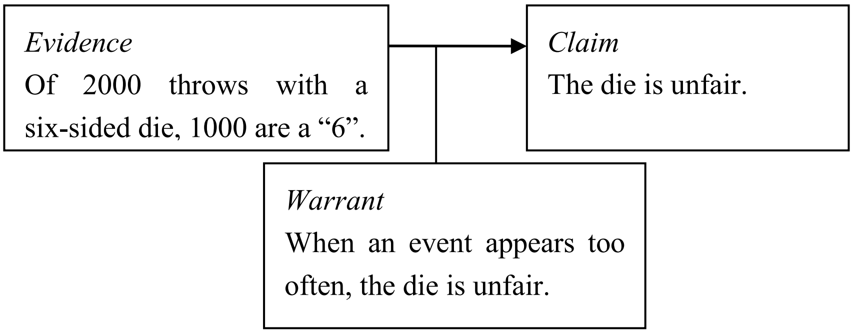

Figure 2: Micro-structure of argument and fictive example.

The example in Figure 2 links a data-focussed Evidence to a Claim about the probability distribution underlying the experiment. Due to page limitations, further elements of arguments such as the *Backing* will not be addressed in this paper (cf. Toulmin 1958).

Applying this model to arguments in a stochastic setting adds a specific reasoning that refers to the random variation of data: "To understand the nature of statistical argument, we must consider what types of explanation qualify as answer to why questions. […] Indeed, statistical inference is rare among scientific logics in being forced to deal with chance explanations as alternatives or additions to systematic explanations" (Abelson, 1995, p. 6).

Types of arguments when dealing with an experiment-based approach to probability

Research in statistics and probability education has uncovered different conceptions and perspectives when dealing with probability that can lead to different arguments. The here presented types of arguments are not supposed to be disjunct; instead the analysis below will show that more complex arguments link different types together.

- *Data-centred arguments*: The Evidence is an observation about the data; for instance, means of data analysis are applied to identify central trends which can then be inferred as a claim.

- *Theoretical arguments*: In this case, the Evidence refers to probabilities (e.g. by determining the ratio of favourable to non-favourable outcomes) and probability distributions from which a Claim then is made, e.g. about expected trends in the data.
- *Non-deterministic arguments*: As Moore (1990) points out, 'random' describes a non-deterministic order. Students might lack words for this, but are sometimes able to find conceptions such as 'unpredictability' (cf. Pratt 2008). These can be used as base for a Claim.
- *(Quasi-)causal arguments*: In this last type of argument, the Evidence is used as a cause to make a Claim why a certain phenomenon occurred. Learners might try to find causal explanations for the result of a single throw of a die, which according to Konold (1989) is a common misunderstanding when dealing with probabilities: Instead of applying stochastic reasoning, people perceive "the goal in dealing with uncertainty [as] to predict the outcome of a single next trial" (p. 61). The (quasi)-causal explanations are often 'magical' (e.g. an animistic nature of the chance device, see Wollring, 1994) or refer to causes that can be manipulated by the students (Wollring 1994). These arguments are important elements of the learners' process of making sense of the interplay of uncertain individual outcomes and regular pattern in the long term perspective (Pratt et al. 2008 and Wollring 1994).

RESEARCH QUESTIONS AND DESIGN

The insights presented here are part of a broader project to investigate students' processes of constructing knowledge when confronted with an experiment-based approach to probability (Schnell, 2014). In this paper, the specific focus lies on the different types of arguments in order to gain deeper insight into the processes of integrating theoretical and data-centred notions of probability:

1. *Which (quasi-)causal and non-deterministic explanations do students use and which role do they play in the learning processes?*
2. *(How) Do students integrate theoretical and experimental aspects of probability in their arguments?*

The teaching-learning-arrangement

To investigate this, a teaching-learning-arrangement was used that works as an experiment-based, informal introduction to probability in grade 6 or 7. The core elements are two consecutive games in which students gain points by betting on the results of a race between four differently-coloured animals. The didactic intention is to provide systematic experience with the empirical law of large numbers. This paper will focus only on the first game (for more details see Prediger & Hußmann, 2014 and Schnell, 2014).

At the core of the game is the repeated throwing of a 20-sided die with the following colour distribution and the corresponding animals: red ant 7 sides, green frog 5, yellow snail 5 and blue hedgehog 3. While students have access to the die at all times in the teaching-learning-arrangement, experience shows that most of them assume an even colour distribution at first and discover the actual colour distribution later on.

The length of the race (i.e. how many times the die is rolled in total) is set before the game start: every number between 1 and 10,000 is possible; longer games take place using a computer simulation. Each throw of the die moves the corresponding animal one step forward on the game board or in the computer simulation; the race is finished and the results are compared when the previously determined number of throws is reached. The animal with the highest absolute frequency is the winner of the race. Motivation for further investigation of the data are the questions "which animal is the best" and "when can you be as sure as possible that this animal will win". To systematically compare the results of short races (e.g. 1, 10 or 20 throws) with each other but also with the results of long races (e.g. 100, 1000, 10 000 throws), the teaching-learning-arrangement provides record sheets and tasks focusing on these comparisons.

Methods

The author conducted design experiments (Cobb et al., 2003) with nine pairs of students (grade 6, German comprehensive school, ages 11 to 13) in a laboratory setting. Each design experiment took across four to six sessions of 60 to 90 minutes each; the game described above was finished within the first session for eight pairs and for one pair (Emily and Leo) within the second session. The data corpus includes videos, screen captures of the simulation, transcripts and all written products such as record sheets.

The research questions were addressed by qualitatively analysing the transcripts and videos turn by turn. In a first step, all arguments were identified, i.e. all statements in which Claims and Evidence were explicitly stated and connected by a(n implicit) Warrant. This paper focusses on arguments with Claims about observations related to data or theoretical aspects (leaving out other arguments, for instance about the quality of a prediction; cf. Schnell 2014 for a broader investigation); in total 49 arguments were identified[1]. Then, these arguments were coded and categorized in terms of the type of argument and Evidence. Selected results of the analysis are presented here.

ANALYSIS AND DISCUSSION OF DIFFERENT TYPES OF REASONING

The role of (quasi-)causal and non-deterministic arguments

(Quasi-)causal arguments: This category shows the most variety in the analysis which is in accordance with the literature (Jones et al. 2007). 17 of the 49 arguments can be

[1] This number refers to arguments that were newly constructed in the course of the design experiments. Not included are numbers for when students repeated a previously used argument (e.g. repeating the superiority of the red ant because of the colour distribution). If two different pairs of students construct the same argument, both are counted.

coded as (quasi)-causal. All arguments with one exception are constructed before the discovery of the colour distribution. By looking more closely at the Evidence, subcategories can be built. Some of these subcategories are:

- *Device-focussed*: Students try to find causes for outcomes by focussing on the die (or the computer simulation, but no participant did that), such as <When the die is manipulated, you get an unwanted outcome> (RS-24:35)[2] or animistic conceptions such as <When the die is evil, you get an unwanted outcome> (DJu8- 31:55). Some students use the physics of throwing the die as a cause for the result. For one pair (Ramona and Sarah), this argument dominates the first 30 minutes: <When the die rolls for a short distance, the outcome is blue> (RS-22:39). All these arguments focus on explaining outcomes of a single throw of the die.
- *Property-focussed*: This subcategory includes two arguments that claim the superiority of the red ant comes from the colour red itself or the specific animal: <When the colour is red, then the animal is on fire and is thus the fastest> (EL-41:42) and <When an animal has long legs, it wins more often> (DeK-87:50). The latter argument is the only causal argument that is built after the colour distribution is discovered.

Non-deterministic arguments: Only two arguments could be identified as solely non-deterministic: They are both created by the same pair of students and use the concept 'luck' as Evidence, for instance: <When bad luck happens, then a series of red is ended by green> (EL-43:38). Both were also constructed before the colour distribution of the die was discovered. Even though other students also refer to good or bad luck, they are not using it explicitly as Evidence for a Claim. In three other cases, non-deterministic Evidence is combined with theoretical insights; these are discussed below.

Addressing the first research question of the role of (quasi-)causal explanations: Looking at the overall picture, six out of nine pairs of students built (quasi-)causal and non-deterministic explanations. The variety of different explanations is in line with findings in literature (cf. Jones et al. 2007). The *device-focussed* arguments are concerned with not only explaining single outcomes, but also with undesired results. This might indicate that students tend to deal with experiences that are opposed to expectations by building these explanations. This observation is in line with Wollring's (1994, p. 136) observations about the behaviour of children in primary school. *Property-focussed* arguments are used to explain the superiority of the red animal, rather than single outcomes. In all but one cases, these (quasi-)causal arguments are built before the discovery of the colour distribution of the die.

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[2] Noted in < > is the reconstructed Warrant of the form 'When Evidence, then Conclusion'; in parentheses is the code for the pair of students and the time-stamp in which the argument was verbalised for the first time.

The role of data-centred and theoretical arguments

Data-centred (8 arguments of 49): Due to the guide question "which coloured animal is the best" and the provision of record sheets, all students focussed on the produced data. The arguments here are those, in which students explicitly used a data-centred Evidence to make a Claim, such as <When red ant has won most often, it is the best animal>. They were built and mainly used before the colour distribution was discovered and seemed to disappear afterwards.

Theoretical arguments (11 arguments): The discovery of the colour distribution of the die is crucial for giving meaning to the patterns in data such as the red ant being more likely to win. Thus, students who don't discover the colour distribution by themselves are prompted by the research teacher (two pairs). Therefore, all pairs of students use arguments like <When red has more faces than the other colours, the red animal is more likely to be rolled> (appears for all pairs of students).

Combination of theoretical and data-centred (8 arguments): The data analysis shows that theoretical- and data-centred arguments were combined in some cases, for instance <When red is more on the die, red is rolled more often and thus red ant wins more often> (EL2-27:10). This argument uses theoretical Evidence to claim the empirically observed superiority of the red ant. Furthermore, it might refer to the connection between the red ant winning a whole race (i.e. having the highest absolute frequency) and the single outcome of one throw of the die.

Combination of theoretical and non-deterministic (3 arguments): Three of these combined arguments could be identified: <When blue hedgehog is lucky, it gets the three blue faces very often and can win> (RS-30:10), <When red ant is unlucky, it loses in races with an even number of throws even though it is superior> (RS-67:16) and <When you are lucky, an animal with fewer chances wins> (RS-41:15). Here, Ramona and Sarah start with (good/bad) luck as Evidence and make a Claim that this might interfere with the chances derived theoretically from the colour distribution. This could be interpreted as the integration of experienced random variation in single outcomes with data-based and theoretical insights.

Addressing the second research question concerning the integration of data-centred and theoretical aspects of probability:

Some arguments could be identified which combine theoretical and data-based aspects. Here, patterns (superior red ant) are related to the colour distribution (7 out of 20 sides on the die). In these arguments, the theoretical insight serves as Evidence to make a claim related to data-centred observations. Looking at the sequence in which the different arguments were built, it is noticeable that after the discovery of the colour distribution, no new, solely data-centred arguments were built. One pair of students also combines theoretical and non-deterministic insights to explain situations in which it encounters random variation (e.g. a losing red ant). A variety of micro-processes of combining different insights were investigated in Schnell & Prediger (2012).

CONCLUDING REMARKS

This paper gives a short insight into the types of arguments that students at the beginning of secondary school use when working on an experiment-based setting introducing probability. The in-depth analysis shows how they not only make connections between theoretical and data-centred aspects, but also integrate non-deterministic arguments in a meaningful way. This supports the claim that informal conceptions are important for individual learning pathways (Pratt et al. 2008; Schnell 2014).

Another observation is that the discovery of the colour distribution seems to lead to a decline in (quasi-)causal and solely data-centred arguments. This raises the question of whether there is some kind of implicit hierarchy between the different types of arguments. The presented data suggests that students might be aware of a superiority of theoretical arguments over other types of arguments. To uncover the relations between different types of arguments, it might be fruitful to take into account further elements of arguments such as the Backing for the Warrant and Rebuttals (Toulmin, 1958).

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EXPLORING STUDENTS' MENTAL MODELS IN LINEAR ALGEBRA AND ANALYTIC GEOMETRY: OBSTACLES FOR UNDERSTANDING BASIC CONCEPTS

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In this paper, we discuss the relevance of 'Grundvorstellungen' (GVs), a didactical category to analyze students' mental models in comparison to the intended mathematical meanings in the context of Linear Algebra and Analytic Geometry. Diagnostic tasks were used to reveal students' conceptual understanding in this field of expertise. In particular, an open item format was chosen to elicit students' individual GVs and to explore how they use them while working on mathematical tasks. 30 students from upper secondary school participated in our study; data was collected by a paper-and-pencil test. The results show that elaborated representations of GVs foster students' understanding of mathematics and facilitate the process of finding problem solving strategies.

INTRODUCTION

Research on students' understanding of mathematical content is huge and varies with respect to constructs and categories employed for analyzing different facets. Some authors elaborate on procedural aspects of knowledge construction and underline the role of abstraction when students delve into mathematics (cf. Dreyfus, 2012). Other research investigates the role of mental models that students build up and to which degree these adequately reflect the mathematical properties of a specific concept (cf. Fischbein 1989; Vinner & Tall, 1981). While introducing the term *concept image*, Tall and Vinner (1981) explicitly accentuate the individual understanding that students develop when trying to make sense of the mathematics they encounter in the classroom.

In German didactics tradition, the construct of *Grundvorstellungen*, abbreviated here as GV, serves as essential tool to capture both normative and intuitive interpretations of mathematics. Vom Hofe, Kleine, Blum and Pekrun (2005) emphasize that the value of the construct lies in interpreting GVs as "elements of connection or as objects of transition between the world of mathematics and the individual world of thinking" (p. 2). In our study we are interested in gaining insight into upper secondary students' GVs in the field of Linear Algebra and Analytic Geometry and how those influence students' performance. In order to reveal what students really know and understand diagnostic tasks were employed.

THEORETICAL BACKGROUND

Exploring the role of intuition for the learning of mathematics has a long tradition in PME research. One essential starting point for subsequent research was provided, for instance, by the seminal work of Fischbein (1989) who differentiates algorithmic, intuitive and formal knowledge. In particular, he stresses:

> To think by manipulating pure symbols which obey only formal constraints is practically impossible. Consequently, we produce models which confer some behavioral, practical, unifying meaning, to this symbols. (p. 9)

These kind of students' models of mathematical concepts and procedures and how their individually constructed knowledge conflicts with the mathematically intended one have been studied in depth. One promising approach lies in analyzing students' *concept images* in relation to the intended *concept definitions*. Here, Tall and Vinner (1981) use the term *concept image* "to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). When working on mathematical tasks, students base their decisions on the *concept image*. To that effect, Vinner (1994) could show how obstacles in calculus occurred since students retain, for instance, a restricted *concept image* of a tangent developed earlier. This *concept image* of a tangent to a circle provokes difficulties in students' learning of calculus when confronted with the analytical definition of a tangent.

Concept images help to identify prototypes that students apply inappropriately in specific situations so that obstacles occur due to incorrect generalization of constructs. Considering GVs, the construct provides a broader scope to analyze students' sense-making and occurring hindrances (cf. Prediger, 2008). Vom Hofe et al. (2005) point out that three significant aspects characterize the process of building up GVs during mathematical concept acquisition:

- constitution of meaning of mathematical concepts based on familiar contexts and experiences,
- generation of generalized mental representations of the concept which make operative thinking (in the Piagetian sense) possible,
- ability to apply a concept to reality by recognizing the respective structure in real life contexts or by modeling a real life situation with the aid of the mathematical structure. (p. 2)

Prediger (2008) uses an explorative item format to access multiple facets of students' GVs when dealing with multiplication of fractions. From the mathematical viewpoint, the following GVs can be activated in the given situation:

- repeated addition, repeated adjoining (temporal-successive interpretation)
- part of interpretation
- scaling up and down
- multiplicative comparison
- area of rectangle. (p. 10)

When comparing to students' GVs, Prediger (2008) showed "that the individual models for multiplication were more heterogeneous and more distant from the mathematically intended models than for addition" (p. 10). Most students still applied the model for multiplication of natural numbers (repeated addition) which cannot be transferred to the multiplication of fractions. Here, analyzing student performances reveals inadequate GVs that occur as implicitly learned rules in another context.

Our study aims at exploring students' GVs in Linear Algebra and Analytic Geometry, a school topic which introduces a great variety of constructs and concepts. Classroom activities in this area are characterized by a dominance of algorithmic procedures and the use of schemata to arrive at solutions (Tietze, Klika & Wolpers, 2000). As a result, such treatment does often not allow students to develop a deep understanding of the mathematical concepts at hand (Malle, 2005).

In particular, we draw on the work by Wittmann (2003) who distinguishes the following three GVs to capture the interplay between Geometry and Algebra:

- *Algebraization*: Students use algebraic expressions to structure a presented (geometric) situation (parametrization, vectorization), and place geometrical objects in the coordinate system.
- *Geometrization*: Students translate algebraic equations into a geometric object to use for further interpretation.
- *Structural Generalization*: Students attend to overriding structural features, and they are involved in abstraction and generalization to bring together concepts on a meta-level.

Mostly teaching of Linear Algebra and Analytic Geometry is restricted to paying attention to developing GVs in *Algebraization* or *Geometrization* (Wittmann, 2003). However, to attain comprehensive understanding that pretends insular knowledge the development of GVs in *Structural Generalization* is decisive (Tietze, Klika & Wolpers, 2000).

RESEARCH QUESTIONS

The research at hand is part of a larger study to survey significance and construction of diagnostic tasks as an instrument to understand students' difficulties with main concepts of Linear Algebra and Analytic Geometry in school (cf. Schueler, 2013). With respect to the theoretical background we pay attention to students GVs on *Algebraization* (GV_A), *Geometrization* (GV_G), and *Structural Generalization* (GV_{SG}). In particular, we pursue the following research questions:

- Do students have preferred GVs (GV_A, GV_G or GV_{SG}) in the field of Linear Algebra and Analytic Geometry?
- How do students deal with mathematical tasks that entail interconnections of different GVs?

METHODOLOGY

Qualitative methods are used for exploring students' GVs while working on specific tasks. During a period of five weeks we observed corresponding lessons and analyzed the teaching material in order to construct a set of diagnostic mathematical tasks implying key aspects of Linear Algebra and Analytic Geometry in school. Data was collected by a one-hour paper-and-pencil test composed of seven diagnostic tasks. In this paper we focus on three tasks to highlight different facets of GVs.

The sample consists of 30 students that range in age from sixteen to eighteen. Among them, 18 female and 12 male students who attend grade 12 of a German high school. In addition to the test we collected some information about students' general performance level in mathematics and their self-assessment compared to the average of the class; these results are not presented in this paper.

RESULTS AND DISCUSSION

For the sake of brevity the presentation of results is limited to exemplary findings which illustrate students' solutions against the background of the three basic GVs discussed in the theory section. In addition, we enrich our presentation by discussing essential mathematical aspects and by reporting typical obstacles.

Task 1

a) Explain with your own words the concept 'vector'.
b) Describe situations of application in which it is essential to use vector algebra.

Introducing vectors in school is based on at least two different approaches, i.e. vectors are considered as equivalence classes of arrows or as n-tuples. In an equivalence class of arrows a vector is defined as an infinite set of arrows with same length, same orientation and same direction. The n-tuple model is based on abstract understanding of a vector as an ordered list of elements.

In task a) we intend to reveal students' prevalent GVs. In addition, task 1a) emphasizes what relevance students' attach to the use of vectors in applications. Table 1 summarizes the answers given by students.

equivalence class of arrows	n-tuple	incorrect answer	no answer
54%	17%	23%	6%

Table 1: Students' answers to problem 1a).

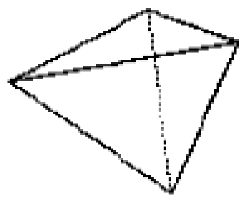
Having observed the lessons, we can confirm that both aspects of the vector concept were introduced in class. However, 54% of students rely on the geometric understanding of an equivalence class of arrows (GV_G) while only 17% of them consider the n-tuple concept (GV_A). Thus, the majority of students are able to define a vector as an equivalence class of arrows. Reviewing relevant teaching material we assume that the preference for a geometric association (GV_G) results from the fact that the majority of the lesson material deals with geometric problems.

The answers given to task 1b) underline this aspect as 87% of students use vector algebra in geometric situations for example by considering the routes of airplanes (GV_G).

In sum, 29 % of students are not able to define the vector concept correctly. About 92% of the incorrect answers result from a deficient geometric interpretation of a vector as single arrow, placed at a concrete position in a three-dimensional coordinate system. Only 12% of students use both GVs to describe the vector concept. The combination of the geometric and the algebraic definition of a vector requires focusing on general mathematical characteristics common to both approaches. These thoughts refer to aspects of structural generalization (GV_{SG}) and present an elaborated understanding of the concept of vectors.

Task 2

The geometric figure is called a regular tetrahedron. It consists of four equilateral triangles.



Draw a figure to illustrate a convenient way to place the tetrahedron in a Cartesian coordinate system. Describe the position of the tetrahedron as accurately as possible.

In task 2 the students were asked to give a possible parameterization of a tetrahedron. This task demands students to activate different facets of GV_A. In the first place, the task strongly refers to GV_A in terms of using algebraic expressions to describe and structure a presented geometric figure. In addition, a correct solution requires the understanding of typical characteristics of a tetrahedron like equal edge length. That is, task 2 furthermore addresses key aspects of studying global features of a geometric figure.

In order to find a solution to this problem the students need to choose a convenient way of placing the Cartesian coordinate system and its point of origin and of translating the geometric characteristics of a tetrahedron into algebraic expressions. Table 2 demonstrates the distribution of the students' answers to task 2.

correct answer	incorrect answer	no answer
47%	33%	20%

Table 2: Students' answers to problem 2.

47% of the students are able to give an adequate visualization of the tetrahedron. However, it is notable that the correct solutions differ with respect to placing the point of origin. The majority of students identify one surface of the tetrahedron with the x1-x2-plane as shown in Figure 1. Figure 2 shows an example of an alternative way that students chose to locate the Cartesian coordinate system.

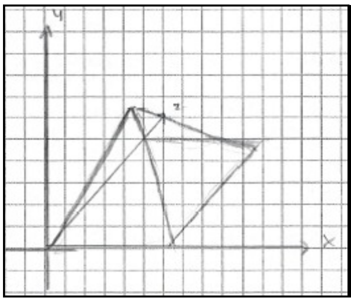


Figure 1

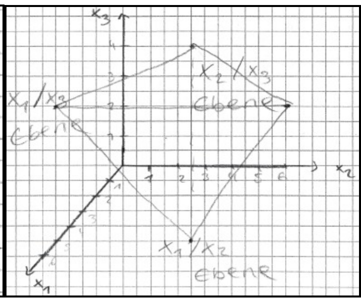


Figure 2

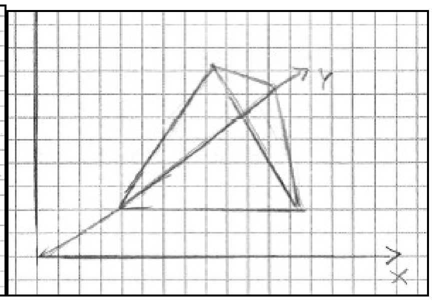


Figure 3

However, 53% of the students are not able to give a correct answer. Analyzing the incorrect answers leads to two major problems. First, we observed that students face difficulties when identifying geometric characteristics of the tetrahedron. On the one hand some students disregard the aspect of equilateral triangles and on the other hand they misinterpret the tetrahedron as a square pyramid as shown in Figure 3. The second difficulty lies in choosing a position of the tetrahedron in the Cartesian coordinate system that facilitates algebraic parameterization. In sum, applying GV_A which capture the process of algebraization, is problematic due to lacking understanding of some basic geometrical features.

Task 3

a) *Describe the position of the planes (i) or (ii) in a Cartesian coordinate system. Draw a figure which illustrates the position of the plane.*

   (i) $x_1 = 4$

   (ii) $x_1 - x_3 = 0$

b) *Give a possible equation of a plane which lies vertical to the $x_1 - x_3$-plane. Explain your choice.*

In-depth understanding of geometric objects in Linear Algebra and Analytic Geometry manifests itself in the ability to switch between a geometric characterization of an object and the corresponding algebraic expression. The ability of combining effectively these different representations is part of GV_{SG}. In task 3 a) the students were asked to give an adequate geometric description of a plane which is presented in coordinate form, whereas subtask b) deals with this problem vice-versa. Table 3 sums up students' answers.

3 a)	correct answer	incorrect answer	no answer
	35%	35%	30%
3 b)	correct answer	incorrect answer	no answer
	44%	40%	16%

Table 3: Students' answers to problem 3a) and 3b).

Our findings show that 35% of the students answer task 3a) correctly, and 44% of them are able to give an adequate solution to task 3b). The relation between correct and incorrect answers for both tasks is hardly different indicating that both GVs (GV_A and GV_G) are equally assessable for students. However, considering the number of students that are not able to provide an answer at all, it appears that algebraization allows more students to approach the mathematical content. From explanations that students wrote to task 3a), we can gather that the missing answers are due to deficient understanding of the coordinate form of a plane (cf. Schüler, 2013). Several students brought forward the argument that the expression $x_1 = 4$ does not describe a plane but a single point in the coordinate system. This argumentation reveals a typical obstacle, i.e. students interpret the missing of a coordinate signifies it to be zero (cf. Wittmann, 2003).

CONCLUSION

The presented problems stress in manifold ways the relevance of GVs in learning Linear Algebra and Analytic Geometry. Considering our exemplary findings we are able to underline the function of GVs as *hinges* which facilitate the transition from students' individual understanding of situations described in tasks to the respective mathematical models.

Regarding research question one and two our findings show that students neither have a preference for GV_A nor GV_G. However, the tasks would allow combing both GVs as required in the category GV_{SG}. Given that GV_{SG} are essential for developing a deep understanding, teaching would profit from using contexts that encourage structural generalization.

Reviewing teaching material and schoolbooks traditionally used in the majority of German high schools shows that the preference of daily practice is to emphasize either GV_A or GV_G, i.e. dealing with characteristics of geometric figures is almost limited to finding an algebraic expression. This proceeding leads to the phenomenon that students learn solution strategies by heart and try to memorize how to fit them to tasks without activating a deeper mathematical understanding (cf. Tietze, Klika & Wolpers, 2000). Such behavior could be seen as well in students' task performance in our study.

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ARE INTEREST AND ENJOYMENT IMPORTANT FOR STUDENTS' PERFORMANCE?

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We conducted an experimental study with 192 ninth graders in which we investigated a connection between performance and students' interest and enjoyment using task-unspecific and task-specific questionnaires. Students were randomly assigned to experimental group 1 or to experimental group 2. In group 1, they were asked about their affective measures after task processing, and in group 2, they were asked before task processing. In both groups, students who achieved higher scores on the performance test reported stronger interest and enjoyment. The connection of performance to the task-unspecific and task-specific affective scales did not differ significantly and ranged between .15 and .47 for problems with and without a connection to the real world.

INTRODUCTION

Affect is highly important for student learning and has been investigated intensively during the last few decades (Zan, Brown, Evans, & Hannula, 2006). The results of previous studies indicate that student achievement measured using students' grades is connected to students' interest and enjoyment. However, only a small number of studies have investigated correlations between students' performance and their affect. In the current study, we examine whether students' performance on problems with and without a connection to the real world is connected (1) to their task-unspecific affect in mathematics or (2) to their task-specific affect when they report on their affect before and after task processing. Further, differences in the correlations between performance and affect were investigated for problems with and without a connection to reality.

THEORETICAL BACKGROUND AND RESEARCH QUESTIONS

Interest and enjoyment

Interest is a motivational variable that characterizes a relation between a person and an object and indicates an individual psychological state of engaging with this object over time (Hidi & Renninger, 2006). Interest develops from situational to individual interest and is important for students' learning. Compared to other motivational constructs, interest is strongly connected to academic achievement. Correlations in mathematics range from .0 and .5 for different achievement tests and tend to decrease from the early to middle secondary level (see summary by Heinze, Reiss, & Rudolph, 2005). Interest is closely connected to emotions such as enjoyment (Schukajlow et al., 2012).

Students' emotions predict their career aspirations and thus influence their current and future lives (Wigfield, Battle, Keller, & Eccles, 2002). A control-value theory of

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 129-136. Vancouver, Canada: PME.

achievement emotions assumes that the value of learning materials and the controllability of learning activities are important for students` emotions (Pekrun, 2006). Although enjoyment is among the most frequently reported positive emotions in the classroom, there are only a few studies that have investigated its connections to academic achievement. Students' grades at school and at universities are positively connected to their enjoyment in mathematics (.22 and .46, respectively, Goetz, Frenzel, Pekrun, Hall, & Lüdtke, 2007; Pekrun, Goetz, Frenzel, Barchfeld, & Perry, 2011). However, we could not find studies that had investigated the relation between students' performance on an achievement test with their enjoyment. As a positive association between students' grades and their enjoyment has previously been found, we expected a positive correlation between performance and students' enjoyment.

Characteristics of affect measurement

Students' affect can be measured before (prospective affect), during (current affect), or after (retrospective affect) activities such as problem solving (Efklides, 2006). Students' prospective interest indicates their level of interest when they begin to solve a problem. Their current affect describes their level of interest while they are trying to solve the problem. Their retrospective affect provides information about their perceptions of mathematical activities after task processing. We argue that students' prospective, current, and retrospective perceptions are important indicators of their affect.

Recently, researchers have demanded several times that subject-specific aspects of affect be taken into account, that multimethod approaches be used, and that new instruments be developed to measure affective variables (Hannula, Pantziara, Wæge, & Schlöglmann, 2009; Zan et al., 2006). Thus, in this study, we used two instruments to measure affect: well-known task-unspecific affective scales that were validated in other studies and a new task-specific approach applied in the study by Schukajlow et al. (2012). In addition, we measured students' affect before and after task processing in order to compare the stability of the connection between performance and affect.

One characteristic of affective measures is their level of subject-specificity. A sample statement may be "I am interested in problem solving" or "I am interested in solving the equation $3 + 2x = -4x$." Although task-specific measures allow researchers to obtain answers about affect with regard to specific topics or kinds of tasks and are more sensitive to the affective changes that occur after intervention programs, they have rarely been used—except for self-efficacy expectations—to measure affect. As task-unspecific and task-specific affect can be used to assess the same construct, we do not expect performance to be more or less strongly correlated with task-specific measures than with task-unspecific measures. However, because of the sensitivity of task-specific measures, correlations between task-specific measures and performance may have greater variability across different types of problems than correlations between task-unspecific measures and performance. Thus, it is possible that the connection between performance and task-specific affect will differ across different problem types.

Problems with and without a connection to the real world

Task-specific measures were used recently to investigate interest and enjoyment regarding to problems with and without a connection to reality (Schukajlow et al., 2012). These problem types were modelling, “dressed up” word and intra-mathematical problems, all three of which are typically distinguished in discussions about modelling and applications (Blum, Galbraith, Henn, & Niss, 2007). To solve modelling problems, students need to construct a situation model of the task, and then they need to simplify that model by structuring and mathematizing it in order to generate a mathematical model that can be solved using mathematical procedures. In the end, mathematical results have to be interpreted and validated. Solving “dressed up” word problems is much simpler because a mathematical model is merely “dressed up” by the situation, and students have to “undress” it, mathematize it, and apply mathematical procedures to solve this type of problem. Intra-mathematical problems are not connected to reality at all.

We assume that there should be no significant differences between correlations of performance and affect for problems with and without a connection to the real world. Students who achieve higher scores on tests should be more interested in the solutions to the problems and should enjoy solving the problems more.

Research questions

The research questions we addressed were:

1. Is students’ performance connected to task-unspecific and task-specific interest and enjoyment in mathematics measured before and after problem solving?
2. Is students’ performance connected more strongly to task-specific than to task-unspecific affect?
3. Are correlations between performance and task-specific affect different for different types of problems (modelling problems, “dressed up” word problems, and intra-mathematical problems)?

METHOD

One hundred and ninety two German ninth and tenth graders from 4 middle-track and 4 grammar school classes (53.6% female; mean age=16.1 years, SD=0.86) were asked about their task-unspecific interest, enjoyment, and boredom as well as about task-specific affect regarding various types of problems. The students were randomly assigned to two experimental groups. Students in group 1 solved problems first and then reported on their task-unspecific affect and on their task-specific interest, enjoyment, and boredom regarding these problems. In group 2, students reported on their task-unspecific and task-specific affect first and then solved tasks that were used in the task-specific part of the questionnaires (see Figure 1). Students in both groups worked on the same tasks and had the same amount of time to solve the problems and to complete the questionnaires.

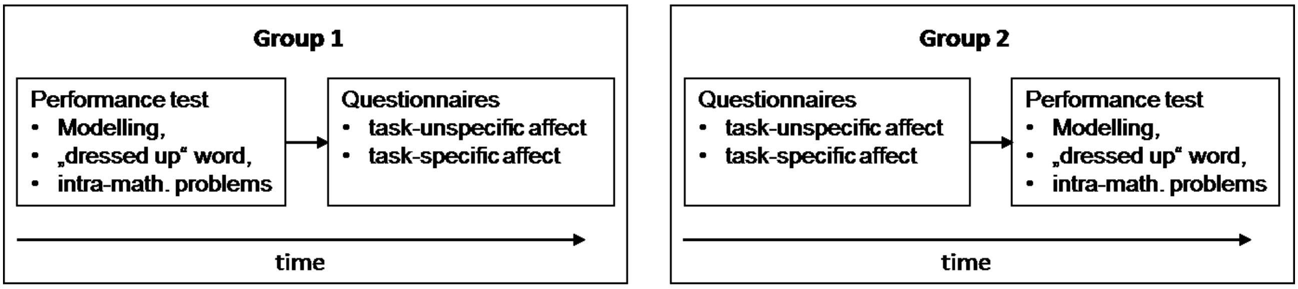


Figure 1: An overview of the study.

Sample problems

Twenty-three problems on the topics Pythagoras' theorem and linear functions—eight modelling, eight word, and seven intra-mathematical ones—were selected for this study and were used to examine students' performance and their task-specific affect. Sample tasks on the topic Pythagoras' Theorem are presented below.

Maypole

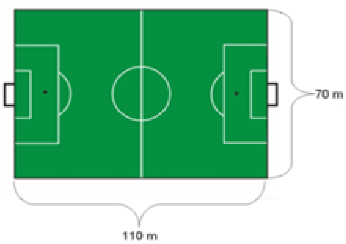
Every year on Mayday in Bad Dinkelsdorf there is a traditional dance around the maypole (a tree trunk approx. 8 m high). During the dance the participants hold ribbons in their hands and each ribbon is fixed to the top of the maypole. With these 15 m long ribbons the participants dance around the maypole, and as the dance progresses a beautiful pattern on the stem is produced (in the picture such a pattern can already be seen at the top of the maypole stem).

At what distance from the maypole do the dancers stand at the beginning of the dance (the ribbons are tightly stretched)?

Figure 2: Modelling problem "Maypole".

The maypole, football pitch, and side c were classified as modelling, "dressed up" word, and intra-mathematical problems, respectively (for more sample tasks and detailed analysis of classification see Krug & Schukajlow, 2013; Schukajlow et al., 2012).

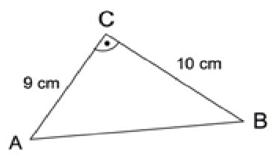
Football Pitch



Trainer Manfred would like to carry out a diagonal run with his team. To do so he would like to know how long the diagonal of the football pitch is. Can you help him?

Calculate the diagonal length of the football pitch.

Side c



Calculate the length of the side c = |AB|.

c= ________

Figure 3: "Dressed up" word and intra-mathematical tasks "Football Pitch" and "Side c".

Performance tests

Three tests with 8, 8, and 7 tasks each were constructed to measure students' performance in solving modelling, "dressed up" word, and intra-mathematical problems, respectively. All tasks that we used were examined in the framework of other projects. The Cronbach's alpha reliabilities were .59, .67, and .52 for the modelling, word, and intra-mathematical tests, respectively, and were acceptable for the small number of items and their diversity (different contexts and/or different mathematical procedures).

Task-unspecific interest and enjoyment

Task-unspecific interest and enjoyment were assessed with scales used in other studies (e.g. Pekrun et al., 2011) and consisted of 6 and 4 statements that were answered on 5-point Likert scales ranging from (1=strongly disagree) to (5=strongly agree). Sample items are "I am interested in mathematics" and "I enjoy being in class." The Cronbach's alpha reliabilities were .88 for interest and .80 for enjoyment.

Task-specific interest and enjoyment

On the task-specific questionnaire, each of the 23 problems was followed by a statement about students' interest and enjoyment. The instructions for both groups (cf. Fig. 1) were: "Read each problem carefully and then answer some questions. **You do not have to solve the problems!**" After task processing, students in group 1 were asked to rate the extent to which they agreed or disagreed with the statements "It was interesting to work on this problem" and "I enjoyed solving the problem shown". Students in group 2, on the other hand, were asked before task processing to rate the statements "It would be interesting to work on this problem" and "I would enjoy solving the problem shown" A 5-point Likert scale was used to record their answers (1=not at all true, 5=completely true). A total of 6 scales that measured either task-specific interest or enjoyment were formed across eight modelling problems, eight "dressed up" word problems, and seven intra-mathematical problems. The Cronbach's alpha reliabilities for the 6 scales were all higher than .83.

Treatment fidelity

To control the treatment fidelity in groups 1 and 2, a five-point Likert item: "Before I agreed or disagreed with the statements (about task-specific affect), I solved the problems" (1=not at all true, 5=completely true) was used. Means and standard deviations were 4.3(1.17) for group 1 and 2.19(1.01) for group 2. An independent *t* test showed a significant mean difference between the two groups ($t(179)=13.07$, $p<.0001$, *Cohen's* $d=1.93$). As intended, students in group 1 solved the tasks significantly more often than students in group 2 *before* they reported their task-specific interest or enjoyment.

RESULTS

Correlations between students' performance and task-unspecific as well as task-specific affect in groups 1 and 2 are presented in Tables 1 and 2, respectively. Students who achieved higher scores on the performance tests reported higher task-unspecific interest in mathematics and enjoyed mathematics classes more than students who received lower scores. Moreover, students who were interested in mathematics and in solving mathematical problems outperformed other students on the achievement tests. Despite finding a low correlation between performance on intra-mathematical problems and task-specific enjoyment in group one (.15) and a low correlation between performance on modelling problems and task-specific interest in group two (.16), a significant positive connection between performance and affect was found using task-specific and task-unspecific affect scales.

		interest			interest	enjoyment			enjoyment
		ma	w	mod	task-unspecific	ma	w	mod	task-unspecific
performance	ma	.18[a]			.25*	.15			.29*
	w		.39*		.39*		.47*		.45*
	mod			.31*	.40*			.27*	.45*

Note: *p<.05; [a]p<.10; ma intra-mathematical, w word, mod modelling problems; sample size N=100

Table 1: Pearson correlations between performance and task-specific and task-unspecific interest and enjoyment in group 1.

To answer the second research question, correlations between performance and task-specific affect were compared with correlations between performance and task-unspecific affect using Fisher's z-test. For example, in group 1, the correlation between performance on intra-mathematical problems and interest in these problems (.18) was compared with the correlation between performance on this problem type and task-unspecific interest (.25). Fisher's z-test showed that the correlations did not differ significantly (p=.61).

		interest			interest	enjoyment			enjoyment
		ma	w	mod	task-unspecific	ma	w	mod	task-unspecific
performance	ma	.19[a]			.21*	.38*			.34*
	w		.24*		.23*		.37*		.31*
	mod			.16	.33*			.37*	.27*

Note: *p<.05; [a]p <.10; ma intra-mathematical, w word, mod modelling problems; sample size N=92

Table 2: Pearson correlations between performance and task-specific and task-unspecific interest and enjoyment in group 2.

Similar results were also found for other correlations between performance and interest. The relations between students' performance and task-specific interest were comparable to the relation between performance and task-unspecific interest in both experimental groups. The comparisons of the correlations between performance and enjoyment revealed similar results. We did not find significant differences between performance on modelling/word/intra-mathematical problems and enjoyment regarding to the respective type of problem and between performance and task-unspecific enjoyment.

The third research question addressed the stability of the connection between performance and task-specific affect across different types of problems. Fisher's z-test did not show any significant differences in performance-interest correlations between groups 1 and 2 for different types of problems. Thus, the relations between students' performance and interest were comparable across intra-mathematical, "dressed up" word, and modelling problems. The connection between performance and task-specific enjoyment[1] was also comparable between problems with and without a connection to the real world. Thus, we could conclude that the relation between performance and affect does not depend on the type of problem.

SUMMARY

In this study, we investigated the relations between performance and students' interest and enjoyment using (1) task-unspecific and task-specific measures as well as (2) different perspectives (prospective and retrospective) in the measurement of affect. The results confirm the importance of interest and enjoyment for students' performance in mathematics. The range of the magnitudes of the correlations between performance and affect in our study was comparable to the range found in other studies (Goetz, Frenzel, Hall, & Pekrun, 2008; Heinze et al., 2005) in which performance was estimated via students' grades.

As expected, correlations between performance and affect were comparable for task-specific and task-unspecific scales. However, we assume that task-unspecific and task-specific measures provide information about different features of interest or enjoyment. Task-specific scales are more unstable than task-unspecific ones and depend on the mathematical topic, the described situation, students' prior knowledge, etc. This issue should be investigated further in future studies.

Finally, we compared correlations for problems with and without a connection to the real world. Although the magnitudes of the correlations between performance and affect varied widely, we found no significant differences in correlations for different types of problems. One open research question involves whether there are different "sources" of interest and enjoyment for different types of problems. We suppose that affect for problems with a connection to reality may depend not only on the mathematical nature of the task but also on the situation described in the task.

[1] As we conducted 12 tests to answer this research question, the significance level was adjusted from 0.05 to 0.005 by using a Bonferroni correction to take into account the accumulation of the alpha-error.

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THE IMPACT OF LEARNING AND TEACHING LINEAR FUNCTIONS PROFESSIONAL DEVELOPMENT

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This study examines the impact of Learning and Teaching Linear Functions (LTLF) professional development materials on teachers' mathematics understanding and teaching practices, as well as students' resulting algebra proficiency, learning, and achievement. Learning and Teaching Linear Functions are modular, video-based professional development materials designed to enable teachers to deepen their specialized content knowledge by understanding ways to conceptualize and represent linear functions within their teaching practice. The intervention consisted of a one-week summer institute and on-line support throughout the academic year.

INTRODUCTION

New directions in mathematics education demand new approaches to professional development. Teacher educators need to help teachers develop richer instructional practices that integrate emphasis on developing students' conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition through mathematical investigation, problem solving, and discourse (Kilpatrick et al., 2001). There is a groundswell of interest in creating and using mathematics professional development materials that focus on helping teachers examine the interplay between mathematical content, teacher, students and context (Smith, 2001). Rooted in the everyday work of teaching, classroom artefacts such as student work, videos or narrative accounts, become invaluable tools for learning teaching in practice-based materials (Lampert and Ball, 1998; Driscoll et al., 2001; Seago et al., 2004). Videos in particular have been found to be a promising tool in supporting teacher learning in professional development (Seidel et al. 2005).

The best practices for supporting such professional development involve providing experiences that are intensive in focus and extensive in duration (Garet et al., 2001) and that are "practice-based"—that is, that offer teachers the opportunity to examine the mathematical skills and understanding that undergird the classroom curriculum, investigate students' mathematical thinking, and explore instructional practices that support student learning (Cohen and Hill, 2001; Thompson and Zeuli, 1999). By focusing on developing the understanding, skills, and dispositions that teachers use in daily practice, this "practice-based" professional development provides a meaningful context for teachers' learning.

The Linear Functions for Teaching study focuses its work on the practice-based video case materials, *Learning and Teaching Linear Functions* (LTLF) (Seago, Mumme and Branca, 2004), which are designed to enable teachers to deepen their understanding of

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 137-144. Vancouver, Canada: PME.

ways to conceptualize and represent algebra content within their teaching practice. LTLF is premised on the idea that using artefacts of practice within a well-structured PD program can promote mathematical knowledge for teaching (Ball & Cohen, 1999). This idea is supported by a variety of learner-centred, inquiry-based theoretical traditions, including constructivist and situative perspectives on learning (Cobb, 1994). These perspectives share the notion that engaging in challenging, problem-based, collaborative, and socially shared activities is likely to promote an expanded knowledge base (Borko, et al., 2005). The *Learning and Teaching Linear Functions* materials were designed with all of these features in mind and include an analytic framework, explicit tasks, teacher learning goals, and facilitation supports.

THEORETICAL FRAMEWORK

The theoretical frame for the LTLF video case materials is adapted from the work of Deborah Ball and colleagues (Ball and Cohen, 1999; Cohen, Raudenbush & Ball, 2003) that incorporates research on both teaching and learning. The content of the video case materials focuses on the interactions between the teacher, the content (in this case, linear functions tasks), and the students, within the context of an authentic classroom environment (see Figure 1, page 3). The materials are designed to be used by a teacher educator who is faced with a similar set of relationships: the interactions between the teacher educator, the content (in this case, teaching and learning of linear functions), and the teachers he/she works with. To assist the teacher educator in using the PD materials productively with teachers, in-depth resource materials are provided to facilitate teachers' knowledge development. Resource materials include: mathematics content information, probing discussion questions, and other facilitation guidance specific to the materials.

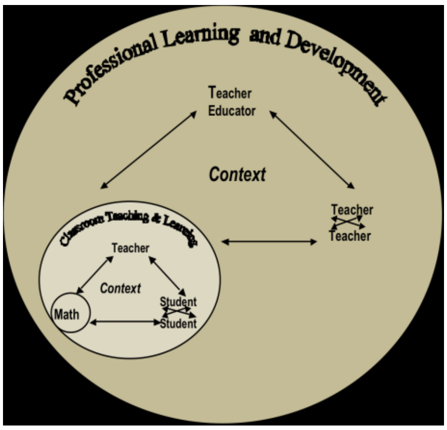

Figure 1: Theoretical Framework (Adapted from Cohen, Raudenbush, & Ball, 2003).

As Ball and her colleagues have noted, teachers' mathematical knowledge for teaching is of central importance with respect to interactions around the content with students (MKT; Ball, Hill & Bass, 2005; Ball, Lubienski & Mewborn, 2001). Their research has shown that MKT relates to the quality of teachers' classroom work and positively predicts gains in their students' mathematical achievement (Hill, 2010; Hill, Rowan & Ball, 2005). MKT can be understood as the knowledge that teachers need to effectively carry out the work of teaching. MKT incorporates subject matter knowledge as well as pedagogical content knowledge (Ball, Thames, & Phelps, 2008).

RESEARCH STUDY

The following research questions guide the study:

- *Do teachers participating in the LTLF professional development program exhibit greater increases in knowledge and skills regarding linear functions*
- *Do teachers participating show greater integration of LTLF-based teaching strategies into their instructional practice than teachers in control classrooms?*
- *Do students in LTLF classrooms demonstrate greater increases in algebra understanding (in particular linear functions) and engagement in mathematics learning than their counterparts in control classrooms?*

The research questions focus on the impacts on teachers and students. For teachers, research on teacher knowledge and instructional practice over two academic years. For students, research focused on students in LTLF classrooms in the year that teachers received the professional development and students in LTLF classrooms in the year subsequent to teacher LTLF professional development.

Study design and timeline

Learning and Teaching Linear Functions was designed to enable teachers deepen their understanding of mathematics content, students' mathematical thinking, and instructional strategies. The study took place from spring 2011 to spring 2013 in 62 schools serving middle grades in California. Schools and teachers were recruited in winter and spring 2011. Participation in the study was voluntary. The intervention involved a one-week summer training course using the LTLF first module, *Conceptualizing and Representing Linear Relationships*, a sequential series of eight 3-hour sessions designed to enrich teachers' ability to teach linear relationships and deepen their own detailed knowledge of the distinctions and linkages among the various representations. Each session has at its core one or two digital video clips of a mathematics classroom. Additionally, participants received academic year online follow-up support in year 1 and year 2 (~20 PD hours).

The efficacy of LTLF was investigated using a pre-test/post-test cluster randomized trial design with one intervention group and one control group. Teachers were randomly assigned to an intervention or control group, in which they remained until the conclusion of the study. The trial was conducted in 43 districts throughout California. A qualitative video study of a smaller sample of six randomly selected teachers is used to examine traceable elements of implementation of the LTLF PD, validate and explain quantitative findings, and to identify factors that influence the success of the pedagogical approach.

A total of 81 teachers in 62 schools were randomly assigned to groups – 41 to intervention and 40 to control. About 77 percent (63) of the original 81 teachers completed the study and provided teacher and/or student test score data. The 63 teachers who were retained in the analytic sample after attrition came from 51 schools in 36 districts. Student quiz data were obtained from 1,645 students (934 intervention and 711 control). There was no evidence to suggest that the experimental groups differed with respect to attrition or missing data patterns.

With an average of 28 students served by each participating teacher, the sample size is sufficient for detecting program impacts on student outcomes of 0.22 standard deviations for primary academic outcomes and 0.31 for item-level data. The estimated minimum detectable effect size for the teacher knowledge assessment (see below) was 0.36 standard deviations.

Key outcomes and measures

Table 1 below lists the study's key outcome variables—teachers' knowledge for mathematical instruction, teacher practice and conceptualization of student work, and student knowledge.

Outcome	Measure
Teacher Knowledge	
Teachers' knowledge for teaching	Learning Mathematics for Teaching Assessment
Teacher Practice	
Teachers' conceptualization of teaching, students and student work	Artefact Analysis
Elements of PD that get used by teachers in their classroom	Videotaped lessons
Student Knowledge	
Knowledge of Algebra I	California Standardized Test – Algebra I
Knowledge of Linear Functions	4 Released NAEP items

Table 1: Outcome measures.

Each outcome measure is described in more detail below.

Learning Mathematics for Teaching Instrument. All participating intervention and control teachers completed the pre-test, post-test, and follow-up assessment of the online version of the Teacher Knowledge Assessment System (TKAS) (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008). Teachers were randomly assigned to complete alternative forms of the assessment. These measures have been used with over 2000 teachers, yielding information about reliability and item characteristics. The reliabilities for these scales range from 0.71 to 0.84.

Artefact Analysis Assessment. All teachers completed an artefact analysis assessment prior to and within one month after the summer institute. The artefact analysis assessment asks teachers to solve a mathematics task and to provide written responses about (a) a 5-minute video clip of 6th grade students presenting solutions to a linear function problem and (b) three specific samples of student work (each representing a different typical student error). Written responses were coded based on the extent to which teacher interpretations focus on students' potential understandings, are backed by evidence, and focus on specific mathematics content.

Videotaped Lessons. Video observations of 56 lessons from a randomly selected subset of teachers, using portable video camcorders and audio equipment, have been completed. Teachers received a package including a flip camera, microphones, tripod, and instructions. Each teacher videotaped two lessons in 2011–2012 and two more lessons in 2012–2013 (one each in fall and spring of each academic year). Coding of the lessons using Studiocode software is currently underway. The purpose of the coding is to identify "traceable elements" from the PD—those elements that were key to the intervention and that we expect to see in classrooms where teachers are implementing what they learned in the institute. Once criteria and coding schemes are finalized, we will score video data to gain scorer reliability of at least 0.8, after which we will code each video for evidence of the key elements and score as high, medium or low fidelity of implementation.

Algebra 1 CST. Students' knowledge of algebra I is assessed using California's end-of-course Algebra I CST. The criterion-referenced CST has been administered annually to all students through 2013. Baseline (pre-test) assessments of mathematics proficiency are used as covariates in the impact analysis models. For this study, data are collected on performance of participating teachers' students in Spring 2011 (prior to the intervention) and again in Spring 2012 and 2013. At this time, these data are still being collected and are not reported on in this paper.

NAEP Items. Students' knowledge of linear functions is assessed with four publicly released NAEP problem-solving items. To date, two of the items have been scored by blinded raters as incorrect, minimal, partial, satisfactory, and extended. Inter-rater agreement on the two items ranged from 0.77 to 0.92.

ANALYSIS AND RESULTS

To estimate program impacts, outcomes for teachers and students in intervention group classrooms were compared with those for teachers and students in control group classrooms. Multilevel regression models were used to analyze the effects of the LTLF program and to account for data clustering by teacher and school (Goldstein 1987; Raudenbush and Bryk 2002; Murray 1998). The impact analyses controlled for baseline (pre-test) measures of outcome variables and other teacher, student-, and school-level covariates.

Estimated impacts

Teacher Knowledge. The results for the Learning Mathematics for Teaching (LMT) assessment suggest that intervention teachers scored about 25 percent of a standard deviation higher than control teachers on the LMT test after the first academic year. This difference, however, is not statistically significant at conventional levels, and the intervention/control group difference was no longer apparent after the second academic year.

Teacher Practice. The results of the artefact analysis suggest the LTLF is associated with changes in teachers' perceptions of student potential and analysis of student work. Although no pre-intervention differences were apparent between intervention and control teachers, at post-test, intervention teachers were substantially more likely to (1) indicate an understanding of students' potential than control teachers on the student work task and (2) focus on the mathematical content of student work than their counterparts in the control group. There was also a greater tendency for intervention teachers to use evidence to justify their inferences with regard to student work and analysis of the classroom video, although these differences were statistically significant at conventional levels.

Student Knowledge. Estimated LTLF impacts on the Algebra I CST are not yet available as collection of state assessment scores is ongoing. Although analyses of the four NAEP items assessing performance on linear functions problems suggest that LTLF is not associated with significant increases in knowledge, there was a tendency for students in intervention classrooms to score higher on the two open-ended items ($p=0.10$ and 0.18).

SYNOPSIS

The impact analyses indicated that LTLF resulted in modest short-term improvements in teachers' knowledge for teaching mathematics, recognition of students' mathematical understanding on student work, and attention to the appropriate mathematics content on student work. However, intervention/control group differences in knowledge for teaching mathematics were completely diminished at the 2^{nd} post-test, as scores of teachers in the control group "caught-up" to their counterparts in the intervention group. We therefore conclude that the year 1 impacts of LTLF on teacher knowledge do not persist in year 2. The impacts (short and long term) on instructional practice are still under investigation.

For student outcomes, only the results the NAEP linear function items are available for analysis at the present time. Although the results favor the intervention group for two of the four items, LTLF is not associated with increases in performance on this measure in a statistically significant manner.

The Learning and Teaching Linear Functions professional development research is one study situated in the larger context of other research on PD interventions. The field is relatively new and has a thin empirical research base (Hill, Beisiegel, & Robin, 2013).

A particular challenge is determining what features of the PD cause an impact on teacher practice and student knowledge. Indeed, there is much to be learned about the development and delivery of effective PD, as well as the research of PD outcomes. The LTLF PD study in the process of developing evidence of impact of a PD intervention and is learning important contributions to the field regarding effective methods and measurement of impact studies.

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MATHEMATICS LEARNING IN MAINLAND CHINA, HONG KONG AND TAIWAN: THE VALUES PERSPECTIVE

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Drawing on 1386 questionnaire responses, 11- and 12-year old primary students in mainland China, Hong Kong, and Taiwan valued the same six orientations in their mathematics learning. These are achievement, relevance, practice, communication, information and communication technologies [ICT], and feedback. Each of these six values was also embraced to different degrees by students across the three regions. These findings shed light on how students' values might be used to support learning, at the same time emphasising that such values are culture-dependent.

INTRODUCTION

Students from several East Asian nations have been consistently performing very well in international comparative tests such as Programme for International Student Assessment [PISA] and Trends in International Mathematics and Science Studies [TIMSS]. These include regions such as Shanghai, Hong Kong, Japan, Korea, Singapore, and Taiwan (Mullis, Martin, Foy & Arora, 2012; OECD, 2013). Given that different cultures (or education systems) embrace and emphasise different approaches to mathematics teaching (Atweh & Seah, 2008), cultural values regarding (mathematics) education constitute a key factor for the students' performance in these international comparative tests (Leung, 2006). A Nuffield Foundation review had found that "high attainment may be much more closely linked to cultural values than to specific mathematics teaching practices" (Askew, Hodgen, Hossain, & Bretscher, 2010, p. 12).

Looking beyond the general characteristics of East Asian nations and the constituent Confucian Heritage Cultures, important differences exist amongst these various education systems. How might these differences affect the people's lifestyles, their outlooks, as well as their views on formal education?

In this light, this paper reports on the analysis of data collected from 11- and 12-year old primary students in mainland China, Hong Kong, and Taiwan which relate to what they value in mathematics learning. These three regions are located close to one another geographically, and most of their populations share the same ethnic roots (i.e. Han Chinese). What do these students value collectively in mathematics learning? To what extent are these valued similarly or differently across each of the three regions?

We will first review generally the recent historical developments across mainland China, Hong Kong and Taiwan. Given that the research being reported here is part of a

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 145-152. Vancouver, Canada: PME.

wider study, this paper will then provide an outline of the background to the study. The quantitative data collected will also be presented, and the findings summarised.

HISTORICAL CONTEXTS OF THE THREE REGIONS

After the Communist party took over mainland China in 1949, the country's education system became very much influenced by that being used in the Soviet Union. Basic computation skills and 'traditional' topics (e.g. Euclidean geometry) were emphasised. The Modern Mathematics movement did not appear to have created any influence on the Chinese mathematics education system. It was not until the mid-1980s when mainland China adopted the open door economic policy that educational ideas from overseas – and from the Western countries in particular – were accepted. School education became available to the general mass of the Chinese population only in the early 2000s.

In those early days, Western missionaries were refused entry to mainland China. They spent their time in Hong Kong instead and established schools there. As a British colony too, the Hong Kong school education system had been British. These had facilitated the introduction into the education system of initiatives stimulated by Nuffield Mathematics and Modern Mathematics. Universal education was implemented in the late 1970s (Wong & Tang, 2012).

Taiwan developed somewhat differently from mainland China after 1949. Since the Nationalist government was set up in Taiwan that year, Taiwan has been in touch with the Western world. Educational ideas from around the world – especially the United States and Japan – were imported. Universal education was implemented in 1968. The Modern Mathematics movement did influence Taiwan and was introduced into the mathematics curriculum around that time.

Recently, in revising their respective mathematics education curricula, mainland China, Hong Kong and Taiwan joined many other education systems around the world in embracing higher order thinking abilities such as collaboration, communication and creativity (Wong, Han, & Lee, 2004). It is in this context that we investigated what students from each of these three regions value in their mathematics learning experiences, and how similar/different these are.

CONTEXTUALISING THIS STUDY

The data reported in this paper were collected for a larger-scale, '**W**hat **I** **F**ind **I**mportant (in mathematics education)' [**WIFI**] study. For us,

> values are the convictions which an individual has internalised as being the things of importance and worth. What an individual values defines for her/him a window through which s/he views the world around her/him. Valuing provides the individual with the will and determination to maintain any course of action chosen in the learning and teaching of mathematics. They regulate the ways in which a learner's/teacher's cognitive skills and emotional dispositions are aligned to learning/teaching. (Seah & Andersson, in press)

The study reported in this paper poses the following research questions:

1. What do primary school students in mainland China, Hong Kong and Taiwan value with regards to mathematics and to mathematics learning?
2. How do primary school students in mainland China, Hong Kong and Taiwan value these aspects of mathematics and mathematics education similarly and differently?

METHODOLOGY

The questionnaire method had been selected, given its appropriateness in values research (Johnson & Christiensen, 2010; Reichers & Schneider, 1990).

The WIFI questionnaire is a 93-item instrument with a combination of 64 Likert-scale items, 10 slider rating scale items, 6 open-ended items, and 13 items which collect demographic information about the respondents. The 74 Likert-scale and slider rating scale items name a list of mathematics learning tasks which reflect Bishop's (1988) 6 mathematical values, 14 mathematics educational values that were identified in a previous Third Wave Project's study (see, for example, Seah & Peng, 2012), and Hofstede's (1997) 6 value continua. The open-ended items include hypothetical situations for students to respond to.

For this paper, only the first section of 64 Likert-scale items of the WIFI questionnaire was analysed. Also, only the responses of the 11- and 12-year old primary school students were selected, even though the same questionnaire was administered to 3814 students in primary and secondary schools in urban areas across the three regions. This translated to a total of 1386 students from Wuhan (mainland China), Hong Kong, and Taipei (Taiwan). The 11- and 12-year old students (typically in the final two years of their primary school education) in the participating schools were invited to take part in the anonymous survey exercise during class time.

RESULTS

What students valued

A Principal Component Analysis [PCA] with a Varimax rotation was used to examine the 64 questionnaire items. The significance level was set at 0.05, while a cut-off criterion for component loadings of at least 0.45 was used in interpreting the solution. Items that did not meet the criteria were eliminated.

The Kaiser-Meyer-Olkin [KMO] (Kaiser, 1970) measure of sampling adequacy was 0.96 and Bartlett's test of sphericity [BTS] (Bartlett, 1950) was significant at the 0.001 level. The factorability of the correlation matrix was thus assumed, which demonstrated that the identity matrix instrument was reliable and confirmed the usefulness of the factor analysis. According to the cut-off criterion, 17 items were removed from the original 64. The analysis yielded six components (see Table 1) with eigenvalues greater than one, which accounted for 45.65% of the total variance.

	Component					
	1	2	3	4	5	6
Q58KnowingWhichFormulaToUse	.706					
Q56KnowingTheStepsOfTheSolution	.678					
Q54UnderstandingConceptsProcesses	.637					
Q13PractisingHowToUseMathsFormulae	.623					
Q14MemorisingFacts	.615					
Q63UnderstandingWhyMySolutionIsIncorrectOrCorrect	.606					
Q59KnowingTheTheoreticalAspectsOfMathematics	.577					
Q32UsingMathematicalWords	.568					
Q49ExamplesToHelpMeUnderstand	.536					
Q38GivenAFormulaToUse	.531					
Q15LookingForDifferentWaysToFindTheAnswer	.522					
Q30AlternativeSolutions	.514					
Q33WritingTheSolutionsStepbystep	.513					
Q55ShortcutsToSolvingAProblem	.504					
Q28KnowingTheTimesTables	.475					
Q5ExplainingByTheTeacher	.473					
Q2Problemsolving	.473					
Q61StoriesAboutMathematicians		.649				
Q18StoriesAboutRecentDevelopmentsInMathematics		.640				
Q17StoriesAboutMathematics		.630				
Q21StudentsPosingMathsProblems		.613				
Q11AppreciatingTheBeautyOfMathematics		.590				
Q60MysteryOfMaths		.586				
Q39LookingOutForMathsInRealLife		.566				
Q20MathematicsPuzzles		.552				
Q52HandsonActivities		.545				
Q29MakingUpMyOwnMathsQuestions		.524				
Q34OutdoorMathematicsActivities		.522				
Q40ExplainingWhereTheRulesFormulaeCameFrom		.506				
Q12ConnectingMathsToRealLife		.491				
Q47UsingDiagramsToUnderstandMaths		.480				
Q25MathematicsGames		.473				
Q19ExplainingMySolutionsToTheClass		.451				
Q36PractisingWithLotsOfQuestions			.791			
Q37DoingALotOfMathematicsWork			.751			
Q57MathematicsHomework			.699			
Q62CompletingMathematicsWork			.600			
Q43MathematicsTestsExaminations			.597			
Q7WholeclassDiscussions				.702		
Q3SmallgroupDiscussions				.581		
Q10RelatingMathematicsToOtherSubjectsInSchool				.454		
Q23LearningMathsWithTheComputer					.789	
Q24LearningMathsWithTheInternet					.777	
Q22UsingTheCalculatorToCheckTheAnswer					.760	
Q4UsingTheCalculatorToCalculate					.673	
Q44FeedbackFromMyTeacher						.726
Q45FeedbackFromMyFriends						.725

Note. Extraction method: PCA; Rotation method: Varimax with Kaiser Normalization, Minimum factor loadings .45; KMO, MSA, Eigenvalues > 1.

Table 1: Rotated component matrix.

We named the six components of the students' set of values as follows: *achievement*, *relevance*, *practice*, *communication*, *ICT*, and *feedback*.

Regional differences amongst the student values

To answer research question (2), a comparison was made of the mean responses for each component for each region. This showed that the structure of the values dimensions was very similar across the regions (see Table 2). (Note that in the questionnaire, a value with a higher mean score means that the items making up the component were considered more unimportant by the students.)

Component	Region			F test	Effect size
	CHN	HKG	TWN		
	M(SD)	M(SD)	M(SD)		
Achievement (C1)	1.44(.37)	1.51(.52)	1.64(.51)	8.045 $p < 0.001$	$\eta 2 = 0.012$; TWN > CHN
Relevance (C2)	1.79(.51)	2.04(.62)	2.23(.74)	78.078 $p < 0.001$	$\eta 2 = 0.102$; TWN> HKG, CHN; HKG > CHN
Practice (C3)	1.72(.62)	1.98(.83)	2.07(.78)	8.412 $p < 0.001$	$\eta 2 = 0.012$; HKG, TWN > CHN
Communication (C4)	1.95(.75)	2.25(.75)	1.94(.72)	49.140 $p < 0.001$	$\eta 2 = 0.067$; HKG > TWN, CHN
ICT (C5)	3.09(.77)	2.69(.93)	3.14(.88)	18.082 $p < 0.001$	$\eta 2 = 0.026$; TWN, CHN > HKG
Feedback (C6)	1.93(.95)	1.92(.82)	2.06(1.0)	13.877 $p < 0.001$	$\eta 2 = 0.020$; TWN > HKG

Note: CHN: Mainland China; HKG: Hong Kong; TWN: Taiwan.

Table 2: Mean comparison among three regions for the six components.

The primary students from each of the three regions valued the same six convictions most. In fact, all of them also valued *achievement* most, since the mean scores of *achievement* (*C1*) in mainland China, Hong Kong and Taiwan (1.44, 1.51, and 1.64 respectively) were the lowest compared to the other five components. *ICT* (*C5*), on the other hand, was valued least by students in all three regions, compared to the other five components. The mean scores were 3.09, 2.69 and 3.14 respectively. However, some differences were identified on closer examination of the results for each region, specifically by examining the sequencing of the mean scores. For mainland China, the sequence of mean scores from lowest to highest was C1-C3-C2-C6-C4-C5. In Hong Kong, the sequence was C1-C6-C3-C2-C4-C5; for Taiwan, it was C1-C4-C6-C3-C2-C5.

A multivariate analysis of variance (MANOVA) with Tukey's HSD Post Hoc multiple comparisons tests was conducted to explore cultural differences for each value dimension by region. We had significant univariate main effects for each of the components at the 0.001 alpha level. There were statistically significant differences amongst the students by region, such that:

- Students in mainland China (CHN) valued *achievement* more than their peers in Hong Kong (HKG) and Taiwan (TWN).
- Students in CHN valued *relevance* more than their peers in HKG, who in turn valued *relevance* more than those in TWN.
- Students in CHN valued *practice* more than those in HKG and TWN
- Students in TWN and CHN valued *communication* more than their peers in HKG
- Students in HKG valued *ICT* more than those in CHN and TWN.
- Students in HKG valued *feedback* more than their peers in TWN.

DISCUSSION

What were valued commonly across the three regions

The WIFI questionnaire was used to identify the value structure of East Asian students in mainland China, Hong Kong and Taiwan. The 1386 11- and 12-year old primary students in these three education systems valued six orientations commonly. These were, in order of importance, *achievement, relevance, practice, communication, ICT*, and *feedback.*

The valuing of *achievement* was the most important to the primary students. *Practice* appeared to be emphasised as a means of doing well in mathematics. The *relevance* of the learning experience was also highly regarded, including its use in daily life and hands-on experience. Students also valued ideas such as *ICT* and *communication* which were advocated in the mathematics curriculum reforms in these regions (and elsewhere). Finally, *feedback* about their learning was highly valued, reflecting the findings of prior studies on students' preferred mathematics learning environment (Ding & Wong, 2012).

The students placed most importance in the valuing of *achievement* in their mathematics learning experience, a cultural trait that has been associated with the ethnic Chinese (see, for example, Bond, 2010). The questionnaire items in this component include knowing, memorizing and using mathematical facts and formulae, emphasizing solutions and seeking different ways to solve problems. On the one hand, this reflects the high value that the ethnic Chinese students place on basic skills. On the other hand, however, when this valuing is considered in the context of the Chinese culture, in which success is often attributed with the efforts made, we can understand how it can create tremendous pressure on the students. This can also be intensified when the students view learning as an obligation, to repay the care given to them by their parents (Wong, 2004). In addition, when the 'basics' progress from computation to other higher order thinking skills, there is a danger that the students will interpret 'memorizing facts' as 'memorizing hands-on skills' and 'memorizing problem solving routines' as well (Wong, Han, & Lee, 2004).

These findings contribute to current knowledge that can further improve our practices in mathematics teaching. It is often suggested that congruence between the students' preferences and the perceived classroom environment is an influential factor for better learning (Fraser, 1998), and current research relating to values alignment reflect this (see Seah & Andersson, in press).

Cross-regional differences

Although each region valued *achievement* most and *ICT* least (comparatively) amongst the six top values, the order of valuing for the other four common top orientations was different in each region.

Statistically significant differences exist amongst the three regions for each of the 6 values. *Achievement*, *relevance* and *practice*, which are closely tied to examinations, were more salient in student values in mainland China. Students in Hong Kong valued *relevance*, *ICT* and *feedback* more than their peers in Taiwan and mainland China, who valued *communication* more than students in Hong Kong. These differences, no matter how subtle they are, show that values are culture dependent. Thus, even if what students in East Asia value might be unique to the area, there can be diversity of value priorities within the area too.

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IS ELIMINATING THE SIGN CONFUSION OF INTEGRAL POSSIBLE? THE CASE OF CAS SUPPORTED TEACHING

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This study explored how the challenges encountered during integral sign determination process change after various learning processes. In this comparative investigation which is based on qualitative data, the students in the CAS group were subjected to technology enhanced teaching whereas the students in the traditional group were subjected to the traditional centered teaching approaches. Sign determination challenges of the students according to the groups were determined by means of pre- and post-application tests, and semi-structured interviews were employed as supportive data. The findings show that the students in CAS group, in comparison to the students in traditional group, had less "negative area" misconception in definite integral after teaching processes. In this investigation, it has also been discussed how teaching technology influences eliminating misconception.

INTRODUCTION

In many studies in the literature, misconceptions and challenges encountered in calculus lessons are mentioned. It is reported in many studies that calculus lesson students whose operational abilities have developed particularly in traditional class environment have difficulty in understanding, associating and interpreting concepts at basic level (Orton, 1983, Cornu, 1991; Rasslan & Tall, 2002; Berry & Nyman, 2003, Sofronas, De Franco, Vinsonhaler et al., 2011). Uniform presentation of information within the learning content and accustoming of students to solve questions in same pattern with mechanical steps are considered as the primary cause of this case. The fact that although students are successful in routine calculation problems requiring operational information they are confused about conceptual level has required revising lesson content and learning approaches. In this context, one of the steps that have been taken for fertilizing learning process is Calculus Reform Movement. As a result of this movement, textbooks and learning programs have been revised and they were rearranged according to the reform approach (Murphy, 1999). Key elements of reform approach are multiple representations. Accordingly, conducting only algebraic operation steps is not adequate to understand calculus subjects; in addition, interpreting inter-conceptual relations and choosing and using representations suitable for specific cases are also necessary (Dreyfus, 1991; Berry & Nyman, 2003). Calculus Reform Movement supporters claiming that teaching content and method must be reorganized in a way to offer opportunity for multiple representations support the process of integration of technology into learning environment (Murphy, 1999; Vlachos & Kehagias, 2000). A number of previous studies suggest that technology support can be benefited for eliminating the challenges encountered during teaching and learning process of calculus (Berry & Nyman, 2003). The definition made by Vlachos and Kehagias (2000) for CAS-supported learning pattern is presentation of teaching contents organized according to multiple representations by means of technology support and this definition is based on in this study. This research is a part of a wider project which is concerned with students' understanding of the first-year calculus and project's pre-findings which is related to "*The role of CAS for concept images of definite integral*" was presented in previous PME conference (Sevimli & Delice, 2013). By means of this study, how misconceptions and challenges about integral observed in traditional calculus classes and that have correspondence in the literature are affected by CAS-supported teaching process was evaluated.

THEORETICAL FRAMEWORK

Integral concept, which is included within the fundamental subjects of higher education and which is the primary subject that students have difficulty in making sense of, is analyzed under definite and indefinite integral topics. Since definite integral involves previous subjects such as limit, derivative and function knowledge and requires solving techniques with various rules, it is considered among the primary and difficult subjects of higher education (Orton, 1983; Rasslan & Tall, 2002). Challenges encountered about definite integral is either associated with the nature of the concept or it can originate from pedagogical reasons. Accordingly, while Cornu (1991, p. 158) mentions about three reasons of cognitive challenges in calculus subjects, he lists them as epistemological, psychological and didactic oriented challenges. Some studies reports that traditional class students that can successfully solve integration problems that are difficult to calculate even with pencil and paper have difficulty in explaining and interpreting concept definitions at basic level (Orton; 1983). In teaching content of traditional classes, more time is allocated for algebraic interpretation of integral subject and more stress is put on calculation sense of integral (Berry & Nyman, 2003; Sofronas et al., 2011). Some cognitive challenges encountered in the class environment in the studies on integral can be listed as follows: limited concept image, lack of awareness of multiple representations, misconception, difficulties in contextual problem, misusing of Fundamental Theorem of Calculus etc. (Orton, 1983; Oberg, 2000; Rasslan & Tall, 2002; Sevimli & Delice, 2013).

One of the first studies on integral concept in the mathematical education literature was conducted to determine the misconceptions of students by Orton (1983). Emphasizing sign determination of students in his study he conducted to determine comprehension levels of students at introduction level of calculus about definite integral, Orton (ibid) expressed that the notion of limit of sums causes confusion in terms of algebra and stated that the biggest problem encountered arose from misconceptions named as 'negative area'. Negative area misconception is caused by interpretation of students the area above x-axis as positive and below x-axis as negative in area calculation problems. However, within [a,b] interval, since heights of the rectangles below the curve will be $-f(x^*_k)$ if $f(x)\leq 0$, ($x^*_k \in [x_{k-1}, x_k]$) area formula will be $\int_a^b -f(x)dx$ (Hughes-Hallet et al., 2008). Oberg (2000) attribute the main problem encountered about sign confusion to lack of integral in geometrical sense. Accordingly, students that can interpret the behavior of a function over a graphic representation have less difficulty in area calculation problems. Rasslan and Tall (2002), embarking with a similar research question, reported that students did not calculate definite integral value by means of sum of positive and negative areas, actually this sign confusion repeated systematically. Although students had sign confusion about definite integral in many studies as it was mentioned in the previous paragraph, a study evaluating the role of teaching processes for encountered misconception and/or challenges was not found. In line with the suggestions of the previous studies, a perspective for the role of use of technology in eliminating misconception/concept challenge in sign determination process was presented in this study.

METHOD

Research Design and Study Group

This study was designed according to multiple case study since teaching processes are assessed with a holistic approach over misconception of integral. The study was carried out in Calculus II during the 2011-2012 spring term. The participants of this study consists of 84 undergraduate calculus students at a state university; out of these students two groups have randomly been assigned, one as traditional group (n=42) and the other as CAS group (n=42). When assessing whether traditional and CAS groups are comparable, their marks in Calculus I in the previous term have been taken as criteria. It has been established that both groups have same scores in Calculus I and that groups are equal to each other in terms of their academic achievement.

Settings

The treatments in traditional and CAS groups in Calculus II are carried out during six weeks. In this period the role of two teaching approaches on eliminating misconception of integral were tested. Both approaches have been followed by the researchers. In the control group, where the course has been delivered in the traditional approach, the course notes from previous students have been made use of, and a traditional calculus textbook which generally emphasizes symbolic representation and focuses primarily on definition, theorem and proof processes has been used. Differing from traditional approach, technology support was benefited to provide different representations for a concept in the CAS-supported teaching. *LiveMath* software embedded textbook which was adjusted as per calculus reform and emphasizes translations between/within representations were used in CAS group (Hughes-Hallet et al., 2008). Teaching activities prepared according to multiple representations for preventing from misconception.

Data Collection Tools

Data collection techniques were test and interviews. Concept Definition Questionnaire (CDQ) used for determining students' misconception of definite integral before and after teaching processes and semi-structured interviews conducted for understanding students' problem-solving process in terms of misconception.

Concept Definition Questionnaire (Pre& post test)

The questions took place in previous studies are used to determine the students' misconception of definite integrals (Orton, 1983; Rasslan & Tall, 2002; Robutti, 2003). CDQ includes misconception and difficulties met during the teaching of integral, particularly "*negative area*" misconception. The questions in CDQ have different characteristic from each other in terms of obstacles at determining sign which might be depending on context of the question and the multiple representations used in the question. While the questions might be relevant to calculation of the integration and area with respect to context, they also might be algebraic and graphical with respect to representations in terms of characteristic. In Figure 1, an example questions from CDQ is presented with area context and algebraic representation. CDQ had been used in prior research (Sevimli & Delice, 2013) and three experts in mathematics (education) evaluated CDQ in terms of face and content validity. CDQ was given to the CAS and traditional groups as pre and post tests.

Semi-structured interviews

After administering post-CDQ, semi-structured interviews were conducted with four participants to get additional knowledge about integration processes and to understand the role of CAS-supported teaching in terms of elimination of misconception. These four participants in the interviews were selected using the purposeful sampling technique. Main selection criteria were that each participant taught by different teaching approach (CAS or traditional) and that they had different integral misconception. The participants were asked to explain their answers to the questions in CDQ.

Data Analysis

Pre & Post CDQ's data was first assessed in terms of students' misconception. To define the difficulties students have in determination of sign at before and after treatments, "*negative area*" confusion and "*positive value*" generalization which are frequently seen in the literature are utilised as categorization (Orton, 1983; Rasslan & Tall, 2002). According to these categorizations the change in determining sign confusion is compared with respect to characteristic of the questions over the study groups. Interview data was tagged for analysis using an open coding method. Participants' arguments when determining integral sign are exemplified as it is.

FINDINGS

Pre & Post CDQ findings

Evaluations were performed over tests that were conducted before (Pre-CDQ) and after (Post-CDQ) teaching application to determine the role of CAS support on eliminating sign confusion. No sign confusions were encountered pre-CDQ and post-CDQ in the problems (Integration/Algebraic) that were delivered by means of algebraic representation and that require operational integral calculation. While operation result was negative in 40% of the answers in the CAS group and 33% of the answers in the traditional group for the problems delivered by means of graphic representation requiring integral calculation, positive results were reached (Table 1). It was observed that, students found positive results by taking the negative values within the integral calculation into absolute value. The challenge encountered in such type of solution is that students interpreting every graphic problem as area problem in integral consider the interval as positive even where the integral function is negative, and generalize operation sign as positive.

Characteristic of Question	**Type of Difficulties**	**Pre-CDQ (%)**		**Post-CDQ (%)**	
		CAS	***Tra***	***CAS***	***Tra***
Integration/Algebraic	-	-	-	-	-
Integration/Graphic	Positive value	40	33	14	31
Area/Algebraic	Negative area	55	50	16	43
Area/Graphic	Negative area	36	29	9	21

Table 1: Distribution of pre-CDQ and post-CDQ sign confusions of the groups according to question type.

Post-CDQ findings showed that the sign that needed to be negative was determined as positive in 14% of the answers in the CAS group and 31% of the answers in the traditional group for integration/graphic characteristic question. When compared to the pre-CDQ findings, it can be suggested that CAS-supported teaching process considerably reduce "*positive value*" confusion encountered in integral problem delivered by means of graphic representation. It was observed that percentage of the students having "*positive value*" confusion was similar in the traditional group.

The questions delivered by means of algebraic or graphic representation in Pre & Post CDQ were applied to both groups to determine the reflections of sign confusion encountered in definite integral onto area calculation problems. Pre-CDQ findings revealed that "*negative area*" confusion is encountered more in area/algebraic characteristic questions when compared to area/graphics characteristic questions. Pre-CDQ findings show that at least one of every two students in both groups had negative area confusion for the questions delivered by means if algebraic representation requiring area calculation in integral. It was observed that post-CDQ negative area confusion encountered in the questions with area/algebraic characteristic decreased for both groups, however CAS support was more determinant in eliminating this challenge. Comparisons between the groups showed that "*negative area*" confusion encountered in area/algebraic characteristic questions was eliminated in CAS group in great extent when compared to traditional group.

Pre-CDQ findings showed that approximately one third of the students in both groups had "*negative area*" confusion in the area/graphic characteristic questions before the application. It was observed that, similar to the area/algebraic characteristic, "*negative area*" confusion encountered in area/graphic characteristic problems was reduced more in the CAS group when compared to the traditional group. When the general situation is considered, it can be stated that one third of the students had difficulty in determining integral sign after traditional teaching process.

Interview findings

Interviews were made with two each participant (CAS-P_1, CAS-P_2, Tra-P_1, Tra-P_2) from each group having "*negative area*" and/or "*positive value*" confusion to determine whether the challenge encountered in the process of determination of integral sign is a kind of misconception. Participants were confronted with their solutions for the question with area/algebraic characteristic and they were asked why they reached negative area when the function was negative. More than half of the participants having difficulty could not visualize the data presented algebraically and could not notice the intervals where the function switches sign. It is remarkable in the solution in Figure 1 that although the Tra-P_2 draw the graph and shaded the area of the region to be calculated, she did not count in negation of the sign in the area of the region below x-axis. It is wonder for what reason the Tra-P_2 used the graph and why she did not benefited from its content and the related analyses were supported by the interview findings.

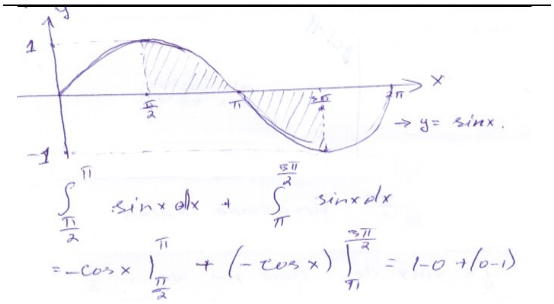
Question	***Solution sample***
Find the areas of the regions enclosed by the function *f(x)=sinx* and x-axis for $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$	

Figure 1: Questions and solution example for area/algebraic characteristic.

Since Tra-P_1 and CAS-P_2 did not draw graph, they noticed that they incorrectly wrote the equation corresponding the related area. CAS-P_1 determined the integral sign correctly for the problem with area/algebraic characteristic and stated that the area cannot be negative just like speed. Tra-P_2 benefited from the argument that, when the function is negative, it will be negative in alteration when she was explaining her solution.

> ...This graphic shows increase when it is above x-axis and decrease when it is below x-axis. Total change will be the sum of positive and negative changes. Therefore, the areas above and below the axis will cancel each other … (Tra-P_2)

DISCUSSION

Limiting definite integral with only area image may cause sign confusion in other algebraic calculations. The participants of the traditional group who stated in the interviews that they considered integrals of positive valued functions as area had sign confusion when the function sign was negative. In this study, this confusion named as "*positive value generalization*" is caused by explication of geometrical interpretation of definite integral as the area only below the curve. Students of CAS group seeing the algebraic and graphical approaches within teaching frequently and as a whole could easily differentiate geometrical sense of integral from algebraic calculation sense. Sevimli and Delice (2013) demonstrated that multiple representation opportunity supports richer and more variable image formation for integral. In this context, it can be remarked that CAS support emphasizes area sense of integral as well as calculation sense and thus provide support for making sense of calculation process.

Another challenged emphasized in this study is negative sign confusion encountered in area calculation problems. Test and interview findings revealed that some of the students in the traditional group did not take the sign of the function into consideration in area calculation problems presented by means of algebra representation and negative sign confusion was actually confusion for some part of the students. It was determined in the interview findings that some students in the traditional group interpreted the area below x-axis as negative and above x-axis as positive. Orton (1983) remarks that the challenge in such solutions is a misconception while he bases the cause of this misconception on the rote teaching that have no conceptual basis. As a matter of fact, answer of a student from the traditional group "*even if the area is negative, I multiply it with minus*" supports the reasoning of Orton (ibid). These misconceptions may be originated from student, information or teaching process (Cornu, 1991). Differently from pedagogical challenges, Orton (1983) reports that integral concept has challenges arising from its own nature, while Dreyfus (1991) integral concept require advance mathematical thinking processes, and they altogether confirm presence of epistemology-originated challenges. The findings of this study are similar to the results of other studies on sign determination process (Oberg, 2000; Rasslan & Tall, 2002), and authentically show that CAS-supported environments create awareness in the process of sign determination in definite integral. The students of CAS group trying to interpret graphic data within the context of algebraic calculation and area senses of integral used analytic and visual judgments together and by means of association, and they were more successful in terms of this respect when compared to the students in the traditional group creating solutions basing on analytic judgment. Area calculation problems in the contents presented by means of *LiveMath* software in CAS group were associated with rectangles sum in Riemann's definition. In teaching applications visualized by means of technology support, the fact that heights of the rectangles below x-axis were $-f(x)$ was stressed and the contents that would provide making sense of sign change by students were employed. These arguments used by the students of CAS group when determining signs can be interpreted as technology being as scaffolding. Namely it may be claimed that technology helps students to construct or reshaped the knowledge and procedures during the integral problem-solving processes.

CONCLUSION

Study results showed that the students in the traditional group could not interpret the graphic data before and after the teaching application, and therefore had "*negative area*" misconception and "*positive value*" confusion. Many students in the traditional group tried to make the transitions between graphic and algebra representation through the rule-based approaches that do not have conceptual basis. After CAS-supported teaching process, the students more frequently benefited from graphic representation in area calculation problems in integral and could correctly interpret graphic data in the problems orientated at integration calculation. Therefore, previous sign confusions of the students in the CAS group were eliminated to a large scale. In the light of the results mentioned above, it is concluded that CAS-supported teaching pattern is more effective in eliminating some misconceptions and challenges encountered before the application or in the literature when compared to the traditional teaching approach.

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THE IMPACT OF A TEACHER DEVELOPMENT PROGRAM ON 7^{TH} GRADERS' LEARNING OF ALGEBRA

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We examine the impact of a teacher development program based on a functions approach to algebra on 7th grade students' understanding of equations. We focus on how students' score gains relate to their teachers' initial level of mathematics performance. Students from participating teachers' and their control peers completed a mathematics assessment at the start and at the end of the school year the teachers were taking the second and third of three courses. Students of participating teachers made greater gains than controls regardless of their teachers' initial level of mathematical understanding.

BACKGROUND

Algebra, a central topic in the mathematics curriculum (Moses & Cobb, 2001; National Council of Teachers of Mathematics, 2000; Common Core State Standards Initiative, 2010), has become a gatekeeper for higher education and a roadblock to access to science careers (Kaput, 1998; Moses & Cobb, 2001). Although requirements across states in the U.S. may differ, all states require that all students take mathematics and science to graduate from high school, at least from the middle school years. However, most students lose interest in mathematics when, in middle school, algebra is first introduced.

Research has repeatedly documented middle and high school students' difficulties with algebra, which are often attributed to the inherent abstractness of algebra and to levels of cognitive development (see reviews by Carraher and Schliemann, 2007 and by Kieran, 2007). Students often view the equals sign as a unidirectional operator, focus on computing specific answers, find difficult to use mathematical symbols to express relationships between quantities, do not use letters as generalized numbers or as variables, and do not operate on unknowns. Generating equations from word problems and using those equations to solve the problem constitutes a major challenge for 11 to 15 year-olds. Even when 6th and 7th grade students can generate the equation to represent a word problem, they often use methods other than algebra syntactic rules for manipulation of symbols to solve the equation.

In contrast, recent studies of early algebra show that, given relevant experiences, elementary school children succeed in understanding basic algebraic principles and representations (Cai & Knuth, 2011; Carraher & Schliemann, 2007; Kaput, Carraher, & Blanton, 2007). Such findings strongly support Booth's (1988) suggestion that students' difficulties with algebra in middle and high school are due to the traditional

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computational approach to algebra in the mathematics curriculum, rather than to developmental limitations.

Mathematics education researchers (Kaput, 1998; Schoenfeld, 1995; Schwartz & Yerushalmy, 1992) have argued that a functions approach to algebra has the potential to better prepare students for a deep understanding of algebra. Within a functions approach, equations are considered as comparisons between two functions. Instead of starting by learning to compute the unknown values in an equation, students are introduced early on to variables and to the analysis of relations between sets of numbers. In doing so, they are introduced to multiple representations of functions, such as verbal statements, number lines, data tables, Cartesian graphs, and algebraic notation. Students move between multiple representations of functions and consider the process of solving equations as the comparison between two functions. Within this approach, we argue that students will become better prepared to solve word problems by representing problem statements as functions and as equations and to understand how the transformations in an equation towards its solution corresponds to transformations of the graphs of the two functions in the Cartesian space.

The implementation of a functions approach for teaching algebra represents a departure from the traditional path of teaching algebra by solely focusing on the manipulation of equations. As such, it requires the preparation of teachers for doing so, as well as close evaluation of the impact of this preparation on students' learning. This study evaluates the impact of a teacher development program focused on a functions approach to algebra on 7th graders' ability to understand algebra equations and to represent and solve verbal problems using equations.

The Poincaré Institute (http://sites.tufts.edu/poincare/), a program supported by the National Science Foundation (grant #0962863), offers three semester-long online graduate level courses to teachers in three New England states (see Teixidor-i-Bigas, Schliemann, & Carraher, 2013 for details). The courses, offered to 5th to 9th grade teachers, covered algebra and functions, their multiple representations, and modelling and applications. Course 1 dealt with functions and relations and the representation of functions on the real line and on the plane. Course 2 focused on fractions and divisibility as they relate to functions, transformations of the line, and transformations of the plane, and on the use of transformations to analyse graphs of functions and for solving equations as the comparison of two functions represented in the plane. Course 3 included the representation of problems as equations, work on linear, quadratic and higher order equations, the relation between factoring and roots of equations, and slope and rate of change. Course 1 alternated weeks of study of mathematical content with weeks on examining classroom lessons related to the topics. Courses 2 and 3 were each structured across four three-week units plus two weeks dedicated to a final project. The first two weeks of each unit were dedicated to mathematics and pedagogical content; in the third week, groups of three to five teachers designed a learning activity (course 2) or interviewed individual students to explore their thinking (course 3). For the final project of courses 1 and 2, each teacher designed, implemented, and analysed a lesson.

The goal was to help teachers deepen their understanding of mathematics and of students' mathematical thinking and to enhance students' mathematical learning in their classrooms. Weekly meetings of teachers contributed to the development of a strong professional learning community to examine and improve their practices.

While, with a few exceptions, previous research more often has described the impact of teacher development programs on changes on teachers' ways of teaching, not many studies have focused on the possible contribution of these programs to the students of participating teachers. This study (a) examines the impact of the Poincaré Institute program on how the students of participating teachers understand and create equations to solve word problems and (b) analyses how students' score gains relate to teachers' initial level of mathematics understanding.

METHOD

We focus on 7th grade teachers and their 12- to 13-year old students because, at this level, the content of the program closely matched their mathematics curriculum.

Of the 56 cohort teachers in the program, only 13 who taught 7th grade were included in this analysis. A total of 319 students of these teachers (cohort students) and 267 from non-cohort teachers were given the mathematics assessment at the start (September 2011) and at the end (June 2012) of the school year during which cohort teachers were taking the second and third courses of the program, respectively. Teachers from both cohorts also took an online mathematics assessment on algebra, functions, and graphs at the start of the program (January 2011).

We focus on four problems (see Figures 1 to 4) from the 15-problem written assessment. The first problem, the "Liam and Tobet problem", is a multi-step algebraic word problem that requires representing two functions, setting them equal, thus generating an equation, solving the equation, and interpreting the equation solution. The second, the "Amusement Park problem", is a multiple-choice problem about the relationship between the dependent and independent variables in an equation. The third, the "Cases problem", is another multiple-choice problem about which equation satisfies three sets of 'x' and 'y' values. The fourth and final problem, the "Finding X" problem, asks students to find the value for 'x' for a given equation.

Each of the five subparts of the Liam and Tobet problem and each of the three other problems was scored as "0" when answers were missing or incorrect and "1" when it was correct; hence, the minimum score a student could receive for the four problems was 0 and the maximum was 8.

Teachers' initial level of mathematics understanding was determined through the 24 problems in their written assessment. Some of the problems had multiple parts, leading to scores that could vary from 0 to a maximum of 47. These scores were then recoded (using the 33.3 and 66.6 percentiles) as three levels, where low corresponded to scores from 0 to 35, medium to 36 to 40, and high scores to 41 to 47.

Liam and Tobet are going to walk in a fund-raising event to raise money for their school.

- Liam's mother promised to donate to the school $4 per mile that Liam walks, plus an additional $30.
- Tobet's father promised to donate to the school $6 per mile that Tobet walks, plus an additional $20.

a) If Liam walks 15 miles during the event, what is the total amount of money his mother will donate? Show or explain how you got your answer.

b) Write an equation that represents *y*, the total amount of money Liam's mother will donate if Liam walks *x* miles during the event.

c) Write an equation that represents *y*, the total amount of money Tobet's father will donate if Tobet walks *x* miles during the event.

After the event, Liam and Tobet compared their results. Liam had walked the same number of miles as Tobet. Liam's mother had donated the same amount of money as Tobet's father.

d) Using your two equations from parts (b) and (c), determine the number of miles Liam and Tobet each walked during the event. Show or explain how you got your answer.

e) Using your answer from part (d), determine the total amount of money Liam's mother and Tobet's father each donated. Show or explain how you got your answer.

Figure 1: Liam and Tobet Problem

Andrea went to an amusement park.

- The cost of admission was $5.
- The cost for each ride was $0.75.

The equation below shows ***c***, Andrea's total cost to go to the amusement park and go on *r* rides.

$c = 5 + 0.75r$

Based on the equation, which of the following statements is true?

A) As the value of *r* increases, the value of *c* increases.
B) As the value of *r* decreases, the value of *c* stays the same.
C) As the value of *c* decreases, the value of *r* increases.
D) As the value of *c* increases, the value of *r* stays the same.

Figure 2: The Amusement Park Problem

	x	y
Case A	0	-3
Case B	1	-1
Case C	2	1

Which equation is true for the values of **x** and **y** shown in Cases A, B, & C above?

A) 3x + 2 = y
B) 3x – 2 = y
C) 2x + 3 = y
D) 2x – 3 = y
E) x – 3 = y

Figure 3: The Cases Problem

Find the value of *x* if 12*x* – 10 = 6*x* + 32

Show your work.

Answer:________________________

Figure 4: The Finding X Problem

RESULTS

The mean number of correct answers for all students in the two groups, for the eight items, increased from 2.88 (SD = 2.285) in September 2011 to 4.53 (SD = 2.541) in June 2012. The initial mathematics scores for all teachers, in January 2011, ranged from 20 to 44, with a mean of 36.92 (SD = 6.726).

The program had a positive impact on the 7th grade students' average mathematics scores for all 15 problems (31 sub-items) in the assessment: the interaction effect of time (September 2011 vs. June 2012) by cohort (Cohort vs. Non-Cohort teachers' students) on all students' mathematics scores, after controlling for any differences between groups at time 1 was statistically significant ($F(1, 583) = 13.63$, $p < .001$,

$\eta^2_p = .023$). The analysis that follows will only deal with the four problems that directly relate to the main content of the courses (see above).

Similar to the general results, students of cohort teachers performed better than the 7th grade students of non-cohort teachers after controlling for group differences at time 1 (Figure 5). The interaction effect of time and cohort on students' mathematics scores was, again, statistically significant ($F(1, 583) = 22.93$, $p < .001$, $\eta^2_p = .038$).

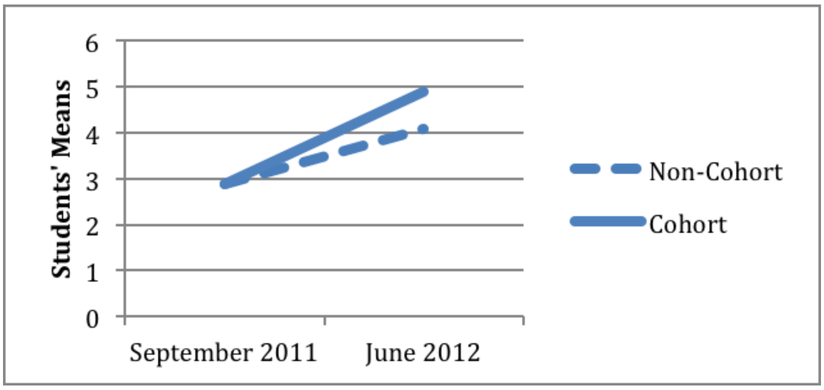

Figure 5: The interaction between Time and Cohort membership on students' scores for the four selected problems.

Students' average score gains in the eight selected items, by cohort and non-cohort teachers' initial mathematics level at the start of the program, were obtained by taking the average difference between students' scores in September 2011 and June 2012. Figure 6 shows that 7th graders' average score gains at the end of the school year were higher for students of cohort teachers, regardless of the teachers' initial levels of mathematics understanding. Here, students of teachers classified at the low level performed nearly as well as those of teachers at the high level. In contrast, the score gains for the non-cohort group closely related to the teachers' initial levels of mathematics, with students of non-cohort teachers in the low initial level showing less gains than students of teachers with high initial mathematics level. The interaction effect of cohort membership and teacher initial mathematics levels on student score gains was statistically significant ($F(2, 585) = 6.53$, $p = .002$). Thus, regardless of cohort teachers' initial mathematics level at the start of the program, their students showed somewhat similar and higher score gains. For the non-cohort group, only students of teachers with high initial mathematical levels showed gains that approached those of students of cohort teachers.

Figure 7 shows the percentage of cohort and non-cohort students who correctly answered each of the eight items in September 2011 and in June 2012. The items can be clustered in three groups in terms of the percentage of students' correct answers in September 2011. Group 1, with more than 50% of students answering correctly, include part a of the Liam and Tobet Problem (LTPa), the Amusement Park problem (APP), and the Cases Problem (CP). Group 2, with an average of 35% of correct answers, include parts b and c of the Liam and Tobet Problem (LTPb and LTPc).

Group 3 includes parts d (LTPd) and e (LTPe) of the Liam and Tobet problem and the Finding X Problem (FXP), with 11 to 16% of students answering correctly.

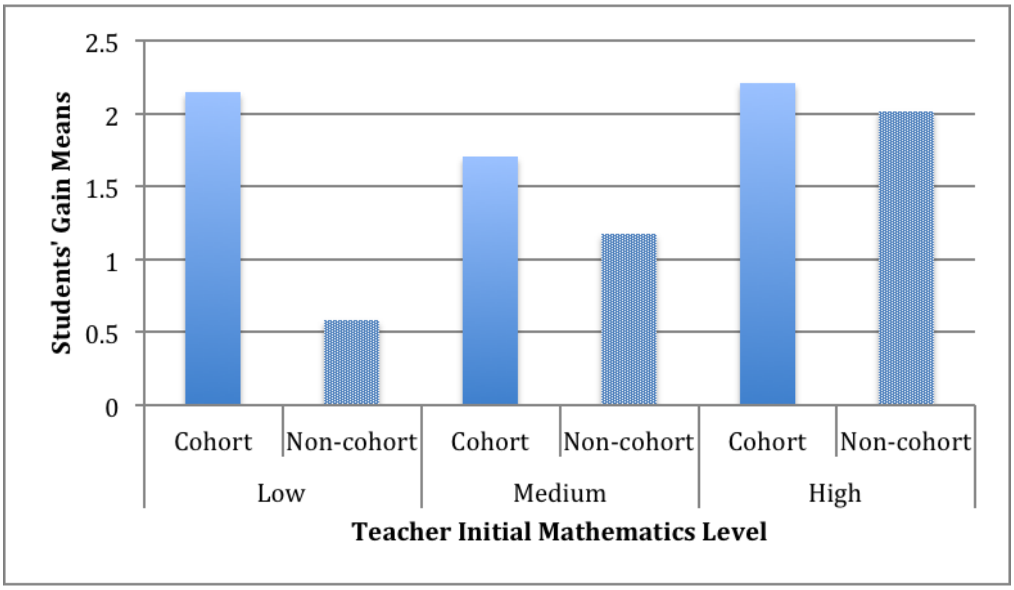

Figure 6: Student gains by Cohort membership and Teacher Initial Mathematics Levels (score gains ranged from -8 to 8).

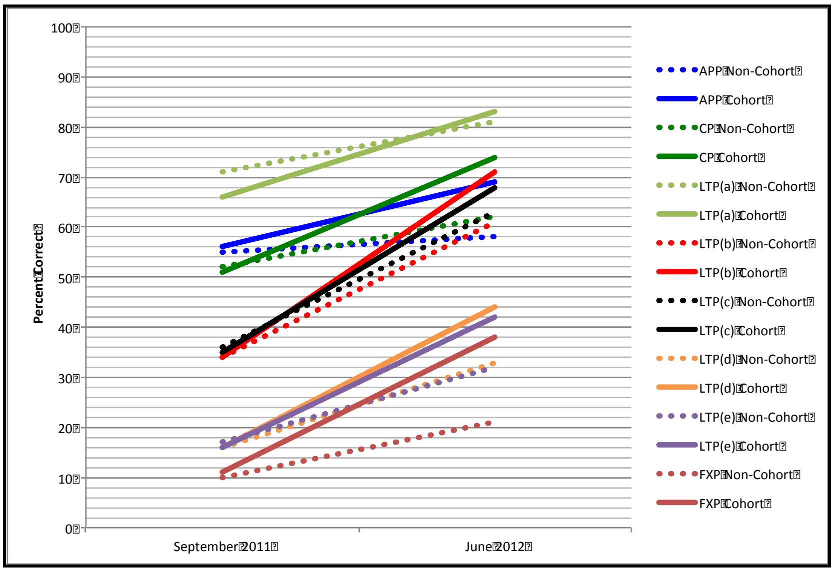

Figure 7: Percentage of students who correctly solved each item by Time and Cohort Membership.

The graphs for each item show that, in June 2012, cohort students did better on each of the items, in comparison to non-cohort students. A Repeated Measure Mixed Design ANOVA showed that the interaction effect of time by cohort membership on the percentage of students giving correct answers was statistically significant ($F(1, 14) = 6.61$, $p = .022$, $\eta^2_p = .321$).

We also investigated, for each of the eight items, the changes in the percentage of students who correctly solved each item, for each cohort and for each group of teacher initial mathematics level.

For teachers with low initial mathematics level, the difference between cohort and non-cohort students' changes were higher for cohort students in all items, varying from 13% to 28%, Students of non-cohort teachers showed modest improvements in seven of the items and a decline in the CP problem (picking an equation that satisfy three sets of x and y values). For teachers with medium initial mathematics level, the change was 5% to 20% higher for students of cohort teachers in five of the eight items (APP, LTPb, LTPc, LTPd, and LTPe), with students of non-cohort teachers showing a decline in the APP problem (finding the relationship between the dependent and independent variables), equal increase in one problem (LTPa), and 2% to 6% higher improvement for the other two problems. For teachers with high initial performance, the change was 4% to 19% higher for students of cohort teachers on five of the eight items (CP, LTPa, LTPd, LTPe, and FXP) and from 3% to 9% higher for non-cohort students in the other problems.

In summary, students of cohort teachers with low initial mathematics level showed higher improvement in their scores on all eight items in comparison to their control counterparts. Moreover, regardless of the initial level of mathematics of their teachers, students of cohort teachers consistently outperformed non-cohort students in items LTPd and LTPe, on finding and interpreting the solution to the Liam and Tobet problem.

DISCUSSION

Our results suggest that teachers' participation in a program focused on a functions approach to algebra and on students' reasoning contributed to 7^{th} grade students' learning of how to represent statements in a word problem as an equation, of how to solve and interpret the solution to equations, and of how the elements in an equation relate to each other.

Students of teachers who had initially demonstrated relatively low levels of mathematical understanding benefited the most, in all the problems analysed here. The program seems to have helped all teachers better contribute to their students' learning about how to create, solve, and interpret algebra equations to solve word problems. These are important achievements, given the well-documented difficulties with algebra among middle and high school students.

Our data contribute to further advance our understanding of the impact of teacher development programs and of teachers' levels of mathematical understanding on their students' learning.

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EXPLORING 'WHAT JAPANESE STUDENTS FIND IMPORTANT IN MATHEMATICS LEARNING' BASED ON THE THIRD WAVE PROJECT

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The present study is an ongoing survey targeting Japanese fifth and ninth grade elementary and junior high school students respectively using the framework of "The Third Wave" international comparative study. The purpose of this research report is to describe the questionnaire survey's results and analyze some similarities and differences between fifth and ninth graders from a value perspective. The main results show that there are five common factors underlying students' valuing and that fifth graders tend to value "process", "effort", "exploration", "fact", "openness" and "progress"; in contrast, ninth graders tend to value "product", "ability", "exposition", "idea", "mystery", and "control".

THE THIRD WAVE: VALUES IN MATHEMATICS EDUCATION

"The Third Wave" is a metaphor from Alvin Toffler's book published in 1980, which implies that cognition is the first wave, affect second, and value third. It is important to note that "the wave metaphor not only encapsulates the energy for change that is generated by the values approach, but it also implies the ongoing relevance of the previous two waves since waves overlap" (Seah & Wong, 2012, p. 1). Under the coordination of the project, initially, the role of values and students' valuing in mathematics learning had been assessed using qualitative data such as interviews, classroom observations, photography or videotapes. Such qualitative data analysis had been important and useful "in a research context in which values studies were relatively new, when it was not known what the scope of values were, and indeed, what they looked like" (Seah, 2013, p. 197). More recently, a new questionnaire survey was designed and validated, due to the qualitative approach's own constraints, such as "the time and skills that are needed to investigate and analyze the values respectively" (ibid., p. 197). The questionnaire survey, called 'What I Find Important (in mathematics learning)' [herein referred to as WIFI], was conceptualized in 2012, and gathered research teams from different countries such as Australia, Brazil, China, Hong Kong, Malaysia, Japan, Singapore, Sweden, Taiwan, Turkey and the United States. (e.g., Kinone *et al*., 2013; Andersson & Österling, 2013; Seah, 2013). This paper intends to investigate the Japanese part of the questionnaire survey based on the unique framework proposed in this project. Thus, there are two research questions as follows: *what Japanese students find important in mathematics learning, and how can we analyze similarities and difference between fifth grade and ninth grade students by means of the questionnaire*.

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Additionally, we would like to reflect on a problematic situation related to the first and second wave in a Japanese educational context, as some other East Asian counties may have similar experiences. According to recent well-known international comparative studies, such as the TIMSS and the PISA, Japanese students' cognitive performance in mathematics has been fairly high when compared to other countries. On the other hand, Japanese students' affective performance in mathematics has been extremely low. For example, the following table shows five high cognitively performing countries and the percentages of respective students who "agree" with the statement "I like math" cited from TIMSS 2011 (cf. Mullis *et al*., 2012; NIER, 2013).

	Grade 4 (%)	**Grade 8 (%)**
Singapore	79.1	77.6
Korea	64.8	41.0
Hong Kong	79.6	62.7
Taiwan	62.4	44.4
Japan	65.9	39.1
TIMSS Ave.	**81.4**	**66.2**

Table 1: Affective performance in mathematics (TIMSS 2011)

There are two problematic gaps, namely the gap between cognitive and affective performance, and between elementary and junior high school students. We believe that the Third Wave project can provide a new framework to understand and/or explain such problematic phenomena in light of the values perspective, since values are "the deep affective qualities which education fosters through the school subject of mathematics" (Bishop, 1999, p. 2).

CONCEPTUAL BACKGROUND OF THE QUESTIONNAIRE STUDY

Research on values in mathematics education began with Alan Bishop's proposal of three pairs of complementary values for (western) mathematics: *rationalism* and *objectivism, control* and *progress*, as well as *mystery* and *openness* (Bishop, 1988). Regarding the term "values" as used in mathematics education, we refer to the following conceptualizations:

> There is clearly a relationship between values, beliefs and attitudes, with the literatures suggesting that values are more deep-seated and personal than attitudes, and less rationalised than beliefs. (Bishop, 2001, p. 238)
>
> Values are the convictions which the individual has internalised as being the things of importance and worth. They regulate the ways in which a learner utilises his/her cognitive skills and emotional dispositions to learning. (Seah, 2013, p. 193)

In a later consideration, Bishop (1998) argued that three categories of values can be encountered in the mathematics classroom: *general educational* values (e.g., *honesty*, *good behaviour*), *mathematical* values (e.g., *rationalism, openness*), and *mathematics educational* values. According to Seah (2013), data analysis by the Third Wave project group specifically identified mathematics educational values continua such as *ability* and *effort*, *wellbeing* and *hardship, process* and *product, application* and *computation,* *facts* and *ideas*, *exposition and exploration, recalling and creating*, as well as *ICT* and *pen-and-paper*.

In developing the WIFI questionnaire, “a diverse range of items that span across the three categories of values in the mathematics classroom – mathematical, mathematics educational, and general educational” were sought after (Seah, 2013, p. 197). Here it is important to note that “children responding to the questionnaire cannot be expected to relate directly to values; hence, the questions posed are about different learning activities, regarded as value indicators. […] The learning activities pictured were treated as value indicators, and the results allowed the researchers to reflect on the problem of marking a difference between a value and a value indicator” (Anderson & Österling, 2013, p. 18). Therefore, the learning activity “learning the proof” is one item in the WIFI questionnaire categorized as an indicator of the mathematical value of rationalism.

METHODOLOGY

Now, let us explain the outline of the questionnaire. The questionnaire consists of four sections. “Section A” consists of 65 questions, 64 of which utilize a five point Likert-scale to indicate the extent that the respondent finds something important in mathematics learning; the final question is for comments. Next, “section B” consists of 10 items in which respondents mark their relative valuation of the complementary values at each end of a horizontal line. Figure 1 shows part of the instructions from section B using a non-math example. A set of ten items in section B is reflective of the conception of the complementary or continua values mentioned above.

Example (non-maths):

Watching a movie ____ ✓ ____ ____ ____ ____ Shopping

Figure 1: Instructions from the section B (excerpt from the WIFI)

“Section C” consists of 4 items and it is “made up of four conceptualised, open-ended items which encourage respondents to write down what they themselves value, given a common scenario of the production of a magic pill the ingestion of which makes one excel at mathematics” (Seah, 2013, p. 198). Finally, “Section D” consists of questions about personal attributes such as nationality, type of school, age, gender, etc. In the present study, the targets of analysis are sections A and B, which are the main part of the WIFI questionnaire.

The questionnaire survey was conducted in different parts of Japan in 2012; seven elementary schools (605 fifth grade students) and seven junior high schools (711 ninth grade students) participated. Although the selection of schools was not random, different types of schools such as national and public from both urban and rural areas in three different prefectures (Hiroshima, Miyazaki, and Osaka) were included. In order for the teachers to understand the aim of the questionnaire survey, we visited each school and explained its purpose. The questionnaire was both distributed to and answered by participating students in their classrooms. In one case a research member was present to observe the students as they completed the questionnaire.

In describing the 64 items in section A, we scored the five choices as follows: “absolutely important” (score: 1), “important” (score: 2), “neither important nor unimportant” (score: 3), “unimportant” (score: 4), and “absolutely unimportant” (score: 5). The construct validity for section A was assessed using a Principal Factor Analysis [PFA] with a Varimax rotation, while a cut-off criterion for factor loadings of at least .35 was used in interpreting the solution. The Kaiser-Meyer-Olkin [KMO] measure of sampling adequacy and Bartlett’s Test of Sphericity [BTS] were also used for validation. As a result, KMO was more than 0.9, and the BTS was significant at 0.001, validating the questionnaire through factor analysis. In the previous study, we compared fifth (G5) and ninth grades (G9) after the extraction of a PFA with a Promax rotation, although the previous analysis was conducted in G5 and G9 data separately (Kinone *et al*., 2013). For the further analysis, in the present study we applied independent sample t-tests to the identified the subscale scores of each factor by calculating the means of the item scores included in each factor respectively.

In describing the 10 items in section B, we scored the five positions on a horizontal line in terms of the semantic differential method, which is a type of a rating scale designed to measure connotative meaning, as follows: (left side) [-2, -1, 0, +1, +2] (right side). Figure 1, for example, would receive a score of “-1”. In order to analyze the difference between G5 and G9, we applied independent sample t-tests to 10 items’ means. The present study analyzes the results of sections A and B separately, because the construction of each section has its own scoring methods. Although there may be some interrelationships between sections A and B, a more complete analysis lies outside the scope of this report, although it is one of our future tasks.

RESULTS

Exploratory factor analysis on students’ valuing

As a result of analyzing the 64 items included in section A, we accepted five interpretable factors after the extraction of principal factor analysis with a Varimax rotation: I) *Ways of understanding and problem-solving*; II) *Mathematical stories and connections*; III) *Collectivism;* IV) *Support from others*; V) *ICT*. Five factors with eigenvalues greater than one explain 45.124% of the variance, with almost 16.645% attributed to the first factor. And seven items were eliminated. Reliability analysis yielded satisfactory Cronbach’s alpha values for each of the five factors, ranging from 0.772 to 0.936, indicating an acceptable degree of internal consistency in each subscale. Although it will take an inordinate amount of space to list data about each of the five accepted factors (such as the factor loading, commonalities, etc.) as Table 2 shows, we shall show this table because of the methodological reasons and of that there are some crucially important results of for the considerations.

items	I	II	III	IV	V	Commonality
58.Knowing which formula to use	**.710**	.168	.120	.172	.007	.576
56.Knowing the steps of the solution	**.661**	.075	.026	.193	.049	.483
63.Understanding why my solution is incorrect or	**.644**	.179	.219	.244	-.028	.555
64.Remembering the work we have done	**.642**	.165	.100	.117	-.002	.463
36.Practising with lots of questions	**.636**	.074	.133	.101	-.052	.441
37.Doing a lot of mathematics work	**.636**	.227	.178	.081	-.006	.494
54.Understanding concepts/processes	**.620**	.207	.260	.149	.030	.518
59.Knowing the theoretical aspects of mathematics	**.597**	.286	.191	.102	-.004	.485
2.Problem solving	**.571**	.109	.148	.029	.018	.361
31.Verifying theorems/hypotheses	**.564**	.229	.316	.082	.042	.479
42.Working out the maths by myself	**.547**	.188	.028	.082	.022	.343
47.Using diagrams to understand maths	**.544**	.285	.195	.252	-.004	.478
62.Completing mathematics work	**.526**	.205	.236	.148	-.074	.401
43.Mathematics tests/examinations	**.507**	.159	.060	.160	-.001	.312
13.Practising how to use maths formulae	**.502**	.263	.162	.068	.025	.352
46.Me asking questions	**.494**	.114	.218	.372	.039	.445
49.Examples to help me understand	**.489**	.229	.178	.342	.112	.453
55.Shortcuts to solving a problem	**.486**	.095	.036	.174	.141	.297
50.Getting the right answer	**.434**	-.006	-.197	.058	.142	.251
32.Using mathematical words	**.423**	**.408**	.172	.110	.021	.387
33.Writing the solutions step-by-step	**.423**	**.354**	.225	.277	.059	.435
8.Learning the proofs	**.420**	.249	.413	.072	-.007	.414
51.Learning through mistakes	**.418**	.159	.191	.274	-.023	.313
26.Relationships between maths concepts	**.415**	**.383**	.288	.132	.131	.437
1.Investigations	**.412**	.338	.327	.027	.029	.393
53.Teacher use of keywords	**.411**	.279	.051	.294	.093	.344
61.Stories about mathematicians	.124	**.709**	.080	.145	.091	.554
18.Stories about recent developments in mathematics	.163	**.698**	.171	.069	.134	.566
17.Stories about mathematics	.179	**.692**	.157	.066	.122	.555
39.Looking out for maths in real life	.195	**.617**	.264	.247	.074	.555
60.Mystery of maths	.264	**.610**	.182	.154	.029	.500
40.Explaining where the rules/formulae came from	.227	**.607**	.163	.115	.088	.468
34.Outdoor mathematics activities	.080	**.603**	.199	.223	.234	.514
11.Appreciating the beauty of mathematics	.215	**.601**	.146	.064	.034	.434
21.Students posing maths problems	.206	**.474**	**.404**	.142	.106	.463
20.Mathematics puzzles	.207	**.471**	.209	.115	.259	.389
12.Connecting maths to real life	.242	**.444**	.299	.161	.017	.371
10.Relating mathematics to other subjects in school	.249	**.441**	.302	.132	.102	.375
29.Making up my own maths questions	**.386**	**.405**	.333	.102	.042	.436
52.Hands-on activities	.209	**.391**	.060	.298	.138	.308
48.Using concrete materials to understand	.242	**.377**	.111	.365	.168	.374
9.Mathematics debates	.189	.276	**.571**	.244	.066	.501
19.Explaining my solutions to the class	.211	.422	**.561**	.147	.049	.562
15.Looking for different ways to find the answer	**.424**	.268	**.538**	.015	.057	.545
7.Whole-class discussions	-.010	.283	**.526**	.346	.121	.491
30.Alternative solutions	**.449**	.317	**.492**	.057	.081	.554
16.Looking for different possible answers	**.402**	.324	**.491**	.064	.026	.513
3.Small-group discussions	.046	.213	**.430**	.301	.116	.337
44.Feedback from my teacher	**.410**	.135	.175	**.592**	.040	.569
45.Feedback from my friends	.199	.185	.232	**.585**	.098	.480
41.Teacher helping me individually	.291	.235	-.041	**.388**	.070	.298
6.Working step-by-step	**.341**	.167	.190	**.384**	.059	.331
5.Explaining by the teacher	**.376**	.090	.097	**.380**	.046	.306
35.Teacher asking us questions	.313	.292	.306	**.349**	.036	.400
23.Learning maths with the computer	.002	.191	.054	.069	**.883**	.823
24.Learning maths with the internet	.005	.226	.059	.063	**.871**	.816
25.Mathematics games	.011	.328	.156	.197	**.503**	.424
Proportion of variance(%)	16.645	12.421	6.960	5.223	3.875	
Comulative proportion(%)	16.645	29.066	36.026	41.250	45.124	

Table 2: The result of the factor analysis (Section A)

How can we conceive the above labeled factors. If we attempt to make some interpretations about factor I, students' learning activities such as knowing, understanding, solving resemble some aspects of problem-solving activities that may be seen as recent Japanese mathematics classroom culture (e.g., Shimizu, 2009). In particular, the following remarks from Stigler and Hiebert (1999) seem pertinent:

> In Japan, teachers appear to take a less active role, allowing their students to invent their own procedures for solving problems. And these problems are quite demanding, both procedurally and conceptually. Teacher, however, carefully design and orchestrate lessons so that students are likely to use procedures that have been developed recently in class. An appropriate motto for Japanese teaching would be "structured problem solving". (p. 27)

Additionally, the factor III, collectivism (in other words, social interactions) can be an essential aspect of "structured problem solving". On the other hand, there are some differences between fifth and ninth graders. By applying independent sample t-tests to the subscale scores included in the five factors in G5 and G9, statistically significance differences between them were found (the significance level was set at .05). Table 3 shows the results of such an analysis for each subscale; "G5 < G9" means that fifth graders' scores are significantly low (a high degree of importance). Interestingly, there is a strong tendency among subscales, for fifth graders to find a high degree of importance when compared to ninth graders.

Factors		**G5**	**G9**	**t-tests**
Factor 1: Ways of understanding and problem-solving	Means SD	1.61 0.45	1.79 0.59	G5 < G9 t=-5.58, df=1282
Factor 2: Mathematical stories and connections	Means SD	2.00 0.62	2.51 0.76	G5 < G9 t=-13.10, df=1284
Factor 3: Collectivism	Means SD	1.63 0.56	2.14 0.73	G5 < G9 t=-13.99, df=1301
Factor 4: Support from others	Means SD	1.70 0.55	1.89 1.66	G5 < G9 t=-5.60, df=1309
Factor 5: ICT	Means SD	2.51 0.96	2.63 0.98	G5 < G9 t=-2.23, df=1279.95

Table 3: Analysis of t-test to the item means of subscale scores (Section A)

Analysis on pairs of complementary values

In the present study, the analysis of section B is rather limited, but some crucial aspects of students' valuing are explicit in terms of their frequency distribution. As a result of an analysis of the 10 items included in section B, Table 4 shows means, SD, modes, medians in total data, as well as means and SD in G5 and G9 data respectively. Here we would like to note that the modes of items 66, 68, and 74 were respectively "-1," "2," and "-2," although other items were "0." Therefore, there is a common disposition among Japanese students to explicitly value *process*, *effort*, and *openness* over *product*, *ability*, and *mystery*. Thus, this is a possible reflection of the reality of Japanese mathematics classrooms. On the other hand, we were surprised by the data for item 72 (*recalling* vs. *creating*) because it is inconsistent with the fact that "creating" is one of the most important values in the development of mathematics education in Japan, closely related to the pedagogical notion of "mathematical thinking" or "mathematical activity" (cf. Baba *et al*., 2012).

Question items	**total**		**G5**	**G9**	**t-tests**
	Means SD	Modes Medians	Means SD	Means SD	t
66. How the answer to a problem is obtained OR What the answer to a problem is	-0.86 1.040	-1 -1	-0.95 1.035	-0.79 1.040	-2.647
67. Feeling relaxed having fun doing maths OR Hardwork needed doing maths	0 1.211	0 0	-0.06 1.209	0.05 1.212	n.s.
68. Leaving it to ability OR Putting in effort	0.77 1.163	2 1	1.00 1.133	0.57 1.153	6.831
69. Applying maths concepts OR Using a rule formula	0.11 1.033	0 0	0.13 1.077	0.10 0.995	n.s.
70. Truths facts which were discovered OR Math ideas practices used in life	0.08 1.050	0 0	0.09 1.076	0.08 1.029	n.s.
71. Someone teaching explaining to me OR Exploring maths myself peers etc	0.04 1.170	0 0	0.31 1.180	-0.20 1.108	8.041
72. Remembering maths ideas etc OR Creating maths ideas etc	-0.58 1.113	0 -1	-0.52 1.155	-0.63 1.074	n.s.
73. Telling me what a triangle is OR Letting me see concrete examples first	0.39 1.155	0 0	0.50 1.180	0.30 1.125	3.058
74. Demonstrating explaining maths to others OR keeping maths magical	-1.19 0.969	-2 -1	-1.35 0.919	-1.06 0.992	-5.479
75. Using maths to predict explain OR Using maths for development progress	-0.04 1.005	0 0	0.09 1.068	-0.16 0.933	4.463

Table 4: Data and analysis of t-tests to each item (Section B)

By applying independent sample t-tests to the means of each item for G5 and G9, statistically significant differences were found between them (the significance level was set at .05) in items 66 (*process* vs. *product*), 68 (*ability* vs. *effort*), 71 (*exposition* vs. *exploration*), 73 (*rationalism* vs. *objectivism*), 74 (*openness* vs. *mystery*), and 75 (*control* vs. *progress*). There is a tendency for fifth graders in their learning activities to value "process", "effort", "exploration", "objectivism", "openness" and "progress"; in contrast, ninth graders tend to value "product", "ability", "exposition", "rationalism", "mystery" and "control." There are no significant differences between G5 and G9 for items 67 (*wellbeing* vs. *hardship*), 69 (*application* vs. *computation*), 70 (*facts* vs. *ideas*), or 72 (*recalling* vs. *creating*). In particular, since the modes of items 69, 70, and 75 in total data were nearly 0, it would mean that both phrases (complementary values) are almost equally important to them. Although further investigation is required concerning the interrelationship between the 10 items in section B, these results imply that different mathematics classroom cultures exist in elementary and junior high schools.

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STUDENTS' SELF-ASSESSMENT OF CREATIVITY: BENEFITS AND LIMITATIONS

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In this paper, we describe the process of students' self-assessment of their creativity and its development in the context of posing mathematical problems, presuming that such a process would support the development of their creativity. Examination of two case studies reveals that self-assessment of creativity may support its development provided that one possesses specific personal recourses; however, this process might suppress the creativity of those lacking the needed resources. Therefore, we suggest that self-assessment of creativity cannot stand on its own, and should be supplemented by teachers' feedback or other environmental 'scaffolding'.

INTRODUCTION

Reviewing over 90 articles with the word "creativity" in the title, Plucker, Beghetto, and Dow (2004) found that only 38% of them explicitly defined what creativity was. Clearly, a lack of agreed-upon definition of creativity makes it difficult to reach any consensus about how to assess creative expressions or creative personality, as well as make a decision about the appropriate design of learning environments intended to realize students' creative potential. Nonetheless, the extensive literature concern with creativity recognizes the importance of nurturing students' creativity and the central role of schools in this regard (e.g. Sternberg & Lubatrt, 1995). Therefore, in order to be able to actualize intentions of nurturing students' creativity, one should first select a preferred definition and approach that would enable to translate ideas about creativity into practice. Thinking over how to design a learning environment that support the development of students' mathematical creativity, we held in mind teachers' difficulties in assessing students' creativity and its development (Shriki, 2010). We found Torrance's (1974) psychometric approach to creativity as appropriate to that end, since it permits 'measuring' creativity using quantitative instruments. In addition, drawing on research that relates to the benefits of students' self-assessment of creativity (e.g. Chamberlin & Moon, 2005), we presumed that allowing students to self-assess the level of creativity expressed in their outcomes might contribute to its development.

In this paper, we present partial results from a study that aimed at examining the effect of students' self-assessment of their mathematical creativity and its development in the context of problem posing on the actual development of their creativity.

LITERATURE BACKGROUND

Approaches to the study of creativity

Various approaches are implemented for studying and assessing creativity. Among them, the **psychometric approach** of Guilford (1967), who distinguished between convergent and divergent thinking, and his follower Torrance (1974), whose battery of tests are still being widely used today; The **cognitive approach** for the study of creativity focuses on cognitive and mental processes, among them the use of different representations, establishment of mental links among ostensibly unrelated objects, solving problems of various fields, and more (Sternberg & Davidson, 2005); The **social-personality approach** refers to emotional as well as socio-cultural aspects. The Investment Theory of creativity (Sternberg & Lubart, 1995) is rooted in this approach. The theory seeks to understand the foundation of creativity, and assumes that creative performance results from a confluence of 6 interrelated resources: Cognitive resources – *Intellectual skills* (synthetic, analytic, and practical), *Knowledge* (which might help or hinder creativity), and *Thinking Styles* (preferred ways of using one's skills); Affective resources – *Thinking Styles* (as before), *Personality* (willingness to overcome obstacles, to take risks, to tolerate ambiguity, and self-efficacy), and *Motivation* (as related to task); and Environmental resources (whether or not support and reward creative ideas). "*In order for these resources to be used effectively, they must converge in a way that capitalizes upon them both singly and in interaction*" (p. 4). In some cases, 'strong' resources may compensate for 'weak' resources.

Self-assessment (SA) of creativity

Students' SA of their learning supports the development of their confidence and individuality and adds reflection and metacognition to learning of mathematics (NCTM, 2000). Therefore, students should be engaged in SA of their progress, and use it for articulating the value of their own study (Brookhart, Andolina, Zuza & Furman, 2004). In order to enable students to assess their performance and progress, teachers should provide them with explicit, easy to understand, guidelines (Enz & Serafini, 1995), and a proper support (Brookhart et al., 2004). Students who are trained in SA outperform those who do not receive such preparation (Enz & Serafini, 1995).

SA of creativity is one of the simplest ways to assess creativity (Kaufman, Plucker & Russell, 2012), which in itself requires creativity and enables to refine one's own products in successive iterations. One approach to SA of creativity is asking people to rate their creative accomplishments or ability (Beghetto, Kaufman & Baxter, 2011). The authors acknowledge that SA of creativity may seem as not reliable as it might not be compatible with estimates of external judges, therefore they suggest measuring it by "creative self-efficacy" (CSE), as it represent not only a subjective appraisal of specific creative ability, but also linked to actual creative behaviour or one's perceived ability to accomplish particular behaviours and tasks. The authors found that on average students tended to rate their creative ability similar to how their teachers rated their creative expression.

THE FRAMEWORK OF THE STUDY

Background

Recognizing the value of students' SA of their own creativity, and realizing that students are able to assess the extent of their creativity, we incorporate a component of SA of creativity as part of our wider educational effort to foster students' mathematical creativity. In Beghetto et al.'s (2011) study, students' CSE was measured by their rank of 5 Likert-type items. As the authors pointed out that some students tended to underestimate their creative ability, it seemed to us that an approach that is based merely on self-perception, without relating to some specific 'evident', may obstruct some students from reliably self-assessing their creativity. Therefore, we presumed that providing students with something "to hold on" while self-assessing their creativity may support their ability to assess it and may also contribute to its development. This presumption was put to the test in the current study.

Assessing students' creativity in the context of problem posing

Acknowledging the centrality of problem posing (PP) processes within the mathematical creative act (Silver, 1997), the current study was part of an experiment in which 6 upper-elementary mathematics teachers engaged their students in a series of 5 PP tasks, aimed at nurturing students' creativity. Given an initial mathematical problem, students were asked to pose as many appropriate problems as possible through employing the "What-if-Not?" strategy suggested by Brown & Walter (1990). Relating to the posed problems, students' creativity and its development were assessed through an instrument developed by Shriki (2013). This instrument considers 4 measurable aspects of creativity: fluency, flexibility, originality and organization, as proposed by Torrance (1974). Fluency is measured by the number of different posed problems; flexibility is measured by the number of different categories of the posed problems; originality is measured by the relative infrequency of the problems, and organization is measured by the number of problems stated as generalizations. The scores given to each aspect are of two types: (1) Total scores: the absolute number of posed problems with respect to each of the 4 aspects, and a final score of creativity (based on predetermined relative weights of each aspect); (2) relative scores: each absolute number is transformed into a number that reflects the relative infrequency of the posed problems in student's reference group. For instance, a student who poses 10 problems receives a total score of 10 for fluency. Suppose that the highest total score for fluency in this student's reference group is 15, the student's relative score for fluency is $10/15 \approx 67$. Providing relative scores rests on the notion that developing students' creativity requires assessment of each student compared to his/her reference group, rather than comparing him/her with professionals in the field (Leikin, 2009).

The SA process

The first task was presented only after the students were informed about the meaning of the 4 measurable aspects of creativity and absolute and relative scores. This was followed by several examples of employing the "What-if-Not?" strategy and clarifying the significance of "an appropriate problem". After every task, each student received a graphical display of his/her relative scores. Starting from the second task, the graphical display included cumulative scores, so that students were able to trace their progress/withdrawal relatively to their classmates. Examining the graphical display of scores, students were asked to reflect on modifications in their relative scores, and try to explain evident changes. As we aimed at examining the feasibility of implementing such a process without any external intervention, the teachers were instructed not to discuss with their students anything that relates to it until its completion.

THE STUDY

The study followed students' perceptions regarding the contribution of the described SA process to the development of their mathematical creativity.

Research questions

Relating to the SA process the following questions were addressed: How do students perceive the contribution of SA to the development of (i) their ability to assess their mathematical creativity; (ii) their self-efficacy as posers of mathematical problems; (iii) their development of mathematical creativity.

Subjects

The study involved 190 students from 6 different regular upper-elementary schools in the northern part of Israel: Two 9th grade classes (high level group of 32 students, low level group of 29 students); one 10th grade class (medium level group of 34 students); two 11th grade classes (high level group of 31 students, medium level group of 36 students); and one 12th grade class (medium level group of 28 students).

Research tools

Data was gathered through weekly questionnaires. Subsequent to each PP task the teachers provided the students with an individual graphical display of their relative scores (cumulative scores starting from the second task) and a questionnaire that included 3 open questions: Observing the graphical display, what can you tell about: (i) your creativity with respect to posing mathematical problems?; (ii) your ability to pose mathematical problems?; (iii) your development of mathematical creativity. Try to explain evident changes or lack of changes.

Methods for analysing the data

Students' responses were analysed by means of analytical induction, aiming to identify the main themes and the typical patterns This process was done through open coding and content analysis in order to form the unifying categories and sub-categories (Strauss & Corbin, 1990).

RESULTS AND DISCUSSION

In this section, we present two case studies, both taken from the medium level group of 11th graders. Given space limitations, to avoid the effects of age group, level of study, and achievements in mathematics, we chose to present examples of students from the same class with a similar average grade in mathematics. The two students, Ruth and Michael, represent two types of students: While Ruth's relative scores consistently increased, Michael's relative scores remained almost unchanged. Ruth's and Michael's average grades in mathematics were 86 and 82, respectively. In Figure 1 appear Ruth's and Michael's cumulative relative scores of fluency, flexibility, originality, organization and creativity for the 5 tasks (t1-t5). The score for creativity was calculated so that each of the four aspects was given an equal weight.

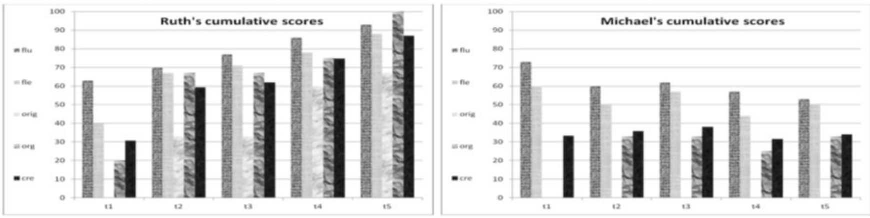

Figure 1: Ruth's and Michael's cumulative scores for task 1 (t1) to task 5 (t5)

Unlike Chamberlin & Moon's (2005) observation, Figure 1 indicates that SA of creativity has a diverse effect on different students. Hence, the question is what are the factors, combined with the SA process, that affect the development of students' creativity? In order to answer this question, we analysed Ruth's and Michael's responses to the questionnaires through the lens of Sternberg & Lubart's (1995) investment theory of creativity. We found the theory as suitable for this purpose since it seeks to understand the interrelations between person and product. Out of the 6 resources, we excluded "knowledge", as both students attended at the same class and their grades were not significantly different. We start with presenting some quotes taken from Ruth's and Michael's responses. Next to each quote appear the number of task (t1-t5) and the number of the questionnaire's question (i-iii) to which it relates. The citations were translated from Hebrew, and we strived to preserve their essence.

Citations taken from Ruth's responses

> "My scores were very disappointing, but I can only blame myself for not giving it enough time…I will surely work harder on the next task" (t1, ii); "I wasn't satisfied with my scores for originality. I think that more than other scores this truly reflects creativity. So I promised myself to think 'big' next time" (t2, i); "I can see that my efforts paid off in all but originality. I think of myself as a creative person, so it's a bit annoying, but I'm not giving up" (t3, iii); "This time I changed my tactic, and it worked! I thought that if I would pose more problems, then I'll increase the chance of being original" (t4, ii); "These tasks truly gave me a chance to think differently. At first, I was afraid to think too wild, because the teacher said that the problem should be appropriate. But when I saw my scores for the three tasks I realized that if I would limit myself to simple problems I will not go far. So I really tried to think of original and generalized problems…and as you can see [the teacher], I am one of the most creative students in the class! Yeeeh!!!" (t5, iii).

Citations taken from Michael's responses

> "I tried to think of many types of problems, and I thought it would be enough. But then I saw my score of creativity, and realized it wasn't enough…O.K., so I am not very original and creative, what does it say about me?" (t1, i); "I tried to prove to you [the teacher[that I can be original, but now I know I'm not…Actually, instead of getting better, I'm getting worse" (t2, iii); "It is the same as before. Perhaps I just don't know how to pose problems. We never did it in class…I'm starting not to like these tasks" (t3, i); "It is not hard to see that other students are much more creative than me, so I give up" (t4, iii); "I understand that what we did was some kind of an experiment, but you [the teacher] probably had to explain it better, or tell me what I was doing wrong. If you had asked me a month ago if I could pose mathematical problems, I would definitely say "yes", but it turned out I'm not very good at it" (t5, ii).

Figure 1 indicates that Michael's starting point was slightly better than Ruth's, and both scores of total creativity were rather low. However, while Ruth exhibited an impressive progress Michael did not. What are the prominent differences between the reactions of both students and how can the investment theory explain them?

Summarizing his extensive research within the area of investment theory, Sternberg (2009) describes the **intellectual skills resource** as relating, among others, to the ability to escape the bounds of conventional thinking, recognize which of one's ideas worth pursuing, and willingness to devote time to think in new ways. Evidently, Ruth's behaviour meets these skills, and she also takes a responsibility for her achievements. At the outset, she realized the need to spend more time working on the tasks in order to achieve the goal she set for herself: improve her relative scores, especially the score of originality (t1, t2). This objective was set following her view of originality as the essence of creativity (t2), and the fact she considered herself a creative person (t3). Through the entire process, Ruth's thinking styles and her personality characteristics facilitated her efforts to improve. **Thinking styles resources** relate to the preferred ways of using one's skills, namely, decisions about how to organize the available skills, and **Personality resources** relate to the willingness to overcome obstacles and take risks, and self-efficacy. These resources are essential for creative functioning (Sternberg, 2009). Apparently, Ruth is able to monitor her actions. Apart from the decision to spend more time, Ruth realized she needed to change her strategy, for example- pose more problems (t4), 'think big' (t3), and think of 'wild problems' (t5). Her self-efficacy as a creative person, combined with her willingness to overcome obstacles and take risks, proved to be what she called 'pay off'. Obviously, the above resources might not be adequate, if Ruth had no motivation. Intrinsic, task-focused **motivation** is essential for creativity. However, motivation is not something inherent in a person, and one decides to be motivated given a certain incentives (Sternberg, 2009). Ruth's motivation to improve was first and foremost intrinsic, driven by her wish to confirm to herself that she was as creative as she believed (t2-t5).

Michael, on the other hand, demonstrated a different pattern of resource exploitation. While, at the first task, he used his **intellectual skills resource** for thinking of various types of problems, believing this was the right approach (t1), realizing it was insufficient for receiving high scores he did not make any attempt to escape the bounds of his conventional thinking, or altered the use of his **thinking styles resource**. In Michael's responses, there is no evidence for pondering, but rather a message of a 'quick waiver' (t1, t3). Evidently, Michael lacks the **Personality resources** that would enable him to overcome obstacles and take risks, and his low self-efficacy is manifested already after t1. Michael's responses (t1-t4) may indicate that he perceived the process as some kind of a 'competition' among his classmates, and it might be that being uncompetitive hindered the development of his creativity, damaged his self-efficacy and suppressed his intrinsic **motivation**. In fact, it appears that Michael's main motivation was to prove something to his teacher, rather than an intrinsic one (t2). As a result, he tended to 'blame' his teacher for not providing him appropriate conditions to succeed (t3, t5). As Sternberg (2009) pointed out, some people may have all the needed internal resources to think creatively, however, if they do not get support from the **environment**, or alternatively – receive negative feedback on their creative thinking, they will find it difficult to demonstrate their creativity. Such people actually decide not to face environmental challenges, thereby blocking their creative output. Unlike Ruth, who is able to resolve her conflicts by herself, Michael needs his teacher's guidance and support.

In summary, the two case studies described above suggest that SA of creativity can be beneficial for students who possess an optimal mixture of resources from the outset. Such students are able to exploit SA of creativity to further develop it. On the other hand, students who lack a certain degree of threshold for some resources might be damaged from the process. It is possible that Michael could realize his creative potential if he had received a personal feedback from his teacher.

Obviously, as we presented only two examples this observation cannot be conclusive, although it implies the impossibility to say that 'one size fits all'. In addition, we based our conclusions merely on one specific approach of SA and one specific method to nurture and assess students' mathematics creativity. However, our initial findings call for the need to a wide scale examination of the advantages and disadvantages of SA of creativity.

Finally, it would be interesting to ask students to respond Beghetto et al.'s (2011) questionnaire prior to engaging them with the described SA process in order identify some interconnections between students' CSE and the impact of SA of creativity on its development.

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PROSPECTIVE TEACHERS' CONCEPTIONS ABOUT A PLANE ANGLE AND THE CONTEXT DEPENDENCY OF THE CONCEPTIONS

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In our study, we looked for an answer to the research question "How prospective teachers comprehend the concept of a plane angle and what kind of variation there is between the conceptions? The study shows that the prospective teachers interpret the concept "plane angle" which in principle is familiar to everybody in many different ways. In this paper, we categorise eight interpretations of a different type, appearing in the future teachers' ideas. We also show some examples of the fact that individuals can use different concept images of a plane angle when performing the tasks of different types even in the same test.

INTRODUCTION

The concept of a plane angle has been in the course of centuries a concept which even the mathematical science community has found hard to define and hard to approach from one single point of view (Matos, 1990; Keiser, 2004). The following three modes of definition have been the ones most frequently applied as the definition of an angle at different times. An angle is defined either (1) as a rotation by which one of two intersecting straight lines is made to merge into the other or (2) as a region defined by two half lines starting from the same point or (3) as the common region defined by two intersecting half planes (Mitchelmore & White, 2000). In fact, the Latin word angulus means literally "a little bending." According to these definition alternatives, an angle can be understood to be either a measurable quantity, a geometric construction or a plane region. Keiser's studies (Keiser, 2004; Keiser et al., 2003), in particular, show that these different interpretations of the concept of an angle reflect the differences discovered in didactic research in the interpretations of individuals to this concept rather well. Our study intends to acquire information on the ways in which prospective class teachers and prospective subject teachers of mathematics grasp and make sense of the concept of a plane angle. We used two different task types in our study. As Tall and Vinner (1981) and after that many others have noted our understanding about concepts is based both to the more holistic and visual concept image and to the more formal concept definition. In our study, we tried to get information of both of these modes of understanding regarding the concept of a plane angle. The study revealed that in different contexts the prospective teachers seem to base their decisions on different concept images although tasks focus on the same concept.

RESEARCH QUESTIONS

The main research question of our study is *how prospective teachers comprehend the concept "plane angle" and what kind of variation there is between the conceptions?*

Primarily, we were here interested in the variation of the conceptions between individuals. However, as a secondary task we also wanted to look as whether student's answers to the definition task and to the point selection task both reflected similar conception of a plane angle or did the answers as well show that individuals may apply different concept images to the same concepts when performing different processes.

METHOD

The way in which an individual grasps a mathematical concept can be examined either by observing how the individual uses the concept spontaneously in speech and action without actually being aware of the observation, or by planning a test situation in which the informant is asked to do something that reveals as much as possible of the way in which this individual interprets the meaning of the concept. In our earlier case study (Joutsenlahti & Silfverberg, 2007) we used the former method of collecting research data from schoolchildren, whereas in Silfverberg and Joutsenlahti (2007) and in the present study the latter method was used for the purpose of examining teacher students' interpretations of an angle.

An analysis of the definitions given by the informants can be done in several ways revealing different aspects the understanding about the concept image and concept definition (Tall & Vinner, 1981). For instance, we can focus on checking (1) how correctly a definition defines the concept in comparison with the normative interpretation; (2) how adequately the form of a given definition meets the formal criteria set for a mathematical definition (Hershkowitz, 1990; Leikin & Winicki-Landman, 2000a, 2000b; de Villiers, 1995); (3) how well a definition given by an informant corresponds with the concept form which this informant seems to have on the basis of the situations in which the concept was actually applied (Tall & Vinner, 1981; Vinner & Dreyfus, 1989; Vinner, 1991); (4) what kind of linguistic form the informant uses in providing a definition (Barnbrook, 2002).

The research data was collected by a questionnaire handed out to 191 Finnish university students. Hundred (100) of them were pursuing studies to become subject teachers in mathematics and science for grades 7 through 12, and 91 to become class teachers for grades 1 through 6. In item 1 of the questionnaire we just asked the prospective teachers to define the concept of a plane angle. In item 2 (Figure 1) the students were asked to choose from a given set of points the ones, which they thought to belong to the given angle. In item 3 (Figure 2) the students picked from the given set of points the ones, which they thought belonged to the angle of the triangle added into the same figure that was used in item 2. Items 2 and 3 were developed from the testing method presented by Hershkowitz et al. (1987). The item where it was asked students to write a definition for a plane angle was given as a first item on a questionnaire but it was on the same paper as the other two items so the answerer could answer the questions in any order and easily correct the definition after answering more concrete items 2 and 3 if she/he felt it necessary.

Which of the points

A, B, C, D, E, F, G, H, and I belong to the angle α ?

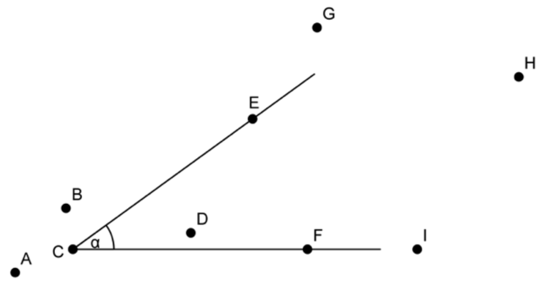


Figure 1: Item 2 of the questionnaire

Which of the points A, B, C, D, E, F, G, H, and I belong to the angle α in a triangle CFE ?

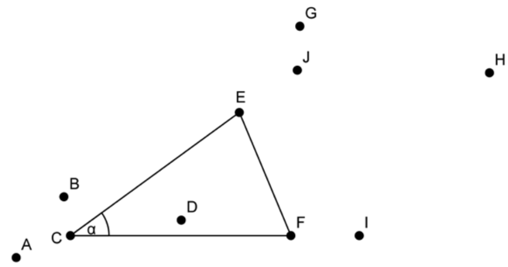


Figure 2: Item 3 of the questionnaire

RESULTS

Based on the earlier research literature we could expect that especially three particular classes of interpretations to the plane angle would be found in our data, namely an angle interpreted as (1) an amount of turning about a point between two lines; (2) a shape, formed by two lines or rays diverging from a common point (the vertex); (3) one of the two regions into which the two sides of the angle (rays) split the plane.

In the following, we call them as turning interpretation (TI), line interpretation (LI) and region interpretation (RI). However, our data revealed that we have to both broaden and specify the range of these three possibilities how the concept of a plane angle can be interpreted. First there were few student teachers, who interpreted an angle to be limited only to a 'corner' of an angle (corner interpretation CI) consisting only the vertex of an angle or/and its 'close surrounding'. There were also different interpretations of LI and RI depending if the angle was considered including only those elements visible in the actual drawn picture of the angle or if the angle was considered continuing endlessly in a direction it specifies. Finally, we classified interpretations into eight categories, namely to TI and CI and to six categories shown in Figure 3.

Point selection task (Item 2)

We will begin by presenting first a summary of student's answers to the point selection task (item 2). In the whole data (n =191) there were two prospective teachers who thought that only the vertex C from the given points belonged to the angle α reflecting the interpretation CI and two prospective teachers who didn't select any of the points to the angle α probably because they interpreted a plane angle in item 1 to be a measure (TI) and not a kind of geometrical line or region construction. Table 1 presents the sub-categorisation we applied to LI and RI in our study. Correspondingly Table 2 shows how the students' answers were distributed into these sub-categories.

Point selection task (Item 3)

When the angle α was placed as an angle of the triangle it did not essentially seem to affect the fact whether an angle was addressed according to the line interpretation or the area interpretation. In the whole data, 63.8 % of the student teachers chose the points in item 2 according to the line interpretation and 27.2 % according to the area interpretation. In the item 3, the corresponding percentages were 64.9 % and 32.5 %. However, as it was considered an angle of the triangle student teachers chose generally only the inner points of the triangle or the points belonging to the sides of the triangle. When 67.5 % of the students chose in the item 2 also the points of the angle belonging outside the part of the drawn angle, in item 3 only 11.5 % of the students did the same.

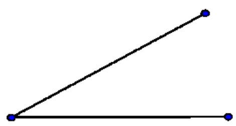
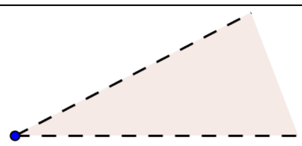
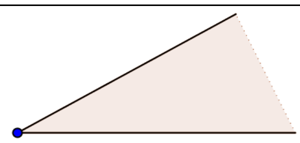
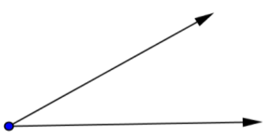
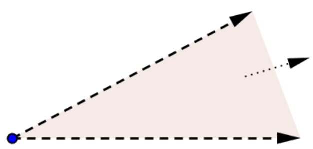
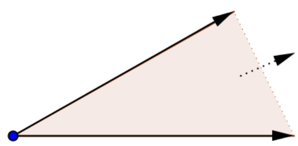
The type of interpretation	The line construction of two line segments or two rays	The region construction with boundaries excluded or	included
Bounded by the drawing (visible)	 CEF	 D	 CDEF
Continuing to the infinity (imaginary)	 CEFI	 DH	 CDFHI

Table 1: Classification of the line and region interpretations to the concept of a plane angle. The combinations of capital letters next to the pictures refer to the corresponding selection of the points in Figure 1.

The type of interpretation: Bounded or infinite/line or region construction	The line construction of two line segments or two rays		The region construction with boundaries excluded		The region construction with boundaries included		
	Class teachers (n=91)	Subject teachers (n=100)	Class teachers (n=91)	Subject teachers (n=100)	Class teachers (n=91)	Subject teachers (n=100)	Total
Bounded by the drawing (visible)	23	33	0	0	3	1	60
	25.3 %	33.0 %	0.0 %	0.0 %	3.3 %	1.0 %	31.4 %
Continuing to the infinity (imaginary)	32	34	1	0	21	26	114
	35.2 %	34.0 %	1.1 %	0.0 %	23.1 %	26.0 %	59.7
Total	55	67	1	0	24	27	174
	60.4 %	67.0 %	1.1 %	0.0 %	26.4 %	27.0 %	91.1 %

Table 2: Distribution of students' responses to item 2 (n=191).

The definition task (Item 1)

The writing of the definition to the concept of the plane angle proved to be a difficult task to many prospective students and especially to the prospective class teachers. Some students left the item 1 totally unanswered. Because of this, we restrict the examination here to the data concerning only prospective subject teachers (n=80). About half of the respondents had attempts to write the description in the form of the definition, such as "An angle is formed by…", "When two lines intersect, …", An angle is a relation between.." etc. Because the answers to this item were so vague we do not here give precise numbers of the occurrence of different types of definitions. Instead of that, we present some general observations from to what kind of ideas and concept images of the angle the attempts of defining an angle concept seemed to be based.

In the point selection task there were very few such students who restricted the angle concept so that only the vertex C would belong to the angle. However, in the defining task (item 1) remarkable many students seemed to have a kind of vertex or "sharp point interpretation" of a plane angle as the following examples show "Point of convergence formed by two line segments", The common point of two lines which forms an acute or an obtuse angle", "An angle is an acute or an obtuse point in a solid" etc. An interesting observation as well was that roughly estimated every fourth of the respondents who gave a written definition in item 1 described it so that it did not seem to base on the same concept image as the point selections in item 2 would reflect. In the following, we will present five examples of the inconsistencies between the definitions students' wrote as an answer to item 1 and the selection of points they made in item 2.

Example 1. A student teacher wrote a definition “Two straight lines intersect each other. An angle stays between the lines”. However, in item 2 the student considered that the points C, E and F only belonged to the angle α and did not choose the points D, H and I. The selection seems to base on the concept image “a combination of line segments intersecting each other” instead of that what was written in the definition.

Example 2. Another student teacher defined an angle as “the region between two straight lines which have a common point” but chose only the points C, D, E and F which corresponds the interpretation “the finite region bounded by two line segments including the boundaries”.

Example 3. The definition given by a student teacher was “An angle consists of two line segments and of their common intersection or of the common point from which the sector opens”. However, she chose only the points C, E, F and I seeming to be applying a concept image “combination of two rays starting from the same point”.

Example 4. The definition which a student teacher wrote was “An angle consists of two line segments and of their common intersection point from which the sector opens”. In item 2, he chose the points: C, E, F and I seeming to apply more like a conception “Angle is formed by two rays starting from the same point”

Example 5. After writing the definition “Two straight lines meet each other at one single point” the prospective teacher chose in item 2 the points D and H reflecting a concept image “An angle is one of the infinite regions between straight lines without boundaries”.

One possible explanation for the fact that the concept images applied in items 1 and 2 do not correspond to each other can be the fact that respondents use the concept straight line when they actually mean the concept line segment or vice versa. But these linguistic inaccuracies do not explain all the incompatibilities of the concept images as one can see also from the examples above. It is important to notice from the point of view of the theory that the concept images which some individuals used seemed to be at least in some extent dependent also on the context where they was taken in use.

DISCUSSION

To summarise the results of our study revealed fairly clearly that prospective teachers interpret the concept of an angle by several ways. Some respondents interpreted an angle as a line consisting of two line segments, some consisting of two rays, and some as a region defined by these elements. On the other hand, interpretations differed as to whether an angle continues outside the part shown in the drawing in the direction determined by the angle, or not. The results of our examination showed that even the adults who have completed their years of mathematics studies at lower and upper secondary school – and many of whom have also pursued the studies of mathematics at university level – still cherish various notions (beliefs) on such basic concepts of elementary mathematics as an angle, and these different notions and beliefs can remain very much alive although we use concepts in our mutual discussions regularly.

Possibly, this fact can partly be explained by school mathematics learning practices. The nature of the exercises typical of school mathematics, like calculate …, draw …, classify …, define the magnitude of … etc., seem to allow communication on the issues to be examined as well as the completion of the exercises even though the basic concepts are understood in ways that are fundamentally different.

In our view, mathematical concept formation could be enhanced by deliberately drawing attention to the differences of the interpretations learners may have even of the basic concepts of mathematics and by critically debating and negotiating the various ways of interpretation in line with the socio-constructivist learning theory. Our research also showed fairly clearly that few Finnish prospective teachers were not at all able to put their idea of the concept definition of the concept plane angle into words. It seems that neither school nor university studies ensure that students are familiar with the idea of mathematical definition or with the requirements of the form and formulation of such a definition. It seems highly likely that the significance of learning to formulate a definition – i.e. being forced to analyse the content and meaning of a concept and to search for an explicit and easily comprehensible way of expressing this meaning – is not held in adequate esteem on any level of school education.

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AN EMERGING THEORY FOR DESIGN OF MATHEMATICAL TASK SEQUENCES: PROMOTING REFLECTIVE ABSTRACTION OF MATHEMATICAL CONCEPTS[1]

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This paper describes an emerging approach to the design of task sequences to promote reflective abstraction. The approach aims at promoting particular mathematical understandings. Central to this approach is the identification of available student activities from which students can abstract the intended ideas. The approach differs from a problem-solving approach. The paper illustrates the approach through data from a teaching experiment on learning of fraction concepts with fourth and fifth graders.

INTRODUCTION

Mathematical tasks are designed for a variety of reasons. The instructional design approach discussed here has the specific focus of promoting particular changes in students' conceptual understanding and, as such, offers an approach to addressing difficult to learn concepts and to working with students who are struggling to learn specific concepts. This task design approach does *not* address other important areas of learning mathematics, particularly the important area of mathematical problem solving. Therefore, the approach is meant to complement existing approaches, not replace them. The emerging task design theory is a product of a research program, Learning Through Activity (LTA, Simon et al, 2010; Simon, 2013), aimed at understanding conceptual learning, particularly the development of abstraction from one's own mathematical activity (activity that occurs in the context of designed sequences of mathematical tasks). Thus our research program involves a spiral approach in which we design task sequences to study learning through student activity, and we use what we come to understand about learning to improve our understanding of task design, and so forth.

What is unique about our instructional approach is that it involves students actively in the developing of new concepts, yet it does *not* depend on the uncertain breakthroughs required in authentic problem solving lessons. Task sequences are designed to elicit the specific activity that will lead to the new conceptualization. I discuss an example of this below.

[1] This research was supported by the National Science Foundation (DRL-1020154). The opinions expressed do not necessarily reflect the views of the foundation.

THEORETICAL BASIS

The theoretical basis of our research program derives from Piaget's (2001) work on reflective abstraction. DiSessa and Cobb (2004) pointed out,

> Piaget's theory is powerful and continues to be an important source of insight. However, it was not developed with the intention of informing design and is inadequate, by itself, to do so deeply and effectively. (p. 81)

Our research program is aimed at building theory that can inform instructional design.

Three characteristics of reflective abstraction are foundational to our work. First, reflective abstraction is not abstraction of properties of objects, but rather abstraction based on the learner's activity and results in a learned anticipation. Second, "activity" refers to *goal-directed* activity, which includes both physical and mental activity. The notion of goal-directed is important, because the learners' goals partially determine both what knowledge they call upon and what they pay attention to and can notice. Third, Piaget (2001) described reflective abstraction as a coordination of actions. We understand the coordination of actions in the following way. Each action is called upon for a particular purpose (i.e., with anticipation of its results). Thus, the actions called upon are each part of existing schemes (the result of prior reflective abstractions). Thus the coordination of actions really is a coordination of schemes. Thinking about coordination of actions as coordination of schemes allows a way of understanding how new knowledge to be constructed from prior knowledge. In our research and theoretical work, consistent with Piaget and others, (c.f., Hershkowitz, Schwarz, & Dreyfus, 2001; Mitchelmore & White, 2008), we consider mathematics conceptual learning as the process of developing new and more powerful abstractions, specifically reflective abstractions.

Our task design approach builds on this theoretical base and involves specifying hypothetical learning trajectories (Simon, 1995) at multiple levels. In this paper, I focus on the level of design for the learning of particular mathematical understandings, not the planning of trajectories for larger mathematical topics. A hypothetical learning trajectory consists of three components (Simon, 1995), (1) a learning goal, (2) a set of mathematical tasks, and (3) a hypothesized learning process. Whereas the specification of the learning goal generally precedes the specification of the tasks and hypothesized learning process, these latter two components necessarily co-emerge. The learning process is at least partially determined by the tasks used and the tasks used must reflect conjectures about the possible learning processes. The design approach outlined here provides a conceptualization of the design process with respect to these two components.

The Design Approach

The first two steps in our design approach are the first two steps in most instructional design that is aimed at conceptual learning. We assess student understanding and articulate a learning goal[2] for the students relative to their current knowledge. It is after these first two steps that our approach diverges.

Our third step is to specify an *activity that students can call on* that can be the basis for the abstraction specified in the learning goal.[3] The consideration of what activity we might elicit begins in a way that is similar to Realistic Mathematics Education (Gravemeijer, 1994), that is a consideration of students' informal strategies. Whereas RME focuses on developing progressively more formal solution strategies, our approach is focused on developing concepts by developing anticipations from those activities.[4] The fourth step is to complete the hypothetical learning trajectory, that is, to design a task sequence and related hypothesized learning process. The task sequence must both elicit the intended student activity and lead to the intended anticipation on the part of the students. The hypothesized learning process must account for not only the overt activity of the students, but also the mental activities that are expected to accompany those overt activities. I will not focus on steps beyond step 4 (e.g., symbolizing, introducing vocabulary, discussing justification), because again they are common to many approaches.

I will now use an example from our current project that focuses on the learning of fraction concepts. Kylie was a fourth grade student (9–10-years old) with whom we worked in a one-on-one teaching experiment. We were using the computer application JavaBars (Biddlecomb & Olive, 2000). In Java Bars, quantities can be represented by rectangles of different lengths. The bars can be partitioned and bars and parts of bars can be iterated. In the example, Kylie is developing a concept of recursive partitioning, the understanding of the size of a particular part of a part.

Task 1: "This is one-third of a unit [pointing to a rectangular bar on the screen], make one-sixth of a unit."

Kylie made clear that the only way she knew how to do the task was by first making the unit. She did not know how to just "cut up" the bar on the screen. She made the whole by iterating the third three times and then cut the first third in half. She indicated that one of the small pieces was one-sixth. Her explanation indicated that she was thinking about the number of subparts that would be created if she subdivided each of the three thirds (i.e., mentally iterating the two subparts three times).

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[2] Articulation of conceptual learning goals is a problematic issue not covered here. It is a theoretical and empirical challenge to specify learning goals in a way and level of specificity that adequately guides instructional design (as well as instruction and assessment).

[3] The learning goal is a researcher/educator construct. There is no assumption that the students, after a successful lesson, will have an identical understanding to that of the instructor. Rather, formative and summative assessment will reveal whether students have a compatible understanding.

[4] Although there are often overlaps in what is learned by students using these two approaches, I emphasize here the differences in the primary aim and the theory built to achieve that aim. We definitely build on aspects of RME, particularly their use of *model of becoming model for* (Gravemeijer, 1994).

Task 2: "This is one-fifth of a unit, make one-tenth of a unit."

Again, Kylie iterated the part to make the whole and then subdivided one of the parts, "Here, you have one-tenth of a unit."

After working three tasks in this way, she spontaneously showed a change in the following task.

Task 3: "This is one-third of a unit, make one-ninth of a unit."

This time Kylie *immediately* divided the third bar into three pieces (without iterating to make the whole).

K: One of those is one-ninth.

R: How do you know?

K: Because, um. How many times does three go into nine? ... Three times. And it's one third! So. Three times three is nine, and one of -- if you cut [the thirds] up into thirds again. That is, um. ... And you take one, it would be ... one-third. … But that's really one-ninth of a unit.

Task 4: "This is one-fifth of a unit, make one-twentieth of a unit."

She immediately cut the fifth into four. She went on to complete two more tasks in this way in this session.

In this example, Kylie learned that she could produce 1/mn from 1/n by partitioning 1/n into m parts. She developed an anticipation that partitioning 1/n into m parts creates a fraction of the unit that is n times smaller than the 1/m (the fraction of the part), that is, a subpart that iterates n times more in the unit than it does in the original 1/n part. Let us look more closely at this transition.

Kylie's partitive fraction scheme (Steffe & Olive, 2010), available at the outset of this set of tasks, included an understanding that 1/p is a part that can be iterated p times to make a unit. Initially, this allowed her to iterate the original part, one-third (Task 1), three times to make the whole. She knew she needed to partition each part into two subparts to make sixths, using her multiplication scheme (# items/group x # groups = # items) along with her partitive fraction scheme.

In Task 3, Kylie was no longer employing the sequence of actions she used in the first two tasks. Rather, she had developed a *new* action that was at a higher level than the sequence from which it was built and allowed her *knowledge at once*. The new abstraction moved Kylie from thinking about iterating a composite unit to thinking about an m split in a part, 1/n, as resulting in a subpart that is n times smaller in relation to the unit than it is in relation to the part. In other words, she knew that the part increased by n times the number of times the subpart would iterate to the unit compared to the number of times it iterated to the part. She now had an anticipation about the relationship of a part of a part to the unit.

What we see from this example is that the learning process began with Kylie setting a goal (e.g., to complete the task) and bringing to bear available schemes (actions) to accomplish the goal. Initially, she used these actions in sequential fashion. However, she came to a *coordination of* those actions. As a result of this coordination, she no longer needed to go through the sequence of actions used previously. The result of the coordination was a structure that was at a higher level than the component schemes. (The reader is referred to Simon et al, 2010 for another example of this instructional design approach with a more in-depth analysis of the learning that took place.)

I now return to the instructional design approach which I summarize as follows:

> **Step 1:** Assess relevant student understanding.
>
> **Step 2:** Identify learning goal.
>
> **Step 3:** Specify an activity, which the students can call on, that could be the basis for the intended abstraction.
>
> **Step 4:** Design a sequence of tasks in conjunction with a hypothesized learning process that accounts for how the students will progress from the initial activity to the intended abstraction.

In the example, the task sequence was quite simple. Of course, this did not represent the full treatment of recursive partitioning in our teaching experiment, but it gives us a straightforward example for discussion. The tasks elicited an initial activity (action sequence):

1. Iterate the part to make the whole.
2. Divide the number of needed subparts by the number of parts to find how many subparts will be in each part.
3. Carry out the appropriate subdivision.

This activity afforded the opportunity for Kylie to make an abstraction. The key was that Kylie's abstraction required no leap of insight or problem-solving breakthrough. Further, it required no input from the teacher or other students. The abstraction emerged from Kylie's activity. Kylie was able to do every one of the tasks without assistance. However, an understanding of recursive partitioning emerged in the course of solving the tasks with her available activity. This seemed to indicate that Kylie was able to use her extant knowledge to develop the intended new understanding.

CONCLUSIONS

Our task design approach is an emerging one. I highlight here two of its features that can be seen in the example above. First, the approach provides a strategy for promoting specific mathematical understandings. It contrasts with strategies in which students must solve novel problems to progress (or hear the solutions of more able peers). Although mathematics teaching cannot cause learning, this is an approach that involves engineering task sequences so that participating students predictably make the new abstraction. (Of course, successful understanding of prerequisite concepts is required.) Second, the learning goal is *not* to learn to solve the tasks, as it is in many approaches. The tasks are made to initially elicit activities that the students already are capable of engaging in. Kylie was able to solve all of the tasks prior to making the intended abstraction. Further, she *was not* consciously trying to find an easier way. Her learning was a product of coordination of actions across a sequence of tasks.

Let us examine some of the possible implications of this approach to task design.

1. This approach potentially provides a way to design task sequences for concepts that students tend to not learn well. These are concepts that many students do not spontaneously reinvent in problem solving situations and of which they do not develop deep understanding by being part of a class discussion with more knowledgeable students. The approach focuses the instructional designers on identifying key activities that are likely to afford the intended abstraction.
2. Small group work using task sequences, of the kind discussed here, can lead to somewhat different class discussions. If students are making the new abstraction as a result of their engagement with the task sequence, discussions can focus more on articulation of the new idea, justification, and establishing the idea as taken-as-shared knowledge.
3. The approach has potential to address issues of equity in two ways. First, many students who have conceptual gaps early on seem to never recover. This design approach provides a general methodology for building up the specific experience, based on students' currently available activities, needed to make particular abstractions. Second, if during small group work, students are generally successful in deriving the new abstractions from their activity with the task sequence, a greater number of students will be able to participate in and benefit from the class discussions that follow. The underlying hypothesis here is that students who abstract ideas through their activity, based on their work with the mathematical tasks, tend to learn the concepts in a more powerful way than those who only *follow* the explanation of their more able peers offered in class discussions.

One final point that was discussed briefly at the beginning of this paper is the relationship of our approach with mathematical problem solving. The approach that I have described and exemplified does not focus on students developing their problem solving abilities. Rather it focuses only on the development of mathematical concepts. Developing problem solving abilities is a key part of mathematics education. One could argue that conceptual understanding and problem solving are the two wings of mathematics education – students cannot fly without effective use of the two together. Students can learn concepts through problem solving lessons. Our approach is in no way intended to minimize the importance of lessons in which that is the case. Rather, our approach provides an additional tool that has the potential for success in areas where mathematics education has been less successful. One open question is how to use this tool in conjunction with the powerful tool of problem solving lessons to maximize the learning of students.

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TWO STAGES OF MATHEMATICS CONCEPT LEARNING: ADDITIONAL APPLICATIONS IN ANALYSIS OF STUDENT LEARNING

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Tzur and Simon (2004) postulated two stages of concept development, participatory and anticipatory. The distinction between the two stages was exemplified by what they termed "the next-day phenomenon" in which learners who could solve a task one day in the context of the activity through which they made the abstraction, could not solve what seemed to be the same task in a subsequent lesson when the students were not engaged in or thinking about the activity. Here we expand the application of this theoretical distinction by providing two examples of use of the distinction in analyses of data segments that are different from the next-day phenomenon.

BACKGROUND

As part of a program of research on conceptual learning of mathematics, Tzur and Simon (2004) postulated two stages of development in learning a mathematical concept: participatory and anticipatory. To illustrate the stage distinction, Tzur and Simon (2004) gave the following example of what they called the "next day phenomenon":

> Consider a teacher who engaged learners for a few lessons in partitioning paper strips to create unit fractions. Toward the end of this hands-on activity, the learners were able to answer questions such as, "Which is larger, 1/6 or 1/8?" The teacher required the learners to explain their answers and most learners could clearly demonstrate with their strips and argue that 1/8 must be smaller than 1/6, because the strip showing eighths was cut into more pieces; so each piece had to be smaller. The next day, … the teacher begins the lesson by attempting to review the ideas generated by the learners during the paper-strip activity. The teacher writes two fractions on the board, '1/7' and '1/5,' and asks which one is larger. To the teacher's surprise, most of the learners claim that 1/7 is larger because 7 is larger than 5. The teacher wonders how learners can "lose" overnight what they learned the day before. Intending to revisit the hands-on experience, the teacher asks the learners to take out their paper strips to set up the problem. Soon after the learners begin manipulating the paper strips, and without completing a paper-strip enactment of the problem posed, many learners, who had earlier claimed that 1/7 was larger, raise their hands to explain how they know that 1/5 is larger than 1/7. (p. 288-289)

Tzur and Simon (2004) argued that it is not a case of learners forgetting what they learned the day before, but rather that there are two distinct stages of abstraction that occur as a learner is developing a new mathematical concept. In the first stage, labelled *participatory*, learners develop an anticipation based on engagement in a particular activity. That is, through engaging in the activity, they develop knowledge of a

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mathematical relationship *and* no longer need to carry out the activity to determine the result. Furthermore, the learner can justify and explain the logical necessity of the result. However, at this first (participatory) stage, this anticipation is limited. Learners have not yet learned to call upon the abstraction (anticipation) when they are not involved with or thinking about the activity through which it was learned. If the learners are presented with a seemingly similar task the next day, outside of the context of the activity through which the anticipation was learned, they are not able to call on the relevant (from the observer's perspective) anticipation. However, in the second stage of abstraction, labelled *anticipatory*, the learner is able to call upon the learned anticipation even when not engaged in the activity through which it was learned.

The distinction between these stages of understanding implies that (for the learner) the next day's task was not the same as the prior day's task, which the learners *were* able to solve, even if the tasks were word-for-word the same. The same question posed in the context of the paper-folding activity was not the same as the question posed unconnected to the paper folding activity. Essentially, the first asked for an anticipation of the results of paper folding (partitioning), whereas the second was a more general question about fractions with no hint of how to approach it. The distinction also suggests that we define *task* not as just the written or oral articulation of the task. Rather the task is in part defined by its place in a sequence of tasks and by the tools available or given to the learner with which to work.

OUR RESEARCH PROGRAM

Before we present data to demonstrate the usefulness of the stage distinction in a long-term teaching experiment, we provide some background on the research program and current project from which these examples are drawn.

We have been engaged in a research program, "Learning through Activity (LTA)" (Simon, 2013; Simon et al., 2010) that builds on Piaget's (2001) theoretical construct of reflective abstraction. The aims of the program are to both explain learners' development of mathematical concepts (a detailed examination of reflective abstraction in mathematics learning) and develop principles for promoting mathematical concept learning. As such, it is a research program intended to develop integrated theory on aspects of mathematics learning and pedagogy.

This paper is based on research conducted during the second and third years of the five-year Measurement Approach to Rational Number (MARN) project[1]. The project is focused on two goals: (1) increasing understanding of how learners learn through their mathematical activity (LTA), and (2) understanding how learners can effectively develop fraction and ratio concepts through activities grounded in measurement. In this article, we present data from the second phase of the project (years 2 and 3), five

[1] This work is supported by the National Science Foundation under Grant No. DRL-1020154. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

one-on-one teaching experiments on fraction and ratio concepts. This phase involved developing and implementing task sequences for fraction and ratio learning and modifying those trajectories based on ongoing analyses. More in depth retrospective analyses followed. The data in this paper comes from a one-on-one teaching experiment with Kylie, during her fourth and fifth grade years (fall, 2011 through spring, 2013) in two one-hour sessions per week.

Much of the students work was done in the software environment, JavaBars (Biddlecomb & Olive, 2000). In JavaBars, quantities are represented by rectangles of different lengths. The bars can be partitioned and bars and parts of bars can be iterated.[2] Although the examples do not contain actual use of JavaBars, the conversations refer to that work.

APPLICATIONS OF THE STAGE DISTINCTION IN RETROSPECTIVE ANALYSES

The participatory-anticipatory distinction is a key theoretical construct in our analysis of data. As demonstrated above, it explains the seeming inconsistency of performance from one session to another (the next-day phenomenon). Because of the frequency of data of this type, this alone is an important function of the construct. However it is not just in comparing learner work from different sessions that the construct has proved useful. Examples 3 and 4 below demonstrate the usefulness of the stage distinction for explaining data that do *not* follow the form of the next-day phenomenon.

Example 1

After not working with Kylie for more than 4 months (mid-May to late September), we started our second year of work by doing an assessment. We discuss here two of the questions that emerged from the analysis of the data generated and how the participatory-anticipatory distinction was useful in postulating answers to those questions.

Midway through the assessment, Kylie was given the following tasks in succession:

Task 1.1: This bar is three-sevenths of a unit long. I repeat it one hundred times, how long is my new bar?

Kylie said three seven-hundredths and then changed her answer to three hundred sevenths. When she was asked for justification, she changed her answer to three hundred seven-hundredths. She could not provide justification for any of the calculations.

Task 1.2: This bar is two-fifths of a unit long. I repeat it four times, how long is my new bar?

[2] Frank Iannucci modified JavaBars for us to include an “iterate” button that creates a new bar the specified number of iterations of the original bar.

Kylie once again multiplied both the numerator and denominator by four resulting in eight twentieths. She was not able to justify her solution and did not seem to have confidence in it.

Task 1.3: This bar is one-sixth of a unit long. I repeat it eleven times, how long is my new bar?

S: Eleven-sixths

R: Eleven-sixths?

S: Yeah.

R: Okay, convince me.

S: Well, I repeated it that many...Oh I know what the other one is [referring to the previous task].

R: Yeah, what?

S: It's eight-fifths.

R: Okay, are you sure?

S: Yes!

Task 1.4: This bar is fourth-ninths of a unit long. I repeat it twenty-five times.

R: What's that one?

S: It's uh...I know, four times twenty-five is a hundred-ninths.

In our analysis, we were initially puzzled by the data. Why could she do Task 1.3 correctly, but not 1.1 and 1.2? Why was she able to do Task 1.2 (and 1.4) after solving 1.3, but not before? The explanation we settled on is the following. In order for Kylie to be able to figure out the bar that would be produced by repeating a two-fifths-unit bar four times, she would have to think about the two-fifths- unit bar as being the result of iterating a one-fifth-unit bars two times. She likely was thinking with a part-whole scheme. Kylie had previously demonstrated that she could *make* two-fifths by partitioning a unit into 5 parts, pulling out 1 part and iterating it twice. However, the data seems to show that Kylie did not have an anticipatory-stage conception of a non-unit fraction as the result of an iteration of a unit fraction. That is, she did not think to call on that idea in the context of Tasks 1.1 and 1.2. Whether or not a student can do so is Steffe and Olive's (2010) distinction between a partitive fraction scheme and an iterative fraction scheme. However, when she was asked the result of stringing together unit fractions in Task 1.3, it caused her to engage in the activity of creating a non-unit fraction through iteration. Following that task, she was able to solve Task 1.2 and Task 1.4. These tasks were now participatory tasks; that is, in the context of thinking about iteration of a unit fraction to make a non-unit fraction, she was able to now think about the non-unit fraction (two-fifths) as the result of iterating a unit fraction (one-fifth). That allowed her to solve the tasks involving the iteration of a non-unit fraction (Tasks 1.2 and 1.4). One might say that she was at the participatory stage of an iterative fraction scheme.

Example 2

In this session below, Kylie had been learning concepts of ratio. The tasks were designed to develop an abstraction of the multiplicative relationship between the two quantities and to understand the invariance of that relationship.

Task 2.1: One of the giant's steps is equal to six of Kylie's steps. If the giant walks 84 miles, how far would Kylie go in the same number of steps?

Kylie needed to anticipate that the relationship between the giant's steps and her steps was multiplicative and that she could use this relationship to determine the number of miles she walked, making use of the invariance of that relationship. (Note, the use of two different units of length in the task, steps and miles, with no conversion factor provided, is meant to pre-empt the task being solved with a simple application of a per-one strategy or a build-up strategy).

K: What's eighty-four divided by six?

I: Fourteen

K: I walk fourteen miles.

I: Why did you divide?

K: I know for each step the giant takes, I take six. … Every time the giant walks eighty-four miles, I walk fourteen. It's like one-sixth of eighty-four.

I: How is that related to you and the giant?

K: Not sure. Oh yeah, every time he takes a step, I have to take six, so if I only take one step that's only one-sixth of his step.

In the task above, Kylie shows an anticipation of the invariance of the multiplicative relationship between how far she walks and how far the giant walks. One can think about this anticipation as having two interrelated parts, comparing the two quantities multiplicatively and knowing that that relationship is invariant across any distance travelled.

Later in the session, we gave her a slightly different task for which Kylie did not exhibit the same anticipation.

Task 2.2: Forty-two of Kylie's steps are equal to twenty-one of Marty's steps, If Kylie walks twenty-five miles, how far does Marty walk taking the same number of steps?

K: One hundred miles. Wait! I have to find out how many steps I take when Marty takes one step. [Note: If the researcher had allowed Kylie to proceed in this manner, she would likely have set herself up for a similar solution to the task above. However, his next question is what changed the task for Kylie.]

I: Can you look at these numbers and tell me the answer?

K: Seven? I thought forty-two, six times seven, and forty-two divided by six is seven.

I: And the twenty-one doesn't matter?

K: Ohhh, yes it does.

I: If I tell you twenty-one of my steps is forty-two of your steps, what do you know about our steps?

K: Yours is bigger

I: How much bigger?

K: Twenty-one steps bigger

The data presented were puzzling. How could we account for the difference in Kylie's anticipation in these two situations? The participatory-anticipatory distinction proved to be useful in allowing us to generate a hypothesis. Based on prior analyses of Kylie's learning and the data described here, we made the following inferences. Kylie had had significant experience considering units and partial units, which was the basis for how she developed her fraction concept: a partial unit (unit fraction) is a part that iterates a certain number of times to make a unit. In Task 2.1, the given information that six of Kylie's steps are equivalent to one giant step likely cued the activity of iterating a partial unit to make a unit. Since, for Kylie, the relationship between a partial unit and the unit is a multiplicative relationship, she was able to anticipate the multiplicative relationship between her steps and the giant's steps and then use her understanding of the invariance of this relationship to determine the number of miles she would walk.[3]

In Task 2.2, Kylie was about to convert this task to the form of the previous task (i.e., the number of Kylie's steps in the larger step) in order to solve it in the same way. However, the researcher prevented that approach. As a result, Kylie was not able to mentally iterate one of her steps to make the larger step, so she did not think to use this relationship between partial units and units and thus did not consider the multiplicative relationship between the quantities. Once again, we see evidence of knowledge that is only at the participatory stage. The task as constrained by the researcher's follow-up question was an anticipatory task. Because it did not cue the activity of iterating a partial unit for Kylie, she was unable to use her anticipation that was tied to that activity (i.e., iterating her step to make a giant step). In lieu of thinking about iteration of a partial unit, she thought only about the additive comparison.

In these two examples of using the participatory-anticipatory distinction to explain puzzling data, we have demonstrated its usefulness in situations that are not of the form of the next-day phenomenon. In the next day phenomenon, the same task can be either a participatory task or an anticipatory task, depending on what came before it (i.e., whether the learner is thinking about the key activity). In this last example, it was not the order of the tasks that was crucial, but rather the extent to which the task evoked thinking about the key activity. The participatory-anticipatory distinction structured

[3] Note, we are not offering an explanation of the basis for her anticipation of the multiplicative invariance for two reasons. First, it is not critical for making our intended point here, and second, we do not yet fully understand it.

our examination of the data to focus on what activity might have afforded the anticipation in Task 2.1 and how Task 2.2 might have not afforded the same access to that activity and the related anticipation.

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NUMBER'S SUBTLE TOUCH: EXPANDING FINGER GNOSIS IN THE ERA OF MULTI-TOUCH TECHNOLOGIES

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*In this paper, we explore a richer sense of finger gnosis with respect to three- and four-year-olds' interactions with a novel iPad application (*TouchCounts*), focusing on their responses to an "inverse subitising" task. The direct and tactile nature of their engagement with* TouchCounts *leads to a striking shift from index finger incrementation to deployment of several fingers all-at-once (in a cardinal touch gesture) to achieve a given target number that is then spoken by the iPad. This form of finger gnosis differs from the more ordinally based differentiation of fingers that is discussed in the psychology literature.*

INTRODUCTION

In nascent numeration with very young children, there is a telling ambiguity concerning the status and nature of fingers in relation to counting. This dual role is well captured by the English expressions 'using fingers to count *with*' and 'using fingers to count *on*'. Fingers can serve as both a physical extension of what Rotman (1987, p. 27) calls the 'one-who-counts' (counting *with* my fingers) as well as the thing-to-be-counted (counting *on* my fingers): fingers are thus simultaneously subject and object, both of the person and of the world. In inhabiting this dual status (being both me and not-me), fingers provide echoes of the analyst Donald Winnicott's notion of 'transitional object': "an intermediate area of experiencing to which inner reality and external life both contribute" (1971, p. 2). (See also Maher, 1994.) When a four-year-old asserts, "Don't do it! I'm just fingering it out!" (Phillips, 1996, p. 82), in that slippage from 'figure' to 'finger' there is a literal as well as metaphorical truth being expressed. In this paper, we explore aspects of fingers' transitional object status with regard to counting by means of three- and four-year-olds working on a novel application, *TouchCounts* (Sinclair & Jackiw, 2011), which makes central use of the iPad's ability to respond to multiple tactile inputs synchronously.

FINGER GNOSIS AND THE DEVELOPMENT OF NUMBER SENSE

Within the field of developmental psychology, subitising (which connects to the mathematical task we report on in this paper) refers to the ability to enumerate the items in a set quickly, without counting. This notion has been claimed to be a core component upon which all other mathematical abilities are built (see Butterworth, 1999; Penner-Wilger *et al*., 2007). For Butterworth, subitising provides initial access to cardinality, allowing children to "categorise the world in terms of numerosities – the number of things in a collection" (p. 6). Such access – and to number sense more generally – appears to be strongly dependent on 'finger gnosis' (literally "finger

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 209-216. Vancouver, Canada: PME.

knowledge"), defined as the ability to differentiate one's own fingers without any visual clues when they are touched. Gracia-Bafalluy and Noël (2008) show that improving children's finger gnosis by training them on finger differentiation tasks increases their numerical performance.

Our research explores the nature of finger gnosis as it relates to children's interaction with *TouchCounts*, which involves the use of various finger gestures (tapping, swiping, pinching, flicking) to produce numbered objects and spoken words. Multi-touch enables direct mediation, allowing children to produce and transform objects with fingers and gestures, instead of acting through a keyboard or mouse. This added sensory input seems to play no role in developmental psychology studies, but may provide a powerful accompaniment to the visual and oral forms of communication that are currently privileged in that research. The word *gesture* has been used by touchscreen interface designers to describe specific configurations and actions of the finger(s) on the screen (swiping, tapping, etc.). These kinds of gesture are different from those typically discussed in the mathematics education literature in two ways: they involve contact with a screen and they *perform an action*. Similar to the performative speech act (Austin, 1962; Searle, 1969), which refers to language that performs on the world, we use the term "performative gesture act" to describe these tangible, input gestures.

DESIGN OF *TOUCHCOUNTS*

Currently, there are two sub-applications in *TouchCounts*, one for Counting (1, 2, 3, …) and the other for Adding (1+2+3+…). Here, we focus exclusively on the former (see Sinclair & Metzuyanim, 2014), for a more complete description). In this world, a user taps her fingers on the screen to summon numbered objects (yellow discs). The first tap produces a disc containing the numeral "1". Subsequent taps produce sequentially numbered discs. As each tap summons a new numbered disc, *TouchCounts* audibly speaks the English word for its number ("one", "two", …). Fingers can be placed on the screen one at a time or simultaneously. With five successive taps, for instance, five discs (numbered 1 to 5) appear sequentially on the screen, which are counted aloud one by one (see Figure 1a). However, if the user places two fingers on the screen simultaneously, two consecutively numbered discs appear at the same time (Figure 1b), but only the higher-numbered one is explicitly named ("two," if these are the first two taps). The entire 'world' can be reset, to clear all numbered discs and return the 'count' of the next summoned disc to one.

The number of taps (made sequentially or simultaneously) is also the number of discs on the screen, which can reinforce the cardinality principle, since the last number "counted" (spoken aloud by *TouchCounts*) is exactly "how many" numbered discs there are. Even after children have counted a set of discs (up to five, say), when they are asked "how many" objects are in a given set, will often count the objects again (Baroody & Wilkins, 1999). The "how many" question seems to provoke a routine of sequential counting. In *TouchCounts*, the child is engaged in a somewhat different routine – rather than counting a *given* set, she is actively *producing* that set with her finger(s) (perhaps to an instructor-given total) and elements of that set count themselves (both aurally and symbolically) as they are summoned into existence.

The Counting world directly supports two of the five aspects of counting identified by Gelman and Meck (1983): (1) when counting, every object gets counted once and only once (one-to-one correspondence principle); (2) the number words should be provided in a constant order. Also, the last number said by *TouchCounts* is always the number of items on the screen, it reflects a third of Gelman and Meck's 'aspects'.

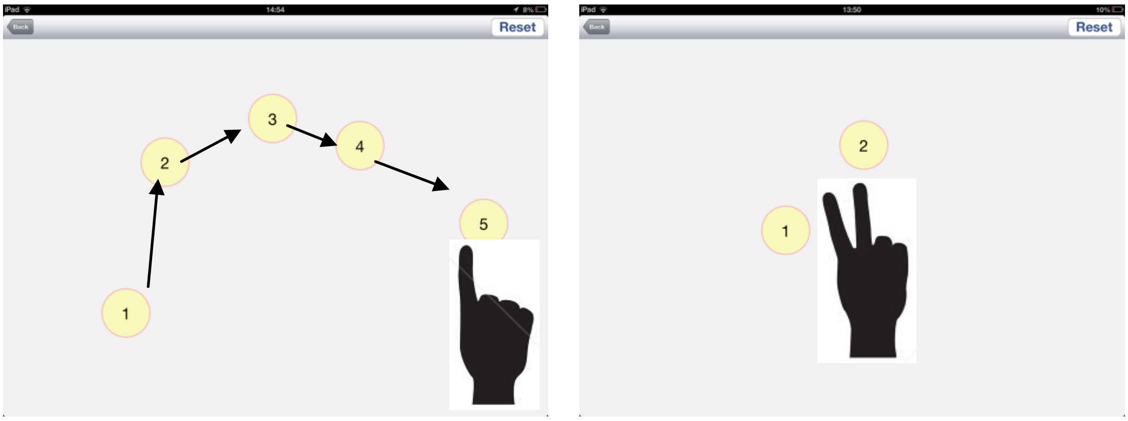

Figure 1(a): Five sequential taps – "one, two, three, four, five" is said;
1(b): A simultaneous two-finger tap – "two" is said.

THEORETICAL FRAMING

Broadly speaking, we take a non-dualistic perspective on thinking and learning. More specifically, we adopt an inclusive materialist approach in which the tool (in this case, *TouchCounts*) is seen as participating in an agential relationship with the user so that the tool and the user mutually constitute each other through interaction (de Freitas & Sinclair, 2013). In so far as the tool 'speaks' (and on occasion moves things) in interaction with the user, it takes on an animate role in the interaction, enabling but also preventing activity. We attend especially to the broad and varied ways of intervening involved in mathematical activity – including, bodily movements gestures and tone of voice. This is in accord with principles of embodied cognition, which posit that cognitive functions are "directly and indirectly related to a large range of sensorimotor functions expressed through the organism's movement, tactility, sound reception and production, perception, etc." (Radford, 2012, p. 4537). However, inclusive materialism insists on dissolving the rigid boundary that usually defines the human body and its sense organs.

An inclusive materialist approach also extends to mathematical concepts, not just to the concrete tools and bodies in the environment. We therefore focus on how the assemblage of finger/tool/number changes over time; how new materialities become part of the activity and affect its progression. The notion of finger gnosis thus strikes us as very interesting since it relates directly to embodiment, while also suggesting a distribution of senses foreshadowed in the introduction, in which the fingers comprise a core presence in the assemblage of counting.

METHOD OF RESEARCH

The study took place over three months in a day care located close to a North American University, which provides a play-based environment in which children are free to choose from a range of activities, each of which offers different sets of materials. In order to fit into the environment, the iPad was placed on the carpet in the corner of the room and children were free to come or leave as they wished. At the beginning of the session we analyse here, many children crowded around the iPad, jostling to get a chance to play, but after about twenty minutes, a small group of four children formed and stayed for the remaining twenty minutes. The analysis begins at the point the group formed, when it was possible to record the interactions and actions of the children. The four children in the group were all three or four years old.

We focus on a five-minute interval because it was the beginning of the group's work together, and there was a clear change in the way they use *TouchCounts* to summon numbers. We offered an 'inverse subitising' task where children were asked to produce a target number by using two or more fingers all-at-once (rather than sequentially). In subitising tasks, students must determine quickly the number of objects in an array, which they then either say or type onto a keyboard. Here, instead of making an spoken or alphanumeric action based on a visual prompt, the children are to make an action based on an oral prompt, a gesture act. Unlike traditional finger counting, which is both ordinal and fixed, such an all-at-once gesture act is neither.

INVERSE SUBITISING

The interviewer (first author, henceforth "I") asked a pair of children (Owen and Ramona) to try to "get four together". They each tapped the screen with one finger once, making *TouchCounts* say "one", "two", then again almost at the same time, thus producing "four" (see Figure 2a). When I asked Katherine and Christine to make four together, they each tapped with one finger, stopping after *TouchCounts* said "eight". They tried again, this time stopping after *TouchCounts* said "sixteen". Thinking that perhaps the girls were having difficulty coordinating their work, I asked Katherine to "get to four by yourself". She placed all five fingers on the screen, which said "five" (see Figure 2b). Prompted by her use of more than one finger, I asked her then to "use lots of fingers to get to four". She placed her whole palm on the screen. When it was her turn, Christine did the same thing.

I then moved the iPad in front of Owen.

I:	You try to use lots of fingers to get to four.
Owen:	*Initially he stretches out his whole right hand, then curls and ripples from pinkie to index finger, then tucks his thumb under, and then straightens the remaining and touches the screen all-at-once (see Figure 2c).*

iPad: Four

Ramona: Ah. (*Very high pitch*) He did it.

I: He did it.

Figure 2: (a) Ramona and Owen working together to get 4; (b) Katherine placing her five fingers on the screen; (c) Owen placing four fingers on the screen.

I offered the iPad to Ramona. She raised her hand in the air and lifted her fingers one by one, then placed four of them on the screen. *TouchCounts* said “six”. Thinking that she had inadvertently tapped other parts of her hand on the screen, I rolled up her sleeve and let her try again. This time she tapped sequentially four times on the screen and *TouchCounts* said “one, two, three, four”. When asked to do it with lots of fingers, Ramona placed her whole palm on the screen, producing “twelve”. She screamed, rolled over and, when asked if that was what she wanted, she exclaimed “no!” Christine was next slapped the iPad with her whole hand, also producing more than four. Christine tried again, as did Katherine, who imitated Christine’s gesture.

I then gave the iPad to Owen and asked him to “use lots of fingers to get to two”. He immediately put out his hand with his index and middle fingers outstretched and placed them on the screen. When it was Christine’s turn, she also extended two fingers, but when she touched the screen, *TouchCounts* said “three” (she had inadvertently touched the screen with another part of her hand). She tried three more times, always holding out her two fingers, but each time *TouchCounts* said a number greater than two. Katherine decided to press Reset and to tap sequentially twice. Then Christine placed two fingers on the screen and *TouchCounts* said “two”. I moved the iPad to Ramona, who lifted her left hand deliberately, extending one finger at a time and placed two fingers on the screen to get “two”. I then asked Owen to “do three with lots of fingers”, which he did successfully, as did Christine. Katherine then successfully placed three fingers on the iPad, as did Ramona. I congratulated the children for all managing to do “three with lots of fingers” and asked them to “do four”. Owen succeeded quickly, as did Christine and Katherine. Ramona stretched out four fingers, but placed her palm on the screen so that *TouchCounts* said “five”. This happened twice, and then she decided to tap successively four times.

EXPANDING THE SENSE(S) OF FINGER GNOSIS

At the very beginning, Ramona and Owen used their fingers to summon numbered discs and hear the count up to four while Christine and Katherine used them simply to summon numbered discs, apparently without attending to the number of discs on the screen or the number words spoken aloud by *TouchCounts*. Ramona and Owen managed to get to four by using the oral/visual feedback from *TouchCounts* to stop tapping once they heard/saw “four/4”, but Katherine and Christine tapped, without listening/looking for four/4. Perhaps the excitement of tapping outweighed the interest in performing the task. However, even on her own, Katherine did not use her fingers to get to four/4. Despite having tapped with one finger previously, both Katherine and Christine tried to get to four/4 by slapping the iPad with their hands, all the while giggling. For them, the request to “make four” seems to have been interpreted as a request to make some big number. While all the children were using their fingers to conjure numbered objects and number words, only Ramona and Owen’s fingers were being used to produce particular ones. For Katherine and Christine, fingers were not yet counting tools (either counting *with* or counting *on*).

Owen’s deliberate gesture introduced a new element to the assemblage; all the children saw his hand, which became joined up to the vocalised four of *TouchCounts*. The children heard that the gesture produced four all-at-once, without passing through other numbers. When it was her turn, Ramona stretched her four fingers out one by one, instead of simultaneously as Owen had done. But the fingers touched the screen simultaneously, perhaps mimicking Owen’s gesture. She had difficulty getting *TouchCounts* to say “four” though, and decided to revert to sequential tapping. Now the verbal sequence “one, two, three, four” had joined the assemblage. It is important to notice though, that each of Ramona’s four fingers touched the screen in order to produce the ordinal sequence 1, 2, 3, 4 so that she was not just counting up to four *on* her fingers, but producing one, two, three, four *with* her fingers – each finger feeling the screen and producing a distinct number word and numbered circle. While it may be argued that Ramona’s lack of manual dexterity got in the way of an Owen-like gesture act, we hypothesise that she may not be feeling the numerosity of her touch. She can present four on her hand, by extending her four fingers, so that her fingers can show four, but not yet use it to make four/4.

Despite seeing/hearing the Ramona-fingers-screen intra-action, both Katherine and Christine stretched out their hands, but slapped the screens. It was as if they were mimicking Owen’s gesture, but without paying attention to the number of outstretched fingers or to the way in which those fingers touched the screen. Ramona and Owen responded by screaming and resetting, respectively, obviously aware that the action was incorrect and that the girls needed to try again.

When a new round of tasks was initiated, one in which the children were asked to use many fingers to make two/2, three/3 and then four/4, Christine and Katherine began to use their fingers very differently. Whereas a few minutes ago, when asked to “do four”, they had slapped the screen almost haphazardly, by the end they both held up and placed four fingers on the screen. The speed at which Christine first held out her two fingers suggested some kind of subitising. She was confident enough about using these two fingers that she was willing to try several times to get *TouchCounts* to do as she wished. Katherine's impatience, and decision to proceed with sequential taps, may have stemmed from a strong ordinal finger sense – with the index finger as the main tool for presenting number (counting *on*). However, the speed and dexterity with which each child made three/3 shows the momentum of the gesture act, with the fingers now used both to present and produce a given quantity. When it came to making four/4 all-at-once, only Ramona extended her fingers one at a time. This reverting from using all three fingers at once to using them sequentially suggests that she was not mimicking the other children's gesture; she knew, however, that counting up to four on her fingers would produce four discs, as well as the sound "four".

DISCUSSION

By the end of this five-minute time span, all the children could use their four fingers all-at-once on the screen, to make *TouchCounts* say "four". Owen was able to do this early on, but not the other three. Significantly, they did this by extending their fingers all at once as well, as a kind of gesture, instead of lifting them up one at a time (as occurred several times earlier in the episode). In this sense, there was a developing finger gnosis about fourness, in that four fingers were being touched to/by the screen. This form of finger gnosis differs from the more ordinally based differentiation of fingers that is discussed in the psychology literature, but seems mathematically significant as a form of 'knowing about and through one's fingers'. Unlike conventional subitising tasks, which rarely extend beyond five, 'inverse subitising' with *TouchCounts* has no upper limit, in the sense that a child may use all her fingers to make ten/10, but can also work collaboratively with other children to make even larger numbers. Our data (not presented here) show that, for numbers between five and ten, children quickly shift from counting on their fingers until they reach the target to a subitised gesture act producing the desired number of fingers all-at-once.

Returning to the notion of fingers as both subject and object for the one-who-counts, each child showed a slightly different relationship between them. Owen was the first to create a fourfold gesture by means first of a brief counting *on* his fingers before counting *with* them, as if a single touch. His subsequent gesture acts reflect this plural resource. Christine and Katherine's work with all-at-once gesture acts is quite distinct, with no independent finger movement (unfurling one by one). They seem to only count *with*. In contrast, Ramona moves back and forth between the newer all-at-once gesture acts and the more familiar single fingering. She seems aware they are different means to reach the same end.

This short episode shows learning occurring in that three of the children were able to do something they could not do at the beginning. We claim that this learning cannot be separated from the materialities and interactions of the situation. *TouchCounts* was centrally involved in the learning. However, what particular role did it play in supporting this learning? Based on the above analysis, three features seem relevant: (1) the children could summon numbers one by one or all-at-once, without having to be previously familiar with the numbers they were creating; (2) the spoken number words could be connected to the tapping, providing feedback that encouraged self-correction, without external prompting; (3) the emotional engagement of the children – the screams, giggles, smiles, as well as the concentration, confusion and cooperation – cannot be overlooked. Further analysis of the affective flow in this episode would provide even greater insight into the assemblages' dynamic nature.

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WHY ANNA LEFT ACADEMIA

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This paper aims to explain why Swedish female mathematicians decide not to work in academia. The stories of five women were merged into one narrative. Anna describes a struggle with her own self-identity in a gendered structure that most often involved implicit power. One of the main reasons for not working in a mathematics department after finishing their PhD was the difficulty in getting a job without support.

INTRODUCTION

In Sweden, more women attend undergraduate higher education and more women than men receive degrees from such studies in most subjects. For graduate studies, the Swedish government has set an official gender policy of 60/40 meaning that no sex constitutes more than 60% of the total numbers of individuals. Nevertheless, there are still some unbalanced areas, and mathematics is one of the most extreme examples (Lindberg, Riis & Silander, 2011). This is despite a 50-50 division at the most mathematical intense upper secondary school programme. Women seem to disappear starting at undergraduate level: one third of all students in mathematics or other mathematics intensive courses including engineer and teacher education are female (Brandell, 2008). This is a similar situation to the USA (Herzig, 2004) and most of the European countries e.g. UK (Burton, 2004). For female post-doctoral fellows, senior lecturers and professors the number decreases even more. In 2007 the number of female post-docs in mathematics in Sweden were 6 %, senior lecturers 21%, and professors 7% (Lindberg, Riis & Silander, 2011). This decreasing pattern is in many ways an international phenomenon with different aspects connected to it (see e.g. Forgasz, Becker, Lee and Steinhorsdottir, 2010). In USA the number of women doing a postdoc is far less the number of graduate students, and considering that a postdoc is a strong factor for the possibility to get a tenure track it affects the number of professors (Nerad & Cerny, 1999). In the UK, in 2002 only 2 % of the professors in mathematics were women (Burton, 2004). Comparing with Sweden and 7% women professors, most of them are in subjects such as mathematical statistics and mathematics education (Wedege, 2011). This means that some areas are (even) more male than others, and this in a society where "gender equity is highly valued at societal and political levels" (Brandell, 2008, p. 659). The main problem seems to be number of women disappearing after undergraduate/graduate level, and the Glass Ceiling Effect in mathematics is in Sweden high (European commission, 2009). If all women doing a PhD in mathematical sciences (one third) stayed on, we would most likely have more female lectures and professors even in the areas 'less female'. So, where do all the women go and why do they leave?

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 217-224. Vancouver, Canada: PME.

Earlier research suggests different potential answers that might explain why this is happening. By interviewing mathematicians, Burton (2004) found that one of the themes that indicated a gender structure was the discourse of power. Women seem to be facing the use of power in many different ways e.g. being disadvantaged as a job applicant only because of being a woman. This use of power can be explicit but also implicit and hidden, so called 'non-events' (Husu, 2013). Non-events can be anything from being ignored at meetings or in the coffee room to not being invited to conferences or selected as keynote speakers. The lack of support and discrimination (both explicit or hidden) are two main factors behind women struggle to advance in their careers both general (Husu, 2005) and in mathematics (Henrion, 1997) or other STEM subjects (Heilbronner, 2013). Women develop several coping mechanism in order to 'survive' (Husu, 2005), especially in areas where a female identity is attached with negative symbols (Volman & Ten Dam, 1998). Solomon (2012) women undergraduate students in mathematics are forced to work with their identity, their self-concept as 'a woman in mathematics', including how they talk about themselves and their situation. Other factors are built in the structure itself, such as norm-controlled self-selections and internal and external factors such as how research grants and other funding are distributed (Lindberg, Riis & Silander, 2011). Solomon (2007) concluded that even when they are successful, women position themselves as not belonging in mathematics. The aim for this paper is to explore women's own stories. The research question posed is: What reasons do female mathematicians give for leaving mathematics as an academic profession?

BACKGROUND

The two main concepts for this paper are gender and self-concept or self-identity. I will here discuss these shortly.

Gender is understood as a social construction more than a consequence of a biological sex (West & Zimmerman, 1987; Damarin and Erchick, 2010). It refers to what is thought of as "feminine and masculine, characteristics and culture dependent traits attributed by society to men and women" (Wedege, 2011, p. 6). The concept gender can be divided into four different aspects (Bjerrum Nielsen, 2003): structural, symbolic, personal, and interactional gender. The structural aspect refers to gender as part of a social structure alongside other factors e.g. ethnicity and class. The number of female PhD students in mathematics in Sweden in relation to male students is an example of structural gender. The second aspect, symbolic gender, stems from these structures. Symbols and discourses are attributed to a specific gender creating norms and trajectories that tell us what is normal and what is deviant. The third aspect is personal gender. It focuses on for instance how the individual perceive the structure with its symbols, e.g. female professional mathematicians description what it is like to work in a mathematics department (Burton, 2004). Sometimes there can be discrepancies between the personal gender and the symbolic gender, e.g. girls are considered insecure in mathematics but don't feel insecure themselves (Sumpter, 2012). The last aspect is interactional gender. Compared to personal gender, which describes gender as something we "are", interactional gender is something we "do" (Wedege, 2011). These four aspects on gender is an analytical tool used to highlight different sides of the same phenomena more than something that occurs in different situations.

Self-identity (or self-concept) is how you view yourself including dimensions, such as capacity or role in different situations (Devos & Banaji, 2003). A broad definition is a person's perceptions of him- or herself (Marsh & Shavelson, 1985). This doesn't need to be conscious knowledge that is explicitly indicated but could also be measured "via unconscious expressions of thought and feeling" (Devos & Banaji, 2003, p. 154). The notion of self-concept should be viewed as a dynamic concept that is developed and changed through interactions and experiences.

METHOD

A written questionnaire was sent out to nine female mathematicians that finished their PhD in a Swedish university in mathematics during the years 2002-2012. Mathematics should here be interpreted as mathematical sciences such as pure and applied mathematics, mathematical statistics, computational mathematics and optimization but not including mathematics history or mathematics education. The author knows four of the women and the other five were found through a mutual contact or the Swedish network 'Women and mathematics', a sub-organisation of IOWME. Since the answers to the questionnaire were kept anonymous, and the respondents were aware of this, the assumption is that the difference connections didn't affect the objectivity or the quality of the replies. The respondents were instructed that they could write as much or as little as they wanted. The main questions posed were: (1) Why did you become a mathematician/ Why mathematics?; (2) How come you did a PhD in mathematics?; (3) How was it to be a (female) PhD student in mathematics?; and, (4) You have a career outside the university. How come? To each question, several optional sub-questions were posed to decompose the main questions. They were also asked to write something about their background. The respondents were requested to send back their answers within two weeks. A first analysis of the data showed that five of them had similar answers with a slightly more negative tone in their responses whereas the remaining four described a slightly more positive view. In this paper, because of the limited space, I will focus on the first five. They constitute the base of for the story here presented as Anna's. This method, to create a collective narrative from several respondents, is a method used by several researchers e.g. Mendick (2002). It is a tool to emphasize meaning of responses, patterns, in a collective context rather than to show individual's replies (Mendick, 2002). In this way, the collective narrative analysis here shares a similar approach to the data as content analysis (Smith, 2000): the aim is to identify characteristics of the material. Here the characteristics are told by a fictive voice. The story presents the most common replies to the questions, where the respondents' written answers have been interweaved into one. This is helpful when you want to increase to possibility of keeping the respondents anonymous especially in a small community. As common in narrative analysis, my own voice is part of the story

(Smith, 2000) although I've tried to minimize it as much as possible by using the respondents' own formulations. Sometimes the formulations have been joined together to one sentence and in some cases I've changed the context (e.g. lessons have become seminars) to make sure that the specific person/ situation can't be identified. The meaning of the replies remains the same in both these cases. This study falls under the personal aspect of gender focusing on women's own explicit self-concept in the role of 'female non-academic mathematician'.

RESULTS

Anna was born and grew up in a middle size town in Sweden. She studied the most mathematical intensive program at upper secondary school (age 16-19), the natural Science Programme. Her parents, although not in mathematics themselves, have always supported Anna in her studies. Anna started her PhD almost right after her undergraduate studies (which was in mathematics/applied mathematics combined with engineering/ physics/ statistics/ computer science; 4-5 years). She was then 25-30 years old. During her PhD, she had children and was on parental leave. Anna finished her PhD when she was around the age of 35. This is now 1-10 years ago. She had both female and male supervisors (where the most common situation was only male supervisors). Anna is now working as a mathematician/ researcher in a private corporation or at a council/governmental institute.

So why did Anna choose mathematics in the first place? Her answer is based in school mathematics and mathematics education:

> I have always found mathematics easy during school. Maybe it wasn't fun to do all the routine stuff, but I really liked problem solving. Math is fascinating!

She studied as much mathematics you can do at upper secondary school. When it was time to enter university, the choice of programme included mathematics. But it wasn't obvious that she would do a Ph.D. in mathematics. Some people encouraged her to become a part of the department, but there were also people who discouraged her:

> During my undergraduate studies I was encouraged to take an amanuensis position. I doubted if I was clever enough to do a Ph.D. in mathematics, and one time during an oral examination, the teacher asked what my plans were. I said I was thinking about Ph.D. but that I wasn't sure that I was clever enough. He said 'yes, I can agree with that.' But in the end, I felt that this was something I really wanted to do. I wanted to do a PhD, I wanted to learn more.

Anna got a PhD position at the same department where she had done her undergraduate studies and worked as an amanuensis. How was it then to be a (female) PhD student in a mathematics department? Anna explains:

> Often you are the only woman (or one of few) when you study mathematics, so I was used to this from my undergraduate studies. Most people were positive and friendly, but it was like an underlying structure that now and again reminded me that I should not think that I was as good or clever as the male students. For instance, one time I took a course and the professor ignored me – he wouldn't answer my questions! In seminars, when I raised

> topics that could be seen as critique, I was told not to be so troublesome. Male students, they were instead praised for their ability to scrutinize and flexible thinking. But this didn't discourage me. I would say I got more determined to show them wrong.

Anna thinks that this might be a reason why female mathematics professor now and again are described as cold and sharp. She says:

> Sometimes you hear that female professor's got sharp elbows because they have to fight their way up, and I assume it is because of the feeling of working in constant headwind.

However, the symbols attributed and used in comments can be tiresome:

> I've heard everything that aims to diminish what I'm doing, such as that it is no point with female PhD students since they are going to end up next to the stove anyway to that the only reason why I got funding is because I'm a woman. Once I was told that you should have higher demands on female PhD students since they need to prove themselves.

The department seemed to be aware of this structure and now and again made some effort to change the work climate. But this was not an easy task:

> When the department tried to implement things to make working life easier for women it was worked against [by some people]. It was like there was a systematic way of opposition against women, but it was made in a very subtle way so no one could really object.

This hidden conscious or not conscious structure got more explicit at some occasions, for instance when scheduling seminars. During her PhD, Anna got pregnant and had children. She was on parental leave and then came back to work, trying to combine parenthood with work:

> The first years being a parent they scheduled seminars late in the afternoon, which meant that I couldn't go since I had to pick up children from nursery. In the beginning there was no understanding about this.

But overall, having children during the PhD was not a problem. The issues were more based on the restricted time of a PhD (related to funding) and the change of view of when to do work.

> Overall, I have never felt I was treated differently or badly for having children. The difference is how you as a parent want to spend your time changes. The working hours are for work, but when the day is over I want to go home to my family. It was though being pregnant, but that was more due to the pregnancy itself. But it was hard to focus when you are tired and math is a subject where you need to be focused the whole time. It was also tough coming back from parental leave because work felt so distance. It was hard to remember and to get going. It took some time and energy before I was back on track so to speak, and now I realise that this cost me a few months.

The choice to leave was easy.

> I started by applying for lot of different positions. I didn't get them. One time, they gave a position to a man who wasn't as qualified as me with the explanation that his work was much more 'developed'. How can you argue against that? I didn't have the network to work for me, to argue for my case. I was alone, and you can't get a position when you don't have support. My supervisor(s) didn't help me at all. So the choice was in some sense easy.

> In my current workplace, I've been extremely well taken care of and they appreciate the work I'm doing. There is a clear plan of how I shall develop and the work I should do. We work in teams and we have the same goals. The sum is greater than the parts. At the department, we sat in our offices with closed doors and that didn't suit me so well. And we are not pushed to work overtime – they respect our working hours. It is very interesting. You get a lot of feedback and you push each other forward. And our product is useful for the society.

But Anna would have stayed if the situation had been different.

> You know, I really wanted to stay in academia. The only thing that was needed was a position that wasn't a short-term temporary one. I both miss the type of research you do at a university – that you can completely focus on one single detail of a problem - and the teaching. I miss going to seminars even in subjects that wasn't your own one. But I do not miss the working climate and I definitely don't miss the stress of applying for grants that you are most likely not to get. I didn't like that everyone should be 'best', the competition and the lack of common goals to strive for.

Anna gives two main reasons for why she is working outside academia. The first reason is the lack of full-time, long-term jobs in academia, or more specifically, how hard it is to get such a job without the right network. The other reason is the atmosphere especially when compared to other workplaces.

DISCUSSION

The purpose of this paper is to try to understand the decreasing numbers of female in positions in higher education in mathematics in Sweden. Anna is one (fictional) voice aiming to give such explanations. Her voice relates to a structure recognised in previous research (e.g. Forgasz, et al, 2010). She describes the subtleness in how women are worked against and the struggle of keeping your self-concept in an environment where you are under pressure just for having the 'wrong' sex. Seminar scheduling and who to praise in a seminar are examples of explicit and implicit power, the latter one including what Husu (2013) refers to as 'non-events'. Anna uses the term 'constant headwind' as an illustration of this struggle.

Anna describes several instances when she is talking about herself in relation to the context and the changes in her self-identity, for instance the change of view of work when having children. The negotiation of the self-identity seems to be an ongoing work (Solomon, 2012). If the efforts to change made by the department were successful depended on the people, not a lack of regulations or policy documents. One of the main reasons why Anna is working out-side academia is the struggle to find a job without support. This has been highlighted in previous studies both in mathematics (Henrion, 1997) and general (Husu, 2005). Anna mentions also the stress of getting grants, which seems to be a filter for further careers (Lindberg, Riis & Silander, 2011). She also compares work environments and stresses how nice it was to find another place where the (collective) effort was appreciated. As Solomon (2007) concluded, there exists a language of a 'not belonging' even when a women is successful. Anna expressed that she belongs to mathematics and mathematics belongs to her. What we need to further understand is why working in a mathematics department is not an option and how we can change this situation.

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TONY'S STORY: READING MATHEMATICS THROUGH PROBLEM SOLVING

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We describe an attempt by a former mathematics teacher to read an undergraduate mathematics proof aided by discussions with a mathematician using the language of mathematical problem solving. The literature on successful approaches to reading mathematics is scarce at the secondary and undergraduate levels. Shepherd, Selden and Selden (2012) offered three possible reasons why undergraduate students find it difficult to read passages from a mathematics textbook. From the ultimately fruitful attempt by the teacher, we postulate how a problem solving approach can successfully negotiate these three difficulties.

INTRODUCTION

Shepherd, Selden and Selden (2012) state that "it appears to be common knowledge that many, perhaps most, beginning university students do not read large parts of their mathematics textbooks in a way that is very useful in their learning". This concurs with our own various experiences as secondary school teachers and university lecturers with regards to our students' mathematics reading proclivities.

Tay(2001) citing Waywood (1992) who noted that the majority of reported work on writing to learn mathematics is focused at a primary level, lamented that little progress had been made at the higher levels and suggested some assessment modes which would encourage good mathematics reading. Others (Bratina & Lipkin, 2003; DeLong & Winter, 2003; Draper, 2002) have also made calls for students to be taught how to read mathematics. Wilkerson-Jerde and Wilensky (2011) investigated how mathematicians make sense of an unfamiliar proof that they read for the first time and try to elicit reading strategies for school students. On the whole however, Osterholm (2008), based on a survey of 199 articles having to do with the reading of word problems, reported that there was little about reading comprehension of more general mathematical text.

Difficulties in reading mathematics textbooks

Shepherd, Selden and Selden (2012) adapted the Constructively Responsive Reading framework (CRR) by Pressley and Afflerbach (1995) to understand the difficulties first-year university students had in reading their mathematics textbooks. They made some significant observations which we shall report in this section.

Their research (Shepherd, Selden, & Selden, 2012) involved eleven precalculus and calculus students who were each asked to read aloud a selected passage from their textbook. They were stopped at intervals during their reading and asked to attempt a

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 225-232. Vancouver, Canada: PME.

task based on what they had read or a textbook example. Overall, the students performed poorly although they were considered good students based on their American College Test (ACT) Reading and Mathematics scores.

> The students' difficulties working tasks all seemed to arise from, or depend largely on, at least one of three main kinds of difficulty: (a) insufficient sensitivity to, or inappropriate response to, their own confusion or error; (b) inadequate or incorrect prior knowledge; and (c) insufficient attention to the detailed content of the textbook. The difficulties working tasks and their origins occurred throughout the passages read and were associated with exposition, definitions, theorems, worked examples, and explorations. Furthermore, most students exhibited all three of these difficulties usually several times. (p. 238)

In spite of the generally poor performance, there were a few students who when they failed to understand a passage, persisted in rereading the passage and reworking the task until they could do it correctly. Shepherd, Selden and Selden (2012) wondered if such students had an unusual feeling or belief that in persisting they could ultimately succeed and if such a feeling of the value of persistence can be engendered by providing supporting experiences.

Placed against the poor reading of most students and the nascent success attributable to persistence, Shepherd, Selden and Selden (2012) offered their observation of why mathematicians appear to be effective readers:

> For sufficiently important reading, we will work tasks or construct examples to check the correctness of our understanding and tend to look for errors. On finding an error, we rework the task, reread the appropriate passage, or construct an example. We each feel that we can benefit from such a process, and suspect that this feeling of self-efficacy arose from our past positive experiences with reworking tasks, rereading associated passages, and constructing examples. We suggest our students lacked the feeling that they can independently rework a task or reread a passage until they ultimately "get it right." (p. 243)

Reading mathematics through problem solving

In this paper, we propose that reading mathematics could be approached as a series of problem solving attempts. Recall that Pólya's (1945) model of problem solving requires firstly that one understands the problem. We conjecture that readers with a problem solving mindset will recognise their lack of understanding of a passage and so will be able to circumvent the difficulty (a) of the students of Shepherd, Selden and Selden (2012) with regard to "insufficient sensitivity to, or inappropriate response to, their own confusion or error". Every statement or phrase that is unclear can be stated as a problem and the relevant problem solving stages of Understand the Problem, Devise a Plan, Carry out the Plan and Look Back, can be employed to gain understanding. While carrying this out, it will not be surprising to find the reader exhibiting a mathematician's behavior of "construct[ing] examples to check the correctness of [his or her] understanding" (Shepherd, Selden, & Selden, 2012, p. 243).

In addition, a reader who is familiar with Schoenfeld's framework of mathematical problem solving (see Schoenfeld, 1985) will realize when his or her resources are inadequate for the passage. We conjecture that this constant realisation may be crucial to overcome the difficulty (b) of inadequate or incorrect prior knowledge.

Weber and Mejia-ramos (2014) contrasts the practice of a mathematician and that of the general undergraduate with respect to the amount of responsibility the reader bears in the comprehension of a proof.

> … the mathematician views his responsibility when reading the proof to be significant. Many new assertions in the proof require the construction of a sub-proof and sometimes understanding them also necessitates the drawing of pictures or the consideration of examples. Indeed, this mathematician suggests that the author of the proof *deliberately* left the responsibility of drawing the appropriate pictures to the reader of the proof, presumably because the reader would gain understanding from engaging in this process. (p. 91)

We conjecture that a problem solving mindset will guide the reader to break down the passage into a series of sub-problems to be worked out so as to gain a deeper understanding both through the stage of Understanding the Problem as well as the stage of Looking Back at the solution. Having a problem solving mindset will thus prepare the reader to scrutinize the passage and give sufficient attention to the detailed content of the textbook (difficulty (c) of the students of Shepherd, Selden and Selden (2012)).

Our research question in our pilot study of one reader is thus:

Can a problem solving approach to reading mathematics overcome the following difficulties encountered by readers who are not professional mathematicians?

a. Insufficient sensitivity to, or inappropriate response to, their own confusion or error.
b. Inadequate or incorrect prior knowledge.
c. Insufficient attention to the detailed content of the textbook.

PARTICIPANT AND METHOD

The reader, pseudonymously referred to as Tony, is a former school teacher who had taught Mathematics and Design and Technology for more than ten years in the secondary school environment. He is a mechanical engineering graduate with limited experience in undergraduate mathematics, having done only two mathematics modules in the university. Tony worked as a research associate with a problem solving project and enthusiastically embraced the Pólya model to problem solving. He found that the approach was very helpful to solving non-routine problems which were pitched at the secondary school level.

We gave Tony a proof that a particular number is transcendental (see Appendix) and asked him to read it. We also suggested that he should view any difficulties he had with the text as non-routine problems and approach them as such with the Pólya model of problem solving (see for example, Pólya, 1945; Toh, Quek, Leong, Dindyal & Tay, 2011).

Four sessions with a mathematician/researcher (the first author) to discuss his progress and difficulties were carried out. Field notes by the researcher and audio recording were taken for the sessions. Protocol analysis (see for example, Ericsson & Simon, 1993) was not used as Tony was not trained to say aloud his thoughts without disturbing his own cognition as he worked through some aspects of the text. Thus, the conversations in the sessions were mostly between the researcher and Tony while there were long stretches of silence as Tony worked on the reading. In addition, Tony was asked to write a reflection of his reading journey (see Tay, Quek, Dindyal, Leong & Toh, 2011) and this short journal was used to corroborate the field notes and audio recordings. Finally, an interview was conducted at the end of the sessions. The interview protocol was semi-structured with the three parts of the research question as the main foci.

TONY'S STORY

Tony was given the two-page text in early November 2013 and the researcher advised him to use a 'problem solving' approach to reading and understanding the text. Discussion I, lasting about 45 minutes, with the researcher was on 6 November. Tony then read the text again sporadically over the next few weeks. At a writing retreat over the period 4-7 December for the project team where there would be time to discuss, Tony continued to work on the text and had three further discussions (II – IV) with the researcher for a period of about an hour each. Tony's story will be told from his point of view through his written reflection interspersed with the researcher's comments on his interaction with Tony during the four discussions.

Tony's reflection of his reading journey

> <u>The first reaction</u> I started reading the entire proof once and I encountered many unfamiliar terms or definitions … "Foreign language!" was the first response … I sort of structure the reading into three main parts. Firstly, understand the definition, then Theorem 1 and its proof and finally Theorem 2 and its proof.
>
> <u>The main definition</u> The approach was to understand the definition line by line … I tried using simple examples to understand or refresh the meanings of the terms like 'complex number', 'root', … it seemed manageable. I … look[ed] for non-examples [unsuccessfully] … The assignment was then left aside for a while. Had the first discussion with [the researcher] to clarify. Explained the approach to him and realized that the non-example is actually defining transcendental number itself (… felt silly).

In Discussion I, the researcher suggested that Tony ignore the rest of the text and focus on the definitions at the beginning, but one at a time.

> … After spending some time trying to understand the definition again, there was some progress (i.e. managed to proceed to the second sentence). What was helpful was to use numbers and do some manipulations … Moving on, was the understanding of algebraic numbers with degree *n*. Again examples were used and it seems understanding this part was fine.

Within Discussion I, we successfully negotiated the definitions of an algebraic number and the degree of the number by making Tony produce concrete instances of each concept. For example, Tony picked out the word complex in the definition and considered if i was algebraic. The researcher suggested that it was the root of $x^2 + 1$, which was readily accepted. Then, he asked Tony to consider if $3+i$ was algebraic and Tony was able to work out the required polynomial starting from $x = 3+i$.

> We moved on with the definition and the next challenge came when I was asked to write out the statement for 'transcendental number'. I thought I understood and knowing 'transcendental number' is just 'not algebraic number' but I was not able to write out the statement. After some discussions, I was troubled by the concept of complement e.g. what is "not some" and what is "not any".

A problem that surfaced was Tony's difficulty with basic propositional logic – the researcher had to explain that the complement of "root of some polynomial" is "not root of any polynomial". Theorems 1 and 2 were tackled at the writing retreat.

> Looking at Theorem 1   There were several terms in Theorem 1 that … I did not manage to understand them on my own … my plan was to proceed on to read the proof hoping that the proof might lead to the understanding of some terms … looking at it closer, I was convinced that [the proof] was just algebraic manipulation. So my plan was to break the long string into smaller parts to understand them. I was successful with the first part of the proof and it can be explained using long division … it appeared that things [in the next part of the proof] are linking back to the earlier part which I've skipped. I was not able to resolve it on my own. [Meanwhile] I [had instinctively] used some heuristics like working backwards and breaking into smaller parts to help myself to understand. In the end I still struggled to understand as [my ability to manipulate expressions with the] absolute [sign] was not strong. With some guidance and scaffolding [in Discussion III] … I was able to [understand] the rest of the proof for Theorem 1.

Although Tony struggled with the inequality

$$\frac{f(r_m)}{r_m - z} < n\,|\,a_n\,|\,(|\,z\,|+1)^{n-1} + (n-1)\,|\,a_{n-1}\,|\,(|\,z\,|+1)^{n-1} + \cdots + 2\,|\,a_2\,|\,(|\,z\,|+1) + |\,a_1\,|,$$

he was able to restate it as a sub-problem:

Show that $a_n(r_m^{n-1} + r_m^{n-2}z + \cdots + r_m z^{n-2} + z^{n-1}) + \cdots + a_3(r_m^2 + r_m z + z^2) + a_2(r_m + z) + a_1$

$$< n\,|\,a_n\,|\,(|\,z\,|+1)^{n-1} + \cdots + 3\,|\,a_3\,|\,(|\,z\,|+1)^2 + 2\,|\,a_2\,|\,(|\,z\,|+1) + |\,a_1\,|.$$

He worked on comparing term by term and was successful for the second and third last terms. But as he reported, he was unable to independently see the general term because of his lack of dexterity with inequalities and the absolute sign.

> [However] I was not able to see the link between the main objective, which is the transcendental number and Theorem 1. [The researcher] had to [explain] to me.

This link is not obvious until one finishes Theorem 2. Yet for Tony to have this realization that he still did not understand the whole picture although he had just understood a major piece, i.e. Theorem 1, is a sign of a good reader.

> <u>Looking at Theorem 2</u> I proceeded on to understand Theorem 2 by myself. The plan was to use some numbers to help in the understanding. I was able to do that. I asked myself if there is a better way to understand it. I proceeded on with using algebraic manipulation. It also ended fine. The part which ended up challenging was the final statement. I tried algebraic manipulation and was still unable to break through. I discussed with [the researcher] again [Discussion IV].

Tony could not see the need for $m > n$ to falsify $\frac{1}{10^{(n+1)m!}} < \frac{1}{10^{(m+1)!-1}}$. Although he capably substituted suitable values of m and n to confirm the inequality, he knew that he did not understand the big picture. Later, he lamented that missing the big picture was a result of not understanding key phrases, in this case “sufficiently large”.

> Then I realized, I was missing the big picture again. Reflecting on that, I realized that I actually did not have thorough understanding of the problem. The understanding of the phrase “sufficiently large” was neglected since the beginning. Little did I realize the importance of understanding this definition. After I was guided through to understand the phrase “sufficiently large”, the understanding of the rest of the proof just fell into place. To stretch the understanding, we went on to ‘check and expand’ and I was able to give an example which I was happy about it.

Tony was keen to produce the ‘next’ transcendental number on his own and following the proof which he was now confident that he understood, he gave this number:

$$\frac{1}{100^{1!}} + \frac{1}{100^{2!}} + \cdots + \frac{1}{100^{n!}} + \cdots$$

DISCUSSION AND CONCLUSION

The proof for the ‘first transcendental number’ is not an easy one. Yet Tony who had only read two mathematics modules in his undergraduate studies was able to successfully understand it. The report in the section above showed that Tony was able to engage with the text for a very long time – in the interview, he estimated that he spent 15 hours working on understanding the proof. This may seem an inordinate amount of time but it pales compared to a colleague of ours who spent five months understanding the first page of a book during his doctoral studies. The report also shows that Tony consistently applied a problem solving approach to his reading – in the interview, he agreed that the first and last stages of Pólya were often applied to understanding definitions and statements in the theorems while the first three stages were useful in working out the algebraic manipulations.

Tony showed great sensitivity to his own confusion and errors – he felt “silly” that he was trying to find a non-example of an algebraic number when the whole proof was about doing that. He was acutely aware of his inadequate or incorrect prior knowledge and sought advice from the researcher. In the interview, he said that in the past (i.e.

without the challenge of a problem solving approach), he would just look for answers from the internet when he was stuck. Finally, Tony paid great attention to the detailed content of the text – in the interview, he agreed that almost every word was important. We think that Tony did not face the same three difficulties that the students of Shepherd, Selden and Selden (2012) had mainly because he had anticipated that the text would be difficult. Also, by adopting the problem solving approach, he had relished the challenge ahead of him as he was confident of making progress.

APPENDIX

The 'first' transcendental number

A complex number z is said to be **algebraic** if z is a root of some polynomial with all integral coefficients.

A complex number z is said to be an algebraic number of degree n if z is algebraic and it is a root of some polynomial of degree n with all integral coefficients but not of any polynomial of degree less than n with all integral coefficients.

A complex number z is said to be **transcendental** if z is not algebraic.

Theorem 1 (Liouville)

Let z be an algebraic number of degree $n > 1$ and let $r_m = \frac{p_m}{q_m}$ be a sequence of rational numbers converging to z. Then, for a sufficiently large M, $\left| z - \frac{p_m}{q_m} \right| > \frac{1}{q_m^{n+1}}$ for all $q_m > M$.

Proof Suppose that z is a solution to the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0.$$

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$. Then

$$\begin{aligned} \frac{f(r_m)}{r_m - z} &= \frac{f(r_m) - f(z)}{r_m - z} \\ &= a_n (r_m^{n-1} + r_m^{n-2} z + \cdots + r_m z^{n-2} + z^{n-1}) + a_{n-1}(r_m^{n-2} + r_m^{n-3} z + \cdots + r_m z^{n-3} + z^{n-2}) + \cdots \\ &\quad + a_3 (r_m^2 + r_m z + z^2) + a_2 (r_m + z) + a_1. \end{aligned}$$

Letting m be such that $|z - r_m| < 1$, we may say that, for sufficiently large m, $\frac{f(r_m)}{r_m - z} <$

$$n|a_n|(|z|+1)^{n-1} + (n-1)|a_{n-1}|(|z|+1)^{n-1} + \cdots + 3|a_3|(|z|+1)^2 + 2|a_2|(|z|+1) + |a_1| = M.$$

Let $q_m > M$. Then $|z - r_m| > \frac{|f(r_m)|}{M} > \frac{|f(r_m)|}{q_m}$.

Now $|f(r_m)| = \left| \frac{a_n p_m^n + a_{n-1} p_m^{n-1} q_m + \cdots + a_1 p_m q_m^{n-1} + a_0 q_m^n}{q_m^n} \right|.$ □

Note that r_m cannot be a solution to $f(x) = 0$ because if it were, we could factor out $(x - r_m)$ and so z would necessarily be of lesser degree. Hence $f(r_m) \neq 0$. Furthermore, the numerator of this fraction is an integer so it must be at least 1. We conclude that

$$|z - r_m| > \frac{1}{q_m} \cdot \frac{1}{q_m^n} > \frac{1}{q_m^{n+1}}.$$

Theorem 2 (Liouville)

The number $z = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots + \frac{1}{10^{n!}} + \cdots$ is transcendental.

Proof Let $r_m = \frac{p_m}{q_m} = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots + \frac{1}{10^{m!}} = \frac{p_m}{10^{m!}}$. Then $|z - r_m| < 10 \cdot \frac{1}{10^{(m+1)!}}$. Now if z is an algebraic number of degree n, then Theorem 1 says that $|z - r_m| > \frac{1}{10^{(n+1)m!}}$ for sufficiently large m. So $\frac{1}{10^{(n+1)m!}} < 10 \cdot \frac{1}{10^{(m+1)!}} = \frac{1}{10^{(m+1)!-1}}$. But this is false for $m > n$, so z is transcendental. □

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MOTIVATING PROSPECTIVE ELEMENTARY TEACHERS TO LEARN MATHEMATICS VIA AUTHENTIC TASKS

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Motivating prospective elementary school teachers (PTs) to learn mathematics in university mathematics content courses remains a constant challenge. While authentic tasks are readily available for students taking methods courses, which generally appear later in students' educational experience, authentic experiences for students enrolled in mathematics content courses are more challenging. We examined the use of a particular kind of authentic task for PTs enrolled in mathematics content course, creating and enacting a mathematics activity with children, and found that PTs were excited about this activity and, knowing they would need to apply the knowledge learned in the course, felt additional motivation to learn the content and engage in the university classroom activities.

RATIONALE AND BACKGROUND

Elementary school children in the United States are not developing acceptable levels of mathematical proficiency (National Center for Education Statistics, 1999). For teachers to teach so that their students develop mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001), teachers must develop deep and flexible understanding of the mathematics they are teaching (Ball, 1990; Ma, 1999; Sowder, Philipp, Armstrong, & Schappelle, 1998). For prospective elementary school teachers (PTs), most colleges and universities in the United States offer specially designed mathematics courses focused on rich mathematical content knowledge, but although such courses have been offered by many universities for decades, teachers' mathematical content knowledge continues to be a major area of concern (Tatto et al., 2012).

Although most PTs and teachers can execute algorithms, many struggle when asked to explain them conceptually (Ball, 1988/1989; Ma, 1999; Thanheiser, 2009) and may be unaware that rationales for the algorithms exist. With recent calls for a focus on having students in the United States develop conceptual understanding (Common Core State Standards, 2010; Kilpatrick et al., 2001; National Council of Teachers of Mathematics, 2000), the fact that most PTs and teachers do not understand the rationales behind the procedures they teach is a major concern for those of us responsible for teaching PTs. However, PTs do not share this concern because many of them hold the beliefs that

1. Knowing how to apply procedures is synonymous with understanding (Graeber, 1999).
2. "If I, a college student, do not know something, then children would not be expected to know it, and if I do know something, I certainly don't need to learn it again" (Philipp et al., 2007, p. 439).

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 233-240. Vancouver, Canada: PME.

Thus many PTs view their mathematics content courses as annoying prerequisites they must endure instead of as opportunities to develop richer mathematical understanding. Of note is that in the United States, PTs taking content courses are typically years removed from teaching, working with children, or even their methods courses, thus they often struggle to see the connection between the university content course and their future careers.

Our interest is in trying to understand and explore ways to motivate PTs to learn the mathematics of their content courses. In prior work (Thanheiser, Philipp, Fasteen, Strand, & Mills, 2013) we have shown that a brief one-on-one content interview with PTs led to the PTs changing their beliefs about mathematics and about their understanding of mathematics, leading to the recognition that (a) there is something to learn beyond procedures, (b) their own knowledge is limited and they need to know more to be able to teach, and (c) engaging in the mathematical activities in their content courses will lead them to learning important content.

In this study we explore a new approach designed to sustain PT motivation and engagement in learning mathematics throughout the course, namely, creating and enacting a family math night (FMN) activity. Such an activity is typically found in methods courses, but we purposefully incorporate it into a content course with the goal to motivate PTs to learn mathematics. The central focus of the activity remains on the mathematics throughout the course.

THEORETICAL FRAMEWORK

Although most researchers studying learning examine the cognitive skills required to solve a task, other factors, such as motivation and engagement (Dweck, 1986; Middleton & Jansen, 2011) and authentic tasks (Newman, King, & Carmichael, 2007) play a major role in learning.

Motivation and Engagement

A student who is not motivated to learn will not engage in a task and thus will miss the chance to learn, whereas students who are motivated to learn and engage in tasks are more likely to learn. We adopt the definition of Hulleman, Durik, Schweigert and Harackiewicz (2008) of motivation as “a motive (e.g., wish, intention, drive) to engage in a specific activity” (p. 298). This is consistent with the theory that engagement in learning activities in the classroom can be seen as the “outward manifestation of a motivated student,” (Skinner, Kindermann, & Furrer, 2009, p. 494) a “visible manifestation” (Skinner & Pitzer, 2012, p. 22), or “the action component of … motivation” (p. 24). Engagement describes the interaction of a student with a task and is easily observable. Engagement, and thus motivation, is an essential element of academic learning as it is “a robust predictor of students’ learning, grades, achievement test scores” (Skinner & Pitzer, 2012, p. 21).

Authentic tasks

Academic tasks have been identified as "especially important determinants of motivation and engagement" (Skinner & Pitzer, 2012, p. 28). To promote engagement, teachers should provide students with tasks that are "authentic, challenging, relevant to students' experiences and concerns, hands-on, project-based, integrated across subject areas, and that allow students some freedom to choose their own direction and to work closely in cooperative groups over long periods of time" (Skinner & Pitzer, 2012, p. 33). One way of making a task more authentic is by connecting the university classroom to the real world (in the case of PTs the K-12 classroom) (Newman et al., 2007). Research has demonstrated the importance of authentic tasks, as "students who experienced higher levels of authentic instruction and assessment showed higher achievement than students who experienced lower levels of authentic instruction" (Newman et al., 2007, p. vii).

PT learning

To help PTs develop mathematical understanding, mathematics teacher educators need to understand three things: (a) the conceptions PTs bring to teacher education because "the key to turning even poorly prepared prospective elementary school teachers into mathematical thinkers is to work from what they do know" (Conference Board of the Mathematical Sciences, 2001, p. 17); (b) how those conceptions can be further developed, by, for example, using a hypothetical learning trajectory (Simon, 1995); and (c) how to motivate PTs to learn mathematics. This paper focuses on the latter point. Prior work has shown that classroom environment can influence learning goals and motivation (Morrone, Harkness, D'Ambrosio, & Caulfield, 2004). We share our results of incorporating a FMN activity into a content course for teachers to motivate them to learn the mathematical content of the class.

METHODS

The FMN consisted of pairs of PTs finding/modifying/developing a mathematical activity to work through with elementary school students. In Week 5 (of a 10 week course), students were asked to pair off and sign up for a topic (one of the topics covered in this course) and then: (a) find what children are expected to know at various grade levels about this topic via the Common Core State Standards (2010), (b) decide on a mathematical goal for their activity, (c) use online resources such as www.nctm.org to find ideas for their topic (including browsing the publication *Teaching Children Mathematics* and navigating through the resources provided, including www.illuminations.org), (d) (re)visit their topic in the textbook used in our course, and finally (e) use the internet and other sources to find additional ideas, if needed. In Week 6 students were asked to send a one-page idea of their activity to the instructor. In Week 7, the pairs of PTs met with their instructor to discuss their ideas and receive feedback. These meetings focussed on clarifying the mathematical goal of the activity and linking the activity to the goal. Once the PTs received feedback on their ideas, they created a draft of their activity, which was then presented in the university classroom to their peers. This allowed the PTs to experience many of the activities and give/receive feedback from their peers and their instructor. The PTs then had a final chance to revise their activity and present it at the FMN event at a local elementary school (at the end of Week 8). The goal of the activity was to allow the PTs to explore one mathematical topic in depth in an authentic setting and motivate them to learn the mathematics of the course.

The authors analysed data from work with 23 PTs in a 10-week (4 hours a week) content course focusing on number and operation at a large state university in the northwestern United States. All students participated in the creation and enactment of a FMN activity and completing three surveys on this experience throughout the term. The surveys were administered before the FMN, immediately after the FMN, and at the end of the term. The surveys were designed to allow the PTs to reflect on and share their experiences (see Table 1 for sample survey items).

Sample Survey Questions for reflections of the FMN activity
(a) How do you think the FMN activity contributes to your learning in this class? (S1)
(b) How do you think that creating and enacting a family math night activity this term affected your learning in this class? Please explain. (S3)
(c) Did anything surprise you? I expect that many things surprised you. I would love to hear at least two things that were surprising to you. (S2)
(d) What did you learn? (S2)

Table 1: Sample Survey Questions (S1=Survey 1, etc.)

Because little is known about PTs' reactions to the creation and enactment of a FMN experience, we used a grounded theoretical approach (Strauss & Corbin, 1990) with open coding (Strauss & Corbin, 1998) to analyse the written responses to the surveys. We read though all PT responses and identified themes while we read the responses. For example after reading Hannah's (all names are pseudonyms), statement "I'm excited about planning and enacting a FMN activity! It will be fun to put what we've learned into practice and spend some time with actual students," we created a category labelled "Excitement/Fun" and when we came across similar statements we placed them into that category and/or adjusted the category as needed (i.e. this category was initially established for survey 1 but reappeared on survey 3, thus the future and past tense were taken out to encompass both. Another instantiation of this category was Alex stating, "It was some of the most fun I've ever had with kids.")

RESULTS AND DISCUSSION

PTs entering the content courses often think their knowledge of mathematics is sufficient to teach K-3 and thus see this class as an inconvenient prerequisite rather than a class in which they can learn something useful. Sixteen of the 23 PTs in this study stated at the beginning of the course that they knew enough math to teach K-3, 2 were ambivalent, and 5 said they did not know enough math to teach K-3. However, these 5 did not refer to their lack of content knowledge, but rather to their lack of knowledge about teaching. For example, one stated, "While I think I can do all of the math in a K-3 class, I'm not sure I have the tools to teach those classes." So in general the PTs feel confident that their content knowledge suffices to teach K-3 and may not see any relevance to the course. The FMN activity made the learning in the university classroom immediately relevant to the PTs and thus has the potential to change the PTs' perception of the usefulness of the course. PTs were very excited about the authenticity of the task, focusing in particular on the fact that rather than simply planning the task, they were actually enacting it. Twenty of the 23 PTs mentioned this in their reflections. For example, Jennifer stated:

> Family Math Night is hands-on. Everything that we have learned about explanations and justification and talking about math are all <u>things I can apply</u> to this night. Reading about it and talking about it is only doing so much for my brain. <u>Actually putting it to real practice with real children</u> … is going to be extremely beneficial to my learning experience. Getting to watch it all play out will help me more.

The PTs were typically nervous but excited in creating and enacting a FMN activity. This excitement motivated them to engage with their tasks and further developed their mathematical understanding. Seventeen of the 23 PTs stated that they held a deeper understanding of the mathematics of their activity through FMN. One PT, Heather, reflected on how her knowledge changed throughout the activity:

> I learned the difference between sharing division and measuring division. It was hard for me to come up with word problems at first for both kinds of division but by the end of preparing the lesson I can do it.

Almost all (21 of 23) of the PTs reported after the FMN that their activity went well and most (16 of the 23) explicitly stated that it was a lot of fun. Alex, for example, stated

> It was some of the most fun I've ever had with kids. … actually seeing all the different ways children solved the problems was really fascinating. It was also interesting to see all the different levels of the children; we weren't expecting such young children, we had some first graders and a kindergartner, but they were able to solve a good portion of the problems.

The FMN activity provided this PT with an *authentic task*, enabling her to engage more deeply in the mathematical activities of the class. In addition, the FMN allowed PTs to realize other important aspects of mathematics teaching. For example, 18 of the 23 PTs commented on the fact that children come in and learn at different levels, which motivated the PTs to dig deeper into the mathematics and create multiple entry levels for students. Sixteen of the 23 PTs seemed genuinely surprised by the fact that children are interested in mathematics. PTs often are scared of mathematics and project their fear onto the children with whom they are working. Experiencing mathematics as fun for all is an essential element of teacher education. When asked whether the PTs would recommend future PTs to take a math class with family math night, all PTs recommended such a class, with 17 stating "definitely take it with the FMN" and 6 stating "take it with the FMN." They argued that "it truly is a practice of what we will be doing each day with students" and "my learning in this class was very focused and determined because I knew that I would need to know the subject well enough to teach it, and that is a whole different level of understanding for me." In later reflections, some PTs explained that it was the immediate applicability of their learning that motivated them to really pay attention in the class. This applicability is especially crucial for content courses, which are typically years removed from PTs working with children.

SUMMARY

In summary, we know that (a) the PTs experienced the FMN as an authentic activity, (b) the FMN activity was a highly motivating activity (the PTs had fun and realized that the children had fun too), (c) the PTs stated that they learned various things through the FMN activity (such as how children do/learn mathematics), and (d) the PTs learned mathematics through the FMN activity. However, we do not want to overstate our claims. We as yet have no data as to the effect the FMN activity might have on PTs' experiences in courses other than this content course. We are also not sure of the extent of the mathematics learning that happened as a result of the FMN activity. We believe that the PTs learned the mathematics of their activity at a deeper level (see Heather's comment about division above). We also believe that, at least for some PTs, the FMN activity affected their learning throughout the course (see Hannah's comment). Some questions that remain for further research: 1. Does the FMN activity change the PTs' stance towards learning mathematics in general (beyond the context of their task)? 2. At what level does the FMN activity affect the PTs' mathematics learning (local to the task, global to the course, global to the sequence of courses, global to mathematics)?

CONCLUSIONS

Learning mathematics in content courses designed for PTs is complicated. We (mathematics educators) are still working on understanding how to motivate the PTs to learn in our courses. Approaches such as working with children in an early field experience (Philipp et al., 2007), a one-on-one content interview (Thanheiser et al., 2013), and the FMN experience described in this paper have been shown to motivate PTs to learn mathematics content. (We want to note that the emphasis of these experiences is not on the "methods" aspect, i.e. *how do I create a lesson?* but rather on the "mathematics" of the activity, i.e. *What mathematics content do I want to work on with the children? What is my mathematical goal?*, etc.) Given the importance of mathematical content knowledge for teaching and the extensive research highlighting the lack of rich teacher content knowledge, the FMN experience described, which may traditionally not have been considered for inclusion in content courses taught in mathematics departments, may be precisely what we need to motivate our students to learn.

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THE ROLE OF TEACHING DECISIONS IN CURRICULUM ALIGNMENT

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The classroom implementation of open-ended mathematics tasks, such as Model-Eliciting Activities (MEAs), can be challenging for teachers. This case study research considers a teacher, Adam, implementing a lesson intended to be an MEA on graphical antiderivative. We describe the lack of alignment of the written, intended and enacted curricula that occurred. An analysis of Adam's conflicting resources, orientations and goals, and how these influenced his pedagogical decision making, enables a description of the reasons for this misalignment. One possible implication for teacher professional development arising from the case study is presented.

INTRODUCTION AND LITERATURE

Although curriculum designers often provide teacher guides on how to implement certain tasks, research shows, and even assumes that "fidelity between written plans in a teacher's guide and classroom action is impossible" (Stein, Remillard, & Smith, 2007, p. 344). Teachers draw on their experiences, goals, knowledge and beliefs to interpret written curricula to form their own implementation plans, which are further transformed upon entering the classroom setting by the actions and thinking from students and the teacher. As a result, the curriculum experienced by the students in a classroom can differ considerably from what the teacher had intended to implement, and what the curriculum designers hoped would be implemented.

This misalignment between written, intended and enacted curricula is often greater for open-ended, non-routine tasks, than for conventional tasks such as procedural exercises (Stein et al., 2007). Open-ended tasks are more dependent on the responses from students and teachers, who may not be used to implementing them. We found great divergence in the ways teachers implemented a particular open-ended task, called a Model-Eliciting Activity (MEA) (Lesh, Hoover, Hole, Kelly, & Post, 2000), involving antidifferentiation in a tramping (hiking) context. In this paper, we report on one case study of a teacher, Adam, who planned to implement the Tramping MEA within a 50 minute lesson, but spent the whole lesson setting up the task, and ran out of time to launch the modelling problem itself. We investigate the research question: What caused the misalignment between Adam's enactment of the Tramping MEA as it was implemented, and his prior intended implementation of the MEA?

The literature suggests that the alignment of written, intended and enacted curricula is influenced by a number of factors, including: teachers' theories of teaching and learning (Biggs, 1996); deep levels of teacher pedagogical and content knowledge (Jaworski, 2012); and the need to deal with a wide range of incumbent educational

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 241-248. Vancouver, Canada: PME.

priorities (Skott, 2001). We analyse how Adam's conflicting goals, orientations and resources (Schoenfeld, 2011) influenced the eventual misalignment between his enacted curriculum and the written and intended curricula regarding the MEA.

THEORETICAL FRAMEWORK

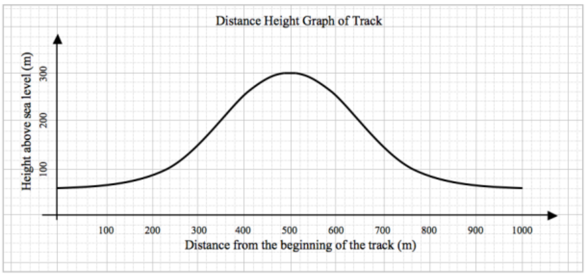
Stein, Remillard and Smith (2007) distinguish between the three broad meanings of curriculum. The *written curriculum* comprises the curriculum materials that are given to teachers, and which may include textbooks, curriculum documents, and specific mathematical tasks. The *intended (or planned) curriculum* is the teacher's interpretation of how they plan to implement the curriculum materials, and the *enacted curriculum* is what is actually implemented in the classroom. A number of factors can affect transitions between these three curricula, which may look quite different, despite being based on the same materials. These include: teacher beliefs, knowledge, orientations, and professional identity; students' capacities and willingness to engage; time, school and classroom culture, and characteristics of the curriculum (Stein, Remillard & Smith, 2007, p. 322). Many of these factors can be incorporated into Schoenfeld's (2011) theoretical framework for decision-making, which is based on *Resources, Orientations*, and *Goals* (ROGs). In this framework a teacher's orientations, which include beliefs, dispositions, attitudes and so forth, determine the goals established in any given situation. The teacher draws on and orchestrates available resources, such as mathematical knowledge for teaching (Ball, Hill & Bass, 2005) and physical artefacts to attain goals. In each lesson a teacher will have a number of competing goals in broad areas such as classroom management, student engagement and student learning outcomes. The manner in which she balances these competing goals and their dynamic relationships, and the extent to which pragmatism is allowed to intervene, will be determined by the relative strength of her orientations.

In this paper, we use Schoenfeld's ROG framework to analyse the misalignment between one teacher's enacted curriculum (the implementation of a particular MEA), his planned curriculum and the written curriculum.

THE TASK: THE TRAMPING MODEL-ELICITING ACTIVITY

The task in this study was a Model-Eliciting Activity, or MEA (Lesh, Hoover, Hole, Kelly & Post, 2000), set in the context of tramping (the term for hiking in New Zealand). MEAs are a class of tasks designed to provide students with authentic experiences of modelling a mathematically rich context. Any given MEA consists of three components: (1) A newspaper article, picture, or video for contextualising the problem, (2) A set of brief warm-up questions, and (3) The modelling problem itself. Correspondingly, the Tramping MEA begins with a newspaper article that describes shortcomings of difficulty ratings for tramping tracks (hiking trails) in New Zealand. After reading the newspaper article, students work on warm-up questions to familiarise themselves with the tramping context and mathematical tools that are necessary to start (but not necessarily successfully finish) the problem. For example, in the Tramping

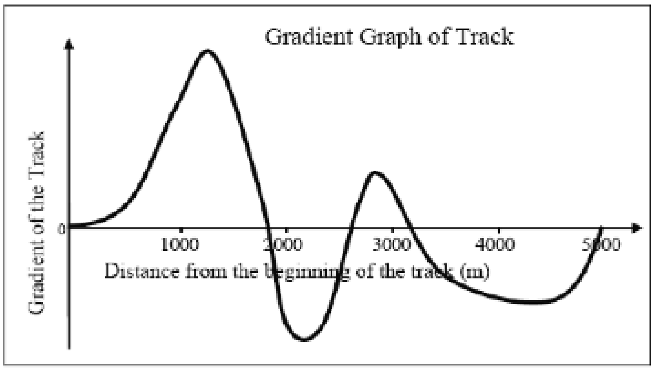
MEA, the warm-up questions ask students to plot the gradient graph (i.e., derivative) of a distance-height graph of a simple tramping track so that students can understand what a gradient graph and distance-height graph are (see Figure 1). However, it does not give them any tips on how to construct a distance-height graph from a given gradient graph, which is the focus of the actual modelling problem.



- How high is the highest part of the track?
- Find the gradient of the track at (a) 400m (b) 600m.
- What does the track look like when the gradient is zero?
- Where is the track steepest uphill? How can you tell?
- Plot the gradients of the track on the axes below and join them with a smooth curve.
- The graph you have drawn is a "gradient graph" of the tramping track. How is it related to the "distance-height graph of the track?

Figure 1: Excerpts from the set of warm-up questions in the Tramping MEA.

The heart of the MEA is the modelling problem itself, which is designed according to six principles (Lesh et al., 2000) to encourage students to express, test and revise their initial mathematical interpretations via multiple modelling cycles. The modelling problem component of the Tramping MEA (Figure 2) asks students to create a method for visualising the terrain of tramping tracks from a graph of the track's gradients—a task that is mathematically equivalent to finding the antiderivative of a function presented graphically. Students are also asked to generalise their method, and to communicate their method in writing.



The O'Neills need your help!

Design a method that the O'Neills can use to sketch a distance height graph of the original track (like the one given in the warm up question). You can assume that the track begins at sea level.

Write a letter to the O'Neills explaining your method, and use your method to describe what the tramping track will be like on the day. In particular, you must clearly show any summits and valleys in the track, uphill and downhill portions of the track, and the parts of the track where the slopes are steepest and easiest.

Most importantly, your method needs to work not only for this tramping track, but also for any other tramping track the O'Neills might consider.

Figure 2: The problem statement for the Tramping Problem MEA.

The modelling problem component of an MEA is intended to be challenging, and students are typically given at least 30 minutes to work on it in groups of three. In contrast, the newspaper article and warm-up components of the MEA are meant to take less than 15 minutes. The warm-up component ensures that students can *start* the modelling problem, but it does not guarantee immediate success; instead, students are likely to begin the modelling problem with primitive mathematical interpretations, and only develop more insightful mathematical models through expressing, testing and revising their ideas within their groups.

METHOD

As part of an international project, seven secondary school teachers implemented the same set of curriculum materials, consisting of four graphical antiderivative tasks, in their classrooms in New Zealand, Israel and Italy. Prior to this study, the tasks had only been implemented in semi-clinical environments using volunteers who worked in pairs, outside of class time, and in the presence of a researcher who was not their teacher. A goal of the international project was to see how teachers could transform and implement these tasks in authentic secondary school classroom environments.

The three New Zealand teachers in the project were instructed to adapt the materials as they saw fit to suit their particular classroom environment. For example, they were encouraged to change the context of the first task (the Tramping MEA) if they wished, and use videos or apps they thought might be useful. They were specifically told to reduce the warm-up questions to fit into 15 minutes (about one-third) of the whole lesson time, and were given concrete suggestions to achieve this, such as reducing the number of calculations students needed to perform, having students gesture rather than plot the gradients, or providing the gradient graph for students. The teachers were also reminded that at least 30 minutes of the lesson time should be allocated to having students working in groups on the modelling problem.

We report on Adam's implementation of the first task, the Tramping MEA. Adam was in his second year of secondary school teaching, and the class was a high ability year 12 (age 16-17) mathematics class in a low socioeconomic school in Auckland, New Zealand, with predominantly Maori and Pacific Island students. The topic of graphical antiderivatives was not part of the New Zealand curriculum (NCEA), although the topic of graphical derivatives was. After each lesson, Adam participated in audiotaped debriefing interviews, in which he described explained his teaching decisions in the lesson and planned for subsequent lessons. These four interviews were transcribed and coded according to Schoenfeld's ROG framework, and used to create descriptions of Adam's overall espoused ROG, and his specific ROGs for each lesson. Adam personally checked and corroborated the coding. The videos of the lessons were transcribed and annotated with photos and descriptions of the teacher's and students' actions, gestures, boardwork and written work.

RESULTS

In accordance with MEA epistemology (Lesh et al., 2000), the primary goal of the Tramping MEA is to have students express, test and revise their initially primitive interpretations of graphical antiderivatives, and develop more powerful ones through modelling cycles over the course of 30 minutes as they worked on the modelling problem. An ancillary goal is to ensure all students can start the modelling problem, by engaging them in a brief 15-minute warm-up beforehand. This means that the entire MEA can fit into one 50-minute lesson. Although Adam also intended to implement the entire Tramping MEA during his 50-minute lesson, he only managed to enact the warm-up questions and ran out of time to enact the modelling problem.

A description of the enacted curriculum

Table 1 describes a timeline of events in Adam's 50-minute lesson. Adam spends the first 4 minutes 50 seconds of his lesson in setting up the context for the MEA. He introduces two contexts—a tramping one, using pictures of mountains accompanied by music, and a rollercoaster one, describing going up and down a rollercoaster and the speed at different points. Next, Adam spends 9 minutes 10 seconds reading out loud the warm-up questions and reviewing the notion of a tangent and its relationship to the gradient at a point on a curve. He then has students work on the first set of warm-up questions in small groups for 15 minutes 37 seconds.

Time	Description
Start—4min 50s	Adam introduces tramping and roller coaster contexts (whole class)
—15min 0s	Adam reviews tangents of gradients (whole class)
—30min 37s	Students work on warm-up questions (group work)
—39min 52s	Adam discusses solutions to first warm-up questions (whole class)
—43min 33s	Students plot warm-up gradient graph (group work)
—50min 0s	Adam discusses gradient graph and features (whole class)
End	Adam tells students to do tramping modelling problem for homework

Table 1: A timeline description of Adam's implementation of the Tramping MEA.

At 30 minutes 37 seconds into the lesson, Adam leads a whole class discussion of possible solutions to the warm-up questions, asking for answers and writing them on the board. During this time, he invites a student to the board to demonstrate his answer to the question, "where is the track steepest uphill", which takes less than 2 minutes. After 9 minutes of teacher led discussion, Adam tells students to plot the gradient graph in the warm-up and proceeds to walk around the classroom observing and helping students. Once again, at 43 minutes and 33 seconds, he invites a student to draw his gradient graph on the whiteboard. The student does so in less than 2 minutes, and although the solution is reasonable Adam says, "Let me just polish this", and redraws the end points of the student's graph to show them trailing off towards the x-axis at the sides, then discusses the concept of asymptotes.

With less than 4 minutes of the lesson remaining, Adam decides to erase the graph he has drawn so far and says "I think I should draw it better." He redraws the graph so that the vertical correspondence of the points with the graph above aligns better with inflection points matching maximums and minimums, and so forth. This is followed by a detailed explanation of the relationship between critical points on each of the graphs, using the terms gradient, increase, maximum, point of inflection, steepest, positive, negative, zero, more negative and less negative. When the bell rings, signalling the end of the lesson, Adam realises that he hasn't yet implemented the modelling problem (Figure 2) so tells students to complete it at home.

ANALYSIS

Adam's failure to implement the modelling problem component of the MEA within the 50-minute lesson can be explained by his adherence to eight, sometimes conflicting, goals, which are summarised in Table 2.

G(A)	**To prepare students for success on future tasks**
G(B)	**To engage in student-centred learning as much as possible**
G(C)	**To complete the entire MEA (warm-up and modelling problem) within the lesson time**
G(D)	**To align the MEA with the national curriculum assessment (NCEA)**
O_1(D)	Belief that although NCEA should drive his teaching, the MEA helps develop understanding so he's happy to make an exception.
O_2(D)	Belief that the NCEA curriculum is the more important, "real curriculum".
O_3(D)	Concern about time spent doing something that is external to the curriculum.
R_1(D)	Knowledge that the content of the MEA lies outside the NCEA curriculum
G(E)	**To cover all the content in a structured and ordered manner**
O_1(E)	Fear of leaving anything out.
O_2(E)	Belief that he has to cover everything he is given.
G(F)	**To make sure the content is in a context meaningful for the students**
O_1(F)	Belief that the context must be meaningful for student understanding
O_2(F)	Belief that a second context will be needed in addition to tramping
R_1(F)	Knowledge that his students are familiar with roller-coasters
G(G)	**To make sure students understand all the content correctly**
O_1(G)	Belief that students need a firm foundation before working on a problem
O_2(G)	Belief that understanding develops over time
R_1(G)	Knowledge that it takes time to develop an understanding of new ideas
G(H)	**To use the MEA to revise previous content on graphical derivative**
O_1(H)	Belief that the modelling problem is too difficult for his students.
O_2(H)	Belief that the warm-up can cover concepts that students didn't "get" previously
R_1(H)	Prior knowledge from students' tests that many couldn't create gradient graphs.

Table 2: The eight goals (A-H) and their corresponding resources and orientations.

Goals A and B arise from Adam's four debriefing interviews, and are described in further detail together with their associated orientations and resources in Thomas and Yoon (2013). Goal C was evident from Adam's lesson plan, in which he stated his intent to complete the entire MEA (warm-up and modelling problem) within the 50-minute lesson. The remaining five goals (D, E, F, G and H) and their associated orientations and resources emerged from Adam's first debriefing interview. Figure 3 shows the core conflict between Adam's desire to implement the entire MEA (goal C) and his desire to prepare students for upcoming tasks (goal A). Goal A was supported from two core directions by a complex, connected network of seven goals. In the first instance, goals B, E and F supported Adam's desire to ensure students understood all the content (goal G) as they contribute to enhanced understanding through student-centred participation, a structured approach to learning provided by the teacher and a meaningful context. The second influence on his decision making came from Adam's desire to align the MEA with the NCEA curriculum (goal D), which also strongly supports his goal to prepare his students well for future tasks. Both of these directional influences on goal A were supported by goal H, to use the MEA for revision purposes, and the cluster of goals caused him to proceed slowly through the warm-up so students would be prepared for the modelling problem itself. This left his other goal, C, to complete the warm-up and modelling problem within the allotted lesson time alone, unsupported and eventually unachieved.

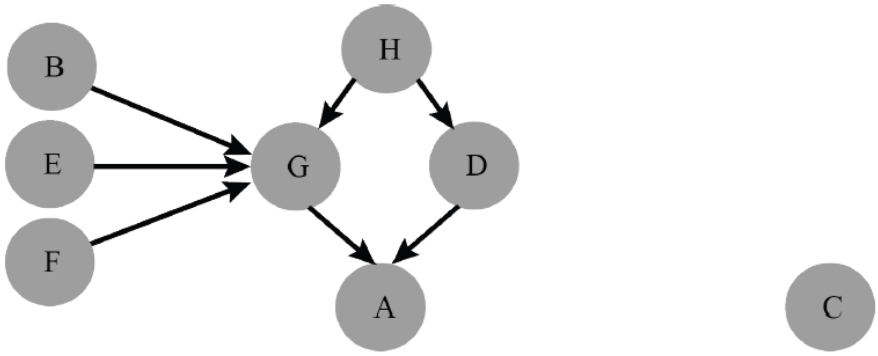


Figure 3: Connections between seven of the goals and the isolation of goal C.

DISCUSSION

The recent ICMI study on task design reminds us that issues surrounding task design and teacher classroom implementation of tasks are complex but crucial to address. The challenges facing the teacher are highlighted in implementation of MEAs due to the greater variability of approach and possible student interpretation. This makes higher demands of the teacher in order to align the written MEA curriculum with the enacted one. Our study supports Stein et al.'s (2007) model of curriculum phases and confirms that teacher orientations play a crucial role in curriculum alignment. A teacher like Adam wants students to feel that they are able to tackle tasks successfully (Smith, 2000) and hence he believes he needs to prepare them thoroughly. However, this desire brought Adam's ROG into conflict with the MEA writer's ROG. The latter includes the belief that the warm-up should enable students to start the modelling phase of the MEA but then they need to struggle with it in order to learn. Adam's goal was to remove the necessity for this struggle by thorough preparation, and hence his decisions led to the lack of alignment. Other contributing factors included Adam's inexperience with the content and approach of the MEA and his unfamiliarity with both the mathematical content and the approach to be employed. Hence, although the time available was sufficient to cover the material in the MEA he did not utilise it in the manner he intended. A second, strong influence on Adam was the constant context of student preparation for the national assessment of the curriculum (NCEA). His belief that he should align the MEA work with this if at all possible led to an inability to separate the lesson from the wider assessment context. Hence, he decided to include as a substantial part of the lesson a revision component that aligned the enacted curriculum with the NCEA one, but caused a misalignment with the MEA curriculum. Thus, out of concern for his students, he was unable to reduce the amount of taught content in the warm-up, and hence the time spent on it, in order to allow the students time to struggle with the modelling activity in the MEA.

This case study of Adam's teaching suggests that including a mechanism for post-implementation lesson discussion in professional development could assist teachers to become more aware of their orientations and goals and the influence these have on decision making and curriculum alignment. Professional development activity that promotes such awareness may be one way to encourage pedagogical change. We believe that outcomes like those presented here could also be used to assist with lesson implementation, to motivate teacher discussion, and hence could supplement a teacher guide.

Acknowledgements

We wish to acknowledge Tessa Miskell for her transcription and video annotation.

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WHEN UNDERSTANDING EVOKES APPRECIATION: THE EFFECT OF MATHEMATICS CONTENT KNOWLEDGE ON AESTHETIC PREDISPOSITION

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This study explored the problem solving experience of pre-service teachers in finding the greatest common factor and the least common multiple using many different approaches. In particular, it examined the effect of pre-service teachers' mathematics content knowledge on how they chose their preferred approach and how they valued the most efficient approach. The findings indicated that the most efficient approach was appreciated only if such approach was reasonably understood by these pre-service teachers.

INTRODUCTION

Aesthetic values play a central role in experts' mathematics problem solving experience (Silver & Metzger, 1989). Typically, a problem solving approach is considered "beautiful" if it is particularly clear, simple, and unexpected. Beginning problem solvers have also demonstrated the ability to develop and favour certain problem solving approaches often considered more efficient than others (Silver, Leung, & Cai, 1995). Nonetheless, little is known about the extent to which beginning problem solvers' mathematics content knowledge influenced how they chose their preferred approach and how they valued the most efficient approach. In particular, does an aesthetic appreciation for the most efficient approach necessitate certain understanding of that approach? Is it possible to appreciate the most efficient approach if one lacks the understanding of problem solving using many different approaches? Does knowing more than one approach allow for more flexibility in problem solving?

The purpose of this article is to investigate the effect of pre-service teachers' mathematics content knowledge on their aesthetic predisposition in their problem solving experience involving problems of finding the greatest common factor (GCF) and the least common multiple (LCM) of two numbers. It begins with the theoretical background on the benefits of problem solving using many different approaches and the mathematics aesthetic aspect of experts' problem solving practices, as well as examples of beginning problem solvers' conceptions of what it means for an approach to be efficient. In connection with the instruments used in the methodology, several approaches for finding the GCF and LCM are discussed. The article continues with the findings and consequent analysis, and concludes with pedagogical recommendations that promote the habit of mind of creative problem solving and the mathematics aesthetic appreciations.

THEORETICAL BACKGROUND

In recent years, mathematics problems solving using many different approaches has drawn more attention than before (Leikin & Levav-Waynberg, 2007). Some researchers, in fact, considered such practice to be beneficial for students' mathematics learning experience.

Silver et al. (2005) believed that students "can learn more from solving one problem in many different ways than [they] can from solving many different problems, each in only one way" (p. 288). They particularly advised students interested in mathematics to obtain more experience in problem solving with many different approaches. They regarded such experience as having "the potential advantage of providing students with access to a range of representations and solution strategies in a particular instance that can be useful in future problem-solving encounters" (p. 288). They also considered the use of many different approaches in order to "facilitate connection of a problem at hand to different elements of knowledge with which a student may be familiar, thereby strengthening networks of related ideas" (p. 288).

Tabachneck, Koedinger, and Nathan (1994) recognized the purpose of adopting many different approaches in problem solving. They argued that on its own, each approach might entail disadvantages and weaknesses. In order to overcome these, they recommended students operate a combination of approaches, instead of counting on only one approach. More specifically, they maintained that students could benefit from employing this learning style in mathematical problem solving. In addition to teaching to solve one problem with many approaches, psychologists encouraged teaching a coherent interrelation among those approaches (Skemp, 1987; De Jong et al., 1998; Van Someren et al., 1998; Bodemer et al., 2004). Equally important, Reeves and Weisberg (1994) recommended showing students many analogical problems or examples concurrently. On the whole, cognitive psychologists took a positive stance on problem solving using many approaches, as did mathematics education researchers.

Given the many possible different approaches to solve the same problem, a decision to choose one approach over the many other approaches may be less than arbitrary. Aesthetic aspects were particularly considered in many studies connected with experts' preference in problem solving approaches.

Silver and Metzger (1989) assessed the role of the aesthetics in a study involving university professors in mathematics. They found that these expert problem solvers displayed signs of aesthetic emotion. On one occasion, a subject resisted the temptation to resort to the use of calculus in solving a geometry problem, acknowledging the possibility of a "messy equation" (p. 66). Only after some unsuccessful attempts to seek a geometric approach did the subject concede to solving the problem using calculus. Although successful, he felt that "calculus failed to satisfy his personal goal of understanding, as well as his aesthetic desire for 'harmony' between the elements of the problem and elegance of solution" (p. 66). On another occasion, having solved another geometry problem algebraically, the same subject appeared unsettled, recognizing that a geometric approach could be "more elegant" (p. 66).

Dreyfus and Eisenberg (1986) were interested in exploring whether students considered aesthetic values of mathematical reasoning in their problem solving approaches. Their study involved college-level mathematics students who had been rigorously prepared in advanced mathematics courses. They were tested on several carefully chosen mathematics problems which involved many different approaches not immediately apparent to average students, yet readily accessible with high school mathematics knowledge. After completing the test, students were presented with approaches that were considered elegant by expert mathematicians.

Dreyfus and Eisenberg (1986) discovered that not only were the students not able to supply elegant approaches in the test as they had been expected to, but they were also not able to recognize the differences between elegant and pedestrian approaches. Furthermore, when presented with elegant approaches, they showed no enthusiasm and found them no more attractive than their own approaches. In other words, they had no sense of aesthetic appreciation. Dreyfus and Eisenberg (1986) concluded that mathematics instruction in classroom settings lacked an emphasis on reflective thinking, especially aesthetic value.

Sinclair (2004) analysed the role of aesthetic values from several conceptual insights. She drew examples from existing empirical findings such as those by Dreyfus and Eisenberg (1986) and Silver and Metzger (1989). In one of her interpretations of their work, she suggested that "mathematicians' aesthetic choices might be at least partially learned from their community as they interact with other mathematicians and seek their approval" (Sinclair, 2004, p. 276). Furthermore, she indicated that mathematical beauty was only feasible in the process "when young mathematicians are having to join the community of professional mathematicians—and when aesthetic considerations are recognized (unlike at high school and undergraduate levels)" (p. 276).

Nevertheless, few studies have demonstrated that beginning problem solvers might actually be capable of recognizing mathematical "beauty" from the standpoint of efficiency. Nesher, Hershkovitz, and Novotna (2003) investigated students' choices of approaches to solving algebra problems. Specifically, they were interested in ninth grade students' use of independent variables when solving algebra word problems. These word problems involved a situation with three unknown quantities whose sum was known. In interviewing the students, the researchers found that the students' choices of independent variables were mainly influenced by the order in which the quantities were described in the word problems. At the same time, students favoured independent variables with the smallest quantity in relation to the other two quantities discussed in the problems. By doing so, students unconsciously revealed their natural inclination to working with whole numbers as opposed to rational numbers. To some extent, students were capable on their own of constructing the notions of the more efficient approach in problem solving.

METHODOLOGY

This study involved 37 pre-service teachers (31 female, 6 male, age 20-24) in an elementary (age 5-12, grade K-6) education program at a large, urban university. These 37 pre-service teachers were enrolled in a mathematics content course in which the researcher was the instructor. Four approaches for the GCF and four approaches for the LCM were introduced to these pre-service teachers during the instruction time of two 50-minutes sessions.

Using an example of finding the GCF and LCM of 24 and 36, the eight approaches are discussed as follows. The first approach for finding the GCF is the Set Intersection Method where given all factors of 24 (e.g., 1, 2, 3, 4, 6, 8, 12, and 24) and 36 (e.g., 1, 2, 3, 4, 6, 9, 12, 18, 36), the common factors of 24 and 36 are 1, 2, 3, 4, 6, and 12, of which 12 is the largest. The second approach for finding the GCF is the Prime Factorization Method where after expressing 24 and 36 in their prime factor exponential forms (e.g., $24 = 2^3 \cdot 3^1$ and $36 = 2^2 \cdot 3^2$), the GCF consists of the prime factors with the smaller exponents (e.g., $2^2 \cdot 3^1 = 12$). The third approach for finding the GCF is the Repeated Subtractions Method where the GCF is obtained by repeatedly subtracting the smaller number from the larger number until both numbers are equal (e.g., GCF(24,36)=GCF(36-24,24)=GCF(12,24)=GCF(24-12,12)=GCF(12,12)=12). The fourth method for finding the GCF is the Euclidean Algorithm Method where the GCF is obtained by repeatedly dividing the larger number by the smaller number until a remainder of zero is obtained (e.g., 36÷24=1R12, 24÷12=2R0, and thus, GCF is 12).

The first approach for finding the LCM is the Set Intersection Method where given some multiples of 24 (e.g., 24, 48, 72, 96, 120, 144, …) and 36 (e.g., 36, 72, 108, 144, …), the common multiples of 24 and 36 are 72, 144, …, of which 72 is the smallest. The second approach for finding the LCM is the Prime Factorization Method where after expressing 24 and 36 in their prime factor exponential forms (e.g., $24 = 2^3 \cdot 3^1$ and $36 = 2^2 \cdot 3^2$), the LCM consists of the prime factors with the larger exponents (e.g., $2^3 \cdot 3^2 = 72$). The third approach for finding the LCM is the Build-up Method where after expressing 24 and 36 in their prime factor exponential forms (e.g., $24 = 2^3 \cdot 3^1$ and $36 = 2^2 \cdot 3^2$), the LCM is obtained by building up the prime factors to the larger exponents (e.g., because $2^2 \cdot 3^2$ has more threes than $2^3 \cdot 3^1$, we build up from $24 = 2^3 \cdot 3^1$ to have the same number of threes as $2^2 \cdot 3^2$, making the LCM $2^3 \cdot 3^2 = 72$). The fourth approach for finding the LCM is using the Theorem Method which states that the product of two numbers is equal to the product of their GCF and LCM (e.g., because the GCF of 24 and 36 is 12, LCM of 72 is obtained by dividing 24×36 by 12).

After the instruction, the pre-service teachers were evaluated by means of a quiz and a survey. In a quiz of 12 problems, problems 1, 2, 3, and 4 involved finding the GCF of 45 and 75 using the first, second, third, and fourth approaches, respectively. Problems 5, 6, 7, and 8 involved finding the LCM of 45 and 75 using the first, second, third, and fourth approaches, respectively. Problem 9 and 10 involved finding the GCF and LCM of 12 and 18 using any method. Problem 11 and 12 involved finding the GCF and LCM of 2,873 and 3,757 using any method. Each problem in the quiz was scored as 1 if the correct answer was supported by clear explanations and logical arguments; otherwise, it was scored as 0. Thus, the quiz score ranged from 0 to 12. In the survey of two questionnaires, they were asked about their preference of finding the GCF and LCM based on their problem solving experience. They were also required to write one or two paragraphs explaining any criteria they identified for their choices of preferred approaches, as well as providing a comparison and contrast analysis of the different approaches for finding the GCF and LCM.

FINDINGS

Based on the survey, the first, second, third, and fourth approaches of finding the GCF were preferred by 2, 7, 6, and 22 pre-service teachers, respectively. The first, second, third, and fourth approaches of finding the LCM were preferred by 3, 20, 5, and 9 pre-service teachers, respectively.

Although the majority of the pre-service teachers recognized that the Set Intersection Method for finding the GCF and LCM was “clunky” and “only works for small numbers,” they agreed that such method was conceptually the more “natural” way of making sense of the GCF and LCM. The Prime Factorization Method for finding the GCF and LCM was the more “familiar” approach that most pre-service teachers “learned in grade school.” The pre-service teachers considered the Euclidian Algorithm Method the most efficient approach for finding the GCF because it “works for any numbers, including large ones” and “simplifies the steps in the Repeated Subtractions Method.” On the other hand, the Build-up Method was not favourable because it was viewed as less efficient than the Prime Factorization Method. While the Theorem Method was not the most popular approach, those who preferred it said it was the most efficient and “easiest” approach “if you figure out the GCF beforehand, especially for big numbers.”

Supposing that the Euclidian Algorithm Method and the Theorem Method were the most efficient approaches for finding the GCF and LCM, respectively, as the pre-service teachers assessed in general, it was apparent that those who preferred either of those two approaches performed well above those who preferred other approaches. The average scores of all problems of the pre-service teachers who preferred the first, second, third, and fourth approaches of finding the GCF were 4, 6.7, 6.6, and 9.4, respectively. The average scores of all problems of the pre-service teachers who preferred the first, second, third, and fourth approaches of finding the LCM were 8, 7.6, 8, and 11, respectively.

In relation to their understanding of the most efficient approaches for finding the GCF and LCM, the pre-service teachers’ performance on problems 4 and 8 (problems involving the most efficient methods for the GCF and LCM, respectively) was highly indicative of their likelihood of preferring those most efficient approaches. The average scores of problem 4 of the pre-service teachers who preferred the first, second, third, and fourth approaches of finding the GCF were 0, 0.1, 0.1, and 0.8, respectively. The average scores of problem 8 of the pre-service teachers who preferred the first, second, third, and fourth approaches of finding the LCM were 0.3, 0.1, 0.2, and 0.9, respectively.

Ten of the 12 pre-service teachers who successfully solved problems 11 and 12 (problems involving finding the GCF and LCM of larger numbers) solved both problems using and chose as their preferred approach the Euclidian Algorithm Method or the Theorem Method. Only twelve of the 25 pre-service teachers who successfully solved problems 9 and 10 (problems involving finding the GCF and LCM of smaller numbers) but not problems 11 and 12 chose as their preferred approach the Euclidian Algorithm Method or the Theorem Method. In other words, the more-mathematically-able pre-service teachers were about twice as likely, in proportion to their group membership, both to solve them using and to prefer the Euclidian Algorithm Method or the Theorem Method to other approaches as the less-mathematically-able pre-service teachers. To this extent, the pre-service teachers' mathematics content knowledge was a determining factor in their appreciation for the most efficient approach.

Nevertheless, like the majority (23) of the 25 pre-service teachers who successfully solved problems 9 and 10 but not problems 11 and 12, 11 of the 12 pre-service teachers who successfully solved problems 11 and 12 was more likely to solve problems 9 and 10 (problems involving finding the GCF and LCM of smaller numbers) using either the Set Intersection Method or the Prime Factorization Method than any other approaches. Evidently, the more-mathematically-able pre-service teachers appeared to be more flexible in choosing problem solving strategy, depending on the level of difficulty of the problems, in particular, the magnitude of the numbers involved in the problems.

This, to some extent, demonstrated, from a point of view of number theory, a similar notion of the "apparently counter-intuitive inverted aptitude-strategy relationship" based on the findings by Roberts, Gilmore, and Wood (1997): the pre-service teachers who were more fluent in the more sophisticated approaches for finding the GCF and LCM (e.g., the Euclidian Algorithm Method and the Theorem Method) ingeniously avoided the use of such sophisticated approaches when solving simpler problems. One explanation to this flexibility might be that these more-mathematically-able pre-service teachers consciously attended to one attribute of an elegant approach, namely, simplicity (Silver & Metzger, 1989): simpler problems could and should be solved using a more elementary strategy (e.g., the Set Intersection Method or the Prime Factorization Method), instead of a more advanced strategy (e.g., the Euclidian Algorithm Method or the Theorem Method).

Indeed, some of them explained that "I only began to see why we need to learn many different approaches until you gave us the last four problems [problems 9, 10, 11, and 12] all at once," while others deduced that "some methods work for some problems, while other methods work better for other problems." Their explanations suggested, to some extent, that they valued the need to study more than one approach in order to better appreciate other approaches. It was clear that the more approaches they understood, the more positive they were towards the practice of problem solving using many different approaches, and thus, the more mathematically mature they became to appreciate the various characteristics of a mathematically “beautiful” approach, including the idea of efficiency and simplicity.

Such flexibility in adapting alternative approaches was yet not observed in the eight pre-service teachers who successfully solved problems 9 and 10 but not problems 11 and 12. These less-mathematically-able pre-service teachers persisted in applying either the Set Intersection Method or the Prime Factorization Method to solve problems 11 and 12, albeit unsuccessfully. While the Set Intersection Method could be viewed as the more “natural” way of conceptualizing the GCF and LCM, this evidence suggested, to some extent, that such approach was realized by these less-mathematically-able pre-service teachers more at the procedural level, rather than at the conceptual level.

CONCLUSIONS AND DISCUSSIONS

The current study explored the relationship between pre-service teachers’ mathematics content knowledge and their preferred approaches for finding the GCF and LCM, as well as their predisposition to favour mathematically “beautiful” approach. Two major findings were observed. First, the more-mathematically-able pre-service teachers were more likely than the less-mathematically-able pre-service teachers to recognize the most efficient approach. An aesthetic appreciation for the most efficient approach appeared to necessitate a certain level of understanding of that approach; to some extent, it was not possible to appreciate the most efficient approach if one lacked the understanding of problem solving using many different approaches. Second, the more-mathematically-able pre-service teachers were more likely than the less-mathematically-able pre-service teachers to adaptively vary their problem solving strategies to accommodate the level of difficulty of the problems. The more tools they could work with to solve a problem, the more options they had when reflecting to decide which tool would be appropriate for which situation.

Two pedagogical recommendations might be proposed. First, mathematics learning experience, perhaps as early as the elementary school level, could involve problem solving using many different approaches. Given sufficient exposure to a variety of different methods to solve the same problem involving the same mathematics concept, beginning problem solvers might become not only fluent in many different problem solving approaches but also creative in looking for novel problem solving approaches and flexible to recognize the appropriateness of utilizing certain problem solving approaches in solving particular situations. Second, aesthetic appreciations towards mathematical “beauty” could be nurtured to young children, even if they might only concern with the idea of efficiency in terms of time and the number of steps to solve a problem. To this end, mathematics teachers could promote classroom discussions that required students to compare and contrast different problem solving approaches.

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DESIGNING TASKS FOR CONJECTURING AND PROVING IN NUMBER THEORY

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The purpose of this study is to develop design principles for crafting tasks that will encourage conjecturing and proving in the context of elementary number theory at the undergraduate level. From the analyses of the written work of 46 prospective mathematics teachers on a task designed according to these principles, we think that there is potential to build on and refine from these principles for other undergraduate mathematics courses.

INTRODUCTION

> Paul Erdös, one of the greatest mathematicians of the twentieth century, and certainly the most eccentric … believed that the meaning of life was to prove and conjecture. (Schechter, 2000)

Most mathematicians would agree that making conjectures and then proving them is an indispensable component of practicing mathematics. The acts of conjecturing and proving also have immense educational value. The NCTM *Principles and Standards for School Mathematics* states that school programs at all levels should enable students to "recognize reasoning and proof as fundamental aspects of mathematics; make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs." (NCTM, 2000, p. 56). This is echoed by Lin et al. (2012, p. 308) who argued that "tasks of conjecturing and proving should be designed to be embedded into any level of mathematics classes in order to enhance students' conceptual understanding, procedural fluency, or problem solving." In addition, it would seem that for these acts of conjecturing and proving to be actualised in schools, it is even more imperative that prospective mathematics teachers should learn them in their mathematics training. This paper describes an attempt to develop design principles for crafting tasks that will encourage conjecturing and proving in an elementary number theory course for undergraduate prospective teachers.

BACKGROUND

It is widely accepted that the act of proving enhances students' mathematical concepts and reasoning (Hanna, 2000), however the enactment in the curriculum sometimes result in students possessing a distorted view of what constitutes a mathematical proof. Selden (2012) suggests that the requirement to construct two-column geometry proofs may be partially responsible for some students' perception that proofs are always constructed in a linear fashion. Hoyles (1997) argues that students see little meaning

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 257-264. Vancouver, Canada: PME.

and purpose in the act of proving mathematical statements, especially those which they already assumed to be true. Schoenfeld, after a series of studies exploring students' understanding of geometry, formulated these erroneous students' beliefs: (1) The processes of formal mathematics (e.g. "proof") have little or nothing to do with discovery or invention. (2) Students who understand the subject matter can solve assigned mathematics problems in five minutes or less. (3) Only geniuses are capable of discovering, creating, or really understanding mathematics (1988, p. 151).

One possible remedy to address these wrong perceptions is to provide students with opportunities to formulate and prove their own conjectures (Lin et al., 2012). As teacher educators, we recognize that correcting these perceptions in prospective teachers is a crucial step in arresting the propagation of these erroneous beliefs. The design of suitable tasks to elicit these dispositions from the prospective teachers is an important step towards this end.

This study arose from the efforts of the first author – henceforth referred to in the first person singular pronoun – to design tasks to promote conjecturing and proving in an undergraduate elementary number theory course for prospective teachers. It is widely accepted that elementary number theory provides an appropriate context for undergraduates to learn proofs and engage in conjecturing (Ferrari, 2002; Selden & Selden 2002; Zazkis & Campbell, 2006). Thus I incorporated into the course problem solving tasks that explicitly required the prospective teachers to engage in conjecturing and proving. The tasks were designed according to these principles: (1) In line with the content emphasis of the course, the problem should require the content and techniques typical to undergraduate number theory courses; (2) The problem should lend itself to the motivation for prospective teachers to actively propose conjectures that is part of the process of solving the problem; in other words, we avoid problems that are too closed-ended – such as the conventional proof problems where the statement to be proven is given and thus there is no room for conjecturing; (3) the problem should be set at the right 'level'; it should not be deemed too inaccessible for most of the students to the point that they do not feel encouraged to even try conjecturing; on the other hand, there should be sufficient cognitive demand in the problem to render the task of solving meaningful; (4) the problem should be unfamiliar to the prospective teachers and not easily found in public media. This is to reduce the likelihood of prospective teachers resorting to duplicating solutions found elsewhere and as such blunt their motivation to attack the problem through conjecturing and proving for themselves.

In crafting these design principles, I relied on prior experiences teaching this course. It is heartening to note that these principles were in line with the characterisation *of open problems* as described by Furinghetti and Paola (2003), as well as several of the principles proposed by Lin et al. (2012). While the principles they stated were generic in nature, my motivation in deriving the design principles were for their specific relevance in the teaching of undergraduate-level number theory.

DATA SOURCE AND THE PROBLEM SOLVING TASK

The aim of our study is to find out whether a problem crafted based on the design principles stated in the previous section will be efficacious, that is, whether it will bring about productive conjectures and motivation for proving these conjectures in the prospective teachers' attempts at the problem.

This study took place in a first year undergraduate number theory course for 46 prospective teachers. The undergraduate programme for these prospective teachers is structured in such a way that they first learn mathematics content before they learn the pedagogical aspects concerning the teaching of the subject. Thus, during their first year, the academic profiles of these prospective teachers are typical to that of an undergraduate mathematics major. Prior to this course, these prospective teachers had already read introductory calculus and introductory linear algebra. In addition, the first two weeks of this 13 week number theory course were devoted to methods of proof.

Our data is taken from the following problem solving task assigned to the prospective teachers near the end of the course:

> Problem: An L-Shaped number is one that can be written as a difference of two squares. For example, $3 = 2^2-1^2$ and $21 = 5^2-2^2$ are L-Shaped numbers but *1* and *2* are not. Note that we do not consider *0* as a square. Can you describe as completely as possible, which natural numbers are L-Shaped numbers? (You should include proofs as necessary.)

A diagram illustrating the geometric interpretation of L-shaped numbers (Figure 1) accompanied the description of the task.

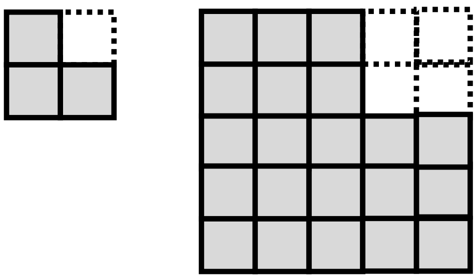


Figure 1: Geometric interpretation of L-Shaped numbers.

The prospective teachers were given two weeks to complete the task on a *practical worksheet*, an instructional scaffold designed to develop problem solving disposition based on Pólya's problem solving model (Pólya, 1945) and Schoenfeld's problem solving framework (Schoenfeld, 1985). It contains sections that explicitly guide students to use the Pólya stages. The practical worksheet was used as part of the prospective teachers' task as it provided a useful aid for their conjecturing and proving in the process of solving the problem. An account of how the practical worksheet was used to develop problem solving disposition in a previous iteration of the same number theory course can be found in Toh et al. (in press).

It is clear that the task coheres with the design principles mentioned in the previous section: (1) an essential step in the complete solution require parity arguments which is a typical technique in number theory courses – and this point will be elaborated in the context of discussion of solutions later; (2) the open nature of the problem requires the subjects' active proposal of conjectures; (3) there are multiple entry levels into the problems – such as proceeding geometrically first, or just listing examples to observe a pattern – and thus encourage the prospective teachers to make good attempts at conjecturing and proving; (4) To the best of my knowledge, this problem is not found in the open media. In fact, this problem is a substantial adaptation from another problem I came across in the Singapore Mathematical Olympiads.

ANALYSIS OF PROSPECTIVE TEACHERS' WORKSHEETS

All except one prospective teacher submitted their solution attempts. Their worksheets were analysed and coded according to whether they made one or more of three conjectures that are productive towards the complete solution of the problem, and whether they were able to provide a valid proof of their conjectures.

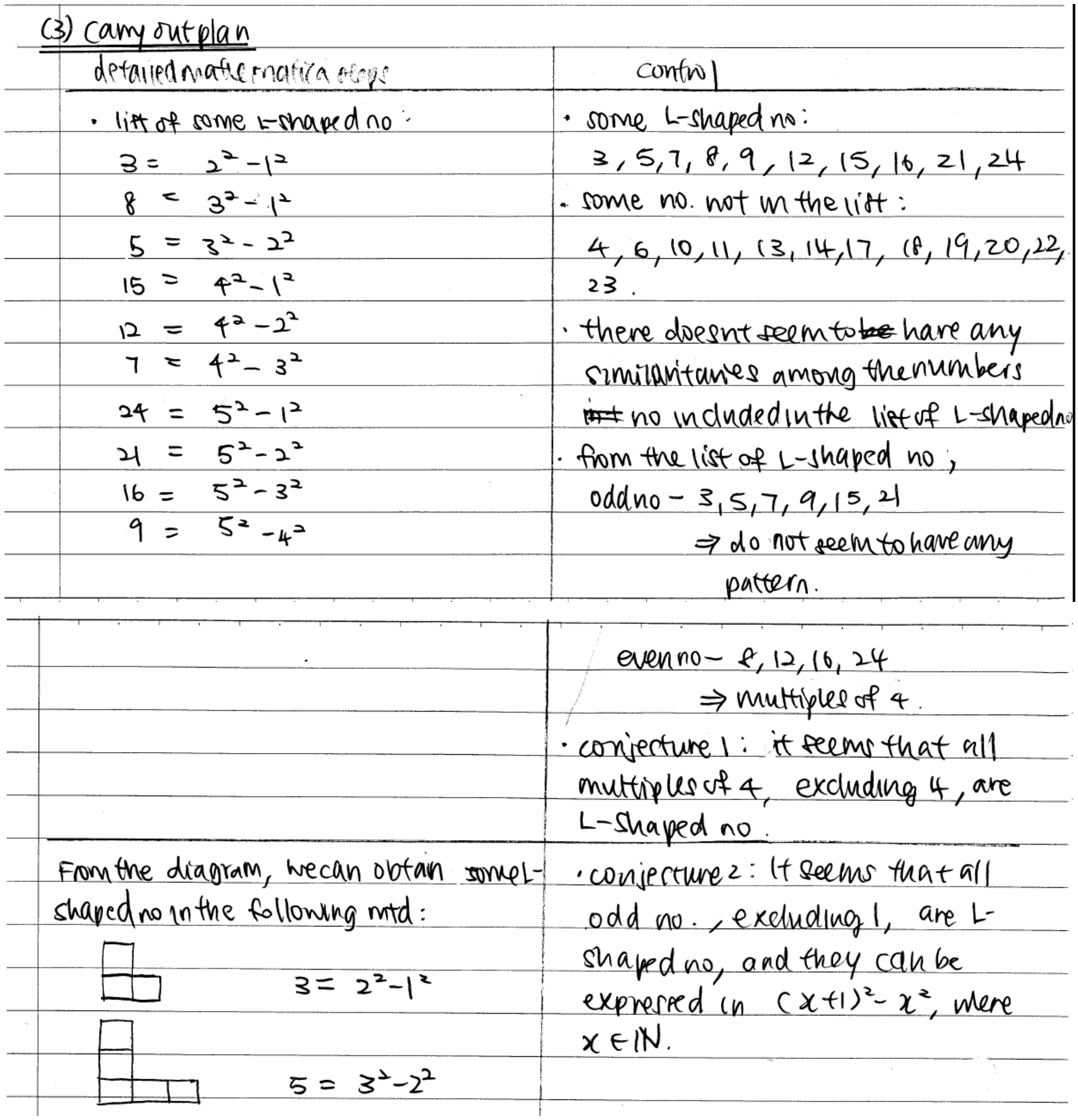
Possible Conjectures	Made the conjecture	Provided valid proof
All odd numbers, with the exception of *1*, are L-shaped numbers	43	34
All even numbers which are multiples of *4*, with the exception of *4*, are L-shaped numbers	39	22
All even numbers which are *not* multiples of *4* are *not* L-shaped numbers	15	3

Table 1: Conjectures made and proved by prospective teachers.

About conjecturing

Most of the prospective teachers began by listing examples of L-shaped numbers and attempted to seek patterns from the list. All but two of the prospective teachers managed to observe that all odd numbers that are greater than *1*, are L-shaped. A total of 39 prospective teachers also made the second conjecture that every even multiple of *4*, with the exception of *4*, are L-shaped.[1] Figure 2 provides an example of a prospective teacher who wrote the two conjectures clearly. It is noteworthy that – as we anticipated under Design Principle (3) – entries made into the problem include the technique of listing and geometrical approaches.

[1] A mathematical point: Both *1* and *4* are not considered L-shaped numbers because of the explicit requirement that *0* is not a square. This additional constraint caused some difficulties for a small number of student teachers. However, since we were interested in the broad conjectures made by the student teachers, we did not make any distinction between students who included or ignored these two exceptional cases.


(3) carry out plan

detailed mathematical steps	control
• list of some L-shaped no:	• some L-shaped no:
$3 = 2^2 - 1^2$	3, 5, 7, 8, 9, 12, 15, 16, 21, 24
$8 = 3^2 - 1^2$	• some no. not in the list:
$5 = 3^2 - 2^2$	4, 6, 10, 11, 13, 14, 17, 18, 19, 20, 22,
$15 = 4^2 - 1^2$	23.
$12 = 4^2 - 2^2$	• there doesnt seem to have any
$7 = 4^2 - 3^2$	similarities among the numbers
$24 = 5^2 - 1^2$	no included in the list of L-shaped no
$21 = 5^2 - 2^2$	• from the list of L-shaped no,
$16 = 5^2 - 3^2$	odd no - 3, 5, 7, 9, 15, 21
$9 = 5^2 - 4^2$	⇒ do not seem to have any
	pattern.
	even no - 8, 12, 16, 24
	⇒ multiples of 4.
	• conjecture 1: it seems that all
	multiples of 4, excluding 4, are
	L-shaped no.
From the diagram, we can obtain some L-	• conjecture 2: it seems that all
shaped no in the following mtd:	odd no., excluding 1, are L-
	shaped no, and they can be
$3 = 2^2 - 1^2$	expressed in $(x+1)^2 - x^2$, where
	$x \in \mathbb{N}$.
$5 = 3^2 - 2^2$	

Figure 2: Prospective teacher formulating two conjectures.

About proofs

Almost 80% of the 43 prospective teachers managed to prove their conjecture that all odd numbers are L-shaped. For the second conjecture, a smaller albeit still significant 56% of those who made the conjecture provided valid proofs. Examples of correct proofs of the two conjectures are given in Figure 3.

Among the prospective teachers who failed to produce a valid proof of their conjectures, more than 75% chose to tackle the problem from the algebraic definition of an L-shaped number as $n = a^2 - b^2$. They then proceeded to consider all the possible parities of a and b which lead to the conclusion that L-shaped numbers are either odd or multiples of 4. These are actually the converses of the first two conjectures shown in Table 1; in other words, instead of proving that *every* odd number and *every* multiple of *4* is an L-shaped number, they showed that an L-shaped number must be *some* odd number or *some* multiple of *4*. It is possible that some prospective teachers were not aware of the differences. Another explanation for this discrepancy between proof and conjecture is perhaps the lack of sufficient resources to prove their conjectures. They focused on the definition of L-shaped numbers and attempted to deduce whatever implications they could, and stopped once they arrived at some plausible conclusions, without checking whether their conclusions were aligned to their conjectures. This is in line with the observations of Selden et al. (2010) of some students' preference for immediately examining the hypothesis without considering the conclusion to be proved.

III Carry out the Plan (1)

Detailed Mathematical steps

Direct proof:

Let n be any odd number >1

$n = 2k+1$

$2k+1 = (k+1)^2 - k^2$

$= k^2 + 2k + 1 - k^2$

$\therefore$ All odd numbers >1 are L-shaped numbers.

To prove: All multiples of 4 can be expressed as a difference of two squares except for the number 4.

All multiples of 4 except 4 can be expressed as $4n = (n+1)^2 - (n-1)^2$ where $n \in \mathbb{N}$, $n \geq 2$

RHS: $(n+1)^2 - (n-1)^2$

$= n^2+2n+1 - (n^2-2n+1)^2$

$= n^2+2n+1 - n^2+2n-1$

$= 4n$

$=$ LHS.

Figure 3: Examples of correct proofs of the two conjectures.

Only 15 out of 45 prospective teachers explicitly stated, in some form or other, the third conjecture that even numbers which are *not* multiples of *4* are *not* L-shaped. Proving this conjecture – together with the previous two – would have completed the solution to the problem. We believe there are two plausible reasons for the relatively small number of students who stated this conjecture: the first is related to the problem of distinguishing a statement and its converse, as discussed earlier; the second was the given instruction to "*… describe as completely as possible, which natural numbers are L-Shaped numbers?*" Prospective teachers may interpret it literally that they need not consider those numbers which are not L-shaped.

DISCUSSION

We set out to study how prospective teachers would respond to a problem that was meant to encourage conjecturing and proving. We crafted the problem based on the design principles as stated in an earlier section of this paper. As seen from Table 1, we note that most of the prospective teachers were able to formulate correctly the first two conjectures and a majority managed to provide a valid proof. We derive encouragement from this result. We interpret this finding to mean that there is potential in these design principles in developing problems that will be helpful for prospective teachers to practise conjecturing and proving. In future research, we intend to replicate these principles and perhaps refine them to elicit better responses from the prospective teachers.

We also notice that the reason a significant proportion of prospective teachers failed to provide a correct proof was due to their attempts at proving the converse instead. This finding reveals a gap in prospective teachers' ability to make a distinction between necessary and sufficient conditions of a mathematical statement. From the perspective of teacher educators, there is a need to respond to this phenomenon. The responses can be in these forms: (1) In the regular teaching of mathematics courses, there should be more opportunities for prospective teachers to make judgments of statements and their converses; (2) in the design of the problems, we should be cognisant of these gaps in their knowledge. The errors made in confusing the necessary and sufficient conditions are opportunities for us to address these deficiencies. In general, we can include this in the list of design principles for problems: We should take into account prospective teachers' errors in the choice of problems so that their solution attempts would reveal the errors and thus provide a motivation for us to address them accordingly.

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USING AN EXPERIMENTAL FRAMEWORK OF KEY ELEMENTS TO PARSE ONE-TO-ONE, TARGETED INTERVENTION TEACHING IN WHOLE-NUMBER ARITHMETIC

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This report focuses on key elements of teaching occurring during intensive, one-to-one instruction with 3rd- and 4th-graders. The study involves six cases in all, each consisting of video records of up to eight lessons, each of 30-45 minutes' duration, resulting in the analysis of about 33 hours of video data. A resulting framework of four stages of analysis of one-to-one teaching is presented and the stage 'During solving a task' is elaborated according to students' responses: correct, partly correct, incorrect, and no response. Three subcategories pertaining to an incorrect response are described and two are exemplified via cases drawn from the data. The cases exemplify how the framework can be applied to analyse or inform intervention instruction and highlight its theoretical and practical importance.

One-to-one tutoring, particularly by expert tutors is widely acknowledged as a powerful method for promoting students' learning gains (Bloom, 1984; Chi, Roy, & Hausmann, 2008; Cohen, Kulik, & Kulik, 1982). However, the reasons for the effectiveness of expert tutors are relatively unexplored. Thus it is worthwhile to research the instructional strategies of expert tutors during highly interactive one-to-one instruction.

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Accordingly, this study focuses on the pedagogical skills that teachers use in one-to-one instruction when responding to particular situations, such as scaffolding and providing explanations (Chi, Siler, Jeong, Yamauchi, & Hausmann, 2001). In this study the one-to-one instruction is conducted by expert teachers who could conceivably be referred to as expert tutors. From here on in this paper we refer to them as teachers. In describing and illustrating key features of intervention instruction, Wright, Martland, Stafford and Stanger (2002) provided a set of 12 teacher behaviours—key elements of one-to-one teaching. These are micro-instructional strategies that teachers use during highly interactive one-to-one teaching. More recently, a set of eight additional key elements has been developed (Wright, 2010). Examples of key elements include *pre-formulating a task*, that is, statements and actions by the teacher, prior to presenting a task to a student, that have the purpose of orienting the student's thinking to the coming task; *post-task wait-time*, that is, the teacher's behaviour in providing sufficient time after posing a task for the student to think about and solve the task; and *within task setting change*, that is, a deliberate action on the teacher's part in changing a setting during the period when the student is attempting to solve a task.

Ewing (2005) documented the characteristics of one-to-one teaching used by four Mathematics Recovery (MR) (Wright, 2003) teachers by analysing videotaped excerpts of their MR teaching sessions. These characteristics include scaffolding, double bind, illusion of competence, pre-formulating and reformulating questions, post question wait-time, vague or ambiguous questioning, questioning and prompting, and communication. Munter (2010) found that key elements such as 'post-task wait-time' and 'child checking' have a significant positive effect on students' learning. In considering teaching in general, as distinct from the teaching of arithmetic in particular, teacher behaviours such as scaffolding, post-task wait-time and child checking are well documented in research literature (e.g. Anghileri, 2006; Bliss, Askew, & Macrea, 1996; Grandi & Rowland, 2013; Hmelo-Silver, Duncan, & Clark, 2007; Van Es & Sherin, 2002).

This study focuses mainly on teaching whole-number arithmetic for the 3rd and 4th grade because most of the teacher behaviours mentioned earlier in the literature (Ewing, 2005; Munter, 2010; Wright et al., 2002) were developed by investigating MR intervention teaching of 1st grade students. The arithmetic content for intervention students at 3rd and 4th grade differs significantly from that at 1st grade. Thus a focus on 3rd and 4th grade students enables a review and extension of the existing framework. In this study, a key element of one-to-one teaching refers to the smallest unit of analysis of teaching with the following distinctive features. It is: (i) purposeful with the intention that it will lead to significant learning; (ii) ubiquitous in one-to-one teaching; and (iii) judged by experts to embody quality teaching.

Research aims and research questions

The study aims to: (i) illuminate the nature of observable teacher behaviours in the interactions between a teacher and a student; and (ii) develop a framework for analysing one-to-one teaching in whole-number arithmetic to 3rd and 4th graders. The study addresses the following research questions. **RQ1**: What are the key elements in one-to-one intervention teaching? **RQ2**: How can a framework of key elements be used to analyse one-to-one teaching?

METHODOLOGY

A qualitative research methodology is used to gain insight into the nature of observable teacher behaviours in teacher-student interactions in intensive, one-to-one teaching. Teacher behaviours are regarded as the central phenomena requiring exploration and understanding (Creswell, 2012). Considering that the nature of this investigation is to target phenomena (i.e., teacher behaviours), a phenomenological approach is adopted (Van Manen, 1990). Grounded theory method (Glaser & Strauss, 1967; Strauss & Corbin, 1994) is also used to discover patterns and theories through analysis of the teacher-student interactions in one-to-one teaching sessions.

The method used for this study is the collective case study (Stake, 2000). The participants consist of four teachers and six students. For two teachers, one student only was selected and for the other two teachers two students were selected. The four teachers were selected from a pool of approximately 50 teachers in the Mathematics Intervention Specialist Project (MISP) (Ellemor-Collins & Wright, 2011) and were regarded by MISP leaders as being particularly competent in intervention teaching. Thus 'purposeful sampling' strategies (Lincoln & Guba, 1985) constituted the basis for selecting the six case studies. The primary data source for this study consists of six sets of videotaped lessons involving one-to-one instruction in whole-number arithmetic. Each set consists of up to eight lessons, each of 30-45 minutes' duration, conducted over a period of 12 weeks resulting in approximately 33 hours of video for analysis. The video data provide a rich corpus of teaching and enables a significant investigation of key elements of one-to-one teaching. The authors systematically observed each teacher-student pair in a context of one-to-one intervention teaching in order to capture the nature of the teacher behaviours.

Data analysis

A standard method of analyzing teaching is to review repeatedly, the recording of teaching sessions and characterise each teaching moment in terms of the teacher's behaviours. Incorporating Van Manen's analytical method (1990), a methodological approach for analysing large sets of videorecordings (Cobb & Whitenack, 1996) and a model for analysis of video data (Powell, Francisco, & Maher, 2003) were adopted in this study. The videos were transcribed and then coded with respect to the key elements of one-to-one teaching by using the NVivo 10 software program. Thus this study involved a systematic study of the teacher behaviours in one-to-one intervention teaching described in the literature review and endeavoured to identify additional teacher behaviours considered to be significant.

RESULTS

The extended list of key elements

Table 1 lists two sets of key elements. Set A were described in research literature prior to the current study and are included in order to test their viability for future analyses of key elements. Set B arose during the analysis phase of the current study and therefore are likely to be useful for future analyses of key elements. Examples of the key elements in Set B include *recapitulating* which refers to a situation where the teacher summarises and states again a student's contribution during solving a task; *stating a goal* refers to a situation where the teacher summarises the recent progress and makes a statement about what needs to be practised more or what needs to be done next; *re-posing the task* refers to a situation where the teacher restates the task in order to help the student fully understand the task. It happens when the student generally shows that they cannot solve the task because they have lost track of some of the details of the task. The key elements in Set B complement the key elements in Set A and this results in a framework for analysing one-to-one teaching as follows.

Set A	Set B
Scaffolding before	Recapitulating
Scaffolding during	Giving a meta-explanation
Introducing a setting	Confirming and highlighting a correct response
Pre-formulating a task	Re-posing the task
Reformulating a task	Rephrasing the task
Post-task wait-time	Stating a goal
Within-task setting change	Referring to a setting
Screening, color-coding and flashing	Asking 'what do you notice?'
Directing to check	Querying an incorrect response
Affirming	
Querying a correct response	
Explaining	
Changing a task format	

Table 1: Extended list of key elements of one-to-one teaching.

Key elements of one-to-one teaching: a reformulation

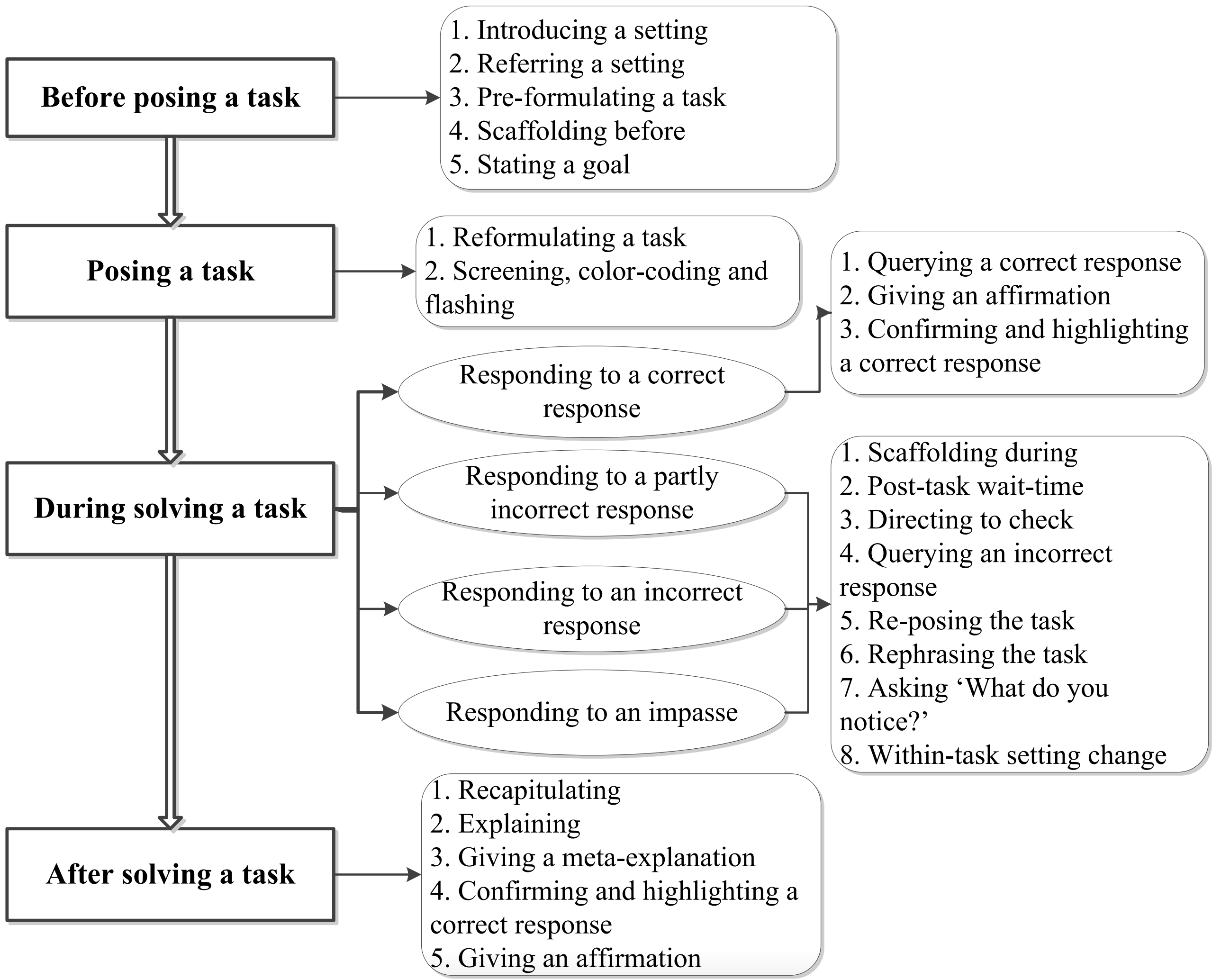

Figure 1: An Experimental Framework for analysing one-to-one teaching.

In teacher-student interactions, the teacher might monitor and respond to the student's response. This is sometimes called a 'teacher move' or 'tutor move' (e.g. Chi et al., 2001; Lu, Eugenio, Kershaw, Ohlsson, & Corrigan-Halpern, 2007). Figure 1 sets out the Experimental Framework that resulted from analysis of the teacher-student interactions in the data. There are four stages of the teacher dealing with a task: A–Before posing a task; B–Posing a task; C–During solving a task; and D–After solving a task. Collectively, these constitute the first or highest level of analysis. As well, the stage of C–During solving a task, is construed as four categories of teacher responses: C1–Responding to a correct response; C2–Responding to a partly correct response; C3–Responding to an incorrect response; and C4–Responding to an impasse. For each category, there are specific key elements that teachers usually use to respond to the student's response.

This report will focus on Category C3. This refers to a teacher's moves in response to an incorrect response on the part of the student, particularly in situations where the student is not using any strategy or seems unable to respond. This results in actions by the teacher relevant to the task and typically has the purpose of helping the student to solve the task. The situation is described as follows. The teacher initially poses a task. The student responds incorrectly. Three cases corresponding to C3 are:

> **Case C31**: The teacher then responds by directly correcting the student's response. This typically applies to answer-focused tasks where the teacher focuses on getting the student's answer but the nature of the task is such that it cannot not be elaborated in terms of a strategy.
>
> **Case C32**: The teacher assists the student indirectly by asking or allowing the student to check their last response. Student checking in this way typically involves a resort to an easier or simpler strategy.
>
> **Case C33**: The teacher provides assistance which results in a less-challenging task. In this situation, the teacher typically uses one or more key elements such as scaffolding during, post-task wait-time, querying an incorrect response, rephrasing the task, re-posing the task, and within-task setting change. This typically applies to strategy-focused tasks (Munter, 2010) where the teacher is interested in a particular strategy that the student used to solve the task. Examples for C31 and C33 follow.

Example C31. Amilia-Karral: Ten pluses

A:	(Opens a workbook on table) What we are going to do in your book is to practice doing some of these (indicates the ten pluses) and some writing. What's ten plus four? (Points at the sum 10+4)
K:	Ten plus four is fourteen.
A:	Can you write that down?
K:	(Writes down the answer in the workbook – 41)

A:	Okay. (Gets the pen from Karral) When we write fourteen, we write a one first, not a four first (corrects the writing in the workbook).
K:	Oh!

Example C33. Amilia-Mia: Jumping forward to a decuple

A:	Okay. (Opens workbook on table) I' m going to tell you a number and I want you to tell me what the next ten is and how many to get there. (Writes in workbook: 48→; 63→; 27→; so on) Does that make sense?
M:	Yep.
A:	Like we were doing yesterday. (Keeps filling page with examples). Okay. (Finishes writing and hands the workbook to Mia). 48. What's the next ten?
M:	Fifty. (Immediately)
A:	Gorgeous! How many to get there?
M:	Um. Eight. (Looks at Amilia)
A:	Forty-eight. Think about what forty-eight would look like. How many more will make fifty?
M:	(After 7 seconds) Wouldn't it be... forty-nine? No… nine? (Looks at Amilia)
A:	Let's have a look. (Takes out some ten-frames)
M:	Oh. It is seven. (Going to writes down the answer in workbook)
A:	No. Stop. Stop. There's my eight (Places out an 8-dot ten-frame). There's my forty (Places out four 10-dot ten-frames)
M:	Oh. Two. (Immediately)
A:	Two more. (Nods)
M:	(Writes down the answer in workbook)

DISCUSSION

The framework of key elements enables micro-analyses of highly-interactive one-to-one instruction. The two cases exemplify how the framework can be applied to analyse or inform intervention instruction. Example C33 illustrates that a teacher can use many key elements effectively in responding to each particular situation and each particular response from the student. Before posing the task, the teacher uses the key element of 'pre-formulating a task' in order to orient the student's thinking to the coming task. When the student responds incorrectly to the task, the teacher first uses 'rephrasing the task' by expressing the task in an alternative way to make the meaning clearer to the student, and then allows the student time to think about the task by using 'post-task wait-time'. The student again gives an incorrect response. The teacher then changes the setting from a formal written task to one using ten frames, by using 'within-task setting change', to help the student solve the task. The student comes up with the right answer and the teacher then uses the key element 'confirming and highlighting a correct response'.

CONCLUSION AND RECOMMENDATIONS

The key elements are of practical importance because they are frequently observed in one-to-one intervention teaching. They are of theoretical importance because understanding them better can lead to more effective ways to characterize the range of instructional methods teachers use. Thus the framework enables a deeper understanding of the teacher-student interactions in particular learning domains. As well, the framework is likely to be applicable across the range of student attainment and also to small group and whole class instruction.

Further research could focus on three questions: (i) to what extent are different key elements prevalent for different teachers, that is, do some key elements occur more frequently for some teachers than others? (ii) to what extent can particular teachers be characterised in terms of the teacher behaviours, that is, to what extent can different teaching styles be determined? and (iii) to what extent are some key elements used more in particular learning domains?

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HOW UNDERGRADUATE STUDENTS MAKE SENSE OUT OF GRAPHS: THE CASE OF PERIODIC MOTIONS

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This study aims to explore how undergraduate students in mathematics and engineering professions make sense out of graphs representing periodic and repeated but non-periodic motions. In this study, making sense out of graphs means interpreting graphical features and describing a situation that could be represented by them. The data was collected by means of a questionnaire administered to 132 participants. Our findings indicated both students' misconceptions, as every repeated motion is periodical, and their strong willingness to assign practical meaning to mathematical entities.

INTRODUCTION

Any motion that repeats itself identically at regular intervals is called 'periodic motion'. As we observe the periodic motion shown on a graph, we are looking at a function that repeats periodically and sinusoidal functions are of this type (King, 2009). The notion of periodicity is very close to students' experiences since it appears in nature all around us (the annual motion of the earth around the sun, the tides etc.). Moreover, periodicity is a considerable part of the scientific culture of every student in his secondary and post-secondary studies. Particularly, students come to terms with this notion in different school subjects such as mathematics and science (oscillations in physics, periodic functions in trigonometry and calculus) and in post-secondary studies (Fourier series, signal processing etc.). Hence, connecting aspects of periodicity from different school disciplines is important for students' future studies in mathematics, science and engineering. Even though periodicity is central in a variety of disciplines, an extensive search of the literature shows that there is a limited number of studies that focus on its understanding. These studies conclude that most students' concept image of periodicity is based on time-dependent variations (Shama, 1998) while usually they consider any repetition as periodical (Buendia & Cordero, 2005).

The present study is part of a research project that intends to take a close look at pedagogical practices adopted in mathematics and physics classrooms in Greek secondary schools on topics that are related to periodicity. To meet the aims of this inquiry, in the first phase of our project we analyzed Greek physics and mathematics textbooks on selected chapters in the topics of periodic motions and periodic functions respectively (Triantafillou, Spiliotopoulou & Potari, 2013). This analysis has indicated that, when aspects of the notion are introduced, physics adopt a holistic perspective on defining periodic motions, whereas mathematics adopt a point-wise perspective on defining periodic functions ($f(x)$ is a periodic function if there is a positive number p, the period, such that $f(x+p)=f(x)$ for all x in the domain of the function f). We also highlighted some common practices in the analysis of the proposed exercises among the subjects, for example, aspects of periodicity are tackled almost exclusively by means of sinusoidal functions and graph related practices were mostly on sketching graphs of particular situations. Furthermore, in physics, functions such as $f(x)=e^{bx}\sin(\omega x)$ that fluctuate in a periodical way on the x-axis, are considered as functions that model periodic motions. This disciplinary understanding of periodicity could encourage incorrect generalizations, such as, any type of repetition is periodical. The aim of the present study is to see if all the above issues will continue to influence undergraduate students' understanding of aspects of periodicity when confronted with the task of making sense of graphical representations of repetitive motions. Our research questions are: (RQ1) How do undergraduate students interpret graphs of periodic motions and do they distinguish them from graphs of repeated but non-periodic motions? (RQ2) What type of examples of motions do they provide that could be represented by graphs of repeated functions? (RQ3) Are there any statistically significant differences between undergraduate students in Mathematics with undergraduate students in Engineering professions, when responding to tasks exploring the above issues?

THEORETICAL FRAMEWORK

We adopt the viewpoint that thinking about physical phenomena could enrich and promote the development of mathematical knowledge (Buendia & Cordero, 2005). Within the school curriculum, graphic competencies are central practices in mathematics and science classrooms (Roth & McGinn, 1997). Different theoretical perspectives have been adopted for analyzing students' making sense of graphs in the mathematical context. From a cognitive perspective, graph sense means "looking at the entire graph (or part of it) and gaining meaning about the relationship between the two variables and, in particular, of their pattern of co-variation" (Leinhardt, Zaslavsky & Stein, 1990, p. 11). Under the embodied cognition perspective, bodily activities are involved in conceptualizing graphical representations as dynamic processes (Nunez, 2007) while from a cultural-semiotic perspective, sensual experiences are important in making sense of motion graphs (Radford, Demers, Guzman & Cerulli, 2004). Moreover, the conceptual movement from graphs to a situation that they represent is termed 'translation' which presupposes the practice of 'making sense out of graphs' (Roth, 2004, p. 77). In the engineering context, translating domain-specific graphs is a central action since graphs mediate collective scientific activities such as communicating and constructing facts (Roth & McGinn, 1997). In the present study, 'making sense out of graphs' means interpreting graphical features and describing a situation that could be represented by them.

METHODOLOGY

The participants were 132 undergraduate students (85 male and 47 female). 19 students were studying Mathematics, 70 were studying Informatics and 43 were studying Electronics. The students were at different stages of the courses (58 were in the second semester, 45 in the fourth semester and 29 in their sixth or remaining semesters). All mathematics students fall into the last case. At undergraduate level, all students in the above fields encounter aspects of periodicity in their first year Calculus and Fourier analysis courses. Fourier analysis is a prerequisite course for studying signal processing in Informatics and Electronics. Thus, for all the participants, periodicity is considered as an important scientific notion not only for their academic studies, but for their professional life as well.

The tasks: The data was collected by means of a questionnaire administered to the participants at the end of the academic year 2012-13. The questionnaire was completed in one teaching hour during a mathematics course in the case of the engineering students and during a course in mathematics education in the case of students in mathematics. The questionnaire was based on three different practices relating to periodicity (exemplifying; making sense out of graphs; and modelling periodic motions). In the present study we analyze participants' responses to the tasks given in making sense out of graphs that represented repeated motions. In this case, four graphs were given to the students that all represent displacement in meters versus time in seconds. Table 1 shows the four graphs and the resources used.

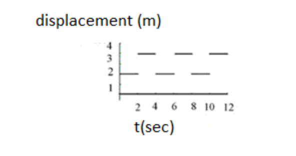
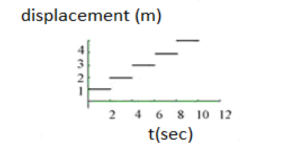
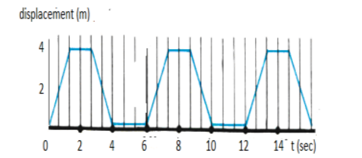
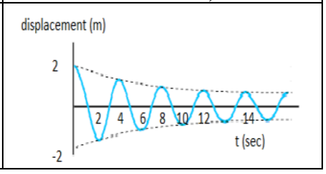
Graph 1 (Buendia & Cordero, 2005)	Graph 2 (Buendia & Cordero, 2005)	Graph 3 (Greek mathematics textbook)	Graph 4 (Greek physics textbook)
			

Table 1: The Graphs

Graph 1 and Graph 3 represent periodic motions while Graph 2 and Graph 4 represent non-periodic motions. Moreover, Graph 1 and Graph 2 represent non-continuous motions. Two tasks were given to the students referring to each graph separately. Task 1: *Does this graph represent a periodic motion? Justify your answer.* In this task, students are asked to focus on how the repetition is accomplished in order to distinguish the periodic from the non-periodic motions, as well as justifying their response. Task 2: *Provide an example that could be described by this particular graph*. In this task the students were asked to assign to each graph a motion that could be represented by it.

Data analysis: Qualitative content analysis has been employed for the analysis of students' responses in both tasks (Mayring, 2000). All the categories emerged from our continuous interrogation of the data. We separated students' responses in distinguishing periodical from non-periodical motions (Task 1a); justifying their responses (Task 1b); analyzing the situations that were created by the students in respect of the salient features of the graphs they took into consideration (Task 2a); and categorizing the type of examples used by the students (Task 2b). Subsequently, we reported the frequency and the valid percentage of students' responses on the categories that emerged from both tasks for the four graphs. Finally, in order to compare mathematics and engineering students' responses on the emerging categories we used the goodness of fit test at 0.05 level of significance.

FINDINGS

We present the results of our analysis of the categories that emerged from the students' responses on the two tasks in the case of each graph and we provide some characteristic examples. Finally, we present the cases of statistical significant differences between the mathematics and the engineering students. In Tables 2, 3, 4 and 5, we present the percentages of students' responses in the categories that emerged from each task across the four graphs and the number of students that responded to the particular item.

Task 1a: Does this graph represent a periodic motion? Two categories emerged from the analysis of this task: the graph represents a *non-periodic motion*; and the graph represents a *periodic motion*.

Categories	Graph 1	Graph 2	Graph 3	Graph 4
	113 Participants	107 Participants	109 Participants	114 Participants
Non-Periodic	23.89	74.77	7.34	32.46
Periodic	76.11	24.30	92.66	67.55

Table 2: % of students' responses across categories and across graphs on Task 1a

Almost three out of four students identified periodicity in Graph 1 and non-periodicity in Graph 2 while this percentage increases in the case of Graph 3 since more than nine out of ten students identified it as a periodic graph. Graph 4, which represents a repeated but a non-periodic motion, seemed to confuse students a lot since almost seven out of ten considered it to be a periodic graph. Comparing mathematics and engineering students' responses the goodness of fit test showed statistical significant differences between them only in the case of Graph 4 (Pearson Chi square value 9.527 and p= 0.009) since more than half of them (11 out of 17) considered it a case of non-periodic motion. However, this result does not change our hypothesis that students have difficulties distinguishing periodic from non-periodic motions.

Task 1b: Justify your answer (in task 1a). The following categories emerged from the analysis of students' responses: *Referring to general patterns of repetition, relating variations in x-y axis, focusing on continuity issues, using the formal definition of periodic functions,* and *reasoning on a specific situation*.

Categories	Graph 1	Graph 2	Graph 3	Graph 4
	59 Participants	46 Participants	58 Participants	63 Participants
Referring to general patterns of repetition	23.73	23.91	32.76	15.87
Relating variations in x-y axis	37.29	26.09	41.38	61.90
Focusing on continuity issues	13.56	15.22	1.72	1.59
Using a formal definition	0.00	2.17	1.72	0.00
Reasoning on a specific situation	25.42	32.61	22.41	20.63

Table 3: % of students' responses in the categories and across graphs on Task1b

In both cases of periodic and non-periodic graphs, students preferred to justify their answers by relating patterns of repetitions between the x-axis and y-axis rather than referring to general patterns of repetition. It is interesting that the same type of the above justifications were used for conflicting answers for the same graph. Two characteristic responses of the type of *General pattern* in the case of Graph1 are the following: "It is periodic because we have a repetition" (st91_elec) and, "It is not periodic since there is not any harmony" (st59_elec). In the following examples we could identify inconsistencies in the students' responses in the case of Graph 4 when *relating patterns of variations in the x-axis and the y-axis*: "It is periodic but we can see that as the time passes it dwindles and we are led to a standstill" (st101_elec); or "It is a periodic motion that decreases (its amplitude diminishes) all the time" (st68_elec). The contradictions in students' responses were not realized by them. *Focusing on the continuity issue* is used as a warrant to take the stance that Graphs 1 and 2 are both non-periodic. For example, st19_math notices: "I do not know if this graph preserves a periodic behaviour because in its second position it has different values from left and right". Only st6_math reasons by *using the definition of periodic functions* in order to accept that Graph 3 is periodic and Graph 2 is non-periodic. For example, Graph 3 "is periodic with period T=6 seconds since f(x+T)=f(x) for every x in the interval [0,14]. In the case of Graph 4, the same student changes his argument as follows: "It is periodic since any sinusoidal function is periodic". The last category is *reasoning on a specific situation*. These situations, in most cases, were the examples they provided in Task 2. This type of situated justification was common in students' responses in all graphs and ranged from 20% to 30%. Some characteristic examples are: (Graph 1) "the body of the graph diverges from the starting point of motion and then always returns within 4 seconds, therefore the graph is periodic (st99_elec); (Graph 2) "the graph shows a person who, as time passes, only draws away from a point 'a,' therefore non-periodic" (st107_inf); and (Graph 4) "it is periodic because it represents the motion of the swing" (st129_inf). This indicates students' need to set up a background for their justifications.

Finally, comparing mathematics and engineering students' responses on the emerging categories the goodness of fit test showed statistical significant differences between them only in the case of Graph 2 (Pearson Chi square value 11.138 and p= 0.025) since they rarely used situated type justifications.

Task 2: Provide an example of motion that could be described by each graph.

Task 2a: The following categories emerged from the analysis of the salient features that were taken into consideration when the students were asked to provide examples that could be represented by each graph: *Enriched repeated motion* when students considering the repeated behaviour and other characteristics emerging from the graphs (periodicity, piece-wise continuity, and the relation between the variables), *Only repeated motions* when students took into consideration only the repeated behaviour, Non-repeated motions when there was no-indication of a repeating motion in students' responses, and *no-motion* when the example was not representing a motion at all.

Categories	Graph 1	Graph 2	Graph 3	Graph 4
	84 Participants	83 Participants	85 Participants	102 Participants
Enriched repeated motions	9.52	21.69	7.06	6.86
Only repeated motions	55.95	31.32	67.06	79.41
Non-repeated motions	15.48	35.55	12.94	0.98
Non-motions	19.05	8.43	12.94	12.75

Table 4: % of students' responses in the categories and across graphs on Task 2a

Creating a motion example of a piece-wise continuous function is very difficult but a few students managed to provide examples that could satisfy all the graphical features in these graphs. In this case, students used their kinesthetic experiences of 'jumping' or 'climbing stairs' in order to respond to this task. Some typical examples of *enriched repeated motions* in the case of Graph 1: "ascending and descending jumps between uneven steps (st1_math)"; in the case of Graph 2 is: "someone who is climbing stairs" (st57_inf). Noticing the resemblance of Graph 2 with stairs and visualize the motions helped almost 22% of the participants to provide enriched examples. However, Graphs 3 and 4 were more complicated since the students had to take into consideration the type of co-variance of the two variables in order to provide enriched examples. Particularly, Graph 3 refers to an object's motion that moves with constant speed in different directions and makes a few seconds stops. A significant example for Graph 3 is: "someone who is using a piece of gym equipment which is going to and fro with constant speed and stops for a few seconds" (st59_elec). Although many students used the swing example for Graph 4, a typical example in their physics classes, only a few managed to specify what the x-axis and the y-axis represent in this graph. This is the reason we have the least percentage of enriched cases.

The number of students who provided *examples of repeated motions* but did not consider other graphical features was high (more than one out of two students) for all graphs besides Graph 2. Two characteristic ideas were met in their answers and the corresponding examples for the graphs follow: (a) discontinuity was not taken into consideration (Graph 1) "it represents an elevator that is trapped going up and down between the second and fourth floor" (st3_math); (b) not specifying the x-y co-variance (Graph 3) "two people who are throwing a ball to each other" (st50_inf). The amount of students who provided examples of *non- repeated motions* was higher in the case of Graph 2, as for example: "a dog that goes hunting and increases its speed" (st71_elec). Some students provided *examples that do not represent motions* at all. These examples are mostly taken from their academic signal processing courses. The goodness of fit test did not indicate any statistical significant differences in mathematics students' responses. Finally, students' high participation in this task indicates their willingness to assign meaning to abstract mathematical entities.

Task 2b: Kinesthetic students' experiences seem to play a significant role in providing examples of repeated motions. So, we further analyzed the type of kinesthetic experiences they refer to and the categories emerged were: *bodily actions* when a human agent performs the motion (an athlete running or a frog jumping), *physical tools motions* (a car is accelerating or a swing is oscillating), and *vibrations of natural objects* (a sea wave or a sound wave

Categories	Graph 1	Graph 2	Graph 3	Graph 4
	70 Participants	81 Participants	74 Participants	90 Participants
Bodily actions	24.29	32.50	13.51	7.78
Physical tools' motions	72.14	63.80	81.09	64.44
Vibrations of natural objects	3.57	3.70	5.40	27.78

Table 5: % of students' responses in the categories and across graphs on Task 2b

Physical tools' motions provided the context used by most students to translate the graphs to situations. The highest percentage of *bodily actions* examples was in the case of Graph 2 (32.5%). The highest percentage of *physical tools' motions* (81%) was in the case of Graph 3. We interpret this result that most students consider that human actions are very difficult to model this type of motion graphs so they have changed the context of their example from bodily actions to physical tools' motions. More than one out of four students used examples of *vibrating natural objects* (e.g. sea waves) in describing the case of Graph 4. The graphical image resemblance with traveling sinusoidal waves was the reason to use them as the context of their examples. We note that waves are functions of two variables, the displacement x and the time t (King, 2013). Finally, the goodness of fit test showed high statistical significant differences in mathematics students' responses in the case of Graph 2 (Pearson Chi square value 13.547 and p= 0.001) since math students exclusively used examples of bodily motions when describing this graph.

CONCLUDING REMARKS

This study aims to explore how undergraduate students in mathematics and engineering professions make sense out of graphs representing periodic and repeated but non-periodic motions. Our findings indicate that conceptions such as "every repeated motion is periodic" or "any sinusoidal graph, even with decreasing amplitude, represents a periodic motion" dominate students' understanding. Even mathematics students seem not to realize the above contradictions in their responses.

However, students' strong willingness to assign meaning to mathematical entities is proved both by their high participation in providing situations that could fit onto motion graphs and by the fact that they use these situations as warrants for their justifications. In this case, the role of students' kinesthetic experiences proved central both when they provided enriched examples of motions represented by the particular graphs and when they take the stance to change the context of the examples according to their perception of the graphical features represented. These findings show the embodied nature of mathematical thinking and the genetic relationship between the sensual and the conceptual in knowledge formation (Nunez, 2007; Radford et al., 2004). Translating graphs into describing situations (Roth, 2004) seems to be an activity that attracts undergraduate students' attention. Maybe such activities help students to cope with the contradictions that arise between their divergent conceptions on periodicity. The formal mathematical tools, as the definition of periodic functions, seem to be not enough to change such perceptions even in the case of students who study mathematics.

Acknowledgment

The current research is carrying out in ASPETE and is part of a research project (SH4_3510 EDARCPADSS) that is implemented within the framework of the Action “Supporting Postdoctoral Researchers” of the Operational Program “Education and Lifelong Learning” (Action’s Beneficiary: General Secretariat for Research and Technology), and is co-financed by the European Social Fund (ESF) and the Greek State.

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EXAMINING MATHEMATICS-RELATED AFFECT AND ITS DEVELOPMENT DURING COMPREHENSIVE SCHOOL YEARS IN FINLAND

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Mathematics-related affect is told to predict choices concerning future studies, to correlate with performance, and to be of importance per se. Unfortunately, the affect towards mathematics is frequently reported to be low in several countries, and this contradiction cannot be solved before knowing more about its development. The objective of this study is to increase our knowledge about the timing of the affective factors getting worse, which is crucial for implementing interventions at a correct phase. We investigated a longitudinal data covering Finnish students' affect during comprehensive school years (n=3502). As a result, it was found that enjoyment of mathematics is most likely to decrease during primary school years, whereas self-efficacy is most likely to decrease during lower secondary school years.

INTRODUCTION

Mathematics-related affect proves to be of importance for a number of reasons. It is told to predict choices concerning future studies, to correlate with performance, and to be of importance per se (Evans, 2006). Unfortunately, the affect towards mathematics is in many studies reported to become low during comprehensive school years in several countries (Lee, 2009). The international trend of decreasing affect is visible as well in Finland in spite of the country's high performance level in recent PISA studies (e.g. OECD, 2010).

Finnish education system is fairly non-authoritarian and non-competitive, and either in spite or because of that in international studies such as TIMMS 2011 and Pisa 2009, Finnish pupils' learning achievements in mathematics have been very good (Mullis, Martin, Foy & Arora, 2012; OECD 2010). Still, regardless of the lack of strictness and the high performance level, in TIMMS 2011 study only one third of Finnish 4th graders' (compared to the international mean of 48%) and one tenth of 8th graders' (compared to international mean 26 %) emotions toward mathematics were positive. In order to understand such a contradiction, we need to acquire more knowledge about the development of affect during comprehensive school years. In other words, though affect is repeatedly reported not to be positive enough, we have not learned what the development that leads into that situation is.

In general, the affect is high when we consider young pupils (Metsämuuronen, Svedlin, & Ilic, 2012; Tuohilampi, Hannula, &Varas, 2013). Yet, this cannot be seen only as a sign of success of the early years of schooling, as the affect is high among young pupils anyway because of the developmental stage they are living through.

According to Harter (1999), the general view of self is typically unrealistically positive in childhood. Interacting with peers, children start to evaluate their skills and appearance according to the reactions of others; this developmental phase places itself into early school years. Harter's view is in line with Op 't Eynde, de Corte, and Verschaffel (2002), who argue that affect becomes from what is "first told". This means that if there is nothing that contradicts with given information (true or false), children tend to take it as true. Only when a contradiction appears, children have a reason to evaluate former affect, as well as given information in the light of former affect. Thus, as children get older, it is normal for the affect to become lower over time (i.e. more realistic) because of the development. Still, when it comes to mathematics, the affect becomes unnecessarily negative nearly worldwide (e.g. Hirvonen, 2012; Lee, 2009).

Chapman (2002) has shown that making changes in affect structure can be hard work. The previous situation needs to be conflicted in a way that is noticeable for the individual before new information can change it. Further, Hannula (2006) argues that the beliefs (i.e. thoughts and conceptions that are true at least for the individual her/himself, but not necessarily logically justified) are more likely to change from positive to negative than vice versa, at least when it comes to mathematics-related beliefs. Thus, it is wiser to concentrate on keeping the affect on a reasonable level in the first place instead of trying to change the situation after letting it get worse. To be able to do that, it is important to be aware of the development of affect that happens accordingly when pupils construct their affect through social responses by significant others during primary school years. In particular, we need to know the separate affective factors, like self-related beliefs (cognitive dimension of affect, see Hannula, 2011), or emotions (emotive dimension of affect, ibid.), distinctive development. So far we lack this information, because though being actively studied, mathematics-related affect has rarely been examined with a help of effective longitudinal data. Thus the dynamics of affect and its components have remained under examined. Yet, earlier studies in Finland have shown that grade 5 students have higher mathematical self-confidence than grade 8 students [n=3057] (Hannula, Maijala, Pehkonen, & Nurmi, 2005) and a longitudinal study indicates a decline in self-confidence from grade 5 to grade 6 and from grade 7 to grade 8 [n=191] (Hannula, Maijala, & Pehkonen, 2004).

When it comes to gender differences regarding mathematics related affect, studies have produced very consistent results that indicate that across age and performance levels, female students tend to have lower self-confidence in mathematics than male students (e.g. Hannula, Maijala, Pehkonen & Nurmi, 2005; Leder, 1995). In Finland, the biggest difference between girls and boys appears regarding self-efficacy feelings despite no differences in achievement: independent of the performance level, girls experienced poorer self-efficacy than boys (Hirvonen, 2012). As this is the case, it seems likely that in Finland girls receive less positive or more negative feedback for their mathematical skills from their social surroundings than boys, independent on their actual capability.

For increasing our knowledge about the development of affect, this study aims to give answers to the following research questions: *1. How do mathematics-related cognitive and emotional affective factors develop from 3rd to 9th grade among Finnish students? 2. Are there differences between girls and boys when it comes to mathematics-related affective components' development?* Carrying out this research will give us crucial information about the dynamics of mathematics-related affect. With that information we will be more capable to address the interventions at a correct phase.

METHOD

Mathematics-related affect can be defined in different ways. In this research we will use a model of Hannula (2011), wherein affect is separated into cognitive, emotional, and motivational dimensions of affect. In particular, we are interested in pupils' beliefs of self-efficacy (cognitive dimension of affect) and enjoyment (emotional dimension of affect) with respect to mathematics.

The data used in this study consists of 3 502 Finnish students (1 702 girls, 1 800 boys) who were followed throughout comprehensive school in its entirety regarding mathematics achievement and mathematics-related affect. The measurements were done at the beginning of third, sixth, and at the end of ninth grade (years 2005, 2008, and 2012, respectively). All students were selected by using the stratified sampling of the comprehensive schools, with a representation of different instruction languages (Finnish/Swedish), provinces and municipal groups (Cities/Population density areas/Rural areas). Not all the students could be followed during the whole data collection process, and the students that dropped out from the study were more commonly weak than high achieving. Thus the data, though being representative of all students in Finland, is slightly biased. This means that the results might be little bit more *positive* than what they would be having included all the weaker students in the following process.

The attitude scale used in the different datasets is a modified version of Fennema-Sherman Mathematics Attitude Scales (Fennema & Sherman, 1976; Metsämuuronen, 2012). In this study, we discuss two factors of the used instrument, i.e. self-efficacy regarding mathematics and enjoyment of mathematics. With respect to different measurements, the wording of the items was slightly modified to fit to the examinees' developmental stage. Spice items were as follows: "*Mathematics is easy*" (self-efficacy, first measurement), "*Mathematics is an easy subject*" (self-efficacy, second and third measurement), "*I like to learn Mathematics*" (enjoyment, first measurement), "*I like to study Mathematics*" (enjoyment, second and third measurement). A 5-point Likert scale was in use, but for the analyses the attitude scores were changed into percentages of maximum score. Hence, as the most positive case, the student would get 100 which is strictly 100% of the maximum score. As the most negative case, the students would get zero which corresponds with 0% of the maximum score. The reliabilities of the attitude scores were high enough for accurate inferences (see Table 1).

α reliability	Grade 3	Grade 6	Grade 9
Overall affect	0.86	0.88	0.92
Self-Efficacy	0.79	0.82	0.88
Enjoyment	0.88	0.89	0.90

Table 1: The reliabilities of the affect scales in the three measurements.

The analyses were done by calculating distributions of overall affect (self-efficacy + enjoyment) and its components (self-efficacy / enjoyment) with respect to different time phases, and by investigating gender differences by t-tests.

Finnish children start going to school normally around the age of 7. Before that most children have a year of pre-schooling. Almost all schools are public and free of charge. The number of school hours per week in Finland is one of the lowest compared to other countries (23 hours per week is the minimum at 3rd grade), and there are about 3 mathematics lessons a week in the 3rd grade. All teachers in Finland need to have a master's degree in education. On primary level, each teacher teaches all or nearly all subjects, and pupils study all subjects in one group, whereas in lower secondary level the teachers are subject teachers, and though the group typically still stays the same all the time, the students move from one class to another depending on the subject. The profession of a teacher is fairly valued in Finland, and the salaries are slightly above the country's medium (OECD 2012). In general, having a childhood without much competition is valued in Finland, but one the other hand the children are given lots of independence: even first graders may walk or cycle to school without an accompanying adult.

RESULTS

In the first measurement the pupils' mathematics-related affect appeared high. The mean for overall affect score was 71%, enjoyment being slightly higher than self-efficacy (mean for enjoyment 72%; mean for self-efficacy 68%). On following years the development turned negative. The overall affect score decreased from 71% to 60% by the second measurement, and to 52% by the last measurement.

The decrease did not happen in similar vein regarding both of the affective dimensions. When examining the dimensions separately, it was found out that the decrease began stronger with respect to enjoyment. In first measurement, this dimension was at a high level (72%), but by the second measurement it had decreased to 54%. The decrease continued after that, but less dramatically: at the last measurement the dimension was at 47%. Regarding self-efficacy, the decrease was very reasonable between the first and the second measurement (from 68% to 66%) despite the decrease of the enjoyment. Instead, by the last measurement the self-efficacy decreased very clearly, becoming 57% of the maximum score (see Figure 1).

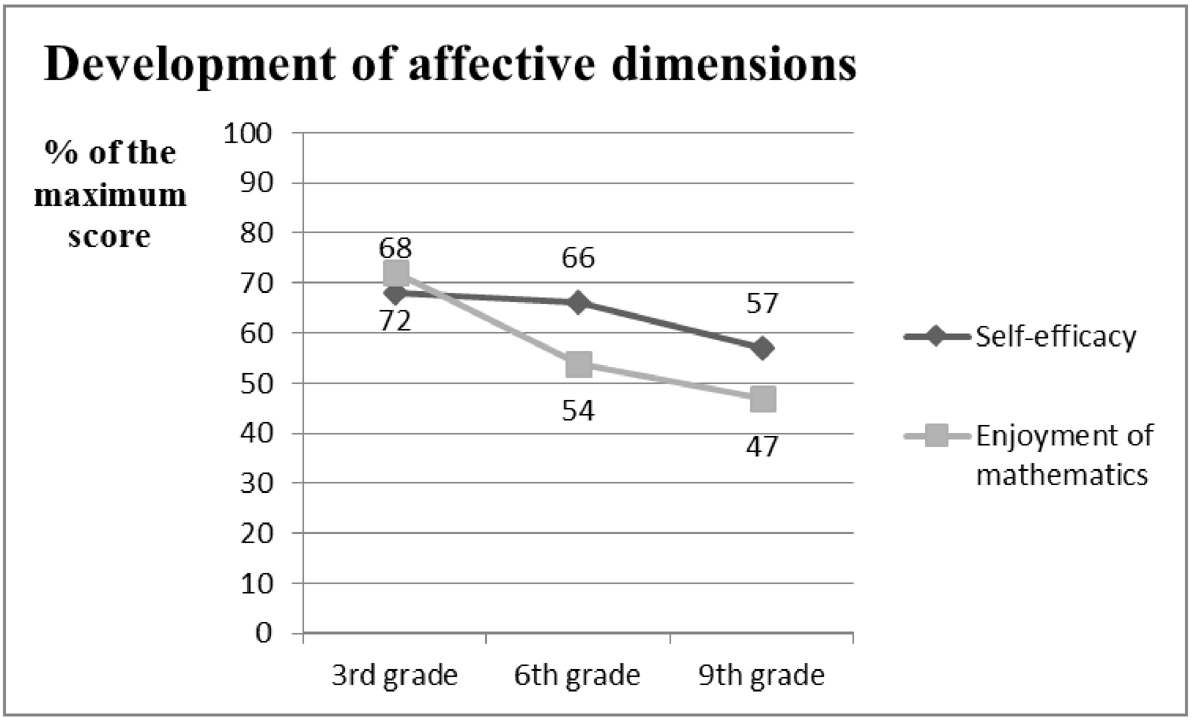

Figure 1: Development of affective dimensions from 3rd to 9th grade

With respect to different genders, the development of affect was similar, but the decrease was more dramatic regarding girls. The degree of overall attitude was statistically significantly different in all the three measurements to the detriment of the girls. At the first measurement, the mean for girls was 68%, whereas the mean for boys was 72%. At the second measurement, the mean for girls was 57%, while for boys it was 64%, and at the third measurement the mean for girls was 50%, whereas the mean for boys was 54%. In all three measurements the difference between genders was statistically significant according to t-tests (the test values were $t = 4.63$, $p < 0.001$; $t = 10.29$, $p < 0.001$; $t = 5.02$, $p < 0.001$ regarding the three measurements respectively).

Examining the gender difference further according to the affective dimensions, it was seen that the difference was greater concerning the cognitive dimension of affect, and it was at its highest at the second measurement. The situation seemed less dramatic regarding the emotional dimension of affect. According to t-tests, the difference was statistically significant in all the three measurement regarding self-efficacy (the test values were $t = 6.4$, $p < 0.001$; $t = 14.1$, $p < 0.001$; $t = 9.5$, $p < 0.001$; covering the three measurements respectively); whereas the difference regarding enjoyment was largest at the second measurement, and disappeared by the last measurement (the test values were $t = 2.3$, $p < 0.05$; $t = 5.2$, $p < 0.001$; $t = 0.2$, $p > 0.05$ covering the three measurements respectively).

DISCUSSION

This study gives confirmation to the previous result that, as is the case internationally, Finnish students' affect decreases rather dramatically during comprehensive school years. Further, perhaps the most important result of this study is that enjoyment of mathematics is most likely to decrease during primary school years, whereas self-efficacy is most likely to decrease during lower secondary school years among Finnish students. There are also clear gender differences: the decrease is more dramatic among girls than it is among boys. As all this happens at the time when students construct their identity according to responses from significant others, it is likely that students in Finland are not getting the right kind of feedback regarding that construction.

The decrease of enjoyment of mathematics during primary school years happens independent of pupils' self-efficacy feelings: boys maintain their self-efficacy during primary school years, girls do not, yet both genders' enjoyment decreases. As earlier studies indicate that student anxiety in Finland is low, it is not likely that enjoyment would be declining due to mathematics being too difficult. On the contrary, we suggest that the declining enjoyment is due to boredom, and this feeling may become socially shared. If looking from the perspective of social responses, it looks like either pupils get negative responses regarding how enjoyable mathematics is, or they do not get enough positive responses of the enjoyability of mathematics independent of their ability to do mathematics. What is the mechanism in Finnish schools that makes this happen? One plausible reason is that no matter the good performance level, teaching seems rather traditional in Finland. According to Joutsenlahti and Vainionpää (2010), teaching practices are largely determined by the textbooks, and the content of the teaching is fairly mechanic. This might lead into needless emphasis on routine tasks, which might further narrow creativity and ability to see mathematics as something interesting. Showing mathematics as a largely mechanical subject, consisting of routines determined by textbooks, may also increase the feeling that one has to have specific univocal skills to be able to work with it.

The decrease of self-efficacy might be connected to the move from primary school to lower secondary school. The teaching is from there on given by subject teachers, and the change might be too challenging for many students. The content of mathematics becomes more abstract, and the students start to notice that some classmates are able to reach the level of abstract thinking easily. Looking from the perspective of social responses, it is possible that because of subject teachers and the most capable students an average student starts to see mathematics unnecessarily challenging. If the students are used to mechanical, routine tasks wherein only one solution is possible, they might value too much finding a solution immediately. In that case it would be important for teachers to start emphasising the value of making progress with the tasks, so that also those students that cannot exceed to the solution easily can have experiences of success because of being able to proceed.

In Finland it has been found that what is most valued by teachers, concerning their teaching, is students' positive affect towards mathematics (Niemi, 2010). Yet, teachers feel they have challenges with class management and they wish to have smaller groups than they do, although the class sizes are fairly small in Finland (the average is 19 at the primary level; OECD, 2012). According to the results of this study, pupils have a positive affect at the beginning of school, and the decrease comes concurrently with the school years. If there is a need for strong class management in Finland despite the fairly small groups and good performance level, we see this as a sign of social responses concerning mathematics learning being poor, affecting to pupils' affect negatively. Thus there is a need for knowing what kind of working methods will help students to construct their affect based on more positive social responses by their significant others.

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MATHEMATICAL ACTIVITY IN EARLY CHILDHOOD: IS IT SO SIMPLE?

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The aim of this paper is to build up an argument about the importance of a mathematical analysis of young children's activity in relevant for the age educational tasks. Most of current approaches (psychological, social, and pedagogical) are limited to the study of the development of children's thinking, paying less attention to the involved mathematical concepts. In the paper these approaches are briefly presented and an attempt is then made to analyse the mathematical activity within and beyond them. Finally, implications and some examples from a program of early mathematics aimed at developing authentic mathematical activity is provided.

INTRODUCTION

Research in early childhood mathematics education highlights its importance; young children, working in appropriate educational and pedagogical environments, show interest and have the potential to develop remarkable mathematical ideas (e.g., Mulligan, & Mitchelmore, 2013; van Oers, 2013; English, 2012; Gisburg et al., 2008; van den Heuvel-Panhuizen et al., 2008; Perry et al., 2008). Most countries provide considerable early mathematics education programs to support children in developing basic mathematical concepts, but also to encourage practice with processes (problem solving, reasoning, etc.), mental skills, routines of mind and creativity (Sarama & Clements, 2009).

There are many perspectives -psychological, social, cultural, pedagogical and recently neurophysiological, which attempt to contribute to the understanding of early mathematics development but there is less reflection and research examining the mathematical nature of this development (Newton & Alexander, 2013). It is documented that children, through a range of relevant experiences, challenges and activities, are enabled to develop interesting ideas, but it remains ambiguous whether these are mathematical ideas and if young pupils reach to a level of *thinking or acting in a mathematical way* (which is the goal of most current curricula). Moreover, it appears that, despite the considerable amount of studies and proposals related to early childhood, there is less progress in school, i.e. teachers' implementation of relevant approaches, tasks and materials.

One of the key factors could be the lack of understanding of the mathematical meaning shaped in the classroom and developed by children. All the aforementioned approaches deal with issues having to do with 'mathematics': mathematical development, mathematical thinking, mathematical activity and so on. But, how do we *define* and how do teachers understand and deal with the 'mathematical' part in these

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 289-296. Vancouver, Canada: PME.

expressions? How can a meaning, an activity or an outcome be characterized as ‘mathematical' and how do young children apprehend it?

In the present paper, pursuing answers to above questions, we attempt to take a more substantial look at the mathematical aspect of several proposals related to early childhood mathematics education. This way we hope to contribute in building up an argument about how mathematics itself is related to both learning and teaching and provides essential answers to early mathematics education. We fist present shortly different approaches (psychological, social, and pedagogical) related to this education and then we attempt to analyze the mathematical activity within and beyond them. Finally, we provide some examples of our proposal concerning a program and tasks aimed at developing authentic early mathematical activity.

THEORETICAL APPROACHES IN EARLY MATHEMATICS EDUCATION

After a long period during which early mathematics education was almost non-existent or was dealing with simplistic activities concerning numbers and shapes, widespread and extensive research gave rise to different scientific and educational approaches that contributed to changes in national curricula with special recommendations for this section of mathematics education.

Starting with Piaget and his psychological approaches, later researchers (Sarama & Clements, 2009) studied systematically young children's mathematical thinking and developed what they call “learning trajectories”. According to the authors:

> Learning trajectories are descriptions of children’s thinking as they learn to achieve specific goals in a mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking. (p. 17)

This approach, based on a theoretical frame that the authors call ‘hierarchic interactionalism’, is focused on children’s’ thinking; thus there are activities and tasks related to the progression of this thinking and its relevant levels. The engagement of children with these tasks is supposed to lead them to some mathematical ideas, but the connection between children’s thinking and relevant mathematical concepts (or aspects of them) don’t appear so clear. For example, while a child recognizes a shape and discusses about it or uses it to compose a larger configuration, what part of the development of geometric knowledge does s/he access? How does s/he draw on the mathematical characteristics of the relevant concepts, objects, properties, relationships, definitions?

On this matter, Levenson, Tsamir and Tirosh (2011), in their work about early childhood geometry, add a ‘mathematical view’ to the development of geometry, proposing the formation of geometrical concepts with the use of the expression ‘working definitions’ that children can use for identifying and showing figures properties, relationships, comparing and communicating. The researchers, based on Fisbein’s and Vinner’s work about concept images and concepts in general, attempt to develop an approach of geometric figures in line with mathematical concept definitions.

Important and systematic work on early mathematics was carried out by English (2012) and Mulligan and Mitchelmore (2013) who also worked on developmental aspects of children's thinking. Their work was not limited to specific mathematical content domains such as arithmetic or geometry, but dealt with the structural elements of mathematics, examining and connecting them with children's mathematical understanding. These studies constituted an important development that opened a new direction to early mathematics education, beyond numbers and shapes. However, they also raise some concerns regarding access to mathematical ideas: working with patterns and common structures isn't only a component of the mathematical activity that has to be combined with other actions to support children's conceptual formation?

From a socio-pedagogical perspective, the 'Learning Mathematics in Play' gave rise to important and interesting suggestions for early mathematics education. Typically, children play joyfully in game situations with mathematical features (Wager, 2013) or mathematical objects (like numbers or shapes), but these applications often end up with the need of the teacher's involvement in order to 'mathematize unintentional mathematical engagement in play' (Van oers, 2013). The later focuses his work on the use of language and communication within the Cultural-Historical Activity Theory perspective. While his approach has a clear orientation to mathematical thinking development, communication is again only a part of the process of mathematization and would also need (undefined) teachers' guidance for the appropriation of the relevant mathematical ideas.

In general, there are still many questions concerning early mathematics education: it is true that important aspects of mathematics can be found all around, in everyday situations and be used to develop children's mathematical learning; children are dealing with mathematical objects and situations and come to school with many mathematical ideas; they are acting in some mathematical content (counting, shape recognizing, measuring etc.) and are involved in actions and tasks that demand serious possesses, like problem solving, testing, explaining, reflecting, etc, using material and technology, with special mathematical features. However, are all these oriented to the development of mathematical thinking, knowing or acting? Do all these 'teach' them mathematics? Which part of what children do or we encourage them to do could be described as a well defined '*genuine mathematical activity'*?

MATHEMATICAL ACTIVITY

Teaching and learning of mathematics is not restricted to the development of mathematical concepts and procedures, but it mainly encourages the development of a human activity within situations and environments, institutionally formed by the educational system in schools. If we are interesting in developing this special human activity we need to define it: What is a *mathematical activity?* Which are its specific characteristics? What *criteria* can be used to evaluate whether an activity developed by the students is or is not mathematical? Which *problems*, tasks or situations guide the development of this activity?

We find many similar or complementary approaches to the issue of what constitutes *mathematical activity* (in early childhood or generally). Most researchers consider as mathematical all the activities that involve specific type of working – processing including problem posing and solving, creative and flexible reasoning, communicating with arguments and documentation, reflecting and generalizing. Freudenthal (1983) understands the mathematical activity as a way of *modelling* to address and deal with real situations, while Brousseau (1997) as finding *appropriate solutions* for situation-problems. However, some researchers point out that learning mathematics overpasses problem solving, modelling and doing mathematics and concerns mainly *obtaining forms of reflection* about the world in a specific historical and cultural way, different from other forms of thinking. For them, acting of solving a problem without further explanation or transfer to a more general framework is only an aspect of the mathematical development (Radford, 2006).

Noss, Healy and Hoyles (1997) argue that mathematical meanings derive from *mathematical connections* that they consider as the important part of a mathematical activity (something that students usually do not learn to do). From another point of view, Ernest (2006) considers Mathematics as that area of human endeavour and knowledge that, more than any other uses a wide and unique range of signs and symbols; thus, he understands the process of *symbolization* as a basic part of mathematical activity and learning. In a different way, Steinbring (2005) addresses it as a dynamic link amongst situations – signs and concepts in his *epistemological triangle*.

In general, different views about mathematical development converge to the view that students need to reach a way of thinking that involves habits and mental routines and forms a high-level processing. Hence, combining different approaches we could argue that *mathematical activity* constitutes a set of (what we can call) *mathematical actions* that, based on the previous references, are summarized in the following (incomplete) list: search for properties and relationships, recognition of patterns and common structures, analysis and synthesis in parts and unit parts, connections, links to language, representations, signs and symbols, explanations / justifications, reflections and generalizations,.... All these actions start with genuine questions, problems, unknown situations, games and involve conjecturing, solving, modelling, use of resources or tools, justification, metacognitive processes and formulations (e.g. Freudenthal, 1983; Brousseau, 1997; Radford, 2006; Perry & Dockett, 2008).

From the previous presentation it becomes clear that the simple engagement of children with mathematical objects does not always evoke relevant mathematical activity; moreover the activation of children alone is not sufficient for the development of a mathematical action. Thus, the study of forms of engagement with actions and tasks that are related to mathematical activity and supports children's mathematical development needs further exploration.

MATHEMATICAL ACTIVITY IN EARLY CHLDHOOD

The idea that simple practice in a concrete and local level does not mean generalizing of mathematical ideas or concepts is an old one (e.g. Nunes & Bryant, 1996). This position becomes more complicated and incoherent for early childhood as at this age children need to work with concrete material in everyday situations. Van oers (2013) analytically highlights:

> Children evidently demonstrate behavior (like counting) that looks mathematical from the outside (as it is fairly in conformity with adult mathematical operations). These children, however, are often unable to apply this 'knowledge' in new situations, or answer questions about numbers…(p. 185)

Young children dispose an impressive amount of intuitive knowledge about space, quantities, patterns, measures, etc. evidenced by research (Sarama & Clements, 2009). This evidence gives an argument about the nature of this knowledge: is it 'mathematical', couldn't it be just general, common or everyday knowledge, perceptual, kinesthetic, social, related to experiences, to needs, etc.? Certainly, this intuitive knowledge as well as the potential of young children to develop ideas and strategies, to find solutions or to communicate and explain could be seen as a base for the development of mathematical ideas. But at this age, if you don't want to reduce mathematical knowledge to other conceptual development, we need to minutely study and analyze children's activity in terms of mathematical work and outcome.

In early mathematics education, one could often wonder about the mathematical nature of tasks or actions carried out by children. A situation, a material, a story or another activity (such as cooking) are frequently presented in the classroom and the teachers ask questions to see if the children know how to count, or to compare bigger or smaller, or to give some location, or find a pattern or compose – decompose figures, accepting all these as mathematical actions and results (e.g. Doverborg, et al., 2011; van den Heuvel-Panhuizen, 2008; Sarama & Clements, 2009). But, these cases could raise questions about the development of authentic mathematical activity.

The special abstract nature of mathematics demands a long term development of each piece of knowledge, sometimes continuous but sometimes discontinuous, during which this knowledge in children's minds is enriched, gets broader and is stabilized in a certain level (Confrey & Kazak, 2006). Thus, their teaching presupposes systematic experiences and activities from early age, during which the research or the teacher needs to follow not only the progress of children's thinking but also the *progress of the knowledge itself* at this level of children's thinking. The example of the use of 'working definition' in approaching geometric figures is very close to this position.

Concerning educational tasks, the suggestions in early childhood mathematics education usually take into account the previous experiences and knowledge of the children, their environment, their interests, their needs and so on. But their design needs also to be orientated by a framework that can connect the mathematical content with the tasks and children's activity. Table 1 presents an example initiated by Keitel (2006) and adapted to early mathematical activity.

Content	Mathematical knowledge / meaning/ idea	What connection with the mathematical knowledge / meaning/ idea that aims to be developed by the task? Does it concern new knowledge, method, approach, reconstruction or widening of an older one? What connections to preexisting knowledge?
Task	Kind of task	Problem, realistic situation, project, research, testing, construction, model, data processing, representation, game, dramatization, implementation?
Tools	Representations/ material/ tools	What kind of language or representation is used for the task? Symbolic, synthetic (common elements), authentic related to the task? What kind of tools can be used? What recourses? What connections or aids?
Actions	Mathematical actions	What actions are proposed? Are there mathematical: search for properties/ relationships, pattern/ structure recognition, analysis and synthesis, connections, links to representations, explanation / justification, reflection and generalization. Do the children look for general solutions, methods, rules, general ideas?
Process	Mathematical processes	What possesses are encouraged? Memorization/ application or imitation? Problem solving, dealing with situations, modeling, justification, metacognitive process, formulation, evaluation, creation?

Table 1: Questions for the design of tasks related to early mathematical activity.

Attempting to implement this approach, we organized a complete mathematical program with relevant content and tasks for ages 5-6 and 6-7 (the whole program is uploaded in www.nured.auth.gr/dp7nured/?q=el/userprofile/42). Following are some examples related to this program.

A PROGRAM DEVELOPING MATHEMATICAL ACTIVITY

The design of the program is based on the study of a coherent progressive development of *mathematical concepts and procedures*, analysed in their structural components and related to children's way of thinking. It aims at putting foundation in the basic concepts of the common mathematics curriculum through relevant tasks that encourage a high level mathematical activity for the target age group. Due to space limitation, we only present an example about Reflection Symmetry from the axis 'Space and Geometry', showing the focus on the mathematical aspects of the concept and the mathematical actions of children.

Preschool children identify quite easily and rather intuitively reflection symmetry in geometric shapes and other situations. Thus, the interest in working with this concept, even at this age, is not its holistic recognition in figures but its 'mathematical' approach through (informal) understanding of its properties in symmetric shapes or symmetrical parts of a shape (same shape and size, equal distance from the axis and reverse orientation), with no formal presentation or teachers' guidance. To achieve this, we suggest tasks in which a transparent paper with a symmetrical part of a drawing is provided and the children have to complete it with the other symmetrical part. The paper is transparent so, after finishing their work, the children can fold the paper and control if their construction is right.

Depending on drawing and paper, the folding activity helps children realize one or more properties of symmetrical parts. For example, Figure 1 makes children understand that they have to draw figures in equal distances from the axis: figures are already drawn, in same size, shape and orientation. If, after folding, there is a mismatch, the children need to reconsider distances. Similarly, Figure 2 helps children understand both equal distances from the axis and change of orientation: figures are given (same size and shape) but they have reverse orientations. Mismatch after folding makes this change apparent.

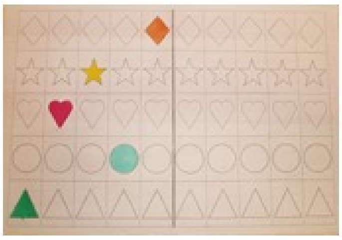

Figure 1

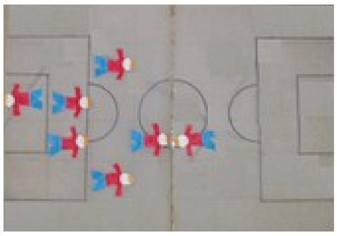

Figure 2

Although the overall teaching approach is far from being completed, systematic implementation and observations of young children have produced important evidence about the development of mathematical activity in them (e.g. Tzekaki & Ikonomou, 2009; Tzekaki & Kaplani, 2013). In the case of symmetry, a set of relevant tasks enabled children to approach the properties of reflection symmetry and 'formulate' them in a way. An ongoing research examines the development of this generalization, as part of mathematical activity, both in symmetry and other contents.

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CONCEPTUAL AND BRAIN PROCESSING OF UNIT FRACTION COMPARISONS: A COGNEURO-MATHED STUDY

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This mixed-method, qualitative/quantitative study examined (a) how a constructivist-based intervention (CBI) effected adults' learning of unit fractions and performance on whole-number (WN) or unit fraction (FR) comparisons and (b) brain circuitry implicated (fMRI) when processing these comparisons. The CBI used unit-iteration based activities to foster a shift in participants' understanding of FR, from the prevalent, limiting "one-out-of-so-many-equal-parts" idea to a multiplicative relation conception and thus inverse magnitude relation among FR (1/n>1/m though m>n). Pre- and two post-intervention tests indicated CBI impact on decreased reaction time in comparing not just FR but also WN and differentiated brain regions implicated for each. Implications for theory testing and CBI impact on WN-FR links are discussed.

BACKGROUNG AND CONCEPTUAL FRAMEWORK

Alluding to President Obama's (White House, 2013) *BRAIN Initiative*, this study examined how task design for brain research and teaching unit fractions, rooted in a constructivist perspective (Piaget, 1985), may impact brain processing when adults compare numbers. It focused on a *milestone shift*—from direct comparison of whole numbers (e.g., 8>3) to the *inverse relationship* among unit fractions (1/3>1/8 while 8>3). At issue was (a) how a *conceptually driven* intervention, used for teaching adults who already knew the "inverse rule", may impact their performance and (b) what brain circuitry would be activated to process the numerical comparisons (i.e., identify the neuronal basis for operating on whole numbers (WN) vs. on unit fractions (FR)).

Cross-disciplinary work of neuroscientists and educators is a new trend. Initially, educators became interested in brain-based research (Westermann et al., 2007). Later, this unidirectional, neuroscience-to-education fertilization, has yielded collaboration and reciprocal scholarship (De Smedt et al., 2011). Five facets of brain research seem of interest to mathematics educators: (a) compare learning/thinking and brain functioning among different groups (e.g., child-adolescent-adult); (b) understand how learners perceive, process, and link symbolic (e.g., Arabic) and non-symbolic quantities; (c) develop/validate observation-based theoretical frameworks of thinking, learning, and teaching; and (d) test effectiveness of practices to promote learning (e.g., critical-yet-intractable domains like fractions). To-date, however, the differences in operating on WN to FR were studied in each discipline separately.

Much brain research has focused on how it represents and processes numerical information. Dehaene's seminal work (Dehaene, 1997; Dehaene et al., 2003) yielded a triple code model of human WN perception. In that model, Arabic numerals are processed and represented in low-level visual cortical regions, numeric words in more anterior and language related cortical areas (lingual gyrus, perysylvian cortex), and analog magnitudes (e.g., a “number-line”) involve the Intraparietal Sulcus (IPS). In contrast, only a few studies focused on how the brain processes fractions (Bonato et al., 2007; Ischebeck et al., 2009; Jacob & Nieder, 2009). One study demonstrated that when adults solve challenging tasks (e.g., 2/3-1/4), the WN triple code model seems to also pertain to FR (Schmithorst & Brown, 2004). However, research has not yet conjoined WN and FR into a single study, let alone used a MathEd conceptual framework to guide research questions and design. The present study addressed this lacuna, to advance knowledge that can explain difficulties and potential affordances provided by (a) common/different brain circuitry used for WN vs. FR and (b) how number recognition (“cue”) and comparison (operation) may impact processing, and hence learning, of FR.

Conceptual Framework

Von Glasersfeld’s (1995) scheme theory grounds this study. A scheme is considered a tripartite conceptual building block: a *situation* into which a person assimilates information (which triggers her goal), an *activity* for accomplishing that goal, and an expected *result*. Extending this work, Simon et al. (2004) proposed (a) *anticipation of activity-effect relationship* as a lens to delineate “conception”—a dyad comprising the last two parts of a scheme, and (b) *reflection* on this relationship (abbreviated as *Ref*AER*) as a mechanism underlying cognitive change. Ref*AER commences with assimilating a task into the situation part of an available scheme, which also sets one’s goal. The mental knowledge system recalls and executes the scheme’s activity. The learner’s goal regulates effects produced by the activity. This enables noticing of discrepancies between one’s goal and the actual effects. Via reflection on solutions to comparable tasks, the learner abstracts a new, invariant relationship between an activity and its anticipated effect(s). This central notion of anticipation, which was developed via observational studies, has been corroborated by recent neuroimaging studies (Schacter et al., 2012; Suddendorf & Corballis, 2007).

Importantly, this framework distinguishes objects on which the mind operates (e.g., number) from operations on those objects (e.g., ordering smaller to larger). This distinction informed task design for this study, so assimilation of cues would be triggered by only one of two possible symbols (number *or* operation) before an entire number-comparison task is presented. Cues that precede number comparisons were expected to differently effect performance due to the brain’s pre-task recognition and ‘pulling the cue’ from long-term into working memory. That is, we hypothesized that distinct patterns of brain activation and/or neuronal circuitry would be recruited when an object is presented before an operation or vice versa.

METHODOLOGY

Participants (N=21), ages 23-36, took a pre-intervention computerized (ePrime) test comprised of 4 runs, each including 90, four-step number comparisons (randomized).

In Step A of each task (1 sec) a symbol of number or operation appeared (e.g., 7, 1/7, >, or =). In Step B (1 Sec) another symbol accompanied the first (e.g., 7>, 1/7=). In Step C the comparison task appeared fully (e.g., 7>8?, 1/7>1/8?), providing up to 2.5 sec to respond by pressing a key on the right for "true" or the left for "false." Step D showed three dots (0.5 sec) to separate tasks (ITI).

A video recorded *teaching* episode (~50 minutes) followed pre-test immediately. First, participants provided, with drawn examples, their definition of fraction. Then, creating their perturbation was promoted via posing a problem for which that definition is inadequate (Figure 1). Next, they were engaged in the challenging task of equally sharing unmarked paper strips among 7 people (then, 11) without folding the paper or using a ruler. Instead, they were taught to use the Repeat Strategy (Tzur, 2000): estimating one person's piece, iterating that piece 7 times, comparing the resulting whole to the given one, adjusting the estimate, etc. Reflection on this activity, promoted by teacher probing into participants' reasons for those adjustments ("make the next shorter/longer? Why?"), aimed to foster a conception of the unique, multiplicative 'fit' between each unit fraction (1/n) and the whole (n times as much of 1/n), and of the inverse relationships among unit fractions (to fit more pieces—each must be smaller). Discussion of why a larger denominator implied a smaller unit fraction *for any FR*, but *no practice* of such comparisons, concluded the episode.

Sticks *A* and *B* are equal in length. *A* contains 4 equal parts. The shaded part on *B* is equal to the part above it on *A*. What fraction, if any, is the shaded part of *B*? Of *A*? Why?

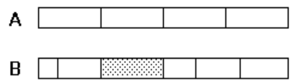

Figure 1

A first post-test as described above was conducted immediately after the intervention, and a post-test took place a few months later during fMRI scanning. To increase fMRI signal, runs were altered to include 140 two-step tasks (eliminating Step B above). Response time (RT) was recorded when subjects pressed a button in the right hand for "true" and the left for "false," but each task ended after exactly 2.5 seconds. Experimental tasks with a true ">" comparison included roughly 90% of all presented, while "=" and false ">" tasks served as control. Runs were organized in a hybrid-block design, including random-length sequence of like-comparisons (e.g., 1/3>1/8, 1/7>1/2, **8=8**, 5>3, 9>7, 4>3, 6>4, etc.).

ANOVA was calculated to determine the impact each independent variable (number type, Step A cue, testing occasion) has on the two dependent variables (RT, ER). Repeated observation and analysis of video recording helped inferring into participants' thinking about fractions before, during, and after instruction.

RESULTS

This section presents data and analysis of change in participants' conception of FR (qualitative), change in their performance of WN or FR comparisons (quantitative – behavioral), and differentiated brain circuitry activated (quantitative – fMRI).

Changing Adults' Conception of Unit Fractions

Upon completion of the pre-test in the computer room, each participant wrote down a definition for fractions (with example of 1/4). Then, s/he was asked to solve the Sticks Problem as a conceptual pre-test. All (100%) participants explained that a unit fraction is, "One out of so many equal parts of a whole," drew a circular figure ("pizza") partitioned into 4 parts and shaded one to show 1/4, and none was able to answer *both* questions about the shaded part on Stick B. Particularly prevalent (>50%) were responses such as, "The shaded part cannot be a fraction of Stick *A* because it is not a part of *A*" and "I cannot determine what fraction is the shaded part of Stick *B* because there are six unequal pieces on it."

Then, asked to equally share a given paper strip among 7 people without folding it or using a ruler, they initially had no solution. When prompted, "Could you estimate the share of one person and then find out?" each either generated the Repeat Strategy independently or was offered by Tzur to use it. Once iterating the first estimated part (say, too long), and asked if the next one had to be shorter/longer, they all knew the direction of change needed (here, shorter), explaining that more pieces had to be "squeezed" into the whole so each should be smaller. After making one piece that's too short and the other too long, they all also used a strategy of estimating the next piece's size between the closest short/long pieces already produced. Once the 7-piece iterated-whole seemed very close to the given whole, they were shown how to use JavaBars to produce an equally partitioned whole (with 7) and how to pull out one of these parts and measure it with the whole as a "Unit Bar" (1/7 shown on piece). Then, when asked if to share the whole among 11 people they would make the first estimate shorter/longer than the pulled-out 1/7-part, all (100%) knew to make it shorter, "because I have to squeeze even more parts into the same whole." At this point, each participant used the Repeat Strategy in JavaBars until the iterated whole was judged close enough to the given whole. Next, in reference to their activity, Tzur provided a definition (while they wrote it): "A unit fraction is a multiplicative relation to the whole; what makes 1/n what it is has to do with how many times it fits in the whole, or that the whole is n times as much of it. For example, your first estimated piece was 1/7 because the whole is 7 times as much of it." He also held one whole "fry" and asked if they could imagine the whole of which this single piece of paper would be 1/5. All explained they "saw" a strip that's 5 times longer.

At this point, Tzur returned to the Sticks Problem. All participants (100%) then explained that the shaded part is 1/4 of Stick *A* and 1/4 of Stick *B* for one and the same reason, namely, "the length of the whole is 4 times as much as the shaded piece's length." These data indicate that the CBI, via the Repeat Strategy, fostered each participant's reconceptualization of what a unit fraction is—not solely or mainly as a part of a whole but rather as a multiplicative relation between two magnitudes. They could thus "see" the shaded part on B as 1/4 in spite of the whole being marked into 6 unequal pieces, or as 1/4 of A although not part of A.

Improvements in Adults' Reaction Time (RT) for Processing WN and FR

Upon completion of each teaching episode, each participant re-took the computerized test (post). Analysis of test data showed that the average error rate in both occasions (pre/post) and for both number types (WN/FR) was very low (3-4%), while average reaction time (RT) significantly improved (p<.001). The latter included consideration of the cue that preceded each comparison trial: operation (>) or number (WN or FR). The chart below shows average RT (**in milliseconds**) for each type of task design, indicating statistically significant improvement (p<.001) from pre to post not only in comparing FR (as expected) but also, surprisingly, for WN. The data also show a cue X number-type interaction: non-significant impact of cue on RT for WN comparison vs. significant impact on RT for FR (p<.05). That is, RT when seeing FR before the comparison was shorter than when seeing ">" and this difference decreased in post-test. These results seem to lend support to the distinction among parts of a thinking process (scheme), as RT needed to recognize and process a mental object to be operated on is effected by how a "situation" is identified in the person's mind.

	Pre		Post	
	Cue: >	Cue: Number	Cue: >	Cue: Number
FR	1208	1144 (-64 = -5.3%)	923	901 (-22 = -2.4%)
WN	925	949 (+24 = 2.6%)	757	763 (+6 = 0.8%)

Table 1

Brain Circuitry Activated to Process Numbers (WN, FR) and Operation (>)

Figure 2 shows adult brain circuitry activated more for WN than FR comparisons (Figure 2a) and more for FR than WN comparisons (Figure 2b). The former shows WN implicated in: (A) the Hippocampus (LTM retrieval) and (B) the Medial Frontal and Anterior Pole (abstract retrieval). The latter shows *substantially greater activation for FR*, implicated in: (A) the bilateral IPS and Angular Gyrus (numerical judgments of denominators) and the Ventral Visual Processing Stream (object-based visual processing), (B) the Dorsal Fronto-Parietal control network (engaged in attention-demanding tasks, e.g., order inversion), (C) the Ventral-Frontal working memory network & Pulvinar (visual object attention/selection), and (D) the Supplementary Motor Area (SMA, preparing response). Combined, these analyses suggest that brain circuitry used by adults to compare FR involves higher activation in some areas used also for WN (e.g., IPS), along with a more widespread use of brain regions.

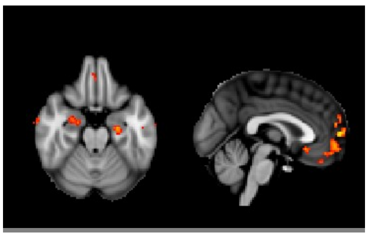

Figure 2a: WN > FR

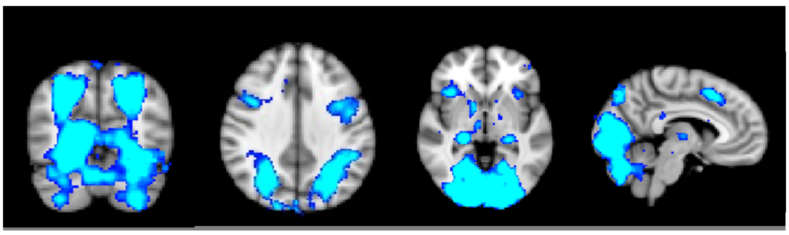

Figure 2b: FR > WN

Figure 3 shows adult brain circuitry activated more for numbers than for the “>” operation (yellow/red colors show this for WN and blue colors for FR). Essentially, when comparing activation of both types of numbers to the operation on these objects (directed by the goal of “find the larger of two numbers”), the same four regions seem to be recruited. The fMRI simulations show more activation for numbers (than “>”) in: (A) the Ventral Visual processing stream/cortex (typical of object-based, visual processing mostly in the right hemisphere); (B) the IPS and Angular Gyrus (numerical judgments); (C) the SMA (preparing for response), and (D) Posterior Dorsolateral PFC (attention-demanding tasks). Combined, these analyses suggest that brain activation employed just for recognizing a “cue,” before any comparison activity of the task is carried out, is markedly different (smaller) for the symbolized operation than for either type of symbolized numbers the brain processes. Not surprisingly, a remarkable overlap can be seen between these regions and those in which greater activation was found for FR than for WN. Both number types activate some similar circuitry much more than the symbolized operation, whereas processing FR comparisons does so to a much greater extent than WN.

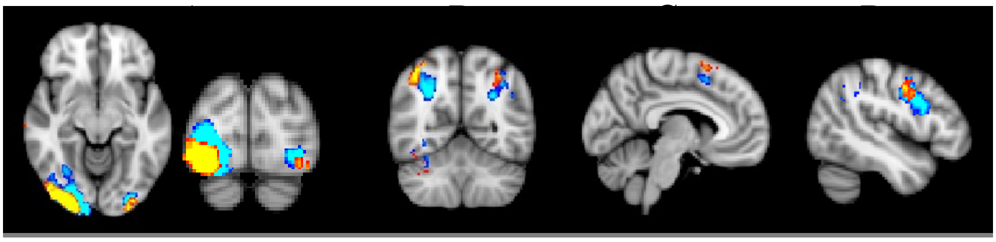

Figure 3: WN > FR

DISCUSSION

We presented three key findings about how a constructivist-based intervention (CBI) impacts adults’ re-learning and performance of whole number (WN) and unit fraction (FR) comparisons, and of brain regions activated to process such comparisons. First, we found a change in participants’ conception of unit fractions, from “part-of-whole” to a multiplicative relation. Second, we found a CBI’s significant impact on their performance of numerical comparisons, not only for FR but *also for* WN. Third, we found significant differences in brain activation: Hippocampus activated more for WN comparisons (long-term memory), whereas IPS (numerical), PFC (task attention and control), Ventral-Frontal and Pulvinar (visual object attention) and SMA (motor response) were substantially more activated for FR comparisons. Combined, these findings entail three contributions to an emerging, cross-disciplinary field at the confluence of mathematics education and cognitive neuroscience.

A first contribution concerns the construction of differentiated brain circuitry to process different types of numerical objects, not identified in previous studies. The limited scope of the fMRI part of our pilot study precludes determining when and how have regions, specialized in recognizing FR and processing comparisons among them, evolved. Moreover, it is not possible to determine if the CBI changed these adults’ previously constructed activation patterns, or the differentiated circuitry evolved when they first learned about FR (as children). While these two issues await future research, distinguishing these regions paves the way for (a) studying such an evolution, (b) figuring out if it depends on the nature of instructional methods, and most importantly (c) appreciating the implied, greater cognitive load involved in making sense of and solving FR comparison tasks. Simply put, FR is not just a simple extension of WN. The brain and mind need to construct circuitry that give rise to these numbers and, by way of extrapolation, likely also for other number types.

A second contribution is of a new way to test, and confirm or disconfirm, conceptual frameworks in mathematics education that were developed through observational studies. This pilot study provided an example of such a research pathway for the constructivist scheme theory (von Glasersfeld, 1995). Comparison tasks we designed capitalized on the distinction between the goal-directed activity and the object on which it operates, and showed differentiated impact on both brain circuitry (Fig. 3) and reaction time (see also, Tzur & Depue, 2014). Our findings seem to support the tripartite notion of a scheme, though more specific measures of brain circuitry that correspond to those parts are needed. Key here is that our study illustrates how a CogNeuro-MathEd collaboration can contribute to a two-way enrichment of research and knowledge, informing CogNeuro by MathEd frameworks and informing (curbing and/or expanding) MathEd by CogNeuro findings of the brain (De Smedt & Verschaffel, 2010).

A third contribution involves the CBI's impact on performance of WN comparisons. At issue is why, and how, would a conceptually driven method for teaching FR effect the comparison of WN—a long-established concept. We hypothesize that a person's focus on the multiplicative relation between a unit fraction and a whole into which it uniquely fits via unit iteration could bring forth reflecting on and re-conceptualizing WN as an iterable magnitude (Steffe, 2010) with direct relationship to other magnitudes. Future research can examine this hypothesis, and alternative ones, to better explain links between WN and FR at both the mind and the brain levels.

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ARGUMENTATION IN UNDERGRADUATE MATH COURSES: A STUDY ON PROBLEM SOLVING

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The purpose of this study is to analyze the complex argumentative structure in undergraduate mathematics classroom conversations during problem solving by taking into consideration students' and teacher' utterances in the classroom using field-independent Toulmin's theory of argumentation. Analyzing students' and teacher' utterances in the class allowed us to reconstruct argumentations evolving in the classroom talk as argumentations in classrooms are generally teacher guided. The analyses contributed to an emerging body of research on classroom conversations.

INTRODUCTION

Problem solving requires argumentation (Cerbin, 1988). Argumentation is a process of making claims and providing justification for the claims using evidence (Toulmin 2003; Mejia-Ramos & Inglis, 2009; Knipping, 2008). On the other hand, argumentation is a verbal and social activity of reason aimed at increasing (or decreasing) the acceptability of a controversial standpoint for the listener or reader, by putting forward a constellation of propositions intended to justify (or refute) the standpoint before a rational judge' (van Eemeren et al., 1996). Argumentation requires problem solvers to identify various alternative perspectives, views, and opinions; develop and select a preferred, reasonable solution; and support the solution with data and evidence (Voss, Lawrence, & Engle, 1991).

Toulmin's model has provided researchers in mathematics education with a useful tool for research, including formal and informal arguments in classrooms (Knipping, 2008) as it is intended to be applicable to arguments in any field. Studies using Toulmin model focused on analyzing students' arguments and argumentations in proving processes in a classroom (Knipping, 2002, 2008; Krummheuer, 1995) and, individual students' arguments in proving processes (Pedemonte, 2007). Toulmin himself noted that his ideas has no finality. Indeed his model has been reshaped in various ways, his claims have been contested by some and in response reformulated by others, and some but not all aspects of his approach have been incorporated in applications in different domains (Hitchcock & Verheij, 2006).

Having established these facts, the goal of our research is to study the argumentation in undergraduate mathematics classrooms during problem solving using Toulmin's theory of argumentation. Specifically, the aim is to analyze the structure of the arguments accomplished in the course of interaction where the teacher and students involvement in this accomplishment. This study is part of a wider study investigating the argumentation generated in undergraduate mathematics classes while proof generation (see Ubuz, et al., 2012), definition construction (see Ubuz et al., 2013), and problem solving. This paper suggests a method by which complex argumentation in problem solving can be reconstructed and analyzed. Analyzing students' and teacher' utterances in the classroom according to Toulmin model allows us to reconstruct argumentations evolving in the classroom talk since arguments are produced by several students together with the guidance of the teacher.

THEORETICAL FRAMEWORK

In the following sections we will expose some theoretical considerations on the Toulmin model, and the problem solving process.

The Toulmin Model

According to Toulmin, an argument is like an organism. It has both a gross, anatomical structure and a finer, as-it-were physiological one (Toulmin, 2003). He is interested in the finer structure. The Toulmin model is differed from analysis of Arisitotle's logic from premises to conclusion. First, we make a claim(C) by asserting something. For the challenger who asks "What have you got to go on?" the facts we appeal to as foundation for our claim is called data (D) by Toulmin. After producing our data, we may being asked another question like "How do you get there?" He notes, at this point we have to show that the step from our data to our conclusion is appropriate one by giving different kind of propositions like rules, principals, inference – licenses or what you will, instead of additional items of information (Toulmin, 2003). A proposition of this form Toulmin calls a warrant (W). He notes that warrants are of different kinds and may confer different degrees of force on the conclusions they justify. We may have to put in a qualifier (Q) such as "necessarily", "probably" or "presumably" to the degree of force which our data confer on our claim in virtue of our warrant. However there may be cases such that the exceptional conditions which might be capable of defeating or rebutting the warranted conclusion. These exceptional conditions Toulmin calls as rebuttal (R). For our challenger may question the general acceptability of our warrant: "Why do you think that?" Toulmin calls our answer to this question our backing (B) (Hitchcock & Verheij, 2006). The diagram of the Toulmin model is as follows:

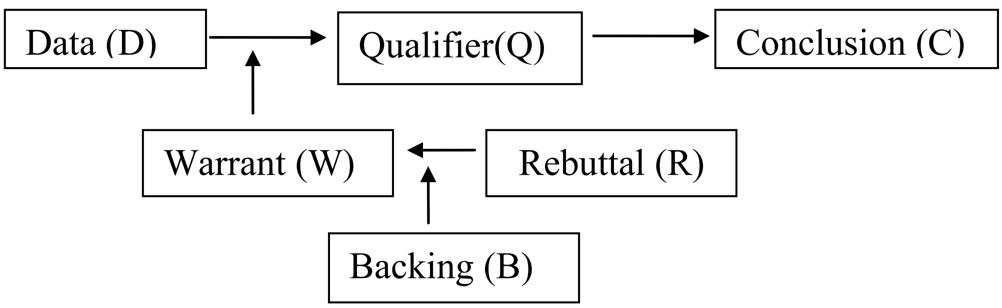

Figure 1: The Toulmin Model

Reconstructing and analyzing the complex argumentative structure in classroom conversations follow their own structure. For example, careful analyses of the types of warrants (and backings) that students and teachers employ in classroom situations allowed two distinctions in the justifications: visual and conceptual (Knipping, 2008). The warrants and backings based on conceptual aspect or deductive are mathematical concepts or mathematical relations between concepts, and make reference to theorems, definitions, axioms and rules of logic. The warrants and backings based on visual or figural aspect make reference to figures as part of the argumentation.

Problem Solving Process

Problems are identified as such if the participant sees a quandary or feels a difficulty or doubt that needs to be resolved (Hiebert et al., 1996). Once a problem has been identified, the participant actively pursues a solution by calling up and searching out related information, formulating hypotheses, interacting with the problem, and observing the results (Hiebert et al., 1996). Eventually some conclusion is reached, some resolution is achieved, some hypotheses are refined. The outcome of the process is a new situation, and perhaps a new problem, showing new relationships that are now understood (Hiebert et al, 1996). So, problem solving has two aspects: (a) the process, or set of behaviors or activities that direct the search for the solution, and (b) the product, or the actual solution. Both the process and the product are essential components of the problem-solving experience (Kantowski, 1977). The teacher bears the responsibility for developing a social community of students that shares in searching for solutions. Analyzing the adequacy of methods and searching for better ones are the activities around which teachers build the social and intellectual community of the classroom (Hiebert et al, 1996).

METHODOLOGY

Data were collected through nonparticipant observations that were videotaped. Observation was conducted 2009-2010 spring semesters in real analysis course for eight weeks, and 2010-2011 spring semesters in advanced calculus course for six weeks, offered to mathematics education student at the third and second years, respectively. These courses were selected as both formal and informal argumentations were at the focus of these courses. In these courses the number of students were 45 and 40, respectively. Formal proof approaches are given to the students at the "Abstract Mathematics I - II" courses provided in the first year. In these courses, students learn what a proof is and how to prove theorems. That is, they learn how to argue mathematically, justify their claims and encounter the cases named "counter example" for the first time which rebuttals their claims.

The analysis of the observations is based on the transcripts. As Toulmin (2003) noted, "an argument is like an organism. When set out explicitly in all its detail, it may occupy a number of printed pages or take perhaps a quarter of an hour to deliver; and within this time or space one can distinguish the main phases marking the progress of the argument from the initial statement of an unsettled problem to the final presentation of a conclusion" (p. 87). Based on this explanation, eleven argumentations were determined and five of them were on problem solving. These five argumentations were observed in real analysis course.

Observations were conducted by the second author. He analyzed the transcripts by marking the progress of the argument from the initial statement to the final conclusion through using Toulmin model components. He noticed that some aspects of observed argumentations were overlooked. He modified the Toulmin model by integrating *guide – backing* and *guide – redirecting* additional components which were observed in almost all argumentations. We called an approval given by teacher to the warrants, backings or intermediate conclusion as *guide – backing*. When the argumentation does not start from a right point or students get stuck on an argument point, teacher intervenes with an example, a question or a suggestion to arrange the argument. We called such intervenes as *guide – redirecting*.

Having discussed with the first author who is a mathematics educator and doing research on proof, it was decided that observed argumentations could be considered into three classes: proof generation, definition construction, and problem solving. She also noted that some components could be classified in itself. After re-analyzing observed argumentations, *warrant* component were divided in two categories: *deductive warrant* and *reference warrant*. Students appeal reasoning like numerical computing, applying a rule to an inequality, creating new ideas from a definition, a theorem or a rule in producing their warrants. We called this kind of warrants as *deductive warrants* as Inglis et al. (2007) did. When a warrant referred to a theorem, a definition, a rule or a problem, we called such a warrant as *reference warrant. Guide – backing* was divided into three categories: *approval, reference* and *terminator*. When teacher just approve the students' warrant, backing or conclusion by saying "good, fine, great, well done" and does not use any mathematical phrase, we called this kind of guide backing as *approval guide backing*. When teacher approve the students' warrant, backing or conclusion by referring a definition, a theorem or a problem recently solved, we called this kind of guide backing as *reference guide backing*. Argumentations come to an end when teacher or students reach the final conclusion to be achieved. In case, teacher reaches the final conclusion, students convince that the conclusion is legitimate. In case, students reach the final conclusion, teacher serves a backing. This backing shows the final conclusion and we called it as *terminator guide backing*. One important point that must be noted here is that argumentations were not analyzed according to their mathematically correctness.

Finally, full transcriptions together with analysis model components explanation are provided to an external auditor who is a researcher in mathematics education field. After a week, the auditor completed her analysis and a complete consensus was reached on analysis of argumentations.

RESULT

Five open-ended problems requiring the search of counter-examples and/or application of the definitions, rules, theorems for the solution constituted five different problem solving argumentation context. In this paper only one of these problems is considered as example because of page restrictions. Here we analyze a transcript of a short argumentation in which *deductive warrant, guide - redirecting and terminator guide – backing* appear. The following argumentation occurred when teacher asked if a boundary point of a set is an accumulation point of that set in R^n.

1	Deniz:	The boundary of a set A is defined as the intersection of its closure with its complement.
2	Teac:	Correct.
3	Stu:	And the closure of A is the union of A with the set of its accumulation points. Eeehm… "Or" operator… I got a mistake! I mean x(a boundary point of A) is in A or in the set of its accumulation points, so x does not have to be in the set of accumulation points of A.
4	Teac:	Well, you are right but it is not the way what is supposed to be. Instead, you should have a set, say A. Then x would be a boundary point of A but would not be a accumulation point of A. I mean you should give a counter example. Got it? Do you have a such example?
5	Alpaslan :	Would it be one-point set?
6	Teac :	Well done! Is it one-point set? Yes, it is.
7	Alpaslan :	The set of its accumulation points is empty set.
8	Teac :	Which means that a boundary point does not have to be an accumulation point.

In line 1, the student defined the boundary of a set. He considered it as a data. In line 3, he realized that he needs to use "or" conjunction. He used this reasoning as a deductive warrant to conclude that a boundary point of A shouldn't be in the set of accumulation points of A. In line 3, teacher intervened with a suggestion to arrange the argument. He clearly stated that he needs to have a counter example. So teacher gave a guide – redirecting. Hereon, Alpaslan suggested one-point set as his data in line 5. In line 6, teacher gave an approval guide backing by using phrases "Well done! Is it one point set? Yes, it is.". Therewith, Alpaslan could produce easily the final conclusion at the end of line 7. In line 8, teacher gave a terminator guide – backing by confirming the final conclusion. Therefore, his conclusion is valid. We observed that student producing deductive and/or reference warrant get easily the final conclusion after getting terminator backing guide. We think that if students have an ability to produce deductive and/or reference warrants and get any kind of guide – backing, then he/she could get easily the final conclusion. The diagram corresponding to the argumentation above is as follows:

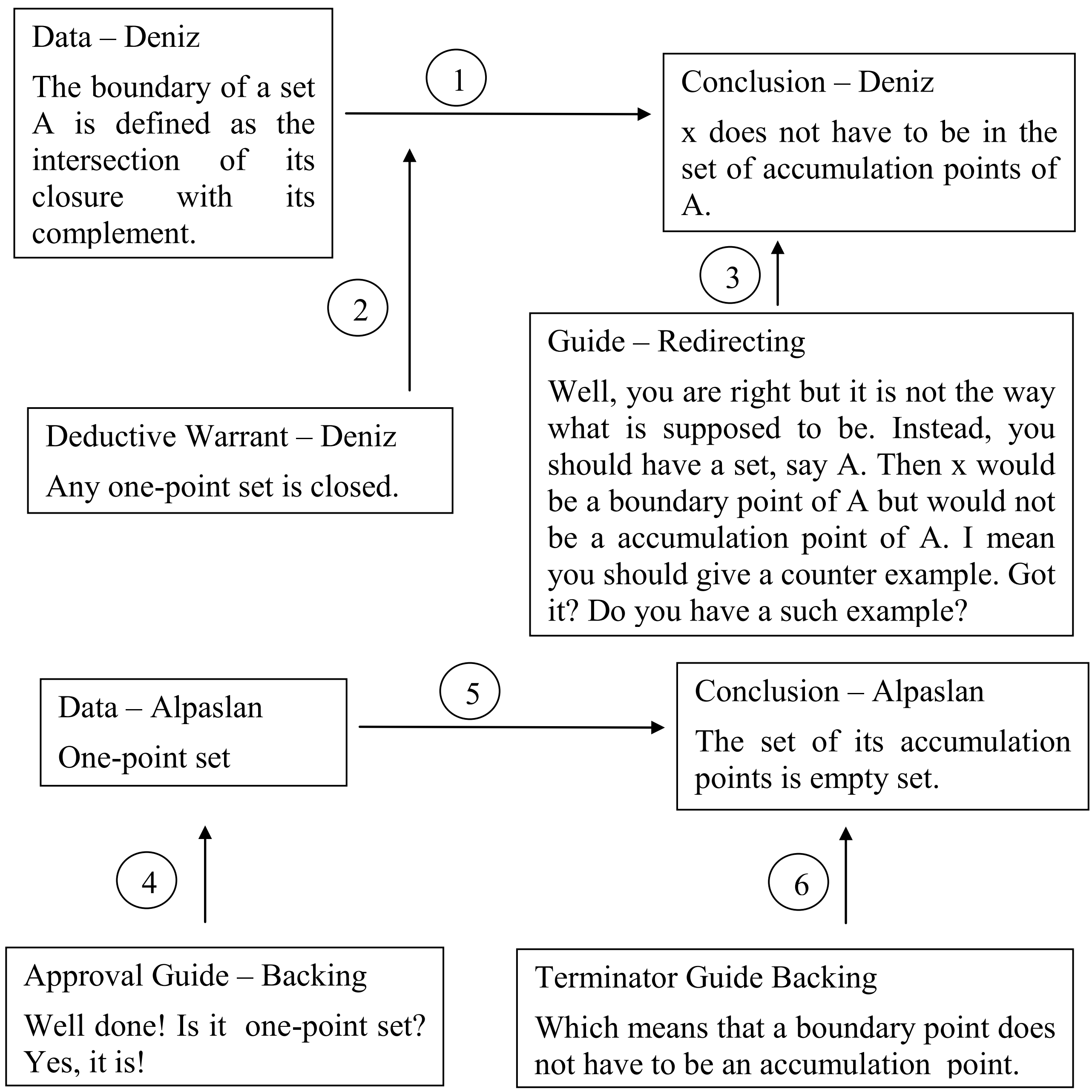

Figure 2

CONCLUSIONS

The model of Toulmin, which is helpful for reconstructing argumentation steps and streams, is not adequate for more complex argumentation structures. Argumentations in classrooms requires a different model for capturing the global structure of the argumentations developed there. Analyzing students' and teacher' utterances in the class according to the Toulmin model allowed us to reconstruct argumentations evolving in the classroom talk. Argumentations in classrooms are generally teacher guided. Teacher acts as a guide who exactly knows the path to follow i.e. where to start and to end the argumentation. Therefore argumentation guided by the teacher in the classroom comes to an end. During the argumentation if students follow the wrong path, get a false intermediate conclusion or get stuck in a point, teacher intervene the students to put them on the path in which they have to follow. If students on their own can manage to get the conclusion of argumentation, then they are sure about the conclusion when they get the terminator guide backing. According to this and based on our observations, teacher played a role in argumentation like guide – backing and guide – redirecting. According to our view, guide – redirecting is an important component for searching out related information, formulating hypotheses, interacting with the problem, and observing the conclusions which are essentials of problem solving process. We also think, guide – backing and guide – redirecting components prevent emergence of qualifier component in argumentations in proof generation, in definition construction and in problem solving process. There are two reasons for that. Firstly, the conclusion needs to be reached is absolute. Secondly, guide – backing and guide – redirecting are components which leads the way to the absolute conclusion.

In sum, new components were identified to be added to the Toulmin argumentation model as well as interactions between them. They were named *guide-backing* and *guide-redirecting*. *Guide backing* was divided into three classes: *approval, reference* and *terminator*. *Approval guide-backing* and *terminator guide-backing* occurred in almost all argumentations related to proof generation, definition construction, and problem solving but *reference guide-backing* occurred in almost all except argumentation on constructing a definition (see also Ubuz, et al, 2012, 2013). Furthermore, *warrants* were divided into two classes: *deductive* and *reference*. *Deductive warrant* occurred in any type of an argumentation but *reference warrant* occurred in any type of an argumentation except in an argumentation for constructing a definition (see also Ubuz, et al, 2012, 2013).

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LEARNING MATHEMATICS WITH PICTURE BOOKS

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This paper describes a field experiment with a pretest-posttest-control group design in which the potential of reading picture books to children for supporting their mathematical understanding was investigated. The study involved 384 children from eighteen kindergarten classes in eighteen schools in the Netherlands. Data analyses revealed that the experimental group showed a significantly larger increase than the control group in their mathematics performance in a project test containing items on a variety of mathematical topics including arithmetic, measurement, and geometry.

PICTURE BOOKS IN MATHEMATICS EDUCATION IN KINDERGARTEN

One way of supporting children's mathematical understanding is making use of children's literature. This approach has become increasingly popular in recent years (Haury, 2001). Even though activities such as reading picture books might not seem very suitable for teaching mathematics, stories narrated in books may contain mathematics, and as such can offer children opportunities to face mathematics (Anderson, Anderson, & Shapiro, 2005). A very important reason why reading picture books to children may help them in learning mathematics has to do with the meaningful context of the stories included in picture books (e.g., Columba, Kim, & Moe, 2005). Research suggests that learning within a story context increases the retention and recall of the learned knowledge (e.g., Mishra, 2003).

Earlier studies about effect of using picture books on mathematics achievement

Several studies have been carried out that investigated the effect of reading picture books on young children's learning of mathematics. In a study by Hong (1996), kindergartners in Korea with highly educated parents were involved. In this study, the intervention was based on mathematics-related storybook reading and play with mathematical materials that were associated with the content of the storybook. Children who received this intervention exhibited a more positive disposition towards mathematics and significantly greater performance in task about classification, number combination and shapes, than children of the control group.

Young-Loveridge (2004) investigated the use of a program including number books and games. She examined the immediate effect of this program as well as its endurance on the improvement in the numeracy of 5-year-old children. The findings of the study showed that the program was highly effective in enhancing the numeracy learning of young children immediately after the intervention. Moreover, although later the performances decreased, children who participated in the intervention still performed significantly better than children who were not involved.

Furthermore, the findings of a study by Casey, Erkut, Ceder, & Mercer Young (2008), which included storytelling instead of story book reading, gave evidence for the advantages of using a storytelling context as a means for improving early geometry learning in children.

A common characteristic of all aforementioned studies is that the book reading or storytelling sessions in class were always combined with other activities such as playing with story-related (mathematical) materials (Hong, 1996; Young-Loveridge, 2004), singing mathematical rhymes (Young-Loveridge, 2004) or composing geometrical puzzles (Casey et al., 2008).

The present study

The present study is meant to gain more knowledge about the effect of the book reading itself, i.e., without inclusion of additional (book-related) mathematical activities. The study was carried out in the Netherlands and was part of the PICO-ma project (PIcture books and COncept development MAthematics).

Our research question was: *Can an intervention involving picture book reading contribute to children's mathematics performance?* Based on earlier research, our prediction was that kindergartners' performance in a mathematics test would increase due to the picture book reading program, i.e., we hypothesized a positive intervention effect.

METHOD

To investigate the effect of reading picture books on young children's mathematics performance, a field experiment was carried out in kindergarten classes based on a pretest-posttest-control-group design with a three-month picture book reading program as an intervention in which each week two books were read in class to the children.

Participants

Our sample was based on a stratified sampling procedure resulting in pairs of schools that were approximately similar regarding urbanization level, school size and average SES of their children. The schools in each pair were assigned randomly to the experimental group or the control group. In total we had 384 four- to-six-year-old kindergartners participating in our study: 199 in the experimental group and 185 in the control group. Both groups were quite similar. They had about the same average class size, proportions of children in Kindergarten year 1 and Kindergarten year 2, proportions of girls and boys, of children with non-Dutch and Dutch home language, and also the children's age did not differ between the experimental and the control group. The same is true for the children's mathematics and language abilities as measured by the Cito mathematics test and the Cito language test before the intervention took place.

The used picture books and reading guidelines

The reading program used in the intervention consisted of 24 trade books of high literary quality which have mathematics-related content. Yet, the authors did not include this content purposely to teach children mathematics. To cover a rich variety of mathematical domains, we chose picture books dealing with arithmetic, measurement, or geometry. Within these domains we focused respectively on numbers and number relations, growth and perspective. Altogether, eight books were selected within each domain on the basis of their learning-supportive characteristics (Van den Heuvel-Panhuizen & Elia, 2012).

For each book a reading guideline was developed that explains how to read the book. In general, the reading guidelines requested the teachers to maintain a reserved attitude and not to take each aspect of the story as a starting point for a class discussion, since lengthy or frequent intermissions could break the flow of being in the story and consequently diminish the book's own power to contribute to the mathematical development of the children. To promote the children's mathematical thinking the teachers were suggested to show behavior such as (1) asking oneself a question out loud about the mathematics, (2) playing dumb, or (3) just showing an inquiring expression at a certain page of a book.

Figure 1: Page 4 of the book *Feodoor has seven sisters*[1]

Figure 1 shows page 4 from the book *Feodoor heeft zeven zussen* [Feodoor has seven sisters] (Huiberts & Posthuma (illustrator), 2006), which is about a man who has seven sisters. The text on page 3, which is left to page 4, says: "At night before he goes to sleep, he doesn't get just one kiss. No, his seven sisters give him, altogether twenty-one kisses. Fourteen arms around him, and he is wrapped up well from head to foot. Then, he is read six stories and one poem. Finally, seven fingers reach for the light-switch."

The reading guideline says the teacher to stop after "altogether twenty-one kisses" and to show an inquiring expression by raising her eyebrows. In one of the classes this led to the following classroom conversation.

All children: [All children react together; look at each other; reactions are mumbled.]

Teacher: Twenty-one kisses!

E:	[Starts counting while tapping her cheek] 3, 4, 5
Y:	On two sides
All children:	[All children react to what Y says; only the word 'two' can be made out]
M:	[Says something inaudible to the teacher]
Y:	.... plus 13?
Teacher:	No, he received twenty-one kisses, and you just said [she looks at Y] he gets a kiss on each side from every sister, right [teacher points at the first sister in the picture in the book] because you were already starting to count. You said 2...
All children:	[The teacher points at the second sister] 4
All children:	[The teacher points at the third sister] 6
All children:	[The teacher points at the fourth sister] 8
All children:	[The teacher points at the fifth sister] 10
All children:	[The teacher points at the sixth sister] 12
All children:	[The teacher points at the seventh sister; children hesitate]
Y:	[Starts, doesn't finish the word] Thir...
Teacher:	F...
All children:	14
Teacher:	Fourteen, but then it's not right. They say twenty-one kisses.
E:	Okay, then it's here, here and here [points at her own face to show where the kisses are placed; one on the left cheek, one on the right cheek, and one on the forehead.]

Then, the teacher invited all children to check whether this is correct by giving Feodoor the kisses as child E suggested. Indeed, then they ended up with twenty-one kisses. All children shouted that Feodoor got three kisses from each of his sisters. This is quite an achievement for a group of kindergartners who have not yet been taught multiplication or division.

The PICO test

To investigate the effect of the picture book reading program we developed the so-called PICO test consisting of multiple-choice items for the domains of arithmetic (including the topics number and number relations), measurement (with the topic of length with emphasis on growth), and geometry (addressing the topic of perspective). Every item covers one page and contains an illustration depicting a situation and four small illustrations that represent the possible answers. After the test instruction of an item was read aloud to them, the children had to answer it by underlining the correct answer. Figure 2 shows two items for the domain of arithmetic.

The PICO test was administered as a pretest before the intervention took place and as a posttest afterwards. At the same time points the PICO test was also administered in the control classes. At the start of the project, the teachers of these classes were not informed about the aim of the study. The teachers were just told that a test would be administered at two time points to gain information about how kindergarten's understanding of mathematics grows over a three-month period in normal school practice.

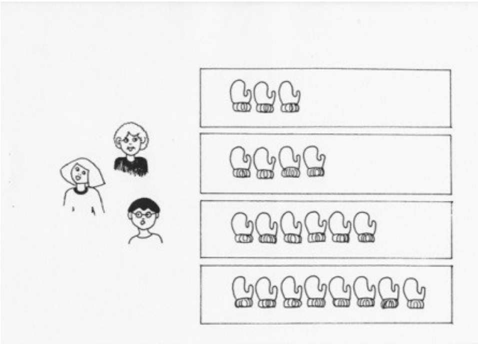
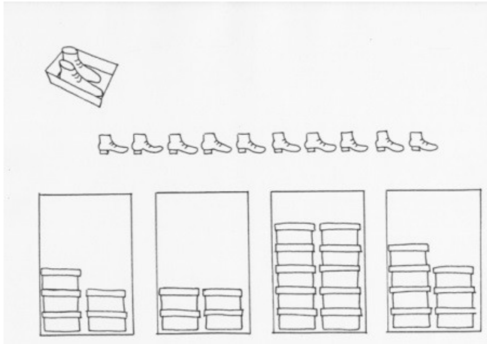
Mittens	Shoe boxes
	
Test instruction: These children have cold hands. They all like to put on the warm mittens. Underline the amount of mittens they need in total.	Test instruction: Two shoes fit into one box. How many boxes are needed for the other shoes? Underline the number of boxes you need for the other shoes.

Figure 2: Two PICO test items

The initial version of the test consisted of 42 items. After calculating the item discrimination based on the pretest data, we removed two items which had negative item discriminations. This led to a test with 40 items in total that all have a positive correlation with the total score. The calculation of the Cronbach's alpha of this final version of the test resulted in a sufficient reliability of $\alpha = .79$ for the whole sample, and $\alpha = .71$ for the sample of K1 children as well as for the sample of K2 children. Furthermore, within the experimental and the control group, we found correlations between the PICO pretest and posttest score ranging from .62 to .83, indicating a high test stability.

To further investigate the properties of the items in the PICO test, we conducted a confirmatory factor analysis at the item level (using WLSMV estimation implemented in Mplus; Muthén & Muthén, 2007) with the three mathematical topics number and number relations, growth, and perspective as dimensions. Due to very large correlations between these dimensions, we treated the test as essentially one-dimensional. Coherent with these findings, a one-dimensional factor analysis resulted in an almost equally well-fitting model (CFI = .96, TLI = .97, RMSEA = .02). Therefore, we used the total score of the PICO test for analyzing the intervention effect.

Statistical analysis

We investigated the intervention effect by using two linear regression models, namely One-Way ANCOVA models. In Model 1, we used the PICO posttest score as a dependent variable and as independent variables the experimental group (as a dummy variable) and the PICO pretest score (as a covariate). In Model 2, further covariates were added, including kindergarten year, age, gender, home language, SES, and Cito mathematics and Cito language.

Despite the nested structure of the data – children belonging to classes which belong to schools – we applied a single-level linear regression model, because our unit of inference was at the level of children. Moreover, our clustered sampling procedure in which matched pairs of schools were randomly assigned to either the experimental or the control group decreased the standard errors of the parameters of interest. Yet, to be sure about using a single-level linear regression model, we calculated the residual intra-class correlation of the PICO posttest score controlling for the pretest score by means of a multilevel random intercept model in lme4 (Bates et al., 2013). It turned out that the residual intra-class correlation was .025. This finding supported the conclusion that ignoring the multilevel structure in our analyses did not lead to notable underestimation of standard errors (Hox, 2010).

RESULTS

In Table 1 the descriptives are presented of the PICO test total scores in the pretest and posttest for the whole sample and specified for the experimental and control group, and for the two kindergarten years. For the PICO pretest score no differences between experimental and control group were found. For the PICO posttest score the experimental group scored slightly, but not significantly higher than the control group.

Kindergarten year	Group	N	Pretest score (total items: 40; max. score: 40)				Posttest score (total items: 40; max. score: 40)			
			M	*SD*	*d*	*p*	*M*	*SD*	*d*	*p*
K1	Experimental	84	14.0	5.5	.08	.59	18.2	6.0	.25	.11
	Control	66	13.6	4.0			16.9	4.5		
K2	Experimental	115	20.2	4.8	.02	.92	24.6	5.3	.19	.17
	Control	119	20.1	5.1			23.6	5.4		
K1 + K2	Experimental	199	17.5	6.0	-.03	.74	21.9	6.4	.11	.26
	Control	185	17.7	5.7			21.2	6.0		
Total sample		384	17.6	5.8			21.6	6.2		

Table 1: Descriptives for PICO pretest and posttest

Table 2 shows the results of the two regression models we used for investigating the intervention effect on the PICO posttest score. Both models gave comparable results. Model 1, in which we had only the PICO pretest score as a covariate, revealed a significant intervention effect ($B = .90$, $p = .01$), while Model 2, in which we controlled for seven additional covariates, resulted in a similar intervention effect ($B = .76$, $p = .02$). In this model, pretest, home language and Cito mathematics did have a significant influence on the PICO posttest score. Due to space limitations further analyses of the intervention effects in subgroups and the differential intervention effects between subgroups cannot be discussed here.

	Model 1					Model 2			
	B^a	*SE*	*p*	β		B^a	*SE*	*p*	β
Intervention	.90	.36	.01	.01	Intervention	.76	.37	.02	.06
PICO pretest [b]	.89	.03	.00	.84	PICO pretest [b]	.69	.05	.00	.64
					Kindergarten year (K2)[c]	.32	.67	.64	.02
					Age	.05	.04	.22	.06
					Gender (girl)	-.10	.35	.79	-.01
					Home language (Dutch)	1.22	.60	.04	.07
					SES (medium/high)	.20	.80	.80	.01
					Cito mathematics	.07	.02	.00	.16
					Cito language	.02	.03	.44	.05
R^2 (Explained variance)	.70					.73			

B: unstandardized regression coefficient of the intervention effect; *SE*: standard error of *B*; *β*: standardized regression coefficient.

[a] Because we expected a positive influence of the picture book reading program the *B* value for the intervention effect was tested in a one-tailed way.

[b] Because the covariates were only treated as control variables, the significance of the *B* value was tested in a two-tailed way.

[c] For the categorical covariates the dummy variables are placed in parentheses.

Table 2: Intervention effect on PICO posttest score

When calculating the effect size *d* by dividing the B-values by the standard deviation of the PICO pretest scores, we found for Model 1 $d = .16$ ($B = .90$ divided by 5.8) and for Model 2 $d = .13$ ($B = .76$ divided by 5.8). Comparing these effect sizes with the effect size of the change from pretest to posttest in the control group (gain score: $M = 3.5$, $SD = 3.5$, $d = .60$, $p = .00$), we found that the influence of the intervention was substantial. In Model 1, the change in the experimental group was 27% (.16/.60 = .27) larger than the change in the control group and in Model 2, the change was 22% (.13/.60 = .22) larger.

CONCLUDING REMARKS

Our study showed that a three-month picture book reading program with picture books containing mathematics-related content, had a positive effect on kindergartners' mathematics performance as measured by the PICO test. Moreover, these positive results were found based on picture book reading without additional mathematical activities. In fact, this gain from a short program is quite a lot taking into account the spurt in cognitive growth children generally make at this age, which is clearly shown by the increase in performance of the children in the control group, and which is also emphasized by other authors (e.g., Bowman, Donovan, & Burns, 2000). In sum, we can conclude that our study provided evidence for giving picture book reading a significant place in the kindergarten curriculum for supporting children's mathematical development.

However, this evidence should be considered with prudence. The participation of schools and teachers was on a voluntary basis which might have caused that only motivated teachers were involved in the study. Another shortcoming of the study was that despite of classroom visits and teachers' logs we could not completely control the implementation of the picture book program and also not what the teachers did as regular mathematics-related activities. Therefore, we cannot be absolutely sure that the picture book reading program as intended was responsible for the effect. Further research should go more in detail at the micro-level of the classroom conversations during the book reading sessions. This would also provide opportunities to identify the specific effective elements of picture book reading that contribute to the mathematical understanding of kindergartners.

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IMPROVING REFLECTIVE ANALYSIS OF A SECONDARY SCHOOL MATHEMATICS TEACHERS PROGRAM

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In this paper we present how the redesign of professional tasks in the teachers' formation of Secondary Mathematics Teachers influences changes in didactical analysis competency of future secondary school teachers. We draw on data collected from 3 groups of prospective teachers, using qualitative methods. We discuss how the training on the use of didactical tools to redesign tasks led prospective teachers to further develop their own professional competence to analyse mathematical tasks from a rigorous didactical point of view.

PRESENTATION AND CONTEXT

In this paper we analyse how a specific mathematics teachers' training program may produce changes in terms of future secondary school teachers' competence of didactical analysis, aiming at the growing and building knowledge for teaching (Zaslavski & Sullivan, 2011). Our general intention in such a program is to lead future teachers to develop the [professional] ability to (re)design sequences of suitable tasks, as well as to make them able to re-design their own designs of school tasks. In our study we call 'professional task' those tasks that we propose to future teachers in order to encourage them doing didactic analysis and developing their didactical analysis competencies. We understand such a competence as the ability for designing, applying and evaluating sequences of learning by means of didactic analysis techniques and quality criteria. It is also assumed that someone may reflect and improve their competence in terms of the analysis of mathematical classrooms, in order to make best use of the opportunities for being a teacher as teacher enquirer (Mason & Johnston-Wilder, 2004).

We want to focus on some immediate effects over the Program. We found them when analysing prospective teachers' thoughts emerging from their feedback [work assignments] with the researchers; and also emerging from our analysis of some impacts of the program itself. Such above mentioned development, it is stated when future teachers incorporate and use tools for the description, explanation and process valuation of mathematical school teacher/learning practices.

THEORETICAL FRAMEWORK

We introduce a teaching project based on an inquiry and reflective practicing framework in which we design and implement diverse teacher training cycles as teaching experiments (Tzur, Sullivan, & Zaslavsky, 2008) for developing transversal competences as citizenship, digital competency, didactical analysis, among others. In

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 321-328. Vancouver, Canada: PME.

particular, in this presentation we discuss a part of a teacher training cycle named "Didactic Analysis" which has been articulated across diverse subjects throughout the courses.

The development of the cycles had been based from the very beginning on the research process including six big types of professional tasks: (a) analysis of practices, objects and mathematical processes in which it is expected to appear and discuss tools for a descriptive and explanatory analysis that serves to answer "what happens in the classroom and why?" (Font, Planas y Godino, 2010); (b) analysis of didactic interactions, conflicts and norms; (c) evaluation of tasks and classroom episodes using criteria of didactic suitability or quality; (d) design and implementation of a lesson in their period of internship; (e) analysis and valuation of the suitability of the didactic implemented unit; (f) improvement of their lessons designs (for future implementation), within the Master's Final Project (MFP).

The analysis and description of the mathematical activity is conducted using the theoretical constructs proposed by the 'Ontosemiotic' approach (OSA). According to this perspective (Godino, Batanero y Font, 2007), the mathematical activity plays a central role and it is modelled in terms of systems of operative and discursive practices. From these practices the different types of related mathematical objects emerge building cognitive or epistemic configurations among them. Problem-situations promote and contextualize the activity; languages (symbols, notations, and graphics) represent the other entities and serve as tools for action; arguments justify the procedures and propositions that relate the concepts. Lastly, the objects that appear in mathematical practices and those which emerge from these practices might be considered from the five facets of dual dimensions. Both the dualities and objects can be analysed from a process-product perspective, a kind of analysis that lead us to the processes shown. During the following type of tasks (c - f), we present theoretical tools (suitability criteria, according Godino, Batanero and Font (2007) to conduct evaluative analysis to answer "what could we improve?" We understand that the study of descriptive and explanatory analysis for a didactical situation is necessary to justify the evaluations (Pochulu & Font, 2011).

METHODOLOGY

The research is mainly qualitative in nature as the purpose is to describe the development of competence in didactic analysis among aspiring secondary school mathematics teachers, from the University of Barcelona (Spain) during the Project development (2010-2013) following Gravemeijer (1998) perspective. The data was collected from video recorded observations, sorting sheets produced by teacher trainers, students' reflections at the end of the workshops and documentation housed in the Moodle platform (slides, reading material, tasks and the students' responses to them, and questionnaires and the students' responses to them). The samples were 3 groups of 24-26 and 25 prospective teachers. This amount of teachers includes almost the totality of students recruited in the Teacher Program in the University. During all these academic years, in general, these students vary in the amount of mathematical knowledge they have, while discussing certain conceptual biases regarding the teaching and learning of mathematics.

During the first year, future teachers did many naïf comments regarding the first tasks (a-b). We conjectured that protocols were static. During the next year we decided to use more videos and transcripts than during the previous one. Prospective teachers designed and implemented tasks (type b), with protocols showing constructs as cognitive and semiotic conflicts, epistemic obstacles, types of norms, patterns of models of management, interaction analysis, and so on. After that, they analysed a lesson focused on equations applying suitability criteria (task type c). Future teachers reflected, improved and refined their analysis by using the notion of 'epistemic suitability' (Font, Planas y Godino, 2010). Nevertheless, observing future teachers' writings, it was still difficult for the students to identify some semiotic mathematical conflicts. Next we proposed them [the prospective teachers] to develop a task of planning and implementing of a lesson in their internship (task type d). When doing the analysis and evaluation of the lesson implemented (task type e), future teachers found that their planning was conditioned by the school plans in which they did their internship. As a consequence it was difficult for them to identify the epistemic consideration implicit in the schoolteacher proposal. We observed that the students focused more on the dialogue than on the mathematics involved in the lesson. For instance, Student 12 said, *"short challenges appear, with follow up questions in order to engage students in brief conversations just to clarify responses"*, and many others as Student 6, talked about *"the teacher remains vigilant in order to ensure that classmates did not distract students."* The future teachers had little autonomy to apply their designed lessons. This aspect was considered a difficult problem to solve during redesign process because of institutional framework for the proposal. The tasks type (e) and (f) are considered activities driving the feedback for future teachers and trainers.

During the second year we decided to implement some tasks type (a), by emphasizing the analysis of processes; and tasks type (b) by using new video sources. In the new tasks (type a) we proposed the observation of three short ways of introducing perpendicular bisector with 12-13 years old students, by observing three different teachers. The main purpose was to present a discussion about the different practices, objects and mathematics processes and to introduce a reflection associated to how each of these classes contributes to introduce different kind of epistemic configurations and objects associated to three different definitions. It was also introduced enough rich episodes which serving to propose different typologies to profit a short time available, instead of using different episodes in each task. It was also observed that some of the final internship reports (task type e) and master's thesis (task type f) were found so rich to be considered as episodes to be incorporated in a later redesign process.

After the second year of experience observing the analysis realized by the future teachers, some difficulties still appear: (1) difficulties to distinguish between concepts and definitions; (2) duplicity between definitions, propositions and procedures; (3) duplicity between propositions and thesis of arguments; (4) the description of practices is overlapped by the configuration of objects and by the description of processes; and (5) difficulties to observe and to catalogue mathematical processes; among others.

As a consequence, the changes proposed for the third year were the following ones: (1) to join the categories for epistemic suitability from OSA with categories from the quality for mathematics instruction given by Hill (2010). In such way, it was introduced new criteria for valuing mathematical quality as it is: mathematical richness, coherence, errors, etc.; (2) to select new case studies from previous years students with more wide and complex explanations than the previous case studies used en year 1 and 2. The aim was to connect echoes and voices to produce more consistent arguments (Garuti & Boero, 2002) when justifying mathematical quality of didactical sequences. We proposed to analyse a lesson presenting a contextualized problem, driving to the division of a desert in a set of regions. Within the works presented by the future teachers we observed interpretation processes, communication of didactical and mathematical meanings, etc. Furthermore it appears a reflection about distinguishing complex processes from simple processes and also a general reflection about the idea of processes itself. During the analysis it was observed that both first and second teachers did classical proposals and management about the content and the classroom. The tasks designed had achieved the effect of improving prospective teachers' analysis of practices, objects and mathematical processes and mainly about processes. In this improvement, it was judged a crucial role of dynamic videotapes to analyse the visualization of professional didactical processes. On the other hand, they were introduced selected episodes of students' from previous years that were considered as a short distance from prospective teachers' perspectives. We still detected that the future teachers applied epistemic suitability criteria, by means of superficial explanations, short justifications, etc. Therefore, it's needed to improve future teachers' justifications about mathematical and didactical quality of their practices as a basis of the second redesign. Epistemic suitability criteria explained for years 1 and 2 were basically sustained in the idea of representativeness, understood as a degree, of representation of learned meanings representing relations to referenced meanings. Due to the superficiality of some students' works during the moment to apply such criteria, it was decided to do an extensive study about how the students have been applied epistemic suitability criteria in their final masters' thesis (to see if they have been used the representativeness criteria, introduced some personal proposals, etc.).

A prototypical example of this new task (type c) was a case based analysis upon a student that planned a sequence with 7th grade (13-14 years old students) for Thales theorem. The main idea was to use the voice of a previous future teacher M that analysed her own practice about Thales Theorem after the school practice during the course 2011-2012 as a new task. We observed that M did a personal final analysis in which she said

> …Additionally, we have tried to establish connections either with the concepts of the unit (relating as an example, Thales with similar triangles; similar triangles with similar figures, and so on) as with other subjects (for example, to compute the measure of a columns with mirrors, Snell's law of refraction, relating physical concepts to mathematical concepts)... So, in conclusion...my epistemic configuration was right. (St. M; final report of practice and master's thesis, 2011).

Some previous examples done by prospective teachers were also introduced as a new tasks (type c) by reflecting about the role of connections, drawing on three documents: (1) tasks proposed by M to explain Thales theorem in her proposal for school practice; (2) the analysis of epistemic suitability about M proposal, and (3) a textbook in which it was ensured the representativeness of epistemic configurations for Thales Theorem having a coherent connection. When doing the task it was promoted a discussion to understand the idea of representativeness and the idea of coherent connection by using triangles in Thales position. The aim of this professional task was to recognize a deep level of analysis from such previous prospective teacher's practices (Choppin, 2011). Thus, the future teachers learn from this analysis, the idea of connecting two epistemic configurations.

SOME RESULTS ABOUT REFLECTIVE ANALYSIS

After first year observations, we found that some future teachers had difficulties to connect didactical analysis to epistemic ideas. For instance, Student 5 claims: *"When I did the didactic unit I didn't contextualized enough the exercises. Now, I think it's important to use activities proposed in the article: 'Algebra for all Junior High School students'*. In these kinds of sentences, we expected to talk more about the specific iterative algebraic approach as an explicit content in the article explained by the student. However, student 5 declined to focus his comment under such approach, and he highlighted the importance of contextualization for designing unit lessons. It's an example of the initial difficulties to accept the role of epistemic and cognitive analysis.

At the end of the third year, we found that students being to present their lesson more carefully, as result of such deeper analysis.

When analysing the final work of those future teachers we found better results than previous years. Here we'll see an example of growing ecological suitability relations among institutional framework, cognitive suitability and epistemic understanding in which the future teacher interpret why his new proposal really improves mathematical meanings. It's the case of a future teacher X (belonging to the 2012 case study group). Let's observe his explanations emerging from his written work compiled after his internship.

> The regular teacher gives to me the opportunity to improve some aspects of the classroom situation, by introducing hypermedia tools with a group of students with mathematical difficulties in another group (not the same as I did my first practice)… Therefore, I proposed "changes" in my initial proposal. In order to achieve the challenge (of expanding enriching, and consolidating the zone of personal geometrical meanings) we devoted more

> than a half of time during the first session to revise their previous mathematical knowledge, and also the techniques, tools, resources and operational knowledge needed (as surfaces formulae, volumes, and so on). My strategy was to create a debate among the students... by using an email-forum in which I could adjust mathematical rhythm to each student. Let's see the dialogue showing the impact of my strategy: For teacher X- Calculate the volume ... Student O- Please, X, I have a doubt, As I must calculate the basis surface, do I need to multiply twice, because I have two basis? ... ... I never heard about such difficulties, because I did group discussion in my first experience. Now, the one-to-one discussion provides the possibility to hear from the students... (Master's work of a future teacher X)

CONCLUSION

As a result of our study, we have analysed in depth what we call professional tasks to promote growing competency of didactical analysis year by year in our Program, considering different students. The levels of didactical analysis proposed by OSA were very useful to illuminate this didactical analysis. We assume the methodological potential of analysing case studies based on the texts coming from the students' works done in previous years. In fact, these practices may explain the complexity of the analysis that teachers should conduct to value his/her own practice to move beyond from narratives and descriptions. We found the importance of some didactical notions as representativeness, connection and coherence. One of our conclusions for prospective teachers enrolled in teacher training programs is the necessity to use theoretical powerful tools to lead them reflect on the mathematical quality of task-design or lesson design (Krainer, 1993).

After three years of experience, we found many evidence suggesting that students really transform their attitude towards using a "didactical approach" to inform their [future] professional work as teachers: *"we had been developed our competence of didactical analysis"*. On the other hand, we recognized the final master degree as the starting point for developing research competency for future teachers. In fact, it gives opportunities for students to learn and recognize problems of their professional context. Following our perspective we intend to see "didactical analysis" beyond the banality, considering classroom situations as an integral and dynamic system evolving in time, promoting autonomous mathematical thinking and independent validation of results as future teachers (Laborde, Perrin-Glorian, Sierpinska, 2005). We found that the "suitability criteria" used for redesigning the tasks (considered as teaching experiments and case studies) has anticipatory purposes as hypothetical trajectories, but also helps to improve didactic training trajectories.

Acknowledgment

The work presented was realized in the framework of the Project EDU2012-32644 Development of a program by competencies in a initial training for Secondary School.

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EXPLORING THE FEASIBILITY AND EFFECTIVENESS OF ASSESSMENT TECHNIQUES TO IMPROVE STUDENT LEARNING IN PRIMARY MATHEMATICS EDUCATION

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In the present study we investigated to what extent workshops aimed at improving teachers' use of classroom assessment techniques had an effect on students' achievement in mathematics. Ten primary school teachers participated in two consecutive small-scale studies, aimed at using and improving different classroom assessment techniques in mathematics education. In total, 214 students were involved. The studies were carried out in the Netherlands. Qualitative and quantitative measures were used to investigate the feasibility and effectiveness of the assessment techniques. In both studies teachers and students reported enjoying the techniques and finding them useful. In terms of mathematics achievement, results indicate students improving considerably; the students' scores increased more than the national mean.

CLASSROOM ASSESSMENT TO IMPROVE STUDENT LEARNING

To gauge student learning classroom assessment by the teacher plays a pivotal role (Cizek, 2010). By using classroom assessment teachers can gather information about their students' skills and level of understanding. Collecting this information on students' learning is primordial for at least two reasons: to find out whether the instruction has had its desired effect and to generate ideas for how to proceed in the subsequent lessons. Based on assessment information teachers can align their teaching to their students' needs, which can result into adapting their teaching, but which can of course also mean not changing anything and continuing with what was planned before.

Many of the characteristics of classroom assessment appear to be part of merely good teaching practice, as Ginsburg (2009) wrote in the context of mathematics education: "Good teaching [...] sometimes involves the same activities as those comprising formative assessment: understanding the mathematics, the trajectories, the child's mind, the obstacles, and using general principles of instruction to inform the teaching of a child or a group of children (p. 126)". Nonetheless, classroom assessment can be performed in many ways, even though most teachers think of externally developed summative assessment instruments such as textbook tests or student monitoring system tests when confronted with the word assessment. Classroom assessment however is much broader than applying these instruments: it comprises all activities that permit teachers to find out where their students are at a particular moment in terms of comprehension of the subject and to give information on what is going right and wrong. Policymakers as well as influential researchers have urged the educational community, and in particular teachers, to embrace (formative) classroom assessment in

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 329-336. Vancouver, Canada: PME.

their practice. For instance, the U.S. National Council of Teachers of Mathematics (NCTM, 2013) recently took the following position on formative assessment in mathematics education: "The use of formative assessment has been shown to result in higher achievement. The National Council of Teachers of Mathematics strongly endorses the integration of formative assessment strategies into daily instruction" (p. 1).

Recently, researchers have critically examined the size of the effectiveness of (formative) assessment on student learning through reviews or meta-analyses of existing studies (e.g., Briggs, Ruiz-Primo, Furtak, Shepard, & Lin, 2012; Kingston & Nash, 2011; 2012; McMillan, Venable, & Varier, 2013). Common to these critical examinations, although their specificities differ, is that they do not contest the positive effects formative assessment is purported to have on student achievement. The matter that is under contention is the size of the effect for formative assessment on student achievement.

The present study

The purpose of the present study was to investigate the feasibility and effectiveness of classroom assessment techniques for mathematics in primary school. These classroom assessment techniques were inspired by research (e.g., Black, Harrison, Lee, Marshall & Wiliam, 2004; Torrance & Pryor, 2001), practice (e.g., Keeley & Tobey, 2011; Wiliam, 2011), and theory on classroom assessment (in mathematics education, e.g., Van den Heuvel-Panhuizen, 1996; Van den Heuvel-Panhuizen & Becker, 2003). The effectiveness of the use of separate assessment techniques (as was done for instance for the 'Muddiest Point' technique by Simpson-Beck, 2011) was not of interest here. Our focus was whether teachers and students were prone to use the techniques and whether the use of an ensemble of techniques would be related to an increase in achievement. Our research question was: *Do teachers like to use classroom assessment techniques* (feasibility/sustainability) *and is this associated with an increase in student achievement* (effectiveness)? To investigate this research question we performed two consecutive small-scale studies with groups of third-grade teachers in the Netherlands. Teachers participated in monthly workshops, consisting of three or four teachers and the first author. In these workshops classroom assessment techniques were presented, discussed, and evaluated.

METHOD

Both studies used the same method. The first part of the research question (feasibility and sustainability) was investigated by conducting regular classroom observations at least once for every teacher in between the workshops. These observations were intertwined with short informal interviews with students on their teacher's assessment practice in mathematics. Teachers were also asked to register their evaluation of the used assessment techniques. These different sources of information are used to determine how teachers performed the classroom assessment techniques in practice, how students reacted to this, and what students and teachers alike thought of the classroom assessment techniques. To answer the second part of the research question (effectiveness), we used a pre-/posttest evaluation of students' mathematics achievement with between the tests professional development for the teachers. The pretest data consisted of the results from the midyear student monitoring system test for Grade 3 and the results from the end of year student monitoring system test for Grade 3 (Cito LOVS; Janssen, Verhelst, Engelen, & Scheltens, 2010) served as posttest data. These tests are administered by the teachers as is common in educational practice in the Netherlands. The scores on these tests are mathematical ability scores, as calculated through item response theory models. Through the use of these test results as pre- and posttest measurement we could evaluate firstly whether the students progressed in their mathematics ability, and secondly whether students of teachers that had participated in the workshops improved more than the national sample of students of teachers that did not participate in the workshops.

Participants

Ten teachers participated in the workshops in two consecutive school years (four in Study 1 and six in Study 2). In Study 1 all teachers were female and their mean age was 38.5 years. In Study 2 two male teachers participated and the mean age was 52.5 years. In total the ten teachers taught 214 Grade 3 students (14 to 29 students per class). The teachers were found through e-mail solicitation and volunteered to participate. The schools were all situated in urbanized areas with highly mixed student populations, and the teachers used four different textbooks.

Material

We used several classroom assessment techniques in this research project. The classroom assessment techniques consisted of short activities of less than 10 minutes, which should help teachers to quickly find out something about their students, providing them with indications for further instruction, and focusing on some of the mathematics content of the second half of Grade 3. Each assessment technique was explicitly introduced as modifiable; teachers could vary the content and/or the form. This was in line with Wilson and Sloan (2000) who noted that:

> [T]eachers must be: (1) Involved in the process of collecting and selecting student work. (2) Able to score and use the results immediately. (3) Able to interpret the results in instructional terms. (4) Able to have a creative role in the way that the assessment system is realized in their classrooms. (p. 191)

The forms of assessment techniques that were used in the two studies were the following: Red/Green cards, Clouds, Hard or easy, Experiment, Find the error(s), and Find problems with the same result. In Figure 1 we illustrate the Red/Green cards. Most of the techniques were centered on the assessment of number sense, mainly in the context of addition and subtraction, but the Red/Green cards could also be used to assess multiplication and division tables. In all workshops attention was paid to giving feedback to students about the assessments, so that students could become aware of their own understanding.

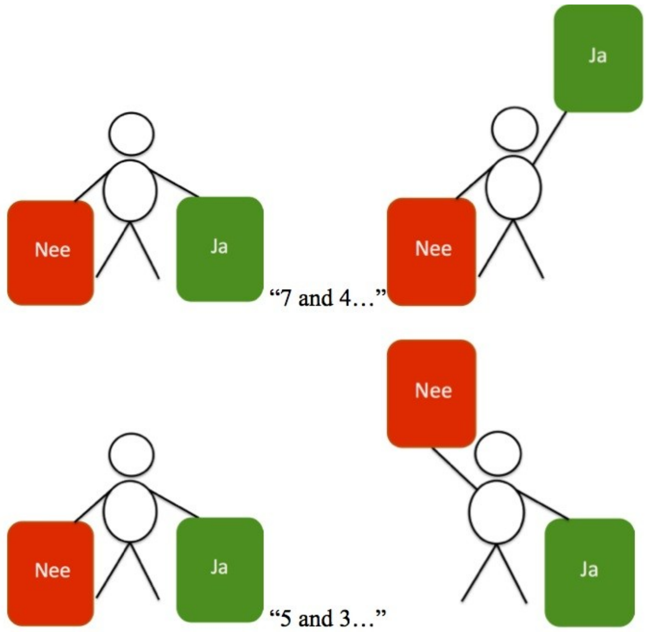

Figure 1: An example of a classroom assessment technique: The Red/Green cards.

Here teachers ask students a question that can be answered with Yes (green) or No (red). The focus is on number sense: the comprehension that two numbers together can be more or less than 10, 100, or 1000. The teacher's question in this particular example is: "Do these numbers together cross ten, yes or no?"

Procedure

Both studies had the same set-up. Teachers used several classroom assessment techniques and in doing so enlarged and reinterpreted their toolbox of assessment techniques. During the second semester of the school year the teachers and the first author convened every three to five weeks in a workshop. These workshops were organized according to the principle of 'practice what you preach' and could be considered teacher learner communities. According to Wiliam (2007) "five principles are particularly important [in establishing and sustaining teacher learning communities]: gradualism, flexibility, choice, accountability, and support (p. 197)", we strived to incorporate all of these in our workshops. As most mathematics classroom assessment techniques were embedded in or inspired by formative assessment ideas, the workshops also had a formative character. Teachers and researchers worked together in order to determine the important content in the weeks between the workshops and ways to find out whether students had learned the prerequisites or not. As such teachers "adopt and integrate these techniques and others into their own practice, they find a new synergy and see their own practice in new ways, which in turn leads to new thinking. In other words, rather than trying to transfer a researcher's thinking straight to the teacher, this new approach to formative assessment emphasizes content, *then* process (Wiliam, 2007, p. 195)". The order of business of every workshop was that first all teachers told what they had done in the preceding weeks: which assessment techniques did they use, why did they use them, in what form, how did the students react, what did they think of them, and what did they do to follow up on what the assessment told them. These same questions were also on a feedback form the teachers were asked to fill out directly following the use of an assessment technique. When every teacher had told how their weeks had been, the researcher shared some observations made in the classrooms. The researcher visited every teacher at least one whole day between two consecutive workshops. In these visits he observed the teachers during mathematics instruction and of course the assessment techniques. As such he was able to reflect in the workshop upon what he had seen and heard in the classrooms. All the while the teachers reacted to each other's stories, they would suggest different approaches or ask for more details; generally discussion in these workshops were very lively and informative. Finally, the focus would switch to the future weeks: the content and the accompanying assessment possibilities. All would discuss these, but the researcher would after some discussion propose several ideas of which the teachers would select some and then the researcher would explain and sometimes show how the assessment technique or activity works, and in particular what could be investigated with them. Then there would be some more discussion about the activities and the researcher would present the discussed techniques on paper so that the teachers could reflect upon them in preparing their lessons.

RESULTS

An overall finding for the first part of the research question (feasibility and sustainability) from the classroom observations, the interviews, and the discussions in the workshops was that every teacher, even though they participated in the same workshop and got the same assessment techniques to work with, interpreted the classroom assessment in their own way and adapted them to their own practice. For instance, the Red/Green cards technique seems quite straightforward on paper; nonetheless there was great variation in how teachers performed this technique in their respective classrooms. A teacher of Study 1 noticed that some students waited to see which card other students held up before choosing their own. She considered this a problem "as it was a testing situation" and decided that students had to be in "testing positions" (separated tables) and even close their eyes so that they could not cheat. Another teacher of Study 1 spend quite some time to ensure that all students were clear about what the colours green and red were, and subsequently in which hand they held each colour. A teacher of Study 2 interpreted the Red/Green cards more as a game, and adapted it to his own practice. He considered it to be "nonsense to be the only one doing the work" and let a student (every time a different one) come up with the problems on the spot. These three short examples show how diversely teachers operated in their classrooms and how flexibly they used this assessment technique. For the second part of the research question (effectiveness) we compared the pre- and posttest data of the student monitoring system tests for every study separately.

Study 1

For the first study, as can be seen in Table 1, the mean ability of students increased from midyear to end of year testing.

	Class	*N*	Mean ability midyear – end of year	Ability gain	Effect size (*d*)
Study 1	I	15	65.3 – 76.7	11.4	1.30
	II	13	62.2 – 68.4	6.2	0.42
	III	14	71.6 – 83.2	11.6	0.73
	IV	24	75.8 – 85.3	9.5	0.78
	Total	*66*	*68.7 – 78.4*	*9.7*	*0.81*
National mean			69.0 – 74.1	5.1	0.36

Table 1: Mean ability, gain score, and effect size per class for the two studies, and the Dutch national mean on the Cito LOVS tests.

It was to be expected that students progress in their mathematical ability whether they have teachers that perform specific assessment activities or not, just as a result of growing older and having more education; the national mean also shows this direction. However, the mean difference between pre- and posttest over the four classes of participating teachers and the effect size (gain score = 9.7 ability points, $d = 0.81$) are notably larger than those of the national norm sample (gain score = 5.1, $d = 0.36$). This means that students of teachers in Study 1 progressed 0.45 SD more in three months' time than students in the national sample.

Study 2

For the second study we had not one workshop, but two separate workshops in two different cities. These workshops were of a slightly different nature than in Study 1, for one the ages of the participating teachers were quite different, also the frequency of meetings was down from every three weeks to once per month or even every five weeks, and finally the assessment techniques were provided in a slightly more definite way. The mean ability gains from midyear to end of the year are displayed in Table 2.

	Class	*N*	Mean ability midyear – end of year	Ability gain	Effect size (*d*)
Study 2	V	22	66.6 – 72.2	5.6	0.32
	VI	17	66.3 – 76.1	7.8	0.44
	VII	26	67.9 – 73.1	5.1	0.51
	VIII	26	72.8 – 82.6	9.8	0.67
	IX	28	74.8 – 83.6	8.9	0.74
	X	27	75.6 – 83.8	8.2	0.61
	Total	*149*	*71.0 – 78.6*	*7.6*	*0.55*
National mean			69.0 – 74.1	5.1	0.36

Table 2: Mean ability, gain score, and effect size per class for the two studies, and the Dutch national mean on the Cito LOVS tests.

Just as for Study 1 we observe that all classes improved on average, and that with an ability gain of 7.6 and effect size of $d = 0.55$. This score gain was bigger than the one in the national sample, which was of 5.1 ability points with an effect size of $d = 0.36$. This means that students of teachers in the second study progressed 0.19 SD more in three months' time than students in the national sample.

DISCUSSION

The feasibility of the classroom assessment techniques combined with an indication that they were effective are the main results of this study. The gain scores of the two small-scale studies showed that students learned considerably more when teachers make effective use of classroom assessment, than students from the national sample. This relative gain was quite large (between 0.19 and 0.45 standard deviation), while the professional development –the intervention– only took between four and six meetings of about an hour. However, given the fact that there was no control group the direct attribution of these learning effects to the sole use of classroom assessment techniques would be too simple.

The second study was performed to fine-tune the techniques as well as to find out whether a lower frequency of workshops and researcher visits would still be associated with a learning effect. In this second study we found approximately the same results as in Study 1: a larger than normal learning effect for the students of participating teachers. As such this provided supplementary evidence for the effectiveness of the techniques; however the same problems as in Study 1 persisted. The difference in Study 2 was slightly smaller than in Study 1. Several things could explain this difference in gains, for one there were less contact moments with other teachers and the researcher, which could have led to less implication in the research in the second study. This came forward in one of the workshops in Study 2, where some teachers admitted that they had only performed the classroom assessment techniques the morning preceding the workshop. The urgency and enthusiasm to use the techniques that they had voiced in the previous workshop had quickly diminished after it had ended. In Study 1 teachers used the techniques generally in the week following the workshop and often repeatedly until the next meeting. Quite understandably this could have caused a differential effect on student learning. An additional explanation as to why some of the teachers in the second study seemed less invested could be their ages. The teachers in Study 2 were older and had as such more experience in teaching than teachers in Study 1. Some of these older teachers did not believe in all the techniques and were less inclined to use some of them, whereas this did not occur with the slightly younger group in Study 1.

These considerations can be taken into account in the design of further research on the effectiveness and the use of classroom assessment techniques.

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CONNECTIONS AND SIMULTANEITY: ANALYSING SOUTH AFRICAN G3 PART-PART-WHOLE TEACHING

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In this paper analysis of Grade 3 mathematics teaching in South Africa shows evidence of associations between teaching and learning outcomes in an adapted learning study. The intervention dealt with partitioning and part-part-whole relations, taking a structural approach within tasks and representations. Our analysis of this teaching emphasizes simultaneity of examples, and connections within and across examples and representations. This analysis indicated differences in enactment of a jointly planned lesson that related to different patterns of learning outcomes between the three classes. Episodes of teaching containing work with representations marked by connections and simultaneity closed gaps in learning outcomes seen in the pre-test.

INTRODUCTION

Difficulties with linking teaching and learning in any direct way have been noted in the literature. Complexity relates to the need to take prior understandings into account, making it hard to directly compare the efficacy of teaching. However, writing also notes the importance of teaching for the possibilities of learning, and acutely so in contexts of disadvantage.

In this paper, we share a micro-analysis of videotaped Grade 3 mathematics teaching in South Africa that shows evidence of associations between teaching and learning outcomes in an adapted learning study intervention (Lo & Pong, 2005). The intervention dealt with part-part-whole relations, taking a structural approach and introducing structural representations – both new to participating teachers and students. Data were collected on students' prior understandings of this topic. Our analysis of teachers' work with part-part-whole representations emphasizes simultaneity of examples and connections within and across examples and representations. This analysis indicated differences in enactment of a jointly planned lesson that related to different trajectories of performance for three classes. Further, this focus suggested that teaching episodes marked by connections and simultaneity could close gaps in pre-test performance.

We begin with an overview of literature on part-part-whole structures and representations within additive relations, noting that operational conceptions are more prevalent in South African curricula.

THE PART-PART-WHOLE RELATIONS AND REPRESENTATIONS

Additive relations is an area with a variety of nomenclatures for problem types, useful representations, and solution strategies. An area of contention relates to directionality

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 337-344. Vancouver, Canada: PME.

in the teaching and learning of additive relations. A significant body of work advocates counting as the fundamental base for addition and subtraction (Carpenter, Fennema, Franke, Levi, & Empson, 1999). Addition and subtraction, in this view, are built on an operational approach. Standing counter is a more structural approach in which addition/subtraction is viewed fundamentally as a relation between parts and wholes (Schmittau, 2003).

Parallel to this discussion are representational options that push more in either operational or structural directions. The empty number line representation (a) advocated in the RME literature (Beshuizen, 1999) tends to align with more operational conceptions, while variations of part-part-whole representations (b) push towards structural relations (Figure 1):

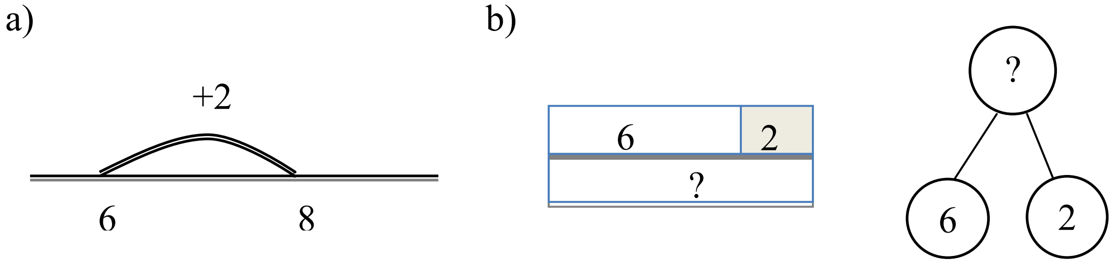


Figure 1: Part-part-whole representations.

Structural orientations to additive relations, in task and representation terms, were taken in this study. Systematicity, equivalence, commutativity, completeness and inverse relations can be dealt with in the context of part-part-whole problems. These ideas require connection between partition examples and help to build generality into specific working (Mason & Johnston-Wilder, 2004).

THEORETICAL FRAME

Variation theory (VT) forms the theoretical base for our analysis. VT argues the need for variation in the midst of invariance, as a condition for learning (Marton & Pang, 2006), necessitating a focus on *what* is simultaneously available and whether, and if so, how, connections between examples are drawn. Schmittau (2003) recognizes part-part-whole relations as the central invariant feature of all additive relation problems – with examples and representations linked to this general theme. Representations can remain invariant across examples, emphasizing their *general* usefulness. Alternately, invariant examples allow for introduction of new representational pathways, providing openings for connections between representations and expanding representation spaces.

RESEARCH DESIGN

Learning studies share common features with Japanese lesson study. As in lesson studies, the teachers were involved in the development, teaching and retrospective analysis of lessons. The broader study involved two sub-study cycles during 2013, each of three weeks' duration, with the three Grade 3 teachers/classes in one suburban school in Johannesburg. In this paper we analyse results from the videorecorded first lesson together with learner performance on two worksheets in the first study. Analysis of student pre-test performance indicated differences between the classes in prior understandings of part-part-whole relations, but performance profiles shifted on the worksheets set after sections of teaching – summarized in Table 1:

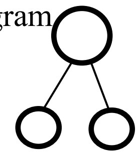
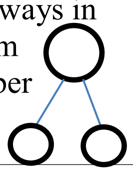
RESULTS PRESTEST				
Task 1: Split 9 marbles in two boxes	No of correct partitions	Class 3.1 (n=40)	Class 3.2 (n=39)	Class 3.3 (n=44)
	10	0%	0%	0%
	4-9	12%	46%	23%
	0-3	88%	54%	77%
Task 2: Split the number 12 in as many different ways as you can 12 = __ + __	No of correct partitions	Class 3.1 (n=40)	Class 3.2 (n=39)	Class 3.3 (n=44)
	10	10%	8%	27%
	5-9	48%	41%	55%
	0-4	42%	51%	18%
RESULTS WORKSHEET 1				
Task: Split number 7 in different ways in a triad diagram 	No of correct partitions	Class 3.1 (n=43)	Class 3.2 (n=43)	Class 3.3 (n=46)
	8	33%	63%	37%
	5-7	44%	18,5%	50%
	0-4	23%	18,5%	13%
RESULTS WORKSHEET 2				
Task: Split number 7 in different ways in triad diagram and in number sentence  7= □ + □	No of correct partitions	Class 3.1 (n=43)	Class 3.2 (n=44)	Class 3.3 (n=46)
	8	25%	50%	66%
	5-7	46%	36%	30%
	0-4	29%	14%	4%

Table 1: Pre-test, worksheet 1 and worksheet 2 results

Pre-test results indicated that while class 3:2 were stronger on the marble splitting activity, class 3:3 were stronger on producing abstract number partitions of 12. Worksheets 1 and 2 followed segments of teaching that are analysed in this paper. Worksheet 1 results showed class 3:2 performing better than the other two classes, in spite of lower performance in abstract number partitioning in the pre-test. In contrast, worksheet 2 data showed class 3:3 outperforming 3:2. Class 3:1 performed weakly throughout.

Our analysis of teaching explored what produced these shifts in performance. In the teaching sections preceding worksheets 1 and 2 we saw differences in the three teachers' work with examples and representations. Salient features of contrast related to which examples were elicited, whether examples were simultaneously visible, and how they were represented and connected within and across examples, and episodes.

FIRST SECTION OF TEACHING

In the planning meeting the teachers had agreed that in the first section they would introduce the idea of splitting a 'whole' into two 'parts'. The triad diagram – a new representation – was to be introduced within the activity of splitting 7 monkeys between two trees (Cobb, Boufi, McClain & Whitenack, 1997). As the descriptive summaries indicate, Teacher 3:1 did not adhere to this plan

Teacher 3:1

Reporting subsequently that she thought 7 would be too easy, teacher 3:1 worked with whole values of 26 and 10 in this section. The first episode consisted of 16 'separate' offers of partitions, 5 of these incorrect. For the first five correct examples, the split offered was represented in a triad diagram. These five triad partitions were then transferred to a table with split values verbally replayed. No gestures or actions emphasized either the connection between representations or the part-part-whole relationship. Thus, the table and triad representations were visible simultaneously but we described the connections between them as 'weak'.

In the next episode a concrete situation with ten monkeys and two trees was visible on the board. Physical splitting actions and table representations of the parts were produced with simultaneous visibility of four partitions in the table, but with each partition produced 'separately' with all monkeys returned to trees after each partition. No explicit connection was made verbally or gesturally by the teacher in support. Her instruction for worksheet 1 was to work on partitions of 30, rather than 7 – the planned whole value.

Teacher 3:2

The teacher introduced the concrete situation visually and orally, and asked students to split the monkeys in different ways. Eight unique partitions of 7 were offered, with some partitions produced by moving monkeys from one partition arrangement in the two trees to another. As the students physically split monkeys between the trees, the teacher verbally 're-played' their actions in numerical terms and subsequently wrote all the different partitions in a table on the board. Across all eight examples offered, the teacher coherently connected students' physical split results to verbal and tabular representations. This coherence between representations, with tabular representation added after the first three examples making all examples visible simultaneously – marked 'strong' connection.

In episode 2 the teacher returned to monkeys to be split between the two trees. She introduced the triad diagram and verbally related it to the concrete situation. Gesturing supported verbal connections between 'monkeys in trees' and 'parts' in the triad model. Three different numerical partitions were produced by learners, without physical actions of moving monkeys. The teacher rubbed out the numbers in the triad when she moved to the next example. The same situation used in episode 1 was thus linked to a 'new' representation, providing an expanded representation space and a pathway to it from a situation that was familiar.

Teacher 3:3

Teacher 3:3 dealt with just one example of splitting number 7. The representation space included, simultaneously, a verbal description of the visible concrete monkeys/trees situation, physical splitting actions with the whole and part values resulting from this action then transferred into a triad diagram. Gestures and verbal descriptions maintained 'strong connections' between the concrete situation, actions and triad diagram. We note gestural and verbal representations repeatedly as research continues to note their salience within mathematics teaching (Alibali et al., 2014). The teacher opened activity to individual working at this point to produce more examples of a split. The inclusion of only one example of splitting 7, in VT terms, provides limited possibility for students to discern other partitions of 7 or to see the invariance of representations across examples. Table 2 summarizes the teaching preceding worksheet 1, including the number and simultaneity of examples, the whole number, the representation space and the nature of connections.

Teacher	No of examples Whole number	Representations	Connections	Simultaneity of examples
3.1	16 (26 –whole)	V Ta Tr	Weak	Different partitions visible
3.1	3 (10 –whole)	V C A Ta	Weak	Different partitions visible
3.2	8 (7 –whole)	V C A Ta	Strong	All partitions visible
3.2	4 (7 –whole)	V C Tr	Strong	Rubs out examples
3.3	1 (7 –whole)	V C A Tr	Strong	One example

Table 2: Section 1 teaching: V = verbal, C = concrete situation, A= physical action, Ta = table and Tr = triad

Teachers 3:1 and 3:2 both present multiple examples of splits of the given number, in contrast to teacher 3:3. In teacher 3:2's work, the table and triad representations record the outcomes of physical splitting of monkeys between the two trees, whereas in teacher 3:1's first episode, the split is enacted on abstract numbers. Further, while no attention was given to systematic production of different splits in any of these teaching episodes, teacher 3:2 did produce a complete set of splits of 7 in her Episode 1, with checks in her questioning ('Is there still another way?'). In class 3:1, the use of 26 is an unwieldy choice for producing completeness, but the production of only four splits of 10 in Episode 2 suggests lack of focus on this aspect anyway.

Verbal descriptions and gesturing connecting between the concrete situation, splitting action and triad representations were consistently present in teacher 3:2 and 3:3's lessons, in contrast to teacher 3:1's lesson. Simultaneous presence of whole and parts could be seen in all three classes, but in class 3:1 the whole faded as the teacher started talking about 'pair of numbers'.

This analysis confirms that multiple examples of split are more useful pedagogically than a single example from the perspective of learner performance, seen in the contrasts in performance on worksheet 1 between classes 3:2 and 3:3. But careful selection of examples and strong and consistent connections between representations are also critical within teachers' handling of sequences of examples.

SECOND SECTION OF TEACHING

The teachers had agreed to continue with partitions of 7, expanding the splitting activity to a number sentence representation, with worksheet 2 following, prior to a final missing part problem task (completed in two classes only and therefore omitted from current analysis). Teacher 3:1 and 3:2 both handled one episode, while teacher 3:3 handled two episodes before worksheet 2. Teacher 3:1 used 9 as the whole instead of 7 and introduced a missing part problem before worksheet 2.

Teacher 3:1

Rather than linking Section 1 representations to number sentences Teacher 3.1 used whole value 9 and dealt with three examples as missing part addition problems. In the first example the concrete situation, triad and number sentence were simultaneously visible. Strong connections were maintained between the teacher´s talk and moves from concrete situation to triad and symbolic form, in contrast to the other two examples where connections became weaker. In these subsequent examples, a triad diagram was presented in one example, and a concrete situation and number sentence in the third example, without connection to the triad. Therefore connections between representations were less consistent. Further, worksheet 2 with whole value 7, was disconnected from the teaching.

Teacher 3:2

Teacher 3:2 returned to the concrete situation and the triad diagram. With the seven monkeys/two trees visible on the board, the teacher verbally linked the concrete situation to the triad, and then transferred the triad partition to a number sentence. Across the four examples dealt with, the different representations were simultaneously present but with more sporadic verbal reference to the concrete situation, but with monkeys/trees remaining visible. Across all four examples verbal descriptions and gesturing connected representations and maintained visibility of the part-part-whole relationship, again marking 'strong connection'. Her rubbing out each example of splitting 7 resulted in a lack of simultaneous representation of instances, and therefore no possibility for linking examples.

Teacher 3:3

Worksheet 1 was followed by students splitting 7 monkeys again using concrete situation, physical action, triad and table. Initially, splitting was demonstrated with physical actions leading to results presented in triad form, but physical actions were dropped in the next three examples with direct moves to triad representations. The next four instances were presented in a table. Across this episode the teacher´s verbal descriptions and gestures connected different partitions and representations. The teacher handled eight partitions of 7, all shown on the board simultaneously. While not all representations were visible for all the partitions, there were always multiple examples using the same representation and multiple representations presented simultaneously across examples, strongly connected through talk and gesture, and additionally, possibilities to discern completeness in the example space.

In the next episode the teacher referred to monkeys while using one partition from the triad to link to a number sentence representation. In this example she connected the 'whole' and 'parts' from the triad to monkeys in trees and transformed the partition to a number sentence with coherent verbal description and gestures. Connections were therefore, again, strong. Table 3 overviews the teaching preceding worksheet 2.

Teacher	No of examples Whole number	Representations	Connections	Simultaneity of examples
3:1	3 (9 –whole)	V C A Tr N	Mostly weak	Fleeting appearance of representations
3:2	4 (7 –whole)	V C Tr N	Strong	Rubs out examples
3:3	8 (7 –whole)	V C A Ta Tr	Strong	All partitions visible
3:3	1 (7 –whole)	V C Tr N	Strong	One example

Table 3: Section 2 teaching: V = verbal, C = concrete situation, A= physical action, Ta = table, Tr = triad and N = number sentence

These descriptions indicate overlap relating to simultaneous presence of parts and whole. Teacher 3:3 produced a complete set of splits of 7 in the second episode, as teacher 3:2 did in the first episode. By leaving the eight splits on the board teacher 3:3 provides opportunity to discern the complete set of partitions of 7, in contrast with teacher 3:2 who rubs out split examples as she proceeds in this section. In class 3:1 and 3:3, some representations were not used across all the presented examples. In class 3:3 though, sporadic representations were strongly connected to each other within examples, compared to fleeting representations connected in more limited ways in class 3:1. Teacher 3:3's episode 2 included only one example, but this example was linked to the previous concrete situation using a split from the triad diagram to provide an expanded representation space. Thus, there were differences in the extent to which teacher talk connected between representations within and across examples. In class 3:2 and 3:3 verbal descriptions and gesturing supporting connections between representations were consistently present, compared to class 3:1. Contrasts related to invariance of the whole value across all episodes in class 3:2 and 3:3, while teacher 3:1 varied the whole several times.

CONCLUDING COMMENTS

The comparatively strong attainment of Class 3:2 on Worksheet 1 and Class 3:3 on Worksheet 2 points to strongly connected representation spaces and simultaneity of examples contributing directly to improved understandings. The 'newness' of the triad representation and the structural approach has, in all likelihood, made it more possible for us to see sharper distinctions in shifting performance patterns between the three classes than would be possible on a more familiar topic where prior understandings would figure. While acknowledging this, these findings point to significant possibilities for progressing learning through attention to simultaneity in the example space, and strong connections between representations and across example spaces and representations.

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PREDICTORS OF FUTURE MATHEMATICS TEACHERS' READINESS TO TEACH: A COMPARISON OF TAIWAN, GERMANY, AND THE UNITED STATES

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This study uses data from TEDS-M to explore and compare possible individual-based and institutional-based predictors of future secondary mathematics teachers' readiness to teach in Taiwan, Germany, and the United States. Across the three countries, future teachers' intrinsic motivation to become teachers and the consistency of courses arrangement in the institutions where they studied were significant predictors of teaching readiness. Future teachers' highest grade level of mathematics studied at secondary school was a predictor of teaching readiness only in Taiwan, whereas the motivation derived from the empathy of prior learning experience was a predictor of teaching readiness in Germany and the United States, but not in Taiwan.

INTRODUCTION

The purpose of teaching is to help students learn (Hiebert, Morris, Berk, & Jansen, 2007), and the purpose of teacher education is to cultivate teachers' ability to help students learn. NCTM (1991), NBPTS(2001), and CCSSO (2010) have set standards to articulate the teaching competencies that mathematics teachers should acquire. These standards delineate not only the knowledge mathematics teachers should possess, but also the actions they should undertake during teaching (for example, performance-based standards of the CCSSO account for actions).

Teacher Education and Development Study in Mathematics (TEDS-M), an international comparison study conducted by the IEA, investigated the readiness of future secondary mathematics teachers to execute tasks central to mathematics teaching in 15 countries (Tatto et al., 2012; hereafter, referred to as "teaching readiness"). TEDS-M measured various facets of teaching readiness, for example, items related to instructional planning (e.g., set up mathematics learning activities to help pupils achieve learning goals), items related to instructional strategies (e.g., use questions to promote higher order thinking in mathematics), and items related to assessment (e.g., develop assessment tasks that promote learning in mathematics). The TEDS-M question was self-reported. Although self-reported data may be skewed because participants' self-impressions may deviate from reality, self-reporting is simple and economical, and allows a large number of respondents, who may belong to various cultures and speak various languages, to be surveyed and compared. Moreover, future teachers' self-evaluation constitutes a pragmatic benefit that is similar to customer evaluation of whether teacher education institutions prepared them well.

Self-evaluation is an effective indicator of readiness since it incorporates components of individual reflection and practical field experience into the survey process.

Tang and Hsieh (2012) indicated that future secondary mathematics teachers in various TEDS-M participating countries reported different levels of teaching readiness. The characteristics that affect future teachers' teaching readiness remain unclear; it also remains unclear whether the characteristics that affect the readiness are the same in various countries. Determining the characteristics that affect the readiness is crucial to enable teacher education institutions to develop their training programs according to reliable references. These characteristics can be considered by such institutions when recruiting or screening future teachers. In this study, TEDS-M data was used to explore and compare possible individual-based and institutional-based predictors of future teachers' teaching readiness in three higher-achieving countries (achieved MCK and MPCK means beyond the international mean of 500) in Asia, Europe, and North America—Taiwan, Germany, and the United States.

RESEARCH METHOD

Conceptual framework

Future teachers' teaching readiness is considered an indicator of the effectiveness of teacher education (Tatto et al., 2012). After a review of studies related to effectiveness of schools and teacher education, individual- and institutional-based characteristics that possibly influenced this readiness were selected for further investigation. Several backgrounds of respondents have been identified to be influential.

<u>Demographics</u>: *Gender*, *home language*, and *socioeconomic status* (SES) are typically considered powerful predictors of future teachers' competence to teach mathematics. Blömeke et al. (2012) revealed that gender was the most critical individual characteristic that affected MCK across TEDS-M participating countries. Language background was known to affect students' achievements in mathematics and was determined to affect knowledge levels among future teachers (Laschke, 2013). SES reflects access to learning resources, such as wealth or education (Stevenson & Baker, 1992). Blömeke et al. demonstrated the relationship between future teachers' MCK and their parents' education levels.

<u>Entrance quality</u>: Two indicators in TEDS-M were designed to measure future teachers' entrance quality: *secondary mathematics level* and *overall grades received in secondary school*. Researches revealed that these two cognitive characteristics affected future teachers' knowledge levels across TEDS-M countries (e.g., Hsieh et al., 2010).

<u>Motivation</u>: Motivation is widely considered as a critical affective characteristic to impact student mathematics achievement (Eklöf, 2010). TEDS-M investigated the factors that motivated future teachers to pursue teaching. Studies found that *intrinsic motivation* and *empathy from prior learning experience* were positively correlated with knowledge achievements; and *extrinsic motivation* was negatively correlated with knowledge achievements (Blömeke et al., 2012; Hsieh et al., 2010; Laschke, 2013).

Predictors of student achievement are often studied by examining the characteristics reflecting the school education quality, like teacher quality and school features (Akyüz & Berberoğlu, 2010). Thus, this study took several indicators of teacher education quality as the institution characteristics to examine.

Teacher education quality: Hsieh et al. (2011) proposed a framework for teacher education quality of programs in institutions, and designed two indicators to measure course quality, *courses arrangement* and *teaching coherence*, measuring the consistency of courses and content within a university, and the continuity between university instruction and practicum instruction respectively. Three indicators were designed to measure person quality: *MR-instructor* and *SB-supervisor* measured the effectiveness of educators responsible for teaching **m**athematics-**r**elated courses and supervising future teachers' school-**b**ased experiences, respectively. The third person quality indicator was future teacher achievement, including *MCK* and *MPCK*.

The framework and the potential predictors of teaching readiness are shown in Figure 1. MCK and MPCK were also analysed as individual characteristics, because teachers' MCK and MPCK are often related to whether they can carry out mathematics teaching or not (Leinhart & Smith, 1985).

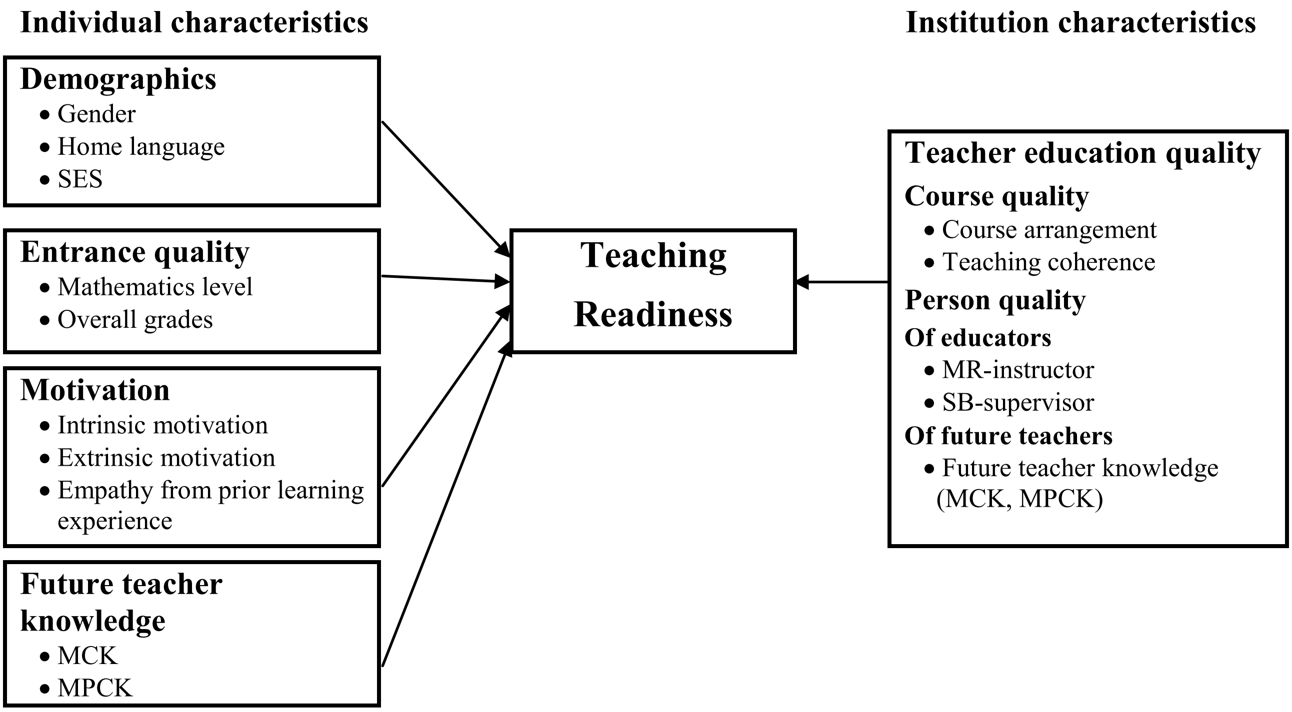

Figure 1: Framework of this study.

Participants

This study uses TEDS-M samples of future secondary mathematics teachers in Taiwan, Germany, and the US. TEDS-M used a stratified multistage probability sampling design, and drew the samples reflecting the distribution of future teachers at the end of their training in each country[1] (Tatto et al., 2012). The samples of this study

[1] The United States limited its participation to public institutions.

include 365 Taiwanese future teachers at 19 institutions, 771 German future teachers at 13 institutions, and 607 American future teachers at 46 institutions. In Taiwan, only one training program is offered: teachers are trained to teach a single subject from Grades 7 to 12. German teachers are trained to teach two subjects from Grades 1 to 9/10, 5/7 to 9/10, or 5/7 to 12/13. In the US, teachers are trained to teach one subject either to Grades 4/5 through 8/9, or 6/7 through 12 (Tatto et al., 2012).

Measures

Teaching readiness was measured by using 11 items graded on a 4-point Likert scale. In TEDS-M, a partial-credit model was used to estimate future teachers' logit scores on the scale; a score of 10 was associated with the neutral position (Tatto et al., 2012). Higher scores indicated greater self-evaluated teaching readiness.

Gender was a dichotomous item. *Home language* measured the frequency of speaking the official language used in teacher education at home on a 4-point Likert scale. A partial-credit model was used to estimate *SES* score for a composite of parental education and home resources. Four TEDS-M questions were included: paternal and maternal education levels (1 = primary to 7 = beyond ISCED 5A), a quantity of items available for education and leisure (e.g., DVD players; 0 to 7 items), and number of books at home (1 = none or few to 5 = enough to fill three or more bookcases).

Secondary mathematics level employed a 5-point Likert scale to measure the highest grade level of mathematics future teachers studied in secondary school (1 = below year 10 to 5 = advanced level of year 12). *Overall grades* were also graded according to a 5-point Likert scale that measured future teachers' secondary school achievements in comparison to their age cohort (1 = generally below average to 5 = always at the top).

TEDS-M measured the factors that motivated future teachers to pursue teaching based on nine items through a 4-point Likert scale (1 = not a reason to 4 = a major reason). In a factor analysis, Hsieh et al. (2010) extracted three aspects: *intrinsic motivation*, *salary and job security*, and *empathy from prior learning experience*, with 4, 3, and 2 items respectively. The average of rating points within an aspect was employed.

TEDS-M measured future teachers' *MCK* and *MPCK* based on 76 and 27 items, applying a balanced incomplete block design with three booklets. Scaled scores were created by using item response theory, and standardized to a mean of 500 and a standard deviation of 100 (Tatto et al., 2012).

Courses arrangement, *teaching coherence*, *MR-instructor*, and *SB-supervisor* were determined based on six, five, six, and four items, respectively, all of which were graded according to 4-point or 6-point Likert scales. In TEDS-M, data were managed by conducting the same statistical analyses as those used for teaching readiness.

Data analysis

This study employed a hierarchical linear model to analyze the data to account for the nested sample structure of TEDS-M (using HLM 6.08). The influence of individual characteristics (level-1) on teaching readiness was examined. These characteristics were introduced using group centering (centered around the arithmetic mean of the institution) to separate level-1 effects from higher-level effects. The effects of institution characteristics were then examined by controlling level-1 predictors.

Institutions with fewer than 6 future teachers were excluded to ensure robust estimates. The adjusted data set contained data collected from 361 Taiwanese future teachers at 18 institutions, 771 German future teachers at 13 institutions, and 563 US future teachers at 32 institutions. Weights of future teachers and institutions provided by TEDS-M were used to reflect selection probabilities and response rates.

RESEARCH FINDINGS

To examine the predictive effects of characteristics on teaching readiness, this study introduced the predictors by block. First, for level 1, demographics were included in the model (M1). Entrance quality (M2), motivation (M3), and future teacher knowledge (M4) were then added in order. Controlling level-1 predictors, the following level-2 predictors were then added to the model: future teacher person quality (M5), educator person quality (M6), and course quality (M7) in order.

Individual characteristics

As shown in M7 for Taiwan (see Table 1), *SES*, *secondary mathematics level*, and *intrinsic motivation* are influential individual characteristics. The effect size of *SES* is much smaller than that of the other two predictors. The German and the US models exhibited predictors different from Taiwan' model, but were similar to each other.

Intrinsic motivation (e.g., I want to have an influence on the next generation) is a common predictor of teaching readiness in all three countries. This corresponds with the idea in the literature that intrinsic motivation is crucial in both East Asia and the West (Zhu & Leung, 2011). However, the connotations of intrinsic motivation differ in the two cultures. Intrinsic motivation among Asian people involves attempts at mastering practices and gaining assurance from others, reflecting "social orientation." People in the West derive intrinsic motivation from individual interest and fulfilment, reflecting "individual orientation" (Laschke, 2013; Markus & Kitayama, 1991).

For the two Western countries, another motivating factor, *empathy from prior learning experience* (e.g., I love mathematics or I was always a good student in school), is predictive of future teachers' teaching readiness. Compared with Taiwan, teaching is a less desirable job and secondary school academic performance requirements for future teachers are less demanding in Germany and the US (Laschke, 2013; Schmidt et al., 2011). Thus, future teachers have various prior learning experiences. Positive experiences may serve as cognitive and affective supports for future teachers' teaching readiness. By contrast, teacher training and teacher jobs are competitive, because of benefits for teachers and social expectations. Typically, only secondary school students competent in mathematics are admitted. *Empathy from prior learning experience* was thus determined to be unimportant to teaching readiness in Taiwan.

In Taiwan, *secondary mathematics level* positively affects teaching readiness. The highest level of mathematics offered in Taiwanese schools is offered in three forms: 12th grade mathematics A (for students with science orientations), 12th grade mathematics B (for students with literature and arts orientations), and 11th grade vocational mathematics. The concepts and skills taught and the difficulty levels differ substantially among these courses (Hsieh et al., 2010). Future teachers who were more competent at secondary mathematics evaluated themselves to be more ready to teach.

	TW						DE	US
	M1	M2	M3	M4	M6	M7	M7	M7
Individual predictor								
Gender	ns							ns
Home language	ns						-0.28^{**}	-0.40^{**}
SES	0.24^{*}	0.23^{*}	$0.15^{\dagger}$	$0.16^{\dagger}$	0.16^{*}	0.16^{*}		
Sec. math level		0.61^{**}	0.61^{**}	0.58^{**}	0.64^{**}	0.64^{**}		
Overall grades		ns					ns	
Intrinsic motivation			0.57^{**}	0.59^{**}	0.60^{**}	0.60^{**}	0.53^{**}	1.11^{**}
Salary and job security			ns					
Empathy			ns				0.21^{**}	0.27^{*}
MCK				$0.001^{\dagger}$				
MPCK				ns				
Institution predictor								
MCK								
MPCK								
MR-instructor,					0.57^{*}	ns		
SB-supervisor					ns			ns
Courses arrangement						$0.39^{\dagger}$	0.65^{**}	0.39^{**}
Teaching coherence						ns	ns	ns
R^2 of level 1	3%	6.4%	17.4%	17.3%	21.4%	20.8%	10.7%	16/8%
R^2 of level 2					43.8%	42.1%	22.4%	64.7%

Note. ns = not significant. $^{\dagger}p < .1$. $^{*}p < .05$. $^{**}p < .01$.

Table 1: Hierarchical linear model for future teachers' teaching readiness.

SES predicts Taiwanese future teachers' teaching readiness. In Confucian culture, SES is often considered to be irrelevant to achievement. Whether SES affects future teachers' teaching readiness from a cognitive or an affective perspective warrants further study. It was unexpected that *home language* produces negative effects in Germany and the US for its representing immigrant status to some degree (Laschke, 2013). Further studies are needed.

Institution characteristics

Courses arrangement was determined to be predictive of future teachers' teaching readiness after controlling for individual predictors in all three countries. This indicated that *courses arrangement* is the most crucial factor to be modified by teacher education institutions in order to improve future teachers' teaching readiness.

In Taiwan, *teaching coherence* and *MR-instructor* were shown to have a significantly positive effect on the readiness when each of them is the single predictor at level 2, indicating that they were also crucial characteristics to modify to improve teaching readiness. When *courses arrangement* was singularly introduced in level 2, the proportions of variance explained by levels 1 and 2 were 20.8% and 39.3%; these percentages were close to those of M7. Most of the variance explained by *teaching coherence* and *MR-instructor* overlaps with that of *courses arrangement*, and *teaching coherence* and *MR-instructor* affect teaching readiness through *courses arrangement*. Germany and the US yielded similar results, but on different institution characteristics.

CONCLUSION

In Taiwan, Germany, and the US, future teachers' intrinsic motivation is a critical individual characteristic predictive of teaching readiness. In Taiwan, whether future teachers are science oriented is a predictor of the readiness, whereas whether future teachers had satisfactory prior learning experiences is predictive of the readiness in Germany and the US. Regarding institution characteristics, the consistency of courses and content arranged by institutions was determined to be most predictive of teaching readiness; thus, this factor is the most crucial factor to enhance in the institutions. Unexpectedly, MCK and MPCK were not observed to be predictive of teaching readiness. A gap exists between future teachers' knowledge and their evaluation of whether they are ready to teach mathematics. Further research is required.

Acknowledgments

We gratefully acknowledge the following: the IEA, the ISC at MSU, the DPC, the ACER, the US NSF, and all TEDS-M national research coordinators for sponsoring the international study and providing information and data.

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PROOF AS A CLUSTER CONCEPT

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Proof is a central concept in mathematics education, yet mathematics educators have failed to reach a consensus on how proof should be conceptualized. I advocate defining proof as a clustered concept, in the sense of Lakoff (1987). I contend that this offers a better account of mathematicians' practice with respect to proof than previous accounts that attempted to define a proof as an argument possessing an essential property, such as being convincing or deductive. I also argue that it leads to useful pedagogical consequences.

PROOF CONCEPTUALIZATION IN MATHEMATICS EDUCATION

It is widely accepted that having students successfully engage in the activity of proving is a central goal of mathematics education (e.g., Harel & Sowder, 1998). Yet mathematics educators cannot agree on a shared definition of proof (Balecheff, 2002; Reid & Knipping, 2010; Weber, 2009). This is recognized as problematic: without a shared definition, it is difficult for mathematics educators to meaningfully build upon each another's research and it is impossible to judge if pedagogical goals related to proof are achieved (e.g., Balacheff, 2002; Weber, 2009). Until now, most mathematics educators have sought to define proof as an argument that possesses one or more desirable properties, such as employing deductive reasoning (Hoyles & Kuchemann, 2002) or being convincing to oneself (Harel & Sowder, 1998) or community (Balacheff, 1987). However, there is not a consensus on which property or properties capture the essence of proof. The main thesis of this paper is that, in mathematical practice, there are no properties that are the essence of proof and viewing proof as a clustered model in the sense of Lakoff (1987) offers a better account of how proof is practiced by mathematicians.

Two approaches to defining proof

There are two approaches that philosophers and mathematics educators have used to define proof (CadwalladerOlsker, 2011). In the analytic philosophical tradition, some have sought to define a proof as a formal object, usually as a strictly syntactic object within a formal theory. Unfortunately, there is little intersection between the objects satisfying definitions of these types and the arguments that mathematicians refer to as proofs. Consequently, such a definition cannot provide a reasonable account of how proofs are produced or how they advance our mathematical knowledge (cf., Pelc, 2009). Further, from an instructional perspective, this can imply the pedagogically dubious suggestion of focusing on the form of proof rather than its meaning.

A second approach to proof is to define proofs as the proofs that mathematicians actually read and write or as the arguments that mathematicians label as proofs.

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However, such a characterization is too broad to do useful philosophical or pedagogical work. What is needed is a sense *what* types of arguments mathematicians recognize as proof. Further, this sense should be philosophically and pedagogically pertinent. For instance, the observation that mathematicians usually publish their proofs is LaTEX will not inform instructional practice. If we accept Larvor's (2012) observation that, "the field [the philosophy of mathematical practice] lacks an explication of 'informal proof' as it appears in expressions such as 'the informal proofs that mathematicians actually read and write'" (p. 716), then it is clear that there is more work to do in this area.

DIFFICULTIES IN FINDING AN ESSENCE OF PROOF

A common approach to defining proof is to locate a characteristic (or set of characteristic) that is shared by all arguments that mathematicians consider to be proofs and not present in all other arguments. If successful, this approach would yield a clear way of characterizing proof. Unfortunately, this approach has not been successful. For instance, a proof has sometimes been defined as an argument that convinces oneself (or one's community) that an assertion is true (e.g., Harel & Sowder, 1998). However, Tall (1989) noted that there are convincing arguments that would not qualify as proofs. For instance, Eccheveria (1996) claims that the empirical evidence in support of Goldbach's Conjecture is so overwhelming that the mathematical community is certain of its truth, but the claim is not proven. Proofs are sometimes defined to be *a priori* deductive arguments that do not depend on one's observations or experience, but Fallis (1997) noted that computer-assisted arguments would not satisfy this description.

It is natural to try to define proof as a category of objects sharing some properties. After all, this is how mathematical concepts are defined (Alcock & Simpson, 2002). However, I argue that proofs are not mathematical concepts, they are *discursive concepts*. And I further argue that there is no property that distinguishes proofs from non-proofs.

Three proofs

To highlight the difficulties of characterizing proofs, consider these three proofs as they appear in the mathematics literature.

Theorem 1: If n is a number of the form $6k$-1, then n is not perfect.

Proof 1: Assume n is a positive integer of the form $6k$-1. Then $n \equiv -1 (\mathrm{mod}\ 3)$ and hence n is not a square. Note also that for any divisor d of n, $n = d\left(\frac{n}{d}\right) \equiv -1 (\mathrm{mod}\ 3)$ implies that $d \equiv -1 (\mathrm{mod}\ 3)$ and $\left(\frac{n}{d}\right) \equiv 1 (\mathrm{mod}\ 3)$ or $d \equiv 1 (\mathrm{mod}\ 3)$ and $\left(\frac{n}{d}\right) \equiv -1 (\mathrm{mod}\ 3)$. Either way, $d + \left(\frac{n}{d}\right) \equiv 0\ (\mathrm{mod}\ 3)$ and $\sigma(n) = \sum_{d|n, d<\sqrt{n}} d + \frac{n}{d} \equiv 0\ (\mathrm{mod}\ 3)$. Computing $2n=2(6k-1) \equiv 1 (\mathrm{mod}\ 3)$, we see that n cannot be perfect. (from Holdener, 2002)

Theorem 2: $\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$.

Proof 2: Here is a proof using *Mathematica* to perform the summation.

FullSimplify[TrigtoExp[FullSimplify[

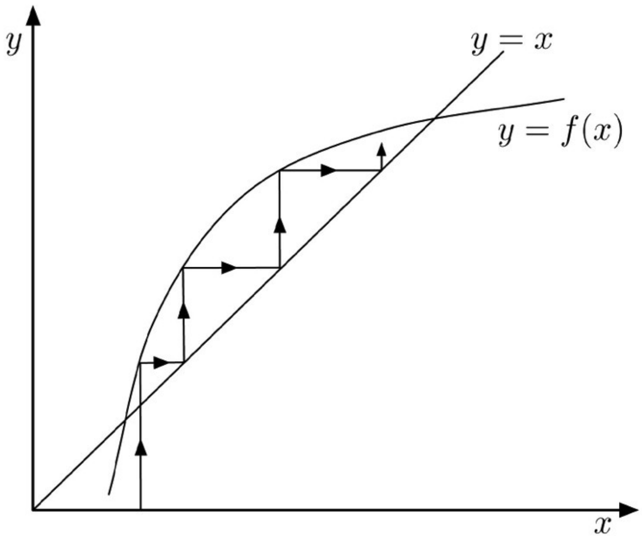
$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$.]]/.

a_Log[b_]+a_Log[c_]:>a Log[b c]].

π (from Adamchik & Wagon, 1997)

Theorem 3: (Fixed Point Theorem) Let *f(x)* be continuous and increasing on [0, 1] such that $f([0,1]) \subseteq [0,1]$. Let $f_2(x)=f(f(x))$ and $f_n(x)=f(f_{n-1}(x))$. Then under iteration of *f*, every point is either a fixed point or else converges to a fixed point.

Proof 3: The only proof needed is:



(from Littlewood, 1957)

These proofs vary widely in terms of the types of inferences that were made, the representation systems used, their level of transparency, and the level of detail they provide. At this point, the reader may want to make three objections: (1) Some of these "proofs" are not really proofs; (2) These proofs are outliers; (3) These proofs are considered controversial.

I do not think (1) is a fair objection. If we were defining what proof *ought* to be, one could say Proof 2 or Proof 3 ought not be considered as a proof. However, if we wish to describe the proofs that mathematicians actually read and write, we must account for Proof 2 and Proof 3 because they were published in the literature by mathematicians as proofs. With (2), Proof 2 and Proof 3 were deliberately chosen to be provocative, yet they are also representative of the wider categories of computer-assisted proofs and visual proofs.

With (3), these proofs *are* controversial. In Adamchik and Wagon's (1997) paper in which their proof was presented, they admitted that, "Some might even say this is not truly a proof! But in principle, such computations *can* be viewed as proofs" (p. 852). In an experimental study, Inglis and Mejia-Ramos (2009) demonstrated that mathematicians collectively find Proof 3 significantly less convincing than more conventional proofs. I accept that these proofs are controversial, but argue this controversy has important consequences for the nature of a descriptive account of proof.

Proof*

Aberdein (2009) coined the term, "proof*", as "species of alleged 'proof' where there is no consensus that the method provides proof, or there is a broad consensus that it doesn't, but a vocal minority or an historical precedent point the other way". As examples of proof*, Aberdein included "picture proofs*, probabilistic proofs*, computer-assisted proofs*, [and] textbook proofs* which are didactically useful but would not satisfy an expert practitioner". As Proof 2 is a computer-assisted proof and Proof 3 is a picture proof, these qualify as proofs*.

Proofs* do not pose a problem for analytic philosophers who attempt to pose *normative judgments* for what should be considered a proof. Recently, there have been arguments that picture proofs, such as Proof 3, are perfectly valid and ought to be on par epistemologically with the more traditional verbal-symbolic proof (e.g., Kulpa, 2009). Granted there may be some mathematicians who disagree, such as those in Inglis and Mejia-Ramos' (2009) experimental study, but the proponents of picture proofs can argue that these mathematicians are simply mistaken.

However, proofs* *do* pose a problem for philosophers and mathematics educators who, as Larvor (2012) put it, wish to describe "the proofs that mathematicians actually read and write". Take picture proofs*, for instance. A proposed criteria of proof must either admit some picture proofs* as proofs or claim that all picture proofs* are not. If the former occurred, one could challenge this claim by citing the large number of mathematicians who do not produce such proofs and reject such proofs when they read them. If the latter occurred, one could rebut the claim by citing the picture proofs in the published literature as well as the large number of mathematicians (or at least the vocal minority) who accept such proofs. Similar arguments could be made for all types of proofs*.

PROOF AS CLUSTER MODEL

Cluster concepts

Lakoff (1987) noted that "according to classical theory, categories are uniform in the following respect: they are defined by a collection of properties that the category members share" (p. 17). This perspective has dominated the way that philosophers have attempted to define proof. However, Lakoff's thesis is that most real-world categories cannot be characterized this way. In particular, he argued that some categories might be better thought of as *clustered models*, which he defined as occurring when "a number of cognitive models combine to form a complex cluster that is psychologically more basic than the models taken individually" (p. 74). I will argue that mathematical proof should be regarded in the same way.

As an illustrative example of a clustered concept, Lakoff considered the category of *mother*. According to Lakoff, there are several types of mothers, including the birth mother, the genetic mother, the nurturance mother (i.e., the adult female caretaker of the child), and the marital mother (i.e., the wife of the father). These concepts are highly correlated-- the birth mother is nearly always the genetic mother and more often than not the caretaker. In the prototypical case, these concepts will converge—that is, the birth mother will also be the genetic mother, the nurturance mother, and so on. And indeed, when one hears that the woman is the mother of a child, the default assumption is that the woman assumes all roles. However, this is not always the case.

Lakoff raised two points that will be relevant to this paper. First, there is a natural desire to pick out the "real" definition of mother, or the true *essence of motherhood*. However, Lakoff rejected this essentialist disposition. Different dictionaries list different conceptions of mother as their primary definition. Further, sentences such as, "I was adopted so I don't know who my *real mother* is" and "I am uncaring so I doubt I could be a *real mother* to my child" both are intrinsically meaningful yet define *real mother* in contradictory ways. Second, in cases where there is divergence in the clustered concept of mother (e.g., a genetic but not adoptive mother), compound words exist to qualify the use of mother. Calling one a birth mother typically indicates that she in not the nurturance mother; calling one an adoptive mother or a stepmother indicates that she is not the birth mother.

Proof as a clustered concept

The main thesis of this paper is that it would be profitable to consider proof as a clustered concept. The exact models that should form the basis of this cluster should be the matter of debate, but I will propose the following models as a working description to highlight the utility of this approach: (1) A proof is a *convincing argument* that convinces a knowledgeable mathematician that a claim is true. (2) A proof is a *deductive argument* that does not admit possible rebuttals. The lack of potential rebuttals provides the proof with the psychological perception of being *timeless*. Proven theorems remain proven. (3) A proof is a *transparent argument where a mathematician can fill in every gap* (given sufficient time and motivation), perhaps to the level of being a formal derivation. In essence, the proof is a blueprint for the mathematician to develop an argument that he or she feels is complete. This gives a proof the psychological perception of being *impersonal*. Theorems are *objectively* true. (4) A proof is a *perspicuous argument that provides the reader with an understanding of why a theorem is true*. (5) A proof is an *argument within a representation system satisfying communal norms.* That is, there are certain ways of transforming mathematical propositions to deduce statements that are accepted as unproblematic by a community and all other steps need to be justified. (6) A proof is an *argument that has been sanctioned by the mathematical community*.

Of course, the criteria above are not original. All have previously been proposed by other philosophers and mathematicians. What is original here is claiming that one cannot demarcate proofs from non-proofs by saying that proofs must satisfy some subset of the criteria above.

I argue that each of these more basic models do not, by themselves, characterize proof completely. I previously argued that (1) fails because there are convincing empirical arguments that are not proofs. Fallis (1997) notes that computer-assisted proofs fail to satisfy (2) and (3), since a possible rebuttal is that the computer software was faulty and since the proof does not give a blueprint for how a human could perform the computer checks for himself or herself. Similar arguments can be given for (4), (5), and (6).

If we accept proof to be a clustered concept as defined above, we would expect the following to occur: (a) proofs that satisfied all of these criteria should be uncontroversial, but some proofs that satisfy only a subset of these criteria might be regarded as contentious; (b) compound words exist that qualify proofs that satisfy some of these criteria but not others; (c) it would be desirable for proofs to satisfy all six criteria.

Regarding (a) and (b), Aberdein's (2009) discussion of proofs* supports these points. He explicitly highlighted compound words delimiting the sense that arguments are proofs. For instance, computer-assisted proofs* are not transparent and it is not clear how a mathematician can fill in every gap of the proof and probabilistic proofs* are not deductive. Not only do these qualifying compound words exist, but as Aberdein (2009) argued, there is not a consensus on their validity amongst mathematicians. For (c), we can consider Dawson's (2006) analysis of why mathematicians re-prove theorems. Dawson's analysis demonstrated that sanctioned proofs are reproven to avoid controversial methods, fill in perceived gaps, become more perspicuous, and increase mathematician's conviction, which correspond to the first four components of the cluster model described above.

IMPLICATIONS FOR PEDAGOGY

If we view proof as a cluster concept, like that of mother, we might expect that this concept is perhaps not best taught by direct instruction, but instead through practice in a community. For instance, Thurston (1994) described how he sought a clear definition in proof in graduate school; he did not find one but through experience, he began to "catch on". Of course, we know that mathematics majors often do not catch on and remain deeply confused about the meaning of proof when they graduate. Here the instructor might help by pointing to features of the argument that make the argument a better or worse example of proof, rather than solely presenting the argument as right or wrong.

At a broad level, the components of the clustered model of proof are correlated with one another. For instance, as an argument becomes more deductive, it often tends to become more convincing, easier to translate into a formal proof, and more likely to be sanctioned by one's peers. Hence, encouraging students to make their arguments more deductive would usually make their arguments more proof-like in other respects as well. However, this is not the case if we take some of these criteria to extremes.

For a first example, suppose we strive to present students with arguments that are as convincing as possible in geometry. In many cases, an exploration on a dynamic geometry package would be entirely convincing, both for mathematicians and for students (de Villiers, 2004). For a student, such explorations would probably be *more* convincing than a complicated deductive argument because the student may worry that he or she has overlooked an error in the argument. If we view the mode of reasoning (deductive vs. perceptual) and the representation system in which an argument is couched as irrelevant, it is difficult to argue why demonstrations on dynamic geometry software packages are not proofs.

A similar claim relates to how formal an argument is. Increasing the formality of an argument usually makes the argument more deductive and more acceptable to the mathematical community. However, it is generally accepted that there is a point where an argument is "formal enough" and making it more rigorous would be detrimental. Filling in *all* the gaps would make the proof impossibly long and unwieldy. The result would be a proof that masks its main ideas. As understanding these ideas is important for determining the validity of the proof, so increasing the rigor of the proof would lessen its persuasive power.

If we want students and teachers to present proofs that satisfy all or most of the criteria above, it would be best not to focus on a single criterion. Not only would the other criteria be ignored, a singular focus on one criterion might actually lessen the possibilities of the other criteria being achieved.

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STUDENTS' USE OF GESTURE AND POSTURE MIMICRY IN DEVELOPING MUTUAL UNDERSTANDING

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In this paper I focus on observations made regarding students mimicking of each other's gestures in face-to-face conversation while problem solving. The data supports the idea that the students may use such gestures to subconsciously signal acceptance. Through talk, gesture, prosody, and intonation, combined with context, the interlocutors may develop a better connection with each other, enabling a belief in having achieved a shared understanding of each other's contribution. In so doing, they are positioned to develop their understanding of the problem. In addition, recordings of students working together on problem solving show evidence of posture mimicking during times of effective collaborative. The results suggest that teachers' recognition of such mimicry may help in knowing when to successfully intervene.

INTRODUCTION

In this report I address the question of what clues a teacher can look for as indicators of when to intervene in student group work. My consideration of the use of mimicked gestures arose on reviewing recordings of students engaged in mathematical problem solving. While not initially looking for such gestures it stood out that the students demonstrated mimicry of both gesture and posture, prompting deeper analysis. My initial question, arising from recognition of this phenomenon, was whether or not there seemed to be any relation between such gesturing and the students' ability to progress with the problem. If so, could this be an indicator of the group's progress? The evidence presented here indicates that a teacher can look for gesture and posture mimicry as guides to appropriate intervention timing.

THEORETICAL FRAMEWORK

The reform-based shift towards a sociocultural approach in mathematics teaching, associated with the Vygotskian school of thought, takes a view of human thinking as being essentially social. There has been a push to replace the traditional classrooms featuring an outspoken teacher and silent students with small groups of learners talking to each other and expressing their opinions in whole class settings (Sfard, Forman, & Kieran, 2001). The need for a teacher to carefully facilitate the discourse in these situations has been noted by many researchers (e.g. Sfard et. al, 1998; Jaworski, 2004). While there is much research on how a teacher can successfully intervene (e.g. Ding et al. 2007), knowing when to intervene has been a less discussed but is an equally important aspect of such facilitation. The close presence of a teacher can stymy the flow of the group, while at other times the teacher needs to intervene in order to encourage and give critical feedback.

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 361-368. Vancouver, Canada: PME.

When students engage in mathematical problem solving in a group situation, there is a clear need for good communication to occur within the group if all participants are to gain from the collective experience. In everyday talk, gestures have been considered to be an integral part of communication (e.g. McNeil, 2005), and linked to speech in a semantic and temporal way. Radford (2009) notes that 'thinking does not occur solely in the head but also in and through a sophisticated semiotic coordination of speech, body, gestures, symbols and tools' (p. 111). Sfard (2009) also considers gestures to be 'crucial to the effectiveness of mathematical communication (...) to ensure that all the interlocutors speak about the same mathematical object' (p. 197). Other researchers (e.g. Goodwin, 2000) have examined the role of gesture on the sequential organization of conversation. Clark and Wilkes-Gibbs (1986) argue that interlocutors in a conversation create meaning jointly, with the aim of creating mutual understanding. The process is considered to be in constant need of attention since, at best, the interlocutors can only believe that they have understood what each other meant. Such a belief, however, may be sufficient to allow the dialogue to continue based on the situation. The impression, then, of students working together on a problem, is one of a continuous need to repair meaning and make connections to each other. If we hold the view that learning mathematics is akin to developing a special type of discourse (Sfard, 2001) then observing students participating in such discourses is important. If, in addition, the important feature of group problem solving is in the activity rather than the end result, then being aware of that activity is a more important outcome than viewing the final answers. If we are interested in the unfolding understanding within the group then we 'must focus on the various forms of signs that speakers make available to others as well as themselves. These signs comprise words, gestures, body positions, prosody, and so on' (Roth & Radford, 2011, p. 55). With this in mind, students taking on, or mimicking, each other's words and gestures may be an important and visible part of the process.

There is evidence that people mimic a wide range of behaviours, including postures and mannerisms (Chartrand & Bargh, 1999). The occurrence of mimicry in physical behaviour during mathematics group work has been noted by Gordon-Calvert (2001). Holler and Wilkin (2011) found that mimicry in co-speech gestures does occur and concluded that 'mimicked gestures play an important role in creating mutually shared understanding' (p. 148). Holler and Wilkin also found that mimicked gestures were used to express acceptance of group members, suggesting that such gestures were an important part of the conversational structure, even when such acceptance was not expressed verbally. Gestures were also found to be important in signalling incremental understanding, something the authors paraphrased as 'I am following what you are saying in an effort to reach shared understanding with you' (p. 145). This view supports that of Roth (2000) who notes that 'the human body maintains an essential rationality and provides others with the interpretive resources they need for building common ground and mutual intelligibility' (p. 1685).

A limitation of many gesture studies, however, is that they are focussed on tangible objects that one party is attempting to describe to another (e.g. in Holler and Wilkin case it is abstract shapes with figure like qualities). A similar limitation can be seen in the work of McNeil (2005), wherein participants are asked to recall scenes from a cartoon they have watched. Students working in a classroom are generally describing or talking about mathematics that is not a recollection of an action but rather an ongoing action. Some of the actions involved may be hard for a student to put an image to in quite such a dynamic way as McNeil's subjects. As a result, it might be expected that the gestures can often be more subtle, especially in the early stages of working together. In the case of mathematical problem solving the participants in the dialogue are trying to create a solution without one member having a privileged informational position (such as would occur if a teacher was present). In addition, any power relations within the group may lead to a particular student being granted a dominant starting position. Mimicked gestures may be an attempt by a student to reflect the mannerisms of his/her interlocutor with the aim of acceptance.

METHODOLOGY

The video clips were taken from a larger study in a school in which two classes of grade 5 students (aged 10-11 years) were videoed over the course of an academic year. A camera was set up and left unattended with the intent that neither researcher nor the classroom teacher was a direct part of, or influence on, the conversation. The school is located just outside of a large city in Canada and reflects a very multicultural population, with several ESL students. Economic background is not considered to be an obvious factor in the school. Recordings were made weekly while the students were engaged in problem solving and transcribed using a framework of Conversation Analysis. A second viewing was made paying attention to gestures and body language. As part of the transcription process the occurrence of mimicked gestures became apparent, and led to this reported study. Going through a collection of clips looking for a particular but different event can bring out common features that were not seen as significant on initial observation. On becoming aware of this mimicry in more obvious cases, a random selection of 20 of the recordings was re-examined explicitly for mimicked gestures and posture. The clips discussed here were selected as exemplary of different forms of observed mimicked gesturing and posture. For the purposes of this report, only clear cases of mimicry were included, where a hand gesture or body position was mimicked either collectively or within two turns at talk. A deeper analysis of smaller gestures over the period of the discourse may prove interesting, but in this case I focussed on what might be seen by a teacher in a classroom setting observing several groups from a distance

RESULTS

Table 1 illustrates a conversation between Gina and Susan. The problem concerns the change in area of a desk reduced to half its length but doubled in width. This example matches several recorded in this lesson and is of interest because, while gestures used differed between groups, there was evidence of gesture mimicry between interlocutors when the students were able to make progress. In examples where the students were unable to make progress, there was no clear evidence of gesture matching. In this example Gina initiated by describing the desk using large gestures. Susan, in her adjacent turn, mimicked the dynamic gesturing of Gina in describing the table.

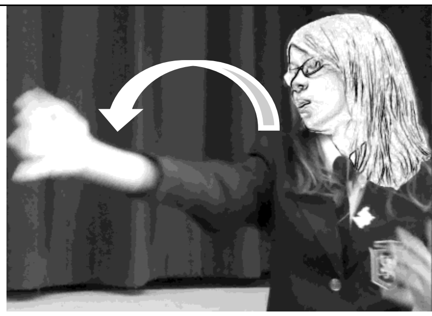
		
1.You're taking it in half	2. (..) and then ….	3. doubling one side, right?
		
4. You take some of it off	5. and you add it to	6. the other side (0.5)

Table 1: Gina and Susan describe the same process.

Table 2 also shows another example of gesture mimicry between two girls working in a group on a problem where they were asked to estimate the size of a bag required to hold a million dollars in $100 dollar notes. Panel 2 shows one girl, Jasmine, making an initial gesture which is then mimicked by Gina (panel 3) as they engaged in conversation. As the conversation develops Jasmine moved gradually closer to Gina until their gesture space became shared. They continued to mimic each other's gestures as they did so. During this time, the conversation was rich, and led to a clear progression in the problem's solution.

Table 2 also shows the group engaging in posture mimicry. The three girls adopted an almost identical posture once they started to work on the problem together. The male member of the group, Jason, seemed to be shut out by this common posture and found it very difficult to gain attention (panel 1) until he adopted a similar posture (panel 3). A male-female dynamic or other social situation, may account for this early barrier to Jason's inclusion, and he may not be aware of his own change in posture during the process, but in order to participate he appears to need to connect through posture first.

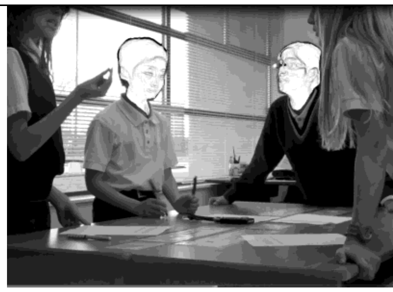
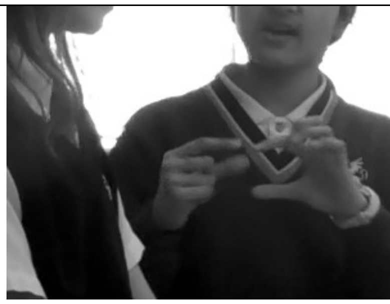
		
Common posture	Initial gesture	Gesture mimic
		
Repeated gesture	Adjacent gesturing	Closing gesture space

Table 2: An example of gesture mimicry within a group.

The group shown in Table 3 also showed signs of gestural mimicry, but in this case it was rare. Panel 5 illustrates the only clear mimicked gesture, a cutting motion used in conjunction with talk of division. A common deictic gesture, as shown in panel 3, seemed to serve the similar purpose of connecting the group while talking. While there were other gestures which were repeated by different members of the group, such as the spread fingers shown by the girl on the left side of panel 5, these may or may not be mimicked gestures since they occurred more than two turns after the initial gesture.

A second example of posture mimicry is illustrated in table 3. Panels 1 and 2 show three of the group have adopted a pose while the fourth student has become disengaged, initially standing while the others leaned, and then a different student sitting while the others stood. Throughout this problem session the group came together in this way, either in pairs, as a threesome, or all together whenever they were successfully sharing something about the problem (as indicated by the conversation transcript). The common posture varied, as shown between panel 1 and 2, but was generally shared by the members of the group. There were occasions when a student stepped back from this shared gesture space, as illustrated in panel 4. This was followed by a return to the group posture, perhaps when the student felt they had something to share, or had given up on an idea.

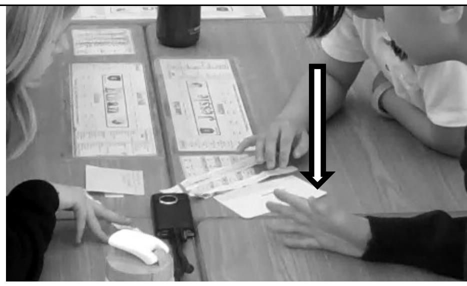
		
Posture mimicry	Participation involves mimicry	Deictic gestures
		
Independent thinking?	Mimicking a cutting gestures related to division	

Table 3: An example of posture mimicry within a group.

SUMMARY

Of twenty recordings analysed there were twenty-one clear incidents of gesture mimicry where students reproduced a given gesture exactly within two turns at talk. In four of the twenty recordings no clear gesture mimicry was observed. Only two recordings demonstrated no posture or gesture mimicry and in both of these recordings the students made little progress with the problem. In all cases gesture mimicry accompanied conversational adjacent pairs rather than an isolated utterance. Groups generally demonstrated several adoptions of posture mimicry and, in all but one case, this coincided with on-task work and resulted in progress with the problem. Gesture mimicry tended to be associated with actions, such as the description of shapes or objects, or mathematical operations such as divide, increase and counting. Very little mimicry was associated with student activities centred on calculating. In seven of the recordings the students were standing and in these recordings gesture mimicry was seen in six cases. These tended to involve a larger gesture space than when the students were seated. There was only one case involving three students mimicking gestures in succession. Generally, only pairs of students mimicked gestures whereas posture mimicking tended to involve more members of the group.

Overall, mimicked gestures clearly occurred but were not seen to be used extensively while students were working on the mathematical processes. Gesture mimicking was predominantly used, and seemed important, in establishing the situation in which the mathematics was framed. When gesture mimicking was observed as related to the actual mathematics, the gestures were seen to represent ‘cutting’ (as in division), ‘framing’ (as in framing a shape such as a circle), ‘counting’ (particularly the action of skip counting using a bouncing motion) and a 'this-and-that' gesture where the flat hand was rotated at the wrist in a back and forth motion (as in referring to two cases). The predominant gesture seen during discussion about mathematical processes was deictic, with students pointing to the pages being working on. While these gestures often looked similar, there is not enough evidence to suggest mimicking, given the limited variations of pointing. Table 3, panel 3, illustrates this type of gesture.

This study indicates that posture imitation is an important part of group work. When students were working productively on a problem, or exploring an idea together, they tended to imitate each other's posture, whether standing or sitting. These common postures shifted throughout the working session and demonstrated enough variation to indicate that it was not merely coincidental. When a student opted out of the common posture they rarely added to the thinking of the group, or their attempted contribution was less well-received. In some cases it appeared that a student removed themselves from the group so that they could think through a situation independently as in these cases the student self-gestured (table 3 panel 4) before re-joining the group. In just over half of such cases the students made a positive contribution to the group. In other situations a student moved out of the group and showed no signs of thinking independently about the problem (i.e. using some kind of self-gesturing or facial expression); in none of these cases did the student return to offer anything new.

The study suggests that mimicked gestures can play a role in creating a mutually shared understanding of the situation within which the problem is set. The mimicked gestures may help to coordinate a mathematical process amongst the group so that mathematic actions are seen to be agreed upon. This communication of acceptance in a process has been seen as a core step in the process of reaching a shared understanding in dialogue (Clark and Wilkes-Gibb, 1986). While gesture-mimicking may not be significant in advancing the mathematical process itself, it may be seen by the interlocutors as an acceptance that the speaker is understood and seen to be making progress. Gesture mimicry is part of the collaborative process but relies on the belief of the interlocutors that they have interpreted each other's' intent in the same way. It must also be noted that such gesturing may be subject to interpersonal relationship issues. Students with a strong rapport with each other may be more likely to mimic gestures.

In conclusion, analysis of the recordings of student work provides evidence that students mimic each other's posture when being collaborative, and also mimic each other's gestures as a means to establish a common process. As such, mimicked gestures may play an important part in helping to establish a shared understanding amongst the interlocutors and assist in progression of the collaborative effort. Given this possibility, there is an opportunity for teachers' observing from afar to recognise good opportunities to intervene in order to best facilitate the group's progress. When a group is seen to mimic each other's posture or gestures then this may be an indication to stay away from the group and allow them to continue to develop their ideas. If there is no evidence of such mimicry then that may indicate a good time to offer support to the group. This result may also tie in with the findings of Gerofsky (2008), in being another observable feature that students who are more confident of their ideas tend to use larger gestures.

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DISMANTLING VISUAL OBSTACLES TO COMPREHENSION OF 2-D SKETCHES DEPICTING 3-D OBJECTS

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This study focuses on potentially misleading information (PMI) and potentially helpful information (PHI) embedded in the 2-D sketch of a 3-D geometric object. Our quest was to discover whether and how PHI and PMI capture visual obstacles, for high-school students, to desirable comprehension of the sketches used in teaching spatial geometry. We compared the anticipated difficulty of 24 sketches of cubes with different auxiliary constructions, according to their orientation and to the ratio #PHI/#PMI, with the actual difficulty reflected in the scores received by 174 high-school students for comprehending these sketches. The findings suggest: (a) deviations from a normative sketch of a cube affect spatial comprehension; (b) the ratio #PHI/#PMI accounts for a significant part of the students' visual difficulty.

THEORETICAL BACKGROUND AND RESEARCH QUESTIONS

Success in learning spatial geometry in high school is frequently attributed to a student's ability to visualize 3-D geometric configurations from 2-D sketches (Gutiérrez, 1996). This attribution is based on the presupposition that human vision and cognition have a high capacity of pattern recognition and synthesis (Gutiérrez, 1996; Christou, Pittalis, Mousoulides, & Jones, 2005). However, as shown by many researchers, this capacity by itself is not enough to enable an easy completion of missing information in the sketches (Parzysz, 1988; Kali & Orion, 1996; Gutiérrez, 1996; Bakó, 2003; Christou et al., 2005). In their study about perception of geological structures from 2-D drawings, Kali and Orion (1996) found that many students rely solely on external visual information, and fail to "penetrate" the 2-D sketches and construct desirable 3-D mental representations. Concurrently, in spatial geometry, Bakó (2003) found that learners mostly consider figural aspects, omitting conceptive didactical inference.

Gutiérrez (1996) suggests that far less information is visible from a 2-D static drawing than from rotating the 3-D object in reality, and learners are not always able to complete the missing information in their minds. Moreover, the learner may be under the illusion that the sketch precisely represents the real object, totally unaware of the loss of information in transit between the real object and the sketch (Parzysz, 1988). As a result, many learners face visual obstacles, often being unaware of their existence.

Further, a quick look on spatial geometry textbooks reveals a trend to orient spatial figures in a particular "normative" way. In planar geometry, existing conflicts between figural and conceptual aspects of geometrical objects may sometimes result in learners' incapacity to recognize a geometric figure when it does not coincide with a prototypical representation or is not placed in a normative position (Maracci, 2001; Larios, 2003). On the other hand, prototypes are not necessarily linked with visual barriers; sometimes the use of familiar prototypes may be advantageous, and permit meaningful learning (Solso & Raynis, 1979). Therefore, the existence of prototypical images has to be taken into consideration when examining spatial perception.

We argue that a better understanding of the visual obstacles' constituents, and the interaction between them, might be the key to improve spatial geometry instruction. Visual perception is undoubtedly influenced by many factors, some intrinsic to the learner, associated with individual knowledge, abilities and experience, while others extrinsic to the learner, related to the geometric problem itself and to the way it is presented (Parzysz, 1988; Arcavi, 2003; Christou et al., 2005).

In our research, we focused on two extrinsic visual aspects, embedded in the 2-D sketch of the 3-D object: (1) Potentially helpful information embedded in the 2-D drawing of a given 3-D geometrical configuration (will be referred to as PHI); (2) Potentially misleading information perceived from the 2-D drawing due to the chosen perspective angle (will be referred to as PMI). As a phenomenon influenced by many factors, visual obstacle investigation requires small and prudent steps. Focusing on simple geometric forms, familiar to high-school students, may increase chances of better understanding visual obstacles. Consequently, our research concentrates on cubes, and on basic shapes such as triangles and quadrilaterals contained in them (hereafter, auxiliary constructions). Perception and visualization undoubtedly consist of complex interactions between many aspects that may not be dismantled into isolated components. However, sometimes a simplistic approach has the power to facilitate and enable comprehension. Therefore, keeping in mind that the whole may be greater than the sum of its parts, we aimed to find answers to the following two questions:

1. Do deviations from the normative images of a cube affect spatial comprehension of auxiliary constructions?
2. Whether and how do the interaction between PHI and PMI capture visual obstacles, for high-school students, to desirable comprehension of 2-D drawings depicting 3-D objects, used in teaching spatial geometry?

DIFFERENT SOURCES OF VISUAL OBSTACLES

Visual obstacles are closely related to the information embedded in a 2-D geometrical sketch of a 3-D object, and to the way this information is perceived. Perception is the process of obtaining awareness, organizing and deriving meaning of sensory visual data, while visualization refers to the cognitive faculty of processing this information and forming an adequate mental image (Kirby, 2008). Perception and visualization are filtered by former experience, prior knowledge and personal expectations (Arcavi, 2003; Kirby, 2008), as well as by individual thinking skills and particular spatial abilities (Parzysz, 1988; Kali & Orion, 1996; Gutiérrez, 1996; Christou et al., 2005).

In spatial geometry, the angle from which a 3-D object is observed has the power to hinder or facilitate visualization. In particular, some projection angles may deform the object displayed almost beyond recognition, and present learners with a substantial visual obstacle. For instance, how can we identify the object in Figure 1? Is it an umbrella from bird's eye view, a right hexagonal-based pyramid seen from above, or a cube? All these interpretations are possible.

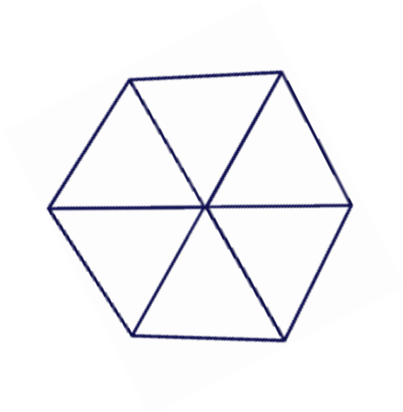


Figure 1: A 2-D sketch that can represent different 3-D objects

Potentially Misleading Information (PMI)

PMI comprises of two categories of geometrical regularities: (1) Hidden correct information (such as hidden from view vertices, edges, surfaces and intersection of edges), and (2) altered or added incorrect information (such as non-existing added intersections of edges, non-existing confluences of edges with a straight line, altered longitudinal ratios, altered angles and edges crossing above the surface and therefore hiding it). For instance, let us contemplate Figure 1 as a drawing of a cube. We may notice one vertex appears to be missing, while some edges appear lying on a beam of straight lines through another vertex. Counting PMI for different sketches of a cube led us to the realization that a cube's PMI values are minimal for the normative sketches frequently used for cubes in school textbooks (see the two examples in Table 1).

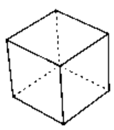
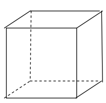
cube's sketch	hidden correct information				altered/added incorrect information			total PMI
	number of hidden vertices	number of hidden edges	number of hidden sides	number of hidden intersections of edges	number of non-existing added intersections of edges	number of non-existing confluences of edges with a straight line	number of geometrically altered sides	
	1	0	0	0	9	3	6	19
	0	0	0	0	2	0	6	8

Table 1: Counting PMI for two different cube sketches

However, typical high-school spatial geometry problems are not as simple, and contain auxiliary constructions. These constructions may add PMI. For example, cube ABCDA'B'C'D' in Figure 2a contains additional PMI: triangle DBB' may look isosceles in the 2-D sketch (DB=DB'), though in the desired comprehension it is not. Moreover, triangle DBB' does not look right-angled in the sketch, although in the desired comprehension it is.

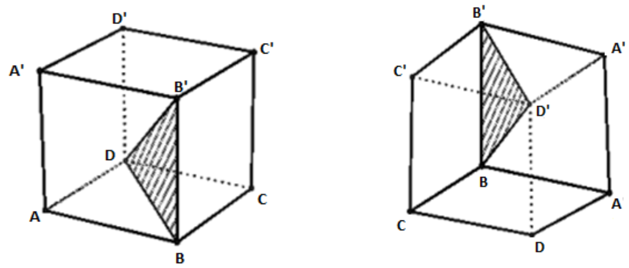


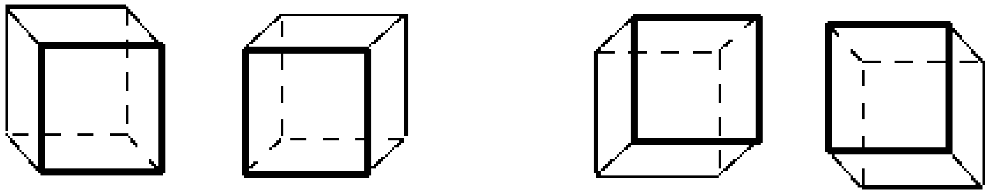
Figure 2: ABCDA'B'C'D' being a cube, what properties does triangle DBB' possess?

Potentially Helpful Information (PHI)

Two-dimensional sketches contain potentially helpful visual information (PHI) as well. Such information may supplement verbal data of the given spatial geometry problem, elicit visualization, and, support deductive reasoning and formal proof (Hadas, Hershkowitz & Schwarz, 2000). PHI comprises of geometrical regularities embedded in the drawing, such as vertices, edges, and diagonals, that auxiliary constructions share with the cube. Consider for example, triangle DBB' in Figure 2a: one side of the triangle coincides with the edge of the cube ABCDA'B'C'D', and the vertices of the triangle are simultaneously the vertices of the cube. Along with prior knowledge about the features of a cube, this visual information might help the learner reason that triangle DBB' cannot represent an isosceles triangle (though its sides on the drawing are equal) and has to be perceived as right-angled ($\angle ADD'$=90º).

Normative and Un-Normative Drawings

PHI and PMI may not be the only objective features of a 2-D drawing influencing one's vision; the orientation of the drawn object may affect perception as well (Larios, 2003). Sketches in spatial geometry textbooks tend to present students with cubes oriented in two out of four possible positions (Figure 3). Consistently with previous findings in planar geometry (Larios, 2003), we expect frequent use of normatively positioned cubes in spatial geometry instruction to form a prototypical image of cubes. Students accustomed to the normative orientation of cubes might find un-normative drawings, obtained by simply turning normative sketches upside down, less familiar and less coherent. Therefore, un-normative sketches may turn out more challenging than normative sketches, even though PHI and PMI remain invariant when changing a cube's orientation (see Figure 2a vs. Figure 2b).



Normative positions
Metaphorically "glued to the floor"

Un-normative positions
Metaphorically "attached to the wall"

Figure 3: Does a certain position of the cube seem more familiar to us?

METHODOLOGY

We started by finding out which geometric elements of 2-D sketches of cube-related configurations function as PMI and PHI. We developed and validated a counting method of PHI and PMI, based on the geometrical elements pointed out by three experienced mathematics high-school teachers. After defining and enumerating the various components of PHI and PMI, an unequivocal counting method was attained and peer-validated by a group of 20 experienced high-school teachers: enumeration was nearly identical (93%), with a kappa coefficient indicating adequate inter-rater reliability (k = .758).

Problem Difficulty Rating Test (PDRT) was constructed to accommodate our goals. To start with, we requested high-school students to draw a cube; since human drawings correspond to their memory representations of frequently encountered patterns (Solso & Raynis, 1979; Larios, 2003), we expected the cubes in the drawings to be normatively positioned, revealing an existing prototype.

In order to find appropriate items to be included in PDRT, auxiliary constructions comprised in 102 initial drawings of normative cubes were sorted out and divided according to #PHI/#PMI into six separate groups of 17 items each. Calculating the ratio #PHI/#PMI (note that #PMI > 0 under parallel projection because parallel projection does not preserve the ratio of lengths of non-parallel sides), seemed a reasonable suggestion for a formula apt to generate a comparative criterion for visual difficulty embedded in various sketches. Average #PHI/#PMI was calculated for each group, and two items having an approximately average #PHI/#PMI were chosen to be included in the test. Attempting to avoid Necker's illusionary dual perception, all drawings used intermittent lines for marking the cubes' hidden edges (Kornmeier & Bach, 2005). These twelve initial items were duplicated by turning the drawings upside-down (see Figure 2). The resulting 24 items were blended throughout the PDRT test. A significant correlation was found between the PDRT scores for corresponding normative and un-normative sketches: r = .931, p<0.0001. Calculating the estimated internal-consistency reliability of the 24-item test rendered high value as well: Cronbach's alpha was a = .887. Table 2 shows an example of #PHI/#PMI calculation for the two corresponding PDRT items presented in Figure 2 (note that calculations in Table 2 are identical for Figure 2a and Figure 2b).

hidden correct information				altered/added incorrect information					PMI derived from cube's perspective
number of hidden vertices	number of hidden edges	number of hidden surfaces	number of hidden intersections of edges	number of non-existing added intersections of edges	number of non-existing confluences of edges with a straight line	number of altered longitudinal ratios	number of altered angles	edges above surfac e	
0	0	0	0	0	0	3	1	0	8
total PMI: 12									
common vertices		common edges		side diagonals		known locations		edges behind surface	
3		1		1		0		1	
total PHI: 6									
#PHI/#PMI = 0.5									

Table 2: Calculating #PHI/#PMI for two corresponding PDRT items

We administered the PDRT test to 174 high-school students, studying mathematics at the highest stream level in the 12th grade, and therefore familiar with cubes and their auxiliary constructions. Three scores were calculated for each respondent: average normative score (12 items), average un-normative score (12 items), and average general score (all 24 items).

RESULTS

Regarding our first question, 97% of the participants (174 out of 180) drew the same image of a cube, which was similar to a normative representation of a cube in the textbooks. We excluded the remaining 3% from our statistical analysis. Moreover, Fisher Test results indicate a significant difference between calculated correlations (see below) of #PHI/#PMI, normative and un-normative 2-D sketches ($z = 2.45$, $p = .014 < .05$), thus sustaining our anticipation that un-normative 2-D drawings, which do not match the prevalent prototype, may alter perceptual difficulty.

As to our second question, the findings show a significant correlation between the ratio #PHI/#PMI and the PDRT scores: $r = .703$, $p < .0001$ for normative drawings, $r = .543$, $p < .0001$ for un-normative drawings, and $r = .612$, $p < .0001$ for all 24 drawings. Thus, our hypothesis that the interaction between PHI and PMI captures the perceptual visual difficulty in spatial geometry for high-school students is highly supported by these significant correlations, not only for learners facing normative 2-D sketches, but also for learners presented with un-normative 2-D drawings: in both cases, spatial perception decreased, as the 2-D sketch exposed less PHI and more PMI.

DISCUSSION AND FURTHER RESEARCH

According to our findings, the ratio #PHI/#PMI accounts for a significant part of the students' obstacles to comprehension of 2-D sketches depicting a cube. This finding is a novelty that suggests a direction for further research, focused on a possibly unconscious mental pattern dominated by extrinsic visual stimuli, placing visual challenges to both, normative and un-normative spatial perception, beyond personal characteristics. Although further investigation is needed, spatial geometry instruction may already take advantage of this cognitive revelation and use #PHI/#PMI as a predictor of the visual difficulty embedded in drawings; different sketches should be adjusted to different educational purposes: minimizing PMI while maximizing PHI in 2-D drafts may help learners comprehend the 3-D geometric situation, and therefore assist visualization, while maximizing PMI and minimizing PHI may serve other pedagogical goals such as training students to cope with high visual difficulty, or spatial ability testing.

However, #PHI/#PMI serves as a better predictor for normative than for un-normative drawing, thus implying the involvement of an additional factor, disrupting vision in un-normative sketches. Evidently, our findings confirm the existence of a prototype representing a cube: the vast majority of the participants drew the same normatively-positioned cube frequently used during spatial geometry instruction. On one hand, the prototypical use of normative drawings of cubes in spatial geometry instruction may form a mental image meant to assist visualization. On the other hand, the prototypical model may not allow enough flexibility, and therefore hinder identification and manipulation of a 3-D geometrical situation in un-normative sketches (Larios, 2003). Further study is needed in order to determine the circumstances under which deviations from standard drawings affect perception. It may also be interesting to further investigate how prototypes influence our perception and whether it is appropriate to enrich the set of prototypes used in spatial geometry instruction. Still, we should denote an immediate instructional, pedagogical implication in classroom: special thought should be assigned to drawings' orientation; two students observing a geometric sketch from opposite directions may encounter different visual difficulty, since one of them is viewing a familiar, normative drawing, while the other is faced with a strange, un-normative sketch.

Next, we intend to examine exploration strategies employed by high-school students when trying to overcome visual obstacles by means of dynamic geometry software. We suggest that when rotating or measuring a computerized 3-D model of a geometric situation, the drawing's orientations, as well as PHI and PMI are altered, and consequently, the change may occur in the problem's difficulty.

We believe that, even though our findings are limited to cubes, and further research is needed for additional generalization to other 3-D geometric objects, the implications may be of interest for both research and practice not just within the area of mathematical education and technology-enhanced learning, but beyond.

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BAKHTINIAN DIALOGUE AND HEGELIAN DIALECTIC IN MATHEMATICAL AND PROFESSIONAL EDUCATION

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We argue that the distinction between dialogue (after Bakhtin) and dialectic (after Hegel, Marx, Vygotsky), that Matusov has previously highlighted, is of key importance to mathematics education. According to Matusov, for Bakhtin these concepts are incommensurable since dialectics implies and the dialogism denies telos (a target). In this essay we argue that mathematical dialogue can and should have teleology Matusov says is implied by dialectics. Thus a good dialogue might involve mathematical (or professional) negation and sublation, providing the dialectic for mathematical (or professional) 'progress' and development. To make this concrete, we illustrate the argument with a lesson study in which progress emerging from dialogues is interpreted in dialectical terms.

INTRODUCTION AND BACKGROUND CONCEPTS

This essay aims to make progress in debates ongoing in the field of socio-cultural theory about dialogism and dialectics, showing how and why this debate is a significant one for mathematics education in particular. In everyday mathematics education terms, we are concerned with dialogue, in mathematics classrooms (between learners and teachers) and in staff rooms (between teachers, for example engaged in lesson study). The concern is whether a dialogue goes beyond the sharing of meanings by those involved, to a point where some sort of progress or development is achieved, for example mathematically, or perhaps professionally. Such developments can be said to have 'telos', a progressive direction.

A case in point is from our lesson study project. Scenario: A group of children are counting the steps (strides) made by Usain Bolt in a video of his Olympic 100 metres winning sprint. The year 2 (6-year-old) children's answers to the question, *How many steps does he make (from the beginning of the video until he crosses the finish line)*?, produce many and varied answers from 18, or 19 (the answer we thought correct) right up to 20s and 30s. In several trials of counting, the children's answers converged somewhat (and gradually), but there were still differences. During the subsequent activity the children modelled the situation physically in the gymnasium, counting the steps laid out for them along number lines across the gym. One of the anticipated issues in children's counting of the steps becomes clear and is discussed as part of the classroom dialogue: should the first foot-print (i.e. that before the start of the race) be counted as zero or one? Another, unanticipated, issue (among others) arises from one of the children counting only up to the last foot-print (before the finishing line) and refusing to cross the line from the 18th to the 19th foot-print (see Figure 1)!

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 377-384. Vancouver, Canada: PME.

Now on the one hand progress in the classroom dialogue here might be the result of the learners coming to agree with the teacher's preferred understanding of the situation and the mathematics, for example a convergence on an agreed answer of 19 steps. On the other hand, one might think progress in the dialogue arises due to the teachers/researchers understanding the children's mathematics and their perspectives. Bakhtinian *dialogism* theorises a dialogue as *monologic* if the authority asserts their preferred, 'correct' answer, but as *dialogic* if an internally persuasive discourse is constructed such that both subjectivities have an opportunity to engage through dialogue with the other's discourse (Bakhtin, 1981). Dialogism might involve progress in each person's subjective understanding, and to that extent we will argue would be subjectively dialectical. The question arises, though, whether this subjective progress represents 'objective' telos in mathematics, i.e. can the dialogue be said to be mathematically more advanced, in an objective, historical-cultural sense?

Figure 1 shows the foot-prints (with numbered marks below them from zero before the start to 19 after the finish line) and the foot-steps (shown by arrows).

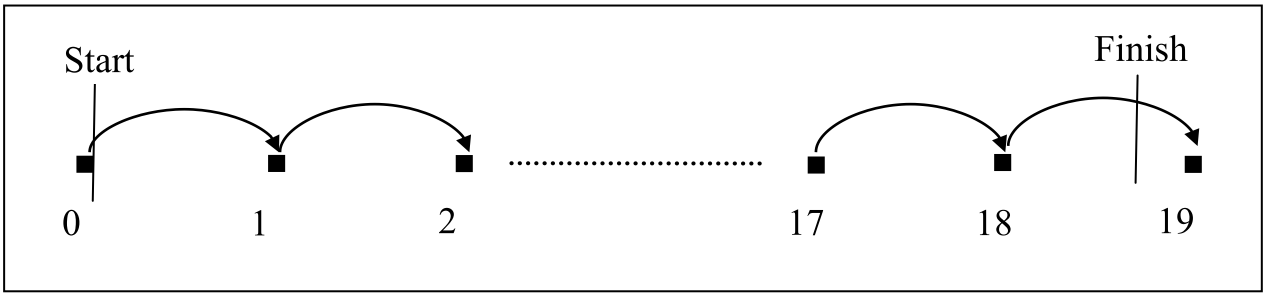

Figure 1: Shows foot-prints and 'arrowed' foot-steps

The simplest answer is that the teachers'/researchers' mathematics – if they are not mistaken in reference to the curriculum (and the curriculum is in turn not mistaken in relation to the cultural-historical state of mathematics) – will be more advanced culturally than the children's mathematics. The telos then is here defined by the definition of the mathematics in the curriculum targeted by the teacher (end of story). Progress is objective if the children (and perhaps the teachers/researchers) make progress towards this target. In this view the dialogue is functional if and only if it allows the correct mathematics to become internally persuasive for the children: this is what Matusov criticises as monologic in Bakhtin's terms.

On the other hand, the constructivist tradition values the children's own mathematics as genuinely mathematical in its own right: their perspective should not be expected to converge with the teachers' or that of the curriculum, negating the above view. Progress is made by the child's mathematics being more (subjectively) adequate in explaining the situation or task, and so on. This is loosely the constructivist position criticised by Radford (2013).

We will now argue that Hegelian dialectics might offer another perspective: in true dialectical terms we seek a new sublation of these two positions.

THEORETICAL ARGUMENT AND DIALECTICS

This situation is addressed by Matusov, though not in the mathematical context. First he characterises Bakhtinian dialogism as ‘intersubjectivity without (necessarily achieving) agreement’ (Matusov, 1996), and more recently in characterising Hegelian dialectics as anathema to dialogism, arguing that it imposes a notion of telos that is more akin to monologue (Matusov, 2011).

We agree that the dialectic implies progress, i.e. a teleological process, though the actual end-target may not be anywhere visible during the process itself. Only from the vantage point of history can one see with some surety where progress was made. Nevertheless, the dialectical process that takes a notion (e.g. mathematics, or ‘counting’) through its negation to its ‘sublation’ in a new notion, can be regarded as in itself progressive or developmental (for ‘notion’ here we might also say ‘idea’ or ‘concept’). It is the character of negation and sublation in Hegel (2009) that characterises the type of progress and teleology: We consider now why this is so.

Sublation requires that the notion and its negation are unified in the new (this is often simplistically presented as a ‘synthesis’ of a thesis and its negation). That is, the original notion and its negation do not disappear, but remain present subsumed within the new form. As such, the sublation represents progress from the old to the new level of thought. In his science of logic, Hegel begins with the notion of ‘being’ as perhaps the simplest notion of ontology; its negation is non-being or pure ‘nothing’. The sublation of being and non-being is then ‘becoming’, i.e. the movement from nothing to being (and additionally from being to nothing – so dying is also a kind of un-becoming, as unbecoming as it sounds).

Here, as in every dialectic, thought (the notion, idea or concept) and its negation do not quite disappear but are sublated in relation to each other in the new thought. Indeed the concept of sublation (one that Hegel does not explicitly address, see Palm, 2009) implies preservation as well as change: in reality, preservation indeed requires adaptation and change. Sublation is thus a self-contradictory concept in this sense.

One can read the notion of number as developing in this way: the notion of whole number is negated in practice by the fact that not all quantities can be counted as whole numbers (maybe not even ‘steps’), and the new number form (maybe fractions or decimals) then sublates the ‘whole number’ and its practical negation. Whole numbers (and their negation) have not disappeared here, but they are preserved in a new form (thus the whole number ‘one’ perhaps becomes the rational number 1.0, which is the same and yet not quite the same as the whole number ‘one’ that was negated).

In this account the concept of practice has been adduced. Hegelian dialectic was originally presented as being the movement of ‘pure thought’ in itself, and the concept of social practice in this engagement is usually stressed as Marxian. Marx’s claim to have turned Hegel dialectics on its head (or feet) is easily misunderstood here: for Marx conscious thought is still a key moment in practice. Thus, in the preceding sublation, the practice of counting in new contexts pushes new thought: the cognitive conflict induced provides one contradictory moment (in the moment of thought), and the contradictory positions in the practice and the dialogue provide others (in the objective and intersubjective moments respectively).

Now Marx and Lenin's readings of Hegel and the dialectic become relevant: we recall that Vygotsky (1986) appealed to both in this famous passage in *Thought and Language*, quoting volume 29 of Lenin's collected works as follows:

> Man's practice, repeated a billion times, anchors the figures of logic in his consciousness. These figures have the strength of prejudice, their axiomatic character, precisely (and only) because of this repetition. (p. 198)

Ultimately, for Marx, Lenin and Vygotsky, the dialectic of theory and practice is developmental to the extent that thought proves efficacious in practice. As such, intersubjective dialogue (for example, between learners and teachers, and the curriculum, and the community of mathematicians) provides a powerful contradictory moment for dialectics in discourse, but material practice (including related discourses) is decisive. Dialogue without a practical context that proves a notion cannot be decisively progressive. Thus, Marx used Hegelian dialectics himself in thinking through his analyses of capital and labour in the *Grundrisse*: labour is the 'negation' of capital, and capital in order to conserve-renew itself must be negated through its investment in labour, and through the surplus value capital-and-labour become sublated in new, expanded capital, otherwise it will die in a pre-capitalist form of money in pure circulation (Marx, 1973). Yet it is the fully produced theory in *Das Kapital* that he publishes – in which dialectics are secondary to data and theory.

In the context of measuring Usain Bolt's footsteps, as we researchers thought about the children's answers (was it 18 or 19 steps?), we saw that both were correct answers to different questions, and that the truth might even be that the answer is somewhere between the two, depending on what we choose to mean by the term 'step'. This is perhaps the sublation of the previously constituted truth (the answer we thought was 19) that we achieved through its negation in our joint lesson experience in practice with the children. In the next section we look at the dialogue of teachers discussing this very issue in their lesson study reflections.

DIALECTICAL ANALYSIS OF LESSON STUDY DIALOGUE

The following dialogue took place in the teacher-researcher meeting following the 'Usain Bolt lesson' referred to above. This lesson was part of our lesson study research project (Williams & Ryan, 2013) where we have been working with staff in a primary school developing mathematical dialogue in their classrooms (Reception to year 6 classes: 4/5- to 10/11-year-olds). The lesson had been planned jointly by a core group of teachers and university researchers for year 2 children. A key point for the team's lesson plan was to make the number line 'come alive' for the children, and in particular to address the problem of counting the 'ticks or the jumps' on the number line, which the team agreed is a key mathematical problem (see Ryan & Williams, 2007, pp. 93-94). This particular lesson event involved the entire school staff in the lesson observation and the extensive after-lesson discussion. The two themes of the discussion that emerged were: how would we teach this lesson again or what would we do in the follow-up lesson? 'Lesson study' follows the Japanese model of research-led continuing teacher professional development cycles, though we have adapted it for local conditions and focus (Williams, Ryan & Morgan, 2013).

In the following transcript, we see a discussion reflecting on the experience of teaching and observing the children and the way the teachers-researchers imagine developing new teaching based on this experience. This can be read as dialogism: the teachers and researchers are engaged in trying to make sense of the experience and to make sense of each others' meanings for what occurred. But is there progress, and is there a dialectic?

Teacher 1: Could you not give two different answers… And so – it could be this answer, it could be this answer … and actually get the children to be involved in deciding why one answer or the other ... and effectively … those who think it's 18, and those who think it's 19 .. .We're not saying this is the definitive answer, we are just saying it is one that could be explained...

Researcher 1: I think we want to get to what Benny [*pseudonym, one of the children*] said where you count, (from) where the zero is ... and historically that's what humankind … had this problem ... so I'd suggest, yes, how did these people get 18, how 19, … so they're engaged in ... so we get: THEY got 18 because they started counting HERE, that's what we want articulation of ... These people call this foot-print "one", and they call it "zero" ... so once you have that out on the table you ... then you can start a debate. So which...

Teacher 1: If you do that, aren't you going to be giving them (an) answer already? If you are saying this is one or zero, aren't you? Whereas if you ask them the reasons for two answers, you aren't explaining why... it's up to them to prove or disprove...

Researcher 1: Yes, I'm saying HOW did these children get 18, how 19... and they have to come back and say ... because *they* started the ONE here and *they* started the ONE there … and then you might get ... Who's the little girl who got down and said that ...? … Because there is miscounting from either one or zero... because it's not a convention, it's sensible: there's the STEP, from a starting point so the one is out there ... (*gestures to the end of the step*).

Teacher 2: Then the, the ...we want the rest of the class to be standing round to watch them do that (*gestures to the circle the class would form*) ... because I had to get Denis [*pseudonym, one of the children*] to come down … and they're not used to having to look and listen to that group that's saying something important... We found when doing the project in the past that it took some lessons for them to get the idea that, 'hold on, I need to stop and listen and look to what they are saying, that has something to do with ME', and I think to get them round one of the group's number line and get them to act out what they were doing again, so that they are there seeing the number line

> together as well, I think that's something I'd want to do in the next lesson as well.

Let us now consider this dialogue as a series of six reflective-imaginative moments in a dialectical process:

"Could you not give two different answers… And so – it could be this answer, it could be this answer …". Here the mathematical answer "19" is negated, that is, confronted directly with its opposite: the previously presumed correct mathematics is negated by an alternative answer, or alternative mathematisations of the task.

" … and actually get the children to be involved in deciding why one answer or the other". Here the children are imagined to participate and engage subjectively with the mathematical alternatives in a classroom dialogue. They are asked to reason about the mathematics with others as subjects, intersubjectively in discourse (in a debate) reflecting the opposition of the mathematical object by its mathematical negation in the measurement practice.

Researcher 1 then imagines that one of the children's (Benny's) mathematics could enter this debate somehow: the key mathematical mediation is the 'zero' reference point for the counting. This explains how the mathematically opposing objects (the counting to get 18 versus 19) become sublated in a new mathematical object-understanding (viz. where you start counting from or the counting of 'foot-prints' rather than 'foot-steps', i.e. how the task is interpreted and modelled).

Teacher 2 says "And then you might get ..." imagining now how this dialectic might come into being in her future classroom: the 'little girl' who showed, from her subjective point of view, why it is sensible to count the foot-print at the end of the first step as "one", justifying this 'correct' mathematical choice/answer.

Researcher 1 participates in the little girl's sense-making with her own gesturing, revealing why (objectifying how) the end of the step should be counted, 'one'. This subjective sharing is offered as a generalisation, and therefore as a plan for the future lesson being imagined.

Teacher 2 accepts this, and starts to envisage concretely how this could work in her future embodied teaching. She imagines how the debate might malfunction (as in past experiences) and she explains how the children need to be gathered to facilitate such a meaningful debate. This is made concrete through the recall of previous lessons where she had done this: her gestures represent the envisaged arrangement in the future class.

In previous work we have described this kind of lesson study dialogue as offering a zone of proximal development for the teachers and for profession's teaching practice, which we called a 'zone of professional development' (Radovic et al., in press). Such a dialogue can only be conceived as developmental in Vygotksyan terms if it is indeed teleological, i.e. if the teacher's professional practice is seen to be making progress towards something better. We do not know what the target professional practice is until after it has developed, but we can perhaps see in the dialogue the dialectic of development that our theoretical argument requires.

Arguably, then, what makes this dialogue progressive is the sublation of notions with their negations in the new imagined practices: we do not know for certain that this will be progressive until practice is confronted in the future, whereupon no doubt new contradictions will arise. Thus, professional development in teaching is a dialectical work of theory and practice; we might play with this dialectic in much the same Grundrisse-sense that Marx did with capital and labour. The classroom provides the teaching-learning practice (labour) in which professional theory (capital) is invested, and which is sublated anew in developed professional theory.

CONCLUSION

Our argument is that dialogue should be (mathematically or professionally) developmental if it is dialectical, and that the dialectic requires a concrete dialogue in which the undeveloped notion is negated in and with practice. Development then can – though of course not with certainty – arise through a genuine sublation in which the undeveloped notion and its practical negation are reconciled but both conserved in the new. The negation may involve a purely discursive, dialogical moment of negation, but at root there lies a negation in practice; its validity in developing and advancing thought is dependent on its relevance and efficacy in practice. Hegel (and Marx) describe this as sublation of the universal (general) notion in the particular, or 'ascending from the abstract to the concrete'.

In conclusion, let us consider the implications for mathematics education and the naïve alternatives put forward in the introduction. Bakhtin's theory of dialogism provides a rationale for the importance of the 'internally persuasive' dialogue with the other: thus the teachers' and the currriculum's mathematics must be made persuasive for effective learning to take place, and we can reject monologism that is based on arbitrary, unequal power relations. In what way can the learners' and teachers' mathematics be 'equal' in the inevitable power relations, given that the teachers' and the curriculum's mathematics has centuries of culturally-historically evolved science behind it?

Materialist dialectics requires that the dialogical persuasion be based in efficacy in practice: the teacher as mediator of the curriculum in practice designs or implements tasks that engage the classroom community in contradictions of given mathematical notions with practice. The teacher and curriculum can thereby arrange for dialogues in which contradictions are developmental because practice negates inadequate notions; new mathematical notions develop in which the old are sublated.

This of course applies equally to the development of the profession in lesson study processes. Well-designed lesson study should confront undeveloped theoretical ideas with challenges through classroom practice: genuine development occurs when, and because professional notions are negated (and so shown inadequate) in practice. We should perhaps celebrate and publish lesson study accounts of such inadequacies in classrooms much more than we usually do: they may be the life blood of real development.

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MAPPING CONCEPT INTERCONNECTIVITY IN MATHEMATICS USING NETWORK ANALYSIS

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This paper reports from a broad investigation of mathematics knowledge as dependent on interconnected concepts. The paper focuses specifically on illustrating how network analysis may be used in examining spatiotemporal relationships between learned mathematics concepts, or curriculum outcomes, and concepts inherent in assessment items. Connections both within and between year levels are shown, based on primary years' multiple-choice assessment items related to measurement. Network analysis provides a potentially powerful tool that may offer educators greater specificity in approaches to the design of revision and intervention through a view of complex rather than linear conceptual connectivity in mathematics learning.

INTRODUCTION

This paper uses analysis based in network theory, a modern development of graph theory, to illustrate connections between measurement items as part of a larger project *MathsLinks: Spatiotemporal Links in Mathematics Learning in Classroom and Online Environments*. A major thrust of this project is an examination of the connections between learned concepts as curriculum outcomes (e.g., Woolcott, 2013) and concepts inherent in assessment items. Network representations of such connections provide a spatiotemporal view of conceptual development in mathematics, with illustration here of complex connectivity in assessment items within and across year levels.

This project is based in a growing awareness that knowledge is interconnected and it utilises the strong groundwork for quantitative and qualitative investigation laid down in approaches using complexity theory (e.g., Davis, Sumara & Luce-Kapler, 2008). Although such approaches have been applied only recently in educational studies, student knowledge of mathematics has been linked specifically to complex and non-linear concept connectivity using network theory, with Mowat & Davis (2010) viewing mathematics in terms of 'complex networks'. Successful learning, in this view, depends on the development of major network junctions, or hubs, that support non-linear conceptual development, as well as the development of weak connections that circumvent hub failures (Khattar, 2010).

BACKGROUND

Network theory is a widely used and powerful tool for representing and examining relationships in terms of system connectivity, and follows a well-established analytical methodology that allows qualitative mapping and quantitative analysis of the

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 385-392. Vancouver, Canada: PME.

relationships between nodes connected in a network (Newman, Barabási, & Watts, 2006). Network analysis has been applied widely across differing disciplines, largely because the rules governing network relationships remain independent of the nature of the subjects being linked (Newman et al., 2006). The main focus of Mowat and Davis (2010) is an argument that mathematics can be integrated through an examination of the complex linkages between mathematics concepts based on the embodied metaphors of Lakoff and Núñez (2000). A sidebar to this argument, however, is that mathematics concepts so integrated must be linked as networks. This seems to have support from the notion of expertise gained through the development of schemas, themselves arguably a type of network (Sweller, van Merriënboer, & Paas, 1998). The idea of knowledge linked as networks implies not only that mathematics concepts are linked together, but also that they are linked to other concepts in what Khattar (2010) considers as bodily experiences that are experienced emotionally.

Contemporary mathematics curricula, however, can be seen as constructs that are, in effect, a sequence of disconnected ‘learned concepts’ (e.g., see Chapter 1 in Glatthorn, Boschee, Whitehead & Boschee, 2012). Devlin (2007) has argued that a mathematics learner may have a functional understanding of a taught concept, as a learned concept, if the learner shows, through assessment, some level of understanding of that concept. A mathematics curriculum concept, in this sense, is a concept being taught that is being defined in terms of what the learner can do with it. A primary school teacher, for example, may consider student knowledge of addition of one-digit numbers to be a concept, but later to consider knowledge of addition of any two-digit numbers to be also a concept. The view of a mathematics concept as determined by a curriculum and its assessment, however simplistic, is useful in that the links between learned concepts may be traceable, using assessment results, in terms of functional understanding (Woolcott, 2013). It may be possible, for example, using a sequence of assessments, to determine if a primary school student, who has answered successfully a question involving knowledge about circles, has knowledge of other mathematics concepts that have led either linearly, or through a network of supporting links, to that knowledge (e.g., Lamb, 1999, in Mowat & Davis, 2010).

METHODOLOGY

Large-scale testing programs, such as the Australian National Assessment Program – Literacy and Numeracy (NAPLAN) (ACARA, 2012) and the Australasian Schools Mathematics Assessment (ASMA) (EAA, 2012) include multiple-choice test items for assessing mathematics curriculum outcomes. Feedback from such testing is limited to assessing student responses against the outcome-based items. Network analysis methodology illustrates here how a more complex view of mathematics learning, generated from item data, may assist educators in understanding how concepts are related and why students find it difficult to make key connections between concepts.

This paper shows examples of representations (maps) based on network analysis of measurement items, about 6-8 items per assessment, from a larger analysis of

2009-2012 ASMA across primary school Years 3-6 in NSW, Australia. The Year 6 network results represent a single class sample of 62 students. The power of this analysis in comparing concepts longitudinally is illustrated using a map generated from results of one student who had completed ASMA in each of the years 2009-2012.

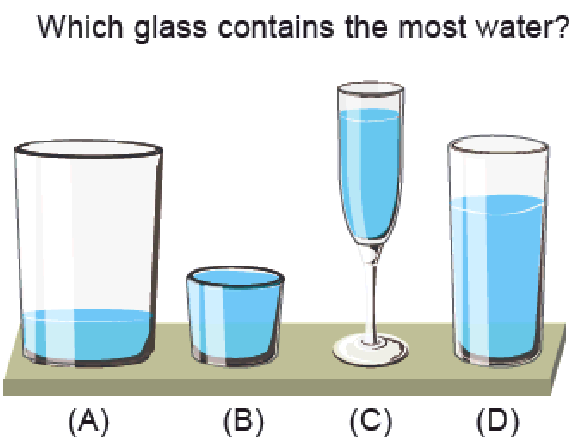
Concept survey and matrix coding

A matrix of coded data generated from a concept survey of all measurement items, was analysed and maps generated using NetDraw (Borgatti, 2002). Each of the items was assigned one or more outcomes from the NSW K-6 Syllabus (BOS, 2012). Adapting Newman's Error Analysis (NEA, see White, 2010), additional inherent concepts were generated as 'access concepts' (Do I understand the question?) and 'answer concepts' (Can I now answer the question?). A limitation in using NEA for multiple-choice items is that analysis of student strategies cannot be used. Words as concepts, however, were included (e.g., Radford, 2003), as well as overarching concepts that allowed interpretation of diagrams (e.g., Lowrie, Diezman & Logan, 2012). An example of the concepts surveyed is shown in Figure 1 for a Year 5 ASMA practice question.

The Measurement Outcomes

MES1.3 Compares the capacities of containers and the volumes of objects or substances using direct comparison

MS1.1 Estimates, measures, compares and records volumes and capacities using informal units



Inherent item concepts

Access

Recognise 3D representation in 2D image
Concept of volume
Object contains a liquid
Objects may contain different volumes

Answer

Select informal unit to describe volumes
Estimate volumes
Compare volumes
Words: Which, contains, most

Figure 1: Concepts determined for a Year 5 multiple-choice measurement item. Practice item used with the permission of EAA.

For each of the ASMA Years 3-6 measurement items, responses and survey results were coded as follows: correct items and associated outcome/concepts as 1; incorrect items and associated outcome/concepts as 0. In network maps constructed using the matrix, nodes are either outcomes/concepts or items. Table 1 shows a sampling of the coded matrix for a Year 6 student with Item 2 correct and Item 5 incorrect.

Outcomes/Concepts	Item 2	Item 5
MES1.5 (NSW K-6 outcome)	1	0
Recognise graph (access concept)	1	0
Read columns in graph (answer concept)	1	0
MS2.4 (NSW K-6 outcome)	0	0
Word: bought (access concept)	0	0
Adds mass in grams (answer concept)	0	0

Table 1: Matrix coding for items and outcomes/concepts, for a Year 6 student with Item 2 correct and Item 5 incorrect. (Sample only - not all concepts shown.)

Direct and inferred network connections

As well as direct relationships between items and outcomes/concepts (Figure 2), analysis utilised two types of inferred relationships: connections between the concepts/outcomes associated with an item; and connections between all concepts/outcomes of two or more items that shared a concept/outcome. The inferred network maps in Figures 3 and 4 are designed to provide additional structural overviews of any concept connectivity. Two types of weightings have been calculated for these network connections: simple weighting based on total numbers of students with correct/incorrect item responses (Figure 2), and; weightings based on class averages of these item responses (Figure 3). Network maps provide a diagnostic tool that can act as a guide to assessed mathematics knowledge and potential interventions and, although associated metrics can also provide additional insights, such as patterns of conceptual linkage, these are included elsewhere in this project.

RESULTS AND DISCUSSION

Individual and class connectivity – Year 6 measurement items

The network map in Figure 2 shows direct connections (lines) between nodes representing items (squares) and their outcomes/concepts (circles) for incorrect items. Each item node is a hub, since all paths from one of its concepts/outcomes to another must pass through the item node. Connection weights were calculated from totals of incorrect item responses across the Year 6 class, with heavier lines representing larger numbers of incorrect responses. The circled node indicates an outcome/concept shared across 3 items, one of several that form the basis for the inferred connectivity maps. Figure 3 shows such an inferred connectivity map for the Year 6 class, focusing again on incorrect responses (concepts excluding words). Network maps such as Figure 3 allow an educator to identify key outcomes/concepts that were, on average, incorrect and that may need to be reinforced for successful future learning, in case of hub failure (see e.g., Khattar, 2010; Mowat & Davis, 2010). The type of inferred analysis in Figure 3 may be particularly useful in its representation of connections between concepts or outcomes which were being used correctly in one context and incorrectly in another, on average (e.g., the connection between the dotted squares). The dotted squares, for example, show concepts that were shared across items that, although incorrect in one item, were correct in other items, more than 50% of the time in this simple illustration.

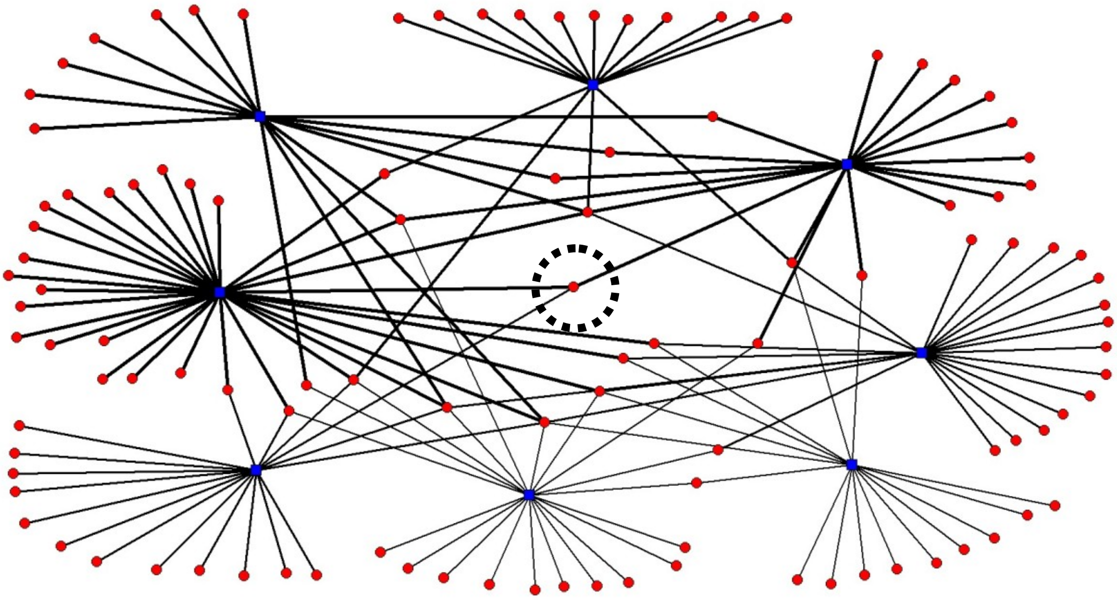


Figure 2: Direct connectivity map for Year 6 students with incorrect item responses

The heavier the line, the larger the number of incorrect responses. Items are indicated by filled squares and outcomes/concepts by filled circles. The dotted circle shows a shared concept, in this case 'a numeral written as a word'.

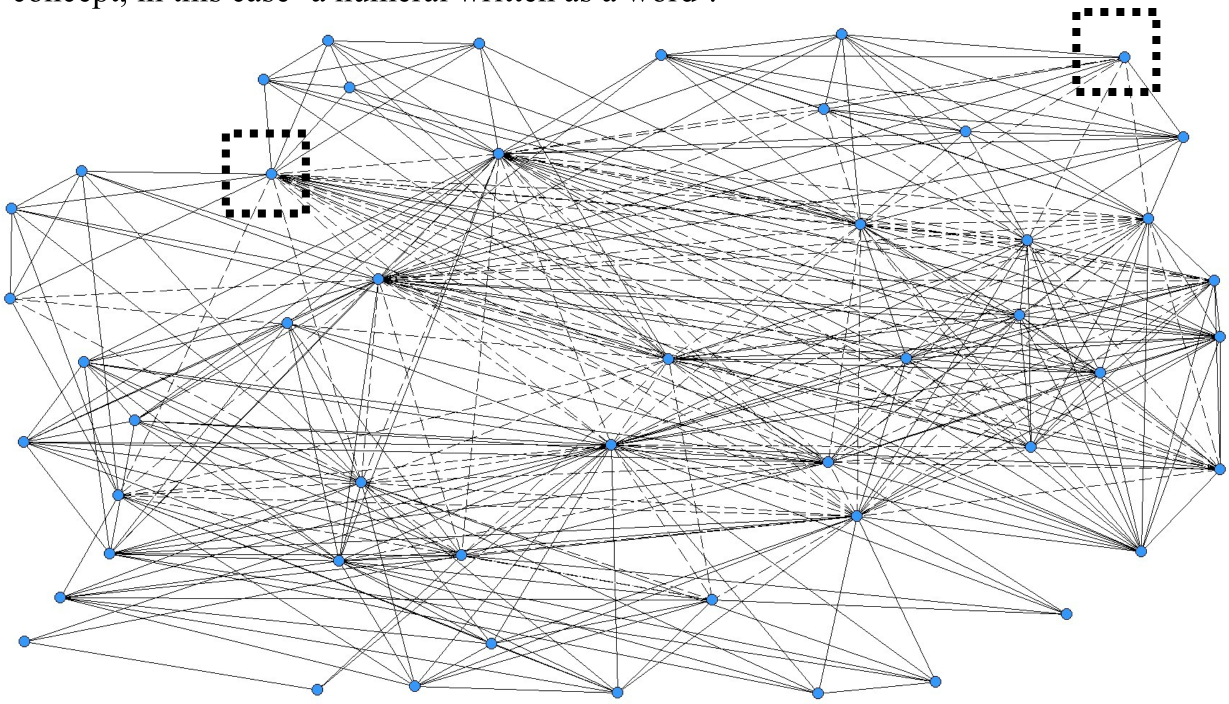


Figure 3: Inferred connectivity map, average weighted, for Year 6 students with incorrect item responses.

Solid lines are based on incorrect to incorrect connections and dashed lines on incorrect to correct connections between inherent item concepts (with word concepts not included for clarity). Concepts are indicated by filled circles. The dotted squares show two of the nodes that, on average, connect these concepts in incorrect and correct items.

The network analysis represented in Figures 2 and 3 (and other analysis not shown here) indicates to the teacher that a number of students in this class have not grasped particular measurement outcomes/concepts in this Year 6 assessment. These outcomes/concepts, therefore, may be a useful target for revision or intervention, even if it is only the centrally located outcomes/concepts that are targeted. The teacher could use such analysis to assist in design of revision or intervention around outcomes/concepts connected to the incorrect item responses for either the entire class or for individuals. Since this type of representation can also show nodes weighted by degree (number of connections), it offers further specificity for the classroom teacher as to relationships between items, outcomes and inherent concepts.

Longitudinal connectivity – Year 3-6 measurement items

Figure 4 shows one of a number of possible inferred relationship maps that can represent longitudinal connectivity. In this case the map shows inferred connections between the incorrect Item 10 in Year 6 and items in Years 3-5. The item connections were inferred from shared outcomes/concepts, effectively reversing the inference process utilised to construct Figure 3. For a focus on curriculum, this analysis could also feature inferred connections between outcomes instead of items, or connections between outcomes and inherent item concepts.

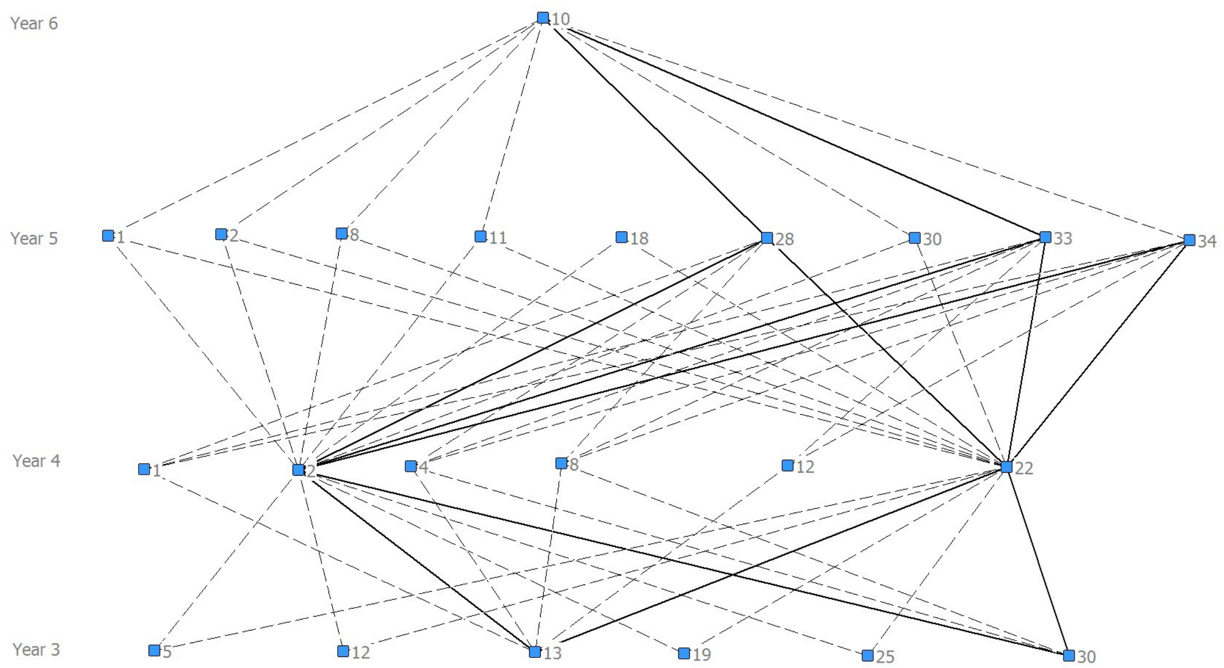


Figure 4: Inferred connectivity of the incorrect Year 6 Item 10 with items in Years 3-5.

Solid lines are based on incorrect to incorrect connections and dashed lines on incorrect to correct items. Items are indicated by filled squares with an item number.

Figure 4 shows how Year 6 items can be connected to items in previous years, effectively a 'concept trail' through past items, indicating which items and associated outcomes/concepts were/were not learned successfully. Analysis exemplified in

Figure 4, used in conjunction with that in Figures 2 and 3, may be useful, therefore, in designing revision or intervention that includes prior knowledge over time, as far back as Year 3 in this case, but over differing periods in such testing systems as NAPLAN. Two of the authors (Woolcott and Chamberlain) are using this longitudinal connectivity to trial an interactive 'App' designed to link curriculum outcomes and inherent concepts to intervention strategies for both multiple-choice and other styles of assessment items. The broader project aims to test the success of such strategies.

IMPLICATIONS AND FUTURE RESEARCH DIRECTIONS

This paper provides an example of the network analysis we have been developing in order to examine the spatiotemporal interconnectivity of mathematical concepts. Although the application of network theory outlined here draws on extensive theoretical research on complex connectivity in mathematics (e.g., Lakoff & Núñez, 2000; Mowat & Davis, 2010), the illustrations aim specifically at an initial examination of whether network analysis is functional in the context of a school mathematics curriculum. This functionality is shown in the exemplar representations here as both direct and inferred connections between inherent concepts and outcomes derived from assessment items. The representation of longitudinal connectivity, in particular, gives a functional picture of conceptual development in mathematics over time. This paper shows examples of novel conceptual connections between outcomes and inherent item concepts, in this case for primary years measurement items, that are not currently utilised in the analysis of such large-scale testing programs as ASMA and NAPLAN.

The analysis here provides support for the view that new mathematics knowledge, even when described in terms of outcomes, requires prior knowledge (Sweller et al., 1998). Longitudinal representations may allow a more extensive analysis of prior knowledge than that undertaken in large-scale testing programs. The analysis here supports also broader analyses we are undertaking at differing conceptual levels, including analyses using embodied conceptualisations (Roth & Thom, 2009) and conceptualisations based on graphic elements in mathematics tasks (Lowrie et al., 2012) and pattern and structure (Mulligan, English, & Mitchelmore, 2013).

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INTERVENTION FOR MIDDLE SCHOOL STUDENTS WITH POOR ACHIEVEMENT IN MATHEMATICS

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Poor mathematics achievement in middle school students is evident in many countries. While some of the difficulties can be attributed to student related factors, there is considerable evidence that computational automaticity is essential for mathematics achievement. A QuickSmart (QS) mathematics intervention program was trialled with a group of students in Grades 7 and 8, matched with a control group of similar underachieving classmates. A statistically significant decrease in mean response latencies was found for QS participants after lessons in multiplication only. Significant differences were also evident between the pre and post scores of the two groups on a standardised test of mathematics. This study confirms and extends previous findings of the efficacy of mathematics intervention for underperforming middle school students.

INTRODUCTION

Poor achievement in mathematics has been investigated in numerous studies in many countries in relation to student factors such as cognitive difficulties, memory, attention, motivation, anxiety and self efficacy. Other studies have focussed on teaching methods and curriculum issues (Vaughn, Bos & Schumm, 2000), with dyspedagogia (Westwood, 2004) or poor teaching cited as having a significant impact on student failure in basic mathematics. While the development of mathematical reasoning depends on students learning appropriate facts, concepts, strategies and beliefs, lack of procedural knowledge of the basic operations for addition, subtraction, multiplication and division is *the most obvious obstacle to academic success in mathematics* (Mayer, 2006, p. 65). The development of computational fluency or the speed with which students can retrieve or calculate answers to simple mathematics problems is a prerequisite to mathematics achievement at all levels (Arroyo, Royer & Woolf, 2011). Cognitive psychologists have established a clear relationship between the development of basic computational automaticity and complex mathematical problem solving skills (Tronsky & Royer, 2002). A multiplicity of studies have found that being able to produce answers to basic number facts rapidly and accurately reduces the load on the working memory and *it is this saving that is a key factor in being able to develop more complex problem solving abilities* (Tronsky & Royer, 2002, p. 118).

A recent report on Australia's performance in the 2012 *Programme for International Student Assessment* (PISA) has raised considerable concerns about the significant decline in the mathematical literacy of 15 year old students in Australia in general and South Australia in particular (Thomson, De Bortoli & Buckley, 2013). Between PISA 2000 and PISA 2012 Australia's mean mathematical literacy performance dropped

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 393-400. Vancouver, Canada: PME.

significantly from fifth to 19th place, with the decline most evident in the mean increase in the proportion of low performing students and decrease in top performers. South Australia has experienced one of the largest deteriorations, with the decrease of 46 score points equivalent to more than a year of schooling and where 12 per cent more students did not reach base Level 2 in 2012 (Thomson, Hillman & De Bortoli, 2013). Significant declines in South Australia are also evident in Grade 8 students in the *Trends in International Mathematics and Science Study* (TIMSS) from 1995 to 2011 (Thomson, Hillman & Wernert, 2012). Results from TIMSS 2011 have highlighted a substantial 'tail' of underperformance in mathematics, with 11 per cent of Australian students not even achieving the Low international benchmark (Thomson et al., 2012).

Students' poor performance in mathematics poses significant pedagogical issues for schools but particularly for middle school teachers who are caught in a "back to basics" dilemma (Yates, 2009a). Over time underachieving students fall increasingly behind their normally achieving peers and by the eighth year of school can be up to five years behind their average achieving peers (Pegg & Graham, 2007). Many students in the middle school years have to expend considerable effort to work on lower level component skills they have encountered many times before (Hattie & Yates, 2014). Practice is essential for students to gain automaticity of basic skills in mathematics content areas (National Mathematics Advisory Panel, 2008), but in middle school classrooms there is much less time and opportunity to develop skills that should have acquired in the early elementary grades (Carr, Taasoobshirazi, Stroud & Royer, 2011).

THE PRESENT STUDY

This research report is part of a larger longitudinal study of student mathematics achievement, self efficacy, anxiety and learned helplessness in a non-government, single sex, elementary and secondary school for boys in Adelaide, South Australia. The *Progressive Achievement Tests in Mathematics Plus* (PATM) (Australian Council for Educational Research, 2010) were administered online to students in Grades 3 to 10 in March (Time 1) (T1) and November (Time 2) (T2), 2013. At T1 students completed the PATM test for their respective previous Grade level and at T2 their current Grade. At T1 a 'tail' of underperformance (Thomson et al., 2012) was most evident in Middle School students, with 33 (53%) of 63 Grade 7 boys and 33 (24%) of 140 Grade 8 boys scoring in the percentile rank range of 1-19 (-1 standard deviation) (SD). A further nine students in Grade 7 and 17 students in Grade 8 scored in the 20-30 percentile rank range. The poor achievement of some students could be accounted for, at least in part, by a verified learning disability or difficulty, but individual PATM profiles for the remaining students indicated poor performance in the numeracy strand.

The school decided to trial the research-based *QuickSmart* (QS) mathematics intervention for the latter group as the program is designed to improve low achieving middle school students automaticity with addition, subtraction, multiplication and division over 30 weeks (Pegg, Graham, & Bellert, 2005; Graham & Pegg, 2010). Previous studies have shown QS participants gained on average two to three years progress (effect size 0.49 - 0.80) measured from PATM pre to post test scores (Pegg & Graham, 2009) and improved on measures of response speed and accuracy compared with average achieving (Bellert, 2009) or high achieving same-age non-participants (Graham, Bellert, Thomas & Pegg, 2007). The present quasi-experimental study used pairs of underperforming students in the same mathematics classrooms to compare pre and post scores on the PATM. Further, the comparisons between QS participants and the paired classmates were undertaken after the completion of the first part of the QS program on multiplication rather than at its conclusion as in the previous studies. Response automaticity and accuracy were examined for both groups prior to QS and for QS students at the completion of the multiplication section of the intervention.

AIMS

1. To investigate the QS intervention program for middle school students with poor achievement in mathematics; and
2. To compare the performance of students participating in the QS program with their paired classmates who received classroom instruction in mathematics only.

METHOD

Participants

Eight Grade 7 students and 12 Grade 8 students with PATM scores in the 1-30 percentile rank range at T1 were nominated by their mathematics teachers on the basis of their attendance, behaviour and poor performance on the PATM numeracy subscale. The 20 students were paired within their mathematics classes, with one student from each pair assigned to the QS group and the other to a control group. QS students were then grouped in pairs by Grade level for the delivery of the program. Students ranged in age from 12.3 years to 13.11 years with a median age of 13.3 years.

The *QuickSmart* Mathematics Intervention Program (Graham et al., 2007)

The multiplication section of QS was delivered to pairs of students in three 30 minute lessons per week over a mean of 16.5 hours.

Procedure

In Terms 3 and 4 a mean of 33 lessons, focussed on multiplication only, were delivered to the 10 pairs of designated QS students by a trained teacher aide, supervised by a registered teacher. These lessons were additional to their classroom instruction in mathematics. Each QS lesson consisted of 5 minute sections of a knowledge and understanding check, flashcards, speed sheet challenge of multiplication number facts, independent work sheet/strategy development, assessment and games. Response speed and accuracy was measured separately for each student in each lesson with the *Cognitive Aptitude Assessment System* (CASS) computer package (Royer, 1996) which is based on the Baddeley model of working memory (Tronsky & Royer, 2002). The CASS times student verbal responses via a microphone to randomised number sentences on a computer screen while the aide scores each response for accuracy.

Results are averaged and graphed automatically, providing each student with feedback to monitor his performance immediately and over time. QS and control students were tested with the CASS prior to the commencement of the intervention in Term 3, but thereafter students in the control group received five 40 minute mathematics lessons per week only. Numeracy achievement data from the *National Assessment Program – Literacy and Numeracy* (*NAPLAN*) administered annually in Australia to students in Grades 3, 5, 7 and 9 was available from 2013 for Grade 7 and 2012 for Grade 8.

Analyses

QS and control group students initial CASS averaged response time and accuracy scores, PATM scaled scores at T1 and T2 and NAPLAN numeracy logit scores were entered into a Statistical Package for the Social Sciences (SPSS) computer program. Speed scores were also entered for the final multiplication lesson for the QS students. The statistical analyses were conducted with nine QS students (program attrition of one boy in Grade 8) and nine control students (incomplete data for a Grade 7 boy).

RESULTS

The median accuracy score measured by the CASS for both groups was 88% prior to the intervention and 100% for QS students at the completion of the multiplication lessons. Initial CASS speed scores for QS students, shown in Figure 1, ranged from 1.77 - 4.80 seconds, with a mean of 2.78 secs and SD of 0.96. Control group scores ranged from 1.50 - 4.79 seconds with a mean of 2.64 (SD = 0.93). Analysis of Variance (ANOVA) revealed no statistically significant differences between the mean speed of the two groups before the QS began [F (1,16) <1]. However, there was a statistically significant relationship between QS students' mean speed score prior to and at the end (Mean = 1.21 secs, SD 0.34) of the multiplication lessons [F (1, 8) = 39.28, p<0.001].

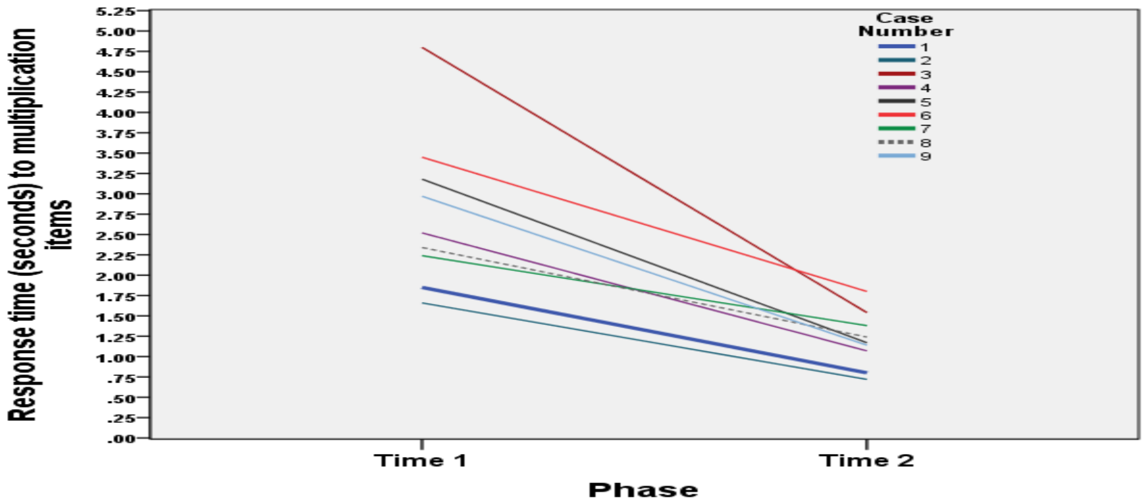

Figure 1: Speed scores for QS students prior to and after the multiplication intervention

Although students were administered different pre and post tests their scaled scores can be validly compared as PATM tests are scaled on a single interval scale of mathematics achievement through the RASCH measurement model (ACER, 2010).

The effect of prior knowledge on PATM scores and QS intervention was tested by using students' NAPLAN scores as a covariate. While NAPLAN scores predicted students' PATM score at T1, it did not serve as a covariate for the treatment effect.

The difference between the mean PATM scaled scores presented in Figure 2 for QS students at T1 of 53.2 (SD = 2.9) and 49.9 (SD = 2.6) for the controls was statistically significant ($p< 0.005$). The difference between the mean PATM scaled scores at T2 of 55.58 (SD = 3.3) for QS and 51.0 (SD = 4.0) for controls was also statistically significant ($p< 0.001$). Covariance analysis which controls for the baseline score at T1 showed the group effect remained significant [$F\ (1,15) = 6.5\ p< 0.022$], with the achievement of the QS students increasing more over time compared with the control group. Repeated measures interaction approached significance [$F(1,15)=6.5, p<0.06$].

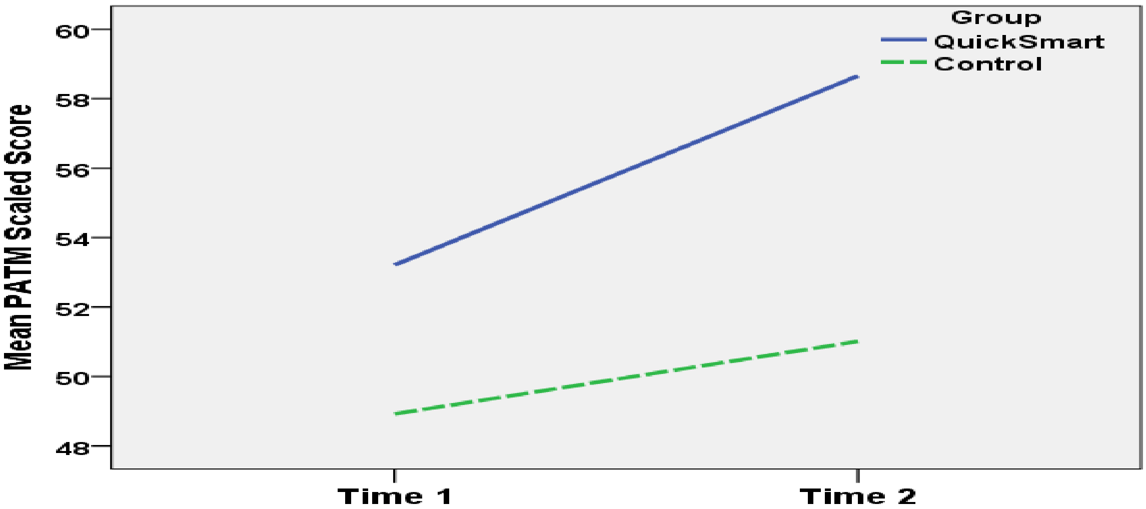

Figure 2: PATM scaled scores for QS and control students at T1 and T2

DISCUSSION

Developing automaticity in cognitive processing is a major goal in mathematics for students in the early elementary grades. Failure to acquire basic mathematics skills by the middle school grades has significant consequences for students who have to employ effortful and costly mental strategies to solve tasks that essentially require low level knowledge (Hattie & Yates, 2014). Lack of automaticity was evident for both groups of students in the initial CASS scores where their mean response latencies were reasonably accurate but slow. There was also considerable variability in the response speeds of both QS and control group students, shown in Figure 1 for the QS group. The statistically significant decrease in the mean response latency for the QS group by the end of the multiplication lessons is an important finding as it extends previous studies which have reported students to be quicker in fact retrieval and smarter at strategy use by the conclusion of the QS intervention (Bellert, 2009; Pegg & Graham, 2009).

In relation to the second aim of the study, students' performance on PATM at T1 was predicted by their numeracy achievement as measured by the NAPLAN. Both the PATM and NAPLAN numeracy test were administered under timed conditions so it is likely that lack of fluency would have influenced the performance of both groups (Arroyo, Woolf, Royer, Tai & English, 2010). Previous studies have found speed of retrieval of mathematics facts to be a significant predictor of middle school students' test performance (Royer, Tronsky, Chan, Jackson, & Marchant, 1999).

The statistically significantly difference in PATM scores between QS participants and control group at T1 is difficult to explain. However, the significant increase in the gap between the achievement of the QS and control groups at T2 is a notable finding which can be attributed to the QS intervention., There was no statistically significant difference in mean CASS response latencies between the two groups prior to the intervention, the effect of baseline performance on PATM was controlled for in the covariance analysis, QS and control students were paired within their respective Grade 7 or 8 classrooms and received the same number of mathematics lessons each week, at the same time, from the same teachers, with the same textbooks and over the same time frame. Further, although the key focus of the additional lessons received by the QS group was to improve their understanding and speedy recall of basic multiplication facts, through the rehearsal of more sophisticated and efficient strategies which foster automatic recall (Bellert, 2009), the increase in their PATM scores is also evidence of generalisation to learning in other domains of mathematics. This finding extends evidence from a previous study in which middle school students had significantly higher PATM raw scores after completing the QS program in the four basic processes (effect size = 0.65) (Bellert, 2009). The current finding also raises the interesting research question of whether it is just as efficacious to administer the multiplication section of the program only rather than in its entirety.

With respect to Aim 1 from the school perspective, the implementation of the intervention program in the middle school grades has considerable response costs associated with the purchase of the QS program, annual licensing fee, teacher aide and supervising teacher training and allocation of teaching time and space within which to operate the program. Further, QS participants have to be withdrawn from three lessons every week which affects their participation in their other subject areas. QS lessons were timetabled to occur on a Monday, Wednesday and Friday to provide the opportunity for spaced rather than massed practice (Carpenter, 2014), but it is interesting to note that over Terms 3 and 4 (of 10 and 9 weeks duration respectively) students completed only a mean of 33 QS lessons over 16.5 hours. While some of the discrepancy can be explained by the time taken with the initial CASS testing with both groups of students and student absence from school, it was noted on several occasions that opportunities for QS lessons to occur were affected by school sanctioned activities such as assemblies, excursions, sports days and other events.

These costs of implementing an intervention in the middle school grades have to be considered against the long term effects of not providing any intervention. There is considerable research evidence that students' ability to retrieve basic number facts will not improve across the elementary school years without intervention (Gersten, Jordan & Flojo, 2005). Further, speed of retrieval is a significant predictor of students' achievement on mathematics tests throughout their secondary schooling and beyond (Royer et al., 1999). While the effects of the numeracy intervention on student work samples, self efficacy, anxiety and learned helplessness in mathematics (Yates, 2009b) will be considered at the completion of the trial of the QS program in 2014, the results thus far nevertheless indicate quite strongly that significant positive changes are evident in response latencies, accuracy and achievement in mathematics when students are provided with the knowledge of and opportunities to practice more efficient and effective basic skills and strategies in a supportive small group environment with motivating feedback.

Acknowledgements

Assistance from Mrs Pamela Holmes and John Santini is gratefully acknowledged.

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VIABLE ARGUMENTS, CONCEPTUAL INSIGHTS, AND TECHNICAL HANDLES

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Findings from an empirical study on prospective elementary teachers' argument productions are reported. In order to analyse the data, a generative study was conducted to develop a framework for expressed actions that afforded the communication of viable arguments for generalizations. Identified are three types of technical handles that appear constructive in communicating viable arguments. Inappropriate and inadequate technical handles are also noted.

STATEMENT ABOUT THE FOCUS OF THIS PAPER

This research builds on and recasts previous work regarding conceptual insights (CI) and technical handles (TH) during argumentation. Raman, Sandefur, Birky, Campbell, and Somers (2009) identify *getting key ideas, discovering THs*, and *culminating the argument information into standard form* as significant "moments" important in creating a proof, in no particular order. Sandefur, Mason, Stylianides, and Watson (2013) recast the framework as "(1) finding a [CI], i.e., a sense of a structural relationship pertinent to the phenomenon of interest that indicates why the statement is likely to be true, and (2) finding some [THs], i.e., ways of manipulating or making use of the structural relations that support the conversion of the CI into acceptable proofs" (p. 328).

Based on empirical data generated by prospective elementary teachers (PSTs), who participated in a teaching experiment designed to improve argumentation skills, I found that I needed to again recast the TH framework. My objective was to describe PSTs' communicated argument as an expression of their "handling" of information, not necessarily to describe the processes PSTs go through or experience when addressing a prompt/task and developing an argument. While the concept of a CI remanded similar to that express in Sandefur et al, I found three related but distinct types of THs important to communicating a viable argument. While the offer of a new framework aligns with a theoretical essay, this article is a research report because it uses empirical data to establish that each of these handles is important to communicating a viable argument.

The presentation of these findings is intended to deepen our understanding of student actions that influence viable argument production and to promote further research in this area. My aim is to make international contacts with other researchers, who might benefit from these findings and who might wish to collaborate on future work.

THEORETICAL FRAMEWORK FOR THE RESEARCH

Viable argument defined for this study

For this study, an argument is defined as viable if it (1) expresses a clear, explicit, and unambiguous claim with explicit conditions and a conclusion, (2) expresses support for that claim that involves acceptable data (foundation), (3) expresses acceptable warrants that link the data to the claim, and (4) identifies the mathematics on which the claim relies.

Criterion 2, acceptable data/foundations, is met by expressing/representing information or insights in a manner that (a) illustrates the conditions in the claim, (b) can be used to represent all cases in the domain of the claim, and (c) can be appealed to when connecting the data to the claim. Acceptable data can include examples, diagrams, prior results, axioms, definitions, narrative descriptions, stories, etc. provided that the three criteria are met.

The connection between the data and claim referred to in acceptable data criterion (c) is the warrant and is deemed "acceptable," viable argument criterion 3, when it expresses how the data is used to support the claim and takes into account all cases to which the claim applies. Using purely empirical warrants to support a generalization that applies to an infinite set is not acceptable. Using logical necessity, referencing prior results, and describing through narrative how the semantic meaning of mathematical objects support the claim can all be acceptable warrants.

Criterion 4 is defined as the meaning of the objects and operations involved in the claim and those meanings are determined by definitions, axioms, and theorems. Viable arguments must express these meanings, at least semantically, for the argument to be viable.

This framework for viable argument was developed by the author from existing frameworks found in Toulmin's (1958/2003) argument analysis scheme and the Common Core State Standards' (CCSS-M, 2010) description of the mathematical practice number 3, construct viable arguments and critique the arguments of others. Consistent with Toulmin, an argument is defined as a claim and its support. Other argument features in Toulmin's scheme include data that provide facts and other information that support the claim, warrants that link data to a claim, backings that give support for why a warrant should be accepted, qualifiers that give the strength of a claim, and rebuttals that give circumstances, cases, or facts under which the claim would not be true. For this study, I focus my attention on claims, data, and warrant, which Krummheuer (1995) calls the core of the argument, and I often combine warrant and backing into a single category labelled warrant, unless the arguer is clearly addressing the acceptability of an existing warrant.

Consistent with CCSS-M (2010), viable arguments can use referents such as objects, drawings, diagrams, actions, and in this study, examples, as the data/foundation of a viable argument. The term viable argument is not explicitly defined in CCSSM (2010), but I make the assumption that this term is used instead of "proof" to note that there are plausible argument types other than formal, mathematically logical ones, and to draw distinctions between proofs and non-proofs. Consequently, my framework for viable argumentation leverages notions of less-than-formal mathematical arguments (e.g., Balacheff's, 1988, notion of generic example proofs), which some authors have called proofs, but I classify them as viable arguments in an effort to value them but not confuse them with mathematical rigorous arguments that are explicit about the logic and prior results used.

Conceptual insights and technical handles

For this study, the term conceptual insights (CI) can refer to any one of the following: developing a sense or belief based in pertinent mathematical structure that a claim is true or false, developing a sense or belief based in pertinent mathematical structure about what might be claimed (stated as true), or developing a sense or belief based in pertinent mathematical structure about why a claim is true or false or what causes the claim to be true or false.

As mentioned earlier, I found it necessary to recast earlier descriptions of THs (e.g., Raman et al, 2009 & Sandefur et al, 2013) to describe my data. My analysis only addresses the articulated argument, not what the arguer intended to write or say. Nor does my analysis attempt to document the processes through which the ideas expressed in the argument are generated. This focus helped me draw clearer distinctions between CI and THs and offered a purer description of what the arguer was able to express appropriately and viably when culminating ideas, findings, and insights into an argument product. From this narrowed focus, I was able to identify three distinct but related "handles." Technical handles of type 1 (TH1) describe the articulated claim in relation to the expressed data, CIs, or warrants. Technical handles of type 2 (TH2) describe how the data or CIs are expressed. Technical handles of type 3 (TH3) describe the expressed link between the data or CIs and the claim (i.e., warrant). THs are first described without any connotation of whether or not the handles are constructive. Adjectives (e.g., appropriate, inappropriate, adequate, inadequate) are applied to note a TH's subtype and potential for viable argumentation. This framework will be further exemplified in the results section.

DATA SOURCES AND JUSTIFICATION FOR ANALYSIS METHODS

Twenty-one PSTs enrolled in a undergraduate mathematics content course for elementary school teachers participated in the study. PSTs were given the definition of viable argument as presented in Section 2 and received two months of training and practice using the definition to construct and critique arguments. Five sources of data were collected and used in this study: (1) students' weekly posts in the online environment (2) teacher/researcher observations during inclass work (3) student writen responses to inclass tasks (4) task-based, clinical interviews, audio taped and transcribed, and (5) responses on paper-and-pencil assessments.

For the online posts, students were instructed that to receive full credit, they must initiate a discussion by making a claim or respond to another student's post. At minimum, a post needed to present at least one argument feature (a claim, data for a existing claim, a warrant, a backing for an existing warrant, a qualifier, or a rebuttal), but they were also told that this was a minimal criterion and that a complete argument was desired in each post. This concept of collective argumentation has been discussed in Krummheur (1995), Lampert (1992), and Yopp and Ely (under review), to name a few.

Data was analysed from the perspective of a *generative study* (Clements, 1999) to develop or recast an existing theoretical model to explain data. Qualitative analysis methods were akin to those described by Miles and Huberman (1994) in which the analyst begins with a theoretical coding framework that is constantly compared to the data until a model that fits the data emerges. A cyclic process of analysis, refinement, and reanalysis was used to test emerging the framework and outcome of the data analysis (similar to the methods described in Goldin, 1999, & Sandefur et al., 2013). As conceptual themes emerged, the themes were verified through triangulation with multiple data sources. For example, as student posts were memoed, emerging themes were triangulated with task-based interview data to confirm TH codes.

SAMPLE OF DATA AND RESULTS

A sample of results are reported for data collected from one of the online tasks:

Task 1: You are teaching a 6th grade class. You ask the class to investigate the sums of consecutive numbers and develop some rules about the types of numbers that are sums of 2, 3, 4, and 5 numbers. After some set time, three students offer rules. Sally says that the sum of two consecutive numbers is odd. Sophia says that the sum of three consecutive numbers is divisible by 3. Isabella says, "I think that the sum of four consecutive numbers is divisible by 4". Write exemplary responses, which include viable arguments.

Adequate and inadequate THs

In this thread we find the same claim coded as an inappropriate TH1 in one post yet coded as appropriate TH1 in another post. We also find a follow-up post in which the arguer expresses appropriate or adequate THs for all three argument features and presents a viable argument.

Charli: Data for Claim #3: My example of the claim: 1+2+3+4=10 or 4+5+6+7=22 supports the claim that says: "The sum of four consecutive counting numbers is not divisible by four" because shown here proves that these consecutive counting numbers are not divisible by four.

Alex: Warrant for Claim 3: The sum of four consecutive counting numbers will not be divisible by four, because there is one multiple of four, but the other numbers do not add to equal a multiple of four. 1+2+3+4= 10[.] 4+6, only the four is divisible by four, not the six[.] 5+6+7+8=26[.] 8+18, only the eight is divisible by 4[.]

Franni: I think that a better claim for this problem would be that for any 4 consecutive numbers added together, the sum will always be divisible by 2. The data supporting this would be: 2+3+4+5=14 which is divisible by 2, 10+11+12+13=46 which is divisible by 2, and 23+24+25+26=98 which is divisible by 2. The warrent [sic] would be: For any 4 consecutive numbers, N, N+1, N+2, N+3 [data] added together, the sum will always be divisible by 2 because when added you have N+(N+1)+(N+2)+(N+3) which can be written as 4N + 6. 4N is divisible by 2 and 6 is divisible by 2 so no matter what number N is, the answer will always be divisible by 2.

In the first post in this thread, Charli labels her post "data" but has presented an argument because both a claim and support are present. She presents empirical support and no CI about the generalization that no sum of four consecutives is divisible by 4. Although her claim is true, it is not appropriate with respect to the data. Charli's use of the word "proves" raises concerns of empiricism. However, follow-up cognitive interviews revealed that Charli wished to express "generality" in her warrant.

I mention "empiricism" and Charli's perspective to again to draw attention to the focus of this paper. The approach here is to analyse what students are able to articulate against a class standard for viable argument, not to analyse their views about adequate support. While it has been established in the literature that a PSTs' beliefs about what constitutes adequate support for a generalization can influence their argument production and their ratings of arguments (Martin & Harel, 1989; Sylianides, G.J. & Stylianides, A.J., 2009), other literature suggests that students who know the limitations of empirical evidence produce empirical evidence when they aren't able to produce better (e.g., more general) arguments (Sylianides, A.J. & Stylianides, G.J., 2009). There is a gap in the literature concerning the skill of argument production as separate from beliefs. The focus of this paper is to examine the argument product from a "handling" point of view as a technical skill, rather than an exploration of what a PST finds convincing. A PST with the appropriate TH training might express a more appropriate TH1 by stating "Based on the examples we can claim that at least some of the sums are not divisible by 4." With this change, Charli's argument would be viable.

This *skill* of making appropriately claims might be unique from the PST's views about sufficient support for claims. This makes this framework unique from the previous studies that gave their participants a claim, typically assumed true, and examined the participants' support for that claim. In this framework, the truth of the claim made by the PST is not the focus. Because PSTs must develop a claim, the focus is the claim's appropriateness relative the data presented.

In contrast to the inappropriate TH1 expressed by Charli, Alex, who responded to Charli's post and wrote basically the same claim, expresses an appropriate TH1. This is because Alex expresses a CI that applies, at least in her mind, to all cases. Her claim is appropriate relative to the general nature of her insight, regardless of whether the insight pans out as adequate support for the claim. Whether or not her insight has potential for an adequate representation (which it does) is not the point when assigning a TH1 code. This code refers only to whether or not the formation of the claim is prudent, and whether or not it is appropriately worded, expressing both the conditions/domain and the conclusion.

Yet, despite Alex's appropriately worded claim and interesting insight, Alex does not handle her insight adequately to communicate a viable argument or a viable argument features. It is noted that Alex labels her post as warrant, but it is also noted that warrants connect data to a claim. By default data must be present for a warrant to exist, either in the current post or a previous post. Thus, an argument is present in Alex's post.

There are at least two ways we might reconstruct the argument. One way is label the Alex's examples and observations about these examples as data and label Alex's statement "because there is one multiple of four, but the other numbers do not add to equal a multiple of four" as the warrant. Another way is to label the "because…" statement as both the data and implicit warrant and the examples as backing for the truth of the data and warrant. (A criticism of Toulmin's analysis is that multiple plausible arguments may be constructed, Aberdein, 2005.) Regardless of how we reconstruct the argument, the CI in the "because" statement is not expressed in a manner that can be appealed to generically. The expression or representation does not reveal how we know all sums of four consecutives have the two mentioned properties: that one summand is a multiple of 4 and the others do not sum to a multiple of 4. Consequently, Alex expresses an inadequate TH2.

Alex need not use a variable to represent these insights. Examples can be useful when crafting the arguments (Balacheff, 1988, Sandefur et al., 2013) and their concreteness makes them particularly accessible to PSTs. An example becomes a referent in an argument if the arguer uses the example to illustrate objects or relationships when supporting a claim. Despite their utility, examples are at times used in ways that do not lead themselves to viable argumentation (Balacheff, 1988; Healy & Hoyles, 2000).

Using examples to construct viable arguments requires that the pre-service teachers' develop CIs and are able to handle the data or insight in appropriate and useful ways (Raman et al., 2009; Sandefur et al., 2013). For a generic example argument, Alex might need several examples to argue all cases (e.g., the multiple of 4 as the first, second, third, or fourth summand). Alex might even develop "sub-arguments" to establishes each of the two properties separately. Never-the-less, the condition "sum of 4 consecutives" must be represented adequately for viable argumentation

In contrast, Franni communicates a viable argument in her response to Alex's post. Franni expresses an appropriate TH1 and adequate TH2 when she expresses a claim about divisibility by 2 and supports this claim using a variable to represent the claim's conditions as N+(N+1)+(N+2)+(N+3). The adequacy of this representation for viable argumentation is affirmed when Franni appeals to its equivalent form 4N+6 and notes that both 4N and 6 are divisible by 2, expressing an adequate TH3. Admittedly, there are parts of the argument that could be improved: there are a few typos and the arguer should explicitly mention the "on the list" prior result that *if both summands are divisible by an integer, then sum is as well*, but the former issue is easily fixed through editing and the later is an issue of community norms about sufficient detail.

Franni offers another opportunity to reflect on the utility of the focus on THs independently from a PST's views about types of argument structure and sufficient support for generalizations. Franni presents several examples that she calls data prior to presenting her generic representation, which she calls the warrant. A cognitive interview revealed that Franni, like her group members, viewed the term "data" as numerical information as in a scientific experiment. This view was different from the way "data" was defined and used in class. Thus, despite her non-canonical expression of the argument structure, globally her data and CI are handled sufficiently well for viable argumentation.

CONCLUSION

The purpose of this work is to establish three types of THs as important for communicating a viable argument in response to a claim. In order to communicate a viable argument, as defined in this study and for this particular community, PSTs need to (1) communicate a claim that is appropriately worded, expressing the conditions and conclusion, and is prudent based on their data or CI; (2) express their data or CI in a manner that can be appealed to appropriate in a warrant that connects the data to the claim; and (3) express a warrant (and possibly backing) that appeals to the data, indicates how all cases in the domain of the claim are considered/expressed, and identifies the mathematics on which the claim relies.

This work is unique from previous work because it focuses solely on what PSTs are able to communicate, not on what PSTs might intend to communicate, and because it demonstrates three distinct, although related, THs. An arguer may adequately or appropriate express any one of the handles without doing so with respect to the other two. This separation allows researchers and educators to focus on students' abilities and deficits during argumentation as technical skills (e.g., given this type of data, examples, or counterexamples, what types of claims are appropriate?).

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DEVELOPING YOUNG CHILDREN'S UNDERSTANDING OF PLACE-VALUE USING MULTIPLICATION AND QUOTITIVE DIVISION

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This paper focuses on selected findings from a study that explored the use of multiplication and division with 34 five- and six-year-old children from diverse cultural and linguistic backgrounds. The focus of instructional tasks was on working with groups of ten to support the understanding of place value. Findings from relevant assessment tasks and children's work highlighted the importance of encouraging young children to move from unitary (counting by ones) to tens-structured thinking.

BACKGROUND

This study has emerged from the findings from both the national numeracy results (Young-Loveridge, 2010) and recent international results (May, 2013). The latest results of the Programme for International Student Assessment (PISA) (a study that assesses and compares how well countries are preparing their 15-year-olds to meet real-life opportunities and challenges) showed that New Zealand's average scores in mathematics have declined since 2009. Compared to countries with a similar average score, New Zealand has a larger proportion of students who can complete only relatively basic mathematical tasks (below Level 2), as well as students who are capable of advanced mathematical thinking and reasoning (Level 5 and above).

Mathematics reform over the past few decades has led to the development of frameworks outlining progressions in number as students acquire increasingly sophisticated ways of thinking and reasoning (Bobis, Clarke, Clarke, Thomas, Wright, Young-Loveridge, & Gould, 2005). Typically, at the lower stages, students solve problems by using counting strategies. As they come to appreciate additive composition, they are able to use strategies that involve partitioning and recombining quantities (part-whole thinking). The initial focus with younger children is often on addition and subtraction before introducing other domains such as multiplication and division, and proportional reasoning. Students are thought to need particular number knowledge in order to apply strategies for solving problems (Ministry of Education, 2008). Such knowledge includes number-word sequences, basic facts, and place value.

Our numeration or place-value system is characterised by four key properties: positional, base-ten, multiplicative, and additive (Ross, 1989). Place value is considered to be an essential foundational concept in mathematics. Fuson, Smith, and Cicero (1997) present a model of two-digit conceptions arranged developmentally, from unitary (count by ones) through an understanding of the ten-based structure, to a multi-unit conception. The shift from unitary counting to these higher stages involves

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 409-416. Vancouver, Canada: PME.

developing an understanding of part-whole relationships. Thompson (2000) argues that place value is too sophisticated for many young children to grasp and this idea is supported by evidence showing that young children have difficulty understanding the place-value system (Kamii, 1988; Ross, 1989). However, others have shown that with carefully planned learning experiences, first grade students can learn the beginnings of place value structure (e.g., Kari & Anderson, 2003; van de Walle, Karp, Lovin, & Bay-Williams, 2014). Mulligan and Mitchelmore's (2009) innovative work on promoting awareness of pattern and structure is consistent with this approach. Grouping and partitioning activities can lay the foundations for developing place value, beginning with tens and ones and extending beyond two digits. Partitioning small numbers, composing wholes from parts, rearranging parts while recognising that the quantity of the whole has not changed, all contribute to developing an understanding of place value and part-whole relationships (Ross, 1989).

It is important for students to develop both counting-based and collections-based approaches to working with numbers (Yackel, 2001). Yang and Cobb (1995, p. 10) have highlighted "an inherent contradiction" in the way that Western children are initially encouraged to count by ones and thus construct unitary *counting-based* number concepts, but are then expected to reorganise these into *collections-based* concepts involving units of ten and units of one when place-value instruction begins. Yang and Cobb contrast the Western counting-based view with the collections-based approach of Chinese mothers and teachers, who emphasize groups (units) of ten. The difference in emphasis on counting versus grouping by tens helps to explain Yang and Cobb's (1995) finding of more advanced mathematical understanding by the Chinese children relative to that of the American children.

The challenge of learning about place value is evident when students in the middle grades show limited understandings of two-digit numbers (Ross, 1989). Language factors have also been shown to influence place value understandings in different cultures. Asian language speakers, for example, have been shown to have a better understanding of place value than English language speakers. The irregularities and inconsistencies in the English language (e.g., '-teen' & '-ty' numbers) contrast with the transparent patterns found in most Asian languages, and research shows more advanced development of place-value understanding in Asian children (Miura, Okamoto, Kim, Steere, & Fayol, 1993).

Although many western mathematics curricula introduce place value before multiplication and division, it has been suggested that multiplication and division provide an important conceptual foundation for understanding place value (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Ross, 1989; van de Walle et al., 2014). Carpenter et al., emphasize the particular importance of quotitive (measurement or repeated subtraction) division problems that require objects to be collected into groups of ten to help develop the base-ten concept.

Place value is a key aspect of number sense, defined as the "understanding of number and operations along with the ability and inclination to use this understanding in flexible ways" (McIntosh, Reys & Reys, 1992, p. 3). A framework for the development of number sense outlined by McIntosh et al., includes knowledge and facility with numbers. Within this component there are four key aspects: a sense of orderliness of numbers (patterns and regularities); multiple representations for numbers (symbols &/or graphical representations); a sense of relative and absolute magnitude of numbers; and a system of benchmarks. Place value is a component of the sense of orderliness of numbers. This framework positions place-value understanding within a broader context and highlights its importance for children learning to engage in mathematical thinking.

The project described here set out to explore the impact of using multiplication and division contexts with five- and six-year-olds on their emerging understandings of number, including part-whole relationships and place value.

THE STUDY

This study was set in an urban school (medium SES) in New Zealand. The participants were 34 five- and six-year-olds (17 girls & 17 boys) in two classes, one designated as Year 1 and the other Year 2. The average age of the students was 6.2 years at the beginning of the study (range 5.6 to 6.9 years). The children were from a diverse range of ethnic backgrounds, with approximately one third of European ancestry, one third Māori (the indigenous people of New Zealand), and other ethnicities including Asian, African, and Pasifika (Pacific Islands people). One third of the children had been identified as English Language Learners [ELL]. At the start of the study, the children were assessed individually using a diagnostic task-based interview designed to explore their number knowledge and problem-solving strategies (April). The assessment interview was completed again after each of the two four-week teaching blocks (June and November). The assessment tasks included: addition, subtraction, multiplication, division, basic facts, incrementing in tens, counting sequences, and place value.

Teaching using Multiplication and Division Contexts

Two series of 12 focused lessons were taught; the first phase was in May and the second in October. In these lessons the children were introduced to groups of two, using familiar contexts such as pairs of socks, shoes, gumboots, jandals, and mittens. Multiplication was introduced using simple word problems, such as:

> Kiri, Sam, and Len each get 2 socks from the bag. How many socks do the 3 children have altogether?

Once children were familiar with working with groups of two, groups of five were introduced using contexts such as gloves focusing on the number of fingers on each glove, and five candles on a cake. The next objective was to introduce groups of ten. For this the context of filling cartons with eggs was introduced with cartons that held exactly ten eggs. Although the emphasis of the study was on multiplication and division, the focus in this paper is specifically on the quotitive division problems, making groups of ten, and considering leftover ones.

A typical problem was:

There are 23 eggs. Each carton holds 10 eggs. How many full cartons are there?

Later problems were posed so that children could self-select numbers, including generating their own 'mystery number' inside the empty brackets.

There are 27 [76] [ ] chocs. Each box holds 10 chocs. How many full boxes are there?

Lesson Structure

A typical lesson began with all students completing a problem together on the mat, using materials to support the modelling process, and sharing ways of finding a solution. The teacher recorded children's problem-solving processes (including use of manipulatives) and discussion in a large scrapbook ('modelling book'). The problem for the day was already written in the book and both drawings and number sentences were recorded, acknowledging individual children's contributions. The children then completed a problem in their own project books, choosing a similar or larger number, and/or selecting a new number. Materials (egg cartons and unifix cubes) were made available and children were encouraged to show their thinking using representations and to record matching equations.

RESULTS

Children's performance on the tasks was examined to look for patterns and progressions. A tens-structure sub-score was calculated using students' responses on 24 tasks related to working with groups of ten (e.g., 60 sticks for grouping into tens; known facts such as 20+7, 10+8, 10+10, 2x10, 60÷10, 80÷10, 200÷10, 23÷10, half of 20; $10 notes in $80, $240; the meaning of "2" in "25"; incrementing in tens such as 15 and 10, 42 and 30; and producing quantities using groups of ten). Children's responses to addition, subtraction, and multiplication problems were weighted according to the sophistication of strategies (counting all = 1, counting on/back or in multiples = 2, known & derived facts = 3).

Children's tens-structure sub-scores ranged from 0 to 24 (1 per task). A comparison of the top 20% of the distribution (n=6) with the bottom 20% (n=6) showed a marked difference in their knowledge of number and relationships, key ideas for tens-structured thinking. The lowest performers were able to complete no more than one task, whereas the top performers completed between 15 and 24 tasks successfully. These children with stronger baseline knowledge of facts, number –word sequences, and counting strategies (e.g., counting on/back), progressed to skip counting and using known or derived facts. They were fluent with incrementing by tens (adding), and also working with multiples of ten (multiplying & dividing). These six children were able to recognise that six groups of ten could be made from 60 objects, justifying their responses by referring to the tens digit.

Children's performance improved on many of the tasks related to tens-structured thinking. For example, when shown an array of 30 cakes in three rows of ten and asked how many cakes altogether, the majority of children (85%) could work out the answer by the end of the project, an increase from 32%. Approximately one-quarter (24%) of these students used known or derived facts, and more than half (56%) used skip counting. More than half of the children were able to combine a multiple of ten (a '-teen' or a '-ty' number) with a single-digit quantity without using a counting strategy. For example, 62 per cent knew 20 + 7 = 27 and 53% knew that 10 + 8 = 18. Children were shown a bag of 60 sticks (labelled with its total) and a bundle of ten sticks, and asked how many bundles could be made from the bag of sticks. Not quite half (44%) of the children were able to work out the answer by looking at the number '60'. More than half of the children (59%) knew the number of $10 notes needed to buy an $80 toy, up from 12 per cent initially. Almost one third (32%) were able to work out the $10 notes needed for an item costing $240. When similar tasks were presented using symbolic expressions as known facts, fewer were able to respond correctly (e.g., 32% knew 60÷10 and 80÷10, while 24% knew 200÷10). One of the most difficult tasks was showing the meaning of the '2' in '25' for a picture of 25 blocks where the two groups of ten were linked. Only nine children (29%) circled the two groups of ten blocks rather than two single blocks. Table 1 presents the inter-correlations for responses to selected tasks at the start and end of the project.

	Nov Add/Sub	Nov Mult	Nov Div	Nov Facts	Nov Tens	Apr A/S	Apr Mult	Apr Div
Nov Mult	0.46							
Nov Div	0.71	0.36						
Nov Facts	0.75	0.66	0.72					
Nov Tens	0.74	0.54	0.75	0.96				
Apr A/S	0.80	0.63	0.68	0.75	0.72			
Apr Mult	0.70	0.50	0.55	0.67	0.59	0.73		
Apr Div	0.65	0.52	0.65	0.71	0.69	0.62	0.61	
Apr Facts	0.83	0.63	0.73	0.86	0.83	0.81	0.73	0.69

Table 1: Correlations among tasks including tens-structured tasks (Nov Tens)

At the start of the study, knowledge of known facts was strongly related to the score on addition and subtraction problem solving ($r = 0.81$). A similar relationship was found for solutions to multiplication ($r = 0.73$), and division problems ($r = 0.69$). Correlations from the start to the end of the project (inside the bordered region) indicate that knowledge of known facts were most predictive of subsequent strategies for addition and subtraction (0.83), and division (0.73). They also predicted the measure of tens-structured awareness (0.83).

Children's representations from both an assessment task and the project books provide evidence that they can represent 2-digit numbers as groups of ten and ones. In the assessment interview, two-thirds (68%) of the children drew accurate diagrams representing 23 eggs in cartons of ten. In the children's individual project books, they constructed their own ways of showing their thinking. However, they were encouraged to use a ten-frame as a representation for quotitive division problems using egg cartons. Figure 1 shows how Nisha solved the following problem:

There are 59 eggs. Each carton holds 10 eggs. How many full cartons are there?

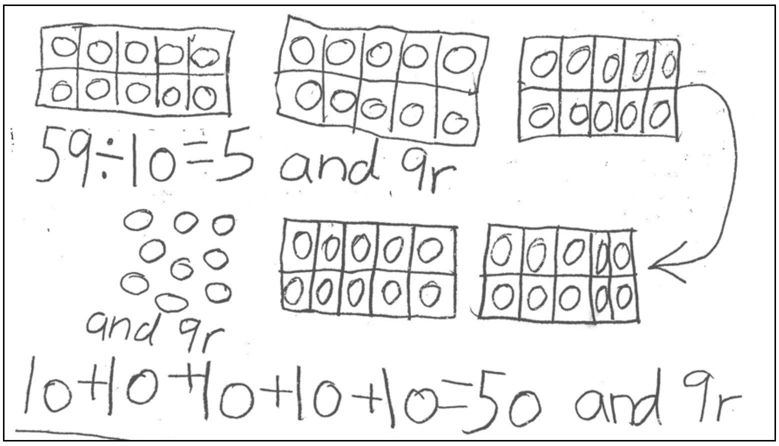

Figure 1: Student's representation of division word problem

Nisha's work shows her clear understanding of groups of ten displayed in ten-frames and recognition of nine ones units as a remainder (9r). Some children referred to these as 'leftovers' whereas others readily adopted the convention of recording this as 'r'. Her second equation was in response to children being asked to justify their solution. In this instance, Nisha has chosen to use an addition equation to show her thinking rather than multiplication.

DISCUSSION

The findings of this study show that even children as young as five and six years of age are able to work with multiplication and division problems, as well as place-value tasks. This is consistent with the work of researchers advocating for the introduction of place-value in the early years (e.g., Fuson et al., 1997; Kari & Anderson, 2003; van de Walle et al., 2014). This provides evidence contrasting with Thompson's (2000) caution about introducing place value to young children, and also challenges international curricula that introduce place value before multiplication and division.

Children were provided with opportunities to solve problems using different contexts (e.g., egg cartons and chocolate boxes) and manipulatives (e.g., unifix cubes). They were also encouraged to use representations such as ten-frames to show their thinking. This enabled children to represent groups of ten as composite units. They were also supported in recording their solutions as equations. The language of place value (e.g., '-teen' and '-ty' numbers) was challenging for the children, consistent with research findings (e.g., (Miura et al., 1993; Yang & Cobb, 1995). One-third of the sample was composed of English language learners. However, it is difficult to know whether these children were advantaged or disadvantaged in learning about place value. Interestingly, three of the top six performers were ELLs whose first languages have transparent tens-structure.

The introduction of multiplication and division prior to formal place-value instruction was beneficial to the students, not just in understanding multiplication and division, but also in developing place-value understanding. The children's knowledge of tens-structure reflected in their recall of known facts and working with groups of ten, is consistent with Yang and Cobb's (1995) argument about the need to move from a counting-based to a collections-based approach for place-value understanding. This assists the transition from counting strategies to part-whole thinking. Providing opportunities to work with 2-digit numbers meant that several children self selected larger numbers (3-digit) for their word problems, moving well beyond expectations at this level (Ministry of Education, 2009).

The early recognition of the underlying patterns and structure of groups of ten in 2-digit numbers provided a foundation from which some children were able to abstract and generalise to larger numbers (Mulligan, 2010; Mulligan & Mitchelmore, 2009; McIntosh et al., 1992). The fact that only a few children were able to demonstrate the meaning of the '2' in '25' indicates the challenge of building a sound understanding of place value. However, this exploratory study has shown that learning experiences using multiplication and quotitive division problems contributes to the development of place-value understanding.

Acknowledgement

This project was made possible by funding from the Teaching and Learning Research Initiative [TLRI] through the New Zealand Council for Educational Research and the interest and support of the teachers and children involved in the project.

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ACTIVITIES THAT MATHEMATICS MAJORS USE TO BRIDGE THE GAP BETWEEN INFORMAL ARGUMENTS AND PROOFS

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In this paper we examine a commonly suggested proof construction strategy from the mathematics education literature—that students first produce an informal argument and then use this as a basis for constructing a formal proof. The work of students who produce such informal arguments during proving activities was analyzed to distill three activities that contribute to students' successful translation of informal arguments into formal proofs. These are elaboration, syntactification, and rewarranting. We analyze how attempting to engage in these activities relates to success with proof construction. Additionally, we discuss how each individual activity contributes to the translation of an informal argument into a formal proof.

INTRODUCTION

Proving is central to mathematical practice. Unfortunately, numerous studies have documented that mathematics majors struggle with proof construction tasks and have documented numerous causes for these difficulties (see Weber, 2003, for a review of some difficulties). However, research on how mathematics majors can or should successfully write proofs has been comparatively sparse. In this paper, we examine one suggestion from the literature—that students base their formal proofs on informal arguments (e.g., Garuti, Boero, & Lamut, 1998; Raman, 2003; Weber & Alcock, 2004).

THEORETICAL PERSPECTIVE

Basing proofs on informal arguments

Boero (1999) observed that a proof must satisfy certain formal constraints, but the reasoning used to generate this proof need not. In particular, the informal arguments that one uses to understand why a proposition is true can be used as a basis for constructing a proof of this proposition (e.g., Bartollini-Bussi et al., 2007; Garuti, Boero, & Lamut, 1998). A number of researchers have advocated that students base their proofs on informal arguments. This is a driving force behind the research program of the Italian school whose proponents endorse proofs having a cognitive unity where, under particular circumstances, there is a continuum between a student's production of a conjecture and how the student proves it (e.g., Garuti, Boero, & Lamut, 1998; Pedemonte, 2007). Support for these recommendations typically comes from the analysis of students successfully basing proofs off of informal arguments (e.g., Garuti, Boero, & Lamut, 1998) and that this is common in authentic mathematical practice (Raman, 2003).

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 417-424. Vancouver, Canada: PME.

To distinguish between an informal argument and a proof in advanced mathematics, we follow Stylianides (2007) who proposed assessing whether an argument is a proof along three criteria: (i) *the representation system* (as opposed to proofs, informal arguments may be expressed in terms of graphs or imprecise language), (ii) *the facts that are taken as the starting points* (in proofs, unjustified statements must be accepted by the mathematical community as true whereas in informal arguments, the individual only needs to believe they are true), and (iii*) inference methods* (the methods employed in a proof must be considered valid by one's mathematical community, whereas in an argument the methods of inference must merely be *plausible* to the individual).

Research on bridging the gap between argumentation and proof

In recent years, researchers concerned about the gap between informal arguments and proofs have begun to look at how this distance is traversed. Much of the research can be divided into two categories: analyzing the types of arguments that are easier to translate into proofs and designing classroom environments that help bridge this gap.

In the first category, researchers such as Pedemonte have conceptualized the *distance* between the informal arguments and the corresponding formal proofs (e.g., Pedemonte, 2002, 2007). Pedemonte observed that if the general method of inference (structural distance) or the mathematical ideas (content distance) used in an informal argument and the corresponding proof differ greatly, students will face difficulties in writing the proof (e.g., Pedemonte, 2002, 2007). The second category of studies examines instructor roles in helping students build proofs of informal arguments. This includes research on creating instructional environments (e.g., Bartollini-Bussi et al., 2007) and teacher moves that may facilitate this behavior (e.g., Stylianides, 2007).

In this paper we explore how mathematics majors bridge the gap between informal arguments and proofs by addressing the following two questions: (1) What activities do mathematics majors engage in when they successfully write a proof based on an informal argument? (2) To what extent can these activities account for their success? The answer to these questions can inform instruction by highlighting what skills and practices students need to learn to write proofs based on informal arguments.

METHODS

We recruited 73 mathematics majors from a large public university in the United States who had recently completed a second linear algebra course. Participants met individually with an interviewer for two sessions, each session lasting approximately 90 minutes. In one session, the participants worked on 7 proof construction tasks in linear algebra; in the other, they completed 7 proof construction tasks in calculus. In each session, participants were presented with proving tasks that could be approached either syntactically or semantically (in the sense of Weber & Alcock, 2004). Participants were asked to "think aloud" as they completed each task, given 15 minutes per task and told to write up a proof as if they were submitting it on a course exam. This corpus yielded a total of 1022 proof attempts (73×14).

We coded a participant's argument as informal whenever it was a multi-inference argument where at least one of the inferences was drawn from the appearance of a graph or a diagram, or the inspection of a specific example. There were 37 informal arguments of this type in our data set. In this paper, we focus on these arguments, and how students attempted to translate these arguments into proofs.

ANALYSIS

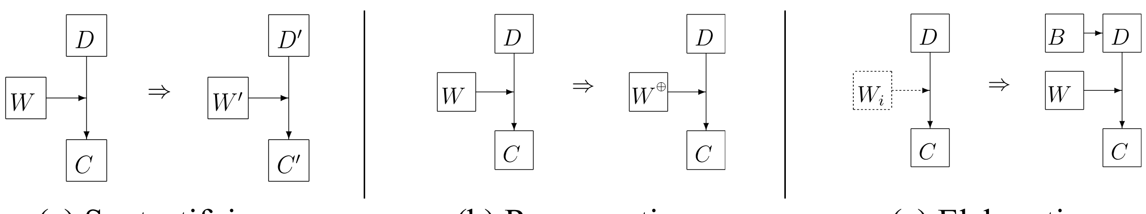
Two research assistants, who are not authors of this paper, coded each proof as valid or invalid. There was 96% agreement on their codings across the data set. Among the 37 proof attempts considered, 14 were coded as valid and 23 were coded as invalid.

Following Pedemonte (2007), we used the basic Toulmin (2003) scheme to analyze each inference that the participant drew in his or her informal argument and final proof. According to the basic Toulmin (2003) scheme, each inference (or sub-argument) contains three parts, the claim (C) being advanced, the data (D) used to support the claim, and the warrant (W) that dictates how the claim follows from the data. In many cases, a warrant was not explicitly stated. In these cases, if possible, we would infer the warrant that the participant was using. This allowed us to notice differences between the participant's initial informal argument and their final proof.

For the 14 successful proof attempts, we used an open coding scheme in the style of Strauss and Corbin (1990) to categorize the ways that the mathematics majors attempted to transform their argument into a proof. This process yielded three categories of activity: *syntactifying, rewarranting*, and *elaborating*. Once these categories were created and defined, we then went through each of the 37 proof attempts, seeking out evidence of participants' attempts to engage in these activities.

RESULTS

In this section, we describe syntactifying, rewarranting, and elaborating, which we illustrate graphically using Toulmin's scheme in Figure 1.



(a) Syntactifying (b) Rewarranting (c) Elaborating

Figure 1: Three translation activities

Syntactifying

Syntactifying occurred when a participant attempted to take a statement in the informal argument that is given in what are perceived to be non-rigorous terms and translate it into what is considered to be a more appropriate representation system for proofs. Such actions included removing references to a diagram used in the informal argument and replacing them with more conventional mathematical terminology, or introducing algebraic or logical notation. In terms of Toulmin's scheme, we can regard syntactifying as translating the data (D), claim (C), and/or warrant (W) of an argument into new data (D'), claim (C'), and/or warrant (W') in another representation system, without intending to change the meaning of D, C, or W. We illustrate this with Figure 1a. The following informal argument occurred in student A's work on when proving the derivative of a differentiable even function is odd.

> Student A: Okay, Like okay, since it's symmetric about the *y*-axis, so it's like a mirror and then all the tangent lines, all the derivatives would be like some values [pointing at negative side of x^2 graph] and then this would just, since it's a mirror would be the negative of them [pointing at positive side of x^2 graph]. So it would be odd.

In the above excerpt Student A argues that since even functions are symmetric about the *y*-axis (D) the *y*-axis acts like a mirror (C). This mirror property is then used as data to justify that $-f'(a) = f'(-a)$ for all a. The warrants used are implicit and perceived visually from the graph of x^2, which is used as an example of a generic even function. Later, Student A syntactifies parts of this argument when she shifts away from discussing tangents in terms of the graph.

> Student A: How do I put that into words? [...] This is what we want *f* prime of negative *x* equals negative *f* prime of *x*. [writes *f'(-x)=-f'(x)*]. Okay, so if we take the derivative at negative, this would be the negative of *f* of *x*'s derivative, which makes sense. So how do we get from *f* of negative *x* equals *f* of *x* [writes *f(-x)=f(x)*]. Use the definition? Okay lets try that. *f* prime *x*. So by the definition of derivative its like as this approaches this point then the tan line of that. This is the limit at *a*. Either way, *f* of *x* minus *f* of *a*. over *x* minus *a*. [writes $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$].

Student A first syntactifies the end points of the argument. She begins with the conclusion, stating that she is trying to show that *f'(-x)=-f'(x)*. This is a syntactification of her claim that the derivatives on one side are the negation of the other. She then syntactifies the initial data when she uses the analytic definition of odd (*f(-x)=f(x)*) to replace the graphical definition used in her informal argument. Although the chain rule can be used to warrant going directly from the data to the claim, she instead begins to build a proof based on her informal argument. Student A's use of the definition of limit at a point can be seen as a syntactification of the tangent part of her argument, since the limit definition is used to find the slope of a tangent at an arbitrary point. By syntactifying she has moved from working with a graphical representation to an analytic representation and in doing so has shifted to a more appropriate representation system for a proof. The completion of her argument is discussed in the subsequent section on rewarranting.

Rewarranting

Many informal arguments employ warrants that are not permissible in a proof. Rewarranting occurred when the participant tried to find a deductive reason for a claim that their informal argument justified in a non-deductive manner. In terms of Toulmin's scheme, we can regard rewarranting as replacing a *plausible warrant* (W) (i.e., a warrant that the participant believes is likely to yield truth) with a *valid warrant* ($W^{\oplus}$) (i.e., a warrant that the participant believes is considered valid by the mathematical community). This is illustrated in Figure 1b. This differs from syntactifying a warrant (W $\Rightarrow$W'), since this involves expressing W more formally but without changing its meaning. Below is the continuation of Student A's work on the odd/even problem from the previous section, syntactification.

Student A: Since *f* of *x* is even then *f* of negative *x* is equal to negative *x*. Now limit as *x* approaches *a* of tangent. I guess that is the right... consider *a*. Then *f* prime of [mumbling]. So *a* should be... then...is equal to some L. [writes $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=L$] so *f* prime of negative *a* equal but this is the same thing as *f* of *x* minus *f* of *a*. over *x* minus minus *a*. [$f'(-a)=\lim_{x\to a}\frac{f(x)-f(-a)}{x-(-a)}$] Which somehow equals negative L. *f* prime of negative *x* equals negative *f* of [writes silently $f'(-a)=\lim_{x\to a}\frac{f(x)-f(-a)}{x-(-a)}=\lim_{x\to a}\frac{f(x)-f(a)}{a-x}=-L$, so *f'(-x)=-f'(x)*].

In the above excerpt student A algebraically manipulates the limit definition of derivative at a point to show that *f'(-x)=-f'(x)*. This changes the nature of the warrant that links the data that *f(-x)=f(x)* to the claim that *f'(-x)=-f'(x)*. The warrant in the original informal argument based on the visual appearance of a graph. It is replaced here by a string of algebraic manipulations. The new warrant is not simply a translation of the previous warrant that leaves the meaning of the warrant unchanged; it is a different route to linking the data and claim.

Elaborating

Elaborating occurred when a participant attempted to add more detail to the proofs that were not present in their informal argument. This occurred in several ways: Participants would justify statements that they took for granted in their informal arguments by making explicit warrants that were initially implicit (W_i) in their informal arguments, or further justify their data (D) (i.e., the participant attempted to justify a fact that was taken for granted). We illustrate this in Figure 1c. The example below is of the first type, justifying claims initially taken for granted. It occurred during a participants work on the problem: Prove that $\int_{-a}^{a}\sin^3(x)dx=0$ for any real number *a*.

Student B: Um it [$sin^3(x)$] must be an odd function. […] Right it'll be symmetrical across the identity line, which would mean that the integral from negative *a* to zero should be the negation of zero to *a*. And so it would be zero.

In this excerpt the participant has an informal argument that $\int_{-a}^{a}\sin^3(x)dx = 0$. Notice that within this argument the assertion that $sin^3(x)$ is odd is treated as a known fact (data). Immediately following this informal explanation the participant begins to *elaborate* this argument by providing a justification for this assertion.

Student B: I'm trying to think how to show that sin of *x* cubed is odd. So basically I'd have to show that *f* of negative *x* has to equal negative *f* of *x*. Is that right... yes. So sin cubed of negative *x*... sine by definition is an odd function [writes *sin(-x)=-sin(x)*]. Uh Yeah. So sin cubed negative is equal to sin negative *x* times sin negative *x* which is equal to sin of *x* times sine of *x* times sin of *x*. Which is sin of *x* cubed. Quantity cubed. [writes: $sin^3(-x)=sin(-x)sin(-x)sin(-x)=(-sin(x))(-sin(x))(-sin(x))=-sin^3(x)$] So it's odd.

In the above excerpt, what was originally taken as data (D) in the argument (*sin(x)* is odd) is now taken to be the claim (C) of a new sub-argument. Student B shifts the starting point for the proof from $sin^3(x)$ is odd to *sin(x)* is odd, which is arguably more mathematically appropriate.

A student may also elaborate by replacing an implicit warrant in their informal argument with an explicit one in their formal proof. The following excerpt is taken from student C's work on the problem: Suppose *f(0)=f'(0)=1* and *f''(x)>0* for all positive *x*. Prove that *f(2)>2*).

Student C: If the second derivative is greater than zero then *f* prime of *x* is increasing.

So we know that *f* prime of zero equals one [draws: ]. So the derivative at zero equals one and the derivative is always increasing then the slope is greater than one after zero. Which means *f* of 1 is greater than one and *f* of 2 is greater than two. Well it makes sense.

In the above student C produced an informal argument that relied on a graph. Notice that he, among other things, argues that *f'(x)* is increasing and *f'(0)=1* (D) implies that *f'(x)>1* for *x>0* (C). The implicit warrant here is the definition of increasing. Later when he writes a formal proof this warrant is no longer implicit:

Student C: [saying what he writes] If *f* double prime of *x* greater than zero, then *f* prime *x* is increasing for all positive *x*. Thus for any *x* sub 1 comma *x* sub two in the interval zero to infinity such that *x* sub 2 is greater than *x* sub 1. *f* prime of *x* sub 2 is greater than *f* prime of *x* sub 1. *f* prime of zero equals one. Thus *f* prime of *x* sub 2 is greater than *f* prime of *x* sub 1 is greater than one. The derivative at any point greater than zero is greater than 1…

Notice that in his proof he explicitly uses the formal definition of increasing ($x_2>x_1 \Leftrightarrow f(x_2)>f(x_1)$), which was an implicit warrant in the informal argument. So elaboration has occurred. However, even though the proof involves taking smaller steps than the informal argument, the path the reasoning follows is unchanged.

Prevalence of these three activities

In Table 1, we present the frequency with which a participant attempted to engage in these activities as a function of whether or not they were able to successfully produce a proof. As Table 1 illustrates, participants who successfully produced proofs were considerably more likely to engage in syntactifying, rewarranting, and elaborating. Those who were successful in writing a proof usually engaged in all three activities, while those who were not successful rarely engaged in all three.

	Total	Syntactifying	Rewarranting	Elaborating	All three
Successful	14	12 (85%)	12 (85%)	11 (79%)	11 (79%)
Unsuccessful	23	15 (65%)	9 (39%)	12 (52%)	4 (17%)

Table 1: Attempted engagement with translation activities and success

Slicing the data another way, there were 15 instances in which a participant engaged in all three activities, and they succeeded in writing a proof 11 times (73% of the time). Among the 22 instances in which a participant did not engage in all three activities, the participants only succeeded in writing a proof three times (14% of the time); in two of those successful instances, the proofs produced differed substantially from the informal argument

It is important to note that Table 1 documented whether a participant *attempted to engage in the activity*, not if this engagement was successful. Consequently, we believe a key factor in determining success in proof writing for these participants was their willingness to try to syntactify, rewarrant, and elaborate.

DISCUSSION

The data in this paper contributes to the literature on bridging the gap between informal arguments and proofs. We highlighted three activities—syntactifying, rewarranting, and elaborating—that contribute to writing a proof based on an informal argument. Syntactifying is used to translate data, claims and/or warrants stated in terms of informal representations and natural language to the representation system of proof. If successful, this results in an argument that uses the appropriate representation system. Elaborating adds additional details to an argument by shifting the starting point of an argument to a more basic and widely accepted statement and making clear how new inferences were derived. Rewarranting seeks to replace plausible warrants with valid ones, changing the meaning of the argumentation into one more acceptable for proof.

We observed that there was a relative scarcity of informal arguments produced across this large data set (37 instances across 1022 proof attempts). In this respect, we support research into the design of instructional environments that encourages students to create proofs based on these informal arguments (e.g., Bossulini-Bussi et al., 2007). We also observed that participants who engaged in syntactifying, rewarranting, and elaborating once their informal arguments were produced enjoyed far greater success in proof-writing than those who did not. Consequently, we hypothesize that some of students' difficulties with bridging the gap between informal arguments and proofs is due to students' inability to successfully engage in these activities. Designing instruction that specifically targets these activities has the potential to improve mathematics majors' abilities to write proofs and would be a useful direction for future research.

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UNCOVERING TEACHER'S VIEWS VIA IMAGINED ROLE-PLAYING

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Role-playing is considered a valuable pedagogical strategy in a variety of fields. However, the use of this strategy in teacher education is underdeveloped. In this study we employ script-writing for a play (which we consider imagined role-playing) as a variation on a role-playing method. We invited teachers participating in a Master of Education professional development program to write an imagined dialogue with a colleague, in which they advocate for their teaching methods. The scripts reveal participants' ideas about teaching as well as their perceptions of nature of criticism and opposition to their ideas. The benefits of the script-writing method are discussed.

ON ROLE PLAYING

Role-playing is an unscripted "dramatic technique that encourages participants to improvise behaviors that illustrate expected actions of persons involved in defined situations" (Lowenstein, 2007, p. 173). In other words, role-playing is "an 'as-if' experiment in which the subject is asked to behave as if he [or she] were a particular person in a particular situation" (Aronson & Carlsmith, 1968, p. 26).

Role-playing is used as an effective pedagogical strategy in a variety of fields, a few of which we mention here. Traditionally it is used in social studies classrooms in order to provide participants with more authentic experiences of historic events and people who experienced them (e.g., Cruz & Murthy, 2006). It is used to explore the complexities of social situations, such as prejudice, and ethical issues (e.g., Lawson, McDonough, & Bodle, 2010; Plous, 2000). Participants, after engaging in role-playing, reported being better prepared to deal constructively with everyday instances of prejudice (Plous, 2000) and generated more effective responses to prejudiced comments (Lawson, McDonough, & Bodle, 2010).

Role-playing is also used in the education of various groups of professionals in organizational research, where, for example, participants assume roles of interviewers of job applicants or performance evaluators (Greenberg & Eskew, 1993). It is also prevalent in the training of health professionals, where the participants play the roles of a care-giver and a patient, practicing their clinical, diagnostic and patient managements skills, and as such developing empathy and tolerance in a low-risk environment (e.g., Joyner & Young, 2006). However, among various uses in developing professionals, the use of role-playing in teacher education is rather rare.

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 425-432. Vancouver, Canada: PME.

ON ROLE-PLAYING IN TEACHER EDUCATION

In considering role-play in teacher education, Van Ments (1983) described it as experiencing a problem under unfamiliar constraints, so that one's own ideas emerge and one's understanding increases. In this sense, role-playing can also be seen as role-training. It is aimed at increasing teachers' awareness of various aspects of their actual work. Despite the known advantages, role-playing in teacher education is underdeveloped. While some authors advocate for this method and report on its implementation, this is most often done in the form of self-reports and anecdotal evidence of participants' experiences. A few examples are below.

Kenworthy (1973) described a method in which one participant takes on a teacher-role while others take on the roles of various students (e.g., a slow student, a gifted student, a disturbing student). He considered this type of role-playing to be "one of the most profitable, provocative and productive methods in the education of social studies teachers" (p. 243). He claimed that engagement in role-playing activities helped participants anticipate difficulties they encounter in their classrooms and as such gain security in their successful experiences should they face similar situations on the job. In a similar fashion, with a particular focus on teaching mathematics, Lajoie and Maheux (2013) used role-playing in courses for prospective elementary school teachers, where participants improvise around mathematical tasks. They suggested that role-playing experience is instrumental in preparing teachers to deal with unpredictability of teaching situation.

ON SCRIPT WRITING

Despite the recognized advantages, time and participation logistics are a significant limitation of role-playing. If we intend to engage our students in role-playing during class time, only a few will be active players and the remainder will serve as an audience. To give all students the opportunity to participate in the role-playing scenario we turned to imagined role-playing, that is, writing a script for a dialogue between characters. We consider this to be imagined (rather than enacted) role-playing.

The use of script writing as an instructional tool has been implemented in prior mathematics education research. For example, Gholamazad (2007) developed the 'proof as dialogue' method. Prospective elementary school teachers participating in her study were asked to clarify statements of a given proof in elementary number theory by creating a dialogue, where one character had difficulty understanding the proof and another attempted to explain each claim. This method was amended and extended by Koichu and Zazkis (2013) and Zazkis (2013) in their work with prospective secondary school teachers. In both studies the participants had to identify problematic issues in the presented proofs and clarify those in a form of a dialogue, referred to as a proof-script. These scripts revealed participants' personal understandings of the mathematical concepts involved in the proofs as well as what they perceived as potential difficulties for their imagined students.

Additionally, the 'lesson play' method was developed and used in teacher education in which participants were asked to write a script for an imaginary interaction between a teacher-character and student-character(s) (Zazkis, Sinclair, & Liljedahl, 2013). 'Lesson play' was juxtaposed with the traditional 'lesson plan' and how the former may account for the deficiencies of the latter was outlined. The method was advocated as an effective tool in preparing for instruction, as a diagnostic tool for teacher educators, and as a window for researchers to studying a variety of issues in didactics and pedagogy. In this study we extend the script-writing method by using it to investigate experienced teachers perceptions of teaching.

THE STUDY

Participants in this study were practicing teachers in Master's of Education professional development program. Towards the end of the program they were asked to write scripts for an imagined conversation in which they explain and argue for their approach to teaching. The interlocutor in this conversation had to be either a colleague, a school principal or a concerned parent. In this paper we analyse the 14 scripts that were dialogues with a colleague.

The participants were provided with the particular setting for such a conversation and a starting prompt:

> It is 8:15 in the morning and you are busy preparing for your classes. A colleague comes to you room and says something like that: "Listen, I know you are doing your Master's and all. But have you thought about what this is doing for the kids?"

The task was to continue this conversation. The morning hour was chosen to keep the conversation rather focused, as 8:30 is the usual time when classes start. It also provided an opportunity to interrupt the conversation 'by the bell' without reaching a conclusion or an agreement, though only a few opted for this choice. The mention of 'kids' was also intentional in order to guide the conversation towards students' activity rather than general teaching strategies. Our analysis focuses on the needs of students that are attended to in the imagined dialogues.

THEORETICAL CONSIDERATIONS: STUDENTS' NEEDS

Sfard (2003) surveyed a variety of theoretical frameworks and identified ten needs of learners, according to these theories, that are "the driving force behind human learning and must be fulfilled if this learning is to be successful" (p. 357). These are: the need for meaning, the need for structure, the need for repetitive action, the need for difficulty, the need for significance and relevance, the need for social interaction, the need for verbal symbolic interaction, the need for a well-defined discourse, the need for belonging, and the need for balance. While the theories that Sfard considered were not specific to learning mathematics, she described how these various needs were featured in the NCTM standards. As such, we use categories identified by Sfard as a theoretical lens for our analysis.

In our analysis we identified the main themes that emerged in teachers' arguments as well as in the arguments of their imaginary interlocutor. We focused on how different intellectual needs of learners were featured in the scripts.

RESULTS AND ANALYSIS

Table 1 indicates what needs of students appeared in the scripts. We note significant overlap among various needs of students and acknowledge the difficulty in discussing them separately. Despite this, we identified the needs that are most evident in the teachers' descriptions of what they do or intend to achieve in their teaching (▯). We analysed each script individually and then compared the analyses and reconciled minor differences. Further, in each script we noted what appeared to be the most prevalent need of students to which the participants attended (¶¶).

Needs/ /Parti-cipant#	meaning	structure	repetitive action	difficulty	significance and relevance	social interaction	verbal symbolic interaction	well-defined discourse	belonging	balance
P#1	¶	¶		¶		¶¶	¶	¶		¶
P#2	¶¶	¶		¶				¶		
P#3	¶¶	¶			¶					¶
P#4	¶	¶		¶	¶	¶¶	¶			¶
P#5	¶¶	¶				¶	¶			¶
P#6	¶					¶¶	¶		¶	¶
P#7	¶	¶					¶			
P#8	¶			¶	¶¶	¶			¶	
P#9	¶	¶¶	¶	¶	¶	¶¶				
P#10	¶					¶				¶
P#11	¶¶			¶		¶				¶
P#12	¶	¶¶	¶			¶				¶
P#13			¶	¶	¶				¶	
P#14	¶		¶	¶		¶				¶

Table 1: Students' needs according to Sfard (2003) identified in the scripts

As is evident in Table 1, the need for meaning/causality and the need for communication or social interaction were featured in almost all the scripts and were of central importance to most script-writers. We exemplify below how these needs were described in the imaginary dialogues. While the participants often used their personal names in the dialogues, we refer to interlocutors anonymously as 'participant' and 'colleague'. While in some scripts one particular need was emphasized, we chose to present the next script because it attended to a variety of students' needs. It was written by a Grade 2 teacher (P4).

1 Colleague: I know you have been busy with your masters, and are very passionate about what you have been doing but have you thought about how confusing this new type of math might be to them, not to mention the fact that when they come to me next year the math will be totally different?

2 Participant: Well Tom I am sure it will be different […]. But I do question why you

would think it would be confusing for the kids?

3 Colleague: I know that the kids in your class do some math but in all honesty it isn't really the math that the rest of the staff is teaching. I don't actually see a lot of them sitting down and doing math like the way they would be doing it in my class and I'm quite concerned that you have not provided them with a strong foundation. […]

4 Participant: Well I do think learning math facts has its place, however we usually do this through math games or some kind of partner activity. Computation is just one part of the math in my classroom. It's important to try and integrate a variety of tasks that challenge students in the area of problem solving and being able to communicate their understanding in mathematics. I know it looks a bit chaotic and its noisy, but I do believe the students are building strong understanding in their mathematics.

5 Colleague: Math should be a quiet time. It should be a time for students to focus on solving problems without all this noise.

6 Participant: Quiet is not always good. It could mean that students are stuck and don't know what to do? I look at talking time in math as a time to share different ways to arrive at an answer. It's a chance for students to learn from each other. […] I try to differentiate instruction through open ended learning tasks. […] Not only are the problems designed for kids with different learning capabilities, but all the kids are helping each other. Sometimes it's not the final answer that's important, but rather the path they took to solve the problem.

7 Colleague: Well sometimes the beauty of math is they either have the right answer or they don't. Math needs to be quiet and kids need to be able to perform the operation correctly so they can solve the problem. Too much talking is a distraction.

8 Participant: It is true my kids are talkative during math, and yes we spend a lot of time on the floor, but that doesn't mean they are distracting others. By having them work on the floor with a friend kids can make connections and communicate with each other -it's a way of getting them to dig deeper into the math. […] They are trying to construct their knowledge so eventually they can move from the concrete to the symbolic. You might see that as play in grade one but it is important that students are able to show their understanding through those concrete materials. So what do you think my kids are doing on the floor, just out of curiosity? [...]

9 Colleague: Well like I said – it really doesn't look like math to me. It looks like they're having a good time, lots of talking.

10 Participant: (chuckle) Yes, you're right they are having fun.

11 Colleague: Well Math needs to be about learning Marie. We've got so much to cover. How can you spend so much time on group work?

12 Participant: […] Students are given a question but looking for different strategies to

		solve it. In their groups, they discuss the different strategies so they know there is more than one way to arrive at an answer. Then together in a gallery walk, the kids get to explain their thinking in numbers, pictures and words as to how they solved that problem. Not only have they shown it, they now have to explain it and its really incredible how some of them arrive at an answer. It is a great opportunity for me to see who really understands the task.
13	Colleague:	How am I supposed to keep track of what they actually learn or what they can do in my class at the end of the day? In my class, I have that workbook that I can look at. I choose a series of questions from the textbook so I can see all the questions and things they have solved. At the end of the day, what do you have to show as evidence as to what they have learned? You've had lots of great discussions again just going by what I see you've got these kids rolling on the carpet with their toys but they are not learning the importance of paying attention during a math lesson and they are not getting used to sitting in their desks quietly! They have all had a lot fun but have they really learned the math?
14	Participant	Well I have their completed work, I have their verbal explanation, I've got their group work mark… […] I'm not sure that the depth of their explanations would be as great if they simply do just worksheets. […] Will they just have to calculate as opposed to showing you? We have really worked hard as a staff at encouraging the use of concrete materials into math. Our math room has all these wonderful manipulatives. […]

We note here a strong connection between students' need for social interaction and the need for meaning making in mathematics ([4], [8], [14]). The script-writer emphasizes the connection between students' understanding and their ability to explain. Furthermore, this excerpt demonstrates attention to other needs of learners. The mention of differentiated instruction and the design of tasks to accommodate learners of different capabilities [6] is consistent with the need for balance. In addition, the need for balance is acknowledged, implicitly, in mentioning a variety of tasks [4] and a variety of solution strategies [6], [12]. The need for difficulty is seen in the reference to tasks that challenge students [4]. The need for structure appears in mentioning connections and in the move from concrete t symbolic [8]. Further, in the repeated mention of concrete materials [8], manipulatives [14], or pictures [12] we recognize this script-writer's attention to students' need for significance and relevance as well as for symbolic interaction.

We acknowledge that while the participant-characters featured in the scripts refer mostly to students' intellectual needs, their colleague-characters refer to the needs dictated by the system, that is, by their understanding of their job description. This includes, for example, classroom management [3], covering the curriculum [11], following the textbook and providing assessment [13]. We also note a significant disagreement between the characters with respect to their views of mathematics. These issues are explored in our subsequent analyses.

IMAGINED ROLE-PLAY: WHAT IS LOST, WHAT IS GAINED

Enhanced interaction among group members, enhanced skills in collaboration and communication, are often considered among the outcomes and benefits of role-playing (e.g., Jackson and Walters, 2000; Mogra, 2012). These outcomes are unlikely to result from writing an imagined dialogue, unless participants collaborate on creating a script. But with this loss there are overwhelming gains, as we discuss below. Role-playing, as an improvisational procedure, requires that the players have a feeling of relative safety. Many unpleasant experiences of participants in role-playing have been attributed to a teacher's failure to "warm up" the group where members learn to know and trust each other. In script-writing the safety concern is marginal as the play is confined to personal imagination.

An important goal that is attributed to role-playing is training professionals to "think on their feet" (Alkin & Christie, 2002). However, one does not necessarily have to think on his/her feet in order to be prepared for it. To the contrary, the script-writing avoids the necessity of an immediate response and as such provides an opportunity for a more-thoughtful and a more-balanced response, that can be redrafted and reconsidered, and – eventually – be relied upon when the opportunity to think on one's feed presents itself.

Many authors agree that simulations and imagined situations can induce learning. In particular, Blatner (2009) described role-playing as "a technology for intensifying and accelerating learning". We add to this that script writing invites a thoughtful and balanced response to an imagined situation and in such can induce learning even further. In addition, it provides researchers with a recorded account of one's imagined scenario from a perspective of both (or several) interlocutors.

In this study the scripts produced by participants demonstrated their views of teaching, in which their emphasis on students' intellectual needs as learners becomes apparent. They also demonstrated participants' perceptions of which traditional views they may be facing in their practice. Script writing appeared a useful way to exemplify and elaborate upon a potential struggle teachers who strive to improve their practice encounter.

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PRE-SERVICE ELEMENTARY TEACHERS' MISCONCEPTIONS OF PROOF AND COUNTEREXAMPLES AND THEIR POSSIBLE INFLUENCES ON THEIR INSTRUCTIONAL DECISIONS

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How might pre-service elementary teachers' misconceptions of proof and counterexamples influence their teaching of proof? To investigate this question, two types of interviews—task-based and scenario-based—were designed to elicit pre-service elementary teachers' (PSTs) conceptions of proof and counterexamples and how those conceptions might impact their instructional decisions. A qualitative analysis of the data revealed that these PSTs had difficulties following or constructing formally presented deductive arguments and understanding how deductive arguments differ from inductive arguments. The data also revealed that the misconceptions that pre-service teachers held played an important role in their instructional decisions.

INTRODUCTION

Proof is considered an essential aspect of mathematics and mathematical reasoning and proof have gained an increasing level of attention in recent attempts to reform mathematics teaching (CCSSM, 2010; NCTM, 2000). More notably, there is a call for an enhanced notion of proof that elevates proof beyond a topic of study in advanced mathematics courses to a tool for studying and learning mathematics at all levels (Stylianides & Ball, 2008). Thus, student understanding of proof should be extended through consistent opportunities to reason about why something is true, make and test conjectures, and build mathematical arguments. Engaging in reasoning and proof enables students to make sense of new ideas and to develop habits that will be of lifelong importance (Hanna, 2000; Martin & Harel, 1989). In order to create such an environment for students, teachers must themselves have a deep understanding of proof. The purpose of this study is twofold: To investigate elementary pre-service teachers' misconceptions of proof and counterexamples, and to examine whether these misconceptions impact their instructional decisions. This study investigates the following two questions: 1) What are pre-service elementary teachers' misconceptions of proof and counterexamples in mathematics classrooms? 2) Do pre-service elementary teachers' misconceptions of proof and counterexamples influence their teaching practices? If so, how?

FRAMEWORKS

Proof Scheme

A fruitful approach to understanding students' difficulties with proof has been to classify these approaches along several dimensions (Balacheff, 1988; Harel & Sowder,

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 433-440. Vancouver, Canada: PME.

1998). Researchers have hypothesized that the development of students' understanding of mathematical justification is likely to proceed from inductive to deductive or from particular cases toward greater generality (Harel & Sowder, 1998; Simon & Blume, 1996) and various proof schemes have been proposed. We reviewed the literature in order to develop a taxonomy for teachers' conception of proof. While many studies have focused primarily on distinctions between inductive and deductive justifications (Chazan, 1993; Martin & Harel, 1989), some researchers have divided inductive and deductive justifications into further subcategories (Balacheff, 1988; Harel & Sowder, 2007; Simon & Blume, 1996). We followed that approach.

The taxonomy of proof schemes, external, empirical, and analytical, proposed by Harel and Sowder (1998), is a fundamental framework for research on students' conceptions of proof. It encapsulates the major categories included in other taxonomies and proposes further sub-categories. However, it is evidenced in the literature that some students may not even need to provide a justification, they may fail to produce a deductive argument even if they start with some deductions, or they may use a particular example—generic example—to express their deductive reasoning (Balacheff, 1988; Simon & Blume, 1996). Since these students do not hold external, empirical, nor fully developed analytical proof schemes, it may be hard to classify these students' proof schemes using Harel and Sowder's taxonomy. We propose Level 0, Level 2, and Level 4, described in Table 1, to be added to Harel and Sowder's taxonomy in order to account for a broader spectrum of proof schemes.

Counterexamples

Zazkis and Chernoff (2008) argue that the existence of a counterexample should fit within an individual's proof scheme, therefore; what is convincing for one may not be convincing for others. They introduce the notions of pivotal and bridging examples to highlight the convincing power of counterexamples within an individual's example space. A pivotal example creates a turning point in the learner's cognitive perception, may introduce a conflict or may resolve it. A bridging example serves as a bridge from the learner's initial conceptions towards more appropriate mathematical conceptions. We use the notions of pivotal and bridging examples in our study of PSTs' conceptions of counterexamples.

METHOD

Participants

To select participants representing a broad spectrum in terms of knowledge and beliefs about proof, a proof questionnaire with open-ended questions was developed and administered to all students in one section of a geometry and measurement course and one section of a mathematics methods course at the beginning of the semester. After administering the questionnaire to all students in both courses, twelve PSTs, including five from the geometry course and seven from the methods course, were selected based on their responses so that there were participants displaying each of the following proof schemes: external, empirical, or deductive.

Categories	Characteristics of Categories	
	Subcategories	Characteristics of Subcategories
Level 0	Responses that do not address justification	
Level 1: External Proof Scheme	Responses appeal to external authority	
	(1) Authoritarian proof	Depends on an authority
	(2) Ritual proof	Depends on the appearance of the argument
	(3) Non-referential symbolic proof	Depends on some symbolic manipulation
Level 2: Naïve Reasoning	Responses usually with incorrect conclusions. Although, provers use some deduction, the arguments start with an analogy or with something that provers remember hearing, often incorrectly.	
Level 3: Empirical Proof Scheme	Responses appeal to empirical demonstrations, or rudimentary transformational frame	
	(1) Naïve Empiricism	An assertion is valid from a small number of cases
	(2) Crucial Empiricism	An assertion is valid from strategically chosen cases of examples
	(3) Perceptual Proof	An assertion is valid from inferences based on rudimentary mental images
Level 4: Generic Example	Responses expressed in terms of a particular instance (examples might be used to generalize the rules, but unlike an empirical proof scheme, the general rules are predicted based on deductive reasoning)	
Level 5: Analytic Proof Scheme	Responses appeal to rigorous and logical reasoning	
	(1) Transformational proof scheme	Involves goal-oriented operations on objects
	(2) Axiomatic proof scheme	Involves statements that do not require justification

Table 1: Taxonomy of proof scheme

Data Collection

The data was gathered in two distinct stages (Part I and Part II) with a different focus, and the primary sources of data were participants' semi-structured interviews. Part I and Part II interviews took place at the beginning and near the end of the semester, in order to detect any possible changes in PSTs' professed way of teaching proof that took place during the course. All twelve participants were interviewed individually, and interviews lasted approximately 60 minutes and were audio-recorded

Part I interviews focused on PSTs' (mis)conceptions of proof, including their way of producing proofs/counterexamples, as well as validations of different types of arguments ranging from empirical to formal. Thus, on PSTs as "knowers" of mathematics. During the interviews, participants were handed four tasks (five tasks during the post interviews), one at a time, and were asked to think out loud when determining the correctness of the tasks. For each task, participants were asked (1) to rate the level of their content understanding of the task using a four-point scale, (2) to determine whether the task is a correct statement or not, and (3) to rate the level of their confidence in terms of the validity of their evaluation using a four-point scale. Then, they were asked to produce a justification in cases where they believed the statements to be always true or to refute the statements where they believed the statements to be never true. After they provided an argument to justify or refute the statement and state their level of confidence in terms of the validity of their arguments, they were presented four brief arguments (five for the post interviews), varying in terms of level of justification; from empirical to deductive, one after the other, and asked to think out loud as they read each one, judge the correctness, and say to what extent each argument is convincing. Incorrect formally written arguments were added for each task for the post interviews. Finally, they were provided "Always," "Sometimes," "Never" cards and asked to assign the appropriate card to each argument presented as well as their own justification.

Part II interviews focused on the participants' usage of their conceptions that emerged from the analysis of Part I data. Part II interviews focused on pre-service elementary teachers as individuals who are going to be teachers of school mathematics. Knuth (2002) criticizes that research on teachers' conceptions of proof has tended to focus exclusively on teachers as individuals who are knowledgeable about mathematics rather than as teachers of school mathematics. Thus, in our study this stage focused primarily on PSTs' conceptions in the context of school mathematics. Participants' responses to questions about classroom scenarios and hypothetical students' questions were used to illuminate the process through which they would (1) validate proofs and counterexamples, (2) verify a statement's veracity, and (3) produce proofs and counterexamples as well as evaluate the validity of students' work. We also examined broader ideas and beliefs about how they plan to teach proofs in mathematics classrooms, including what types of arguments to incorporate in elementary classrooms.

RESULTS

Task-based interview results

The findings of this study outline a mixed picture of what constitutes proof and counterexample in the eyes of those twelve pre-service elementary teachers. The arguments that the participants constructed to justify the statements as well as the arguments presented to the participants after each task were coded according to the frameworks explained above. We now present our findings.

When asked to define proof, it was clear that pre-service teachers had some experience with proof and were using this to inform their judgments about what constituted a good proof. They had experience of seeing a proof being performed and were quoting these as examples of what was required. However, despite their experience seeing proofs in their classrooms, the majority of the participants failed to produce and/or recognize a proof. For instance, when given Task A—A kite is a quadrilateral with two distinct pairs of adjacent sides that are equal. Given this definition, justify whether or not the following statement is true. "*In a kite, one pair of opposite angles is the same*."—Only three out of seven students from the methods course were able to reproduce the proof that they learned in their previous geometry course correctly. Four students attempted to use triangle congruency to prove the statement as they learned in their geometry course. However, they either started with incorrect assumptions, such as trying to prove the wrong pair of angles as congruent, or they used incorrect reasoning to reach a correct conclusion. Only one out of 5 students from the geometry course was able to construct an argument that was coded as a deductive argument. The other four students came up with empirical arguments to justify the statement.

Not surprisingly, empirical approaches were by far the most common strategy employed by participants. Seven out of twelve students who participated in the study found empirical arguments as sufficient proof. Overall, pre-service teachers who were using an empirical approach to justify the statements recognized that they needed to test multiple examples. However, we should also note here that the participants tended to test fewer examples when they were familiar with the statement or the statement was initially believed to be true.

The fact that a generalization is found to be true in some cases does not guarantee − and thus does not prove − that it is true for all possible cases is a fundamental distinction between empirical and deductive arguments. However, we found that this distinction was not clear to the participants who constructed empirical arguments or found empirical arguments sufficient to prove. This is a fundamental difference between an empirical argument and the notion of proof in mathematics (Stylianides, 2007) and we believe it is necessary to learn it in order to move from an empirical proof scheme to a deductive proof scheme. We also found that some pre-service teachers failed to recognize that a proof always holds true.

Moreover, if participants could not make the distinction between empirical and deductive arguments, they tended not to recognize incorrect reasoning presented in formally written arguments and claimed that the argument would suffice as a proof. Similarly, some of the participants claimed that a counterexample could be found even after a proof was presented. In other words, some of the participants seemed to believe that a proof and a counterexample could exist for the same situation.

The participants also demonstrated various misconceptions refuting wrong mathematical statements, for example the belief that providing more counterexamples would make an argument more convincing. We also found that a counterexample, when presented to or created by the learner, may not create a cognitive conflict or result in refuting the statement. Instead, it may be simply dismissed or treated as an exception and as a result the need of seeing more counterexample may occur.

Scenario-based interview results

In the scenario-based interviews, it was evident that the misconceptions described above played an important role when the pre-service teachers evaluated the classroom scenarios. We found that PSTs' decisions of whether an argument was a proof were influenced by the context, and PSTs'conceptions of proof differed when they switched from discussing proof from their own perspective to examining proof in the context of evaluating student work. We believe that this speaks to deep theoretical and practical concerns. The participants demonstrated the tendency of accepting empirical arguments as sufficient proofs in the context of elementary school, even if they did not display an empirical level of thinking about proofs.

Watson and Mason (2005) argued that examples could be seen as instances of a more general class or objects. In this study, PSTs treated examples as representation of a bigger class. In other words, the majority of the participants stated the importance of providing examples of different types to justify a statement in order to ensure the generality of the justification, thus, highlighting the importance of example space.

If students view proof as sufficient evidence to support a conjecture, one would expect the students' reasoning to end after generating a valid proof. While this was the case for the majority of the PSTs, some tested examples after generating/seeing a proof. It should also be noted that the majority of the PSTs stated that providing additional empirical checks could be helpful for students to better understand the proof and/or statement. Thus, almost all participants claimed that additional empirical checks were necessary. We interpret this finding in two possible ways: as a result of the conversation between the interviewer and the participant or it can be considered to be evidence that the students were not convinced by the generality of proofs.

CONCLUSION AND DISCUSSION

Despite the growing emphasis on justifying and proving in school mathematics, a large body of research shows that students of all levels of experience use empirical arguments to prove statements in mathematics and/or they accept empirical arguments as valid proofs and that many students fail to understand the nature of what counts as evidence and justification. We found confirmation for these results as the majority of the participants in our study failed to recognize that testing examples is not sufficient for proof.

Several researches have focused on why many students possess these invalid proof techniques. Recio and Godino (2001) note that many such invalid proof techniques would be appropriate in non-mathematical domains. Reid and Knipping (2010) observe that reasoning about a concept using a prototypical example is common in our everyday experience. In this study, it was evident that some participants were over generalizing what they learned in other courses to mathematics. We believe that unless pre-service teachers realize the limitations of empirical arguments as methods for validating generalizations, they are unlikely to appreciate the importance of proof in mathematics (Stylianides & Stylianides, 2009). In order to achieve this learning objective, however, teachers must have good knowledge in the area of proof, for the quality of learning opportunities that students receive in classrooms depends on the quality of their teachers' knowledge (Ball, Thames, & Phelps, 2008).

Elementary teaching practices that promote or tolerate a conception of proof as an empirical argument may instill mental habits in students that significantly deviate from conventional mathematical understanding in the field. Martin and Harel (1989) state that if elementary teachers lead their students to believe that a few well-chosen examples constitute a proof, it is natural to expect that the idea of proof in high school geometry and other courses will be difficult for the students (pp. 41-42). It was clear in this study that those PSTs tend to believe that empirical arguments could be tolerated as proofs in elementary levels while they cannot be accepted as proofs in higher grade levels. Additionally, we found that the distinction between empirical arguments and deductive arguments was not clear for many of the participants. Thus we argue that unless teachers at all levels of schooling develop a good understanding of this distinction, it is unlikely that large numbers of students will overcome their misconception that empirical arguments are proofs.

There has been relatively little attention paid to the way PSTs conceptions of proof may depend on the particular context in which proof is being utilized. The results in this study indicate that this is an area worthy of further investigation as teachers' conceptions of proof in the context of teaching may be, and perhaps should be, different from the way they engage with proof in other settings.

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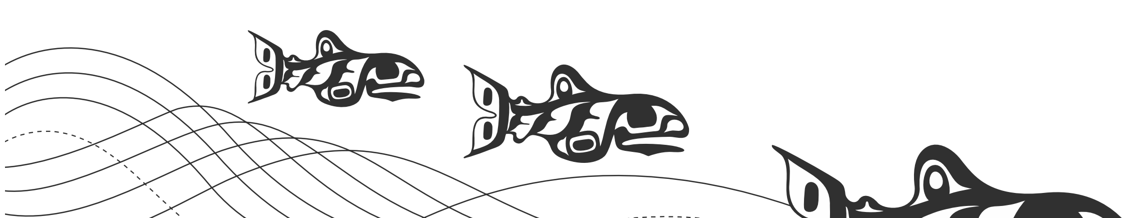

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